Chapter 6
Motion of a Rigid Body About a Fixed Point in the Field of a Distant Newtonian Centre and Brun's Problem

In the preceding chapters, we concentrated on the study of motion of a rigid body about a fixed point in the uniform gravity field. This field serves also as a good approximation for the forces acting on the body in most purposes. However, for certain applications, like precise surveying instruments, the slight variations in the Earth's gravitational field from point to another must be taken into consideration. These variations play a decisive role in the determination of rotational motion of artificial satellites of Earth, especially those whose tasks demand high precision of orientation, for terrestrial, cosmic or astronomical purposes.

In the present chapter, we study in detail another approximate model for one rigid body moving in the gravitational field of a fixed body. In this model, we make two assumptions:

(a) The fixed body is spherically symmetric. This enables us to treat it as a point mass concentrated at the centre of the body.

(b) The diameter of the moving body is very small compared to the distance between the fixed point of the moving body and the centre of the fixed body.

Let a body of mass M be in motion about its fixed point O at a distance R from a fixed Newtonian attraction centre of mass M' at O'. Denote by $\boldsymbol{\gamma}$ the unit vector in the direction $\overrightarrow{O'O}$ and by V the potential of the body in the field of the centre. To calculate the moment of gravitational forces about O, we take a mass element dm of the body whose position vector referred to O is $\mathbf{r}$:

The potential of this element will be $dV = -\frac{\mu dm}{|R\boldsymbol{\gamma}+\mathbf{r}|}$, the force exerted on it by the attraction centre is $d\mathbf{F} = -\frac{\mu(R\boldsymbol{\gamma}+\mathbf{r})dm}{|R\boldsymbol{\gamma}+\mathbf{r}|^3}$ and its moment about O is $d\mathbf{L} = -\frac{\mu R\boldsymbol{\gamma}\times\mathbf{r}dm}{|R\boldsymbol{\gamma}+\mathbf{r}|^3}$, where μ is Gauss' constant of the centre ($\mu = M' \times$Newton's gravitational constant). Integrating over the whole mass of the body, we get

$$V = -\mu \int \frac{dm}{|R\boldsymbol{\gamma}+\mathbf{r}|}, \tag{6.1}$$

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$$\mathbf{L} = -\mu R\gamma \times \int \frac{\mathbf{r}dm}{|R\gamma+\mathbf{r}|^3} \tag{6.2}$$

and recalling that $|R\gamma+\mathbf{r}| = \sqrt{R^2 + 2R\gamma\cdot\mathbf{r}+r^2}$, one can write (2) in the form

$$\mathbf{L} = \gamma\times\frac{\partial V}{\partial\gamma}. \tag{6.3}$$

6.1 Approximate Form of the Potential

It is evident from (6.3) that the knowledge of the potential V is sufficient for complete determination of the moment $\mathbf{L}$. However, in most cases it is difficult to calculate V from (6.1) due to the complexity of the shape of the body, its mass density or because the mass distribution in the body is unknown. In those cases, an efficient solution is to expand the potential in powers of $\frac{1}{R}$ and keeping terms up to the third degree in this parameter. First we write the expansion of the function

$$\begin{aligned}
\frac{1}{\sqrt{R^2 + 2R\gamma\cdot\mathbf{r}+r^2}} &= \frac{1}{R}[1 + \frac{2\gamma\cdot\mathbf{r}}{R} + \frac{r^2}{R^2}]^{-\frac{1}{2}} \\
&= \frac{1}{R}[1 - \frac{1}{2}(\frac{2\gamma\cdot\mathbf{r}}{R} + \frac{r^2}{R^2}) + \frac{3}{8}(\frac{2\gamma\cdot\mathbf{r}}{R} + \frac{r^2}{R^2})^2 + o(\frac{1}{R^3})] \\
&= \frac{1}{R} - \frac{\gamma\cdot\mathbf{r}}{R^2} - \frac{r^2}{2R^3} + \frac{3}{2R^3}(\gamma\cdot\mathbf{r})^2 + o(\frac{1}{R^4}).
\end{aligned}$$

Inserting this expression into (6.1), and neglecting terms of degrees higher than the third, we obtain

$$\begin{aligned}
V &= -\mu[\frac{1}{R}\int dm - \frac{1}{R^2}\gamma\cdot\int \mathbf{r}dm - \frac{1}{2R^3}\int r^2 dm + \frac{3}{2R^3}\int(\gamma\cdot\mathbf{r})^2 dm] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(3\sum\gamma_i\gamma_j\int x_i x_j dm - \frac{1}{2}tr(\mathbf{I}))] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(3\gamma\bar{\mathbf{I}}\cdot\gamma - \frac{1}{2}tr(\mathbf{I}))] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(3\gamma(\frac{1}{2}tr(\mathbf{I})\delta - \mathbf{I})\cdot\gamma - \frac{1}{2}tr(\mathbf{I}))] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(tr(\mathbf{I}) - 3\gamma\mathbf{I}\cdot\gamma)].
\end{aligned} \tag{6.4}$$

In expanded form that is

$$V = -\frac{\mu M}{R} + \frac{\mu M}{R^2}(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3)$$

$$-\frac{\mu}{2R^3}tr(\mathbf{I}) + \frac{3\mu}{2R^3}(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2). \tag{6.5}$$

As from (6.3) only derivatives of V with respect to components of $\boldsymbol{\gamma}$, the constant terms may be discarded and thus the potential can be written as

$$\begin{aligned} V &= Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) + \frac{1}{2}\lambda(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) \\ &= Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} + \frac{1}{2}\lambda\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma}, \end{aligned} \tag{6.6}$$

where g is the gravity field intensity of the centre at the fixed point O of the body and $\lambda = \frac{3g}{R}$. Formula (6.6) gives the approximate form of the potential of the body by the knowledge of its centre of mass and moments of inertia, without need to completely specify the distribution of mass in the body.

Remark: Note that for a Newtonian centre of attraction both parameters g and λ are positive. In certain physical problems, the Newtonian attraction is replaced by Coulomb's electric interaction (e.g. [58]). In that case g and λ can be either positive or negative and $\mathbf{I}, Mg\mathbf{r}_0$ are replaced, respectively, by the inertia matrix of the electric charges and the moment of those charges multiplied by the intensity of the electric field of the central charge at O.

6.2 Brun's Problem

Brun considered the motion of the following model [47]:

Let a rigid body be in motion about a fixed point O, while each of its mass elements is influenced by a force proportional to its mass and its distance from a fixed plane at O in the direction perpendicular to that plane. In the usual notation, the axes $Oxyz$ are taken as the system of principal axes of the body at the fixed point and $OXYZ$ as the inertial frame, with the Z-axis orthogonal to the fixed plane and the unit vector $\boldsymbol{\gamma}$ along the Z direction.

The potential of the body

$$\begin{aligned} V &= \frac{1}{2}N\int Z^2 dm = \frac{1}{2}N\int (\mathbf{r}\cdot\boldsymbol{\gamma})^2 dm \\ &= \frac{1}{2}N\boldsymbol{\gamma}\bar{\mathbf{I}}\cdot\boldsymbol{\gamma} = -\frac{1}{2}N\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma} + const \end{aligned} \tag{6.7}$$

where N is proportionality constant. This potential is just the quadratic part of the potential (6.6), with λ replaced by $-N$.

Thus, Brun's problem is a special version of the general problem of motion of a rigid body in an approximate Newtonian field, namely, the case when the body is fixed from its centre of mass.

6.3 Equations of Motion and Integrals of Motion

Taking (6.6) into account, one can write the equations of motion of the body in the approximate field of a Newtonian centre of attraction in the form

$$\begin{aligned} &\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I} = \boldsymbol{\gamma}\times(Mg\mathbf{r}_0+\lambda\boldsymbol{\gamma}\mathbf{I}),\\ &\dot{\boldsymbol{\gamma}}+\boldsymbol{\omega}\times\boldsymbol{\gamma}=\mathbf{0}. \end{aligned} \tag{6.8}$$

Or in expanded form

$$\begin{aligned} A\dot{p} + (C-B)(qr-\lambda\gamma_2\gamma_3) &= Mg(z_0\gamma_2 - y_0\gamma_3),\\ B\dot{q} + (A-C)\,(pr-\lambda\gamma_1\gamma_3) &= Mg(x_0\gamma_3 - z_0\gamma_1),\\ C\dot{r} + (B-A)(pq-\lambda\gamma_1\gamma_2) &= Mg(y_0\gamma_1 - x_0\gamma_2), \end{aligned}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0,\, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{6.9}$$

This is a closed system of six first-order differential equations. For this system, we have three obvious general integrals of motion. They are the energy, areas and geometric integrals

$$\begin{aligned} &I_1 = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega}+Mg\mathbf{r}_0\cdot\boldsymbol{\gamma}+\frac{1}{2}\lambda\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}=h,\\ &I_2=\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\gamma}=f,\\ &I_3=\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}=\mathbf{1}. \end{aligned} \tag{6.10}$$

6.4 Integrable Cases

As will be seen later in this book, the problem under consideration is in fact a special version of the much more general problem considered in Chap. 10, dealing with the motion of a body in a liquid or a magnetized and electrically charged body about a fixed point. However, in virtue of the importance of this version in applications, we present here its integrable cases. For the system (6.9) to be integrable, just as in the problem of motion in the uniform gravity field, we have to find a fourth integral independent of those three. It turned out that this can be achieved in two cases.

6.4.1 Brun's Case [47] (Analog of Euler's Case)

In this case $\mathbf{r}_0 = 0$, i.e. the body is fixed from its centre of mass. To obtain the complementary integral, we write the Euler–Poisson equation

$$\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I}=\lambda\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I}$$

multiplying scalarly by $\boldsymbol{\omega}\mathbf{I}$ to get

$$\boldsymbol{\omega}\mathbf{I}\cdot\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I})=\lambda\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I}).$$

That is

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2&=-\lambda\boldsymbol{\gamma}\cdot(\boldsymbol{\omega}\mathbf{I}\times\boldsymbol{\gamma}\mathbf{I})\\&=-\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot(\boldsymbol{\omega}\times\boldsymbol{\gamma}).\end{aligned}\tag{6.11}$$

The last relation can be verified very easily by writing the triple scalar product in the form of a determinant and using the determinant's properties. Using Poisson's equation in this relation, we write

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2&=\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\dot{\boldsymbol{\gamma}}\\&=\frac{d}{dt}\frac{1}{2}\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}\end{aligned}$$

and hence finally we get

$$|\boldsymbol{\omega}\mathbf{I}|^2-\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}=c,\tag{6.12}$$

or in expanded form

$$I_4=A^2p^2+B^2q^2+C^2r^2-\lambda ABC\left(\frac{\gamma_1^2}{A}+\frac{\gamma_2^2}{B}+\frac{\gamma_3^2}{C}\right)=c.\tag{6.13}$$

This integral is an obvious generalization of the fourth integral in Euler's case (the square of the modulus of the angular momentum). Nevertheless, complete solution of the equations of motion (6.9) for Brun's case with the integrals (6.10) and (6.13) is much more complicated than in Euler's case. Kobb [223] expressed the Hamiltonian of the problem using Euler's angles as generalized coordinates and the momenta conjugate to them. By writing Hamilton–Jacobi equation and constructing a complete solution for it, he reduced the solution to certain quadratures, but he has not tried to solve those quadratures explicitly for the coordinates in terms of time. A similar approach was adopted by Kharlamova in [204] using different coordinates, the sphero-conic coordinates on the Poisson sphere (See Chap. 9).

6.4.2 The Generalization of Lagrange's Case

Let $A = B$ and $x_0 = y_0 = 0$, i.e. the body admits axial dynamical symmetry about the z-axis passing through the fixed point and the centre of mass of the body lies on the axis of dynamical symmetry. Under those conditions, it is easy to write the third equation of (6.9) in the form

$$C\dot{r} = 0$$

and thus the integral is the same as in Lagrange's case

$$I_4 = r = r_0. \tag{6.14}$$

To obtain the solution of the equations of motion, one can proceed exactly as in Lagrange's case in Chap. 4 (Sect. 4.2). The relation between γ_3 and time t has the same form as in (4.42)

$$t = \int \frac{d\gamma_3}{\sqrt{F(\gamma_3)}}, \tag{6.15}$$

but with

$$F(\gamma_3) = (1-\gamma_3^2)(E - a\gamma_3 - a_1\gamma_3^2) - (b - cr_0\gamma_3)^2, \tag{6.16}$$

i.e. $F(\gamma_3)$ is here a polynomial of the fourth degree. Thus, γ_3 is expressible in terms of elliptic functions of time. This procedure was pointed out in [250], where the Eulerian angles are given expressions in terms of Weierstrass' elliptic functions.

Arkhangelsky [14] proved that the equations of motion (6.8) do not admit a complementary single-valued integral in any more cases than the above two. Note also that the case of axially symmetric body can be readily generalized by adding a gyrostatic moment, i.e. a rotor along the axis of symmetry of the body. The integral I_4 becomes $Cr + k_3$, i.e. $I_4 = r$ is still a constant of the motion.

6.4.3 The Place of Brun's Potential

At present we know very little about the answer to an important question: for which potentials are the Euler–Poisson equations

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I} = \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}},\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0} \tag{6.17}$$

integrable? Even for the simplest case of a quadratic integral we know a single partial result, obtained in [381] using reduced equations in isometric coordinates on the inertia ellipsoid. We deduce it here in a direct way from Euler–Poisson's equations. We formulate it as

Theorem 6.1 *Equation (6.17) is integrable with a quadratic complementary integral for arbitrary A, B, C, and for all initial conditions only for the potential*

$$V = \frac{1}{2}\lambda(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\ \ \lambda \text{ arbitrary constant} \tag{6.18}$$

and their integral is

$$I_4 = A^2p^2 + B^2q^2 + C^2r^2 - \lambda(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2). \tag{6.19}$$

Proof Let for a general potential $V(\boldsymbol{\gamma})$ and $\boldsymbol{\mu} = \mathbf{0}$ the equations of motion have an integral quadratic in the angular velocities. This integral must have the form

$$I_4 = A^2p^2 + B^2q^2 + C^2r^2 + F(\gamma_1, \gamma_2, \gamma_3), \tag{6.20}$$

so that $F \equiv 0$ when $V \equiv 0$. Differentiating this integral we get

$$2\boldsymbol{\omega}\mathbf{I}\cdot\dot{\boldsymbol{\omega}}\mathbf{I} + \frac{\partial F}{\partial \boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} = \mathbf{0},$$

and using (6.17) this becomes

$$2\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I}) + \frac{\partial F}{\partial \boldsymbol{\gamma}} \cdot (-\boldsymbol{\omega}\times\boldsymbol{\gamma}) = \mathbf{0},$$

and finally we obtain the relation

$$2\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}}) - \boldsymbol{\omega} \cdot (\boldsymbol{\gamma}\times\frac{\partial F}{\partial \boldsymbol{\gamma}}) = \mathbf{0}, \tag{6.21}$$

which is satisfied for arbitrary vector $\boldsymbol{\omega}$.

Now we replace $\boldsymbol{\omega}$ by $\boldsymbol{\gamma}$ in Eq. (6.21), to obtain an equation for the potential V

$$2\boldsymbol{\gamma}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}}) = 0,$$

whose solution is readily

$$V = V(\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}, \boldsymbol{\gamma}^2).$$

Since we have $\boldsymbol{\gamma}^2 = 1$, the last expression can be written as

$$V = V(\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}). \tag{6.22}$$

In a similar way, we replace $\boldsymbol{\omega}$ by $\boldsymbol{\gamma}\mathbf{I}^{-1}$ in Eq. (6.21), to obtain the equation for F

$$\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot (\boldsymbol{\gamma}\times\frac{\partial F}{\partial \boldsymbol{\gamma}}) = \mathbf{0}.$$

It follows that

$$F = F(\gamma\mathbf{I}^{-1}\cdot\gamma). \tag{6.23}$$

Now, inserting (6.22) and (6.23) into (6.21), one gets

$$4V'(\gamma\mathbf{I}\cdot\gamma)\boldsymbol{\omega}\mathbf{I}\cdot(\gamma \times \gamma\mathbf{I}) - 2F'(\gamma\mathbf{I}^{-1}\cdot\gamma)\boldsymbol{\omega} \cdot (\gamma \times \gamma\mathbf{I}^{-1}) = \mathbf{0}. \tag{6.24}$$

Comparing the structures of the two terms in the last equation and noting that $\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma}$ and $\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot \boldsymbol{\gamma}$ are independent functions for a tri-axial body ($A \neq B \neq C$), The two derivatives V' and F' must be constants. Neglecting an insignificant constant in each of V and F, we write

$$V = \frac{1}{2}\lambda(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\ F = N_1(\gamma_1^2/A + \gamma_2^2/B + \gamma_3^2/C). \tag{6.25}$$

Again, inserting the last expressions into (6.24), we obtain

$$\lambda\gamma \cdot (\boldsymbol{\omega}\mathbf{I} \times \boldsymbol{\gamma}\mathbf{I}) + N_1\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot (\boldsymbol{\omega} \times \gamma) = \mathbf{0}.$$

Applying the identity (A.2) in Appendix A, we rewrite the last relation as

$$(\lambda ABC + N_1)\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot (\boldsymbol{\omega} \times \gamma) = \mathbf{0}$$

and thus we obtain

$$N_1 = -\lambda ABC.$$

This completes the determination of the integral as in (6.13). □

6.5 Exercises

1- In the classical problem of motion of a rigid body about a fixed point in the constant uniform gravity field two equilibrium positions are possible. In the problem of motion in approximate Newtonian field, show that in any equilibrium position one of the generators of Ampère's cone (Staude's cone) must be vertical (passes through the fixed point and the centre of attraction).

2- Noting that λ is positive in (6.8) for a centre of attraction, show that the vertical generator of Staude's cone in an equilibrium position must be one of the axes inadmissible for Staude's rotation in the uniform gravity field. [Compare the equations of equilibrium to equations of Staude's rotation and note that ω^2 in Staude's rotation is replaced by $-\lambda$]

3- For a body fixed from its centre of mass in a uniform gravity field, any position is a possible equilibrium position. Show that for a body fixed from its centre of mass in the approximate field of a Newtonian centre there are only six equilibrium positions and find them.

4- For a body fixed from its centre of mass, show that a uniform rotation is possible only in two cases:

(a) The rotation about a principal axis with arbitrary angular velocity.
(b) The rotation with angular velocity $\pm\sqrt{\lambda}$ about an arbitrary axis.

5- Show that all the axes of uniform rotation of the body in approximate field of a Newtonian centre are generators in Ampère's cone and that possible axes of Staude's rotation constitute a subset of possible axes of uniform rotation in the Newtonian field.

6- Using the terminology of the present chapter, **F** is the exact resultant force exerted on the body by the centre of attraction. Show that

(a) $\mathbf{F} = -\mu \int \frac{(R\gamma+\mathbf{r})}{|R\gamma+\mathbf{r}|^3} dm$,

(b) The component of this force in the direction of γ is $\mathbf{F}\cdot\gamma = -\frac{\partial V}{\partial R}$, V being the exact potential ($V = -\mu \int \frac{dm}{|R\gamma+\mathbf{r}|}$).

(c) The resultant force **F** can be written in terms of the potential in the form

$$\mathbf{F} = -\frac{\partial V}{\partial R}\gamma + \frac{1}{R}\gamma \times (\gamma \times \frac{\partial V}{\partial \gamma}).$$

Note that $\mathbf{F} \cdot \mathbf{L} = 0$, which agrees with the fact that the resultant attraction must be a single force passing through the centre of attraction whatever be the position of the body.

(d) The magnitude of the resultant force **F** is

$$|\mathbf{F}| = \sqrt{(\frac{\partial V}{\partial R})^2 + \frac{1}{R^2}|\gamma \times \frac{\partial V}{\partial \gamma}|^2}.$$

(e) The reaction of the fixed point on the body at any equilibrium position must be vertical.

7- A rigid body is fixed from a point and its centre of mass lies on a principal plane for that point, say, the xy-plane. The body is acted upon by the gravitational force due to a distant Newtonian attraction. Show that the body can perform a plane pendulum-like motion about a horizontal axis, described by the conditions $p = q = \gamma_3 = 0$, and determine the variables r, γ_1, γ_2 as functions of time.

8- Brun's model problem is modified, so that each mass element of the body is influenced by a force proportional to its mass and its distance from an arbitrary fixed plane and in the direction perpendicular to that plane. Show that the potential is equivalent to the general potential (6.6).

9- Investigate the stability of the uniform rotation of a body fixed from its centre of mass O in the approximate Newtonian field of a far centre at O', determined in Exercise 4-a. Show that the rotation with angular speed r_0 about the z-axis, which is directed to pass through the centre of attraction and the fixed point of the body, is

(a) unstable when the z-axis is the middle principal axis of inertia of the body at O,

(b) stable for all values of r_0, when C is the least principal moment of inertia, and
(c) when C is the largest principal moment of inertia, the rotation is unstable if

$$r_0 < \sqrt{\frac{3g}{R}} \frac{\sqrt{a} + \sqrt{b}}{1 + \sqrt{ab}},$$

and stable if

$$r_0 > \sqrt{\frac{3g}{R}} \frac{\sqrt{a} + \sqrt{b}}{1 + \sqrt{ab}},$$

where $a = \frac{C-B}{A}$, $b = \frac{C-A}{B}$, g is the acceleration of gravity at O and $R = |\overrightarrow{O'O}|$.

[Beletsky [20]. See also [256], Sect. 18.4]