Chapter 3
The Classical Problem: The Motion of a Heavy Rigid Body About a Fixed Point

In the present chapter, we present detailed analysis of the classical problem of motion of a rigid body about a fixed point under the action of its own weight. This problem has a long history that began with the work of Euler and continued to the present day. Various powerful methods belonging to eminent specialists in mechanics and mathematics were applied to this problem without stopping, sometimes successfully and sometimes with less success. The list of basic contributors to this problem from our perspective, the construction of integrable cases, will be clearly presented as we proceed through this preliminary chapter and the following two chapters, after we make clear the meanings of general integrable, conditional integrable and particular solvable cases of the classical problem. This chapter is mainly concerned with basic concepts and various forms of the equations of motion, each of which would be more suited for use in certain investigations of the classical problem.

3.1 Equations of Motion

In this section, we derive the equation of rotational motion of the rigid body about a fixed point, under the action of arbitrary forces, which are not necessarily conservative or even having a potential. For such general setting, the Lagrangian approach is not a suitable choice and it is preferable to use ordinary vector mechanics. Denote by $\mathbf{r}$ the position vector of a mass element dm and by $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ its velocity. The angular momentum of the body, denoted here by $\mathbf{G}$, is the sum of moments of momenta of the elements about the origin O, the fixed point of the body,

$$\mathbf{G} = \int \mathbf{r} \times (\mathbf{v}dm), \tag{3.1}$$

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where the integral is taken over the whole mass of the body. Differentiating the last relation, we get

$$\frac{d\mathbf{G}}{dt} = \int \mathbf{v} \times \mathbf{v} dm + \int \mathbf{r} \times \frac{d\mathbf{v}}{dt} dm. \tag{3.2}$$

The first integral vanishes, and from the equation of motion of the mass element dm, we have

$$dm \frac{d\mathbf{v}}{dt} = d\mathbf{F} + d\mathbf{F}',$$

in which $d\mathbf{F}$, $d\mathbf{F}'$ are, respectively, the resultant external and internal forces exerted on that element. Inserting this into (3.2), we write

$$\frac{d\mathbf{G}}{dt} = \int \mathbf{r} \times (d\mathbf{F} + d\mathbf{F}'). \tag{3.3}$$

Since the internal forces appear only in equal and opposite pairs, their overall moment vanishes, i.e. $\int \mathbf{r} \times d\mathbf{F}' = 0$. Thus, we finally have

$$\frac{d\mathbf{G}}{dt} = \mathbf{L}, \tag{3.4}$$

where $\mathbf{L} = \int \mathbf{r} \times d\mathbf{F}$ is the resultant moment of all the external forces acting on the body about the fixed point. This is the equation of rotational motion of the rigid body about a fixed point, under the action of arbitrary forces with moment $\mathbf{L}$. It is curious that this equation is similar to the equation of motion of a particle $\frac{d\mathbf{P}}{dt} = \mathbf{F}$, but replacing the Linear momentum $\mathbf{P}$ and the force $\mathbf{F}$ by the angular momentum and the moment of forces about the fixed point.

3.2 The Heavy Rigid Body

Equation (3.4) is quite general. It is valid for an arbitrary rigid body subject to arbitrary system of forces. In this chapter, we are concerned with the simplest case of motion of a body subject only to its own weight. For such a body, let $\mathbf{g}$ be the intensity of the gravity field directed vertically downwards. We have

$$\begin{aligned} \mathbf{L} &= \int \mathbf{r} \times (\mathbf{g} dm) \\ &= \int \mathbf{r} dm \times \mathbf{g}. \end{aligned}$$

Recalling the definition of the centre of gravity (the centre of mass) of the body in Chap. 1, we write the last relation in the form

$$\mathbf{L} = M\mathbf{r}_0 \times \mathbf{g}. \tag{3.5}$$

Without loss of generality, for most applications, one can take the Z-axis in the vertical direction upwards, so that the gravity field becomes

$$\mathbf{g} = -g\boldsymbol{\gamma}. \tag{3.6}$$

Thus, in the case of a heavy rigid body, the equation of motion takes the form

$$\frac{d\mathbf{G}}{dt} = -Mg\mathbf{r}_0 \times \boldsymbol{\gamma}. \tag{3.7}$$

3.3 The Angular Momentum of a Rigid Body

The mass element at the point $\mathbf{r}$ has velocity

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}. \tag{3.8}$$

Recalling the definition (3.1), we write

$$\begin{aligned}\mathbf{G} &= \int \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r})dm \\ &= \int [\mathbf{r}^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r})\mathbf{r}]dm.\end{aligned}$$

The i-th component is

$$\begin{aligned}G_i &= \omega_i \int \mathbf{r}^2 dm - \int \sum_{j=1}^{3} \omega_j r_j r_i dm \\ &= \sum_{j=1}^{3} \omega_j \int [\mathbf{r}^2\delta_{ij} - r_i r_j]dm \\ &= \sum_{j=1}^{3} \omega_j I_{ij},\end{aligned}$$

where $\mathbf{I} = (I_{ij})_{i,j=1}^{3}$ is the inertia matrix in the system of axes at the fixed point O. Making use of the symmetry of the inertia matrix, we write the last expressions in the form

$$\mathbf{G} = \boldsymbol{\omega}\mathbf{I}. \tag{3.9}$$

Remark 15 In most textbooks, the last relation is usually written as

$$\mathbf{G}' = \mathbf{I}\boldsymbol{\omega},$$

where $\mathbf{G}'$ is a column vector. We use the notation (3.9) to express $\mathbf{G}$ as a normal row vector. This is especially convenient in applying rules of vector algebra to that vector.

The relation (3.9) can be written in the expanded form as

$$\begin{aligned} \mathbf{G} &= (p, q, r) \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} \\ &= (I_{11}p + I_{12}q + I_{13}r, I_{12}p + I_{22}q + I_{23}r, I_{13}p + I_{23}q + I_{33}r). \end{aligned} \tag{3.10}$$

In the inertial system of axes at O, the body moves and its orientation changes with time. Equation (3.7) then involves 12 variable quantities: six moments and products of inertia, three coordinates of the mass centre and three components of the angular velocity. This makes the equations of motion quite complicated and impractical to use.

The system of axes fixed in the body with origin at the fixed point O enjoys the advantage that the inertia matrix is constant and also the position vector of the centre of mass. This makes it plausible to use a coordinate system fixed in the body to express Eq. (3.7) in it. A question arises, how to express that equation which is derived in the inertial system of axes in the body system? The answer will be given soon.

3.4 Kinetic Energy of a Moving Body

Summing the kinetic energy of mass elements and making use of the formulas of the last subsection, we get

$$\begin{aligned} T &= \int \frac{1}{2}\mathbf{v}^2 dm \\ &= \frac{1}{2}\int \mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}) dm \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \int [\mathbf{r} \times \mathbf{v}] dm \\ &\equiv \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega}. \end{aligned} \tag{3.11}$$

In expanded form, this means

$$T = \frac{1}{2}(I_{11}p^2 + I_{22}q^2 + I_{33}r^2 + 2I_{12}pq + 2I_{23}qr + 2I_{13}pr). \tag{3.12}$$

3.5 Equations of Motion in the Moving Coordinate System

Let us now write Eq. (3.7) in the body system. It takes the form, called Euler's equation,

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0. \tag{3.13}$$

In addition to the vector $\boldsymbol{\omega}$, this equation involves the vertical unit vector $\boldsymbol{\gamma}$, which has variable components in the body system $Oxyz$. Being constant in space, the vector $\boldsymbol{\gamma}$ satisfies $\frac{d\boldsymbol{\gamma}}{dt} = 0$ in the inertial frame. In the body system, this is equivalent to

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{3.14}$$

which bears the name of Poisson's equation.

The pair of vector Eqs. (3.13) and (3.14), known as the *Euler–Poisson* equations, constitute a closed system of six scalar first-order differential equations in six variables, which can be chosen as either $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$ or $\mathbf{G}$, $\boldsymbol{\gamma}$.

3.5.1 The Use of the Variables $\boldsymbol{\omega}, \boldsymbol{\gamma}$. Special Axes Related to the Inertia Matrix

In that case, using (3.9), we write the Euler–Poisson equation as

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0, \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{3.15}$$

which is the most commonly used form of those equations. For arbitrary choice of the body axes, they have the expanded form

$$\begin{aligned}
I_{11}\dot{p} + I_{12}\dot{q} + I_{13}\dot{r} + (I_{33} - I_{22})qr + I_{23}(q^2 - r^2) + p\,(I_{13}q - I_{12}r) &= Mg(z_0\gamma_2 - y_0\gamma_3), \\
I_{12}\dot{p} + I_{22}\dot{q} + I_{23}\dot{r} + (I_{11} - I_{33})\,pr + I_{13}\left(r^2 - p^2\right) + q\,(I_{12}r - I_{23}p) &= Mg(x_0\gamma_3 - z_0\gamma_1), \\
I_{13}\dot{p} + I_{23}\dot{q} + I_{33}\dot{r} + (I_{22} - I_{11})pq + I_{12}\left(p^2 - q^2\right) + r\,(I_{23}p - I_{13}q) &= Mg(y_0\gamma_1 - x_0\gamma_2),
\end{aligned} \tag{3.16}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{3.17}$$

In this form of equations, the inertia matrix has six elements. The system is not solved for the derivatives, a situation that is not in favour of a process of solution.

3.6 Integrals of Motion

Equations of motion (3.13), (3.14) or in expanded form (3.16), (3.17) are essentially nonlinear. For their solution, in the sense of reduction to quadratures, the application of Jacobi's theorem about the last integrating multiplier (See, e.g. [305]) requires the knowledge of four integrals of motion.

The first step is to see how much general integrals the above system admits in its most general form.

3.6.1 *The Energy Integral*

The rigid body is assumed to be smoothly fixed at O and moving in the uniform field of gravity, whose potential is

$$V = Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma}. \tag{3.18}$$

Regarding expressions (3.11), (3.12) and (3.18), one can immediately write the energy integral as

$$\begin{aligned} I_1 &\equiv T + V \\ &\equiv \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{G} + Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} \\ &\equiv \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} + Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} = h, \end{aligned} \tag{3.19}$$

h being the arbitrary constant of conserved total energy of the motion. In the expanded form, we can write

$$\begin{aligned} I_1 \equiv\ & \frac{1}{2}(I_{11}p^2 + I_{22}q^2 + I_{33}r^2 + 2I_{12}pq + 2I_{23}qr + 2I_{13}pr) \\ & + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) \\ =\ & h. \end{aligned} \tag{3.20}$$

3.6.2 *The Area's Integral*

Now we rewrite the equation of rotational motion (3.7)

$$\frac{d\mathbf{G}}{dt} = -Mg\mathbf{r}_0 \times \boldsymbol{\gamma},$$

and note that on multiplying scalarly by the vector $\boldsymbol{\gamma}$ on both sides, we get

$$\gamma \cdot \frac{d\mathbf{G}}{dt} = 0,$$

which may be now written as

$$\frac{d}{dt}(\mathbf{G} \cdot \gamma) = 0,$$

so that we obtain the second general integral of motion

$$I_2 \equiv \mathbf{G} \cdot \gamma = f, \tag{3.21}$$

f being an arbitrary parameter. In the moving axes, it has the form

$$I_2 \equiv \boldsymbol{\omega}\mathbf{I} \cdot \gamma = f. \tag{3.22}$$

In a general body system, it may be written as

$$I_2 = (I_{11}p + I_{12}q + I_{13}r)\gamma_1 + (I_{12}p + I_{22}q + I_{23}r)\gamma_2 + (I_{13}p + I_{23}q + I_{33}r)\gamma_3 = f.$$

The integral of motion (3.21) or (3.22) is linear in the components of the angular velocity. In accordance with the tradition prevailing in celestial mechanics, it is called the *areas integral*.

3.6.3 The Geometric Integral

The vector γ is defined as the unit vector directed vertically upwards. From this definition, it directly follows that its square is a constant of motion, with its constant value normalized to 1:

$$I_3 \equiv \gamma^2 \equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{3.23}$$

3.6.4 Exercise

Use Euler–Poisson's equations of motion in the vector form (3.15) to directly obtain the three general integrals of motion.

3.7 Special Axes Associated with the Inertia Matrix

Equation (3.16) can be somewhat simplified, by a suitable choice of the body axes. For example, we can take the z-axis as the one joining the fixed point with the centre of mass, so that $\mathbf{r}_0$ can be written as

$$\mathbf{r}_0 = (0, 0, z_0). \tag{3.24}$$

Moreover, we still have the freedom to rotate the x, y-axes in their plane to a position in which

$$I_{12} = 0. \tag{3.25}$$

We shall call the final set of axes *the special axes associated to the inertia matrix*. Those axes are most convenient in describing some particular solutions of the classical problem and other problems in rigid body dynamics, such as the regular precessions. This will be made clear later on.

The Euler equations now take the form

$$\begin{aligned} I_{11}\dot{p} + I_{13}\dot{r} + (I_{33} - I_{22})qr + I_{23}(q^2 - r^2) + I_{13}pq &= Mgz_0\gamma_2, \\ I_{22}\dot{q} + I_{23}\dot{r} + (I_{11} - I_{33})\,pr + I_{13}\left(r^2 - p^2\right) - I_{23}pq &= -Mgz_0\gamma_1, \\ I_{13}\dot{p} + I_{23}\dot{q} + I_{33}\dot{r} + (I_{22} - I_{11})pq + r\,(I_{23}p - I_{13}q) &= 0, \end{aligned} \tag{3.26}$$

while Poisson's equations still have the form (3.17), i.e.

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0.$$

The general integrals of motion can be written in the given coordinate system in the form

$$\begin{aligned} I_1 \equiv\ & \frac{1}{2}(I_{11}p^2 + I_{22}q^2 + I_{33}r^2 + 2I_{23}qr + 2I_{13}pr) \\ & + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) \\ =\ & h, \\ I_2 \equiv\ & (I_{11}p + I_{13}r)\gamma_1 + (I_{22}q + I_{23}r)\gamma_2 + (I_{13}p + I_{23}q + I_{33}r)\gamma_3 = f, \\ I_3 \equiv\ & \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{3.27}$$

3.8 The Use of Principal Axes of Inertia of the Body

In the special case, when the body axes are chosen to be the principal axes of the body at O, we have

$$\mathbf{I} = \mathrm{diag}(A, B, C), \mathbf{G} = (Ap, Bq, Cr), \mathbf{r}_0 = (x_0, y_0, z_0). \tag{3.28}$$

The equations of motion take the form

$$
\begin{aligned}
A\dot{p} + (C-B)qr &= Mg(z_0\gamma_2 - y_0\gamma_3),\\
B\dot{q} + (A-C)\,pr &= Mg(x_0\gamma_3 - z_0\gamma_1),\\
C\dot{r} + (B-A)pq &= Mg(y_0\gamma_1 - x_0\gamma_2),\\
\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 &+ r\gamma_1 - p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0,
\end{aligned}
\tag{3.29}
$$

and the integrals of motion become

$$
\begin{aligned}
I_1 &\equiv \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h,\\
I_2 &\equiv Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f,\\
I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.
\end{aligned}
\tag{3.30}
$$

The equations of motion acquire in (3.29) their simplest and most symmetric form. The most favoured form is in scientific and technical literature. Those equations involve six parameters: three principal moments of inertia and three quantities formed by multiplying three coordinates of the centre of mass by the body weight. Unlike Eq. (3.16), Eq. (3.29) is readily solved in the derivatives $\dot{p}, \dot{q}, \dot{r}$, which is quite an advantage.

In the sequel, we shall mostly adhere to this form of the equations of motion. Only in exceptional occasions, we find other forms more appropriate or easier to use.

3.9 Determination of Euler's Angles

Solving the system of six equations of motion (3.29), we determine the vectors $\boldsymbol{\omega}(t)$ and $\boldsymbol{\gamma}(t)$as functions of the time t and only five arbitrary constants of integration, since the initial values of $\boldsymbol{\gamma}$ satisfy the geometric integral without arbitrary constant. This determines the Eulerian angles of nutation and proper rotation θ and φ as

$$
\theta = \cos^{-1}\gamma_3,\ \varphi = \tan^{-1}\frac{\gamma_1}{\gamma_2}. \tag{3.31}
$$

To complete the solution of the dynamical problem, i.e. to determine the orientation of the body in space, we should also determine the precession angle ψ. To this end, we use (2.39) of Chap. 2 to write

$$
\dot{\psi} = \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2}, \tag{3.32}
$$

so that we finally obtain

$$\psi = \psi_0 + \int_0^t \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2} dt, \tag{3.33}$$

ψ_0 is the sixth integration constant of the solution. This completes the solution of the problem of motion about a fixed point.

3.10 The Movable and Immovable Hodographs

Applied to the angular velocity vector the relation (2.37) gives

$$\frac{d\boldsymbol{\omega}}{dt} = \dot{\boldsymbol{\omega}}, \tag{3.34}$$

i.e. the angular velocity has the same rate of change in space as in the body. This formula, noted by Poisson, means that the infinitesimal change in the angular velocity at any moment of time is the same in both space and body reference frames. This has a very useful interpretation. Let the spatial curves Γ, named as the movable angular velocity hodograph, and Γ_0, the immovable angular velocity hodograph, be the loci of the angular velocity vector in the body and space system of axes, respectively. The two curves have the same tangent at every moment of time. The motion of the body in space can be represented as rolling the movable hodograph Γ without slipping on the immovable hodograph Γ_0 (fixed in space). The hodograph motion was studied as a way of geometric visualization of the motion in solvable cases. A voluminous literature exists on this topic. Interested readers may see, e.g. [108, 121] for several concrete examples.

3.11 The Use of the Variables G, γ. Special Axes Associated with the Gyration Ellipsoid

Let $\mathbf{G} = (P, Q, R)$ denote the angular momentum of the body and its components referred to the body axes. In that case, inverting the relation (3.9), we write

$$\boldsymbol{\omega} = \mathbf{GA}, \mathbf{A} = \mathbf{I}^{-1}, \tag{3.35}$$

so that (3.13) and (3.17) take the form

$$\dot{\mathbf{G}} + \mathbf{GA} \times \mathbf{G} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0, \dot{\boldsymbol{\gamma}} + \mathbf{GA} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{3.36}$$

and the integrals of motion become

$$
\begin{aligned}
\frac{1}{2}\mathbf{G}\mathbf{A}\cdot\mathbf{G}+Mg\mathbf{r}_0\cdot\boldsymbol{\gamma} &= h,\\
\mathbf{G}\cdot\boldsymbol{\gamma} &= f,\\
\boldsymbol{\gamma}^2 &= 1.
\end{aligned} \tag{3.37}
$$

The main advantage of Eq. (3.36) is that they are solved for the derivatives, in the sense that each of the six equations involves only one derivative of one component of **G** or $\boldsymbol{\gamma}$. In this form, also the areas integral takes its simplest form. The situation can be made more advantageous by using the so-called "*Special axes associated with the gyration ellipsoid*", introduced and extensively used by Kharlamov [191]. They are formed in the following way: Choose the z-axis as the one joining the fixed point with the centre of mass, so that $\mathbf{r}_0$ can be written as

$$
\mathbf{r}_0 = (0, 0, z_0), \tag{3.38}
$$

and then rotate the x, y-axes in their plane to a position in which

$$
A_{12} = 0. \tag{3.39}
$$

In those special axes, the angular velocity

$$
\boldsymbol{\omega} = (A_{11}P + A_{13}R,\ A_{22}Q + A_{23}R,\ A_{33}R + A_{13}P + A_{23}Q), \tag{3.40}
$$

and Euler–Poisson's Eq. (3.36) become

$$
\begin{aligned}
\dot{P} + (A_{22} - A_{33})QR - A_{13}PQ + A_{23}(R^2 - Q^2) &= Mgz_0\gamma_2,\\
\dot{Q} + (A_{33} - A_{11})PR + A_{23}PQ + A_{13}(P^2 - R^2) &= -Mgz_0\gamma_1,\\
\dot{R} + (A_{11} - A_{22})PQ + (A_{13}Q - A_{23}P)R &= 0,\\
\dot{\gamma}_1 + (A_{22}Q + A_{23}R)\gamma_3 - (A_{33}R + A_{13}P + A_{23}Q)\gamma_2 &= 0,\\
\dot{\gamma}_2 + (A_{33}R + A_{13}P + A_{23}Q)\gamma_1 - (A_{11}P + A_{13}R)\gamma_3 &= 0,\\
\dot{\gamma}_3 + (A_{11}P + A_{13}R)\gamma_2 - (A_{22}Q + A_{23}R)\gamma_1 &= 0.
\end{aligned} \tag{3.41}
$$

As to the integrals of motion in the special axes, we note that the areas and geometric integrals still have the form as in (3.37), but the energy integral takes the form

$$
I_1 = \frac{1}{2}(A_{11}P^2 + A_{22}Q^2 + A_{33}R^2 + 2A_{23}QR + 2A_{13}PR) + Mgz_0\gamma_3 = h. \tag{3.42}
$$

3.12 Equations of Motion in Generalized Coordinates

The Euler–Poisson form is mostly preferred in the study of rigid body motion. Nevertheless, in certain situations, it is advantageous to write the Lagrangian form of the equations of motion, sometimes using the Eulerian angles as generalized coordinates and other times using different coordinates or some redundant coordinates, for example, the components of the vector $\boldsymbol{\gamma}$ or the quaternions. This formalism turns out to be most useful in the case of a dynamically symmetric body, but we shall not impose this condition for the time being.

The Lagrangian can be written in arbitrary coordinate system fixed in the body, but to obtain a more tractable form, we use the principal axes of inertia of the body at the fixed point as body axes. We write

$$L = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) - Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma}. \tag{3.43}$$

Using Eqs. (2.39) and (14.1), the Lagrangian takes the form

$$\begin{aligned} L = &\frac{1}{2}[A(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)^2 \\ &+B(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi)^2 + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\ &-Mg(x_0\sin\theta\sin\varphi + y_0\sin\theta\cos\varphi + z_0\cos\theta). \end{aligned} \tag{3.44}$$

We note at once two properties of the Lagrangian leading to two integrals:

(1) The system is conservative and hence admits the energy integral

$$\begin{aligned} I_1 \equiv &\frac{1}{2}[A(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)^2 \\ &+B(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi)^2 + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\ &+Mg(x_0\sin\theta\sin\varphi + y_0\sin\theta\cos\varphi + z_0\cos\theta) = h. \end{aligned} \tag{3.45}$$

(2) The angle of precession ψ is a cyclic coordinate and this leads to the cyclic integral

$$\begin{aligned} I_2 &\equiv \frac{\partial L}{\partial \dot{\psi}} \\ &= A\sin\theta\sin\varphi(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi) \\ &\quad +B\sin\theta\cos\varphi(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi) + C\cos\theta(\dot{\psi}\cos\theta + \dot{\varphi}) \\ &= (A\sin^2\theta\sin^2\varphi + B\sin^2\theta\cos^2\varphi + C\cos^2\theta)\dot{\psi} \\ &\quad +(A - B)\sin\theta\sin\varphi\cos\varphi\dot{\theta} \\ &= f. \end{aligned} \tag{3.46}$$

It is evident that those integrals are the same as the two in (3.30). Note that the geometric integral in (3.30) turns into an identity in the Euler angles as coordinates. In fact, $\gamma_1, \gamma_2, \gamma_3$ are redundant coordinates, i.e. they are dependent coordinates subject to the geometric integral as a constraint.

3.13 Canonical Equations of Motion in Euler's Angles

For certain important applications, such as different perturbation procedures, it may be advantageous to use the Hamiltonian formalism. We shall give now the Hamiltonian function and canonical equations of motion in Euler's angles and their conjugate momenta p_ψ, p_θ, p_φ. From (3.44), we get

$$\begin{aligned} p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = D\dot{\psi} + (A - B)\sin\theta\sin\varphi\cos\varphi\dot{\theta} + C\cos\theta\dot{\varphi}, \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = (A - B)\sin\theta\cos\theta\sin\varphi\dot{\psi} + (A\cos^2\varphi + B\sin^2\varphi)\dot{\theta}, \\ p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = C(\dot{\psi}\cos\theta + \dot{\varphi}), \end{aligned} \tag{3.47}$$

where

$$D = A\sin^2\theta\sin^2\varphi + B\sin^2\theta\cos^2\varphi + C\cos^2\theta. \tag{3.48}$$

Then, after solving (3.47) for $\dot{\psi}$, $\dot{\theta}$, $\dot{\varphi}$, we calculate the Hamiltonian

$$\begin{aligned} H &= \dot{\psi}p_\psi + \dot{\theta}p_\theta + \dot{\varphi}p_\varphi - L \\ &= \frac{(A\cos^2\varphi + B\sin^2\varphi)}{2AB\sin^2\theta}(p_\psi - p_\varphi\cos\theta)^2 + \frac{(A\sin^2\varphi + B\cos^2\varphi)}{2AB}p_\theta^2 \\ &\quad - \frac{(A - B)\sin\varphi\cos\varphi}{AB\sin\theta}(p_\psi - p_\varphi\cos\theta)p_\theta + \frac{p_\varphi^2}{2C} \\ &\quad + Mg(x_0\sin\theta\sin\varphi + y_0\sin\theta\cos\varphi + z_0\cos\theta). \end{aligned} \tag{3.49}$$

The equations of motion can be written in the form

$$\begin{aligned} \dot{p}_\psi &= -\frac{\partial H}{\partial \psi}, \dot{\psi} = \frac{\partial H}{\partial p_\psi}, \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta}, \dot{\theta} = \frac{\partial H}{\partial p_\theta}, \\ \dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi}, \dot{\varphi} = \frac{\partial H}{\partial p_\varphi}. \end{aligned} \tag{3.50}$$

Since H does not depend on ψ, we have $\dot{p}_\psi = 0$, i.e. $p_\psi = \text{const}$. In conformity with (3.46), we take this constant to be f, so that

$$p_\psi = f. \tag{3.51}$$

The second and third pairs of equations give

$$\begin{aligned}
\dot{p}_\theta &= \frac{(B-A)\sin\varphi\cos\varphi}{AB\sin^2\theta}(f\cos\theta - p_\varphi)p_\theta \\
&\quad + \frac{(A\cos^2\varphi + B\sin^2\varphi)}{AB\sin^2\theta}(f - p_\varphi\cos\theta)(f\cos\theta - p_\varphi) \\
&\quad - Mg\,(x_0\cos\theta\sin\varphi + y_0\cos\theta\cos\varphi - z_0\sin\theta)\,, \\
\dot{\theta} &= \frac{(B-A)\sin\varphi\cos\varphi}{AB\sin\theta}(f - p_\varphi\cos\theta) + \frac{(A\sin^2\varphi + B\cos^2\varphi)}{AB}p_\theta, \\
\dot{p}_\varphi &= \frac{(A-B)}{AB}\Big[\frac{\sin\varphi}{\sin\theta}(f - p_\varphi\cos\theta) + p_\theta\cos\varphi\Big] \\
&\quad \times\Big[\frac{\cos\varphi}{\sin\theta}(f - p_\varphi\cos\theta) - p_\theta\sin\varphi\Big] \\
&\quad - Mg\sin\theta\,(x_0\cos\varphi - y_0\sin\varphi)\,, \\
\dot{\varphi} &= \frac{p_\varphi}{C} + \frac{(A-B)\sin\varphi\cos\varphi\cos\theta}{AB\sin\theta}p_\theta \\
&\quad - \frac{(A\cos^2\varphi + B\sin^2\varphi)\cos\theta}{AB\sin^2\theta}(f - p_\varphi\cos\theta).
\end{aligned} \tag{3.52}$$

If a solution is obtained for the last system giving θ, φ, $\dot{\theta}$, $\dot{\varphi}$ as functions of time, the precession angle ψ can be then determined by integrating the second equation in (3.50), which is now written as

$$\dot{\psi} = \frac{(A\cos^2\varphi + B\sin^2\varphi)}{AB\sin^2\theta}(f - p_\varphi\cos\theta) - \frac{(A-B)\sin\varphi\cos\varphi}{AB\sin\theta}p_\theta. \tag{3.53}$$

3.14 The Routhian Reduction

From (3.46), we find

$$\dot{\psi} = \frac{f - (A-B)\sin\theta\sin\varphi\cos\varphi\dot{\theta} - C\cos\theta\dot{\varphi}}{(A\sin^2\theta\sin^2\varphi + B\sin^2\theta\cos^2\varphi + C\cos^2\theta)}. \tag{3.54}$$

One can now use Routh's procedure to ignore the cyclic coordinate ψ and reduce the problem of motion to a system of two degrees of freedom. The Routhian of the system is

$$
\begin{aligned}
R &= L - f\dot{\psi} \\
&= R_2 + R_1 - V_1,
\end{aligned} \tag{3.55}
$$

where

$$
\begin{aligned}
R_2 &= \frac{1}{2D}\{C\sin^2\theta(A\sin^2\varphi + B\cos^2\varphi)\dot{\varphi}^2 - \\
&\quad -\frac{1}{2}C(A-B)\sin 2\theta \sin 2\varphi\dot{\theta}\dot{\varphi} \\
&\quad +[D(A\cos^2\varphi + B\sin^2\varphi) - (A-B)^2\sin^2\theta\sin^2\varphi\cos^2\varphi]\dot{\theta}^2\}, \\
R_1 &= \frac{f}{D}[C\cos\theta\dot{\varphi} + (A-B)\sin\theta\sin\varphi\cos\varphi\dot{\theta}], \\
V_1 &= V + \frac{f^2}{2D}.
\end{aligned} \tag{3.56}
$$

The function V_1 is called the reduced potential, while V is the original potential of the problem. The equations of motion are

$$
\frac{d}{dt}\frac{\partial R}{\partial \dot{\theta}} - \frac{\partial R}{\partial \theta} = 0, \; \frac{d}{dt}\frac{\partial R}{\partial \dot{\varphi}} - \frac{\partial R}{\partial \varphi} = 0. \tag{3.57}
$$

We shall not write them down in the expanded form because they lack symmetry and they are not easy to use in general. However, they can be used much easily in case of a dynamically symmetric body. Such concrete applications are not in the focus of the present book and can be found in several books on perturbation problems. Those are two second-order equations in the two variables θ and φ. After solving those equations and expressing the two angles in terms of time, one can determine the ignored angle ψ by integrating (3.54) with respect to time.

Remark 1: The Lagrangian (3.44) (and equations of motion derived from it in any generalized coordinates) is time-reversible, i.e. the Lagrangian and equations remain invariant if the sign of time t is changed. On the contrary, the Routhian and Routhian equations of motion are not time-reversible. They are invariant only on the simultaneous change of signs of t and f.

When $f = 0$, the Routhian becomes

$$
R = R_2 - V,
$$

and the Routhian equations of motion are time-reversible.

Remark 2: In the case of axial dynamical symmetry $B = A$, a significant simplification occurs in the Routhian (3.55). It renders to the form

$$
\begin{aligned}
R &= \frac{A}{2D}(C\sin^2\theta\dot{\varphi}^2 + D\dot{\theta}^2) + \frac{f}{D}C\cos\theta\dot{\varphi} - V_1, \\
V_1 &= V + \frac{f^2}{2D}.
\end{aligned} \tag{3.58}
$$

Finally, when $A = C$, i.e. in the case of complete dynamical symmetry, $D = A$ and we have

$$R = \frac{A}{2}(\sin^2\theta\dot{\varphi}^2 + \dot{\theta}^2) + f\cos\theta\dot{\varphi} - V(\theta, \varphi). \tag{3.59}$$

This is the Lagrangian of a particle moving on a smooth sphere with θ and φ as polar coordinates on that sphere. The particle is subject to forces with potential V and gyroscopic forces represented by the term linear in $\dot{\varphi}$. Note that the last term is proportional to f and thus vanishes when $f = 0$.

3.15 Exercises

(1) A heavy rigid body is moving about a fixed point, which is not coincident with the centre of mass of the body. Use the energy integral and Euler's equations to express the vector $\boldsymbol{\gamma}$ in terms of the angular velocity $\boldsymbol{\omega}$ and its time derivative $\dot{\boldsymbol{\omega}}$ in the form

$$\boldsymbol{\gamma} = \frac{1}{Mg|\mathbf{r}_0|^2}[\mathbf{r}_0 \times (\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I}) + (h - \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega})\mathbf{r}_0], \tag{3.60}$$

where h is the energy constant and $\mathbf{r}_0 \neq \mathbf{0}$ is the position vector of the centre of mass of the body with respect to the fixed point.

(2) Use the last result to reduce the equations of motion of the classical problem to the form of three autonomous first-order differential equations in the components of the angular velocity with respect to an arbitrary system of axes fixed in the body in time as independent variable to the form[1]

$$\begin{aligned}
(\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I}) \cdot \mathbf{r}_0 &= 0,\\
(h - \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega})^2 + (\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I})^2 &= M^2g^2|\mathbf{r}_0|^2,\\
\mathbf{r}_0 \cdot [(h - \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega})\boldsymbol{\omega}\mathbf{I} + \dot{\boldsymbol{\omega}}\mathbf{I} \times \boldsymbol{\omega}\mathbf{I} - |\boldsymbol{\omega}\mathbf{I}|^2\boldsymbol{\omega}] &= Mgf|\mathbf{r}_0|^2,
\end{aligned} \tag{3.61}$$

where f is the areas constant and other parameters as defined above.

Hint: Use the following equations:

$$\begin{aligned}
(\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I}) \cdot \mathbf{r}_0 &= 0,\\
|\boldsymbol{\gamma}|^2 &= 1,\\
\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\gamma} &= f.
\end{aligned}$$

[1] For this form, or (3.63), to be equivalent to the original Euler–Poisson system, a condition on the motion must be satisfied (See the two theorems in Sect. 8.1).

(3) Show that in terms of $\mathbf{G}$, the formula (3.60) and reduced Eq. (3.61) take the following form:

$$\gamma = \frac{1}{Mg|\mathbf{r}_0|^2}[\mathbf{r}_0 \times (\dot{\mathbf{G}} + \mathbf{GA} \times \mathbf{G}) + (h - \frac{1}{2}\mathbf{GA} \cdot \mathbf{G})\mathbf{r}_0], \tag{3.62}$$

and

$$\begin{aligned} (\dot{\mathbf{G}} + \mathbf{GA} \times \mathbf{G}) \cdot \mathbf{r}_0 &= 0, \\ (h - \frac{1}{2}\mathbf{GA} \cdot \mathbf{G})^2 + (\dot{\mathbf{G}} + \mathbf{GA} \times \mathbf{G})^2 &= M^2 g^2 |\mathbf{r}_0|^2, \\ \mathbf{r}_0 \cdot [(h - \frac{1}{2}\mathbf{GA} \cdot \mathbf{G})\mathbf{G} + \dot{\mathbf{G}} \times \mathbf{G} - |\mathbf{G}|^2 \mathbf{GA}] &= Mgf|\mathbf{r}_0|^2, \end{aligned} \tag{3.63}$$

where $\mathbf{A} = \mathbf{I}^{-1}$.