Chapter 11
The General Problem of Motion of a Rigid Body Acted upon by a Coaxial Combination of Potential and Gyroscopic Forces

11.1 Introduction

In the last chapter, we have seen that the problem of motion of a body in a liquid or, more precisely, the alternative problem of motion of a body about a fixed point, while acted by magnetic, electric and Lorentz forces, lies on the top of a hierarchy of problems, each of which generalizes the one below it. In this chapter, we extend this hierarchy upwards, by allowing general axi-symmetric potential and gyroscopic forces to act on the body. The fact that problems on that level of complication were not treated in the literature in no way means that such problems have little physical significance. A natural reason is that the grave theoretical difficulties met in as simple as the classical problem gave the impression that difficulties will grow with the degree of complication of forces applied to the body. Fortunately, it turned out that certain symmetries grow with the complication, opening wide chances to achieve far-reaching results. In fact, one can go along the line of thinking that led to the precession transformation in the last chapter, but this time replacing the constant precession speed n with a function $\nu(\gamma)$. Under different circumstances, this type of transformation keeps the equations of motion of the new problem form-invariant, leading to construct new integrable/solvable cases from all known cases of the previous chapters. To this end in this chapter, we shall use two different types of transformations which can be applied to all the known integrable cases to generate from them new ones of the most complicated structure ever seen, while preserving integrability either general or restricted to a certain level of the areas integral. Some of the new cases can be given definite and non-trivial physical interpretation. In this respect a word of warning is due. As stressed in previous chapters, we are dealing with physical models, which have their obvious limitations. Both relativistic effects and the radiation from accelerated electric charges are permanently neglected.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_11

11.2 Equations of Motion

Now we assume a rigid body moving about a fixed point, while subject to conservative (time-independent) potential and gyroscopic forces of the most general form with a common axis of symmetry OZ fixed in space and passing through the fixed point O of the body. The Lagrangian has the form

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\omega + \boldsymbol{l}.\omega - V, \tag{11.1}$$

in which $V = V(\gamma_1, \gamma_2, \gamma_3), \boldsymbol{l} = \boldsymbol{l}(\gamma_1, \gamma_2, \gamma_3)$. The precession angle ψ is a cyclic coordinate. The corresponding cyclic integral is

$$\frac{\partial L}{\partial \dot{\psi}} = \frac{\partial L}{\partial \boldsymbol{\omega}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{\psi}} = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l}) \cdot \boldsymbol{\gamma} = f. \tag{11.2}$$

To write down the dynamical (Euler-like) equations of motion in the body System, we first deduce the equation corresponding to the angle φ (the proper rotation angle around the z-axis fixed in the body):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0.$$

That is

$$\frac{d}{dt}(Cr + l_3) - [(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\frac{\partial \boldsymbol{\omega}}{\partial \varphi} + \boldsymbol{\omega}\cdot\frac{\partial \boldsymbol{l}}{\partial \varphi} - \frac{\partial V}{\partial \varphi}] = 0,$$

and after expressing derivatives w.r.t. φ in terms of derivatives w.r.t. $\boldsymbol{\gamma}$, it can be written as

$$C\dot{r} + (B - A)pq + p[\frac{\partial(l \cdot \gamma)}{\partial \gamma_2} - \gamma_2\frac{\partial}{\partial \boldsymbol{\gamma}} \cdot \boldsymbol{l}] - q[\frac{\partial(\boldsymbol{l} \cdot \boldsymbol{\gamma})}{\partial \gamma_1} - \gamma_1\frac{\partial}{\partial \boldsymbol{\gamma}} \cdot \boldsymbol{l}]$$

$$-(\gamma_1\frac{\partial V}{\partial \gamma_2} - \gamma_2\frac{\partial V}{\partial \gamma_1}) = 0.$$

The last equation can be given the form

$$\mathbf{k} \cdot \{\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times[\boldsymbol{\omega}\mathbf{I} + \frac{\partial(\mathbf{l} \cdot \boldsymbol{\gamma})}{\partial \gamma} - (\frac{\partial}{\partial \gamma} \cdot \boldsymbol{l})\gamma] - \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}}\} = 0.$$

Now, we note that nothing in the curly bracket depends on the unit vector **k** figuring before that bracket. This vector can be replaced in the last equation by any of the other two unit vectors **i** and **j**. Thus, we can write the dynamical equation in the final vector form

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\mu}) = \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}}, \tag{11.3}$$

where

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{l}+(\boldsymbol{\gamma}\times\frac{\partial}{\partial \boldsymbol{\gamma}}) \times \boldsymbol{l} \\ &\equiv \frac{\boldsymbol{\partial}}{\boldsymbol{\partial\gamma}}(\boldsymbol{l}\cdot\boldsymbol{\gamma}) - (\frac{\boldsymbol{\partial}}{\boldsymbol{\partial\gamma}}\cdot\boldsymbol{l})\boldsymbol{\gamma}.\end{aligned} \tag{11.4}$$

Equation (11.3) and Poisson's equation constitute the system

$$\begin{aligned}\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\mu}) &= \boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} &= \mathbf{0},\end{aligned} \tag{11.5}$$

of six first-order equations in 6 unknowns, which generalizes the equations of motion in all problems considered in the previous chapters. We shall refer to V and $\boldsymbol{l}$ as the scalar and vector potentials, respectively, and to $\boldsymbol{\mu}$ as the gyroscopic vector.

It is easy to check that the system (11.5) satisfies Jacobi's condition for the last integrating multiplier

$$\frac{\partial\dot{\boldsymbol{\omega}}}{\partial\boldsymbol{\omega}}+\frac{\partial\dot{\boldsymbol{\gamma}}}{\partial\boldsymbol{\gamma}} \equiv 0.$$

Hence, for its integration one needs a single additional integral of motion I_4 besides the three general integrals, which we write in the form

$$\begin{aligned}I_1 &\equiv \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + V = h, \\ I_2 &= \boldsymbol{\gamma}^2 = 1, \\ I_3 &= (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\boldsymbol{\gamma} = f.\end{aligned} \tag{11.6}$$

Those are the energy integral or, more precisely, Jacobi's integral, the geometric integral and the cyclic integral corresponding to the coordinate ψ. The last can be found as

$$I_3 = \frac{\partial L}{\partial\dot{\psi}} = \frac{\partial L}{\partial\boldsymbol{\omega}}\cdot\frac{\partial\boldsymbol{\omega}}{\partial\dot{\psi}} = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\boldsymbol{\gamma}.$$

The solution of the system (11.5) determines only $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ as functions of t. This completely determines only the angles $\theta = \cos^{-1}\gamma_3$ and $\varphi = \tan^{-1}\frac{\gamma_1}{\gamma_2}$. To obtain ψ, one has to use the cyclic integral (11.2) together with formulas of Chap. 2 to express $\dot{\psi}$ in the form

$$\dot{\psi} = \frac{1}{\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}}[f - \boldsymbol{l}\cdot\boldsymbol{\gamma} - \frac{(A-B)\gamma_1\gamma_2\dot{\gamma}_3 - C(\gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2)}{1-\gamma_3^2}]. \tag{11.7}$$

The angle of precession is found by integrating this relation with respect to time, and this completes the solution.

Remark: It must be noted here that the gyroscopic vector $\boldsymbol{\mu}$, which enters the equations of motion (11.5), is unique for any physical problem, but the vector potential $\boldsymbol{l}$ is not. In fact, as was noted before in Chap. 10, a term of the type

$$-\frac{d\chi(\gamma)}{dt} = -\frac{d\chi}{d\gamma} \cdot \frac{d\gamma}{dt} = \boldsymbol{\omega} \cdot (\gamma \times \frac{d\chi}{d\gamma})$$

can be added to the Lagrangian without changing the equations of motion. Thus, the vector $\boldsymbol{l}$ can be determined only up to a term of the form

$$\boldsymbol{l}_0 = \gamma \times \frac{d\chi}{d\gamma}, \tag{11.8}$$

in which χ is an arbitrary function of γ.

11.3 Relation to Grioli's and Kharlamov's Equations

11.3.1 Grioli's Equations

Grioli [139] considered the system of equations of motion

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times [\boldsymbol{\omega} I + m(\boldsymbol{\omega}, \gamma)] &= \gamma \times \frac{\partial V(\gamma)}{\partial \gamma}, \\ \dot{\gamma} + \boldsymbol{\omega} \times \gamma &= \mathbf{0}, \end{aligned} \tag{11.9}$$

as a generalization of the classical problems of motion of a rigid body about a fixed point including a general potential function $V(\gamma)$ and a general gyroscopic term $\mathbf{m}(\omega, \gamma)$. He answered the question: for which $\mathbf{m}$ does this system admit an areas integral? In fact, one can use (11.9) to deduce the relation

$$\frac{d}{dt}(\boldsymbol{\omega}\mathbf{I} \cdot \gamma) + \mathbf{m} \cdot \dot{\gamma} = \mathbf{0}. \tag{11.10}$$

If $\mathbf{m}$ is expressible in the form

$$\mathbf{m} = \frac{\partial F(\gamma)}{\partial \gamma} + \Phi(\boldsymbol{\omega}, \gamma)\gamma, \tag{11.11}$$

where F and Φ are scalar functions of their arguments, then (11.9) admits the areas integral

$$\boldsymbol{\omega}\mathbf{I} \cdot \gamma + F(\gamma) = f \text{ (arbitrary constant)}. \tag{11.12}$$

Grioli did not study further the system (11.9) with $\mathbf{m}$ as in (11.11). Although such system preserves the sum of kinetic and potential energies, it cannot be presented in Lagrangian or Hamiltonian form. The powerful techniques of analytical dynamics are inapplicable to that system.

11.3.2 M. Kharlamov's Equations

M. Kharlamov considered the same question of existence of areas integral, but demanded that the system (11.9) had Lagrangian structure [173]. He was led to the same form of the gyroscopic function as (11.11), but with velocity-independent Φ, so that

$$\boldsymbol{\mu}=\frac{\partial F(\gamma)}{\partial \gamma}+\Phi(\gamma)\gamma, \tag{11.13}$$

where F, Φ are two arbitrary functions of $\boldsymbol{\gamma}$.

We now prove that gyroscopic terms in the equations of motion (11.5) can be determined in two equivalent ways, either by giving the vector $\boldsymbol{l}(\gamma)$ or the pair of scalar functions F and Φ :

(1) Let $\boldsymbol{l}(\gamma)$ be given, then

$$F=\boldsymbol{l}\cdot\gamma, \tag{11.14a}$$

$$\Phi=\frac{\partial}{\partial\gamma}\cdot\boldsymbol{l}. \tag{11.14b}$$

Note that a gauge-term vector $\boldsymbol{l}_0$ in the form (11.8) gives no contribution to any of those functions.

(2) Let $\boldsymbol{l}(\gamma)$, $\boldsymbol{l}'(\gamma)$ be two solutions of (11.14a) and (11.14b) for given F and Φ. The difference

$$\boldsymbol{\lambda}=\boldsymbol{l}'-\boldsymbol{l} \tag{11.15}$$

satisfies the equations

$$\boldsymbol{\lambda}\cdot\gamma=\mathbf{0},\frac{\partial}{\partial\gamma}\cdot\boldsymbol{\lambda}=0.$$

The general solution of the first equation is

$$\boldsymbol{\lambda}=\gamma\times\mathbf{s}(\gamma), \tag{11.16}$$

and inserting this into the second equation we get

$$\frac{\partial}{\partial\gamma}\cdot(\gamma\times\mathbf{s}(\gamma))=-\gamma\cdot(\frac{\partial}{\partial\gamma}\times\mathbf{s})=0.$$

This is a single under-determined linear partial differential equation in the three components of $\mathbf{s}$. Its solution involving two arbitrary functions χ and N is

$$\mathbf{s} = \frac{\partial \chi}{\partial \gamma} + N(\gamma)\boldsymbol{\gamma}. \tag{11.17}$$

Inserting this expression into (11.16) and using (11.15), we can write

$$\begin{aligned} \boldsymbol{l}' &= \boldsymbol{l} + \gamma \times [\frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{\gamma}} + N(\gamma)\boldsymbol{\gamma}] \\ &= \boldsymbol{l} + \gamma \times \frac{\partial \boldsymbol{\chi}}{\partial \gamma}. \end{aligned} \tag{11.18}$$

Thus, replacing $\boldsymbol{l}$ by $\boldsymbol{l}'$ in the Lagrangian (11.1) adds to L a term of the form

$$\boldsymbol{\omega} \cdot (\gamma \times \frac{\partial \boldsymbol{\chi}}{\partial \gamma}) = \frac{\partial \chi}{\partial \gamma} \cdot (\boldsymbol{\omega} \times \gamma) = -\frac{d\chi}{dt},$$

which is a nugatory term, having no contribution to the equations of motion. Kharlamov's form (11.13) for the vector $\boldsymbol{\mu}$ is equivalent to our form (11.4).

11.4 Potential of, and Torques on, a Heavy, Magnetized and Electrically Charged Body

The model of an absolutely rigid body as such is a purely mathematical model. All ordinary materials suffer deformation under stresses applied to them. Nevertheless, this model has proved practical, useful and comfortable in the study of a wide spectrum of physical and mechanical problems. In this section, we formulate the equations of motion of a rigid body about a fixed point in a much wider physical setting, taking into account classical interactions, all at a time. In addition to its mass distribution acted upon by gravitational forces, assume that the body has some magnetized parts and carries some electric charges. The body is also subject to electric and magnetic fields.

The picture to be drawn here for the rigid body and physical effects on it should not be taken as literally describing a real body with usual properties as electric insulation or conductivity, magnetic permeability or other properties that change its physical characteristics when its orientation changes under the action of external fields. Our aim here is to construct a mathematical model that would lead to tractable equations of motion of the rigid body in the presence of all the classical physical interactions. To this end, we make some necessary simplifying assumptions:

1- The main part of the body (the carrier body), which is fixed from the origin O, has neither electrical nor magnetic properties, so that it does not interfere with the interaction between the external fields and the magnets and electric charges carried by the body.

2- The physical characteristics of the rigid body are constant in it. They do not change with time, with the change of the body's orientation in space, nor with the change of internal forces in the body. Thus, the body may carry a distribution of immovable electric charges and some permanently magnetized parts, also fixed in it. Magnetization of the body can also arise due to the presence of steady electric circuits in the body. An electric motor whose axis is fixed in the body generates in its working mode a magnetic moment due to electric current in its coil, equivalent to a permanent magnet, and a constant gyrostatic moment due to the steady rotation of the coil.

3- It is well known that, according to the laws of classical physics, an accelerated electric charge emits electromagnetic radiation. This was established by Larmor [254] in 1897 (see also [161]). The total energy of motion of the body decreases with time. The maximum acceleration attained by a point of the body will be assumed small enough to justify neglecting this effect.

Under those conditions, the following effects on the body will be taken into account:

(1) A torque arises due to the gravitational field $\mathbf{g}$ of a certain distribution of gravitating sources, fixed in the inertial system of axes $OXYZ$, O being the fixed point of the body. Gravitational forces are derivable from a scalar potential $V_g(X, Y, Z)$ by the relation $\mathbf{g} = -\nabla V_g$. The gravitational potential is harmonic, i.e. satisfies Laplace's equation in the inertial coordinate system outside gravitating sources. The potential of the body, due to the gravitational field, has the form

$$V_G = \int V_g(X, Y, Z)dm,$$

where dm is the mass element at the point $\mathbf{r}(X, Y, Z)$ of the body and integration is extended on the space domain occupied by the body. Referring to the system of axes $Oxyz$ fixed in the body, we have $\mathbf{r} = (x, y, z)$ and hence the potential can be written as

$$V_G = \int V_g(\mathbf{r} \cdot \boldsymbol{\alpha}, \mathbf{r} \cdot \boldsymbol{\beta}, \mathbf{r} \cdot \boldsymbol{\gamma})dm, \tag{11.19}$$

$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ being the basic unit vectors in the inertial space.

(2) The external electric field $\mathbf{E}$, derived from the potential V_e by the relation $\mathbf{E} = -\nabla V_e$, acts on the electric charges on the body in a similar way. The electric potential of the body is

$$V_E = \int V_e(\mathbf{r} \cdot \boldsymbol{\alpha}, \mathbf{r} \cdot \boldsymbol{\beta}, \mathbf{r} \cdot \boldsymbol{\gamma})de. \tag{11.20}$$

(3) The external magnetic field $\mathbf{H} = -\nabla V_m$ acts on the magnetized parts of the body. Note that we use the magnetic field $\mathbf{H}$ rather than the magnetic induction $\mathbf{B}$,

since the body is considered as having unit permeability. Also, for simplicity, we consider the magnetized part of the body as composed of a set of short magnets (dipoles). If $\boldsymbol{m}_i$ is the magnetic dipole moment at the point $\mathbf{r}_i(X, Y, Z)$, the potential of the body due to the scalar magnetic interaction is

$$\begin{aligned} V_M &= \Sigma \boldsymbol{m}_{\mathbf{i}} \cdot \nabla V_m(\mathbf{r}_i \cdot \boldsymbol{\alpha}, \mathbf{r}_i \cdot \boldsymbol{\beta}, \mathbf{r}_i \cdot \gamma) \\ &= -\Sigma \boldsymbol{m}_{\mathbf{i}} \cdot \mathbf{H}(\mathbf{r}_i \cdot \boldsymbol{\alpha}, \mathbf{r}_i \cdot \boldsymbol{\beta}, \mathbf{r}_i \cdot \gamma). \end{aligned} \tag{11.21}$$

(4) The external magnetic field also exerts the velocity-dependent Lorentz forces on the electric charge distribution in the body. The moment of those forces about the origin is[1]

$$\mathbf{M}_H = \int \mathbf{r} \times [(de\frac{d\mathbf{r}}{dt}) \times \mathbf{H}],$$

where the velocity of the point of the body in space $\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$. We have

$$\begin{aligned} \mathbf{M}_H &= \int \mathbf{r} \times [(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{H}]de \\ &= \int (\mathbf{r} \cdot \mathbf{H})\boldsymbol{\omega} \times \mathbf{r}de \\ &= \boldsymbol{\omega} \times \int (\mathbf{r} \cdot \mathbf{H})\mathbf{r}de. \end{aligned} \tag{11.22}$$

This means that the vector $\boldsymbol{\mu}$ in the equations of motion (11.3) may be written in the form

$$\boldsymbol{\mu} = \boldsymbol{\kappa} - \int (\mathbf{r} \cdot \mathbf{H})\mathbf{r}de. \tag{11.23}$$

For certain purposes, e.g. to construct a Lagrangian for the problem of motion, the magnetic field can also be derived from a vector potential $\mathcal{A}$, which is also assumed time-independent, according to the formula $\mathbf{H} = \nabla \times \mathcal{A}$. The vector potential $\boldsymbol{l}$ of the body may be written as

$$\boldsymbol{l} = \boldsymbol{\kappa} + \int \mathbf{r} \times \mathcal{A}de, \tag{11.24}$$

while $\boldsymbol{\mu}$ can be derived from $\boldsymbol{l}$ according to (11.4).

For the purpose of giving a concrete example, let us consider the following physical situation.

Let the principal body of a gyrostat be carrying a permanent distribution of electric charges and the system be subject to

(1) A uniform magnetic field $\mathbf{H}$ in the Z-direction, i.e. $\mathbf{H} = H\boldsymbol{\gamma}$.

[1] Here MKS units are used. In Gaussian units de should be divided by the velocity of light c (e.g. [44]).

(2) An electric field whose potential is $a_1 Z + \frac{1}{2}a_2 Z^2$.
(3) A gravitational field with another quadratic potential $b_1 Z + \frac{1}{2}b_2 Z^2$.

It should be noted that those forms of the electric and gravitational potentials appear as a second approximation of the potentials of a general rigid body (or gyrostat) in arbitrary coaxially symmetric electric and gravitational fields, by including the second harmonics. The same applies for the case of fields due to a distant axi-symmetric and symmetrically situated invariable body.

According to (11.23), we write

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{\kappa} - H\int(\mathbf{r}\cdot\gamma)\mathbf{r}de \\ &= \boldsymbol{\kappa} - 2H\gamma\bar{\mathbf{I}}_e,\end{aligned} \tag{11.25}$$

where $\bar{\boldsymbol{I}}_e = \frac{1}{2}(tr\mathbf{I}_e)\delta - \mathbf{I}_e$, $\mathbf{I}_e$ is the inertia matrix of the distributions, and δ is the unit matrix. The corresponding vector potential is

$$\boldsymbol{l} = \boldsymbol{\kappa} + \frac{1}{2}H\gamma\mathbf{I}_e. \tag{11.26}$$

On the other hand, the total potential of the system is (ignoring an insignificant constant)

$$\begin{aligned}V &= \int[a_1\mathbf{r}.\gamma + \frac{1}{2}a_2(\mathbf{r}.\gamma)^2]de \\ &\quad + \int[b_1\mathbf{r}.\gamma + \frac{1}{2}b_2(\mathbf{r}.\gamma)^2]dM \\ &= \mathbf{a}.\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma\end{aligned} \tag{11.27}$$

where $\mathbf{J} = -a_2\mathbf{I}_e - b_2\mathbf{I}$, $\mathbf{a} = a_1\int\mathbf{r}de + b_1 M\mathbf{r}_c$, M the mass of the system and $\mathbf{r}_c$ its centre of mass. As seen in Chap. 10, formulas (11.25)–(11.27) characterize the problem of motion of a body in a liquid.

The effect of Lorentz forces on the motion of a rigid body was considered only in very few works (e.g. [22, 139, 140, 378, 382]). In a number of more recent works, similar problems were considered, repeating to a great extent previous results as introducing the inertia matrix of electric charges, e.g. [430–433].

Expressions analogous to the above ones can be derived for more complicated forms of the magnetic field. In the case when the scalar potential of the external magnetic field can be expressed as a second-degree harmonic polynomial,

$$V_M = a_1 Z + a_2(3Z^2 - r^2). \tag{11.28}$$

The vector $\boldsymbol{\mu}$ can be expressed as

$$\boldsymbol{\mu} = \int [a_1 \mathbf{r} \cdot \boldsymbol{\gamma} + 2a_2(3(\mathbf{r} \cdot \boldsymbol{\gamma})^2 - r^2)]\mathbf{r} de. \tag{11.29}$$

In expanded form, one may write

$$\begin{aligned}
\mu_1 &= -2a_2(I_{xxx} + I_{xyy} + I_{xzz}) + a_1(I_{xx}\gamma_1 + I_{xy}\gamma_2 + I_{xz}\gamma_3) \\
&\quad +6a_2(I_{xxx}\gamma_1^2 + I_{xyy}\gamma_2^2 + I_{xzz}\gamma_3^2 + 2I_{xxy}\gamma_1\gamma_2 + 2I_{xxz}\gamma_1\gamma_3 + 2I_{xyz}\gamma_2\gamma_3) \\
\mu_2 &= -2a_2(I_{xxy} + I_{yyy} + I_{yzz}) + a_1(I_{xy}\gamma_1 + I_{yy}\gamma_2 + I_{yz}\gamma_3) \\
&\quad +6a_2(I_{xxy}\gamma_1^2 + I_{yyy}\gamma_2^2 + I_{yzz}\gamma_3^2 + 2I_{xyy}\gamma_1\gamma_2 + 2I_{xyz}\gamma_1\gamma_3 + 2I_{yyz}\gamma_2\gamma_3) \\
\mu_3 &= -2a_2(I_{xxz} + I_{yyz} + I_{zzz}) + a_1(I_{xz}\gamma_1 + I_{yz}\gamma_2 + I_{zz}\gamma_3) \\
&\quad +6a_2(I_{xxz}\gamma_1^2 + I_{yyz}\gamma_2^2 + I_{zzz}\gamma_3^2 + 2I_{xyz}\gamma_1\gamma_2 + 2I_{xzz}\gamma_1\gamma_3 + 2I_{yzz}\gamma_2\gamma_3)
\end{aligned} \tag{11.30}$$

where, for example, $I_{xx} = \int x^2 de$, $I_{xyz} = \int xyzde$ and so forth are the second- and third-degree moments of the charge distribution.

11.5 On General and Conditional Integrable Cases in Rigid Body Dynamics

As explained in previous chapters of this book, we call a problem *general integrable* if I_4 exists for arbitrary initial conditions and *conditional integrable* if it admits a fourth integral I_4 only on a single level f of the cyclic integral I_3 (in many cases $f = 0$) but for all initial conditions compatible with that level. In both types of integrable problems, the solution can be reduced to quadratures through the application of Liouville's theorem or Jacobi's theorem to the reduced two-dimensional Hamiltonian system. It is thus sufficient to point out the fourth integral to ensure integrability in those cases. In some cases, it becomes possible to construct a quantity constant only under other restrictions on the initial state of motion, which do not fit as conditions on the integral level of I_3. Then one cannot apply Liouville's theorem to construct the solution and a procedure for accomplishing this task should be indicated separately. In such cases, we deal with *particular solutions* of the problem.

Equations of motion of the form (11.5) cover a wide range of applications in rigid body dynamics. Special cases are the classical problem of motion of a heavy body, its generalizations to the case of a gyrostat moving under potential and Lorentz forces. We recall that they cover also the Routhian reduction of the problem of motion of a body in a liquid, in which the body has no fixed point. In many cases, Eq. (11.5) with reasonably behaving functions V can be interpreted as characterizing gravitational, electric and magnetic interactions and $\boldsymbol{\mu}$ as the Lorentz force exerted by the magnetic field on some electric charges resting on the body. However, this is not always the case. In some problems that happen to be integrable, such interpretation is not possible, due to the presence of singularities that cannot be exhibited by the potentials of real bodies. Detailed examples of integrable problems of both types will be considered in the next two chapters, Chap. 12 and Chap. 13.

11.6 Transformation of the Equations of Motion

In the preceding chapter, we have applied the transformation $\boldsymbol{\omega} = \boldsymbol{\omega}' + \nu\boldsymbol{\gamma}$, where ν is a constant, to a system of the type (11.5) and its form-invariance is used to generate integrable cases containing ν as a parameter. Here we shall develop this idea, by replacing the constant ν by a function $\nu(\boldsymbol{\gamma})$. In fact, the substitution

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \nu\boldsymbol{\gamma}, \quad \nu = \nu(\gamma_1, \gamma_2, \gamma_3) \tag{11.31}$$

leaves the invariant form of the Poisson equation in (11.5), transforming it to

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0}, \tag{11.32}$$

while the areas integral in (11.6) takes the form

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l} + \nu\boldsymbol{\gamma}\mathbf{I}) \cdot \boldsymbol{\gamma} = f. \tag{11.33}$$

Substituting in the Eulerian part of the equations of motion, using (11.32) and rearranging terms, we get

$$\begin{aligned}&\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\mu} + 2\nu\boldsymbol{\gamma}\mathbf{I} - \nu(tr\mathbf{I})\boldsymbol{\gamma} + \boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma}\frac{\partial \nu}{\partial \boldsymbol{\gamma}} - (\boldsymbol{\gamma}\mathbf{I} \cdot \frac{\partial \nu}{\partial \boldsymbol{\gamma}})\boldsymbol{\gamma}) \\ &= \boldsymbol{\gamma} \times [\frac{\partial V}{\partial \boldsymbol{\gamma}} - \nu\,\boldsymbol{\mu} - \nu^2\boldsymbol{\gamma}\mathbf{I} + (\boldsymbol{\omega}'\mathbf{I} \cdot \boldsymbol{\gamma})\frac{\partial \nu}{\partial \boldsymbol{\gamma}}].\end{aligned} \tag{11.34}$$

On the level $I_3 = f$ (say), we substitute $\boldsymbol{\omega}'\mathbf{I} \cdot \boldsymbol{\gamma}$ from (11.33) and after some manipulations write the equations of motion in the final form:

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\mu}') &= \boldsymbol{\gamma} \times \frac{\partial V'}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} &= \mathbf{0},\end{aligned} \tag{11.35}$$

where

$$\begin{aligned}\boldsymbol{\mu}' &= \boldsymbol{\mu} + \frac{\partial}{\partial \boldsymbol{\gamma}}(\nu\mathbf{I} \cdot \boldsymbol{\gamma}) - [\frac{\partial}{\partial \boldsymbol{\gamma}} \cdot (\nu\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma}, \\ &\equiv \boldsymbol{\mu} - 2\nu\boldsymbol{\gamma}\bar{\boldsymbol{I}} + \boldsymbol{\gamma}\mathbf{I} \times (\frac{\partial \nu}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma})\end{aligned}$$

$$V' = V + \nu(f - \boldsymbol{l}\cdot\gamma) - \frac{1}{2}\nu^2\gamma\mathbf{I}\cdot\gamma, \tag{11.36}$$

and $\bar{\boldsymbol{I}}=\frac{1}{2}tr(\mathbf{I})\boldsymbol{\delta}-\mathbf{I}$. From the first of Eq. (11.36) and comparing with (11.4), we can also write the transformation law for the vector $\boldsymbol{l}$ as

$$\boldsymbol{l}' = \boldsymbol{l}+\nu\gamma\mathbf{I}. \tag{11.37}$$

Thus, the transformation (11.31) preserves the form of the equations of motion on a fixed level of I_3, changing only V, $\boldsymbol{\mu}$ (or $\boldsymbol{l}$) to V', $\boldsymbol{\mu}'$ (or $\boldsymbol{l}'$). The value f of I_3 enters in the potential V' as a parameter. The solution of the transformed equations of motion (11.35) can be obtained from that of (11.5) through the substitution (11.31).

The system of Eq. (11.35) admits the linear integral

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l}')\cdot\,\gamma = f,$$

equivalent to (11.33), and also the energy (Jacobi's) integral

$$\frac{1}{2}\boldsymbol{\omega}'\mathbf{I}\cdot\boldsymbol{\omega}' + V' = h.$$

On the one hand, the transformed system (11.36) can be viewed as the equations of motion of the original system as in (11.5), as seen by an observer resting in the reference frame moving with the position-dependent angular velocity $\nu(\gamma_1, \gamma_2, \gamma_3)$. The new terms that appeared in the transformed system are the inertial forces due to the rotation of the frame.

On the other hand, there is a different and more constructive way of looking at (11.36). We shall make use of the situation that the transformation preserves the form of the equations of motion to understand the transformed equations on their own as describing the motion of a second body in the inertial frame under the forces determined by V', $\boldsymbol{\mu}'$. In other words, we consider the system (11.36) as formally generalizing (11.5) to which it reduces when $\nu = 0$. However, this will not prevent us from relating the solutions of the two systems by the (formal) transformation (11.31). This duality in interpretation is the key to understanding the present method. From now on, we will mostly regard the system (11.36) as a generalization of (11.5) rather than a transformed form of it.

Remark 17 A curious note may be in place here. In certain cases, it is possible from Eq. (11.36) to choose the function ν so that V' vanishes. This means that in those cases, when the resulting $\nu(\gamma)$ is a real function, the original forces with potential V can be replaced by purely gyroscopic forces in a properly chosen rotating coordinate frame. However, we shall not follow this line, since it seemingly has no practical consequences.

11.7 Maximal Reduction of the Order of the Equations of Motion

The method used in Chap. 9 Sect. 9.2 and an exercise of Chap. 10 can be used here in the most general case of potential and gyroscopic forces to obtain a second-order orbital equation connecting two of the geometric variables γ_i. The Lagrangian of the problem of motion will be taken in the form (11.1), namely

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \boldsymbol{l}\cdot\boldsymbol{\omega} - V, \tag{11.38}$$

in the redundant configurational variables $\psi, \gamma_1, \gamma_2, \gamma_3$, subject to the holonomic condition

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{11.39}$$

The angular velocity may be written as

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{\gamma} + \mathbf{N}, \tag{11.40}$$

where

$$\begin{aligned}\mathbf{N} &= \dot{\theta}\mathbf{n} + \dot{\varphi}\mathbf{k}\\ &= -\frac{\dot{\gamma}_3}{\sqrt{1-\gamma_3^2}}(\cos\varphi, -\sin\varphi, 0) + \frac{\gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2}{\gamma_1^2+\gamma_2^2}\mathbf{k}\\ &= \frac{(-\gamma_2\dot{\gamma}_3, \gamma_1\dot{\gamma}_3, \gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2)}{1-\gamma_3^2}.\end{aligned} \tag{11.41}$$

As a result of cyclicity of the Lagrangian in the variable ψ, we have the integral

$$\frac{\partial L}{\partial\dot{\psi}} = \frac{\partial L}{\partial\boldsymbol{\omega}}\cdot\frac{\partial\boldsymbol{\omega}}{\partial\dot{\psi}} = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\boldsymbol{\gamma} = f. \tag{11.42}$$

Multiplying (11.40) scalarly by $\boldsymbol{\gamma}\mathbf{I}$ and using (11.42), we obtain

$$\dot{\psi} = \frac{1}{D}(f - \boldsymbol{l}\cdot\boldsymbol{\gamma} - \boldsymbol{\gamma}\mathbf{I}\cdot\mathbf{N}),\ D = \boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}. \tag{11.43}$$

Then, ignoring ψ we construct the Routhian

$$R = \frac{ABC}{2D}\dot{\boldsymbol{\gamma}}\mathbf{I}^{-1}\cdot\dot{\boldsymbol{\gamma}} + \boldsymbol{l}^*\cdot\dot{\boldsymbol{\gamma}} - V^*, \tag{11.44}$$

where

$$V^* = V(\gamma) + \frac{1}{2D}(f - \boldsymbol{l}\cdot\boldsymbol{\gamma})^2,$$
$$\boldsymbol{l}^* = \frac{1}{D}[\boldsymbol{\gamma}\mathbf{I}\times\boldsymbol{l} + f\frac{\partial}{\partial\dot{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}\mathbf{I}\cdot\mathbf{N})]. \quad (11.45)$$

Just as in the preceding chapters, applying Maupertuis' principle to (11.44) and eliminating γ_2, we arrive at the following second-order differential equation in $\gamma_3(\gamma_1)$, to which the equations of motion of the problem are reduced on the integral level $\{I_1 = h, I_2 = 1, I_3 = f\}$[2] [384]:

$$\begin{aligned}
&D(1-\gamma_1^2-\gamma_3^2)\gamma_3'' + C\gamma_3(1-\gamma_3^2)\\
&-\gamma_1[A-(A+2C)\gamma_3^2]\gamma_3' + \gamma_3[C-(C+2A)\gamma_1^2]\gamma_3'^2\\
&-A\gamma_1(1-\gamma_1^2)\gamma_3'^3\\
&-\frac{\rho}{ABCD}\{C\gamma_3[(A-B)(A+B-C)\gamma_1^2 + B(B-C)(1-\gamma_3^2)]\\
&\qquad +A\gamma_1[(B-C)(B+C-A)\gamma_3^2 + B(A-B)(1-\gamma_1^2)]\gamma_3'\}\\
&+\frac{\rho}{2ABC(h-V^*)}[\frac{\partial V^*}{\partial\gamma_3}(\lambda+\mu\gamma_3') - \frac{\partial V^*}{\partial\gamma_1}(\mu+\nu\gamma_3')]\\
&+\frac{\rho^{3/2}}{ABC\sqrt{aD^3(h-V^*)}}\\
&\times\{f[(A-B)(A+B-C)\gamma_1^2 - B(A-B+C) + (C-B)(B+C-A)\gamma_3^2]\\
&\quad +\Lambda\}\\
=\ &0,
\end{aligned} \quad (11.46)$$

where

$$\begin{aligned}
\rho &= \lambda + 2\mu\gamma_3' + \nu\gamma_3'^2,\\
\lambda_1 &= C[B(1-\gamma_3^2) + (A-B)\gamma_1^2],\\
\lambda_2 &= AC\gamma_1\gamma_3,\\
\lambda_3 &= A[B(1-\gamma_1^2) + (C-B)\gamma_3^2],
\end{aligned} \quad (11.47)$$

and

$$\begin{aligned}
V^* &= V(\gamma) + \frac{1}{2D}(f - \boldsymbol{l}\cdot\boldsymbol{\gamma})^2,\\
\Lambda &= D^2\boldsymbol{\gamma}\cdot[\frac{\partial}{\partial\boldsymbol{\gamma}}\times(\frac{\boldsymbol{l}\times\boldsymbol{\gamma}\mathbf{I}}{D})]\\
&\equiv D^2\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot[\frac{1}{D}\gamma\times(\boldsymbol{\gamma}\mathbf{I}s\times\boldsymbol{l})].
\end{aligned} \quad (11.48)$$

[2] The positive sign of the square root in (11.46) corresponds the choice of positive sign of the root in (11.49). If this choice is reversed, Eq. (11.46) is not changed, provided the signs of f and $\boldsymbol{l}$ are reversed. This is a consequence of the invariance of the system (11.5) with respect to the replacement $t, \boldsymbol{\omega}, \boldsymbol{\mu} \to -t, -\boldsymbol{\omega}, -\boldsymbol{\mu}$.

As should be expected, one can verify that a gauge term $\boldsymbol{l}_0$ (11.8) does not contribute to the two functions V^* and Λ.

Now, having a solution $\gamma_3 = \gamma_3(\gamma_1)$ of the orbital Eq. (11.46), one can obtain the dependence of γ_1 on time by inverting the integral

$$t = \int \sqrt{\frac{\lambda_1 + 2\lambda_2\gamma_3' + \lambda_3\gamma_3'^2}{2D(h - V^*)(1 - \gamma_1^2 - \gamma_3^2)}} d\gamma_1, \tag{11.49}$$

and substituting in γ_3, the last is determined in terms of time and then γ_2 is found from the geometric integral. This completes determination of $\boldsymbol{\gamma}$ and hence the two Eulerian angles θ and φ as functions of t. Thus, we have shown the equivalence of the reduced Eq. (11.46) to the equations of motion (11.5) on the integral level $\{h, f\}$, provided γ_3' and γ_3'' are well defined, i.e. excluded are only trajectories along which γ_1 takes a constant value.

It should be noticed here that the three functions V^*, $\boldsymbol{l}^*$ and Λ, which occur in (11.44) and (11.46), are all invariant with respect to the transformation (11.31). This can be easily verified by replacing the pair $(V, \boldsymbol{l})$ in them by the pair $(V', \boldsymbol{l}')$. This means that, on the integral level $\{h, f\}$, the Routhian (11.44), the orbital Eq. (11.46) and the expressions of $\boldsymbol{\gamma}$, θ and φ do not change by the variable rotation transformation (11.31). We shall use this property later in several situations.

To completely determine the position of the body in space, one has to find an expression for the precession angle ψ by integrating (11.43), which involves the vector potential $\boldsymbol{l}$. Using (11.40), one can express the angular velocity $\boldsymbol{\omega}$ in the form

$$\boldsymbol{\omega} = \frac{1}{D}[\dot{\boldsymbol{\gamma}}\times\boldsymbol{\gamma}\mathbf{I} + (f - \boldsymbol{l}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}]. \tag{11.50}$$

Not only all the Euler–Poisson variables are thus determined as functions of time, but also the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$. For the transformed system (11.35), regarding (11.37), this process gives

$$\begin{aligned} \dot{\psi}' &= \frac{1}{D}(f - \boldsymbol{l}' \cdot \boldsymbol{\gamma} - \boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{N}) \\ &= \dot{\psi} - \nu, \\ \boldsymbol{\omega}' &= \boldsymbol{\omega} - \nu\boldsymbol{\gamma}, \end{aligned} \tag{11.51}$$

which coincides with (11.31).

11.7.1 The Case of Complete Dynamical Symmetry

For the purpose of future use, we now write down in expanded form the Routhian (11.44) in the special case when the inertia ellipsoid of the body at the fixed point becomes a sphere. Then, from (11.44) we have

$$R = \frac{1}{2}A(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2)$$
$$-[l_1(\gamma_2\dot{\gamma}_3 - \gamma_3\dot{\gamma}_2) + l_2(\gamma_3\dot{\gamma}_1 - \gamma_1\dot{\gamma}_3) + (l_3 + \frac{f\gamma_3}{\gamma_1^2 + \gamma_2^2})(\gamma_1\dot{\gamma}_2 - \gamma_2\dot{\gamma}_1)]$$
$$-[V(\boldsymbol{\gamma}) + \frac{1}{2A}[f - l_1\gamma_1 - l_2\gamma_2 - l_3\gamma_3]^2]. \tag{11.52}$$

11.8 Extensions of Integrable Problems

As a direct application of the transformed equations, we can readily deduce the following theorems which construct integrable extensions of the known integrable problems and connect the solutions of the generalized systems to those of the original problems.

Theorem 11.1 *Let the system (11.5) with certain $V(\gamma)$ and $\mu(\gamma)$ corresponding to vector potential $l(\gamma)$, be general integrable, for arbitrary initial conditions, with the complementary integral $I_4 = F(\omega, \gamma)$. Then, upon replacing V, μ by*

$$V' = V + \nu(b - \boldsymbol{l} \cdot \boldsymbol{\gamma}) - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma},$$
$$\boldsymbol{\mu}' = \boldsymbol{\mu} + \frac{\partial}{\partial\boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}) - [\frac{\partial}{\partial\boldsymbol{\gamma}} \cdot (\nu\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma} \tag{11.53}$$

where $\nu = \nu(\gamma)$ is an arbitrary function and b a new parameter, the new system is integrable on the level

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l} + \nu\boldsymbol{\gamma}\mathbf{I}) \cdot \boldsymbol{\gamma} = b. \tag{11.54}$$

This theorem allows one to generate from an unconditional case (integrable for arbitrary initial conditions) a conditional case integrable on a single level of the areas integral I_3, but with additional potential and gyroscopic forces involving an arbitrary function $\nu(\boldsymbol{\gamma})$ and an arbitrary parameter b more than the original integrable problem. To illustrate the feasibility of the generalized problem, one can calculate for it the reduced potential. One gets

$$V^* = V + (b - f)\nu + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2. \tag{11.55}$$

The extra-parameter b enters in Eq. (11.35), in the equations of motion derived from the Routhian (11.44) as well as in the orbital Eq. (11.46). The extended problem may not be integrable for arbitrary initial conditions. However, on the single level $f = b$, the reduced potential reduces to that of the original problem. The extended problem involves one more physical parameter b and under the dynamical condition $f = b$, it becomes integrable and its solution has the same number of parameters as in the solution of the original problem.

A quick example can be readily given by the simplest extension of Euler's case of the motion of a body under no torques. Let us take $V = 0, \boldsymbol{l} = 0$ and choose $\nu = n + n_1\gamma_1$, so that

$$V' = b\nu - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I}\cdot\gamma, \boldsymbol{l}' = \nu\boldsymbol{\gamma}\mathbf{I}. \tag{11.56}$$

A family of solutions of the transformed problem can be written down generalizing formulas (10.111), (10.112) of Chap. 10 by replacing n by ν in that solution. The resulting solution is valid on the level $f = b$.

Theorem 11.2 *Let the system (11.5) with certain* $V(\gamma)$ *and* $\boldsymbol{\mu}(\gamma)$ *corresponding to vector potential* $\boldsymbol{l}(\gamma)$*, be general integrable (for arbitrary initial conditions). Let also* V *have the structure*

$$V = V_0 + b_1V_1 + ... + b_kV_k, \tag{11.57}$$

where $V_i, i = 0...k$ *and* $\boldsymbol{l}$ *are functions of* $\boldsymbol{\gamma}$ *not involving any of the parameters* $b_1, ..., b_k$ *and the complementary integral be*

$$I_4 = F(\boldsymbol{\omega}, \gamma; b_1, ..., b_k). \tag{11.58}$$

Then, upon replacing $V, \boldsymbol{\mu}(\boldsymbol{l})$ *by*

$$\begin{aligned} V' &= V_0 + b_1V_1 + ... + b_kV_k - \nu\mathbf{l}\cdot\gamma - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I}\cdot\gamma,\\ \boldsymbol{\mu}' &= \boldsymbol{\mu} + \frac{\partial}{\partial\boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\mathbf{I}\cdot\gamma) - [\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot(\boldsymbol{\nu}\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma},\\ (\boldsymbol{l}' &= \boldsymbol{l} + \nu\boldsymbol{\gamma}\mathbf{I}), \end{aligned} \tag{11.59}$$

where $\nu = n_1V_1 + ... + n_kV_k$ *and* n_i *are new constants, the new system is unconditionally integrable with the areas integral*

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l}')\cdot\gamma = f, \tag{11.60}$$

and for it the complementary integral is

$$I_4 = F(\boldsymbol{\omega}' + \nu\boldsymbol{\gamma}, \boldsymbol{\gamma}; b_1 - n_1I_3, ..., b_k - n_kI_3). \tag{11.61}$$

In fact, comparing the reduced potentials for the original problem characterized by the pair $\{V, \boldsymbol{l}\}$ with that of the extended problem characterized by the pair $\{V', \boldsymbol{l}'\}$ in (11.59), we find

$$\boldsymbol{l}'^* = \boldsymbol{l}^*,$$

$$
\begin{aligned}
V^* &= V_0 + b_1 V_1 + \dots + b_k V_k + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2, \\
V^{'*} &= V_0 + b_1 V_1 + \dots + b_k V_k - f(n_1 V_1 + \dots + n_k V_k) + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2 \\
&= V_0 + (b_1 - f n_1) V_1 + \dots + (b_k - f n_k) V_k + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2. \qquad (11.62)
\end{aligned}
$$

The potentials V^*, V'^* in (11.62) are identical in form. The only difference is that each b_i is replaced by $b'_i = b_i - f n_i, i = 1, ..., k,$ and hence follows integrability and the form of the integral (11.61). The set of solutions of the extended problem is the same as that of the original problem. Notable here is the coupling between the constants which characterize the physical problem, and hence appear in the equations of motion, and a dynamical constant of motion I_3,which appears in the process of integrating those equations. In fact, the phase portrait and phase trajectories of the new integrable problems are different from their original counterparts provided $f \neq 0$.

Theorem 11.2 generates from an unconditional case integrable for arbitrary initial conditions another unconditional case also integrable for arbitrary initial conditions. The new system involves $k + 1$ parameters $n_0, n_1, ..., n_k$ more than the old one and renders to it when one puts $n_0 = n_1 = ... = n_k = 0$. According to the problem setting, the new parameters invoke additional forces in the equations of motion, which can be given concrete physical interpretation.

The presence of I_3 in the expression for I_4 in the transformed problem may lead in certain cases to notable changes in the structure of the integral. For example, we shall see below a case in which the degree of the quadratic I_4 is raised to 3 because of the appearance of I_3 in the coefficients of the quadratic terms.

Theorem 11.3 If $\{\omega = \Omega(t, \omega^\circ, \gamma^\circ), \gamma = \Gamma(t, \omega^\circ, \gamma^\circ)\}$, is the general solution of the first system satisfying the arbitrary initial conditions $\{\omega = \omega^\circ, \quad \gamma = \gamma^\circ\}$, then for arbitrary $\nu(\gamma)$ the solution of the second system, satisfying the initial conditions $\{\omega' = \omega'^\circ, \quad \gamma = \Gamma^\circ\}$, is

$$
\begin{aligned}
\{\boldsymbol{\omega}' = \boldsymbol{\Omega}&(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ) \\
&- \nu(\boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ))\boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ), \\
\gamma = \boldsymbol{\Gamma}&(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\boldsymbol{\gamma}^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ)\}. \qquad (11.63)
\end{aligned}
$$

Theorem 11.4 If the first system admits any particular solution $\{\omega = \Omega(t), \quad \gamma = \Gamma(t)\}$, then for arbitrary $\nu(\gamma)$ the second system admits the solution $\{\omega' = \Omega(t) - \nu(\Gamma(t))\Gamma(t), \quad \gamma = \Gamma(t)\}$.

The last theorem follows from the fact that the solution of the second system for the Poisson variables $\boldsymbol{\gamma}$ is not affected by the function $\nu(\boldsymbol{\gamma})$.

In the following chapter, we discuss the consequences of the above theorems in application to known solvable problems of rigid body dynamics. Theorem 11.1 ensures the integrability of the problem (11.35) on the level f of the cyclic integral and for arbitrary $\nu(\boldsymbol{\gamma})$ whenever the corresponding problem (11.5) is integrable, either for arbitrary initial conditions or only on a fixed level of the cyclic integral. Theorem 11.2 relates the explicit solutions of the two problems. Theorem 11.3 enables the generalization, by means of including the function ν, of particular solutions of (11.5), i.e. solutions not involving any arbitrary constants or involving a number of constants of motion less than needed to guarantee integrability.

11.9 Transformations of Cyclic Variables

In Sects. 11.6 and 11.8, we have introduced the variable precession transformations that leave the invariant form of the Euler–Poisson equations of motion. We were also able to use this transformation, specially designed for rigid body dynamics under the influence of axi-symmetric forces, to construct integrable extensions of known cases. It turns out that the same transformation can be attained in a completely different way, applicable to any system whose structure involves cyclic coordinates. The basic idea is that for such system to be integrable, all that matters is the structure of its Routhian equations of motion after ignoring the cyclic coordinates. We use a simple observation that several Lagrangian mechanical systems that have different Lagrangian and Routhian functions can be reduced to one and the same set of Routhian equations in the palpable part of the generalized coordinates. Clearly, this will be the case if the Routhians of those systems differ only by constant terms that may depend only on the cyclic constants, but not on any of the palpable coordinates or velocities.

Consider the mechanical system of $n + k$ degrees of freedom, of which k degrees are cyclic, characterized by the time-independent Lagrangian

$$L = L(q_1 ..., q_n, \dot{q}_1 ..., \dot{q}_n, \dot{Q}_1, ..., \dot{Q}_k). \tag{11.64}$$

The system admits the cyclic integrals

$$\frac{\partial L}{\partial \dot{Q}_i} = f_i, \qquad i = 1, ..., k. \tag{11.65}$$

Let us consider another system with the Lagrangian

$$L' = L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, \dot{Q}'_1 + \nu_1, ..., \dot{Q}'_k + \nu_k) - \sum_{i=1}^{k} \beta_i \nu_i (q_1 ..., q_n), \tag{11.66}$$

where β_i are certain constants and ν_i are certain functions of the palpable coordinates $q_1 ..., q_n$. We notice that the system (11.66) is time-independent with the cyclic variables $Q'_1, ..., Q'_k$. This system can be considered as a transformation of (11.64) through the linear time-independent transformation of the cyclic variable rates

$$\dot{Q}_i = \dot{Q}'_i + \nu_i(q_1, ..., q_n). \tag{11.67}$$

Consider the motion of the system (11.66) on the same level of the cyclic integrals as in (11.65), i.e.

$$\frac{\partial L'}{\partial \dot{Q}'_i} = f_i, \qquad i = 1, ..., k. \tag{11.68}$$

This is the transformed form of (11.65) according to (11.67).

Now, let R and R' be the Routhians of the two systems, then their difference

$$\begin{aligned} R' - R &= L - \sum_{i=1}^{k} \beta_i \nu_i - \sum_{i=1}^{k} \dot{Q}'_i f_i - (L - \sum_{i=1}^{k} \dot{Q}_i f_i) \\ &= \sum_{i=1}^{k} (\dot{Q}_i - \dot{Q}'_i) f_i - \beta_i \nu_i \\ &= \sum_{i=1}^{k} (f_i - \beta_i) \nu_i. \end{aligned} \tag{11.69}$$

The Routhian equations of motion (see, for example, [305, 368]) of the system characterized by (11.64), (11.65) will be identical to those obtained for the transformed system (11.66), (11.68) if we set $\{f_i = \beta_i, i = 1, ..., k\}$. In other words, under the last conditions, the arbitrary functions ν_i do not affect the solution for the non-cyclic coordinates.

From the above considerations we draw the following theorems:

1. For constant $\{\nu_i = n_i, i = 1, ..., k\}$. In this case the right-hand side of (11.69) is constant, and one can take $\{\beta_i = 0\}$. Equations for $q_1 ..., q_n$ are identical from R' and R.

Theorem 11.5 *If the Lagrangian (11.64) is general integrable (for arbitrary initial conditions), then the Lagrangian*

$$L' = L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, \dot{Q}'_1 + n_1, ..., \dot{Q}'_k + n_k) \tag{11.70}$$

is also integrable for arbitrary initial conditions.

It is not hard to see that this theorem applied to the problem of motion of a body about a fixed point under the action of axi-symmetric fields, i.e. with one cyclic coordinate ψ (the angle of precession), leads to the uniform precession transformation introduced in Chap. 10. Note that in this method, we have not used the property of invariance of the form of Euler–Poisson equations.

Exercise [405]: Apply the last theorem to exercise 5 of Chap. 9, using the transformation $\dot{\psi} \to \dot{\psi} + n$, $\dot{\varphi} \to \dot{\varphi} + N$, n, N constants. Show that the transformed integrable Lagrangian is

$$L' = \frac{1}{2}[(A+mz^2)(\dot\theta^2+\sin^2\theta\dot\psi^2)+C(\dot\psi\cos\theta+\dot\varphi)^2+\dot z^2]$$
$$+n[\left(A+mz^2\right)\sin^2\theta\dot\psi+C\cos\theta(\dot\psi\cos\theta+\dot\varphi)]+CN(\dot\psi\cos\theta+\dot\varphi)$$
$$-\{V(z)-\frac{n^2}{2}[\left(A+mz^2\right)\sin^2\theta+C\cos^2\theta]-nNC\cos\theta\}. \tag{11.71}$$

Note that the transformation engenders, among other effects, the presence of a gyrostatic momentum CN along the axis of symmetry and uniform field potential $-nNC\gamma_3$.

2. For variable $\{\nu_i = \nu_i(q_1..., q_n), i = 1, ..., k\}$

Theorem 11.6 *If the system with the Lagrangian*

$$L = L(q_1..., q_n, \dot q_1..., \dot q_n, \dot Q_1, ..., \dot Q_k) \tag{11.72}$$

is integrable for arbitrary initial conditions, then the system whose Lagrangian is

$$L' = L(q_1, ..., q_n, \dot q_1, ..., \dot q_n, \dot Q'_1+\nu_1, ..., \dot Q'_k+\nu_k) - \sum_{i=1}^{k}\beta_i\nu_i(q_1..., q_n) \tag{11.73}$$

is integrable for arbitrary functions ν_i and arbitrary constants $\{\beta_i\}$ on the level

$$\{\frac{\partial L'}{\partial \dot Q'_i} = \beta_i, \qquad i = 1, ..., k\} \tag{11.74}$$

of the cyclic integrals.

It should be stressed again that the integrability of the system with Lagrangian (11.73) in the last theorem is conditional, i.e. valid only for initial conditions consistent with the restriction (11.74), even though the original system (11.72) is integrable for arbitrary initial conditions. In application to dynamics of a rigid body about a fixed point in an axi-symmetric field, this theorem reproduces Theorem 1 of the previous section, which generates a conditional integrable extension from a general one.

There are, however, very important situations when the new system can be made integrable for all initial conditions. This depends on the structure of the potential part of the Lagrangian.

For the sake of clarity and for future applications, we consider in detail the case of a generalized natural system with three degrees of freedom, of which one is cyclic. Let

$$L = \frac{1}{2}(a_{11}\dot q_1^2+2a_{12}\dot q_1\dot q_2+a_{22}\dot q_2^2)+(c_1\dot q_1+c_2\dot q_2)\dot Q+\frac{1}{2}c_3\dot Q^2$$
$$+b_1\dot q_1+b_2\dot q_2+b_3\dot Q-V, \tag{11.75}$$

where a_{ij}, b_i, c_i, V depend only on q_1, q_2, so that Q is a cyclic variable. On an arbitrary level of the cyclic integral

$$c_1\dot{q}_1 + c_2\dot{q}_2 + c_3\dot{Q} + b_3 = f \tag{11.76}$$

the Routhian has the form

$$R = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2 \\ +b_1\dot{q}_1 + b_2\dot{q}_2 - V. \tag{11.77}$$

Now we perform in (11.75) the transformation

$$\dot{Q} = \nu + \dot{Q}', \nu = \nu(q_1, q_2). \tag{11.78}$$

According to the last theorem, we get the new Lagrangian

$$L' = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) + (c_1\dot{q}_1 + c_2\dot{q}_2)(\dot{Q}' + \nu) + \frac{1}{2}c_3(\dot{Q}' + \nu)^2 \\ +b_1\dot{q}_1 + b_2\dot{q}_2 + b_3(\dot{Q} + \nu) - V, \tag{11.79}$$

integrable on the level of the cyclic integral

$$c_1\dot{q}_1 + c_2\dot{q}_2 + c_3(\dot{Q}' + \nu) + b_3 = f. \tag{11.80}$$

Now, ignoring the cyclic coordinate Q' in (11.79) with the aid of this integral, one obtains the Routhian

$$R' = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2 \\ +b_1\dot{q}_1 + b_2\dot{q}_2 - V + f\nu. \tag{11.81}$$

Note that

$$R' = R + f\nu. \tag{11.82}$$

Let the system with the Lagrangian (11.75) be integrable. This implies the integrability of the system described by the Routhian (11.77), which should admit a complementary integral, independent of the Jacobi integral (the Hamiltonian). The transformed system with Lagrangian L' is not necessarily integrable. This is clearly seen from the relation (11.82) between the Routhians R and R'. When ν is not a constant, the two systems have different Routhian equations for the palpable coordinates. The following curious situation arises, which enables us to construct a wide class of extended integrable problems.

Let the potential V in (11.75) have the structure

$$V = V_0 + \sum a_i v_i \tag{11.83}$$

where $\{a_i\}$ are arbitrary constants and V_0, v_i are certain functions in the palpable generalized coordinates q_1, q_2. Let, further, the system (11.75) be integrable for arbitrary initial conditions. This means that, besides the three general integrals, the Routhian equations of this system admit a complementary general integral, which will depend on the set of constants $\{a_i\}$, say

$$I_4 = F(q_1, q_2, \dot{q}_1, \dot{q}_2, f; a_1, a_2, ...). \tag{11.84}$$

If, moreover, we choose ν in the transformation (11.78) in the form

$$\nu = \sum n_i v_i \tag{11.85}$$

and substitute this and (11.83) in (11.77) and (11.81), we put the two Routhians in the form

$$\begin{aligned} R = &\frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2 \\ &+ b_1\dot{q}_1 + b_2\dot{q}_2 - V_0 - \sum a_i v_i, \end{aligned} \tag{11.86}$$

and

$$\begin{aligned} R' = &\frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2 \\ &+ b_1\dot{q}_1 + b_2\dot{q}_2 - V_0 - \sum A_i v_i, \end{aligned} \tag{11.87}$$

where $A_i = a_i - f n_i$. The only difference between the two is that $\{a_i\}$ are replaced by $\{A_i\}$.

The Routhian equations of motion for the transformed problem are also generally integrable. They admit the integral

$$\begin{aligned} I_4' &= F(q_1, q_2, \dot{q}_1, \dot{q}_2, f; A_1, A_2, ...) \\ &= F(q_1, q_2, \dot{q}_1, \dot{q}_2, f; a_i - f n_i, a_2 - f n_2, ...). \end{aligned} \tag{11.88}$$

Moreover, the corresponding Lagrangians are also integrable. The integrals I_3 and I_3' can be obtained by substituting the parameter f from (11.76) and (11.80), respectively. It is remarkable that this substitution replaces some constant coefficients of the complementary integral of the transformed problem by ones depending on f, which can be replaced by its expression involving the velocity variables. The presence of the added parameters $\{n_i\}$ changes the structure of the integral. Naturally, the transformed system is a physical generalization of the original one and when all $\{n_i\}$ vanish one goes back to the original system.

The above three ways of generalizing integrable systems with cyclic coordinates can be applied to rigid body dynamics. In the particular case of axi-symmetric fields, they give the same results as described in the theorems of the last section. The method using cyclic variables furnishes a great advantage. It does not require invariance of the Euler–Poisson equations and it will be used in various situations in the sequel, while dealing with the motion of the body in an asymmetric combination of fields.