Chapter 2
Description of Rotation of a Rigid Body About a Fixed Point

Although in motion of a rigid body about a fixed point, we deal with continuous change of position, i.e. with a sequence of infinitesimal rotations, it turns out that the study of finite rotations is essential in understanding several concepts concerning infinitesimal ones. In this chapter, we present a brief theory of finite rotations to elucidate their properties and relation to infinitesimal rotations. We shall concentrate on the use of Euler's angles and the Euler–Rodrigues or quaternion coordinates as the most relevant to the Lagrangian approach. For space considerations, some other alternative descriptions, like Cayley–Klein, are not considered. We also use the most common notation. For a useful historical survey, the reader may consult [332] and several references therein.

2.1 The Position of a Rigid Body. Euler's Angles

The position of a rigid body moving about a fixed point O is completely determined by the position of a Cartesian coordinate system $Oxyz$ fixed in the body and moving with it with respect to the system $OXYZ$ fixed in space. The number of parameters necessary for the description must be three, the number of degrees of freedom of the rotational motion of the body. Several types of angles are used to this end, and their choice depends mainly on the suitability for the concrete problem of motion under consideration. For example, some angles are most suitable to use in the study of motion of a ship and others are suitable for describing the flight of a plane. Although, of course, different sets of angles must be equivalent for arbitrary position of the body, each of them can have specific privileges or drawbacks in regard to certain application.

One of the most frequently used sets of angles in many applications and especially for theoretical purposes are Euler's angles ψ, θ and φ. They are defined as follows (see Fig. 2.1):

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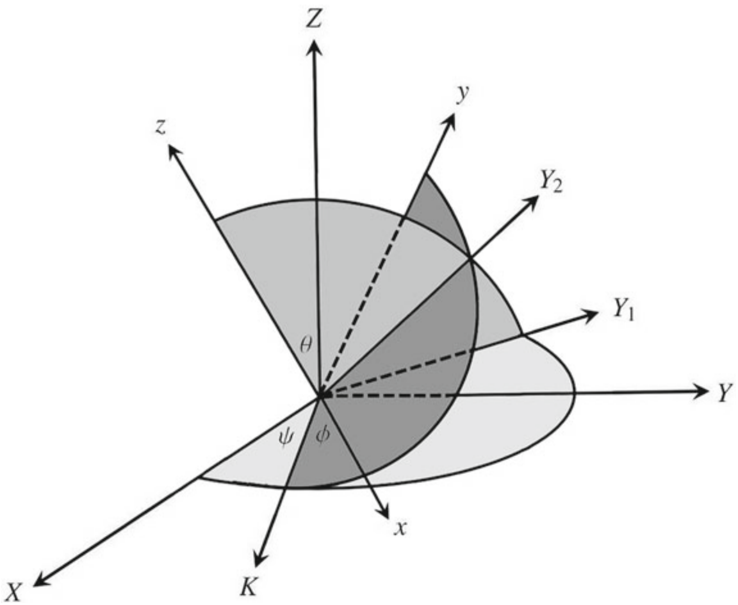

Fig. 2.1 Euler's angles

We begin by the body axes $Oxyz$ coinciding with the space-fixed axes $OXYZ$, and from that position, we give the body system a rotation by an angle ψ (the *precession* angle) about the Z-axis, so that the body system takes the position OKY_1Z. Then, the last system is given a rotation by an angle θ (the angle of *nutation*) about OK. This brings the body system to the position OKY_2z. We now fix the z-axis in the body and give the body system a rotation by an angle φ (the angle of *proper rotation*) about the z-axis to reach its final position $Oxyz$, fixed in the body. In this way, ψ is the angle of rotation of the body about the space axis Z, θ is the angle between z and Z and φ is the angle of rotation about z. The line OK is the intersection of the two planes Oxy and OXY. It is called the *line of nodes*.

Now, let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the space axes XYZ and the body axes xyz, respectively. Let also $\mathbf{n}$ be a unit vector along the nodal line OK and $\mathbf{j}_1, \mathbf{j}_2$ be unit vectors along OY_1 and OY_2, respectively. One can express the components of the fixed unit vectors with respect to the moving axes. For example,

$$\begin{aligned}
\boldsymbol{\alpha} &= \cos\psi\mathbf{n} - \sin\psi\mathbf{j}_1 \\
&= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, \sin\theta\sin\psi), \\
\boldsymbol{\beta} &= (\sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, -\sin\theta\cos\psi), \\
\boldsymbol{\gamma} &= (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta).
\end{aligned} \tag{2.1}$$

We can also write the unit vector $\mathbf{n}$ in the body and space axes, respectively, as

$$\mathbf{n} = \cos\varphi\mathbf{i} - \sin\varphi\mathbf{j} = \cos\psi\boldsymbol{\alpha} + \sin\psi\boldsymbol{\beta}. \tag{2.2}$$

Conversely, we can express the moving unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the fixed (space) basis. They have the form

$$\begin{aligned}
\mathbf{i} &= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, \sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, \sin\theta\sin\varphi),\\
\mathbf{j} &= (-\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, \sin\theta\cos\varphi),\\
\mathbf{k} &= (\sin\theta\sin\psi, -\sin\theta\cos\psi, \cos\theta).
\end{aligned} \tag{2.3}$$

2.2 The Rotation Matrix

Let (X, Y, Z) be the coordinates of a point P in the system of axes XYZ. When the system XYZ is rotated by an angle Φ around Z-axis, the point P is displaced to the new point $P'(X', Y', Z')$. One can easily write

$$\begin{aligned}
X' &= X\cos\Phi + Y\sin\Phi,\\
Y' &= -X\sin\Phi + Y\cos\Phi,\\
Z' &= Z,
\end{aligned}$$

which can be put in the matrix form

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{R}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

where

$$\mathbf{R} = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{R}$ is called the rotation matrix. Note that $\mathbf{R}$ is an orthogonal matrix (all rows are orthogonal and also columns), its inverse is its transpose ($\mathbf{R}^{-1} = \mathbf{R}^T$) and its determinant is 1.

Now we consider the characteristic equation of $\mathbf{R}$. That is

$$\begin{vmatrix} \cos\Phi - \lambda & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0.$$

In the final form, this becomes

$$(1-\lambda)(\lambda^2 - 2\lambda\cos\Phi + 1) = 0. \tag{2.4}$$

As we see, one of the characteristic roots equals 1. It corresponds to the eigenvector $(0, 0, 1)$, which coincides with the axis of the rotation. The other two roots are complex

$$\cos\Phi \pm \sqrt{\cos^2\Phi - 1} = \cos\Phi \pm i\sin\Phi = e^{\pm i\Phi}.$$

Thus, the eigenvector vector corresponding to the unit eigenvalue of the rotation matrix coincides with the axis of the rotation and the argument of the complex pair of eigenvalues directly expresses the angle of rotation.

Now we apply the same conception to an arbitrary rotation. Let $\mathbf{r}$ be a vector whose components are (X, Y, Z) in the space-fixed axes and (x, y, z) in the body-fixed axes. We can find the relations between the components of the vector in the two systems as follows:

The components of $\mathbf{r}$ in the system OKY_1Z after a rotation by an angle ψ around the Z-axis

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \mathbf{R}_\psi \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2.5}$$

and in the system OKY_2Z after a rotation by an angle θ around OK

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \mathbf{R}_\theta \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}. \tag{2.6}$$

Finally, after a rotation by an angle φ around the z-axis, we find the components of $\mathbf{r}$ in the body system

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathbf{R}_\varphi \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\ &= \mathbf{R} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \mathbf{R} &= \mathbf{R}_\varphi \mathbf{R}_\theta \mathbf{R}_\psi \\ &= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi & \sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi & \sin\theta\sin\varphi \\ -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi & -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi & \sin\theta\cos\varphi \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}. \tag{2.8}$$

Comparing with (2.1), we conclude that

$$\mathbf{R} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

and again we note that $\mathbf{R}$ has all rows and columns orthogonal unit vectors. Its inverse is its transpose $\mathbf{R}^{-1} = \mathbf{R}^T$ (check that $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \boldsymbol{\delta}$) and its determinant equals 1. Note also that the columns of are the components of the fixed unit vectors referred to the moving (body) axes while its rows are the components of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, in the directions of the movable axes xyz, referred to the fixed axes XYZ.

2.2.1 The Angle of Rotation

We now proceed to form the characteristic equation of the rotation matrix (2.8). After some manipulations and factorization, we get

$$\begin{aligned} |\mathbf{R} - \lambda\boldsymbol{\delta}| &= (1-\lambda)\{\lambda^2 + [(1-\cos\theta) - (1+\cos\theta)\cos(\psi+\varphi)]\lambda + 1\} \\ &= (1-\lambda)\{\lambda^2 + 2[1 - 2\cos^2\frac{\theta}{2}\cos^2\frac{\psi+\varphi}{2}]\lambda + 1\} = 0. \end{aligned} \tag{2.9}$$

From here, we see that one of the characteristic roots of the rotation matrix is 1. The eigenvector $\mathbf{v}$ corresponding to that root satisfies the equation

$$\mathbf{R}\mathbf{a} = \mathbf{a},$$

i.e. the rotation represented by the matrix $\mathbf{R}$ leaves that vector unchanged. This vector coincides with the axis of the rotation. On the other hand, to obtain the rotation angle of the rotation Φ, we compare the quadratic factors in (2.9) and (2.4). We get

$$\cos\Phi = 2\cos^2\frac{\theta}{2}\cos^2\frac{\psi+\varphi}{2} - 1. \tag{2.10}$$

This can be also written in the form

$$\cos\frac{\Phi}{2} = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2} \tag{2.11}$$

in which we have chosen positive sign.

Remark: In spite of their simplicity, Euler's angles suffer from the defect that the two angles ψ and φ lose their independence when the third angle θ takes one of the values 0 or π. In those cases, the rotation matrix takes the form

$$\begin{pmatrix} \cos(\psi \pm \varphi) & \sin(\psi \pm \varphi) & 0 \\ -\sin(\psi \pm \varphi) & \cos(\psi \pm \varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which involves not the two angles, but their sum when $\theta = 0$ and difference when $\theta = \pi$.

2.3 Description of Finite Rotation

We have just seen that the finite rotation can be always and completely represented by a rotation matrix. On the other hand, it is evident that such rotation can be completely determined by giving the axis of rotation and the angle of rotation about that axis, i.e. one can say the rotation is determined by a scalar quantity and a direction. Nevertheless, it cannot be represented in the full sense by a vector, since an essential rule of vector algebra, the commutation rule, is not followed by matrices. This means that performing two consequent rotations R_1 and then R_2 gives different resultant from that of reverse order R_2 and then R_1. This can be clearly illustrated by the following

Example: Let us perform to the body in Fig. 2.2a two consecutive rotations, each by a right angle,[1] the first about the x-axis and the second about the y-axis. Figure 2.2b shows what we get in this case, but Fig. 2.2c shows the completely different result of performing the rotations in the reverse order.

Analytically, R_1 and R_2 can be represented by the matrices

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \mathbf{R}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Resultant of the first sequence is

$$\mathbf{S} = \mathbf{R}_2\mathbf{R}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the second sequence

[1] We mean rotation by an angle described in the positive sense about an axis, i.e. counterclockwise as viewed from the positive end of that axis.

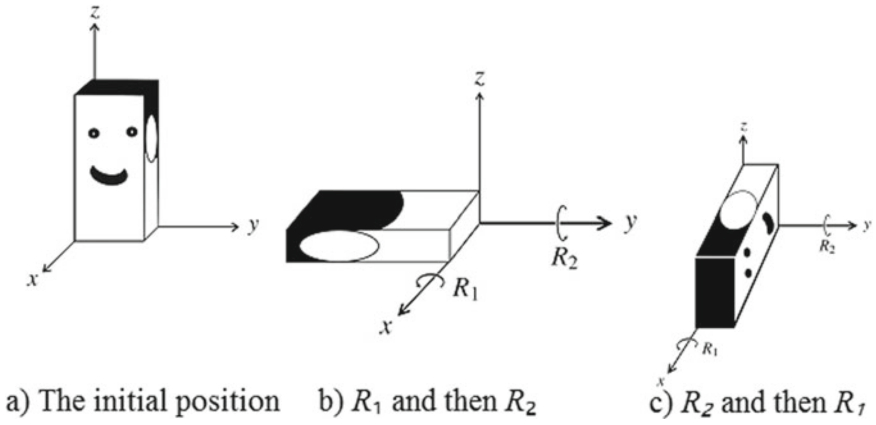

Fig. 2.2 Finite rotations are not commutative

$$\mathbf{S}' = \mathbf{R}_1\mathbf{R}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

The two matrices **S** and **S**′ are obviously different. We can go a little further and construct the characteristic equation for **S**. That is

$$1 - \lambda^3 = (1 - \lambda)(\lambda^2 + \lambda + 1) = 0,$$

and its roots are $1, e^{\pm i\frac{2\pi}{3}}$. This means that the angle of the rotation **S** is equal to $\frac{2\pi}{3}$. To find the axis of the rotation, we solve the equations

$$\mathbf{S}\mathbf{v}' = \mathbf{v}',$$

which gives the column vector

$$v' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, **S** is a rotation by angle $\frac{2\pi}{3}$ around the axis in the direction parallel to the vector $\mathbf{v} = (1, 1, 1)$.

Similarly, one can show that **S**′ is a rotation by angle $\frac{2\pi}{3}$ around the vector $\mathbf{v} = (1, 1, -1)$.

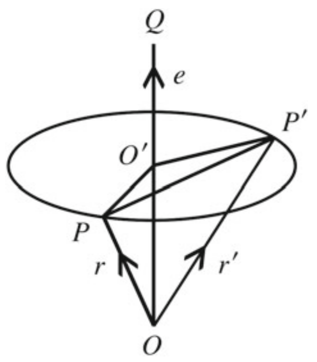

Fig. 2.3 Geometry of a rotation vector

2.4 Representation of Finite Rotation by Means of a Vector

The above example shows that finite rotation cannot be completely represented by a vector as we represent the position vector or the velocity of a particle. Thus, we shall try now to find the formula that expresses the finite rotation as a vector quantity and to find a suitable rule for the resultant of two rotations, a rule that must account for the non-commutation of rotations.

2.4.1 The Rotation Vector

Let us begin with some vector $\mathbf{r} = \overrightarrow{OP}$ and an axis $\overline{OQ}$ with a unit vector $\mathbf{e}$ in its direction. The rotation of $\mathbf{r}$ by an angle Φ around OQ in the positive direction carries $\mathbf{r}$ to its new position $\mathbf{r}' = \overrightarrow{OP'}$ and the point P along the circular arc $\widehat{PP'}$ to P' (4.1). Let also O' be the centre of PP'. Our aim now is to express $\mathbf{r}'$ in terms of $\mathbf{r}$ and the angle and direction of the rotation (see Fig. 2.3).

The plane Fig. 2.4 shows the circle $O'PP'$. $O'R$ is orthogonal to PP' and RS is orthogonal to $O'P$. Note that $\mathbf{e}$ is the outward unit vector normal to the plane of the figure and $SR = O'P \sin\frac{\Phi}{2}\cos\frac{\Phi}{2}$, so that $SR = \mathbf{e} \times O'P \sin\frac{\Phi}{2}\cos\frac{\Phi}{2}$.

From simple geometry, we find that

$$PP' = 2PR = 2(PS + SR) = 2[-O'P\sin^2\frac{\Phi}{2} + \mathbf{e} \times O'P \sin\frac{\Phi}{2}\cos\frac{\Phi}{2}].$$

But since

$$O'P = \mathbf{r} - (\mathbf{r}\cdot\mathbf{e})\mathbf{e} = (\mathbf{e}\cdot\mathbf{e})\mathbf{r} - (\mathbf{e}\cdot\mathbf{r})\mathbf{e} = -\mathbf{e} \times (\mathbf{e} \times \mathbf{r}),$$

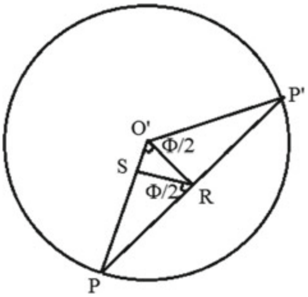

Fig. 2.4 Rotation angle

then we obtain

$$PP' = 2[\sin^2 \frac{\Phi}{2}\mathbf{e} \times (\mathbf{e} \times \mathbf{r}) + \sin \frac{\Phi}{2} \cos \frac{\Phi}{2}\mathbf{e} \times \mathbf{r}],$$

so that we can finally write

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + PP' \\ &= \mathbf{r} + 2[\sin \frac{\Phi}{2} \cos \frac{\Phi}{2}\mathbf{e} \times \mathbf{r} + \sin^2 \frac{\Phi}{2}\mathbf{e} \times (\mathbf{e} \times \mathbf{r})].\end{aligned} \tag{2.12}$$

This is the Rodrigues formula, which expresses the rotated vector $\mathbf{r}'$ in terms of the initial vector $\mathbf{r}$, the angle of rotation Φ and the direction $\mathbf{e}$ of the rotation axis. It can be written also in the form

$$\mathbf{r}' = \mathbf{r} + (\sin \Phi)\mathbf{e} \times \mathbf{r} + (1 - \cos \Phi)\mathbf{e} \times (\mathbf{e} \times \mathbf{r}). \tag{2.13}$$

It is valid for arbitrary angle and arbitrary direction of the rotation. As expected, for a point on the axis of rotation, $\mathbf{r} = \mathbf{e}$ and $\mathbf{r}' = \mathbf{r}$. Also, a rotation with an angle 2π brings all points of space to their initial positions.

To push forward the concept of a vector representing a finite rotation, we assume that the rotation angle $\Phi \neq \pi$, i.e. $\cos \frac{\Phi}{2} \neq 0$. Then we can write (2.12) in the form

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + 2\cos^2 \frac{\Phi}{2}[\tan \frac{\Phi}{2}\mathbf{e} \times \mathbf{r} + \tan^2 \frac{\Phi}{2}\mathbf{e} \times (\mathbf{e} \times \mathbf{r})] \\ &= \mathbf{r} + \frac{2}{1 + \tan^2 \frac{\Phi}{2}}[\tan \frac{\Phi}{2}\mathbf{e} \times \mathbf{r} + \tan^2 \frac{\Phi}{2}\mathbf{e} \times (\mathbf{e} \times \mathbf{r})].\end{aligned}$$

Introducing the notation

$$\boldsymbol{\rho} = \tan \frac{\Phi}{2}\mathbf{e}, \tag{2.14}$$

then we rewrite the last formula as

$$\mathbf{r}' = \mathbf{r} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times [\mathbf{r} + \boldsymbol{\rho} \times \mathbf{r}]. \tag{2.15}$$

Using (2.14), one can verify that

(1) $\boldsymbol{\rho}(\Phi + 2\pi, \mathbf{e}) = \boldsymbol{\rho}(\Phi, \mathbf{e})$.
(2) $\boldsymbol{\rho}(-\Phi, -\mathbf{e}) = \boldsymbol{\rho}(\Phi, \mathbf{e})$.
Those properties are geometrically obvious.
(3) Formula (2.14) is not suitable for expressing any rotation with an angle π about any axis. That is the singular point of the function tan . This is not related to the rotation itself, but due to the way of representing the rotation as a vector in (2.14). The previous formulas (2.12), (2.13) are still valid for the angle $\Phi = \pi$.
(4) Finite rotation is thus represented by a vector, which may be written in terms of its components as $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$ or $\boldsymbol{\rho} = \rho_1\mathbf{i} + \rho_2\mathbf{j} + \rho_3\mathbf{k}$, but this determines the magnitude and direction of the rotation and does not mean at all that the rotation is the resultant of its parts, or equivalent to any sequence of those parts.
(5) It is evident that rotation vectors do not commute and cannot be summed according to rules of vector algebra. However, it can be easily shown that infinitesimally small rotations do commute and obey the rule of summation of vectors.
Let $\boldsymbol{\rho}_1$ be a rotation by a small angle Φ_1 the rotation vector $\boldsymbol{\rho}_1 = \tan\frac{\Phi_1}{2}\mathbf{e}_1 = \frac{\Phi_1}{2}\mathbf{e}_1$. After neglecting nonlinear terms in the rotation vector, formula (2.15) takes the form

$$\mathbf{r}' = \mathbf{r} + 2\boldsymbol{\rho}_1 \times \mathbf{r}. \tag{2.16}$$

If $\boldsymbol{\rho}_1$ is followed by another small rotation $\boldsymbol{\rho}_2 = \frac{\Phi_2}{2}\mathbf{e}_2$, the vector $\mathbf{r}'$ is transformed to

$$\begin{aligned}\mathbf{r}'' &= \mathbf{r}' + 2\boldsymbol{\rho}_2 \times \mathbf{r}' \\ &= \mathbf{r} + 2\boldsymbol{\rho}_1 \times \mathbf{r} + 2\boldsymbol{\rho}_2 \times (\mathbf{r} + 2\boldsymbol{\rho}_1 \times \mathbf{r}).\end{aligned} \tag{2.17}$$

Neglecting the nonlinear term, we get

$$\mathbf{r}'' = \mathbf{r} + 2(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2) \times \mathbf{r}. \tag{2.18}$$

Small rotations are summed according to vector addition rule, and their sum does not depend on the order of the rotations.

Now we return to formula (2.14) to see how a rotation $\boldsymbol{\rho} = \tan\frac{\Phi}{2}\mathbf{e}$ acts on the unit vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ fixed in the directions of XYZ and bring them to be coincident with the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively. According to (2.15), we have

$$\mathbf{i} = \boldsymbol{\alpha} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times (\boldsymbol{\alpha} + \boldsymbol{\rho} \times \boldsymbol{\alpha}),$$
$$\mathbf{j} = \boldsymbol{\beta} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times (\boldsymbol{\beta} + \boldsymbol{\rho} \times \boldsymbol{\beta}),$$
$$\mathbf{k} = \boldsymbol{\gamma} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times (\boldsymbol{\gamma} + \boldsymbol{\rho} \times \boldsymbol{\gamma}). \tag{2.19}$$

One immediately notices that

$$\boldsymbol{\rho} \cdot \mathbf{i} = \boldsymbol{\rho} \cdot \boldsymbol{\alpha}, \boldsymbol{\rho} \cdot \mathbf{j} = \boldsymbol{\rho} \cdot \boldsymbol{\beta}, \boldsymbol{\rho} \cdot \mathbf{k} = \boldsymbol{\rho} \cdot \boldsymbol{\gamma},$$

i.e. the components of the rotation vector are the same in the directions of the initial and final axes. We shall denote those components by $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$.

We now express the rotation matrix in terms of the components of the rotation vector

$$R = \begin{pmatrix} \mathbf{i}\cdot\boldsymbol{\alpha} & \mathbf{i}\cdot\boldsymbol{\beta} & \mathbf{i}\cdot\boldsymbol{\gamma} \\ \mathbf{j}\cdot\boldsymbol{\alpha} & \mathbf{j}\cdot\boldsymbol{\beta} & \mathbf{j}\cdot\boldsymbol{\gamma} \\ \mathbf{k}\cdot\boldsymbol{\alpha} & \mathbf{k}\cdot\boldsymbol{\beta} & \mathbf{k}\cdot\boldsymbol{\gamma} \end{pmatrix}$$
$$= \begin{pmatrix} 1 - 2\frac{\rho_2^2+\rho_3^2}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_1\rho_2+\rho_3}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_1\rho_3-\rho_2}{1+\boldsymbol{\rho}^2} \\ 2\frac{\rho_1\rho_2-\rho_3}{1+\boldsymbol{\rho}^2} & 1 - 2\frac{\rho_1^2+\rho_3^2}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_2\rho_3+\rho_1}{1+\boldsymbol{\rho}^2} \\ 2\frac{\rho_1\rho_3+\rho_2}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_2\rho_3-\rho_1}{1+\boldsymbol{\rho}^2} & 1 - 2\frac{\rho_1^2+\rho_2^2}{1+\boldsymbol{\rho}^2} \end{pmatrix}. \tag{2.20}$$

2.5 Hamilton–Rodrigues' Parameters

We now introduce the four quantities $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to express the three components of the rotation vector, such that

$$(\rho_1, \rho_2, \rho_3) = \frac{(\lambda_1, \lambda_2, \lambda_3)}{\lambda_0}. \tag{2.21}$$

As we have one redundant parameter, we assume that the new parameters satisfy the condition

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1, \tag{2.22}$$

so that the end of the four-dimensional vector $\boldsymbol{\Lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ lies on a three-dimensional sphere of unit radius. This implies the relation

$$\lambda_0\dot{\lambda}_0 + \lambda_1\dot{\lambda}_1 + \lambda_2\dot{\lambda}_2 + \lambda_3\dot{\lambda}_3 = 0. \tag{2.23}$$

From (2.21), we calculate

$$1+\rho^2 = 1+\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{\lambda_0^2} = \frac{1}{\lambda_0^2}, \tag{2.24}$$

and substituting (2.21) in (2.20) and using the last relation, we obtain the expression of the rotation matrix

$$\mathbf{R} = \begin{pmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & 2(-\lambda_0\lambda_2+\lambda_1\lambda_3) \\ 2(-\lambda_0\lambda_3+\lambda_1\lambda_2) & \lambda_0^2+\lambda_2^2-\lambda_1^2-\lambda_3^2 & 2(\lambda_0\lambda_1+\lambda_2\lambda_3) \\ 2(\lambda_0\lambda_2+\lambda_1\lambda_3) & 2(-\lambda_0\lambda_1+\lambda_2\lambda_3) & \lambda_0^2+\lambda_3^2-\lambda_1^2-\lambda_2^2 \end{pmatrix}. \tag{2.25}$$

This form of the rotation matrix is more symmetric than that in terms of the rotation vector or in terms of Euler's angles. Moreover, it does not have the problem of degeneration of Euler's angles at $\theta = 0$ or π, nor the singularity of the rotation vector corresponding to a rotation by an angle π. This makes Euler–Rodrigues' parameters in certain problems appropriate for use as variables describing motion and finite rotations of the rigid body.

On the other hand, one can readily notice that the two sets of the Hamilton–Rodrigues parameters $\pm\Lambda$ correspond to the same rotation matrix. Thus, any expression designating a quantity of physical meaning should contain only even terms in Λ, otherwise it will be double-valued on the group of rotations $SO3$. This remark will have some implications in later chapters.

Remark 12 The expression (2.25) for the rotation matrix can be decomposed into three parts of simpler structure (two symmetric and one antisymmetric):

$$\mathbf{R} = (2\lambda_0^2-1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{pmatrix}$$
$$+2\lambda_0\begin{pmatrix} 0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & 0 \end{pmatrix} \tag{2.26}$$

or in the shorter tensor form

$$R_{ij} = (2\lambda_0^2-1)\delta_{ij} + 2\lambda_i\lambda_j + 2\lambda_0\epsilon_{ijk}\lambda_k, \tag{2.27}$$

where δ is the Kronecker delta and ϵ is the Levi-Civita tensor.

Remark 13 It is clear from (2.25) that the points $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ and $(-\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3)$ represent the same rotation matrix. The sphere (2.22) covers the configuration space of the rotating body twice. The configuration space can, thus, be represented by one half of that sphere, say, the half on which $\lambda_0 \geqq 0$.

Remark 14 From (2.24) and (2.14), we have

$$\frac{1}{\lambda_0^2} = 1+\rho^2 = 1+\tan^2\frac{\Phi}{2} = \sec^2\frac{\Phi}{2},$$

so that

$$\lambda_0 = \cos\frac{\Phi}{2}, \tag{2.28}$$

and hence

$$\begin{aligned}(\lambda_1, \lambda_2, \lambda_3) &= \lambda_0 \boldsymbol{\rho} \\ &= \cos\frac{\Phi}{2}\tan\frac{\Phi}{2}\mathbf{e} \\ &= \sin\frac{\Phi}{2}\mathbf{e}.\end{aligned}$$

Thus, we can write the following expression for the Euler–Rodriguez parameters

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (\cos\frac{\Phi}{2}, \sin\frac{\Phi}{2}\mathbf{e}). \tag{2.29}$$

2.6 The Angular Velocity Vector

During motion of the body, Euler's angles change with time t, and also the rotation matrix. To express the time derivatives of an arbitrary quantity, one may first check the following relations using expressions (2.1)

$$\begin{aligned}\tfrac{\partial\boldsymbol{\alpha}}{\partial\psi} &= \boldsymbol{\alpha}\times\boldsymbol{\gamma},\ \tfrac{\partial\boldsymbol{\beta}}{\partial\psi} = \boldsymbol{\beta}\times\boldsymbol{\gamma},\ \tfrac{\partial\boldsymbol{\gamma}}{\partial\psi} = 0,\\ \tfrac{\partial\boldsymbol{\alpha}}{\partial\theta} &= \boldsymbol{\alpha}\times\mathbf{n},\ \tfrac{\partial\boldsymbol{\beta}}{\partial\theta} = \boldsymbol{\beta}\times\mathbf{n},\ \tfrac{\partial\boldsymbol{\gamma}}{\partial\theta} = \boldsymbol{\gamma}\times\mathbf{n},\\ \tfrac{\partial\boldsymbol{\alpha}}{\partial\varphi} &= \boldsymbol{\alpha}\times\mathbf{k},\ \tfrac{\partial\boldsymbol{\beta}}{\partial\varphi} = \boldsymbol{\beta}\times\mathbf{k},\ \tfrac{\partial\boldsymbol{\gamma}}{\partial\varphi} = \boldsymbol{\gamma}\times\mathbf{k}.\end{aligned} \tag{2.30}$$

From those, we get

$$\dot{\boldsymbol{\alpha}} = \frac{\partial\boldsymbol{\alpha}}{\partial\psi}\dot{\psi} + \frac{\partial\boldsymbol{\alpha}}{\partial\theta}\dot{\theta} + \frac{\partial\boldsymbol{\alpha}}{\partial\varphi}\dot{\varphi} \tag{2.31}$$

$$\begin{aligned}&= \boldsymbol{\alpha}\times\boldsymbol{\gamma}\dot{\psi} + \boldsymbol{\alpha}\times\mathbf{n}\dot{\theta} + \boldsymbol{\alpha}\times\mathbf{k}\dot{\varphi}\\ &= \boldsymbol{\alpha}\times(\dot{\psi}\boldsymbol{\gamma}+\dot{\theta}\mathbf{n}+\dot{\varphi}\mathbf{k}).\end{aligned} \tag{2.32}$$

Let us now introduce the notation

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{\gamma}+\dot{\theta}\mathbf{n}+\dot{\varphi}\mathbf{k}. \tag{2.33}$$

The vector $\boldsymbol{\omega}$ is called the angular velocity of the body and it is, in fact, the usual vector sum of the three vectors $\dot{\psi}\boldsymbol{\gamma}, \dot{\theta}\mathbf{n}, \dot{\varphi}\mathbf{k}$, which represent the angular velocities $\dot{\psi}$, $\dot{\theta}$ and $\dot{\varphi}$ about the axes Z, K and z, respectively. In a similar way, we can get two expressions for the derivatives of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, so that we can write

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}. \tag{2.34}$$

Those equations satisfied by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are called Poisson's equations and they express the constancy of those vectors in space, as we shall see soon. They play an important role in the dynamics of rigid body as will be seen in due course.

2.7 Space and Relative Time Rates of Change of a Vector

During the motion of the body, the three unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ fixed in the body along the axes x, y, z change with time. The rate of change of one of them, $\mathbf{i}$, say, is the velocity $\frac{d\mathbf{i}}{dt}$ of its end point in space. Hence, we have

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i}, \frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j}, \frac{d\mathbf{k}}{dt} = \boldsymbol{\omega} \times \mathbf{k}, \tag{2.35}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the body.

Now, let $\mathbf{u}$ be a vector given by its components (u_1, u_2, u_3) in the body system of axes, so that we write

$$\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}. \tag{2.36}$$

The time derivative of this vector is

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{i}+\frac{du_2}{dt}\mathbf{j}+\frac{du_3}{dt}\mathbf{k}+u_1\frac{d\mathbf{i}}{dt}+u_2\frac{d\mathbf{j}}{dt}+u_3\frac{d\mathbf{k}}{dt}.$$

Using (2.35) in the last three terms, we get

$$\begin{aligned}\frac{d\mathbf{u}}{dt} &= \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k} + u_1\boldsymbol{\omega} \times \mathbf{i}+u_2\boldsymbol{\omega} \times \mathbf{j} + u_3\boldsymbol{\omega} \times \mathbf{k} \\ &= \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k} + \boldsymbol{\omega} \times (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}).\end{aligned}$$

Now we introduce the notation $\dot{\mathbf{u}} = \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k}$, i.e. $\dot{\mathbf{u}}$ is the time derivative of the vector $\mathbf{u}$ as if the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ were constant vectors, or as seen by an observer fixed in the body and moving with it. This derivative will be called the relative derivative or the relative rate of change of $\mathbf{u}$. The last relation becomes

$$\frac{d\mathbf{u}}{dt} = \dot{\mathbf{u}}+\boldsymbol{\omega}\times\mathbf{u}. \tag{2.37}$$

Thus, we have split the space derivative into two terms: the relative derivative $\dot{\mathbf{u}}$ in the body system and the term $\boldsymbol{\omega} \times \mathbf{u}$ resulting from the rotation of the body system. When $\boldsymbol{\omega} = \mathbf{0}$ the two derivatives coincide.

As an example, we apply this rule to the three vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ fixed in space. As $\frac{d\boldsymbol{\alpha}}{dt} = \frac{d\boldsymbol{\beta}}{dt} = \frac{d\boldsymbol{\gamma}}{dt} = 0$, we have

$$\dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}, \dot{\boldsymbol{\beta}} + \boldsymbol{\omega} \times \boldsymbol{\beta} = \mathbf{0}, \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{2.38}$$

so that we again obtain Poisson's equations (2.34).

2.7.1 Components of the Angular Velocity in the Body Axes and Space Axes

The direction of the angular velocity at the fixed point determines a line called the instantaneous axis of rotation. Points of the body lying on that line at any moment of time are instantaneously at rest. The magnitude of $\boldsymbol{\omega}$ is a measure of the angular speed of rotation of the body. In case of rotation about a fixed axis, the angular velocity is the time rate of change of the angle of rotation about that axis, but it is not possible in general to write a rotation angle such that the angular velocity is represented as its rate of change.

We shall denote by p, q, r and p', q', r' the components of the angular velocity $\boldsymbol{\omega}$ in the moving and in the fixed axes, respectively. Using (2.2) and (2.3) together with (2.33), we have

$$\begin{aligned} p &= \dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi, \\ q &= \dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi, \\ r &= \dot{\psi} \cos\theta + \dot{\varphi}, \end{aligned} \tag{2.39}$$

and similarly, we write

$$\begin{aligned} p' &= \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \\ q' &= -\dot{\varphi} \sin\theta \cos\psi + \dot{\theta} \sin\psi, \\ r' &= \dot{\varphi} \cos\theta + \dot{\psi}. \end{aligned} \tag{2.40}$$

2.7.2 The Use of the Euler–Rodrigues Parameters

From (2.11) and (2.28), we have $\lambda_0 = \cos\frac{\theta}{2} \cos\frac{\psi+\varphi}{2}$. One can easily obtain expressions for the other three parameters by comparing corresponding elements of the rotation matrix $\mathbf{R}$ in formulas (2.8) and (2.25). It is even easier to compare the antisymmetric parts, e.g. one can see from (2.26) and (2.8) that

$$\lambda_1 = \frac{R_{23} - R_{32}}{4\lambda_0} = \frac{\sin\theta(\cos\varphi + \cos\psi)}{4\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}}$$
$$= \sin\frac{\theta}{2}\cos\frac{\psi - \varphi}{2}$$

and so on, so that we get the expressions

$$\lambda_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2},\ \lambda_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2},$$
$$\lambda_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2},\ \lambda_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}. \tag{2.41}$$

The angular velocity has the expression

$$\begin{aligned} p &= 2(\lambda_0\dot\lambda_1 - \lambda_1\dot\lambda_0 + \lambda_3\dot\lambda_2 - \lambda_2\dot\lambda_3),\\ q &= 2(\lambda_0\dot\lambda_2 - \lambda_2\dot\lambda_0 + \lambda_1\dot\lambda_3 - \lambda_3\dot\lambda_1),\\ r &= 2(\lambda_0\dot\lambda_3 - \lambda_3\dot\lambda_0 + \lambda_2\dot\lambda_1 - \lambda_1\dot\lambda_2). \end{aligned} \tag{2.42}$$

This may be written in the vector form

$$\boldsymbol\omega = 2[\lambda_0\dot{\boldsymbol\lambda} - \dot\lambda_0\boldsymbol\lambda - \boldsymbol\lambda \times \dot{\boldsymbol\lambda}], \tag{2.43}$$

where $\boldsymbol\lambda$ denotes the three-dimensional vector $(\lambda_1, \lambda_2, \lambda_3)$. In this notation, (2.22) and (2.23) take the form

$$\lambda_0^2 + \boldsymbol\lambda^2 = 1, \tag{2.44}$$
$$\lambda_0\dot\lambda_0 + \boldsymbol\lambda\cdot\dot{\boldsymbol\lambda} = 0. \tag{2.45}$$

The formula (2.43) together with (2.44), (2.45) can be used to obtain a remarkable expression for the square of the angular velocity. Squaring both sides of (2.43) and noting that the third term on the right-hand side is orthogonal to the other two, we write

$$\begin{aligned} \omega^2 &= 4[\lambda_0^2\dot{\boldsymbol\lambda}^2 - 2\lambda_0\dot\lambda_0\boldsymbol\lambda\cdot\dot{\boldsymbol\lambda}\dot{\boldsymbol\lambda} + \dot\lambda_0^2\boldsymbol\lambda^2 + |\boldsymbol\lambda\times\dot{\boldsymbol\lambda}|^2]\\ &= 4[\lambda_0^2\dot{\boldsymbol\lambda}^2 - 2\lambda_0\dot\lambda_0\boldsymbol\lambda\cdot\dot{\boldsymbol\lambda} + \dot\lambda_0^2\boldsymbol\lambda^2 + \boldsymbol\lambda^2\dot{\boldsymbol\lambda}^2 - (\boldsymbol\lambda\cdot\dot{\boldsymbol\lambda})^2]\\ &= 4[(\lambda_0^2 + \boldsymbol\lambda^2)\dot{\boldsymbol\lambda}^2 + 2\lambda_0^2\dot\lambda_0^2 + \dot\lambda_0^2\boldsymbol\lambda^2 - (\lambda_0\dot\lambda_0)^2\\ &= 4[(\lambda_0^2 + \boldsymbol\lambda^2)\dot{\boldsymbol\lambda}^2 + (\lambda_0^2 + \boldsymbol\lambda^2)\dot\lambda_0^2]\\ &= 4(\lambda_0^2 + \boldsymbol\lambda^2)(\dot\lambda_0^2 + \dot{\boldsymbol\lambda}^2)\\ &= 4(\dot\lambda_0^2 + \dot{\boldsymbol\lambda}^2), \end{aligned}$$

so that, finally, we have

$$\omega^2 = p^2 + q^2 + r^2 = 4(\dot{\lambda}_1^2 + \dot{\lambda}_2^2 + \dot{\lambda}_3^2 + \dot{\lambda}_0^2) = 4\dot{\mathbf{\Lambda}}^2. \tag{2.46}$$

That is four times the square of the speed of the point $\mathbf{\Lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ moving on the unit sphere (2.22).

2.8 Quaternions and Representation of Finite Rotation

Quaternions, or hypercomplex numbers, discovered by Hamilton, are a generalization of the ordinary complex number system. A quaternion is composed of one real component and three imaginary ones. A general quaternion can be written in the form

$$Q = (a, A_1, A_2, A_3) = a + A_1 i + A_2 j + A_3 k, \tag{2.47}$$

where i, j, k are imaginary units satisfying the multiplication rules

$$\begin{aligned} &i^2 = j^2 = k^2 = 1, \\ &ij = -ji = k, \; jk = -kj = i, ki = -ik = j. \end{aligned} \tag{2.48}$$

In (2.47), the first part a is an ordinary real part and the remaining parts can be viewed as a vector $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, and thus, we write

$$Q = a + \mathbf{A}. \tag{2.49}$$

Now it is easy to check that the product of the quaternions Q and $Q' = a' + \mathbf{A}'$ according to the rules (2.48) can be put in the usual form using scalar and vector products of vectors as

$$QQ' = aa' - \mathbf{A} \cdot \mathbf{A}' + a\mathbf{A}' + a'\mathbf{A} + \mathbf{A} \times \mathbf{A}', \tag{2.50}$$

and if we define the conjugate quaternion $\bar{Q} = a - \mathbf{A}$, we easily note that the quantity

$$Q\bar{Q} = \bar{Q}Q = a^2 + \mathbf{A} \cdot \mathbf{A} \tag{2.51}$$

is a positive real number which we adopt as the squared magnitude of the quaternion

$$|Q| = \sqrt{Q\bar{Q}} = \sqrt{a^2 + A_1^2 + A_2^2 + A_3^2}. \tag{2.52}$$

From the last, we get that for a non-zero quaternion Q

$$Q\frac{\bar{Q}}{|Q|^2} = 1. \tag{2.53}$$

That is the multiplicative inverse of Q (which satisfies $QQ^{-1} = Q^{-1}Q = 1$) is $Q^{-1} = \frac{\bar{Q}}{|Q|^2}$.

Consider now the quaternion formed by Euler–Rodrigues' parameters $Q = \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k$. It can be put in the form

$$Q = \lambda_0 + \lambda_0 \boldsymbol{\rho} = \cos\frac{\Phi}{2}(1 + \boldsymbol{\rho}). \tag{2.54}$$

Note that

$$|Q| = 1,\ Q^{-1} = \cos\frac{\Phi}{2}(1 - \boldsymbol{\rho}). \tag{2.55}$$

Let $\mathbf{r}$ be a vector, i.e. a quaternion with zero real part. Applying (2.50), we calculate the product

$$\begin{aligned}
Q\mathbf{r}Q^{-1} &= \cos^2\frac{\Phi}{2}(1+\boldsymbol{\rho})[(0+\mathbf{r})(1-\boldsymbol{\rho})] \\
&= \cos^2\frac{\Phi}{2}(1+\boldsymbol{\rho})(\boldsymbol{\rho}\cdot\mathbf{r} + \mathbf{r} + \boldsymbol{\rho}\times\mathbf{r}) \\
&= \cos^2\frac{\Phi}{2}[\boldsymbol{\rho}\cdot\mathbf{r} + \mathbf{r} + \boldsymbol{\rho}\times\mathbf{r} - \boldsymbol{\rho}\cdot(\mathbf{r}+\boldsymbol{\rho}\times\mathbf{r}) \\
&\quad +(\boldsymbol{\rho}\cdot\mathbf{r})\boldsymbol{\rho} + \boldsymbol{\rho}\times(\mathbf{r}+\boldsymbol{\rho}\times\mathbf{r})] \\
&= \cos^2\frac{\Phi}{2}[\mathbf{r} + 2\boldsymbol{\rho}\times\mathbf{r} + (\boldsymbol{\rho}\cdot\mathbf{r})\boldsymbol{\rho} + \boldsymbol{\rho}\times(\boldsymbol{\rho}\times\mathbf{r})] \\
&= \cos^2\frac{\Phi}{2}[\mathbf{r} + 2\boldsymbol{\rho}\times\mathbf{r} + 2\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\mathbf{r}) + \boldsymbol{\rho}^2\mathbf{r}] \\
&= \cos^2\frac{\Phi}{2}[(1+\boldsymbol{\rho}^2)\mathbf{r} + 2\boldsymbol{\rho}\times\mathbf{r} + 2\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\mathbf{r})] \\
&= \cos^2\frac{\Phi}{2}(1+\boldsymbol{\rho}^2)[],
\end{aligned}$$

and using (2.14), we finally get

$$Q\mathbf{r}Q^{-1} = \mathbf{r} + \frac{2\boldsymbol{\rho}}{1+\boldsymbol{\rho}^2}\times(\mathbf{r} + \boldsymbol{\rho}\times\mathbf{r}). \tag{2.56}$$

Comparing this formula with (2.15), we note that the rotation $\boldsymbol{\rho}$ transfers the vector $\mathbf{r}$ to

$$\mathbf{r}' = Q\mathbf{r}Q^{-1}, \tag{2.57}$$

so that the rotation $\boldsymbol{\rho}$ is completely determined by the quaternion Q of unit magnitude.

Also, we have

$$
\begin{aligned}
Q^{-1}\mathbf{r}'Q &= Q^{-1}(Q\mathbf{r}Q^{-1})Q \\
&= (Q^{-1}Q)\mathbf{r}(Q^{-1}Q) \\
&= \mathbf{r},
\end{aligned}
$$

so that the inverse of the rotation is given by the quaternion $Q^{-1} = \bar{Q}$.

2.9 Composition of Two Rotations

Formula (2.57) is due to Cayley. Although equivalent to (2.15), it is much simpler in dealing with finite rotations. We use it now to obtain a formula for the composition of two rotations.

Consider a rotation through an angle Φ_1 around the axis in the direction $\mathbf{e}_1$. This rotation is completely described either by the rotation vector $\boldsymbol{\rho}_1 = \tan\frac{\Phi_1}{2}\mathbf{e}_1$ or by the quaternion $q_1 = \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k$. Let the rotation vector $\boldsymbol{\rho}_2 = \tan\frac{\Phi_2}{2}\mathbf{e}_2$ and the quaternion $q_2 = \mu_0 + \mu_1 i + \mu_2 j + \mu_3 k$ correspond to another rotation through an angle Φ_2 around $\mathbf{e}_2$. We have

$$
q_1 = \cos\frac{\Phi_1}{2}(1 + \boldsymbol{\rho}_1),\ q_2 = \cos\frac{\Phi_2}{2}(1 + \boldsymbol{\rho}_2). \tag{2.58}
$$

The vector $\mathbf{r}$ is transformed by the first rotation to

$$
\mathbf{r}' = q_1 \mathbf{r} q_1^{-1} \tag{2.59}
$$

and then by the second rotation to

$$
\mathbf{r}'' = q_2 \mathbf{r}' q_2^{-1} = q_2 q_1 \mathbf{r} q_1^{-1} q_2^{-1} = (q_2 q_1)\mathbf{r}(q_2 q_1)^{-1}. \tag{2.60}
$$

Thus, the resultant rotation corresponds to the quaternion

$$
\begin{aligned}
Q &= q_2 q_1 \\
&= (Q_0, Q_1, Q_2, Q_3),
\end{aligned} \tag{2.61}
$$

where

$$
\begin{aligned}
Q_0 &= \lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3, \\
Q_1 &= \lambda_1\mu_0 + \lambda_0\mu_1 + \lambda_3\mu_2 - \lambda_2\mu_3, \\
Q_2 &= \lambda_2\mu_0 + \lambda_0\mu_2 + \lambda_1\mu_3 - \lambda_3\mu_1, \\
Q_3 &= \lambda_3\mu_0 + \lambda_0\mu_3 + \lambda_2\mu_1 - \lambda_1\mu_2.
\end{aligned} \tag{2.62}
$$

If we like to express the resultant rotation in terms of the rotation vectors, we use (2.58) to write

$$
\begin{aligned}
Q &= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}(1+\boldsymbol{\rho}_2)(1+\boldsymbol{\rho}_1) \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}[(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)+\boldsymbol{\rho}_1+\boldsymbol{\rho}_2+\boldsymbol{\rho}_2\times\boldsymbol{\rho}_1] \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)[1+\frac{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2+\boldsymbol{\rho}_2\times\boldsymbol{\rho}_1}{(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)}].
\end{aligned} \tag{2.63}
$$

Comparing this with (2.58), we can write the resultant rotation quaternion in the form

$$
Q = \cos\frac{\Phi}{2}(1+\boldsymbol{\rho}). \tag{2.64}
$$

We find

$$
\boldsymbol{\rho} = \frac{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2+\boldsymbol{\rho}_2\times\boldsymbol{\rho}_1}{(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)} \tag{2.65}
$$

and

$$
\begin{aligned}
\cos\frac{\Phi}{2} &= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2) \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\mathbf{e}_1\cdot\mathbf{e}_2 \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\cos\chi.
\end{aligned} \tag{2.66}
$$

Rodrigues's formula (2.65) gives the resultant rotation vector and (2.66) gives the angle of the resultant rotation in terms of the two rotation angles and the angle χ between the two axes of the rotations.

Note that Rodrigues's formula (2.65) is not valid when $\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2 = 1$. In that case, from (2.66), we see that the rotation angle $\Phi = \pi$. This is expected whenever we deal with vectors of rotation.

2.10 Exercises

1- Show that the resultant of two half turns around different axes intersecting at an angle θ is equivalent to a rotation at an angle 2θ around the axis orthogonal to the two axes.

2- Put the formula (2.42) for the angular velocity in the quaternion form

$$
\boldsymbol{\omega} = 2q^{-1}\dot{q}, \tag{2.67}
$$

and hence prove the formula (2.46).