# Chapter 13
# Miscellaneous Cases Integrable on a Single Level of the Areas Integral

In this chapter, we collect certain sets of conditional integrable cases of different origins and of various characters, which do not belong to one of the problems discussed in previous chapters that have definite physical interpretation, but they are unified only by being valid on a single level of the areas integral, and mostly on $f = 0$. The first set consists of the separable cases investigated in Chap. 9 above. Another set began to appear in a work of Goryachev [117], who used an inverse method to find potentials that admit existence of a complementary integral of the third or fourth degrees in the components of the angular velocity, as modifications of the known cases at that time. Those are Kowalevski's case of a heavy body with a fixed point and Chaplygin's case of a rigid body in a liquid and Goryachev–Chaplygin's case of the classical problem. The search led in [117] to the new cases:

(1) A conditional case, on the level $f = 0$, under the condition $A = B = 2C$, with the potential

$$V = a_1\gamma_1 + a_2\gamma_2 + b_1(\gamma_1^2 - \gamma_2^2) + 2b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}.$$

This adds the singular term $\frac{\lambda}{2\gamma_3^2}$ to a former result of Chaplygin [53], which combines the potentials of Kowalevski's classical case and Chaplygin's case of a body in a liquid. The complementary integral for this case is of the fourth degree.

(2) Another case under the condition $A = B = \frac{4}{3}C$, with the potential

$$V = \frac{a + b\gamma_1}{\gamma_3^{\frac{2}{3}}},$$

which admits a cubic integral.



H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_13

A variety of modifications and generalizations of known conditional integrable cases have accumulated in the last three decades, mainly in the works of the author and coworkers, as a result of the introduction of a new method of construction of two-dimensional integrable systems living on Riemannian manifolds and whose integrals are polynomials in velocities [381]. The resulting systems usually involved a large number of parameters, which could be adjusted to identify the metric of the problem with that of the reduced system of rigid body dynamics after ignoring the precession angle as in Chap. 9. Here we shall not try to make any physical interpretation of the potential and gyroscopic terms in each case, even though they are mostly generalizations of some of the cases presented in the previous chapters in natural physical settings.

The most natural and comfortable classification of the relatively large number (22) of known conditional integrable cases is the classification by the degree of the polynomial complementary integral in every case. We shall follow this classification here. We also give full-time sequence of each hierarchy of overlapping cases.

Although most of those cases do not have physical interpretation, due to strange singularities in the potentials, we give full up-to-date list of them. As known cases are scattered in the literature, we believe this step, made here for the first time, can play a definite role in future development of the subject. As will be seen below, some of those cases have already stimulated further studies on the separation of variables and also on topological classification.

**Remark 18** We have used the uniform and variable precession transformations in some previous chapters. In the present one, those transformations will not be applied. The reason is that in conditional cases on the level $f = 0$ this transformation may be easily applied using an arbitrary function $\nu(\gamma)$ as discussed in Sect. 12.2 of the preceding chapter.

## 13.1 Cases with a Quadratic Integral

It is well-known that a natural (time-reversible) mechanical system of two degrees of freedom which admits an integral quadratic in velocities and independent of the energy integral must be Liouville separable system in some generalized coordinates. The dynamics of a rigid body acted upon by pure potential forces is time-reversible on the zero-level of the cyclic integral. When gyroscopic forces are present, equations of motion become irreversible and Liouville separability is lost. In the present section, we list the three known types of potentials that admit a complementary quadratic integral and hence admit Liouville separation and also three non-separable cases with a quadratic integral.

### 13.1.1 Separable Integrable Potentials

From the Minkowski analogy between the motion of a rigid body about a fixed point and the motion of a material point on the inertia ellipsoid of the body at the fixed point, it follows that certain potentials exist, which allow separation of variables on the zero level of the areas integral. Those are of three types:

(1) Potentials separable in elliptic coordinates on the tri-axial ellipsoid ($A \neq B \neq C$) Chap. 9 Sect. 9.7.2.
(2) Potentials separable in spherical coordinates for a dynamically symmetric body ($A = B$), including the case of dynamical spherical symmetry at the fixed point Chap. 9 Sect. 9.7.1.
(3) Potentials separable in sphero-conic (elliptic) coordinates on the sphere of inertia in the case of complete dynamical symmetry ($A = B = C$) Chap. 9 Sect. 9.7.3.

For all three types of separable cases $\boldsymbol{\mu} = \mathbf{0}$, $f = 0$ and $u, v, F, G, g$ are arbitrary functions of their arguments (Table 13.1).

**Table 13.1** Conditional integrable cases. The first 20 cases are valid on the level $f = 0$

| | |
|---|---|
| 1 | Separable in elliptic coordinates on the ellipsoid of inertia [31, 224, 381] |
| | $V = \frac{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}{\sqrt{\beta}}[u(\alpha+\sqrt{\beta}) + v(\alpha-\sqrt{\beta})]$, $\alpha = AB+BC+CA-ABC(\frac{\gamma_1^2}{A}+\frac{\gamma_2^2}{B}+\frac{\gamma_3^2}{C}) = ABC[tr(\mathbf{I}^{-1}) - \boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}]$, $\beta = \alpha^2 - 4ABCD, \quad D = A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2 \equiv \boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}$. |
| | $I_4 = A^2p^2 + B^2q^2 + C^2r^2 + \frac{1}{\sqrt{\beta}}[(\alpha-\sqrt{\beta})v(\alpha-\sqrt{\beta}) + (\alpha+\sqrt{\beta})u(\alpha+\sqrt{\beta})]$ |
| 2 | Separable in spherical coordinates on the Poisson sphere $B = A$ |
| | $V = F(\gamma_3) + \frac{A-(A-C)\gamma_3^2}{A(1-\gamma_3^2)}G(\frac{\gamma_1}{\gamma_2}) \equiv F(\gamma_3) + \frac{A-(A-C)\gamma_3^2}{A\gamma_2^2}g(\frac{\gamma_1}{\gamma_2}), \qquad G(\frac{\gamma_1}{\gamma_2}) = \frac{g(\frac{\gamma_1}{\gamma_2})}{1+(\frac{\gamma_1}{\gamma_2})^2}$ |
| | $I_4 = Cr^2 + 2G(\frac{\gamma_1}{\gamma_2})$ |
| 3 | Separable in sphero-conic coordinates on the Poisson sphere [390, 391] $A = B = C$, |
| | $V = \frac{[u(\alpha' - \sqrt{\beta'}) + v(\alpha' + \sqrt{\beta'})]}{\sqrt{\beta'}}$, $\alpha' = a+b+c-(a\gamma_1^2+b\gamma_2^2+c\gamma_3^2)$, $\beta' = \alpha'^2 - 4abc(\frac{\gamma_1^2}{a}+\frac{\gamma_2^2}{b}+\frac{\gamma_3^2}{c})$. |
| | $I_4 = (ap^2+bq^2+cr^2) + \frac{1}{\sqrt{\beta'}}[(\alpha'+\sqrt{\beta'})u(\alpha'-\sqrt{\beta'}) + (\alpha'-\sqrt{\beta'})v(\alpha'+\sqrt{\beta'})]$ |

A special case of this separable potential, equivalent to $F(\gamma_3) = 0$, was pointed out also by Kolossov in [224], but the general potential seems to be unnoticed in the literature. This type of potential appears as a part of the potential in some integrable generalizations of Kowalevski's case, which admit an integral quartic in velocities. The quartic integral in those cases can be written as the square of the quadratic integral in the separable

### 13.1.2 Non-separable Cases with a Quadratic Integral

| | |
|---|---|
| 4 | Yehia [414]<br>Separable ($K = 0$),<br>Subcase of Steklov's ($s = 0$) |
| | $V = \frac{sS}{2ABC\sqrt{S^2-4ABCD}}$,<br>$S = A(B+C)\gamma_1^2 + B(C+A)\gamma_2^2 + C(A+B)\gamma_3^2$<br>$\boldsymbol{\mu} = -K(\frac{\gamma_1}{A}, \frac{\gamma_2}{B}, \frac{\gamma_3}{C})$,<br>$\boldsymbol{l} = \frac{K}{2ABC}(A(B+C)\gamma_1, B(C+A)\gamma_2, C(A+B)\gamma_3)$ |
| | $I_4 = \frac{1}{2}(A^2p^2 + B^2q^2 + C^2r^2) - K(p\gamma_1 + q\gamma_2 + r\gamma_3) + \frac{s}{\sqrt{S^2-4ABCD}}$ |
| 5 | Yehia [394] |
| | $V = n\frac{A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2}{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}$<br>$\boldsymbol{\mu} = \frac{\partial F}{\partial \boldsymbol{\gamma}} - \Phi\boldsymbol{\gamma}$<br>$F = J\frac{A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2}{\sqrt{ABC(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)}}$ |
| | $\Phi = J\frac{2(A+B+C)v_0-3v_0^2+\frac{2ABC}{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}}{\sqrt{ABC(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)}}$ |
| | $I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 + F$<br>$I_4$ constructed in elliptic coordinates and not in Euler-Poisson variables. See [394] |
| 6 | Yehia [401]<br>Separable ($N = 0$)<br>Subcase of Lyapunov's ($K = 0$) |
| | $A = B = C$<br>$V = C[-\frac{N^2abc}{2}(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}) + \frac{K}{\sqrt{\beta'}}]$<br>$\mathbf{l} = -\frac{1}{2}C((b+c)\gamma_1, (c+a)\gamma_2, (a+b)\gamma_3)$<br>$\mu = C(a\gamma_1, b\gamma_2, c\gamma_3)$ |
| | $I_4 = \frac{1}{2}[(b+c)\,p^2 + (c+a)\,q^2 + (a+b)\,r^2]$<br>$-Nabc(p\frac{\gamma_1}{a} + q\frac{\gamma_2}{b} + r\frac{\gamma_3}{c}) + \frac{1}{2}K\frac{a+b+c+a\gamma_1^2+b\gamma_2^2+c\gamma_3^2}{\sqrt{\beta'}}$ |

## 13.2 Cases with a Cubic Integral

The list of such cases comprises only three items:

| | | |
|---|---|---|
| 7 | Yehia and Elmandouh 2016 [424] ($c_1, c_2$ added) | |
| | Yehia 2002 [409] | $c_1 = c_2 = 0$ |
| | Sokolov and Tsiganov 2002 [337] | $e_0 = e_1 = \lambda = 0$ |
| | Yehia [395] 1996 (Independently of [226]) | $e_0 = e_1 = c_1 = c_2 = 0$ |
| | Komarov and Kuznetsov [226] 1987 | $e_0 = e_1 = c_1 = c_2 = a_2 = 0$ |
| | Sretensky [341] 1963 | $e_0 = e_1 = c_1 = c_2 = \lambda = 0$ |
| | Goryachev [118] 1916<br>Goryachev-Chaplygin [115] 1900, [52] 1901 | $e_0 = e_1 = c_1 = c_2 = k = 0$<br>$e_0 = e_1 = c_1 = c_2 = k = \lambda == 0$ |

| | |
|---|---|
| | $A = B = 4C,$<br>$\boldsymbol{l} = C\left(0, 0, k + c_1\gamma_1 + c_2\gamma_2 + e_0(\frac{2}{\gamma_3^4} - \frac{1}{\gamma_3^2}) + \frac{e_1}{\gamma_1^2+\gamma_2^2}\right),$<br>$\mu_1 = C[c_1\gamma_3 + 2e_0\frac{\gamma_1(4-\gamma_3^2)}{\gamma_3^5} - \frac{2e_1\gamma_1\gamma_2}{(\gamma_1^2+\gamma_2^2)^2}],$<br>$\mu_2 = C[c_2\gamma_3 + 2e_0\frac{\gamma_2(4-\gamma_3^2)}{\gamma_3^5} - \frac{2e_1\gamma_1\gamma_2}{(\gamma_1^2+\gamma_2^2)^2}],$<br>$\mu_3 = C[k + c_1\gamma_1 + c_2\gamma_2 + e_0\frac{\gamma_2(2-\gamma_3^2)}{\gamma_3^4} + \frac{e_1}{\gamma_1^2+\gamma_2^2}],$<br>$V = C\,\{a_1\gamma_1 + a_2\gamma_2 - c_1c_2\gamma_1\gamma_2$<br>$\quad + \frac{c_2^2}{2}\gamma_1^2 + \frac{c_1^2}{2}\gamma_2^2 + \frac{\lambda}{\gamma_3^2}$<br>$\quad + e_0\left(\gamma_3^2 - 2\right)\frac{c_1\gamma_1+c_2\gamma_2}{\gamma_3^4} - e_0^2\frac{4-8\gamma_3^2+5\gamma_3^4}{2\gamma_3^8}$<br>$\quad + e_1\frac{k+e_0-\gamma_1c_1-\gamma_2c_2}{\gamma_1^2+\gamma_2^2} - \frac{1}{2}e_1^2\frac{4\gamma_1^2+4\gamma_2^2+1}{\left(\gamma_1^2+\gamma_2^2\right)^2}\}$ |
| | $I_3 = 4p\gamma_1 + 4q\gamma_2$<br>$\quad + [r + k + c_1\gamma_1 + c_2\gamma_2 + \frac{e_1}{\gamma_1^2+\gamma_2^2} + e_0(\frac{2}{\gamma_3^4} - \frac{1}{\gamma_3^2})]\gamma_3 = 0,$ |
| | $I_4 = \left[r - k + c_1\gamma_1 + c_2\gamma_2 + \frac{e_0(2-\gamma_3^2)}{\gamma_3^4} - \frac{e_1(8\gamma_1^2-1)}{(\gamma_1^2+\gamma_2^2)}\right]\Big\{[p + \frac{c_1}{2}\gamma_3]^2$<br>$\quad + [q + \frac{c_2\gamma_3}{2}]^2 + \frac{\lambda}{2\gamma_3^2} + k(\frac{e_0}{\gamma_3^4} - \frac{e_1}{2}) - (\frac{e_1}{2} + \frac{e_0(2-\gamma_3^2)}{2\gamma_3^4})r$<br>$\quad + (c_1\gamma_1 + c_2\gamma_2)\,[\frac{e_1}{2} + \frac{e_0(2-\gamma_3^2)}{2\gamma_3^4}] - \frac{e_0^2(3\gamma_3^4-6\gamma_3^2+4)}{2\gamma_3^8}$<br>$\quad + \frac{e_1}{\gamma_1^2+\gamma_2^2}[\frac{e_1(1-8\gamma_1^2)}{2} + \frac{8\gamma_1^2(\gamma_3^2-2)-2\gamma_3^4+9\gamma_3^2-8}{2\gamma_3^4}]\Big\}$<br>$\quad -\gamma_3[(2e_1c_1 - c_1k + a_1)(p + \frac{c_1\gamma_3}{2})$<br>$\quad + (2e_1c_2 - c_2k + a_1)(q + \frac{c_2}{2}\gamma_3)]$<br>$\quad + k\Big\{(c_1\gamma_1 + c_2\gamma_2)[\frac{e_0}{\gamma_3^2} + \frac{e_1(1-2\gamma_3^2)}{\gamma_1^2+\gamma_2^2}] + \frac{4e_1\gamma_1^2(e_0-2e_1\gamma_3^2)}{\gamma_3^2(\gamma_1^2+\gamma_2^2)}$<br>$\quad - \frac{4e_1\gamma_3}{\gamma_1^2+\gamma_2^2}(p\gamma_1 + q\gamma_2)\Big\}$<br>$\quad - \frac{8e_0e_1c_2\gamma_2\gamma_1^2}{\gamma_3^2(\gamma_1^2+\gamma_2^2)} + (a_1\gamma_1 + a_2\gamma_2)[\frac{e_0}{\gamma_3^2} - \frac{e_1}{\gamma_1^2+\gamma_2^2}] + 4e_1\gamma_1^2[\frac{2e_0^2}{\gamma_3^6} + \frac{\lambda}{\gamma_3^2(\gamma_1^2+\gamma_2^2)}]$<br>$\quad - \frac{8e_1c_1\gamma_1^3}{\gamma_1^2+\gamma_2^2}[e_1 + \frac{e_0}{\gamma_3^2}] + \frac{4e_0e_1^2\gamma_1^2}{\gamma_3^4(\gamma_1^2+\gamma_2^2)^2}[8\gamma_1^2(\gamma_3^2 - 2) - (\gamma_1^2 + \gamma_2^2)(\gamma_3^2 - 4)]$<br>$\quad + \frac{e_1(c_1^2+c_2^2)}{9(\gamma_1^2+\gamma_2^2)}[18\gamma_1^2\gamma_3^2 - 9\gamma_3^4 + 13\gamma_3^2 - 4] + \frac{8e_1^3\gamma_1^3(1-4\gamma_1^2)}{(\gamma_1^2+\gamma_2^2)^2}$<br>$\quad + \frac{2e_1\gamma_3}{\gamma_1^2+\gamma_2^2}\{q[c_2(5\gamma_1^2 - \gamma_2^2) - 2c_1\gamma_1\gamma_2]$<br>$\qquad + p[c_1(3\gamma_1^2 + \gamma_2^2) - 2c_2\gamma_1\gamma_2]\}$<br>$\quad - \frac{16e_1\gamma_1\gamma_2}{\gamma_1^2+\gamma_2^2}\left[pq - \frac{e_1c_2}{2}\gamma_1\right] + \frac{8e_1(\gamma_1^2-\gamma_2^2)}{\gamma_1^2+\gamma_2^2}q^2$ |

The second case with a cubic complementary integral is due to Goryachev [117]. It is characterized by the following

| | |
|---|---|
| 8 | Goryachev 1915 [117]. |
| | $A = B = \frac{4}{3}C,$ |
| | $\boldsymbol{l} = \boldsymbol{\mu} = \mathbf{0},$ |
| | $V = \frac{a\gamma_1+b\gamma_2+c}{\gamma_3^{\frac{2}{3}}},$ |
| | $I_3 = \frac{4}{3}(p\gamma_1 + q\gamma_2) + r\gamma_3 = 0,$ |
| | $I_4 = 2r(p^2 + q^2) + r^3 - 2a\gamma_3^{\frac{1}{3}}p + r\frac{a+b\gamma_1}{\gamma_3^{\frac{2}{3}}}.$ |

Although having no obvious physical meaning, this case has received a growing interest in the last years [45, 361]. It turns out to be the first example of a mechanical system whose complex invariant varieties are strata of Jacobians of a non-hyper-elliptic curve, here a trigonal curve of genus 3 [45].

Goryachev's case 8 has been generalized to the following one involving two (vector and scalar) potentials (Yehia 2002 [409]):

| | |
|---|---|
| 8* | $\boldsymbol{l} = (0, 0, \kappa + \frac{1}{\gamma_1^2+\gamma_2^2}[3\kappa + e_0\gamma_3^{\frac{2}{3}} + \frac{e_1(2+\gamma_3^2)}{\gamma_3^{\frac{2}{3}}}]),$ <br> $V = \frac{a\gamma_1+b\gamma_2+c}{\gamma_3^{\frac{2}{3}}}$ <br> $+\frac{1}{(\gamma_1^2+\gamma_2^2)^2}[\frac{e_0^2(4-7\gamma_3^2)}{6\gamma_3^{\frac{2}{3}}} - \frac{e_1^2(13\gamma_3^4-8\gamma_3^2+4)}{2\gamma_3^{\frac{4}{3}}}$ <br> $-e_0e_1(5\gamma_3^2-2) + \frac{3\kappa^2\gamma_3^2}{2}(\gamma_3^2-4)$ <br> $-3e_0\kappa\gamma_3^{\frac{8}{3}} - 3\kappa e_1\gamma_3^{\frac{4}{3}}(\gamma_3^2+2)].$ |

where $\kappa$, $e_0$ and $e_1$ are arbitrary constants. Note that the constant gyrostatic momentum $\kappa$ is a coupling constant for some potential and gyroscopic terms.

For this generalization, one can easily write

$$I_3 = \frac{4}{3}(p\gamma_1 + q\gamma_2) + \{r + \kappa + \frac{1}{\gamma_1^2+\gamma_2^2}[3\kappa + e_0\gamma_3^{\frac{2}{3}} + \frac{e_1(2+\gamma_3^2)}{\gamma_3^{\frac{2}{3}}}]\}\gamma_3 = 0,$$

but the complementary cubic integral is not yet expressed in the Euler–Poisson variables (See [409]).

Cases 7 and 8 were constructed as special cases of an integrable many-parameter system with a cubic integral under some restrictions on those parameters.

| | |
|---|---|
| 9 | Yehia 2000 [404]. <br> Gaffet 1998 [96, 97] (The equivalent problem for a particle on a sphere). |
| | $A = B = C$, <br> $V = \frac{K}{(\gamma_1\gamma_2\gamma_3)^{\frac{2}{3}}}. \mathbf{l} = \mathbf{0}.$ |
| | $I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 = 0,$ <br> $I_4 = Apqr - 2K(\gamma_1\gamma_2\gamma_3)^{\frac{1}{3}}(\frac{p}{\gamma_1} + \frac{q}{\gamma_2} + \frac{r}{\gamma_3}).$ |

## 13.3 Cases with a Quartic Integral

The thirteen presently known cases with a quartic integral are characterized by the Kowalevski configuration $A = B = 2C$. They are mostly (but not all) generalizations on the level $f = 0$ of the two classical cases: Kowalevski's case of a heavy body and Chaplygin's case of a body in a liquid. It is curious to note that the main potential in most of those cases is composed of the basic potential present in Kowalevski's or Chaplygin's cases or both of them and some additional terms that belong to separable potentials discussed in Chap. 9 Sect. 9.7.1. When alone, the last terms give rise to a quadratic integral, instead of the quartic one.

Those cases are divided into five types, presented in the following subsections.

### *13.3.1 Cases Stemming from Kowalevski's Case*

Four cases of this type are listed here:

| | |
|---|---|
| 10 | Yehia 1996 [396],<br>$\lambda = 0$ Yehia-Bedwehy 1987, (unconditional case) [419] (unconditional case),<br>$\lambda = \varepsilon = 0$ Kowalevski 1888 [238] (unconditional case). |
| | $A = B = 2C$,<br>$V = C[a_1\gamma_1 + a_2\gamma_2 + \frac{\varepsilon}{\sqrt{\gamma_1^2+\gamma_3^2}} + \frac{\lambda}{\gamma_3^2}]$,<br>$\boldsymbol{l} = \boldsymbol{\mu} = \boldsymbol{0}$. |
| | $I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 = 0$,<br>$I_4 = [p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{2\gamma_3^2}]^2$<br>$+[2pq - a_1\gamma_2 - a_2\gamma_1 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$<br>$+\varepsilon[\frac{(p+n\gamma_1)^2+(q+n\gamma_2)^2}{\sqrt{\gamma_1^2+\gamma_3^2}} + \frac{\varepsilon}{\gamma_1^2+\gamma_3^2} + \frac{2\lambda\sqrt{\gamma_1^2+\gamma_3^2}}{\gamma_3^2}]$ |
| 11 | Yehia 2006 [413]<br>$\nu_1 = \delta_2 = 0$: Yehia-Bedwehy 1987 [419]<br>$A = B = 2C$.<br>$V = C[a_1\gamma_1 + \frac{\lambda}{\gamma_3^2} + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}} + \frac{2-\gamma_3^2}{\gamma_2^2}(\delta_2 + \frac{\nu_1\gamma_1}{\sqrt{1-\gamma_3^2}})]$.<br>$\boldsymbol{l} = \boldsymbol{\mu} = \boldsymbol{0}$. |
| | $I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 = 0$,<br>$I_4 = [p^2 - q^2 - a_1\gamma_1 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}]^2 + [2pq - a_1\gamma_2 - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$<br>$+[\delta_2\frac{\gamma_3^2}{\gamma_2^2} + \frac{\varepsilon+\nu_1\gamma_1\frac{\gamma_3^2}{\gamma_2^2}}{\sqrt{1-\gamma_3^2}}][2p^2 + 2q^2 + \delta_2\frac{\gamma_3^2}{\gamma_2^2} + \frac{\varepsilon+\nu_1\gamma_1\frac{\gamma_3^2}{\gamma_2^2}}{\sqrt{1-\gamma_3^2}}]$<br>$+\frac{2\lambda\varepsilon\sqrt{1-\gamma_3^2}}{\gamma_3^2} - \frac{2}{\gamma_2^2}(a_1\gamma_3^2 + \lambda\gamma_1)(\delta_2\gamma_1 + \nu_1\sqrt{1-\gamma_3^2})$ |

The main result of [413] was the construction of an integrable system of two degrees of freedom living on a Riemannian (or pseudo-Riemannian[1]) manifold and admitting an integral of degree four in velocities. Cases 11, 14 were obtained as special cases under suitable restrictions of this twenty-one-parameter system that render the metric to that of the Routhian of the rigid body dynamics. Case 18 below was obtained by further development of the method of [413].

Separation variables and expressions of the dynamical variables in terms of them are constructed for case 11 in [218] (See also [137]), without treating the inversion of the resulting quadratures.

[1] In differential geometry, that is a manifold whose metric is not necessarily positive-definite

| | |
|---|---|
| 12 | Yehia, Elmandouh 2011 [422]<br>$c = 0$. Yehia-Bedwehy (unconditional case) [419]<br>$\lambda = 0$. Sokolov (unconditional case) |
| | $A = B = 2C,$<br>$V = C[a\gamma_1 + \frac{c^2}{2}(\gamma_1^2 - \gamma_3^2) + \frac{\varepsilon}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{\lambda}{\gamma_3^2}],$<br>$\boldsymbol{\mu} = Cc(0, \gamma_3, \gamma_2),$<br>$I_4 = (p^2 - q^2 - a\gamma_1 - cr\gamma_2 - c^2\gamma_1^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2})^2$<br>$+(2pq - a\gamma_2 + cr\gamma_1 - c^2\gamma_1\gamma_2 - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2})^2$<br>$+\varepsilon[\frac{2(p^2-q^2)}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{\varepsilon}{\gamma_1^2+\gamma_2^2} + 2\sqrt{\gamma_1^2 + \gamma_2^2}(c^2 + \frac{\lambda}{\gamma_3^2})]$ |

| | |
|---|---|
| 13 | Yehia, Elmandouh 2011 [422],<br>$\lambda = 0$: the integral $I_4$ becomes unconditional and gives Sokolov's [336]. |
| | $A = B = 2C,$<br>$V = C[\kappa c\gamma_1 + a_2\gamma_2 + \frac{c^2}{2}(\gamma_2^2 - \gamma_3^2) + \frac{\lambda}{\gamma_3^2}],$<br>$\boldsymbol{l} = C\,(0, 0, k + c\gamma_1)\,,$<br>$\boldsymbol{\mu} = C(c\gamma_3, 0, \kappa + c\gamma_1),$<br>$I_4 = \left[p^2 - q^2 + a_2\gamma_2 + c^2\gamma_2^2 - c\gamma_1(\kappa - r) - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}\right]^2$<br>$+\left[2pq - a_2\gamma_1 - c^2\gamma_1\gamma_2 - c\gamma_2(\kappa - r) - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}\right]^2$<br>$+2\kappa(r - \kappa + c\gamma_1)[p^2 + q^2 + 2cp\gamma_3 + \lambda(1 + \frac{1}{\gamma_3^2}) - \gamma_3\left(2qa_2 + c^2\kappa\gamma_3\right)]$<br>$-2\kappa c^2[4\gamma_3(p\gamma_1 + q\gamma_2) - (1 - \gamma_3^2)(r + c\gamma_1)].$ |

## 13.3.2 Cases Stemming from Chaplygin's Case

Four cases of this type are listed in the following table:

| | |
|---|---|
| 14 | Yehia 2006 [413]<br>$\delta_1 = \delta_2 = 0$, Yehia 2003 [411]<br>$\delta_1 = \delta_2 = \rho = 0$, Goryachev [118]<br>$\delta_1 = \delta_2 = \rho = \lambda = 0$, Chaplygin [53] |
| | $A = B = 2C,$<br>$V = C[b_1(\gamma_1^2 - \gamma_2^2) + \frac{\lambda}{\gamma_3^2} + \rho(\frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6})$<br>$+(2 - \gamma_3^2)(\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2})],$<br>$\boldsymbol{l} = \boldsymbol{\mu} = \mathbf{0},$<br>$I_3 = 2p\gamma_1 + 2q\gamma_2 + r\gamma_3 = 0,$<br>$I_4 = [p^2 - q^2 + b_1\gamma_3^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}]^2 + 4[pq - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$<br>$+2(p^2 + q^2)\{\rho[\frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6}] + \gamma_3^2[\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2}]\} + \rho\frac{(\gamma_1^2+\gamma_2^2)^2}{\gamma_3^{12}}(\rho - 2\lambda\gamma_3^4)$<br>$+\frac{2\rho b_1(\gamma_1^2-\gamma_2^2)}{\gamma_3^4} - 2b_1\gamma_3^4[\frac{\delta_1}{\gamma_1^2} - \frac{\delta_2}{\gamma_2^2}] + \gamma_3^4[\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2}]^2$<br>$-2(\rho + \lambda\gamma_3^4)\frac{1-\gamma_3^2}{\gamma_3^4}[\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2}].$ |

Together with case 11, this case was constructed in [413] (See comment next to case 11). Separation variables and quadratures were constructed for this case (and for case 11) in [137].

| | |
|---|---|
| 15 | Yehia-Elmandouh 2013 [423]<br>$K=0$ Yehia [411] (Sect. 4.2.3) 2003<br>$K=\rho=0$ Goryachev [118] 1916<br>$K=\rho=\lambda=0$ Chaplygin [53] 1903 |
| | $\begin{aligned} &A=B=2C\\ &V=C\{k\left[2d\gamma_1\gamma_2+c(\gamma_1^2-\gamma_2^2)\right]\\ &\qquad +K^2[2cd\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2)+\tfrac{d^2}{2}(\gamma_3^4+4\gamma_1^2\gamma_2^2)\\ &\qquad -c^2(\gamma_3^2(\gamma_1^2+\gamma_2^2)+2\gamma_1^2\gamma_2^2)]+\tfrac{\lambda}{\gamma_3^2}+\rho(\tfrac{1}{\gamma_3^4}-\tfrac{1}{\gamma_3^6})\},\\ &\mathbf{l}=C(0,0,K[d(\gamma_2^2-\gamma_1^2)+2c\gamma_1\gamma_2]),\\ &\boldsymbol{\mu}=C(2K\gamma_3(c\gamma_2-d\gamma_1),2K\gamma_3(d\gamma_2+c\gamma_1),\\ &\qquad K[d(\gamma_2^2-\gamma_1^2)+2c\gamma_1\gamma_2]). \end{aligned}$ |
| | $\begin{aligned} &I_3=2p\gamma_1+2q\gamma_2+\{r+K\left[d(\gamma_2^2-\gamma_1^2)+2c\gamma_1\gamma_2\right]\}\gamma_3,\\ &I_4=\{p^2-q^2+ck\gamma_3^2+\gamma_3^2[Kdr+cK^2(c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2)]\\ &\qquad -\tfrac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}\}^2\\ &\qquad +\{2pq+dk\gamma_3^2+[dK^2(c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2)-Kcr]\gamma_3^2\\ &\qquad -\tfrac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}\}^2\\ &\qquad +\rho\{\tfrac{2(\gamma_3^2-1)}{\gamma_3^6}[p^2+q^2]\\ &\qquad -\tfrac{2Kr}{\gamma_3^4}[2c\gamma_1\gamma_2+d(\gamma_2^2-\gamma_1^2)]\\ &\qquad +\tfrac{(1-\gamma_3^2)^2}{\gamma_3^{12}}(\rho-2\lambda\gamma_3^4)\\ &\qquad +K^2[2c^2(\tfrac{1}{\gamma_3^4}-\tfrac{2}{\gamma_3^2})+8\tfrac{(d^2-c^2)\gamma_1^2\gamma_2^2+cd\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2)}{\gamma_3^4}]\\ &\qquad +\tfrac{2k}{\gamma_3^4}[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2]\}. \end{aligned}$ |

| | |
|---|---|
| 16 | Yehia and Elmandouh 2016 [425]<br>$\kappa=0$ : Special case of Yehia and Elmandouh [423] |
| | $\begin{aligned} &A=B=2C\\ &V=C\{\kappa[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2]+\kappa K[2c\gamma_1\gamma_2-d(\gamma_1^2-\gamma_2^2)]\\ &\qquad +K^2\{2cd\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2)-c^2[\gamma_3^2(\gamma_1^2+\gamma_2^2)+2\gamma_1^2\gamma_2^2]\\ &\qquad +\tfrac{d^2}{2}(\gamma_3^4+4\gamma_1^2\gamma_2^2)\}+\tfrac{\lambda}{\gamma_3^2}\}, \end{aligned}$ |
| | $\mathbf{l}=C(0,0,\kappa+K[2c\gamma_1\gamma_2-d(\gamma_1^2-\gamma_2^2)]),$ |
| | $\boldsymbol{\mu}=C(2K\gamma_3(c\gamma_2-d\gamma_1),2K\gamma_3(c\gamma_1+d\gamma_2),\kappa+K[2c\gamma_1\gamma_2+d(\gamma_2^2-\gamma_1^2)]),$ |
| | $I_3=2p\gamma_1+2q\gamma_2+(r+\kappa+K[2c\gamma_1\gamma_2+d(\gamma_2^2-\gamma_1^2)])\gamma_3,$ |
| | $\begin{aligned} &I_4=\Big\{p^2-p^2+ck\gamma_3^2+cK^2\gamma_3^2[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2]\\ &\qquad +dK[2\kappa-\gamma_3^2(3\kappa-r)]-\tfrac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}\Big\}^2\\ &\qquad +\Big\{2pq+kd\gamma_3^2+dK^2\gamma_3^2[2d\gamma_1\gamma_2+c(\gamma_1^2-\gamma_2^2)]\\ &\qquad +cK[\gamma_3^2(3\kappa-r)-2\kappa]-\tfrac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}\Big\}^2\\ &\qquad +2\kappa[r-\kappa-K(2c\gamma_1\gamma_2+d(\gamma_2^2-\gamma_1^2))]\Big\{p^2+q^2+\lambda(1+\tfrac{1}{\gamma_3^2})\\ &\qquad\qquad +\gamma_3^2[K^2(c^2+d^2)(\gamma_3^2-1)-2d\kappa K+\kappa c]\Big\} \end{aligned}$ |

| | |
|---|---|
| | $-4\kappa\gamma_3\{[2K\kappa(c\gamma_1+2d\gamma_2)-k(2c\gamma_2-d\gamma_1)](q+n\gamma_2)$ <br> $+\gamma_2(p+n\gamma_1)(2c\kappa K+dk)\}$ <br> $-8\kappa\{c^2K\gamma_3^2[\kappa K(\gamma_3^2-1)-k\gamma_1\gamma_2]$ <br> $+c\{2\kappa dK^2\gamma_1\gamma_2\gamma_3^2+K[\frac{1}{2}kd\gamma_3^4-2\lambda\gamma_1\gamma_2$ <br> $+dk\gamma_3^2(\gamma_1^2-\frac{1}{2})]-\frac{1}{2}k\kappa\gamma_3^2\}$ <br> $-2\kappa d^2K^2\gamma_1^2\gamma_3^2+K[2\lambda d\gamma_1^2+d(\kappa^2+\lambda)\gamma_3^2]\}$ |

| | |
|---|---|
| 17 | Yehia 2003 [411] <br> Goryachev 1916 [118], $\rho=0$. <br> Chaplygin 1903 [53] $\lambda=\rho=0$. |
| | $A=B=2C$, |
| | $V=C[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2+\frac{\lambda}{\gamma_3^2}+\rho(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})]$, |
| | $\boldsymbol{l}=\boldsymbol{\mu}=\mathbf{0}$, |
| | $I_3=2p\gamma_1+2q\gamma_2+r\gamma_3=0$, |
| | $I_4=[p^2-q^2+c\gamma_3^2-\frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}]^2+[2pq+d\gamma_3^2-2\frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$ <br> $+2\rho[(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})\left(p^2+q^2\right)+c\frac{(\gamma_1^2-\gamma_2^2)}{\gamma_3^4}+2d\frac{\gamma_2\gamma_1}{\gamma_3^4}-\lambda\frac{(1-\gamma_3^2)^2}{\gamma_3^8}]$ <br> $+\rho^2\frac{(1-\gamma_3^2)^2}{\gamma_3^{12}}$. |

Elmandouh (2015) [73] introduced a two-parameter generalization of this case by adding singular terms into the vector and scalar potentials:

| | |
|---|---|
| 17* | $V=C[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2+\frac{\lambda}{\gamma_3^2}+\rho(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})+\frac{(\gamma_3^2-2)\gamma_3^2}{2\gamma_1^2}(\frac{\nu_1}{\gamma_1}+\frac{\nu_2}{\gamma_2})^2]$, |
| | $\boldsymbol{l}=C(0,0,\frac{2-\gamma_3^2}{\gamma_1}(\frac{\nu_1}{\gamma_1}+\frac{\nu_2}{\gamma_2}))$. |

The fourth integral is also given in the Euler–Poisson variables in [73].

### 13.3.3 Cases Combining Kowalevski's and Chaplygin's Cases

Two cases are listed in the following table:

| | |
|---|---|
| 18 | Yehia 2012 [415], <br> $\delta=0$ Goryachev 1917 [118], <br> $\delta=\lambda=a_1=0$ Chaplygin 1903 [53], <br> $\delta=\lambda=a_2=0$ Kowalevski 1888 [238], |
| | $A=B=2C$, <br> $V=2C[a_1\gamma_1+a_2(\gamma_1^2-\gamma_2^2)+\frac{\lambda}{\gamma_3^2}+\delta\frac{2-\gamma_3^2}{\gamma_2^2}]$, <br> $\boldsymbol{l}=\boldsymbol{\mu}=\mathbf{0}$, |
| | $I_3=2p\gamma_1+2q\gamma_2+r\gamma_3=0$, <br> $I_4=[p^2-q^2-a_1\gamma_1+a_2\gamma_3^2-\frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2})^2$ <br> $+(2pq-a_1\gamma_2-\frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2})^2$ <br> $+\frac{\delta}{\gamma_2^2}\{2[(p^2+q^2]\gamma_3^2-2a_1\gamma_1\gamma_3^2-2\lambda\gamma_1^2+2a_2+\frac{\delta\gamma_3^4}{\gamma_2^2}\}$. |

The Goryachev subcase ($\delta = 0$) of case 18 has led to complex separation variables in [337, 359].

| | |
|---|---|
| 19 | Yehia 1996 [396],<br>$\lambda = 0$. Yehia 1987 [386],<br>$k = 0$. Goryachev [118],<br>$k = a_2 = \lambda = 0$ Chaplygin 1903 [53],<br>$k = b_1 = b_2 = \lambda = 0$ Kowalevski 1888 [238], |
| | $V = C[a_1\gamma_1 + a_2\gamma_2 + b_1(\gamma_1^2 - \gamma_2^2) + b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}]$,<br>$\boldsymbol{l} = \boldsymbol{\mu} = C(0, 0, k)$. |
| | $I_3 = 2p\gamma_1 + 2q\gamma_2 + (r + k)\gamma_3 = 0$,<br>$I_4 = [p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2 + b_1\gamma_3^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{2\gamma_3^2}]^2$<br>$+[2pq - a_1\gamma_2 - a_2\gamma_1 + b_2\gamma_3^2 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$<br>$+k\{(r - k)[2(p^2 + q^2) + \lambda(1 + \frac{1}{\gamma_3^2})]$<br>$-4\gamma_3[(a_1 + b_1\gamma_1 + b_2\gamma_2)p + q(a_2 + b_2\gamma_1 - b_1\gamma_2)]\}$. |

Elmandouh (2015) [74], added a parameter $e$, which engenders singular terms in the vector and scalar potentials, as follows:

| | |
|---|---|
| 19* | $V = C\{a_1\gamma_1 + a_2\gamma_2 + b_1(\gamma_1^2 - \gamma_2^2) + b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}$<br>$-\frac{e\gamma_3^2}{\gamma_1^2}[k - \frac{e(2\gamma_2^2+\gamma_3^2)}{2\gamma_1^2}]\}$,<br>$\boldsymbol{l} = C\left(0, 0, k + \frac{e(1+\gamma_2^2)}{\gamma_1^2}\right)$,<br>$\boldsymbol{\mu} = C\left(\frac{-2e\gamma_3}{\gamma_1^3}(1 + \gamma_2^2), \frac{2e\gamma_2\gamma_3}{\gamma_1^2}, k + \frac{e(1+\gamma_2^2)}{\gamma_1^2}\right)$, |

The complementary integral was also provided in [74].

Separation of variables for the Chaplygin level of the above hierarchies was attained by Chaplygin himself. For detailed solution see Chap. 10 Sect. 10.16. The version $\kappa \neq 0$ of this case was considered in Tsiganov's work [358], where an assertion is made that separated variables are constructed for what the author calls "the Kowalevski–Goryachev–Chaplygin gyrostat". However, the proposed separated variables are complex functions of the physical variables. It remains an open problem how to construct real solutions using complex hyper-elliptic quadratures [358]. **Note that** the title and references in that work have brought certain confusion, which was commented in our note [410].

In [416], it was shown that the problem of motion of a rigid body, with $A = 2C$ and arbitrary $B$, subject to forces with potential containing one Kowalevski term, one Chaplygin term together with the singular Goryachev term

$$V = a_1\gamma_1 + b_1(\gamma_1^2 - \gamma_2^2) + \frac{c_1}{\gamma_3^2}, \tag{13.1}$$

under the additional restrictions $q = 0$, $f = 0$, is solvable in elliptic functions of time. The solution is the same in case 17, when $A = B = 2C$, $n = \kappa = a_2 = b_2 = 0$

under the additional restriction $q = 0$. Without this restriction, the solution corresponding to the last potential (13.1) is not known at this moment.

The subcase with the potential

$$V = b_1(\gamma_1^2 - \gamma_2^2) + \frac{c_1}{\gamma_3^2}, \tag{13.2}$$

common between hierarchies #14-19, has attracted more attention. It is called by some authors the "Goryachev system". Ryabov found real separation variables for this case in [323], reduced its integration to hyper-elliptic quadratures and studied the phase topology for positive values of the parameters, i.e. when the integral surfaces are compact. The case of negative values of the parameters, when the integral surfaces become non-compact, is treated by Nikolaenko [297], who has also shown that Goryachev's system is Liouville equivalent to other integrable cases in rigid body dynamics, according to the value of the energy parameter on the admissible energy interval $[h_{min}, \infty)$ [296].

### *13.3.4 A Case with a Quartic Integral Outside the Above Classification*

| | |
|---|---|
| 20 | Yehia 2003 [411], $f = 0$, |
| | $A = B = 2C$,<br>$V = \frac{a\gamma_3}{(\gamma_1^2+\gamma_2^2)^{\frac{3}{4}}} + \frac{b}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{\gamma_3\sqrt{c\gamma_1+d\gamma_2+\sqrt{(c^2+d^2)(\gamma_1^2+\gamma_2^2)}}}{\sqrt{\gamma_1^2+\gamma_2^2}},$<br>$\boldsymbol{\mu} = \mathbf{0}$. |

This is a case of algebraic potential, which involves three singular terms of different fractional powers. It reminds the case of fractional power potential and third-degree integral due to Goryachev. The fourth integral for this case can be expressed in terms of Euler's angles, using formulas provided in [411], but it is not constructed yet in the Euler-Poisson variables.

### *13.3.5 Two Conditional Cases Valid on a Single, Not Necessary Zero, Level of the Linear Integral [421]*

This case adds to Yehia's gyrostat the parameter $m$, figuring in potential and gyroscopic terms, and turns into it when $m = 0$. The quartic integral is expressed in terms of Euler's angles, but not in Euler-Poisson variables [421] (Table 13.2).

**Table 13.2** Conditional cases on a single level of the linear integral $f = \alpha(\alpha arbitrary)$

| | |
|---|---|
| 1 | $A = B = 2C$, |
| | $V = a\gamma_1 + b\gamma_2 - \frac{m}{2(\gamma_1^2+\gamma_2^2)}\left[2(k-m) - 2\alpha\gamma_3 + \frac{m}{\gamma_1^2+\gamma_2^2}\right]$, |
| | $\boldsymbol{l} = (0, 0, k + \frac{m}{\gamma_1^2+\gamma_2^2})$ |
| 2 | $V = a\gamma_1 + b\gamma_2 - \frac{k}{2(\gamma_1^2+\gamma_2^2)}\left[-2\alpha\gamma_3 + \frac{k}{\gamma_1^2+\gamma_2^2}\right] + \frac{\lambda+\gamma_3\sqrt{\frac{c^2+d^2}{2}\sqrt{\gamma_1^2+\gamma_2^2}+\frac{c^2-d^2}{2}\gamma_2+cd\gamma_1}}{\sqrt{\gamma_1^2+\gamma_2^2}}$, |
| | $\boldsymbol{l} = (0, 0, k + \frac{k}{\gamma_1^2+\gamma_2^2})$. |

Case 2 generalizes the case of Yehia and Bedwehi. In both cases, the added new terms are all singular at the two positions $\boldsymbol{\gamma} = (0, 0, \pm 1)$.

## 13.4 Integrable Extensions of Conditional Integrable Cases

As remarked in the beginning of this chapter, the method of transformation with an arbitrary function $\nu(\gamma_1, \gamma_2, \gamma_3)$ used in Sect. 12.2 of the last chapter is also applicable to all conditional integrable cases, valid on the zero level of the cyclic integral, i.e. to the 20 cases of this type listed in the last three sections.

We shall not give here a list of generalizations of the conditional cases. Most of those cases involve singular terms that are not likely to get acceptable physical interpretation. Physical effects of the transformation are here immaterial and will remain at present just as parts of mathematical models. Moreover, unlike the generalized cases introduced in Chap. 12, the flexibility offered by the presence of the areas constant as an arbitrary parameter is here lost. The transformed integrable problems and their original counterparts share the same Hamiltonian. To this kind of extension of conditional integrable cases applies the argument of Borisov and Mamaev [41], mentioned in the last section of the preceding chapter and they need not to be considered unless for some reason it becomes necessary to use a concrete form of the function $\nu$ in the transformation.