Chapter 12
The Most General Integrable Cases in Rigid Body Dynamics

In this chapter, we present a small, but most exotic, set of general and conditional integrable cases, which constitute currently the uppermost level of the hierarchy of integrable cases in rigid body dynamics. That level is inaccessible for all direct methods used in mechanics in the past. Methods which investigate the existence of analytical or polynomial integrals and the existence of single-valued solutions of the equations of motion are equally hopeless in facing such a wide class of problems. Here, we speak about several-parameter generalizations of six of the seven integrable cases in the dynamics of a body in a liquid in two different ways, general and conditional, by applying Theorems 2 and 3 of the last chapter.

12.1 General Integrable Cases

Listed in Table 12.1 are the most general and most exotic integrable cases known up-to-date of the problem of motion of a rigid body about a fixed point under the action of an axi-symmetric combination of conservative potential and gyroscopic forces. Their generality results from the extra number of parameters (an arbitrary function in case 7) included in their structure. The first five of the seven general integrable cases, namely cases occupying positions 1–5 in Table 12.1, are obtained by applying Theorem 2 of the preceding chapter to construct unconditional integrable generalizations of all but one of the integrable cases of Chap. 10 concerning the motion of a body in a fluid. Depending on the structure of the potential, a number of additional parameters, ranging up to 4, is added to the structure of each case. The case number 6 of Table 12.1 is obtained by applying the same Theorem 2 to a general integrable case found by Yehia and Bedwehy. The latter generalizes the classical Kowalevski case by adding a singular term $\frac{\varepsilon}{\sqrt{1-\gamma_3^2}}$ to the heavy body potential.

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Table 12.1 General integrable extensions of general integrable cases

1	Yehia [398] (1997), Oreshkina [301], Clebsch [55]: $n_1 = 0$, Brun [47]: $n = n_1 = 0$, Tisserand [354]: $n = n_1 = 0$
	$V = (A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)\{b - \frac{1}{2}[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]^2\}$, $\nu = n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$, $\boldsymbol{l} = [n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\boldsymbol{\gamma}\mathbf{I}$, $\mu_1 = \gamma_1\{(A - B - C)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$ $+2n_1[A(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\}$, $\mu_2 = \gamma_2\{(B - C - A)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$ $+2n_1[B(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\}$, $\mu_3 = \gamma_3\{[(C - A - B)[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$ $+2n_1[B(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\}$
	$I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3$ $+[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)](A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$
	$I_4 = A^2\{p + \gamma_1[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2$ $+B^2\{q + \gamma_2[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2$ $+C^2\{r + \gamma_3[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2$ $-2(b - n_1 I_3)(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$
	$I_3 = M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3$, $H = 1/2\,(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}) + b\left(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2\right)$ $-\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$
	$I_4 = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2)$ $-ABC(b - n_1 I_3)(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$

The last of those cases, number 7, is the ultimate generalization of Lagrange's case to the most general case in which the proper rotation angle φ is a cyclic coordinate and the fourth integral is the cyclic integral.

We also give relevant supplementary information and some characteristic properties of the cases provided in the table of the last section. Some clarifications are made about the present status of the explicit solution of each of the generalized cases as per the progress made in solving their primitive counterparts at the lower levels of the hierarchy.

12.1.1 Table of General Integrable Extensions of General Integrable Cases

This case includes one parameter n_1 more than Clebsch's case and two parameters n, n_1 more than Brun's problem. As established in the last chapter, the explicit solution for this case in terms of time can be obtained by the variable precession transformation (11.31) from the basic solution $n = n_1 = 0$, given by Kötter in terms of theta functions of two arguments for the first integrable case of Clebsch (see Chap. 10).

In [301], an integrable case of M. Kharlamov's equations was constructed. It admits a fourth integral quadratic in the angular velocities, with coefficients depending on $\boldsymbol{\gamma}$. To this end, the author used an ansatz for the quadratic integral, and used consistency conditions with the equations of motion. The resulting expressions are quite complicated and lack transparency. The author has neither noted the possibility of transforming this case to Clebsch's case of motion of a body in a liquid, nor even any relation to Clebsch's case as a special case. Consequently, to the end of her paper, the author states that the existence of the fourth algebraic integral "means the possibility, in principle, to reduce the problem to quadrature". Our method gives an effortless constructive way to build the explicit solution by applying the variable precession transformation to Kötter's solution.

2	Yehia [398] (1997), Kharlamova L. [214] (1990): $\frac{n_1}{c_1} = \frac{n_2}{c_2} = \frac{n_3}{c_3}$ Clebsch's case of spherical symmetry [55]: $n_1 = n_2 = n_3 = 0$.
	$B = C = A$, $V = \frac{1}{2}A[c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2 - (n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)^2]$, $\nu = n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2$, $\boldsymbol{l} = A\nu\boldsymbol{\gamma}$, $\mu_1 = -A\gamma_1[n + n_1\gamma_1^2 + \gamma_2^2\,(3n_2 - 2n_1) + \gamma_3^2\,(3n_3 - 2n_1)]$, $\mu_2 = -A\gamma_2[n + \gamma_1^2\,(3n_1 - 2n_2) + n_2\gamma_2^2 + (3n_3 - 2n_2)\,\gamma_3^2]$, $\mu_3 = -A\gamma_3[n + \gamma_1^2\,(3n_1 - 2n_3) + \gamma_2^2\,(3n_2 - 2n_3) + n_3\gamma_3^2]$
	$I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 + n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2$,
	$I_4 = (c_1 - 2n_1 I_3)\,(p + \nu\gamma_1)^2 + (c_2 - 2n_2 I_3)\,(q + \nu\gamma_2)^2$ $+ (c_3 - 2n_3 I_3)\,(r + \nu\gamma_3)^2$ $-[(2n_2 I_3 - c_2)\,(2n_3 I_3 - c_3)\,\gamma_1^2 + (2n_3 I_3 - c_3)\,(2n_1 I_3 - c_1)\,\gamma_2^2$ $+ (2n_1 I_3 - c_1)\,(2n_2 I_3 - c_2)\,\gamma_3^2]$
	$I_3^* = M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3$, $H = \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2)$ $-(n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)$
	$I_4 = \left(Ac_1 - 2n_1 I_3^*\right) M_1^2 + \left(Ac_2 - 2n_2 I_3^*\right) M_2^2 + \left(Ac_3 - 2n_3 I_3^*\right) M_3^2$ $-A[\left(2n_2 I_3^* - Ac_2\right)\left(2n_3 I_3^* - Ac_3\right)\gamma_1^2 + \left(2n_3 I_3^* - Ac_3\right)\left(2n_1 I_3^* - Ac_1\right)\gamma_2^2$ $+ \left(2n_1 I_3^* - Ac_1\right)\left(2n_2 I_3^* - Ac_2\right)\gamma_3^2]$

In this case, the body has spherical dynamical symmetry. The basic case $n_1 = n_2 = n_3 = 0$ is a case of the motion of a body in a liquid (Case 3 of Table 10.1 of Sect. 10.15). It is closely related to the other Clebsch's integrable case with a tri-axial body in the same problem. The solution of this case can be expressed in terms of theta functions of two variables [233] (see also [71]) and so, in principle, will be the present generalization. However, this point needs a closer examination, as the present case presents some unusual and rarely met characteristic properties.

(1) The presence of the three extra parameters n_1, n_2, n_3, which we assume different, raises the degree of the polynomial potential $V(\boldsymbol{\gamma})$ from 2 to 4 and the degree of the components of $\boldsymbol{\mu}$ from 1 to 3. The combination of forces acting on the body has turned into a much complicated one, compared to the original.

(2) The presence of the same parameters raises the degree of the complementary polynomial integral I_4 as a function of the angular velocity components in the six-dimensional phase space $\{\boldsymbol{\omega}, \boldsymbol{\gamma}\}$ of the three-dimensional problem from 2 to 3. However, on every level of the integral I_3, say, $I_3 = f$, the complementary integral I_4 becomes of the second degree. This may be reformulated in the language of analytical dynamics in the following way: The reduced equations of motion of the problem under consideration after ignoring the cyclic coordinate ψ admit a quadratic complementary integral I_4 in the other two Eulerian angles $\{\dot{\theta}, \dot{\varphi}\}$.

(3) An exceptional case arises, when the two sets of constants are proportional

$$\frac{n_1}{c_1} = \frac{n_2}{c_2} = \frac{n_3}{c_3} = \lambda \text{ (say).}$$

Then I_4 takes the form

$$\begin{aligned} I_4 = (1 - 2\lambda I_3)\, \{&c_1(p + \nu\gamma_1)^2 + c_2(q + \nu\gamma_2)^2 + c_3(r + \nu\gamma_3)^2 \\ &- (1 - 2\lambda I_3)\,[c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2]\}. \end{aligned}$$

For arbitrary value of I_3, one can cancel the first factor $(1 - 2\lambda I_3)$ to obtain for I_4 the expression

$$\begin{aligned} I_4 = {} &c_1(p + \nu\gamma_1)^2 + c_2(q + \nu\gamma_2)^2 + c_3(r + \nu\gamma_3)^2 \\ &- (1 - 2\lambda I_3)\,[c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2], \end{aligned}$$

which is quadratic in the velocities, but has some linear terms. This is the case of a quadratic integral found in [214] in a more complicated and less transparent way.

3	Yehia [398] (1997), Rubanovsky [317]: $n_1 = n_2 = n_3 = 0$ Lyapunov [267]: $n_1 = n_2 = n_3 = a_1 = a_2 = a_3 = 0$
	$B = C = A$, $\nu = n + n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, $V = A\{a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 - \frac{1}{2}(bc\gamma_1^2 + ca\gamma_2^2 + ab\gamma_3^2)$ $\quad + \frac{1}{2}\nu[(b + c)\,\gamma_1^2 + (c + a)\,\gamma_2^2 + (a + b)\,\gamma_3^2] - \frac{1}{2}\nu^2\}$, $\boldsymbol{l} = A[-\frac{1}{2}((b + c)\gamma_1, (c + a)\gamma_2, (a + b)\gamma_3) + \nu\boldsymbol{\gamma}]$ $\mu_1 = A[n_1 + \gamma_1(a + n - 2\nu)]$, $\mu_2 = A[n_2 + \gamma_2(b + n - 2\nu)]$, $\mu_3 = A[n_3 + \gamma_3(c + n - 2\nu)]$
	$I_3 = (p\gamma_1 + q\gamma_2 + \gamma_3 r) - \frac{1}{2}[\,(b + c)\,\gamma_1^2 + (c + a)\,\gamma_2^2 + (a + b)\,\gamma_3^2] + \nu$ $I_4 = \frac{1}{2}\{\,(b + c)\,(p + \nu\gamma_1)^2 + (c + a)\,(q + \nu\gamma_2)^2 + (a + b)\,(r + \nu\gamma_3)^2\}$ $\quad + (-n_1 I_3 + a_1)\,(p + \nu\gamma_1 + a\gamma_1)$ $\quad + (-n_2 I_3 + a_2)\,(q + \nu\gamma_2 + b\gamma_2)$ $\quad + (-n_3 I_3 + a_3)\,(r + \nu\gamma_3 + c\gamma_3)$ $\quad -(bcp\gamma_1 + caq\gamma_2 + abr\gamma_3) - \nu(bc\gamma_1^2 + ca\gamma_2^2 + ab\gamma_3^2)$

	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3,$ $H = \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}[(b+c)M_1\gamma_1 + (c+a)M_2\gamma_2 + (a+b)M_3\gamma_3]$ $+[(Aa_1 - n_1 I_3^*)\gamma_1 + (Aa_2 - n_2 I_3^*)\gamma_2 + \gamma_3(Aa_3 - n_3 I_3^*)]$ $-\frac{A}{8}[(a^2+2bc)\gamma_1^2 + (b^2+2ca)\gamma_2^2 + (c^2+2ab)\gamma_3^2]$
	$I_4 = (b+c)[M_1 + \frac{A}{2}(b+c)\gamma_1]^2 + (c+a)[M_2 + \frac{A}{2}(c+a)\gamma_1]^2$ $+(a+b)[M_3 + \frac{A}{2}(a+b)\gamma_1]^2$ $+(Aa_1 - n_1 I_3^*)[2M_1 + A(2a+b+c)\gamma_1]$ $+(Aa_2 - n_2 I_3^*)[2M_2 + A(a+2b+c)\gamma_2]$ $+(Aa_3 - n_3 I_3^*)[2M_3 + A(a+b+2c)\gamma_3]$ $-A\{bc\gamma_1[2M_1 + A(b+c)\gamma_1] + ca\gamma_2[2M_2 + A(c+a)\gamma_2]$ $+ab\gamma_3[2M_3 + A(a+b)\gamma_3]\}$

Lyapunov's case $s_1 = s_2 = s_3 = n_1 = n_2 = n_3 = 0$ [267] was solved by Kötter, as well as the related Steklov case, in terms of theta functions of two arguments [235]. This solution will cover the case $s_1 = s_2 = s_3 = 0$ for arbitrary n_1, n_2, n_3. It is obvious that to express the solution in the most general case by applying the variable precession transformation, it suffices to obtain the solution for the basic case $n = n_1 = n_2 = n_3 = 0$, $s_1 s_2 s_3 \neq 0$. This was not done up to the present time.

4	Yehia [398] (1997), Yehia [383] $n_1 = n_2 = 0$, Yehia [380]: $n = n_1 = n_2 = 0$, Kowalevski [238]: $k = n = n_1 = n_2 = 0$
	$A = B = 2C,$ $V = C[a_1\gamma_1 + a_2\gamma_2 - \kappa\gamma_3\nu - \frac{1}{2}\nu^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)],$ $\nu = n + n_1\gamma_1 + n_2\gamma_2.$ $\mu_1 = C\left(-n\gamma_1 - n_1\gamma_1^2 + 2n_1\gamma_2^2 + n_1\gamma_3^2 - 3n_2\gamma_1\gamma_2\right),$ $\mu_2 = C\left(-\gamma_2 n + 2n_2\gamma_1^2 - n_2\gamma_2^2 + n_2\gamma_3^2 - 3n_1\gamma_1\gamma_2\right),$ $\mu_3 = C(\kappa - 3n\gamma_3 - 5n_1\gamma_1\gamma_3 - 5n_2\gamma_2\gamma_3)$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + (r+\kappa)\gamma_3 + \nu\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)$
	$I_4 = [(p+\nu\gamma_1)^2 - (q+\nu\gamma_2)^2 - (a_1 - n_1 I_3)\gamma_1 + (a_2 - n_2 I_3)\gamma_2]^2$ $+[2(p+\nu\gamma_1)(q+\nu\gamma_2) - (a_1 - n_1 I_3)\gamma_2 - (a_2 - n_2 I_3)\gamma_1]^2$ $+2\kappa(r + \nu\gamma_3 - \kappa)[(p+\nu\gamma_1)^2 + (q+\nu\gamma_2)^2]$ $-4\kappa\gamma_3[(a_1 - n_1 I_3)(p+\nu\gamma_1) + (a_2 - n_2 I_3)(q+\nu\gamma_2)]$
	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3,$ $H = \frac{1}{2C}(\frac{M_1^2 + M_2^2}{2} + M_3^2) - kM_3 + C(a_1\gamma_1 + a_2\gamma_2)$ $-(n + n_1\gamma_1 + n_2\gamma_2)(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$
	$I_4 = [\frac{M_1^2 - M_2^2}{4C^2} - (a_1 - \frac{n_1}{C}I_3^*)\gamma_1 + (a_2 - \frac{n_2}{C}I_3^*)\gamma_2]^2$ $+[\frac{M_1 M_2}{2C^2} - (a_1 - \frac{n_1}{C}I_3^*)\gamma_2 - (a_2 - \frac{n_2}{C}I_3^*)\gamma_1]^2$ $+\frac{k}{C^2}(\frac{M_3}{2C} - k)(M_1^2 + M_2^2)$ $-\frac{2k}{C^2}\gamma_3[(a_1 - \frac{n_1}{C}I_3^*)M_1 + (a_2 - \frac{n_2}{C}I_3^*)M_2]$

In view of the presence of the coefficients a_1, a_2 in the linear terms of V and n_1, n_2 in the function ν, it is evident that by rotating the xy axes one can eliminate one of those four coefficients. Suppose we have eliminated a_2, then n_2 will remain there, although it will not appear if we apply our method to the basic potential without a_2. Thus, we shall keep the four terms and the resulting case will neither repeat nor be included in the other generalization of Kowalevski's case in the line leading to case number 5.

The Euler–Poisson variables in Kowalevski's case were expressed by Kowalevski herself in terms of hyper-elliptic functions of time [238]. The solution was somewhat simplified and systematized by Kötter [232, 234] (see also [256]). Explicit expressions for the six variables in terms of the separation variables can also be found in [113, 256]. Many qualitative and global properties of motion are discussed in [240] The same problem was treated in a large number of recent works using methods of modern algebraic geometry and the inverse scattering method (e.g. [71, 145] and references cited therein). Of special interest is the work [152], which relates the Kowalevski case to a special version ($f = 0$) of Clebsch's case by means of a rational transformation and thus explicit solutions for the first can be obtained from that of the other. The same idea was realized for our generalization of Kowalevski's case to the gyrostat by Gavrilov, who related it to the full case of Clebsch ($f \neq 0$) solvable in Theta functions of two arguments [110]. Thus, it becomes evident that the generalized case under discussion here is, in principle, solvable in terms of the same class of functions. However, a direct separation of Yehia's gyrostat is not achieved yet (see Chap. 5, Sect. 5.6).

5	Yehia [411] Sokolov [336] $n = n_2 = 0$ Yehia [380] $n = n_2 = c = 0$ Kowalevski [238]$n = n_2 = c = \kappa = 0$
	$A = B = 2C$, $\nu = n + n_2\gamma_2$, $V = C[\kappa c\gamma_1 + a_2\gamma_2 - \nu(\kappa + c\gamma_1)\gamma_3 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2)$ $\quad -\frac{\nu^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)]$, $\boldsymbol{l} = C(2\nu\gamma_1, 2\nu\gamma_2, \kappa + \nu\gamma_3 + c\gamma_1)$, $\mu_1 = C(c\gamma_3 - n\gamma_1 - 3n_2\gamma_1\gamma_2)$, $\mu_2 = C[-n\gamma_2 + n_2\left(2\gamma_1^2 - \gamma_2^2 + \gamma_3^2\right)]$, $\mu_3 = C(\kappa + c\gamma_1 - 3n\gamma_3 - 5n_2\gamma_2\gamma_3)$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + (r + \kappa + c\gamma_1)\gamma_3$ $\quad +(n + n_2\gamma_2)(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)$, $I_4 = \{(p + \nu\gamma_1)^2 - (q + \nu\gamma_2)^2 + (a_2 - n_2 I_3)\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r + \nu\gamma_3 - \kappa)\}^2$ $\quad +\{2(p + \nu\gamma_1)(q + \nu\gamma_2) - (a_2 - n_2 I_3)\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r + \nu\gamma_3 - \kappa)\}^2$ $\quad +2\kappa(r + \nu\gamma_3 - \kappa + c\gamma_1)[(p + \nu\gamma_1)^2 + (q + \nu\gamma_2)^2 + 2c(p + \nu\gamma_1)\gamma_3]$ $\quad -4\kappa\gamma_3(a_2 - n_2 I_3)(q + \nu\gamma_2)$ $\quad -2\kappa c^2[2\gamma_3 I_3 - \kappa\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + \nu\gamma_3 + c\gamma_1)]$
	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3$, $H = \frac{1}{2C}(\frac{M_1^2 + M_2^2}{2} + M_3^2) - (\kappa + c\gamma_1)M_3 + C(a_2\gamma_2 + 2c\kappa\gamma_1 - c^2\gamma_3^2)$ $\quad -\nu(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$
	$I_4 = [\frac{M_1^2 - M_2^2}{4C^2} + (a_2 - \frac{n_2}{C} I_3^*)\gamma_2 + c(\frac{M_3}{C} - 2\kappa)\gamma_1 - c^2(\gamma_1^2 - \gamma_2^2)]^2$ $\quad +[\frac{M_1 M_2}{2C^2} - (a_2 - \frac{n_2}{C} I_3^*)\gamma_1 + c(\frac{M_3}{C} - 2\kappa)\gamma_2 - 2c^2\gamma_1\gamma_2]^2$ $\quad +\kappa(\frac{M_3}{C} - 2\kappa)[\frac{M_1^2 + M_2^2}{2C^2} + 2c\gamma_3\frac{M_3}{C}] - 2\kappa\gamma_3(a_2 - \frac{n_2}{C} I_3^* + 2c^2\gamma_2)\frac{M_2}{C}$ $\quad -\frac{2\kappa c^2}{C}[2\gamma_1\gamma_3 M_1 - (\gamma_1^2 + \gamma_2^2)M_3]$

Variable separation for the basic Sokolov's case, $n = n_2 = 0$, was obtained in [227] and explicit expressions for dynamical variables are constructed in [70], in terms of two intermediate variables, which are expressed in genus-2 Theta functions. At the present level of the hierarchy, after the introduction of the parameters n, n_2, the solution is obtained by applying the relevant precession transformation.

6	Yehia [398] (1997), Yehia–Bedwehy [419]: $n = n_1 = n_2 = N = 0$, Kowalevski [238]: $\varepsilon = n = n_1 = n_2 = N = 0$
	$A = B = 2C$, $\nu = n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}$
	$V = C[a_1\gamma_1 + a_2\gamma_2 + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}} - \frac{1}{2}\nu^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)]$
	$\boldsymbol{l} = C\nu(2\gamma_1, 2\gamma_2, \gamma_3)$, $\mu_1 = C[-n\gamma_1 + n_1\left(2\gamma_2^2 - \gamma_1^2 + \gamma_3^2\right) - 3n_2\gamma_1\gamma_2 - \frac{N\gamma_1}{(1-\gamma_3^2)^{\frac{3}{2}}}]$, $\mu_2 = C[-n\gamma_2 + n_2\left(2\gamma_1^2 - \gamma_2^2 + \gamma_3^2\right) - 3n_1\gamma_1\gamma_2 - \frac{N\gamma_2}{(1-\gamma_3^2)^{\frac{3}{2}}}]$, $\mu_3 = -C[(3n + 5n_1\gamma_1 + 5n_2\gamma_2)\gamma_3 + \frac{N\gamma_3}{\sqrt{1-\gamma_3^2}}]$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + r\gamma_3 + \nu(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)$
	$I_4 = [(p + \nu\gamma_1)^2 - (q + \nu\gamma_2)^2 - (a_1 - n_1 I_3)\,\gamma_1 + (a_2 - n_2 I_3)\,\gamma_2]^2$ $+[2(p + \nu\gamma_1)(q + \nu\gamma_2) - (a_1 - n_1 I_3)\,\gamma_2 - (a_2 - n_2 I_3)\,\gamma_1]^2$ $+2\frac{(\varepsilon - N I_3)}{\sqrt{1-\gamma_3^2}}[(p + \nu\gamma_1)^2 + (q + \nu\gamma_2)^2] + \frac{(\varepsilon - N I_3)^2}{1-\gamma_3^2}$
	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3$, $H = \frac{1}{2C}(\frac{M_1^2 + M_2^2}{2} + M_3^2) - \nu(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$ $+C[a_1\gamma_1 + a_2\gamma_2 + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}}]$
	$I_4 = [\frac{M_1^2 - M_2^2}{4C^2} - (a_1 - \frac{n_1}{C} I_3^*)\gamma_1 + (a_2 - \frac{n_2}{C} I_3^*)\gamma_2]^2$ $+[\frac{M_1 M_2}{2C^2} - (a_1 - \frac{n_1}{C} I_3^*)\gamma_2 - (a_2 - \frac{n_2}{C} I_3^*)\gamma_1]^2$ $+\frac{M_1^2 + M_2^2}{2C^2}[\frac{\varepsilon - \frac{N}{C} I_3^*}{\sqrt{1-\gamma_3^2}}] + \frac{[\varepsilon - \frac{N}{C} I_3^*]^2}{1-\gamma_3^2}$

Separation variables and expressions of the dynamical variables in terms of them are constructed in [218] for the conditional case 11 of Table 13.1 of Chap. 13, which covers the Yehia–Bedwehy only on the level $f = 0$. To cover the present full general case, explicit solution of the full Yehia–Bedwehy case for $f \neq 0$ is needed. This was not achieved until now. Note that in cases 1–6, the constant I_3(or I_3^*) figuring in the expression for I_4 can be substituted by its expression in each case as a function in the components of $\boldsymbol{\omega}$(or $\mathbf{M}$) and $\boldsymbol{\gamma}$.

Remark: *It would be highly interesting to study how this phenomenon of coupling constants changes many results and conclusions obtained for all integrable cases of motion of a body in a liquid and, in particular, the portrait of the integrals of motion, bifurcation diagrams and the topological classifications of integral manifolds. Those questions are presently open for all the above six cases in Table 12.1.*

7	Generalization of Lagrange's case. Two cyclic coordinates ψ and φ. $B = A$, $V = V_0(\gamma_3)$, $\boldsymbol{l}=(\ell\gamma_1, \ell\gamma_2, l_3)$, $\boldsymbol{\mu} = (-l_3'\gamma_1, -l_3'\gamma_2, l_3 - 2\gamma_3\ell + (\gamma_1^2+\gamma_2^2)\ell')$, $V_0(\gamma_3), \ell(\gamma_3), l_3(\gamma_3)$ arbitrary functions, and l_3', ℓ' denote derivative w.r.t. to γ_3
	$I_3 = A(p\gamma_1 + q\gamma_2) + Cr\gamma_3 + (\gamma_1^2+\gamma_2^2)\ell + l_3\gamma_3$
	$I_4 = Cr + l_3$
	$I_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3$, $H = \frac{1}{2}(\frac{M_1^2+M_2^2}{A} + \frac{M_3^2}{C}) - [\frac{\ell}{A}(M_1\gamma_1 + M_2\gamma_2) + \frac{l_3}{C}M_3]$ $+V_0(\gamma_3) + \frac{1}{2A}(\gamma_1^2+\gamma_2^2)\ell^2 + \frac{l_3^2}{2C}$
	$I_4 = M_3$

This is the most general case of Lagrange's type. The body admits not only dynamical but also physical symmetry about its z-axis. The equations of motion can be easily reduced to quadratures. In fact, the integrals I_3 and I_4 can be written as

$$
\begin{aligned}
I_3 &= A(p\gamma_1 + q\gamma_2) + \ell(\gamma_1^2+\gamma_2^2) + (Cr + l_3)\gamma_3 = f,\\
I_4 &= Cr + l_3 = j,
\end{aligned}
\tag{12.1}
$$

where f, j are the integral constants. The first can be reduced to the relation

$$
\dot{\psi} = \frac{1}{A}[\frac{f - j\gamma_3}{1-\gamma_3^2} - \ell], \tag{12.2}
$$

while I_4 gives

$$
C(\dot{\psi}\gamma_3 + \dot{\varphi}) + l_3 = j. \tag{12.3}
$$

From here we find, using (12.2),

$$
\dot{\varphi} = \frac{1}{C}(j - l_3) - \frac{\gamma_3}{A}[\frac{f - j\gamma_3}{1-\gamma_3^2} - \ell]. \tag{12.4}
$$

Thus, we have expressed $\dot{\psi}$ and $\dot{\varphi}$ in terms of γ_3. To obtain the relation with time, we use the energy (in fact, Jacobi's) integral

$$
\frac{1}{2}[A(p^2+q^2) + Cr^2] + V = h.
$$

That is

$$
A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + \frac{(j - l_3)^2}{C} = 2(h - V),
$$

which can be given the final form

$$\dot{\gamma}_3^2 = \frac{1}{A}(1-\gamma_3^2)[2(h-V) - \frac{(j-l_3)^2}{C}] - \frac{1}{A^2}[f - j\gamma_3 - \ell(1-\gamma_3^2)]^2. \quad (12.5)$$

Denoting the right-hand side of this relation by $F(\gamma_3)$, one can make a separation of variables and find the quadrature

$$t = \int \frac{d\gamma_3}{\sqrt{F(\gamma_3)}}, \quad (12.6)$$

which may be used to express $\gamma_3 = \cos\theta$ as a function in time, and hence (12.2) and (12.4) can be integrated to obtain the angles ψ and φ, respectively.

The above formulas are direct generalization of their lower counterparts in the hierarchy, beginning from Lagrange's top to Kirchhoff's case of the motion of a body in a liquid (Case 1 of Table 10.1 of Sect. 10.15). For Lagrange's top, as we have seen in Sect. 4.2, $F(\gamma_3)$ is a cubic function and γ_3 can be expressed in elliptic functions of time. In the case of a multi-connected symmetric body in a liquid, we have

$$\begin{aligned} V &= a_3\gamma_3 + \frac{1}{2}[b_1(1-\gamma_3^2) + b_3\gamma_3^2], \\ \ell &= K_1, l_3 = K_3\gamma_3 + \kappa, \end{aligned} \quad (12.7)$$

so that

$$\begin{aligned} F(\gamma_3) = &\frac{1}{A}(1-\gamma_3^2)[2h + b_1 - 2a_3\gamma_3 + (b_1 - b_3)\gamma_3^2 \\ &- \frac{1}{C}(j - \kappa - K_3\gamma_3)^2] - \frac{1}{A^2}[f - j\gamma_3 - K_1(1-\gamma_3^2)]^2 \end{aligned}$$

is a polynomial of the fourth degree and hence γ_3 is also an elliptic function of time.

Kirchhoff reduced the case of simply connected body ($a_3 = \kappa = 0$) to an elliptic quadrature and expressed some particular motions in terms of elliptic functions [219]. Detailed analysis of the general solution of the last special case in elliptic functions was performed by Greenhill [135]. This solution was not extended to the case of multi-connected body, but it can be noted that the presence of the constant gyrostatic term κ and the parameter a_3 changes the distribution of the roots of F and hence affects the picture of motion. As far as we know, this case was not studied in detail.

Thus, of all known results remains without the present type of generalization only one case, namely the case of a body in a liquid found by Rubanovsky [317] that includes as special versions an earlier case due to Steklov [345] and the case of a torque-free gyrostat considered by Joukovsky [163] and Volterra [366]. Due to the situation that in this case the basic potential is zero, no more terms can be added by the present method. However, as we have seen in Chap. 11, a generalization involving an arbitrary function and a parameter has been applied to this case in the next section, but to produce a conditional integrable case from it.

12.1.2 About the Hamiltonian Formulation

We have shown in Chap. 10 that, when $\nu(\boldsymbol{\gamma}) = n =$ const, the transformed problems are generalizations of the original ones, without bringing any mathematical complication to the solution process. The original and transformed problems are described by one and the same set of Hamiltonian equations. This is a consequence of the fact that the Hamiltonian after transformation is $H' = H + n\mathbf{M}\cdot\boldsymbol{\gamma}$, i.e. a combination of the original Hamiltonian and the areas integral.

We shall show now that when $\nu(\boldsymbol{\gamma})$ is chosen according to one of the two procedures described in Theorems 11.1–2, the Hamiltonian flow of the generalized problem is really different from that of the original problem and hence the first problem is a genuine generalization of the second.

In the tables of extended integrable cases, we have adopted the choice to identify every case by the expressions of functions V and $\boldsymbol{\mu}$, which are unique and completely characterize the physical setting for that case. An expression for the vector potential $\boldsymbol{l}$ is also given in the tables, so that the Lagrangian for each case can be readily constructed. The Hamiltonian of a problem can be obtained as described in previous chapters. To this end one can write

$$H = \boldsymbol{\omega}\cdot\frac{\partial L}{\partial \boldsymbol{\omega}} - L, \tag{12.8}$$

and eliminate $\boldsymbol{\omega}$ using the momentum variables

$$\mathbf{M} = \frac{\partial L}{\partial \boldsymbol{\omega}}. \tag{12.9}$$

The equations of motion take the form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M}\times \frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma}\times \frac{\partial H}{\partial \mathbf{M}}, \end{aligned} \tag{12.10}$$

and their integrals are

$$\begin{aligned} I_1 &= H, \\ I_2 &= \boldsymbol{\gamma}^2 = 1, \\ I_3 &= \mathbf{M}\cdot\boldsymbol{\gamma} = f. \end{aligned} \tag{12.11}$$

In order to illustrate our point of view described above, we now give examples of the extended cases in the Hamiltonian formalism.

(1) The first example is case 1 of Table 12.1, which involves one parameter n_1 of the new type in addition to the uniform precession parameter n. The Hamiltonian for that case is

$$
\begin{aligned}
H &= 1/2\,(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}) + b\left(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2\right) \\
&\quad - (M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)] \\
&= H_c - (M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)], \qquad (12.12)
\end{aligned}
$$

where H_c is the original Hamiltonian of the Clebsch case. The equations of motion are

$$
\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M} \times \{\frac{\partial H_c}{\partial \mathbf{M}} - [n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\boldsymbol{\gamma}\} \\
&\quad + \boldsymbol{\gamma} \times \{\frac{\partial H_c}{\partial \boldsymbol{\gamma}} - [n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\mathbf{M} - 2n_1(\mathbf{M}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}\mathbf{I}\}, \\
&= \mathbf{M} \times \frac{\partial H_c}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times [\frac{\partial H_c}{\partial \boldsymbol{\gamma}} - 2n_1(\mathbf{M}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}\mathbf{I}], \qquad (12.13)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \{\frac{\partial H_c}{\partial \mathbf{M}} - [n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\boldsymbol{\gamma}\} \\
&= \boldsymbol{\gamma} \times \frac{\partial H_c}{\partial \mathbf{M}}. \qquad (12.14)
\end{aligned}
$$

The fourth integral is

$$
\begin{aligned}
I_4 &= \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) \\
&\quad - ABC[b - n_1\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)](BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2) \\
&= I_{4c} + n_1 ABC\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2). \quad (12.15)
\end{aligned}
$$

From (12.13), we see that when $f \neq 0$ and $n_1 \neq 0$, then H_c and I_{4c} are no more integrals of motion. The Hamiltonian flow is deformed and the overall picture of the trajectories in the phase space of the new problem is different from that of Clebsch's case.

On the other hand, when $n_1 = 0$ ($\nu = n$)

$$
H = H_c - nI_3,
$$

so that the Hamiltonian is a linear combination of the two integrals H_c and I_3 with constant coefficients. In that case, from (12.13), (12.14), we see that the flow defined by the Hamiltonian (12.12) is identical with the flow corresponding to Clebsch's Hamiltonian H_c and the integral takes the form of Clebsch. The same holds also if consideration is restricted to the level $I_3 = f = 0$.

(2) The second example is the extension of Clebsch's spherically symmetric case. That is case 2 in Table 12.1. Let H_s and I_{4s} be the original Clebsch spherical Hamiltonian and the corresponding fourth integral. We have

$$\begin{aligned} H_s &= H_s(\mathbf{M},\boldsymbol{\gamma}, c_1, c_2, c_3) \\ &= \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2), \\ I_{4s} &= c_1 M_1^2 + c_2 M_2^2 + c_3 M_3^2 - (c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2). \end{aligned} \tag{12.16}$$

For the extended integrable system with three different parameters n_1, n_2, n_3,

$$\begin{aligned} H &= H_s - (n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,, \\ &= H_s(\mathbf{M},\boldsymbol{\gamma}, c_1 - 2n_1 I_3, c_2 - 2n_2 I_3, c_3 - 2n_3 I_3). \end{aligned} \tag{12.17}$$

The equations of motion have the form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M}\times\frac{\partial H_s}{\partial \mathbf{M}} + \boldsymbol{\gamma}\times[\frac{\partial H_s}{\partial \boldsymbol{\gamma}} - 2(\mathbf{M}\cdot\boldsymbol{\gamma})(n_1\gamma_1, n_2\gamma_2, n_3\gamma_3)], \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma}\times\frac{\partial H_s}{\partial \mathbf{M}}, \end{aligned} \tag{12.18}$$

and the complementary integral is

$$\begin{aligned} I_4 &= (c_1 - 2n_1 I_3)\,M_1^2 + (c_2 - 2n_2 I_3)\,M_2^2 + (c_3 - 2n_3 I_3)\,M_3^2 \\ &\quad - A[(2n_2 I_3 - c_2)\,(2n_3 I_3 - c_3)\,\gamma_1^2 + (2n_3 I_3 - c_3)\,(2n_1 I_3 - c_1)\,\gamma_2^2 \\ &\quad + (2n_1 I_3 - c_1)\,(2n_2 I_3 - c_2)\,\gamma_3^2] \\ &= I_{4s}(\mathbf{M},\boldsymbol{\gamma}, c_1 - 2n_1 I_3, c_2 - 2n_2 I_3, c_3 - 2n_3 I_3). \end{aligned} \tag{12.19}$$

The deformation caused by the presence of the three parameters n_1, n_2, n_3 and I_3 in the Hamiltonian flow on any level $I_3 = f \neq 0$ is obvious. The solution of the new problem of motion is obtained from that of the original case by replacing the original physical parameters c_1, c_2, c_3 by their new values involving three new physical parameters and the dynamical parameter f.

It is remarkable that the fourth integral (12.19) is cubic in the momenta in the whole phase space, due to the presence of I_3 in the coefficients but becomes quadratic in momenta on any fixed level of I_3. It also reduces to Clebsch's quadratic integral when $n_1 = n_2 = n_3 = 0$.

12.2 Conditional Integrable Deformations of General Integrable Cases

By Theorem 1 in Chap. 11, Sect. 11.6, all the general and conditional integrable cases of integrability of the previous hierarchy of problems admit a generalization using the transformation (11.31) to conditional cases involving the arbitrary function $\nu(\gamma_1, \gamma_2, \gamma_3)$. The explicit solution of the equations of motion in each case can be obtained from the solution of the original case, if the last is known, by Theorem 3. We apply this procedure here to all the general integrable cases, which were the subject of generalization in the preceding chapter, but this time including the case of Steklov–Rubanovsky, which was not amenable to the generalization as an unconditional case. As the structure of potential in the basic cases is not significant for the present type of generalization, we consider only one branch of the Kowalevski–Sokolov hierarchy, so that the total number of cases is still 7.

12.2.1 Table of Cases

In Table 12.2, we list the deformations of the known general integrable cases as conditional cases on a fixed level of the areas integral. For each case we provide

– conditions, if any, on the inertia matrix of the body,

– the pair of scalar potential V and vector $\boldsymbol{\mu}$, figuring in the equations of motion,

– the vector $\boldsymbol{l}$ which enters in the Lagrangian or Hamiltonian of the problem and in the structure of the cyclic integral,

– the cyclic integral I_3 itself,

– and, finally, the complementary integral.

– Each of the systems in the following table is integrable on the level $I_3 = \beta$. For each case, we also add the forms of the Hamiltonian function and the complementary integral in terms of momenta. In the Hamiltonian formulation, the cyclic integral has the same form for all integrable cases

$$I_3 = M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3 = \beta.$$

Table 12.2 Conditional integrable extensions of general integrable cases, valid on the level $I_3 = \beta$. $\nu = \nu(\gamma_1, \gamma_2, \gamma_3)$ is an arbitrary function

1	Case of Clebsch's type
	$V = (b - \frac{1}{2}\nu^2)(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) + \beta\nu$, $\mathbf{l} = \nu\boldsymbol{\gamma}\mathbf{I}$, $\boldsymbol{\mu} = \frac{\partial}{\partial\gamma}(\nu\boldsymbol{\gamma}\mathbf{I}\cdot\gamma) - [\frac{\partial}{\partial\gamma}\cdot(\nu\boldsymbol{\gamma}\mathbf{I})]\gamma$
	$I_3 = \boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\gamma} + \nu(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) = \beta$
	$I_4 = \frac{1}{2}[A^2(p + \nu\gamma_1)^2 + B^2(q + \nu\gamma_2)^2 + C^2(r + \nu\gamma_3)^2$ $-b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$
	$I_3 = M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3$, $H = 1/2\,(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}) + b\left(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2\right)$ $+[\beta - (M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)]\nu$
	$I_4 = M_1^2 + M_2^2 + M_3^2$ $-b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$

In all the cases under consideration, one can show that

$$\frac{dI_4}{dt} = (I_3 - \beta)\Phi(\boldsymbol{\omega}, \boldsymbol{\gamma}),$$

where Φ is a different function for each case. Thus, I_4 becomes an integral under the condition $I_3 = \beta$. As examples we give explicit results for two cases from the table.

Remark: *For the conditional integrable cases presented in Table 12.2, the Hamiltonian of the extended system is*

$$H' = H + \nu(\beta - M \cdot \gamma),$$

where H is the original Hamiltonian (before the transformation). The equations of motion are

$$\dot{\mathbf{M}} = \mathbf{M}\times\frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}} + (\beta - \mathbf{M}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma} \times \frac{\partial \nu}{\partial \boldsymbol{\gamma}},$$
$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}}. \tag{12.20}$$

Although the equations of motion in all cases of Table 12.2 depend on the parameter β and the (non-constant) function $\nu(\gamma)$, on the single level $f = \beta$ the Hamiltonian flow of the new conditional integrable problem becomes identical with the flow in the original unconditional integrable problem. In the last problem, the parameter f is arbitrary, while in the former the additional parameter β is present but regarding the dynamical condition $f = \beta$ both problems have the same number of parameters.

2	Case of Clebsch's type of spherical symmetry
	$A = B = C$, $V = \beta\nu + \frac{1}{2}(b_1\gamma_1^2 + b_2\gamma_2^2 + b_3\gamma_3^2) - \frac{1}{2}C\nu^2$, $\mathbf{l} = C\nu\boldsymbol{\gamma}$, $\boldsymbol{\mu} = C[\frac{\partial \nu}{\partial \boldsymbol{\gamma}} - (\nu + \boldsymbol{\gamma} \cdot \frac{\partial \nu}{\partial \boldsymbol{\gamma}})\boldsymbol{\gamma}]$
	$I_3 = C(p\gamma_1 + q\gamma_2 + r\gamma_3 + \nu) = \beta$
	$I_4 = C[b_1(p + \nu\gamma_1)^2 + b_2(q + \nu\gamma_2)^2 + b_3(r + \nu\gamma_3)^2]$ $-(b_2b_3\gamma_1^2 + b_1b_3\gamma_2^2 + b_1b_2\gamma_3^2)$
	$H = \frac{1}{2C}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(b_1\gamma_1^2 + b_2\gamma_2^2 + b_3\gamma_3^2)$ $+[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]\nu$
	$I_4 = b_1M_1^2 + b_2M_2^2 + b_3M_3^2 - C(b_2b_3\gamma_1^2 + b_1b_3\gamma_2^2 + b_1b_2\gamma_3^2)$

3	Rubanovsky–Lyapunov type [317]
	$A = B = C$ $V = C\{a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 - \frac{1}{2}(bc\gamma_1^2 + ca\gamma_2^2 + ab\gamma_3^2)$ $+\nu\{\beta + \frac{1}{2}[(b+c)\gamma_1^2 + (c+a)\gamma_2^2 + (a+b)\gamma_3^2]\} - \frac{1}{2}\nu^2\}$ $\boldsymbol{l} = C[\nu\boldsymbol{\gamma} - \frac{1}{2}((b+c)\gamma_1, (c+a)\gamma_2, (a+b)\gamma_3)]$, $\boldsymbol{\mu} = C[(a\gamma_1, b\gamma_2, c\gamma_3) + \frac{\partial \nu}{\partial \boldsymbol{\gamma}} - (\nu + \boldsymbol{\gamma} \cdot \frac{\partial \nu}{\partial \boldsymbol{\gamma}})\boldsymbol{\gamma}]$
	$I_3 = (p\gamma_1 + q\gamma_2 + r\gamma_3 + \nu) - \frac{1}{2}[(b+c)\gamma_1^2 + (c+a)\gamma_2^2 + (a+b)\gamma_3^2] = \beta$

	$I_4 = \frac{1}{2}[(b+c)(p+\nu\gamma_1)^2 + (c+a)(q+\nu\gamma_2)^2 + (a+b)(r+\nu\gamma_3)^2]$ $+a_1[p+(\nu+a)\gamma_1] + a_2[q+(\nu+b)\gamma_2] + a_3[r+(\nu+c)\gamma_3]$ $-[bc(p+\nu\gamma_1)\gamma_1 + ca(q+\nu\gamma_2)\gamma_2 + ab(r+\nu\gamma_3)\gamma_3]$
	$H = \frac{1}{2C}(M_1^2+M_2^2+M_3^2) + \frac{1}{2}[(b+c)M_1\gamma_1 + (c+a)M_2\gamma_2 + (a+b)M_3\gamma_3]$ $+C(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3)$ $-\frac{C}{8}[(a^2+2bc)\gamma_1^2 + (b^2+2ac)\gamma_2^2 + (c^2+2ab)\gamma_3^2],$ $+[\beta - (M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)]\nu$
	$I_4 = (b+c)M_1^2 + (c+a)M_2^2 + (a+b)M_3^2$ $+C\{[(b^2+c^2)\gamma_1 + 2a_1]M_1 + [(a^2+c^2)\gamma_2 + 2a_2]M_2 + [(a^2+b^2)\gamma_3$ $+2a_3]M_3\} + \frac{C^2}{4}[(b+c)(b-c)^2\gamma_1^2 + (c+a)(c-a)^2\gamma_2^2 + (a+b)(a-b)^2\gamma_3^2]$ $+C^2[(a+b+c)(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3) + 2(a_1 a\gamma_1 + a_2 b\gamma_2 + a_3 c\gamma_3)]$

4	Case of Steklov–Rubanovsky's type
	$V = \nu(\beta - \boldsymbol{\kappa}\cdot\boldsymbol{\gamma} + a[tr(\mathbf{I}^{-1})\vert\boldsymbol{\gamma}\vert^2 - \boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}]) - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma},$ $\boldsymbol{l} = \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J} + \nu\boldsymbol{\gamma}\mathbf{I},\ \mathbf{J} = [tr(\mathbf{I}^{-1})\boldsymbol{\delta} - \mathbf{I}^{-1}]$ $\boldsymbol{\mu} = \boldsymbol{\kappa} + 2a\boldsymbol{\gamma}\mathbf{I}^{-1} + \frac{\partial}{\partial\boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}) - [\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot(\nu\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma}$
	$I_3 = [\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J} + \nu\boldsymbol{\gamma}\mathbf{I}]\cdot\boldsymbol{\gamma} = \beta$
	$I_4 = \frac{1}{2}\vert\,\boldsymbol{\omega}\mathbf{I} + \nu\boldsymbol{\gamma}\mathbf{I} + \boldsymbol{\kappa}\,\vert^2 + 2a(\boldsymbol{\omega}\cdot\boldsymbol{\gamma} + \nu)$
	$H = \frac{1}{2}(\mathbf{M} - \boldsymbol{\kappa} - \boldsymbol{\gamma}\mathbf{J})\mathbf{I}^{-1}\cdot(\mathbf{M} - \boldsymbol{\kappa} - \boldsymbol{\gamma}\mathbf{J})$ $+[\beta - (M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)]\nu$
	$I_4 = \frac{1}{2}\vert\,\mathbf{M} - a\boldsymbol{\gamma}\mathbf{J}\,\vert^2 - a\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot(\mathbf{M} - \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J})$

For this case, one can show that

$$\frac{dI_4}{dt} = (I_3 - \beta)[(\boldsymbol{\omega}\mathbf{I} + \nu\boldsymbol{\gamma}\mathbf{I} + \boldsymbol{\kappa})\cdot(\frac{\partial\nu}{\partial\boldsymbol{\gamma}} \times \boldsymbol{\gamma}) + a\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot(\frac{\partial\nu}{\partial\boldsymbol{\gamma}} \times \boldsymbol{\gamma})].$$

(1) For any function ν, I_4 is an integral on the level $I_3 = \beta$ and the dynamics is conditionally integrable. On the other hand, when $\nu(\boldsymbol{\gamma}) = n$ (a constant) the terms in the square bracket vanish and this case becomes integrable for arbitrary initial conditions and coincides with the Rubanovsky–Steklov case of motion of a body in liquid.

5	Case of Kowalevski–Yehia–Sokolov type
	$A = B = 2C,$ $V = \beta\nu + C[\kappa c\gamma_1 + a_2\gamma_2 - \nu\kappa\gamma_3$ $-\frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2) - \nu c\gamma_1\gamma_3$ $-\frac{\nu^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)]$
	$\boldsymbol{l} = C(2c\gamma_3 + 2\nu\gamma_1, 2\nu\gamma_2, \kappa - c\gamma_1 + \nu\gamma_3),$ $\boldsymbol{\mu} = C\{(c\gamma_1, 0, \kappa + c\gamma_3) + \frac{\partial}{\partial\boldsymbol{\gamma}}[(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)\nu]$ $-[5\nu + 2\gamma_1\frac{\partial\nu}{\partial\gamma_1} + 2\gamma_2\frac{\partial\nu}{\partial\gamma_2} + \gamma_3\frac{\partial\nu}{\partial\gamma_3}]\boldsymbol{\gamma}\},$ $I_3 = C[2p\gamma_1 + 2q\gamma_2 + (r+\kappa)\gamma_3 + c\gamma_1\gamma_3 + (2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)\nu] = \beta$
	$I_4 = \left[(p+\nu\gamma_1)^2 - (q+\nu\gamma_2)^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r+\nu\gamma_3-\kappa)\right]^2$ $+\left[2(p+\nu\gamma_1)(q+\nu\gamma_2) - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r+\nu\gamma_3-\kappa)\right]^2$ $+2\kappa(r+\nu\gamma_3-\kappa+c\gamma_1)\left[(p+\nu\gamma_1)^2 + (q+\nu\gamma_2)^2 + 2c(p+\nu\gamma_1)\gamma_3\right]$ $-4a_2\kappa(q+\nu\gamma_2)\gamma_3$ $-2\kappa c^2\{2\gamma_3[2(p+\nu\gamma_1)\gamma_1 + c\gamma_1\gamma_3 + 2(q+\nu\gamma_2)\gamma_2 + (r+\nu\gamma_3)\gamma_3]$ $+\kappa\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r+\nu\gamma_3+c\gamma_1)\}$

	$I_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 = \beta,$ $H = \frac{1}{2C}(\frac{M_1^2+M_2^2}{2} + M_3^2) - (\kappa + c\gamma_1)M_3 + C(a_2\gamma_2 + 2c\kappa\gamma_1 - c^2\gamma_3^2)$ $+\nu[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]$
	$I_4 = [\frac{M_1^2-M_2^2}{4C^2} + a_2\gamma_2 + c(\frac{M_3}{C} - 2\kappa)\gamma_1 - c^2(\gamma_1^2 - \gamma_2^2)]^2$ $+[\frac{M_1M_2}{2C^2} - a_2\gamma_1 + c(\frac{M_3}{C} - 2\kappa)\gamma_2 - 2c^2\gamma_1\gamma_2]^2$ $+\kappa(\frac{M_3}{C} - 2\kappa)[\frac{M_1^2+M_2^2}{2C^2} + 2c\gamma_3\frac{M_3}{C}] - 2\kappa\gamma_3(a_2 + 2c^2\gamma_2)\frac{M_2}{C}$ $-\frac{2\kappa c^2}{C}[2\gamma_1\gamma_3M_1 - (\gamma_1^2 + \gamma_2^2)M_3]$

6	Yehia [398], Yehia–Bedwehy [419]: $\nu = 0$, Kowalevski [238]: $\nu = a = 0$
	$A = B = 2C$
	$V = \beta\nu + C[a_1\gamma_1 + a_2\gamma_2 + \frac{a}{\sqrt{1-\gamma_3^2}} - \frac{1}{2}\nu^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)]$
	$\boldsymbol{l} = C\nu(2\gamma_1, 2\gamma_2, \gamma_3),$ $\boldsymbol{\mu} = C\{\frac{\partial}{\partial\gamma}[\nu(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2] - [5\nu + \frac{\partial\nu}{\partial\gamma}\cdot(2\gamma_1, 2\gamma_2, \gamma_3)]\boldsymbol{\gamma}\}$
	$I_3 = C[2p\gamma_1 + 2q\gamma_2 + r\gamma_3 + \nu(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)] = \beta$
	$I_4 = [(p+\nu\gamma_1)^2 - (q+\nu\gamma_2)^2 - a_1\gamma_1 + a_2\gamma_2]^2$ $+[2(p+\nu\gamma_1)(q+\nu\gamma_2) - a_1\gamma_2 - a_2\gamma_1]^2$ $+2\frac{a}{\sqrt{1-\gamma_3^2}}[(p+\nu\gamma_1)^2 + (q+\nu\gamma_2)^2] + \frac{a^2}{1-\gamma_3^2}$

7	Case of Lagrange's type
	$B = A,$ $V = V_0(\gamma_3) + \beta\nu - \nu[(\gamma_1^2 + \gamma_2^2)\ell + l_3\gamma_3] - \frac{1}{2}\nu^2[A + (C - A)\gamma_3^2],$ $\boldsymbol{l} = ((\ell + A\nu)\gamma_1, (\ell + A\nu)\gamma_2, l_3 + C\nu\gamma_3),$ $\boldsymbol{\mu} = \frac{\partial}{\partial\gamma}(\boldsymbol{l}\cdot\boldsymbol{\gamma}) - (\frac{\partial}{\partial\gamma}\cdot\boldsymbol{l})\boldsymbol{\gamma},$ $V_0(\gamma_3), \ell(\gamma_3), l_3(\gamma_3), \nu(\gamma_1, \gamma_2, \gamma_3)$ arbitrary functions
	$I_3 = A(p\gamma_1 + q\gamma_2) + (Cr + l_3)\gamma_3 + (\gamma_1^2 + \gamma_2^2)\ell + \nu[A + (C - A)\gamma_3^2] = \beta$
	$I_4 = C(r + \nu\gamma_3) + l_3$
	$I_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3,$ $H = \frac{1}{2}(\frac{M_1^2+M_2^2}{A} + \frac{M_3^2}{C}) - [\frac{\ell}{A}(M_1\gamma_1 + M_2\gamma_2) + \frac{l_3}{C}M_3]$ $+V_0(\gamma_3) + \frac{1}{2A}(\gamma_1^2 + \gamma_2^2)\ell^2 + \frac{l_3^2}{2C} + \nu[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]$
	$I_4 = M_3$

Note that the precession angle ψ is cyclic and the corresponding generalized momentum is the integral of motion I_3. The proper rotation angle φ is no longer cyclic in general, but becomes cyclic on the level $I_3 = \beta$. In fact, one may calculate

$$\frac{\partial L}{\partial\varphi} = (I_3 - \beta)\frac{\partial\nu}{\partial\varphi}.$$

The cyclic integral I_4 is conditional on the level $I_3 = \beta$. One can easily find

$$\frac{dI_4}{dt} = (I_3 - \beta)(\gamma_2\frac{\partial\nu}{\partial\gamma_1} - \gamma_1\frac{\partial\nu}{\partial\gamma_2}). \tag{12.21}$$

This expression vanishes when either $I_3 = \beta$ or $\nu = \nu(\gamma_1^2 + \gamma_2^2)$, which is equivalent to $\nu = \nu(\gamma_3)$. Case 7 is integrable for arbitrary function $\nu(\gamma_1, \gamma_2, \gamma_3)$ on the level $I_3 = \beta$, but becomes unconditional when $\nu = \nu(\gamma_3)$.

12.2.2 Example of Physical Application

Consider the simple original case of Kowalevski, obtained from case 5 of Table 12.2 by setting $\kappa = c = 0$. We shall transform this case using

$$\nu = \lambda\gamma_3, \beta = C\lambda a.$$

The generalized case will be characterized by

$A = B = 2C$
$V = C[a_1\gamma_1 + \lambda a\gamma_3 - \frac{\lambda^2}{2}\gamma_3^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)]$
$\boldsymbol{l} = C\lambda\gamma_3(2\gamma_1, 2\gamma_2, \gamma_3),$ $\boldsymbol{\mu} = C\lambda(-2\gamma_1\gamma_3, -2\gamma_2\gamma_3, 2 - 3\gamma_3^2)$
$I_3 = C[2p\gamma_1 + 2q\gamma_2 + r\gamma_3 + \gamma_3(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)] = Ca$
$I_4 = [(p + \lambda\gamma_1\gamma_3)^2 - (q + \lambda\gamma_2\gamma_3)^2 - a_1\gamma_1]^2$ $+[2(p + \lambda\gamma_1\gamma_3)(q + \lambda\gamma_2\gamma_3) - a_1\gamma_2]^2$
$H = \frac{1}{4C}(M_1^2 + M_1^2 + 2M_1^2) + C(a_1\gamma_1 + \lambda a\gamma_3)$ $+\lambda\gamma_3[Ca - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)],$ $I_4 = [M_1^2 - M_2^2 - 4C^2a_1\gamma_1]^2$ $+4[M_1M_2 - 2C^2a_1\gamma_2]^2$

The potential is modified by the addition of two terms. The first $C\lambda a\gamma_3$ means a displacement of the centre of mass of the body in the z-direction (normal to the equatorial plane) to the point ($x_0 = Ca_1/Mg, 0, z_0 = \frac{C\lambda a\gamma_3}{Mg}$) and the second term is quartic in $\boldsymbol{\gamma}$. The last may be written as

$$-\frac{C\lambda^2}{2}\gamma_3^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right) = -\frac{C\lambda^2}{2}\gamma_3^2\left(2 - \gamma_3^2\right).$$

The new problem involves also the gyroscopic moments described by the vector $\boldsymbol{l}$ and $\boldsymbol{\mu}$ and it is integrable on the level $I_3 = Ca$. On setting $\lambda = 0$, one recovers the general integrable case of Kowalevski.

Note that the parameter λ does not appear in the Hamiltonian form of I_4. Note also that from the Hamiltonian function of the new problem one cannot recognize the acting potential and gyroscopic forces, which are clearly defined by the potential V and the vector $\boldsymbol{\mu}$.