Chapter 5
The Motion of a Heavy Gyrostat

Originally, the gyrostat, as the terminology was coined by Lord Kelvin, is a heavy rigid body with a rotor or a fly-wheel spinning with a constant angular speed about its axis of symmetry. The subject gained a great interest at the first two decades of the twentieth century. Examples are the two books by Crabtree [59] (1909) and Gray [133] (1918), devoted exclusively to describing gyroscopic phenomena, specially the stabilizing effects of rotors, and the ways to make use of them in warfare of World War I. Today, gyroscopic apparatuses are indispensable in cell phones, in so many applications in terrestrial and cosmic navigation and in technology. Most useful is the stabilizing effect of fast rotors on normally unstable motions and equilibria.

In this chapter, different types of mechanical systems having the same equations of motion as the gyrostat are presented. General and conditional integrable cases of motion are presented. In fact, these are generalizations of the relevant integrable cases in the classical problem, and reduce to them when the gyrostatic momentum vanishes.

At present, several particular solutions to the problem of motion of a gyrostat are known, namely, 13 solutions. Some of them are generalizations of classical counterparts by adding a gyrostatic momentum. Other cases lose their meaning when the gyrostatic moment vanishes.

5.1 Models of the Gyrostat

5.1.1 The Classical Model

Consider a system S, composed of two joint rigid bodies. The first, S_0, which we shall call the carrier (or the main) body, is fixed in the inertial space from its point

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O. The second body, the rotor S_1, is an axially symmetric body fixed from its axis of symmetry in the main body. Its centre of mass, O_1, lies on its axis of symmetry. Usually, such a symmetric body is called gyroscope. Because of the symmetry of the rotor, its rotation does not change the distribution of mass in the system. Let $\mathbf{I}$ and $\mathbf{r}_0$ be the inertia matrix and the position vector of the centre of mass of the system, referred to the system of axes $Oxyz$ fixed in the main body. Let also $\mathbf{J}$ be the inertia matrix of the rotor with respect to a system of axes $O_1x_1y_1z_1$ fixed in it with z_1 along its axis of symmetry. From symmetry, it is clear that $O_1x_1y_1z_1$ is a system of principal axes of the rotor and hence we can write $\mathbf{J} = \mathrm{diag}(J_1, J_1, J)$.

Let the rotor be set and kept in motion about its axis with a constant angular velocity Ω, by means of some device. Let $\mathbf{r}_1 = \overrightarrow{OO_1}$ and denote by $\mathbf{r}$ the position vector of a mass element dm of the system with respect to O. The velocity of that element is $\boldsymbol{\omega} \times \mathbf{r}$ if it belongs to S_0 and $\boldsymbol{\omega}\times\mathbf{r}+ \Omega\mathbf{e} \times \mathbf{r}'$ for elements of S_1, where $\mathbf{r}'$ is the position vector of the mass element of the rotor with respect to O_1. The angular momentum of the system can be written as

$$\begin{aligned}
\mathbf{G} &= \int_{S_0} \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r})dm + \int_{S_1} \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r}+\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \int_{S_0} \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r})dm + \int_{S_1} \mathbf{r} \times (\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \boldsymbol{\omega}\mathbf{I}+ \int_{S_1} (\mathbf{r}_1 + \mathbf{r}')\times(\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \boldsymbol{\omega}\mathbf{I} + \mathbf{r}_1\times(\Omega\mathbf{e} \times \int_{S_1} \mathbf{r}'dm) + \int_{S_1} \mathbf{r}'\times(\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \boldsymbol{\omega}\mathbf{I} + \mathbf{0}+\Omega\mathbf{eJ} \\
&= \boldsymbol{\omega}\mathbf{I}+\Omega J\mathbf{e}.
\end{aligned} \tag{5.1}$$

Here we have used $\int_{S_1} \mathbf{r}'dm = \mathbf{0}$. The last expression will be written as

$$\mathbf{G} = \boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}, \tag{5.2}$$

where $\boldsymbol{\kappa} = \Omega J\mathbf{e}$ is the gyrostatic momentum, the angular momentum of the rotor relative to the carrier body. It is directed along the axis of symmetry of the rotor.

Now we write down the equation of motion of the system. The mutual forces between the main body and the rotor are internal forces in the system and do not appear in this equation. One has

$$\dot{\boldsymbol{G}} + \boldsymbol{\omega} \times \boldsymbol{G} = Mg\boldsymbol{\gamma}\times\mathbf{r}_0.$$

Since $\boldsymbol{\kappa}$ is kept constant in the body, $\dot{\boldsymbol{\kappa}} = \mathbf{0}$, and the last equation reduces to

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa}) = Mg\boldsymbol{\gamma}\times\mathbf{r}_0. \tag{5.3}$$

This is the final form of the dynamical equation of motion of the gyrostat. Together with Poisson's equation

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{5.4}$$

one obtains a closed system which we now write in the following scalar form of six first-order differential equations:

$$\begin{aligned} A\dot{p} + (C - B)qr + \kappa_3 q - \kappa_2 r &= Mg(z_0\gamma_2 - y_0\gamma_3), \\ B\dot{q} + (A - C)pr + \kappa_1 r - \kappa_3 p &= Mg(x_0\gamma_3 - z_0\gamma_1), \\ C\dot{r} + (B - A)pq + \kappa_2 p - \kappa_1 q &= Mg(y_0\gamma_1 - x_0\gamma_2), \end{aligned} \tag{5.5}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{5.6}$$

This system admits the general integrals:

$$\begin{aligned} &I_1 \equiv Ap^2 + Bq^2 + Cr^2 + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h, \\ &I_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ &I_3 = (Ap + \kappa_1)\gamma_1 + (Bq + \kappa_2)\gamma_2 + (Cr + \kappa_3)\gamma_3 = f. \end{aligned} \tag{5.7}$$

The first integral is usually termed Jacobi's integral for the system, since it is different from the total energy of the system, which contains terms linear in the components of $\boldsymbol{\omega}$.

When the angular speed Ω of the rotor vanishes, gyrostatic momentum $\boldsymbol{\kappa} = \mathbf{0}$, and equations (5.5) and the integrals (5.7) reduce to their counterparts of the classical problem.

5.1.2 The Free Rotor Model

In the previous model, the angular velocity of the rotor was kept constant relative to the carrier body. In an interesting alternative, due to Levi-Civita [261], the rotor is left to move freely around its axis of symmetry fixed in the body, so that the system will have an additional rotational degree of freedom. Let χ be the angle of rotation of the rotor relative to the body. Using the same symbols as in the previous subsection, the kinetic energy of the system is expressed as the sum of two parts

$$\begin{aligned} T &= \frac{1}{2}\int_{S_0} (\boldsymbol{\omega}\times\mathbf{r})^2 dm + \frac{1}{2}\int_{S_1} (\boldsymbol{\omega}\times\mathbf{r}+\dot{\chi}\mathbf{e} \times \mathbf{r}')^2 dm \\ &= \frac{1}{2}\int_{S_0} (\boldsymbol{\omega}\times\mathbf{r})^2 dm + \dot{\chi}\int_{S_1} (\boldsymbol{\omega}\times\mathbf{r}) \cdot (\mathbf{e} \times \mathbf{r}')dm + \frac{1}{2}\dot{\chi}^2 \int_{S_1} (\mathbf{e} \times \mathbf{r}')^2 dm \end{aligned}$$

$$= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \dot{\chi}\int_{S_1}(\mathbf{r}_1+\mathbf{r}')\times(\mathbf{e}\times\mathbf{r}')dm\cdot\boldsymbol{\omega} + \frac{1}{2}\dot{\chi}^2\int_{S_1}\mathbf{r}'\times(\mathbf{e}\times\mathbf{r}')dm\cdot\mathbf{e}.$$

Noting that

$$\mathbf{r}_1\times(\mathbf{e}\times\int_{S_1}\mathbf{r}'dm) = 0, \int_{S_1}\mathbf{r}'\times(\mathbf{e}\times\mathbf{r}')dm = \mathbf{eJ} = J\mathbf{e},$$

we obtain

$$T = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + J\dot{\chi}\mathbf{e}\cdot\boldsymbol{\omega} + \frac{1}{2}J\dot{\chi}^2$$

and hence the Lagrangian of the system may be written as

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + J\dot{\chi}\mathbf{e}\cdot\boldsymbol{\omega} + \frac{1}{2}J\dot{\chi}^2 - \mathbf{a}\cdot\boldsymbol{\gamma} \tag{5.8}$$

where $\mathbf{a} = Mg\mathbf{r}_0$. Obviously, the angle χ is a cyclic variable. The corresponding cyclic integral is

$$\frac{\partial L}{\partial\dot{\chi}} = J(\mathbf{e}\cdot\boldsymbol{\omega} + \dot{\chi}) = \kappa, \tag{5.9}$$

κ is an integration constant. Note that this integral means that the component of the total angular velocity of the rotor along its axis of symmetry remains constant during motion, i.e.

$$\mathbf{e}\cdot\boldsymbol{\omega} + \dot{\chi} = \frac{\kappa}{J}.$$

Now, ignoring the cyclic coordinate, we obtain the Routhian

$$\begin{aligned} R &= L - \kappa\dot{\chi} \\ &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} - \frac{1}{2}J(\boldsymbol{\omega}\cdot\mathbf{e})^2 + \kappa\mathbf{e}\cdot\boldsymbol{\omega} - \mathbf{a}\cdot\boldsymbol{\gamma} - \frac{\kappa^2}{2J} \\ &= \frac{1}{2}\boldsymbol{\omega}\tilde{\mathbf{I}}\cdot\boldsymbol{\omega} + \kappa\mathbf{e}\cdot\boldsymbol{\omega} - \mathbf{a}\cdot\boldsymbol{\gamma} \end{aligned} \tag{5.10}$$

where

$$\tilde{I}_{ij} = I_{ij} - Je_ie_j \tag{5.11}$$

and a constant $\frac{\kappa^2}{2J}$ has been ignored. In the way described in Chap. 3, the Euler dynamical equation derived from this Routhian are

$$\dot{\boldsymbol{\omega}}\tilde{\mathbf{I}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\tilde{\mathbf{I}} + \boldsymbol{\kappa}) = \boldsymbol{\gamma}\times\mathbf{a}, \tag{5.12}$$

where $\boldsymbol{\kappa} = \kappa\mathbf{e}$.

This equation has the same structure as (5.3), but the matrix $\tilde{\mathbf{I}}$ is not simply the matrix of inertia of the system, but depends on the axial moment of inertia of the rotor and on the orientation of its axis relative to the main body. The meaning of the vector $\boldsymbol{\kappa}$, the gyrostatic momentum is different in the two equations. Moreover, it should be noted that, in view of (5.11), the matrix $\tilde{\mathbf{I}}$ may not satisfy conditions, normal to ordinary inertia matrix of the simple body, like positivity of diagonal elements, triangle inequalities, etc.

5.1.3 Joukovsky's Model

Generalizing previous particular cases considered by Stokes and Neumann, Joukovsky established that "a fluid mass with an initial velocity in a multiply-connected cavity in the rigid body performs an action that is similar to the action of some rotor attached to the rigid body" [163] (see also [41, 286]).

The gyrostatic moment can also be due to internal cyclic degrees of freedom such as circulation of fluid in tubes inside the body or to forced stationary motions as motors, whose axes are fixed in the body.

As will be seen in a later chapter, terms in the equations of motion similar to gyrostatic momentum appear in problems of motion of a perforated rigid body (a body bounded by a multi-connected surface) in a liquid, as a result of the presence of circulations through perforations.

5.2 Equations of Motion in Hamiltonian Form

As in the classical problem (Chap. 3), one may use some generalized coordinates like Euler's angles, to construct the Hamiltonian function and the canonical equations of motion, involving those coordinates and momenta conjugate to them. This form of the equations of motion is rarely used in applications and is left as an exercise. Non-canonical equations

$$\mathbf{M} = \frac{\partial R}{\partial \boldsymbol{\omega}} = \boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}, \tag{5.13}$$

so that

$$\boldsymbol{\omega}= (\mathbf{M}-\boldsymbol{\kappa})\mathbf{I}^{-1}. \tag{5.14}$$

Also, the Hamiltonian corresponding to the same Routhian as a function in $\mathbf{M}$ and $\boldsymbol{\gamma}$ is

$$H = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{a}\cdot\boldsymbol{\gamma}$$

$$
\begin{aligned}
&= \frac{1}{2}(\mathbf{M}-\boldsymbol{\kappa})\mathbf{I}^{-1}\cdot(\mathbf{M}-\boldsymbol{\kappa}) + \mathbf{a}\cdot\boldsymbol{\gamma} \\
&= \frac{1}{2}\mathbf{M}\mathbf{I}^{-1}\cdot\mathbf{M}-\boldsymbol{\kappa}\mathbf{I}^{\mathbf{-1}}\cdot\mathbf{M} + \mathbf{a}\cdot\boldsymbol{\gamma},
\end{aligned}
\tag{5.15}
$$

so that the equations of motion can be written as

$$
\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M}\times\ \frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{\gamma}}, \\
\dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma}\times\ \frac{\partial H}{\partial \mathbf{M}}.
\end{aligned}
\tag{5.16}
$$

or, in the expanded form,

$$
\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M}\times(\mathbf{M}-\text{ˇ})\mathbf{I}^{-1} + \boldsymbol{\gamma}\times\mathbf{a}, \\
\dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma}\times(\mathbf{M}-\boldsymbol{\kappa})\mathbf{I}^{-1}.
\end{aligned}
\tag{5.17}
$$

5.3 Tables of Integrable Cases

Equations (5.5 and 5.6) have three known general and one conditional integrable cases, which generalize the four cases of a simple heavy body. Those are listed in the following Tables 5.1 and 5.2.

Table 5.1 Unconditional cases

	Author	Conditions
1	Joukovsky 1885 [163] and Volterra 1899 [366] Euler $\kappa_1 = \kappa_2 = \kappa_3 = 0$. $I_4 = (Ap+\kappa_1)^2 + (Bq+\kappa_2)^2 + (Cr+\kappa_3)^2$.	$g\mathbf{r}_0 = \mathbf{0}$.
2	Axially symmetric case (Generalization of Lagrange's top) $I_4 = Cr + \kappa_3$.	$B = A$, $x_0 = y_0 = 0$, $\kappa_1 = \kappa_2 = 0$.
3	Yehia 1986 [380], [383]* Kowalevski 1889 ($\kappa = 0$) [238] $I_4 = (p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2)^2 + (2pq - a_1\gamma_2 - a_2\gamma_1)^2$ $+2\kappa\,(r-\kappa)\,(p^2+q^2) - 4\kappa\gamma_3(a_1p + a_2q)$,	$A = B = 2C$, $z_0 = 0$, $\kappa_1 = \kappa_2 = 0$.

where $\kappa = \kappa_3/C$, $a_1 = Mgx_0/C$, $a_2 = Mgy_0/C$

*The case (3) was rediscovered in 1987 by Komarov [225] and Gavrilov [109]. In the monograph [41], it is attributed to Yehia, Komarov and Gavrilov, but in the Russian literature it is mostly called *Kowalevski–Yehia's* case

Table 5.2 Conditional cases $f = 0$

1	Sretensky (1963) [341]. Goryachev–Chaplygin 1900–1901 ($\kappa = 0$).	$A = B = 4C$, $z_0 = 0$, $\kappa_1 = \kappa_2 = 0$, $\kappa_3 = C\kappa$.
	$I_4 = (r - \kappa)(p^2 + q^2) - \gamma_3(a_1 p + a_2 q)$.	

where $\kappa = \kappa_3/C$, $a_1 = Mgx_0/C$, $a_2 = Mgy_0/C$

5.4 The Case of Joukovsky and Volterra

The first integrable case, which generalizes Euler's case, i.e. a balanced gyrostat or a gyrostat under no external torques, was noted in 1885 by Joukovsky in his study of the motion by inertia of a body containing liquid-filled cavities [163]. He also devised a geometric-mechanical interpretation of the motion in that case. Independently, in a trial to explain the displacement of Earth's poles by adding a rotor to the model of rigid Earth, Volterra gave in 1899 the full solution of the equations of motion in terms of Weierstrass' elliptic functions σ_i of time [366]. Those functions are complex in general. An alternative but real solution in terms of Jacobi's elliptic functions was constructed by Wittenburg [369]. Volterra's solution and stability analysis of the permanent rotations were reconsidered in [18].

5.5 The Case of Axially Symmetric Gyrostat

The axi-symmetric gyrostat is a trivial generalization of Lagrange's top and the solution of the equations of motion for it is practically the same as that of Lagrange's case. In a later Chap. 12, we will show a much richer generalization of this case.

We now prove the following

Theorem 5.1 *Any integrable case of an axi-symmetric body in a potential field, in which both ϕ and ψ are cyclic variables, can be generalized by the addition of a rotor aligned with the axis of symmetry.*

Theorem 5.2 *Consider the motion of an axi-symmetric gyrostat, with a gyrostatic momentum κ aligned along the axis of symmetry of the carrier body. The motion of the axis of the gyrostat is identical with the motion of a simple body with the same moments of inertia of the carrier body and moving in the same potential. The gyrostatic momentum is compensated by an additional angular speed κ/C given to the body about its axis, C being the axial moment of inertia of the body.*

Conversely, the motion of the axis of a simple axi-symmetric body, which given an additional angular speed Ω around that axis is identical to the motion the axis of a similar body, which carries a rotor with gyrostatic momentum $\kappa = C\Omega$.

Those two theorems can be proved by writing the Lagrangian of the simple body in a field with potential $V(\theta)$. Let the moments of inertia of the body be C about its z-axis of symmetry and A about any axis orthogonal to it. For such body, we have from (3.44)

$$\begin{aligned} L &= \frac{1}{2}[A(p^2+q^2)+Cr^2]-V(\theta) \\ &= \frac{1}{2}[A(\dot{\theta}^2+\sin^2\theta\dot{\psi}^2)+C(\dot{\psi}\cos\theta+\dot{\varphi}^2)-V(\theta). \end{aligned} \tag{5.18}$$

We now study the motion in another reference frame, which is rotating abut the z-axis with a constant angular rate Ω. This can be achieved by a substitution

$$\varphi = \varphi' + \Omega t,$$

which preserves the holonomicity of the system. The Lagrangian transforms to

$$\begin{aligned} L &= \frac{1}{2}\{A(\dot{\theta}^2+\sin^2\theta\dot{\psi}^2)+C[\dot{\psi}\cos\theta+(\dot{\varphi}'+\Omega)^2)]\}-V(\theta) \\ &= \frac{1}{2}\{A(\dot{\theta}^2+\sin^2\theta\dot{\psi}^2)+C(\dot{\psi}\cos\theta+\dot{\varphi}')^2\} \\ &\quad +C\Omega(\dot{\psi}\cos\theta+\dot{\varphi}')-V(\theta)+\frac{1}{2}C\Omega^2. \end{aligned}$$

Ignoring the last constant term, this can be regarded as describing the motion of the body referred to fixed axes in it, but with the coordinate φ' instead of φ and with the same potential V and additional gyroscopic term $C\Omega(\dot{\psi}\cos\theta+\dot{\varphi}') = C\Omega r'$. The last term is the contribution of a gyrostatic momentum $\boldsymbol{\kappa}$ directed along the z-axis

$$\boldsymbol{\kappa} = (\mathbf{0}, \mathbf{0}, C\Omega).$$

This proves Theorem 5.1 and the first part of Theorem 5.2. The second part of Theorem 5.2 follows naturally.

Those results may be used to express the quadrature resulting from separation of variables in the case 2 of Table 5.1, i.e. the gyrostatic generalization of Lagrange's case from the quadrature (4.42) by replacing r_0 by $r_0 + \frac{\kappa_3}{C}$, so that it becomes

$$t = \int \frac{d\gamma_3}{\sqrt{(1-\gamma_3^2)(E-a\gamma_3)-\frac{1}{A^2}[f-(Cr_0+\kappa_3)\gamma_3]^2}}.$$

5.6 Yehia's Case

The history of the third case in Table 5.1 has experienced some confusion and misunderstandings. It was direct and easy, guided by the same principle of conservation of angular momentum, to obtain the integrable case of Joukovsky, as a generalization of

Euler's case in the classical problem, by adding a constant gyrostatic momentum. The generalization of Lagrange's case of a symmetric body was even easier. Nevertheless, the search of a gyrostatic generalization of Kowalevski's case was so futile, that it was generally believed that, unlike Euler's and Lagrange's cases, Kowalevski's case does not admit generalization by the addition of a gyrostatic momentum. This trend may have been augmented by three contradicting results published in the mid-sixties by Keis:

(1) In the first of those works [167] (1963), Keis claimed having obtained generalizations to the gyrostat problem for four known integrable and particular cases of the classical problem, namely, Lagrange's, Hess', Bobylev–Steklov's and Delone's cases. The first case, the generalization of Lagrange's top, is trivially simple and the next two cases will be commented on in the relevant section on particular solutions. The last case (Delone's) is a special case of Kowalevski's, when Kowalevski's integral takes a zero value and splits into two invariant relations (See Sect. 4.3). Keis added to the body a constant gyrostatic momentum, aligned with the centre of mass in the equatorial plane of the inertia spheroid and claimed that the resulting system admits two invariant relations generalizing those of Delone's case. This claim was cited as being true in the review book [256].

(2) In the second paper [168] (1964), may be after realizing the flaw in his 1963 paper (cited in [168]), Keis used the method of Husson [154, 155] to give another theorem asserting that the equations of motion of the heavy gyrostat with Kowalevski's configuration $A = B = 2C$ admit an algebraic complementary integral only when the gyrostatic momentum vanishes ($\kappa_1 = \kappa_2 = \kappa_3 = 0$). He then formulated it as[1] "If $x_0^2 + y_0^2 + z_0^2 \neq 0$ and $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \neq 0$ a fourth algebraic integral is possible only when $A = B, x_0 = y_0 = 0,\ \kappa_1 = \kappa_2 = 0$", i.e. only in the case of Lagrange when both the centre of mass and the gyrostatic momentum are directed along the axis of dynamical symmetry of the body. This meant that Kowalevski's case has no extension to the gyrostat problem.

(3) In the third paper [170] (1965), Keis used Golubev's method [113] (In fact, Poincaré's method of small parameter) to establish a new result. The search for all cases, when all the solutions of the equations of motion of a heavy gyrostat are single-valued, reduces to investigation of the solution in three cases:

    (a) The torque-free gyrostat, or gyrostat fixed from its centre of mass, ($x_0 = y_0 = z_0 = 0$).
    (b) The axi-symmetric gyrostat ($A = B, x_0 = y_0 = 0,\ \kappa_1 = \kappa_2 = 0$).
    (c) The "Kowalevski gyrostat" ($A = B = 2C, y_0 = z_0 = 0, \kappa_1 = \kappa_2 = 0$).

    In this paper, the presence of the third component of the gyrostatic momentum does not give rise to multi-valued solutions in any of the three cases, up to the second degree of a small parameter. However, the conditions for case c are considered as necessary. The only way to ensure integrability is to find the

[1] Here we use the notation adopted in the present book.

complementary fourth integral of motion, a step which was not considered by Keis. Strangely, in the third paper the author does not refer to any of the other two papers, each of which announces a conflicting result.

Probably, influenced by the result in the second paper of Keis, published in the most influential Russian mechanics journal PMM, Kharlamov and coworkers concentrated on the search for particular solutions of the equations of motion. In this respect, they have succeeded in constructing the most part of the cases of that type known up to date. To this end, they used equations of motion in the form of Euler–Poisson and various modified forms. Kharlamov [198] obtained a particular solution of the heavy gyrostat with the Kowalevski configuration $A = B = 2C$, $y_0 = z_0 = 0$, involving a gyrostatic moment along the axis of dynamical symmetry of the body under certain restrictions on the initial motion. His case characterized by the existence of an invariant relation quadratic in the angular velocities fits as a special case of the third general integrable case in Table 5.1.

The full generalization of Kowalevski's case by the addition of a rotor to the body came out, in our work [380], in almost a century (exactly 98 years) after the publication of Kowalevski's case (See also [383]). Actually, it was not found as a solution of Euler–Poisson equations or any of their modifications. It was one of the first results obtained by the completely new method devised by the author of the present book to construct integrable 2D conservative mechanical systems, which admit a complementary integral polynomial in the velocities. After constructing a several-parameter integrable time-irreversible system of the above type, the parameters of the system are given certain values, such that the metric of the system could be identified with that of the Routhian reduction of the rigid body dynamics and then potential and gyroscopic forces could be identified and only then the appropriate Euler–Poisson equations are verified and the presentation of the new case in [380] was made in the last context.[2] The details of the method will not be presented here for space considerations, but the reader can get some acquaintance with it from the early papers [381, 419]. This method has proved fruitful and still gives new integrable cases of much more complicated problems in particle and rigid body dynamics (see, e.g. [411, 413, 422, 423]). All cases pertaining to rigid body dynamics obtained in this way will be described later in this book. They form the most part of the list of conditional integrable cases in Chap. 13.

The question of integrability of Eqs. (5.5 and 5.6) did not attract as much interest as the problem of a simple heavy body. Only one result on this aspect is known. It generalizes the above-mentioned theorem of Husson to the present problem of motion of a gyrostat.

Theorem 5.3 (Gavrilov [110, 111]) *The equations of motion of heavy gyrostat (5.5 and 5.6) possess an additional algebraic first integral only in the three cases of Joukowsky, Lagrange and Yehia.*

[2] This result was announced at the International Conference on Mechanics held in Moscow University, January 1986.

5.6.1 Separation of Variables

For greater clearness we first write down the equations and integrals of motion in the present case, after adding a simplifying condition $y_0 = 0$, which can be attained by a coordinate rotation. The equations have the form

$$\begin{aligned} 2\dot{p} - q(r-\kappa) &= 0, \\ 2\dot{q} + p(r-\kappa) &= a\gamma_3, \\ \dot{r} + a\gamma_2 &= 0, \end{aligned}$$

$$\dot{\gamma_1} + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma_2} + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma_3} + p\gamma_2 - q\gamma_1 = 0, \tag{5.19}$$

in which $a = \frac{Mgx_0}{C}$, $\kappa = \frac{\kappa_3}{C}$. The three general integrals of motion may be written as

$$\begin{aligned} I_1 &= 2(p^2+q^2) + r^2 + 2a\gamma_3 = 2h, \\ I_2 &= 2(p\gamma_1 + q\gamma_2) + (r+\kappa)\gamma_3 = f, \\ I_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ I_4 &= (p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2)^2 + (2pq - a_1\gamma_2 - a_2\gamma_1)^2 \\ &\quad +2\kappa\,(r-\kappa)\,(p^2+q^2) - 4\kappa\gamma_3(a_1 p + a_2 q) \\ &= K \end{aligned} \tag{5.20}$$

and here we retained the same names for h and f after dividing by C and K is an arbitrary constant.

Unlike the case of Kowalevski's top (with $\kappa = 0$), the explicit solution of the equations of motion in terms of time in Yehia's case is still unsuccessful. Separation coordinates analogous to Kowalevski's s_1, s_2 (Chap. 4 Sect. 4.3) were not found, even on the zero level of the areas integral. However, there is an indirect indication about the class of functions needed to describe this solution. An idea of special interest was presented in [145], which relates Kowalevski's case to a special version ($f = 0$) of an integrable case of the problem of motion of a rigid body in a liquid, known as Clebsch's first case (Case 2 of Table 10.1. Chap. 10) A bi-rational complex transformation was found relating the two sets of variables describing the two integrable problems, so that explicit solution for one of the cases can be obtained from that of the other. In the meantime, Clebsch's case is known to be solvable in terms of Theta functions with two arguments.

As was established by Gavrilov [110]: "The gyrostat of Yehia can be realized in a similar way as (the full $f \neq 0$) Clebsch's geodesic motion on E_3. This leads, in particular, to formulas for its explicit solution in terms of genus-two hyper-elliptic Theta functions [233]". Of course, this construction is not practical as a method of solution and there must be another direct way to obtain the solution. This way has not been found yet.

Komarov and Tsiganov [229] (See also [228]) considered the trajectory isomorphism of what they call "the Kowalevski gyrostat" and the Clebsch problem.

Although appeared fifteen years later, the last works do not contain any reference to Gavrilov's work.

In a recent work [326], Ryabov shows that the separated equations of the Yehia (Kowalevski–Yehia) case, *on its zero level of area's integral,* can be formally written in the Abel–Jacobi form analogous to (4.58) with $\Phi(s)$ as a polynomial of degree five in the variable s. However, the relation of the original variables of the problem to separated ones are not obtained, so that the problem of separation of variables cannot be considered complete yet, even on the level $f = 0$.

An earlier work relying on the Lax pair representation of the equations of motion constructs separation variables that are simultaneously suitable, *on the zero level of the areas integral*, for what the author names as gyrostatic generalizations of Kowalevski's and Goryachev–Chaplygin's cases [249]. It is claimed there that the given separation is "much simpler than the Kowalevski separation". However, the impact of that situation on the solution of the equations of motion was not considered.

Further (unpublished) results are announced by Fedorov et al. concerning the case of a gyrostat in two constant uniform fields, which includes the present case as a special version. This will be commented later in Chap. 14 Sect. 14.2.1.1.

The gyrostatic generalizations of Appelrot's classes of motions are discussed in Appendix D.

5.7 The Conditional Case of Sretensky

In [341], Sretensky found the modification of the complementary integral in the generalization of Goryachev–Chaplygin's case. He also generalized the procedure due to Chaplygin for explicit solution (See Chap. 4 Sect. 4.4) by changing the definitions of the three quantities r, U and V in (4.70)–(4.72) to be

$$\begin{aligned} r &= u - v - \kappa, \\ U &= u(u-\kappa)^2 - 2Eu - 4G, \\ V &= v(v+\kappa)^2 - 2Ev + 4G. \end{aligned} \tag{5.21}$$

The ultra-elliptic quadratures remain the same as in (4.77). The investigation of the critical sets and bifurcation diagrams in Sretensky's case is performed in [172, 174] (See also [183]). The results generalize relevant ones for Goryachev–Chaplygin's case, but they are much more complicated in view of the presence of three significant parameters. For more detail about Sretensky's case see Appendix E.

5.8 Some Applications of the Gyrostat Motion

We have seen that the presence of the gyrostatic momentum leads to appearance of a gyroscopic moment $\kappa \times \omega$ in the equations of motion.

5.9 Exercises

1- Show that the Lagrangian

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \boldsymbol{\kappa}\cdot\boldsymbol{\omega} - V(\boldsymbol{\gamma}),$$

describes the motion of a gyrostat with gyrostatic momentum $\boldsymbol{\kappa}$ about a fixed point, while acted upon by axially symmetric forces with potential $V(\boldsymbol{\gamma})$ ($\boldsymbol{\gamma}$ is the unit vector along the axis of symmetry of the forces). Deduce the equations of motion in the form

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}) &= \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} &= \mathbf{0}. \end{aligned}$$

2- In the previous problem, show that all possible axes of stationary motions lie on the cone, with vertex at the origin and generators passing through the points of the spherical curve

$$\begin{aligned} [\boldsymbol{\gamma} \cdot (\boldsymbol{\gamma}\mathbf{I}\times\frac{\partial V}{\partial \boldsymbol{\gamma}})]^2 - [\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{I})][\boldsymbol{\kappa} \cdot (\boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}})] &= 0, \\ \boldsymbol{\gamma}^2 &= 1, \end{aligned}$$

and the angular velocity of the gyrostat about the axis in the direction of $\boldsymbol{\gamma}$ is given by any of the expressions

$$\omega = \frac{\boldsymbol{\gamma}\cdot(\boldsymbol{\gamma}\mathbf{I} \times \frac{\partial V}{\partial \boldsymbol{\gamma}})}{\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{I})} = \frac{\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}})}{\boldsymbol{\gamma}\cdot(\boldsymbol{\gamma}\mathbf{I}\times\frac{\partial V}{\partial \boldsymbol{\gamma}})}.$$

3- An axially symmetric gyrostat, moving about a fixed point under its own weight, has its centre of mass on its axis of symmetry and the gyrostatic momentum is collinear with that axis. Show that the upper equilibrium position of the gyrostat, which is unstable in the absence of gyrostatic moment, can always be stabilized and find the minimum angular velocity necessary for that effect.

4- Write down the Hamiltonian and Hamiltonian equations of motion of a heavy gyrostat moving about a fixed point, using Euler's angles as generalized coordinates.

5- A system composed of a main body S_0 fixed from one point O, while carrying another body S_1 whose axis $O'P$ is fixed in S_0 by means of a smooth cylindrical hinge and freely rotates about this axis. Let $OO' = \mathbf{r}_1$ and $\mathbf{e}$ is the unit vector in the direction of $O'P$. Show that the kinetic energy of the system is

$$T = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \dot{\chi}\mathbf{e}\mathbf{J}\cdot\boldsymbol{\omega} + \frac{1}{2}\dot{\chi}^2\mathbf{e}\mathbf{J}\cdot\mathbf{e}+M_1\dot{\chi}\boldsymbol{\omega}\cdot[\mathbf{r}_1\times(\mathbf{e}\times\mathbf{r}_0')],$$

where $\mathbf{I}$ is the inertia matrix of the system at the fixed point, $\mathbf{J}$ is the inertia matrix of the second body at O', M_1 is the mass of S_1 and $\mathbf{r}'_0$ is the position vector of its centre of mass.