Chapter 10
The Problem of Motion of a Body in a Liquid

The present chapter is devoted to the investigation of the problem of motion of a rigid body by inertia in an ideal incompressible fluid, infinitely extending in all directions and at rest at infinity. Strictly speaking, this problem belongs to the field of fluid dynamics. The problem evolved namely in this way. The ordinary differential equations of motion of the solid are simultaneously solved with partial differential equation governing the motion of the liquid under boundary conditions satisfied on the surface of the moving solid. In this process, the pressure of the liquid had to be explicitly calculated at every point of the surface of the body. Nevertheless, after the study of some simple cases, and mainly in the works of Thomson and Tait [352] and of Kirchhoff [219], it became clear that the body and the liquid can be treated as forming together one dynamical system of six degrees of freedom, so that the detailed picture of the pressure of the fluid on the surface of the body is completely avoided. This system, composed of the body and liquid, was reduced to the motion of a rigid body with modified characteristics to compensate the motion of the liquid. When referred to a coordinate frame fixed in the body, the kinetic energy of this system is expressed as a quadratic form of the components of the angular and linear velocities of the body with constant coefficients. This step was decisive in the evolution of the subject along the next few decades.

In this setting, the present problem has six degrees of freedom: three for the rotational motion and three for the translation of a point of the body and is traditionally described for a simply connected body by Kirchhoff's equations [219] (see also [220]) or by their Hamiltonian form, mostly used by mathematicians, which are due in their final form to Clebsch [55]. For a perforated body (a body bounded by a multi-connected surface) the equations of motion are usually taken in the form due to Lamb [253], or in the equivalent Hamiltonian form (see e.g. [41]).

Research in the problem of motion of a body in a liquid passed through a period of vigorous activity in the last decades of the nineteenth century. After the formulation of the equations of motion in their final most general form by Kirchhoff, Clebsch and Lamb, a lot of significant results was obtained by several eminent, and mostly

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Russian, scientists, including Minkowsky [284], Lyapunov [267], Chaplygin [53] and Steklov [344, 345, 348]. For almost half a century, the research in the problem entered a state of stagnation. As stated by Aref and Jones [12] "*The Kirchhoff equations present a most remarkable simplification of a problem that, in principle, involves an infinite number of degrees of freedom. Surprisingly, the literature exploring these equations from the point of view of dynamical systems theory is rather sparse*". Half a century later, the first significant results concerning the integrable cases were obtained by Rubanovsky [317–320] (See also books: [121, 125]) using a modified form, due to Kharlamov P., of Clebsch's equations of motion. In most works outside Russia, the form of Clebsch (also Hamiltonian) was mostly used for some qualitative studies of the motion, e.g. [151, 263] (See also references of the last paper).

It turned out that the form of equations of motion involving the variables of Euler–Poisson type, rather than those of Hamiltonian type, enjoy some privileges that will be explained later in this chapter. Those are equations formulated, for the first time, in their most general form in [383]. They are in fact a form of Lagrangian equations, using redundant non-Lagrangian variables. Such Lagrangian equations are not completely new. Similar equations were used by Minkowski, in the special case of Kirchhoff's equations, to establish his brilliant theorem about the isomorphism between the reduced problem of rigid body motion and the motion of a particle on a smooth ellipsoid through a time transformation [284].

In this chapter, equations of motion are presented in their original forms of Kirchhoff, Clebsch and Lamb. Our new equations of Lagrangian, in fact Routhian, form [383] are also presented. This form turned out to be so effective that they put the problem in a unified context with other problems considered in this book. Those problems form a hierarchy, ascending from the classical problem to the one of the present chapter. This hierarchy is extended in the next part of this book to include the most general problem of motion of a rigid body under the action of conservative potential and gyroscopic forces which have a common axis of symmetry through the fixed point. The last problem reduces under some restrictions on the forces, to the problem, equivalent to that of motion of a body in a liquid. Going lower in the hierarchy, we note that every problem in it contains all the problems considered before it as a special case. As a result of this representation of the equations of motion, a striking property of the equations of motion of a rigid body in a liquid is revealed. It is the first problem which is closed under the regular precession transformation. Referring the equations to a coordinate frame precessing with a uniform speed with respect to the inertial frame, results in the same equations, as if in the inertial frame, but with changed characteristics of the body. Thus, this transformation generates from any solution in the present problem or any problem lower in the hierarchy, a new solution that contains the precession speed as an extra-parameter. This situation helped to re-organize the known information about the subject and to fill gaps in it. Some recently discovered integrable cases are generalized. Tables are given for all integrable cases, general and conditional. The most important known families of particular solutions to the problem are discussed on different levels of detail.

In our presentation of the subject, the problem of motion of a body in a liquid plays a rather unusual role. Results obtained in this problem by various methods and accumulated along a century have grown into a core for the advancement of some other problems of motion of a rigid body under more sophisticated forces. In later chapters, we shall use some transformations to obtain new integrable extensions which are more general from the physical and mathematical aspects and which were not subjected to any studies before.

10.1 Equations of Motion

10.1.1 Kirchhoff's Equations

Consider a rigid body moving in an ideal incompressible liquid, extending to infinity in all directions and at rest at infinity. Assume that the body is bounded by a simply connected surface and is moving by inertia, i.e. under no forces, except those exerted on it by the pressure of the liquid on its surface. Let O' and O, respectively, be the origins of the inertial coordinate system $O'XYZ$ and another system $Oxyz$ fixed in the body and let $\mathbf{r} = O'O$. Denote by $\boldsymbol{\omega}$ the angular velocity of the body and by $\mathbf{u}$ the velocity of O with respect to O', so that $\mathbf{u} = \frac{d\mathbf{r}}{dt}$. The equations of motion were derived in Lagrangian form using the Lagrangian function L (kinetic energy T, since no external forces are present):

$$L = T = \frac{1}{2}(\omega\mathbf{A} \cdot \omega + 2\mathbf{uB} \cdot \omega + \mathbf{uC} \cdot \mathbf{u}) \tag{10.1}$$

in which $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are constant 3×3 real matrices; $\mathbf{A}, \mathbf{C}$ symmetric and $\mathbf{B}$ is not necessarily symmetric. Here, the state variables $\boldsymbol{\omega}$ and $\mathbf{u}$ and all quantities (parameters of the problem) are referred to the body system. Of course, as a quadratic form, T must be positive definite in the six variables. For this the three matrices must satisfy certain inequalities.

We shall not go through the explicit derivation of the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ from the underlying hydrodynamical problem, because that would increase the size of this chapter beyond preassigned limits. All this material can be found in the treatise of Lamb [253]. It will be helpful in dealing with motion of bodies with certain symmetry properties to borrow the cases presented in the following table from that treatise. A similar table is presented in [41].

Table 0:

	Symmetry	Matrices	Examples
1	Plane of symmetry xy	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \begin{pmatrix} 0 & 0 & B_{13} \\ 0 & 0 & B_{23} \\ B_{13} & B_{23} & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}$.	
2	Two orthogonal planes of symmetry xy, xz	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B_{23} \\ 0 & B_{23} & 0 \end{pmatrix}$, $\mathbf{C} = \text{diag}(C_1, C_2, C_3)$	
3	Three orthogonal planes of symmetry xy, xz, yz	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = \text{diag}(C_1, C_2, C_3)$	Tri-axial ellipsoid, Parallele-piped.
4	Rotation through an angle π about axis Oz.	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}$.	Two-bladed ship screw.
5	Rotation through an angle $\pi/2$ about axis Oz.	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \text{diag}(B_1, B_2, B_3)$, $\mathbf{C} = \text{diag}(C_1, C_2, C_3)$.	Four-bladed ship screw.
6	Helicoidal symmetry about axis Oz.	$\mathbf{A} = \text{diag}(A, A, A_3)$, $\mathbf{B} = \text{diag}(B, B, A_3)$, $\mathbf{C} = \text{diag}(C, C, C_3)$.	Helicoid.
7	Oz is axis of symmetry (or rotation through an angle $\frac{2\pi}{n}$, $n \notin \{2, 4\}$ around z-axis).	$\mathbf{A} = \text{diag}(A, A, A_3)$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = \text{diag}(C, C, C_3)$.	Spheroid, Three-bladed ship screw.
8	The body is similarly related to each of the coordinate planes.	$\mathbf{A} = A\boldsymbol{\delta}$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = C\boldsymbol{\delta}$.	Cube, sphere.

It is usually argued that the origin of the movable coordinate system can always be shifted so that O coincides with a certain point of the body, called the central point, at which the matrix **B** becomes symmetric. It is also usually assumed that the axes of the body system are rotated to the principal axes of the matrix **A**, so that the matrix **A** becomes diagonal. However, we shall see soon that there is no need for those steps for the time being, if one is not concerned in using the original variables $\boldsymbol{\omega}$ and **u**.

The equations of motion are [219]

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} + \boldsymbol{\omega} \times \frac{\partial L}{\partial \boldsymbol{\omega}} + \mathbf{u} \times \frac{\partial L}{\partial \mathbf{u}} = 0,$$
$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{u}} + \boldsymbol{\omega} \times \frac{\partial L}{\partial \mathbf{u}} = 0. \tag{10.2}$$

Explicitly, Kirchhoff's equations can be written in vector form

$$\dot{\omega}\mathbf{A} + \dot{\mathbf{u}}\mathbf{B} + \omega \times (\omega\mathbf{A} + \mathbf{u}\mathbf{B}) + \mathbf{u} \times (\omega\mathbf{B}^T + \mathbf{u}\mathbf{C}) = 0,$$
$$\dot{\omega}\mathbf{B}^T + \dot{\mathbf{u}}\mathbf{C} + \omega \times (\omega\mathbf{B}^T + \mathbf{u}\mathbf{C}) = 0 \tag{10.3}$$

or, if one introduces the notation

$$\mathbf{M} = \frac{\partial L}{\partial \boldsymbol{\omega}} = \omega\mathbf{A} + \mathbf{u}\mathbf{B}, \tag{10.4}$$

and

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = \omega\mathbf{B}^T + \mathbf{u}\mathbf{C}, \tag{10.5}$$

in the alternative form

$$\dot{\mathbf{M}} + \omega \times \mathbf{M} + \mathbf{u} \times \mathbf{p} = 0,$$
$$\dot{\mathbf{p}} + \omega \times \mathbf{p} = 0. \tag{10.6}$$

Equation (10.3) are quite complicated. An obvious disadvantage is that they are not solved with respect to the derivatives. Every scalar equation of motion may contain the six components of the derivatives $\dot{\omega}$ and $\dot{\mathbf{u}}$. Following Kirchhoff, we also note that those equations admit three integrals of motion:

1. The energy integral, as the Lagrangian is a homogeneous quadratic polynomial of the velocities

$$I_1 = \frac{1}{2}(\omega\mathbf{A} \cdot \omega + \mathbf{2uB} \cdot \omega + \mathbf{uC} \cdot \mathbf{u}). \tag{10.7}$$

2. From the second equation in (10.6), it follows that the magnitude of the vector $\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}}$ is conserved.

$$I_2 = |\mathbf{p}|^2 = |\omega\mathbf{B}^T + \mathbf{u}\mathbf{C}|^2. \tag{10.8}$$

3. Also, using both Eq. (10.6), we get

$$I_3 = \mathbf{M} \cdot \mathbf{p} = (\omega\mathbf{A} + \mathbf{u}\mathbf{B}) \cdot (\omega\mathbf{B}^T + \mathbf{u}\mathbf{C}). \tag{10.9}$$

The system of Eq. (10.3) was used in the treatment of certain simple cases.

10.1.2 Example: Permanent Translational Motions

For example, Kirchhoff investigated the possibility that the body performs uniform translational motion without rotation. From (10.3), setting $\boldsymbol{\omega} = \mathbf{0}$, it turns out that the condition for this motion is

$$\mathbf{u} \times \mathbf{uC} = \mathbf{0}.$$

That is, the vector $\mathbf{u}$ must be directed along one of the principal axes of the matrix $\mathbf{C}$. Thus, a body of an arbitrary shape (with a tri-axial ellipsoid of the matrix $\mathbf{C}$) always has three mutually orthogonal axes such that if the body is set in motion parallel to one of them along any direction in space and then left to itself, it will permanently continue this motion with constant velocity. In case of two equal principal axes, all axes at the equatorial plane are possible axes of permanent translation and also the polar axis, and in case of spherical symmetry all directions in the body are possible for permanent translation. Note that actual spherical symmetry of the body is not necessary. It is a dynamical property that the matrix C has three equal eigenvalues. This condition is satisfied by cubes as well as by spheres [253].

10.1.3 Clebsch's Form of Kirchhoff's Equations

Clebsch [55] transformed Eq. (10.6) to Hamiltonian form using the variables $\mathbf{M}$, $\mathbf{p}$ and the Legendre transformation

$$\begin{aligned} H(\mathbf{M}, \mathbf{p}) &= \mathbf{M} \cdot \omega + \mathbf{p} \cdot \mathbf{u} - L \\ &= \frac{1}{2}(\mathbf{M}\tilde{\mathbf{a}} \cdot \mathbf{M} + 2\mathbf{M}\tilde{\mathbf{b}} \cdot \mathbf{p} + \mathbf{p}\tilde{\mathbf{c}} \cdot \mathbf{p}) \end{aligned} \tag{10.10}$$

where

$$\begin{aligned} \tilde{\mathbf{a}} &= (\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1}, \\ \tilde{\mathbf{b}} &= -\mathbf{C}^{-1}\mathbf{B}(\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1} \\ \tilde{\mathbf{c}} &= \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}(\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{C}^{-1}. \end{aligned}$$

Note that $\tilde{\mathbf{a}}$, $\tilde{\mathbf{c}}$ are symmetric but $\tilde{\mathbf{b}}$ is not.

The equations of motion acquire the Hamiltonian form due to Clebsch

$$\dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}}, \; \dot{\mathbf{p}} = \mathbf{p} \times \frac{\partial H}{\partial \mathbf{M}} \tag{10.11}$$

or, in expanded form,

$$\dot{\mathbf{M}} = \mathbf{M}\times(\mathbf{M}\tilde{\mathbf{a}}+\mathbf{p}\tilde{\mathbf{b}}^T) + \mathbf{p}\times(\mathbf{M}\tilde{\mathbf{b}}+\mathbf{p}\tilde{\mathbf{c}}),$$
$$\dot{\mathbf{p}} = \mathbf{p}\times(\mathbf{M}\tilde{\mathbf{a}}+\mathbf{p}\tilde{\mathbf{b}}^T), \tag{10.12}$$

which is used, usually assuming symmetry of the matrix $\tilde{\mathbf{b}}$, until recently, e.g. [41, 246, 263, 327, 328]. For the Hamiltonian form of the Kirchhoff equations see also [12, 257, 258, 280].

The general integrals of motion take their simplest form in the variables $\mathbf{M}$, $\mathbf{p}$:

$$I_1 = H,$$
$$I_2 = \mathbf{M}\cdot\mathbf{p}$$
$$I_3 = \mathbf{p}^2. \tag{10.13}$$

10.2 Thomson-Lamb's Equations

By the words of the contemporary of Thomson and Lamb, A.B. Basset [19] "the general theory of motion of a ring in an infinite liquid, when there is cyclic irrotational motion through its aperture, was first given by Sir William Thomson in the *Philosophical Magazine* (1871), and his theory has been subsequently developed by Professor Lamb, in his *Treatise on the motion of fluids*" [252]. Hence, and although the equations of motion are direct generalization of Kirchhoff's equations, I will give the name "Thomson-Lamb's theory" to the theory of equations of motion of a multi-connected (perforated) rigid body in a liquid. This problem was not considered in its generality in the western literature for about a century. In fact, after the works of Basset and Fawcett on the motion of perforated bodies in liquid (e.g. [19, 83]) in the last two decades of the nineteenth century, no significant results are seen in this area until the equations of motion were reformed by Kharlamov in the sixties. The deduction of the Lagrangian (Euler–Poisson type) equations appeared in 1986, a whole century later. This may have been caused by the historical nature of that period at the beginning of the twentieth century. The period of birth of new physical theories: atomic physics, relativity, old quantum and then quantum theories. Research in branches of classical mechanics was significantly retarded.

It may be noted here that the generalization of Kirchhoff equations for perforated body was given by some authors the name "Kirchhoff–Poisson equations". As examples, see [121, 125]. This name seems to us irrational, since Poisson had no relation at all to the present circle of problems.

Let O' and O, respectively, be the origins of the inertial coordinate system and the system fixed in the body, and let $\mathbf{r} = \overrightarrow{O'O}$. Denote by $\boldsymbol{\omega}$ the angular velocity of the body and by $\mathbf{u}$ the velocity of O with respect to O', so that $\mathbf{u} = \frac{d\mathbf{r}}{dt}$. The equations as in [253] are derived from a Lagrangian function (kinetic energy, since no external forces are present):

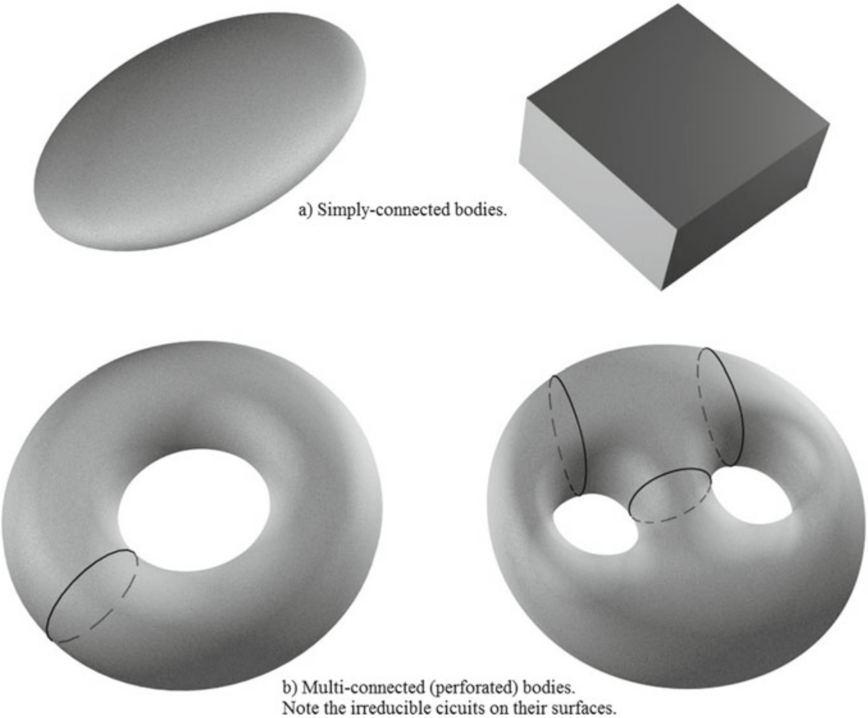


Fig. 10.1 Simple and perforated bodies

$$T = \frac{1}{2}(\omega\mathbf{A}\cdot\omega + \mathbf{2uB}\cdot\omega + \mathbf{uC}\cdot\mathbf{u}) + \bar{\alpha}\cdot\omega + \bar{\beta}\cdot\mathbf{u} \tag{10.14}$$

in which $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are constant 3×3 real matrices; $\mathbf{A}, \mathbf{C}$ symmetric and $\mathbf{B}$ is not necessarily symmetric and $\bar{\alpha}, \bar{\beta}$ are constant vectors, which characterize the multi-connectedness of the body and the circulations of the fluid on irreducible circuits drawn on its surface (Fig. 10.1b). Here, the state variables ω and $\mathbf{u}$ and all quantities (parameters of the problem) are referred to the body system. For a body bounded by a simply connected surface the vectors $\bar{\alpha}, \bar{\beta}$ vanish and the Lagrangian turns into the one used by Kirchhoff and Clebsch.

It is usually argued that the origin of the movable coordinate system can always be shifted to a certain point of the body, called the central point, at which the matrix $\mathbf{B}$ becomes symmetric if it is not so at O, and hence it is also usually assumed that the axes of the system are rotated, so that the matrix $\mathbf{A}$ becomes diagonal. However, we shall see soon that there is no need for those steps for the time being.

The equations of motion are [253]

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \boldsymbol{\omega}} + \boldsymbol{\omega} \times \frac{\partial T}{\partial \boldsymbol{\omega}} + \mathbf{u} \times \frac{\partial T}{\partial \mathbf{u}} &= 0, \\
\frac{d}{dt}\frac{\partial T}{\partial \mathbf{u}} + \boldsymbol{\omega} \times \frac{\partial T}{\partial \mathbf{u}} &= 0.
\end{aligned} \tag{10.15}
$$

Explicitly, Lamb's equations can be written in vector form

$$
\begin{aligned}
\dot{\boldsymbol{\omega}}\mathbf{A} + \dot{\mathbf{u}}\mathbf{B} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{A} + \mathbf{u}\mathbf{B} + \bar{\boldsymbol{\alpha}}) + \mathbf{u} \times (\boldsymbol{\omega}\mathbf{B}^T + \mathbf{u}\mathbf{C} + \bar{\boldsymbol{\beta}}) &= 0, \\
\dot{\boldsymbol{\omega}}\mathbf{B}^T + \dot{\mathbf{u}}\mathbf{C} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{B}^T + \mathbf{u}\mathbf{C} + \bar{\boldsymbol{\beta}}) &= 0
\end{aligned} \tag{10.16}
$$

or, if we introduce the notation

$$
\mathbf{M} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \boldsymbol{\omega}\mathbf{A} + \mathbf{u}\mathbf{B} + \bar{\boldsymbol{\alpha}}, \tag{10.17}
$$

and

$$
\mathbf{p} = \frac{\partial T}{\partial \mathbf{u}} = \boldsymbol{\omega}\mathbf{B}^T + \mathbf{u}\mathbf{C} + \bar{\boldsymbol{\beta}}, \tag{10.18}
$$

in the alternative form

$$
\begin{aligned}
\dot{\mathbf{M}} + \boldsymbol{\omega} \times \mathbf{M} + \mathbf{u} \times \mathbf{p} &= 0, \\
\dot{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p} &= 0.
\end{aligned} \tag{10.19}
$$

Equation (10.16) are quite complicated. An obvious disadvantage is that they are not solved with respect to the derivatives. Every scalar equation of motion may contain the six components of the derivatives $\dot{\boldsymbol{\omega}}$ and $\dot{\mathbf{u}}$. Following Lamb, we also note that those equations admit three integrals of motion:

1. Jacobi's integral, the homogeneous quadratic part of the Lagrangian

$$
I_1 = \frac{1}{2}(\boldsymbol{\omega}\mathbf{A} \cdot \boldsymbol{\omega} + \mathbf{2}\mathbf{u}\mathbf{B} \cdot \boldsymbol{\omega} + \mathbf{u}\mathbf{C} \cdot \mathbf{u}). \tag{10.20}
$$

2. From the second equation in (10.19), it follows that the magnitude of the vector $\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}}$ is conserved.

$$
I_2 = |\mathbf{p}|^2 = |\boldsymbol{\omega}\mathbf{B}^T + \mathbf{u}\mathbf{C} + \bar{\boldsymbol{\beta}}|^2.
$$

3. Also, using both Eq. (10.19), we get

$$
I_3 = \mathbf{M} \cdot \mathbf{p} = (\boldsymbol{\omega}\mathbf{A} + \mathbf{u}\mathbf{B} + \bar{\boldsymbol{\alpha}}) \cdot (\boldsymbol{\omega}\mathbf{B}^T + \mathbf{u}\mathbf{C} + \bar{\boldsymbol{\beta}}).
$$

The system of Eq. (10.16) was used in the treatment of certain simple cases and is usually transformed to the Hamiltonian variables. Using a Hamiltonian (see, e.g. [41]):

$$\begin{aligned} H &= \mathbf{M}\cdot\boldsymbol{\omega} - T \\ &= \frac{1}{2}(\mathbf{M}\tilde{\mathbf{a}}\cdot\mathbf{M} + 2\mathbf{M}\tilde{\mathbf{b}}\cdot\mathbf{p} + \mathbf{p}\tilde{\mathbf{c}}\cdot\mathbf{p}) + \tilde{\boldsymbol{\alpha}}\cdot\mathbf{M} + \tilde{\boldsymbol{\beta}}\cdot\mathbf{p} \end{aligned} \tag{10.21}$$

the equations of motion acquire the form

$$\dot{\mathbf{M}} = \mathbf{M}\times\frac{\partial H}{\partial \mathbf{M}} + \mathbf{p}\times\frac{\partial H}{\partial \mathbf{p}},\ \dot{\mathbf{p}} = \mathbf{p}\times\frac{\partial H}{\partial \mathbf{M}}, \tag{10.22}$$

or in explicit form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M}\times(\mathbf{M}\tilde{\mathbf{a}} + p\tilde{\mathbf{b}}^T + \tilde{\boldsymbol{\alpha}}) + \mathbf{p}\times(\mathbf{M}\tilde{\mathbf{b}} + \mathbf{p}\tilde{\mathbf{c}} + \tilde{\boldsymbol{\beta}}), \\ \dot{\mathbf{p}} &= \mathbf{p}\times(\mathbf{M}\tilde{\mathbf{a}} + p\tilde{\mathbf{b}}^T + \tilde{\boldsymbol{\alpha}}). \end{aligned} \tag{10.23}$$

Integrals of motion take the simple form:

$$\begin{aligned} I_1 &= H, \\ I_2 &= \mathbf{p}^2, \\ I_3 &= \mathbf{M}\cdot\mathbf{p}. \end{aligned}$$

The last form of equations is used in most recent works, e.g. [41].

10.3 On Different Forms of the Equations of Motion

The traditional equations of Kirchhoff and Lamb suffer some disadvantages that in most cases lead to their treatment for most of their history in isolation from other problems of rigid body dynamics. They also involve the non-symmetric matrix **b**. Although this matrix can be reduced to symmetric form by shifting the origin to the central point of the body, the presence of non-symmetry complicates the equations either in Lagrangian or Hamiltonian forms. In most recent works some simplifying restrictions on the parameters are assumed, such as $\mathbf{b} = \mathbf{0}$ (e.g. [151, 263]). Equation (10.11) has also the disadvantage that their solution gives the vector quantity **M**, which has no direct interpretation in terms of the motion unless transformed to an expression involving the angular velocity $\boldsymbol{\omega}$ and the vector **p** constant in space.

If Eqs. (10.2) (or (10.11)) are written in the frame of reference attached to the principal axes of a matrix **A** (or $\tilde{\mathbf{a}}$), they involve 15 parameters characterizing the shape of the body.

If Eqs. (10.15) (or (10.22)) are written in the frame of reference attached to the principal axes of a matrix **A** (or $\tilde{\mathbf{a}}$), they involve 24 parameters characterizing the shape of the body and, for a perforated body, circulations of the fluid along irreducible contours on its surface.

10.4 A New Form of the Equations of Motion

Here, we present with minor modification a new form of the equations of motion of a general body in a liquid, which was derived in our work [383]. We note first that in the Lagrangian (10.1), the Cartesian coordinates (X, Y, Z) of the origin of the system of axes fixed in the body relative to the inertial system are cyclic variables, since the resultant of forces acting on the body-and-fluid system vanishes. We now ignore those coordinates using the vector cyclic integral

$$\frac{\partial T}{\partial \mathbf{u}} = \boldsymbol{\omega}\mathbf{B}^T + \mathbf{u}\mathbf{C} + \tilde{\boldsymbol{\beta}} = \mathbf{p}, \tag{10.24}$$

where p is a vector whose components are constant in space and hence satisfies the Poisson equation

$$\dot{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p} = \mathbf{0}. \tag{10.25}$$

Now, solving the relation (10.24) in $\mathbf{u}$ we obtain

$$\mathbf{u} = (\mathbf{p} - \bar{\boldsymbol{\beta}} - \boldsymbol{\omega}\mathbf{B}^T)\mathbf{C}^{-1} \tag{10.26}$$

and we proceed to form Routh's function

$$\begin{aligned} R &= T - \mathbf{u} \cdot \frac{\partial L}{\partial \mathbf{u}} \\ &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} + (\boldsymbol{\kappa} + \mathbf{p}\tilde{\mathbf{K}}) \cdot \boldsymbol{\omega} - \mathbf{a} \cdot \mathbf{p} - \frac{1}{2}\mathbf{p}\mathbf{J} \cdot \mathbf{p} \end{aligned} \tag{10.27}$$

where $\mathbf{I}, \tilde{\mathbf{K}}, \mathbf{J}$ are the constant 3×3 matrices and $\boldsymbol{\kappa}$, $\mathbf{a}$ are the constant vectors given by

$$\begin{aligned} \mathbf{I} &= \mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B}, \\ \mathbf{J} &= \mathbf{C}^{-1}, \\ \tilde{\mathbf{K}} &= \mathbf{C}^{-1}\mathbf{B}, \end{aligned} \tag{10.28}$$

$$\mathbf{a} = -\bar{\boldsymbol{\beta}}\mathbf{C}^{-1}, \tag{10.29}$$

$$\boldsymbol{\kappa} = \bar{\boldsymbol{\alpha}} - \bar{\boldsymbol{\beta}}\mathbf{C}^{-1}\mathbf{B}. \tag{10.30}$$

As seen from (10.30), the matrices $\mathbf{I}, \mathbf{J}$ are symmetric but $\tilde{\mathbf{K}}$, in general, is not. Let $\tilde{\mathbf{K}}_s$ and $\tilde{\mathbf{K}}_a$ be the symmetric and antisymmetric parts of $\bar{\mathbf{K}}$, so that

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_s + \tilde{\mathbf{K}}_a, \tag{10.31}$$

$$\tilde{\mathbf{K}}_s = \frac{1}{2}[\mathbf{C}^{-1}\mathbf{B} + (\mathbf{C}^{-1}\mathbf{B})^T] \equiv -\frac{1}{2}\mathbf{K}, \tag{10.32}$$

$$\tilde{\mathbf{K}}_a = \frac{1}{2}[\mathbf{C}^{-1}\mathbf{B} - (\mathbf{C}^{-1}\mathbf{B})^T]. \tag{10.33}$$

Here we introduced a constant matrix $\mathbf{K} = -[\mathbf{C}^{-1}\mathbf{B} + (\mathbf{C}^{-1}\mathbf{B})^{\mathbf{T}}]$. Inserting (10.31) into (10.27), we can write

$$R = R_0 + \mathbf{p}\tilde{\mathbf{K}}_a \cdot \boldsymbol{\omega}, \tag{10.34}$$

where

$$R_0 = \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} + (\boldsymbol{\kappa} - \frac{1}{2}\mathbf{pK}) \cdot \boldsymbol{\omega} - \mathbf{a} \cdot \mathbf{p} - \frac{1}{2}\mathbf{pJ} \cdot \mathbf{p}. \tag{10.35}$$

We now show that the antisymmetric part $\mathbf{K}_a$ does not contribute to the equations of motion. In fact, the last term of (10.34) is

$$\begin{aligned}\mathbf{pK}_a \cdot \boldsymbol{\omega} &= (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)\begin{pmatrix} 0 & -K_{a3} & K_{a2} \\ K_{a3} & 0 & -K_{a1} \\ -K_{a2} & K_{a1} & 0 \end{pmatrix} \cdot \boldsymbol{\omega} \\ &= (\mathbf{p} \times \mathbf{k}_a) \cdot \boldsymbol{\omega}\end{aligned}$$

where we introduced the vector $\mathbf{k}_a = (K_{a1}, K_{a2}, K_{a3})$ constant in the body axes. Thus, we have

$$\begin{aligned}\mathbf{pK}_a \cdot \boldsymbol{\omega} &= \mathbf{k}_a \cdot (\boldsymbol{\omega} \times \mathbf{p}) \\ &= -\mathbf{k}_a \cdot \dot{\mathbf{p}} \\ &= \frac{d}{dt}(-\mathbf{k}_a \cdot \mathbf{p}).\end{aligned}$$

Thus, the extra term in (10.34) is a nugatory term and has no contribution to the equations of motion (e.g. [305]). The Routhian R_0 gives full description of the rotational motion of the body. Euler's equation for this motion can be deduced in a simple way. With an eye on future applications, we present that in detail. In fact, the equation of motion about the third axis of the body system is

$$\frac{d}{dt}(\frac{\partial R_0}{\partial \dot{\varphi}}) - \frac{\partial R_0}{\partial \varphi} = 0. \tag{10.36}$$

This gives

$$\frac{d}{dt}(\frac{\partial R_0}{\partial \boldsymbol{\omega}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{\varphi}}) - \frac{\partial R_0}{\partial \boldsymbol{\omega}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \varphi} - \frac{\partial R_0}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial \varphi} = 0.$$

But from formulas of Chap. 2, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial \dot{\varphi}} = \mathbf{k}, \frac{\partial \boldsymbol{\omega}}{\partial \varphi} = -\mathbf{k} \times \boldsymbol{\omega}, \frac{\partial \mathbf{p}}{\partial \varphi} = -\mathbf{k} \times \mathbf{p}, \tag{10.37}$$

and thus we get

$$\mathbf{k} \cdot [(\frac{\partial R_0}{\partial \boldsymbol{\omega}}\dot{)} + \boldsymbol{\omega} \times \frac{\partial R_0}{\partial \boldsymbol{\omega}} + \mathbf{p}\times \frac{\partial R_0}{\partial \mathbf{p}}] = 0,$$

so that the vector equation of motion can be written as

$$(\frac{\partial R_0}{\partial \boldsymbol{\omega}}\dot{)} + \boldsymbol{\omega} \times \frac{\partial R_0}{\partial \boldsymbol{\omega}} + \mathbf{p}\times \frac{\partial R_0}{\partial \mathbf{p}} = \mathbf{0}.. \tag{10.38}$$

Now, inserting the expression (10.35) for the Routhian, we obtain

$$(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}\dot{)} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}) + \mathbf{p} \times [-\frac{1}{2}\boldsymbol{\omega}\mathbf{K} - (\mathbf{a} + \mathbf{p}\mathbf{J})] = \mathbf{0}, \tag{10.39}$$

As $\mathbf{I}, \boldsymbol{\kappa}$ and $\mathbf{K}$ are constants in the body, the last equation becomes

$$\dot{\boldsymbol{\omega}}\mathbf{I} - \frac{1}{2}\dot{\mathbf{p}}\mathbf{K}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}) - \frac{1}{2}\mathbf{p}\times\boldsymbol{\omega}\mathbf{K} = \mathbf{p} \times (\mathbf{a} + \mathbf{p}\mathbf{J}),$$

and using Poisson's equation in the second term

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}) + \frac{1}{2}(\boldsymbol{\omega}\times\mathbf{p})\mathbf{K} + \frac{1}{2}\boldsymbol{\omega}\mathbf{K} \times \mathbf{p} = \mathbf{p} \times (\mathbf{a} + \mathbf{p}\mathbf{J}). \tag{10.40}$$

Here, using the identity

$$(\boldsymbol{\omega}\times\mathbf{p})\mathbf{K}+\boldsymbol{\omega}\mathbf{K} \times \mathbf{p} = \boldsymbol{\omega}\times(\mathbf{p}[tr(\mathbf{K})\boldsymbol{\delta}-\mathbf{K}]),$$

valid for any two vectors $\boldsymbol{\omega}$, $\mathbf{p}$ and symmetric matrix $\mathbf{K}$, we write the final form of the equations of motion

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} + \mathbf{p}\bar{\mathbf{K}}) = \mathbf{p} \times (\mathbf{a} + \mathbf{p}\mathbf{J}), \\ \dot{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p} = \mathbf{0}. \end{aligned} \tag{10.41}$$

where $\bar{\mathbf{K}} = \frac{1}{2}tr(\mathbf{K})\boldsymbol{\delta} - \mathbf{K}$, which is the same as the relation between $\mathbf{I}$ and $\bar{\mathbf{I}}$ in Chap. 1.

Equation (10.41) admit three first integrals:

$$
\begin{aligned}
I_1 &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{a}\cdot\mathbf{p} + \frac{1}{2}\mathbf{pJ}\cdot\mathbf{p},\\
I_2 &= \mathbf{p}^2,\\
I_3 &= (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\mathbf{pK})\cdot\mathbf{p}.
\end{aligned}
\tag{10.42}
$$

The vectors $\boldsymbol{\kappa}$ and $\mathbf{a}$, resulting from the circulation of the fluid in the body perforations vanish for a simply connected body, in which case Eq. (10.41) reduce to a form equivalent to Kirchhoff's equations.

When referred to principal axes of the matrix $\mathbf{I}$, Eq. (10.41) in the general case involve only 21 parameters, compared to 24 in (10.22). The parameters of the original problem can be expressed by inverting the relations (10.30) as:

$$
\begin{aligned}
\mathbf{A} &= \mathbf{I} - (\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a)\mathbf{J}^{-1}(-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{K} + \tilde{\mathbf{K}}_{\mathbf{a}}),\\
\mathbf{B} &= \mathbf{J}^{-1}(-\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\mathbf{C} &= \mathbf{J}^{-1},\\
\tilde{\mathbf{K}} &= -\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a,\\
\bar{\boldsymbol{\alpha}} &= \boldsymbol{\kappa} + \mathbf{aJ}^{-1}(-\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\bar{\boldsymbol{\beta}} &= -\mathbf{aJ}^{-1},
\end{aligned}
\tag{10.43}
$$

so that we retain, if we like, the three elements of the antisymmetric matrix, and thus also the full set of 24 (18) parameters of the original Lamb (Kirchhoff) formulation.

Remark *The observation that the antisymmetric part K_a of the matrix $C^{-1}B$ has no contribution to the equations of motion, except entering into the symmetric matrix A and the vector $\bar{\alpha}$, eliminates the necessity in several works to translate the origin O fixed in the body to the so-called central point of the body, or to assume the symmetry of the matrices and thus, unnecessarily, restricting the possible forms of the body. Calculation of the coefficient matrices can be done at a suitable point from the point of view of calculation and the characteristics of the rotational motion are then constructed free of the choice of the origin.*

Remark *The same observation resolves once for all a situation that the Hamiltonian equations based on the original Kirchhoff and Lamb equations that one can need to perform a canonical transformation to the Hamiltonian, to the equations of motion and to the integrals of motion, so that after the transformation the integrals of motion take a relatively simpler form. An example of such situation is the case found originally by Sokolov in [335]. A canonical transformation introduced by Borisov and Mamaev in [39] was used to simplify the Hamiltonian and to give the integral a simpler form.*

10.5 Steklov and Kharlamov Analogies and Their Generalization

The problem of motion of a rigid body in a liquid has been considered for a part of its history in complete isolation of other problems of motion of a rigid body about a fixed point.

As will be seen in more detail in the coming chapters, the equations of motion in their full form (10.41) derived from the Routhian (10.27) can be interpreted as equations of motion about a fixed point of a heavy, magnetized and electrically charged body bearing a rotor and influenced by an axially symmetric combination of three classical fields. More precisely, the second equation of (10.41) resembles Poisson's equation met in several previous chapters. This equation describes the space time derivative of a vector $\mathbf{p}$ constant in space, referred to the body system. Let us take a unit vector γ in the direction of $\mathbf{p}$, so that $\mathbf{p} = p_0\gamma$ and thus equations (10.41) take the form

$$\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}+p_0\gamma\bar{\mathbf{K}})=\gamma\times(p_0\mathbf{a}+p_0^2\gamma\mathbf{J}),$$
$$p_0(\dot{\gamma}+\boldsymbol{\omega}\times\gamma)=\mathbf{0}. \tag{10.44}$$

Let us first assume that $p_0 \neq 0$. In that case one can absorb this constant in the definitions for $\mathbf{a}$, $\mathbf{J}$ and $\bar{\mathbf{K}}$.

10.5.1 The Equivalent Problem of Motion About a Fixed Point

Alternatively, one can choose the units of measurement so that p_0 becomes unity. Finally, Eq. (10.41) can be written in the form of equations of motion of a rigid body about a fixed point with the vector γ fixed in space, i.e.

$$\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}+\gamma\bar{\mathbf{K}})=\gamma\times(\mathbf{a}+\gamma\mathbf{J}),$$
$$\dot{\gamma}+\boldsymbol{\omega}\times\gamma=\mathbf{0}. \tag{10.45}$$

This system of equations may be obtained from (10.41) by the replacement

$$(\mathbf{p},\bar{\mathbf{K}},\mathbf{a},\mathbf{J})\rightarrow(p_0\gamma,\bar{\mathbf{K}}/p_0,\mathbf{a}/p_0,\mathbf{J}/p_0^2), \tag{10.46}$$

so that if for some parameters $\mathbf{I},\boldsymbol{\kappa},\bar{\mathbf{K}},\mathbf{a},\mathbf{J}$ one has a solution $\boldsymbol{\omega}=\boldsymbol{\omega}(\mathbf{t})$ and $\gamma=\boldsymbol{\Gamma}(t)$ of the equivalent system (10.45), then one can obtain a solution $\boldsymbol{\omega}=\boldsymbol{\omega}(\mathbf{t})$, $p=p_0\gamma$ of (10.41) through the replacement of parameters

$$(\bar{\mathbf{K}}, \mathbf{a}, \mathbf{J}) \to (p_0\bar{\mathbf{K}}, p_0\mathbf{a}, p_0^2\mathbf{J}). \tag{10.47}$$

In this way, in the solution of the problem (10.41) an additional parameter p_0 is added. Returning to the problem of motion of a body in a liquid in the original formulation, we obtain a solution containing five parameters more than the solution of the equivalent problem.

Let us now turn to the excluded case $p_0 = 0$. In that case (10.44) reduces to

$$\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}) = \mathbf{0}, \tag{10.48}$$

which are the equations of motion of a free gyrostat fixed from one point. Those are the integrable equations already discussed in Chap. 5 under the name of Joukovsky and Volterra. This justifies the use of Eq. (10.45) in the generic case.

Equation (10.45) can be derived from the Lagrangian

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + (\boldsymbol{\kappa}-\frac{1}{2}\gamma\mathbf{K})\cdot\boldsymbol{\omega}-\mathbf{a}\cdot\gamma-\frac{1}{2}\gamma\mathbf{J}\cdot\gamma, \tag{10.49}$$

which is the last form of the Routhian R in (10.27). They admit the set of three integrals corresponding to (10.42), which are now written as

$$\begin{aligned} I_1 &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{a}\cdot\gamma+\frac{1}{2}\gamma\mathbf{J}\cdot\gamma = h,\\ I_2 &= \gamma^2 = 1,\\ I_3 &= (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} - \frac{1}{2}\gamma\mathbf{K})\cdot\gamma = f \end{aligned} \tag{10.50}$$

where h, f are arbitrary integration constants.

The six-dimensional problem of motion of the rigid body in the liquid is thus reduced to another problem of motion of a body about a fixed point, having only three degrees of freedom. This problem is described by Eq. (10.49) and has the integrals (10.50). The Lagrangian of the new problem (the Routhian R of the original problem) involves the angular velocity $\boldsymbol{\omega}$ and the vector $\boldsymbol{\gamma}$ constant in space. The forces acting on this virtual body can be interpreted as having a scalar potential

$$V = \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma, \tag{10.51}$$

and a vector potential

$$\mathbf{l} = \boldsymbol{\kappa}-\frac{1}{2}\gamma\mathbf{K}. \tag{10.52}$$

From now on, to conform with the previous simpler problems and with future study of more complex problems, we shall write the Lagrangian (10.49) as

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{l}\cdot\boldsymbol{\omega} - V, \tag{10.53}$$

and Eq. (10.45) as

$$\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\mu}) = \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}},$$
$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0}. \tag{10.54}$$

where

$$V = \mathbf{a}\cdot\boldsymbol{\gamma}+\frac{1}{2}\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma}, \tag{10.55}$$

$$\boldsymbol{\mu} = \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}. \tag{10.56}$$

In this form, each of the terms appearing in the equations of motion (10.54) of a rigid body in a liquid can be given concrete alternative interpretation:

(a) The vector $\mathbf{a}$ constant in the body, compared with formulas in Chap. 3, can be interpreted as the term $Mg\mathbf{r}_0$, the product of the weight of the equivalent body in a uniform gravity field g in the direction of $(-\boldsymbol{\gamma})$ and the position vector $\mathbf{r}_0$ of the centre of mass of that body.

(b) The vector $\boldsymbol{\kappa}$, also constant in the body, can be interpreted as a gyrostatic momentum of a symmetric rotor fixed from its axis of symmetry and rotating about it with a constant angular rate (Compare with Chap. 5).

(c) The potential term $\frac{1}{2}\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma}$ has a form, similar to that of the potential of a far Newtonian centre of attraction (Compare with Chap. 6), but can be interpreted in that way only when the matrices $\mathbf{J}$ and $\mathbf{I}$ are proportional $\mathbf{J} = \lambda\mathbf{I}$. For an arbitrary matrix $\mathbf{J}$, this term can be given interpretation as partially due to an attraction centre and partially as due to the electric interaction of a far Coulomb centre on the line parallel to $\boldsymbol{\gamma}$ and passing through the origin O, fixed in the present analogy, on a set of electric charges fixed in the equivalent body. In this interpretation, the matrix $\mathbf{J}$ is proportional to the inertia matrix of the electric charges on the equivalent body.

(d) The term $\boldsymbol{\gamma}\bar{\mathbf{K}}$ of the vector $\boldsymbol{\mu}$ can be interpreted as a result of the Lorentz effect of a uniform magnetic field parallel to $\boldsymbol{\gamma}$ on the electric charge distribution on the body (see e.g. [139]). This effect will be considered in more detail later in this book.

Conclusion: *The above considerations show that the overall effect of the hydrodynamic forces exerted by the fluid on the body can be replaced, as to their effect on the rotational motion of the body, by a set of relatively simple gravitational and electromagnetic interactions.*

By analogy or equivalence between the two problems here we mean full isomorphism of their equations of motion.[1] This analogy, pointed out in 1986 [383], generalizes the limited earlier analogies due to Steklov and Kharlamov:

---

[1] A weaker type of equivalence will be treated below involves isomorphism on the level of Routh-reduced equations of motion. The full Lagrangian systems are not isomorphic to each other, but any integrable case of one of them leads to an integrable case of the other.

10.5.2 Steklov's Analogy

In [345] (1895) and [348] (1902) noted that if in Kirchhoff's equations in Clebsch's form (10.12) one sets $\tilde{\mathbf{b}} = \mathbf{0}$, $\tilde{\mathbf{c}} = \epsilon\tilde{\mathbf{a}}$ those equations become identical with the equations of motion of a rigid body about a fixed point while acted upon by approximate Newtonian field in the integrable case when the body is fixed from its centre of mass (Case 2 of Chap. 6). In the terminology of Eq. (10.45) Steklov's analogy concerns the case $\mathbf{J} = \epsilon\mathbf{I}$, $\bar{\mathbf{K}} = \mathbf{0}$, $\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$.

10.5.3 Kharlamov's Analogy

In 1963, Kharlamov [192] generalized Steklov's analogy to the case of a perforated body, allowing non-zero vectors $\tilde{\alpha}$, $\tilde{\beta}$ in (10.23) and requiring only that $\tilde{\mathbf{b}} = \mathbf{0}$, $\tilde{c} = \epsilon\tilde{\mathbf{a}}$. For Eq. (10.54) Kharlamov's analogy requires that:

$$\mathbf{J} = \epsilon\mathbf{I}, \bar{\mathbf{K}} = \mathbf{0},$$

under which the problem of motion of a body in a liquid is analogous to the motion of a gyrostat about a fixed point, under the action of approximate Newtonian field of a centre (See Chap. 6).

10.6 Completing the Solution

10.6.1 Solution of the Equivalent Problem

Solving the system of equations of motion (10.45) we determine, as functions of the time t, the vectors $\boldsymbol{\omega}(t)$ and $\boldsymbol{\gamma}(t)$. In the alternative problem we regard the vector $\boldsymbol{\gamma}(=\frac{\mathbf{p}}{p_0})$, constant in space, as the unit vector pointing vertically upwards, take the Z-axis in that direction and measure the angle of precession ψ in the plane orthogonal to it. As in the classical problem (see 3.9), this determines the Eulerian angles of nutation and proper rotation θ and φ as

$$\theta = \cos^{-1}\gamma_3, \varphi = \tan^{-1}\frac{\gamma_1}{\gamma_2} \tag{10.57}$$

while the precession angle ψ is expressed by the quadrature

$$\psi = \psi_0 + \int_0^t \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2}dt, \tag{10.58}$$

ψ_0 is an integration constant. This completes the solution of the equivalent problem of motion about a fixed point, which is also the solution of the rotational part of the body in a liquid.

10.6.2 Solution of the Original Problem

Suppose that for the parameters $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}, \mathbf{a}, \mathbf{J}$ the equivalent problem (10.45) has a solution $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{t})$ and $\boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)$. The rotational motion of the body in the liquid is the same as in the previous subsection.

Conditions on the parameters of the original parameters are obtained by applying (10.47) to (10.43). This gives

$$\begin{aligned}
\mathbf{A} &= \mathbf{I} - (\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a)\mathbf{J}^{-1}(-\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\mathbf{B} &= \frac{1}{p_0^2}\mathbf{J}^{-1}(-\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\mathbf{C} &= \frac{1}{p_0^2}\mathbf{J}^{-1},\\
\tilde{\mathbf{K}} &= -\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a,\\
\bar{\boldsymbol{\alpha}} &= \boldsymbol{\kappa} + \frac{1}{p_0}\mathbf{a}\mathbf{J}^{-1}(-\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\bar{\boldsymbol{\beta}} &= -\frac{1}{p_0}\mathbf{a}\mathbf{J}^{-1}.
\end{aligned} \tag{10.59}$$

The velocity of the point O taken as origin is determined from (10.26) through the formula

$$\begin{aligned}
\mathbf{u} &= \mathbf{a} + \boldsymbol{\Gamma}\mathbf{J} - \boldsymbol{\Omega}\tilde{\mathbf{K}}^T\\
&= \mathbf{a} + \boldsymbol{\Gamma}\mathbf{J} - \boldsymbol{\Omega}(\frac{1}{2}\mathbf{K} - \mathbf{K}_a)\\
&= \mathbf{a} + \boldsymbol{\Gamma}\mathbf{J} - \frac{1}{2}\boldsymbol{\omega}\mathbf{K} + \boldsymbol{\Omega} \times \mathbf{k}_a.
\end{aligned} \tag{10.60}$$

In the last formula, one can easily recognize the term $\boldsymbol{\omega} \times \mathbf{k}_a$ as the only origin-dependent term. It represents the velocity of a unique point of the body whose position vector relative to O is $\mathbf{k}_a$. In the sequel, this point will be called the *proper* central point of the body. In contrast to the settled notation of the central point as the point at which the matrix B is symmetric, the proper central point has direct dynamical significance. If we take this point of the body as the origin, the matrix $\tilde{\mathbf{K}}$ would be symmetric. Taking (10.47) into account, the velocity $\mathbf{u}_{cp}$ of the central point is

$$\mathbf{u}_{cp} = p_0\mathbf{a}+p_0^2\boldsymbol{\Gamma}\mathbf{J}-p_0\boldsymbol{\Omega}\mathbf{K}. \tag{10.61}$$

That is origin-independent and depends only on the angular velocity and the orientation of the body.

For a given solution of the equations of motion, the position vector of the central point of the body can be found by quadratures

$$\begin{aligned}\mathbf{r}' &= (X', Y', Z') = \mathbf{r}'_0 + \int_0^t \mathbf{u}_{cp} dt \\ &= p_0\mathbf{a}t+p_0^2(\int_{\mathbf{0}}^{\mathbf{t}} \boldsymbol{\Gamma}(t)dt)\mathbf{J}-p_0(\int_{\mathbf{0}}^{\mathbf{t}} \boldsymbol{\Omega}(t)dt)\mathbf{K}.\end{aligned} \tag{10.62}$$

This yields the projections of the position vector of the origin of the body system relative to the origin of the inertial coordinate system on the axes of the body system. To express the position vector of the central point of the body referred to the inertial system, we write

$$\mathbf{r} = (\mathbf{r}' \cdot \boldsymbol{\alpha}, \mathbf{r}' \cdot \boldsymbol{\beta}, \mathbf{r}' \cdot \boldsymbol{\gamma}).$$

That is

$$\mathbf{r} = \mathbf{r}'\mathbf{R} \tag{10.63}$$

in terms of the rotation matrix $\mathbf{R}$, which can be constructed using the expressions (10.57) and (10.58) as shown in Chap. 2.

In the rest of this chapter we shall deal with the equivalent problem, returning to the original problem only occasionally, when some important assertions are to be made concerning the original problem. This is made here as a way of accommodating the problem of motion of a body in a liquid in the hierarchy on the top of problems of the previous chapters. A higher level in this hierarchy will be added in the next chapter.

10.7 Uniform Translational-Rotational Motion of a Body in a Liquid (Permanent Rotations of a Body with a Fixed Point About a Vertical Axis)

We now put forward a more general motion than that of Sect. 10.1.2, to find all possible permanent stationary (time-independent) motions. That is all solutions of (10.45) with the pair $(\boldsymbol{\omega}, \boldsymbol{\gamma})$ constant in the body and also in space. Substituting $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\gamma}} = \mathbf{0}$ in (10.45), we get

$$\begin{aligned}\boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}) &= \boldsymbol{\gamma}\times(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J}), \\ \boldsymbol{\omega} \times \boldsymbol{\gamma} &= \mathbf{0}.\end{aligned} \tag{10.64}$$

From the second equation, we can express the angular velocity in the form

$$\boldsymbol{\omega} = \omega_0 \boldsymbol{\gamma}, \tag{10.65}$$

where ω_0 is some proportionality constant and inserting this in the first equation, we obtain

$$\boldsymbol{\gamma}\times[\omega_0^2\boldsymbol{\gamma}\mathbf{I}+\omega_0(\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}}) - (\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})] = \mathbf{0}. \tag{10.66}$$

This condition determines the vector $\boldsymbol{\gamma}$, which characterizes the possible stationary motion in the sense that at any moment the body rotates about an axis parallel to $\boldsymbol{\gamma}$ and passing through the proper central point.

Equation (10.66) has an obvious and direct geometric meaning:

For each, arbitrarily given, real ω_0, the vector $\boldsymbol{\gamma}$ characterizing the possible stationary motion lies along one of the lines drawn from the proper central point to intersect at right angle the surface

$$\boldsymbol{\Phi} = \frac{1}{2}\gamma(\omega_0^2\mathbf{I}+\omega_0\bar{\mathbf{K}}-\mathbf{J}) \cdot \gamma + (\omega_0\boldsymbol{\kappa}-\mathbf{a}) \cdot \gamma{=}{=}const. \tag{10.67}$$

Here *fl* is an inhomogeneous quadratic function. The surface is a quadric referred to an origin (the proper central point of the body) different from its centre. In the case of a simply connected body $\boldsymbol{\kappa}= \mathbf{a} = \mathbf{0}$ and then γ becomes one of the eigenvector of the matrix $\omega_0^2\mathbf{I} - \mathbf{2}\omega_0\bar{\mathbf{K}}-\mathbf{J}$, which are known to be three in number and orthogonal to each other (See Sect. 10.1.2). The same conclusion can be reached also when $\boldsymbol{\kappa}$, $\mathbf{a}$ are non-zero parallel vectors and ω_0 is chosen such that $\omega_0\boldsymbol{\kappa}-\mathbf{a} = \mathbf{0}$. In the general case of a Multiply connected (perforated) body no such general rule can be stated. When this surface is an ellipsoid, for arbitrary ω_0, $\boldsymbol{\kappa}$ and $\mathbf{a}$ only two lines are guaranteed to be drawn from the origin to intersect the surface orthogonally. Those are points on the surface, nearest and farthest from the origin. Only one such line is guaranteed when the surface is one-sheeted and extending to infinity.

The vector Eq. (10.66) can be written in the form of three scalar equations, but only two of those equations are independent. In fact, multiplying (10.66) scalarly by each of the vectors $\boldsymbol{\gamma}\mathbf{I}$ and $(\mathbf{a} + \boldsymbol{\gamma}\mathbf{J})$, one obtains two different expressions for the angular speed

$$\omega_0 = \frac{\gamma \cdot [\boldsymbol{\gamma}\mathbf{I} \times (\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})]}{\gamma \cdot [\boldsymbol{\gamma}\mathbf{I} \times (\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]} = \frac{\gamma \cdot [(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J}) \times (\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]}{\gamma \cdot [\boldsymbol{\gamma}\mathbf{I} \times (\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})]}. \tag{10.68}$$

Equality of the two expressions for ω_0 determines the locus of the vector γ in the form

$$\begin{aligned} &\{\gamma \cdot [\boldsymbol{\gamma}\mathbf{I} \times (\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})]\}^2 - \\ &- \{\gamma \cdot [\boldsymbol{\gamma}\mathbf{I} \times (\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]\}\{\gamma \cdot [(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J}) \times (\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]\} \\ = {}& 0. \end{aligned} \tag{10.69}$$

This equation is non-homogeneous of degree six, and it represents a surface fixed in the body. This surface intersects the Poisson sphere in some spherical curve. The line joining the fixed point to each point of that spherical curve generates the cone of possible axes of permanent rotations. One readily recognizes the following special cases:

(1) From (10.66) we find that pure translations ($\omega_0 = 0$) are possible if and only if γ is a generator of the cone $\mathbf{a} \cdot (\gamma \times \gamma\mathbf{J}) = 0$. This equation resembles that of Staude's cone, except for replacing the inertia matrix $\mathbf{I}$ by the matrix $\mathbf{J}$, that appears in the potential.
(2) For a simply connected body ($\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$) Eq. (10.69) of degree six in γ becomes homogeneous, and hence represents a cone. This result was obtained by Minkowski [284] in 1888.
(3) For a gyrostat moving about a fixed point in a uniform gravity field, $\mathbf{J} = \bar{\mathbf{K}} = \mathbf{0}$, (10.69) becomes, as already seen in Chap. 5,

$$[\mathbf{a} \cdot (\gamma \times \gamma\mathbf{I})]^2 - [\boldsymbol{\kappa} \cdot (\gamma \times \gamma\mathbf{I})][\mathbf{a} \cdot (\boldsymbol{\kappa} \times \gamma)] = 0.$$

(4) In the special case collinear gyrostatic momentum and centre of mass and proportional matrices $\bar{\mathbf{K}}$, $\mathbf{J}$ such that $\bar{\mathbf{K}} = \epsilon\mathbf{J}$, $\boldsymbol{\kappa} = \epsilon\mathbf{a}$, the cone of permanent rotation axes reduces to Staude's cone for the classical problem. Shortly below, we shall see that this is a result of certain symmetry of the equations of motion, which allows for a rotation transformation.
(5) For the classical problem of motion of a body ($\mathbf{J} = \bar{\mathbf{K}} = \boldsymbol{\kappa} = \mathbf{0}$) it gives Staude's cone described by the equation $\mathbf{a} \cdot (\gamma \times \gamma\mathbf{I}) = 0$ [343].

Remark The above analysis applies mostly to the equivalent problem of motion of a rigid body about a fixed point under the action of potential and gyroscopic forces, described by the equations of motion (10.45) or (10.54). In the problem of motion of a body in a liquid, as explained above, the body rotates with the constant angular speed ω_0 about an axis parallel to γ and passing through the proper central point, while the latter moves with the uniform velocity

$$\mathbf{u}_{cp} = p_0\mathbf{a} + p_0^2\gamma\mathbf{J} - p_0\omega_0\gamma\mathbf{K}). \tag{10.70}$$

The position vector of the central point of the body can be expressed in the form

$$\begin{aligned} \mathbf{r}_{cp} &= \mathbf{r}_{cp0} + \int_0^t \mathbf{u}_{cp} dt\mathbf{R} \\ &= \mathbf{r}_{cp0} + \mathbf{u}_{cp} \int_0^t dt\mathbf{R} \\ &= \mathbf{r}_{cp0} + \frac{1}{\omega_0}[\mathbf{a} + \gamma(\mathbf{J} - \omega_0\mathbf{K})] \begin{pmatrix} \sin(\omega_0 t) & 1 - \cos(\omega_0 t) & 0 \\ \cos(\omega_0 t) - 1 & \sin(\omega_0 t) & 0 \\ 0 & 0 & \omega_0 t \end{pmatrix}. \end{aligned} \tag{10.71}$$

10.8 Stationary Motions About an Axis Inclined to the Vertical

Unlike the classical problem and its generalization to the heavy gyrostat, the problem of motion of a body in a liquid admits another type of motions in which $\boldsymbol{\omega}$ is constant (in space and in the body), but in a direction different from that of $\boldsymbol{\gamma}$. Let us take the z-axis of the body coordinate system along that direction. One can write

$$\boldsymbol{\omega} = \Omega \mathbf{k}, \, \Omega = \text{const}\,. \tag{10.72}$$

Equations of motion give

$$\begin{aligned} \Omega \mathbf{k} \times (\Omega \mathbf{k} \mathbf{I} + \boldsymbol{\kappa} + \gamma \bar{\mathbf{K}}) &= \gamma \times (\mathbf{a} + \gamma \mathbf{J}), \\ \dot{\gamma} + \Omega \mathbf{k} \times \gamma &= \mathbf{0}. \end{aligned} \tag{10.73}$$

The motion can be described by Euler's angles in the usual notation: $\psi = \psi_0, \theta = \theta_0, \phi = \Omega t$. In virtue of the symmetry of the problem about the Z-axis and without loss of generality, one can take $\psi_0 = 0$. The unit vector $\boldsymbol{\gamma}$ can be expressed as

$$\gamma = (\sin\theta_0 \sin\Omega t, \sin\theta_0 \cos\Omega t, \cos\theta_0). \tag{10.74}$$

This can be easily shown to satisfy the second equation in (10.73). Substituting in the first equation and equating coefficients of similar terms in powers of $\sin\Omega t$ and $\cos\Omega t$, we arrive at the following set of conditions:

$$\begin{aligned} J_{12} = J_{13} = J_{23} = 0, \, J_{22} &= J_{11}, \\ a_1 = a_2 = 0, \, a_3 + \Omega \tilde{K}_{11} - (J_{11} - J_{33}) \cos\theta_0 &= 0, \\ \bar{K}_{12} = 0, \, \bar{K}_{22} &= \bar{K}_{11}, \\ \kappa_1 + \Omega I_{13} + \cos\theta_0 \bar{K}_{13} = 0, \, \kappa_2 + \Omega I_{23} + \cos\theta_0 \bar{K}_{23} &= 0. \end{aligned} \tag{10.75}$$

In the generic case, we can write

$$\begin{aligned} \mathbf{J} &= \begin{pmatrix} J_{11} & 0 & 0 \\ 0 & J_{11} & 0 \\ 0 & 0 & J_{33} \end{pmatrix}, \bar{\mathbf{K}} = \begin{pmatrix} \bar{K}_{11} & 0 & \bar{K}_{13} \\ 0 & \bar{K}_{11} & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33} \end{pmatrix}, \\ \mathbf{a} &= (0, 0, -\Omega \bar{K}_{11} + \cos\theta_0 (J_{11} - J_{33})), \\ \boldsymbol{\kappa} &= (-\Omega I_{13} - \cos\theta_0 \bar{K}_{13}, -\Omega I_{23} - \cos\theta_0 \bar{K}_{23}, \kappa_3). \end{aligned} \tag{10.76}$$

where $J_{11}, J_{33}, \bar{K}_{11}, \bar{K}_{33}, \bar{K}_{13}, \bar{K}_{23}, I_{13}, I_{23}, \kappa_3$ are arbitrary parameters. Note that the axis of a permanent rotation of the present type must be a principal axis of the matrix $\mathbf{J}$, while the eigenvalues corresponding to the other two principal axes are equal. The virtual centre of mass (the vector $\mathbf{a}$) should also lie on the axis of rotation.

The last two Eq. (10.76) determine the pair of vectors $\mathbf{a}, \boldsymbol{\kappa}$ depending on the angular velocity Ω and the angle θ_0, which can be given arbitrary values. On the other hand, if one regards $\mathbf{a}, \boldsymbol{\kappa}$ as given parameters, the last equations reduce to three equations in two unknowns Ω and θ_0, for the solution of which a condition on the parameters of the body should be imposed. The parameters of the body must also satisfy the obvious restriction $|\cos\theta_0| \leq 1$. The solution of (10.75) exhibits several special and degenerate cases, some of which will be summed up in the exercises.

10.9 A Several-Parameter Particular Solution

A result, which will be presented in the next chapter, was obtained in [389] in the context of the problem of motion of a body about a fixed point under the action of an axially symmetric combination of forces. A special case of this result reduces to a case of motion of a body in a liquid (in fact, the alternative problem) and gives a quite general particular solution of that problem, in the sense of the number of parameters retained in it. That is a solution satisfying three invariant relations. It can be formulated as the following

Theorem 10.1 *Let in (10.45)*

$$\begin{aligned}
\mathbf{J} &= -\mathbf{MIM} + \alpha\mathbf{M} + \varepsilon\boldsymbol{\delta}, \\
\bar{\mathbf{K}} &= \beta\mathbf{M} - \alpha\boldsymbol{\delta} + \mathrm{tr}(\mathbf{MI})\boldsymbol{\delta} - \mathbf{IM} - \mathbf{MI}, \\
\boldsymbol{\kappa} &= \mathbf{m}(\beta\boldsymbol{\delta} - \mathbf{I}), \\
\mathbf{a} &= \mathbf{m}(\alpha\boldsymbol{\delta} - \mathbf{IM}).
\end{aligned} \tag{10.77}$$

where $\mathbf{M}, \mathbf{m}$ *are a constant real* 3×3 *symmetric matrix and a vector, respectively,* $\alpha, \beta, \varepsilon$ *are constants. Then,*

(1) (10.45) admits a solution, which satisfies the relations

$$\boldsymbol{\omega} = \boldsymbol{\gamma}\mathbf{M} + \mathbf{m}, \tag{10.78}$$

and γ *is a solution of Poisson's equation, which now takes the form*

$$\dot{\boldsymbol{\gamma}} + (\boldsymbol{\gamma}\mathbf{M} + \mathbf{m}) \times \boldsymbol{\gamma} = \mathbf{0}. \tag{10.79}$$

(2) In the generic case, the solution ($\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$) is expressed in terms of elliptic functions of time.

Proof (1) On substituting (10.77)–(10.79) into (10.45) and using the identity in Appendix 11.1, the first equation turns into identity.

(2) Assuming that $\det(\mathbf{M}) \neq \mathbf{0}$ and using the relation inverse to (10.78), one can write Eq. (10.79) as

$$\dot{\omega}\mathbf{M}^{-1} + \omega \times (\omega\mathbf{M}^{-1} - m\mathbf{M}^{-1}) = \mathbf{0}.$$

This is the equation of motion of a free gyrostat with inertia matrix $\mathbf{M^{-1}}$ and gyrostatic momentum $-\mathbf{mM^{-1}}$. This characterizes the case discussed in Chap. 5 under the name of Joukovsky–Volterra's case. Referring equations to the principal axes of $\mathbf{M^{-1}}$ by a suitable rotation, the solution is determined in terms of elliptic functions of time. ■

This completes building for the alternative problem (10.45), which involves 21 parameters, a solution depending on 15 of those parameters. One can also go back through (10.43) to build the relevant solution of the problem of motion of a body in a liquid (Eq. (10.41)) and including the parameter p_0 and the three elements of the anti-symmetric part of **B**.

The solution established by theorem 1 generalizes by the presence of several extra-parameters a former solution obtained by Kharlamov in [197]. It also generalizes another solution obtained by Kharlamova [205], while studying the motion of a rigid body about a fixed point in an approximate Newtonian field without gyroscopic forces, except the constant gyrostatic momentum. The choices in both works [197, 205] correspond to a matrix M which is diagonal and hence commuting with **I**. The relation of the variables to time was established only in some special cases, where very restrictive conditions were imposed on the parameters. Much older partial results were obtained by Steklov, who considered the case $\mathbf{m} = \mathbf{0}$ and **M** diagonal in the principal axes of inertia [345].

Solutions on invariant relations of the general form (10.79) (with non-diagonal **M**) were considered in the much later papers [128, 129] (See also [125]). In those papers, no reference is made to our relevant result in [389], published 14 years earlier. Conditions that the dynamical equations of motion are satisfied along with the given invariant relations are obtained by the (brute force) method of solving algebraic equations. Expressions obtained in [129] are not transparent, and there is no comparison with previous results.

10.10 Alternative Hamiltonian Formulation

Equations of motion (10.45) can be put in Hamiltonian form in two ways. On one hand, using canonical variables, such as Euler's angles and momenta conjugate to them is hopelessly complicated for analytical considerations. On the other hand, one can introduce the angular momentum of the system described by the Routhian (10.27)

$$\mathbf{M} = \frac{\partial R}{\partial \boldsymbol{\omega}} = \boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} + \gamma\bar{\mathbf{K}}, \tag{10.80}$$

as phase variable instead of $\boldsymbol{\omega}$, so that

$$\boldsymbol{\omega} = (\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{\mathbf{K}})\mathbf{I}^{-1}. \tag{10.81}$$

The Hamiltonian corresponding to the Lagrangian (10.49) as a function in $\mathbf{M}$ and γ is

$$\begin{aligned} H &= \frac{1}{2}\omega\mathbf{I}\cdot\omega + \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma \\ &= \frac{1}{2}(\mathbf{M}-\kappa-\gamma\bar{\mathbf{K}})\mathbf{I}^{-1}\cdot(\mathbf{M}-\kappa-\gamma\bar{\mathbf{K}}) + \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma \\ &= \frac{1}{2}\mathbf{M}\mathbf{I}^{-1}\cdot\mathbf{M} - (\kappa+\gamma\bar{\mathbf{K}})\mathbf{I}^{-1}\cdot\mathbf{M} \\ &\quad + (\mathbf{a}+\kappa\mathbf{I}^{-1}\bar{\mathbf{K}}^T)\cdot\gamma + \frac{1}{2}\gamma(\mathbf{J}+\bar{\mathbf{K}}\mathbf{I}^{-1}\bar{\mathbf{K}}^T)\cdot\gamma, \end{aligned} \tag{10.82}$$

so that the equations of motion can be written as

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M}\times\frac{\partial H}{\partial\mathbf{M}} + \gamma\times\frac{\partial H}{\partial\gamma}, \\ \dot{\gamma} &= \gamma\times\frac{\partial H}{\partial\mathbf{M}}. \end{aligned} \tag{10.83}$$

Note that

$$\begin{aligned} \frac{\partial H}{\partial\mathbf{M}} &= (\mathbf{M}-\kappa-\gamma\bar{K})\mathbf{I}^{-1} = \omega, \\ \frac{\partial H}{\partial\gamma} &= \mathbf{a}+\gamma\mathbf{J} - (\mathbf{M}-\kappa-\gamma\bar{\mathbf{K}})\mathbf{I}^{-1}\bar{\mathbf{K}}^T, \end{aligned}$$

or, in the expanded form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M}\times(\mathbf{M}-\kappa-\gamma\bar{\mathbf{K}})\mathbf{I}^{-1} + \gamma\times\frac{\partial H}{\partial\gamma}, \\ \dot{\gamma} &= \gamma\times(\mathbf{M}-\kappa-\gamma\bar{\mathbf{K}})\mathbf{I}^{-1}. \end{aligned} \tag{10.84}$$

For the Hamiltonian equations, the integrals of motion take the simplest form

$$\begin{aligned} I_1 &= H = h, \\ I_2 &= \mathbf{M}\cdot\gamma = f, \\ I_3 &= \gamma\cdot\gamma = 1. \end{aligned} \tag{10.85}$$

This situation makes use of the Hamiltonian form of equations favourable in certain situations. However, in other situations and for most of our purposes, the Lagrangian formalism of the equations of motion remains the favourable choice.

Throughout this book, we adhere to the use of Lagrangian formalism. We owe the reader some explanation for that. In early times of Hamiltonian mechanics, the formulation of mechanical problems stemmed directly from the physical setting. In the Hamiltonian describing the motion of a particle under the action of certain forces, each term of the Hamiltonian usually had its definite and unambiguous meaning. The situation in modern days is different. Some integrable Hamiltonians of the structure (10.82) were recently obtained by searching the relevant coefficients in a general ansatz. Two factors come into play even in the simpler cases when the description of motion is given a priori in Hamiltonian form:

(1) The Hamiltonian and the equations of motion derived from it are not unique for one and the same physical problem. This makes classification of integrable cases in Hamiltonian formulation rather problematic. As a matter of fact, to decide whether two Hamiltonians describe the same mechanical system practically reduces to computing the functions V and $\boldsymbol{\mu}$ as the quantities that remain invariant under all gauge transformations (canonical transformations linear in momenta in Hamiltonian terms).
(2) In the Euler–Poisson variables, it is possible to tell about physical interpretation of various terms of the potential. For example, terms linear in $\boldsymbol{\gamma}$ represent the potential of the heavy body in a constant gravity field. The centre of mass of the body is uniquely determined by terms in the Lagrangian linear in $\boldsymbol{\gamma}$. Other terms can be identified as a result of gravitational, electric or magnetic potential, but in the transformed Hamiltonian form terms of various degrees are totally disguised.

To illustrate the above points, we use as an example the case introduced by Sokolov [336] with the Kowalevski configuration $A = B = 2$, $C = 1$. The original Hamiltonian describing this case is

$$\begin{aligned} H_1 = &\frac{1}{4}(M_1^2 + M_2^2 + 2M_3^2) + \frac{1}{2}M_3(c_1\gamma_1 + c_2\gamma_2) + \frac{1}{2}\gamma_3(c_1M_1 + c_2M_2) \\ &+ (c_1\gamma_2 - c_2\gamma_1)^2 - (c_1^2 + c_2^2)\gamma_3^2. \end{aligned} \tag{10.86}$$

Calculating the Lagrangian corresponding to this Hamiltonian (10.86) and using the Legendre transformation

$$\begin{aligned} \boldsymbol{\omega} \equiv (p, q, r) &= \frac{\partial H_1}{\partial \mathbf{M}} \\ &= (\frac{1}{2}M_1 + \frac{1}{2}c_1\gamma_3, \frac{1}{2}M_2 + \frac{1}{2}c_2\gamma_3, M_3 + \frac{1}{2}(c_1\gamma_1 + c_2\gamma_2)), \end{aligned} \tag{10.87}$$

we find

$$\begin{aligned} L_1 = &\mathbf{M} \cdot \frac{\partial H_1}{\partial \mathbf{M}} - H_1 \\ = &p^2 + q^2 + \frac{r^2}{2} \\ &- c_1(p\gamma_3 + \frac{1}{2}r\gamma_1) - c_2(q\gamma_3 + \frac{1}{2}r\gamma_2) \\ &- \frac{9}{8}[(c_1\gamma_2 - c_2\gamma_1)^2 - (c_1^2 + c_2^2)\gamma_3^2] + \frac{1}{8}(c_1^2 + c_2^2)(\gamma_1^2 + \gamma_2^2 + \gamma_3^2). \end{aligned} \tag{10.88}$$

In the last expression one can eliminate its last term, since the spherically symmetric term does not contribute to the equations of motion. Thus, the Lagrangian L_1 has a potential part

$$V = \frac{9}{8}[(c_1\gamma_2 - c_2\gamma_1)^2 - (c_1^2 + c_2^2)\gamma_3^2]. \tag{10.89}$$

On the other hand, the gyroscopic terms of L_1 correspond to the choice of the vector $\boldsymbol{l}$ as

$$\boldsymbol{l} = -(c_1\gamma_3, c_2\gamma_3, \frac{1}{2}c_1\gamma_1 + \frac{1}{2}c_2\gamma_2).$$

This uniquely determines the vector

$$\begin{aligned}\boldsymbol{\mu} &= -\nabla[\frac{3}{2}(c_1\gamma_1 + c_2\gamma_2)\gamma_3] \\ &= -\frac{3}{2}(c_1\gamma_3, c_2\gamma_3, c_1\gamma_1 + c_2\gamma_2).\end{aligned} \tag{10.90}$$

The mechanical system under consideration is completely characterized by the pair of functions V and $\boldsymbol{\mu}$. The complementary integral of motion in the Euler–Poisson variables can be written as

$$I_4 = Z_1 Z_2, \tag{10.91}$$

$$Z_1 = (r - 1/2\, a_1\,\gamma_1 - 1/2\, a_2\,\gamma_2)\,,$$

$$\begin{aligned}Z_2 &= 1/4\,(2r - a_1\,\gamma_1 - a_2\,\gamma_2)\,[4\,p^2 + 4\,q^2 + (2r - a_1\,\gamma_1 - a_2\,\gamma_2)^2] \\ &+ 2\,(2\,pa_1 + 2\,qa_2)\,(2\,p\gamma_1 + 2\,q\gamma_2) + (a_1\,\gamma_1 + a_2\,\gamma_2)\,(2r - a_1\,\gamma_1 - a_2\,\gamma_2)^2 \\ &+ 1/2\,(a_1\,\gamma_1 + a_2\,\gamma_2)^2\,(2r - a_1\,\gamma_1 - a_2\,\gamma_2) \\ &- 1/2\left(a_1^2 + a_2^2\right)\gamma_3\,[8\,p\gamma_1 + 8\,q\gamma_2 + \gamma_3(2r - a_1\,\gamma_1 - a_2\,\gamma_2)].\end{aligned} \tag{10.92}$$

where we have set $a_1 = 3c_1, a_2 = 3c_2$.

An alternative Hamiltonian describing the same system was introduced by Borisov and Mamaev [39], using a linear transformation of the phase variables, which preserves the Poisson brackets and simplifies the Hamiltonian to

$$H_2 = \frac{1}{4}(M_1^2 + M_2^2 + 2M_3^2) + \frac{1}{2}M_3(a_1\gamma_1 + a_2\gamma_2) + \frac{1}{4}(a_1^2 + a_2^2)(\gamma_1^2 + \gamma_2^2). \tag{10.93}$$

The relation between $\boldsymbol{\omega}$ and $\mathbf{M}$ for this Hamiltonian is

$$\begin{aligned}\boldsymbol{\omega} &= \frac{\partial H_2}{\partial \mathbf{M}} \\ &= (\frac{1}{2}M_1, \frac{1}{2}M_2, M_3 + \frac{1}{2}(a_1\gamma_1 + a_2\gamma_2)),\end{aligned} \tag{10.94}$$

and by direct calculation of the corresponding Lagrangian

$$
\begin{aligned}
L_2 == {} & p^2 + q^2 + \frac{r^2}{2} \\
& - \frac{1}{2} r (a_1 \gamma_1 + a_2 \gamma_2) \\
& + \frac{1}{8} [(a_1 \gamma_2 + a_2 \gamma_1)^2 + 2(a_1^2 + a_2^2)\gamma_3^2].
\end{aligned} \tag{10.95}
$$

Note that its second line gives

$$
\boldsymbol{l} = (0, 0, \frac{1}{2}(a_1 \gamma_1 + a_2 \gamma_2)),
$$

which leads to the same $\boldsymbol{\mu}$ as in (10.90). The two Lagrangians $L_{1,2}$ are in fact equivalent. Their difference is

$$
\begin{aligned}
L_1 - L_2 &= c_1(r\gamma_1 - p\gamma_3) + c_2(r\gamma_2 - q\gamma_3) - (c_1^2 + c_2^2)(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \\
&= \frac{d}{dt}(c_2\gamma_1 - c_1\gamma_2) - (c_1^2 + c_2^2)(\gamma_1^2 + \gamma_2^2 + \gamma_3^2),
\end{aligned}
$$

i.e a gauge term and a central potential term. Both terms do not contribute to the equations of motion.

In contrast to the clarity and the physical relevance of the Lagrangian approach, none of the Hamiltonians H_1 and H_2 reflects the real nature of the potential (The terms quadratic in γ_i). Physical characteristics of the mechanical system are disguised in Hamiltonian form.

On the other hand, different Hamiltonian equations of motion are obtained using the Hamiltonians H_1, H_2. Also, each form of the Hamiltonians corresponds to a different form of the complementary integral, which can be constructed by substituting (10.87) and (10.94), respectively, in (10.91).

The change of the phase variables $\{\mathbf{M}, \boldsymbol{\gamma}\} \to \{\bar{\mathbf{M}}, \boldsymbol{\gamma}\}$ which transforms H_1 into H_2 can be obtained by comparing (10.94) with (10.87), in the form

$$
\bar{M}_1 = M_1 + c_1\gamma_3, \bar{M}_2 = M_2 + c_2\gamma_3, \bar{M}_3 = M_3 - c_1\gamma_1 - c_2\gamma_2. \tag{10.96}
$$

This is identical to the (canonical) transformation given by Borisov and Mamaev in [39] (See also [336]).

10.11 The Uniform Precession Transformation [383]

In its full final form (10.45), the problem of motion of a body in a liquid is at the top of a hierarchy, consisting of the problems considered in the previous chapters, involving the gyrostatic effect, the Newtonian potential term and the uniform gravity field as special cases. Consequently, every integrable case or solution of the above problems may have a generalization in the frame of the present one. This situation will be made clear in the tables of integrable cases provided below in this chapter.

A remarkable feature of Eq. (10.45) for the body in liquid, which is not enjoyed by any of the three simpler problems of Sects. 10.3–10.6, is their invariance under the uniform precession transformation, which we are going to describe now. This transformation was firstly introduced for the problem of motion of a body in a liquid in [383].

10.11.1 Direct Derivation

In the equations of motion (10.45), we perform the transformation of the variables $\boldsymbol{\omega}$ to a new set of variables $\boldsymbol{\omega}'$ by the relation

$$\boldsymbol{\omega} = \boldsymbol{\omega}' - n\boldsymbol{\gamma}, \tag{10.97}$$

containing the free real parameter n. Substituting in (10.45) we obtain

$$\begin{aligned}(\dot{\boldsymbol{\omega}}' - n\dot{\boldsymbol{\gamma}})\mathbf{I} + (\boldsymbol{\omega}' - n\boldsymbol{\gamma}) \times [(\boldsymbol{\omega}' - n\boldsymbol{\gamma})\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}) = \boldsymbol{\gamma} \times (\mathbf{a} + \boldsymbol{\gamma}\mathbf{J}),\\ \dot{\boldsymbol{\gamma}} + (\boldsymbol{\omega}' - n\boldsymbol{\gamma}) \times \boldsymbol{\gamma} = \mathbf{0}.\end{aligned} \tag{10.98}$$

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}} - n\boldsymbol{\gamma}\mathbf{I}) = n(\dot{\boldsymbol{\gamma}}\mathbf{I} + \boldsymbol{\gamma} \times \boldsymbol{\omega}'\mathbf{I}) + \boldsymbol{\gamma} \times (\mathbf{a} + \boldsymbol{\gamma}\mathbf{J})\\ + n\boldsymbol{\gamma} \times (-n\boldsymbol{\gamma}\mathbf{I} + \boldsymbol{\kappa} - 2\boldsymbol{\gamma}\bar{\mathbf{K}}),\\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0}.\end{aligned} \tag{10.99}$$

Using Poisson's equation to express $\dot{\boldsymbol{\gamma}}$ and noting that

$$(\boldsymbol{\omega}' \times \boldsymbol{\gamma})\mathbf{I} + \boldsymbol{\omega}'\mathbf{I} \times \boldsymbol{\gamma} = \boldsymbol{\omega}' \times \boldsymbol{\gamma}[tr(\mathbf{I})\delta - \mathbf{I}]$$

we give (10.99) the form

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}} + 2n\boldsymbol{\gamma}\bar{\mathbf{I}}) = \boldsymbol{\gamma} \times [\mathbf{a} + n\boldsymbol{\kappa} + \gamma(J - 2n\bar{\mathbf{K}} - n^2\mathbf{I})],\\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0},\end{aligned} \tag{10.100}$$

which can be readily put in the final form

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}') = \boldsymbol{\gamma} \times (\mathbf{a}' + \boldsymbol{\gamma}\mathbf{J})\\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0},\end{aligned} \tag{10.101}$$

after introducing the notation

$$\mathbf{K}' = \mathbf{K} + 2n\mathbf{I}, \bar{\mathbf{K}}' = \bar{\mathbf{K}} + 2n\bar{\mathbf{I}},$$
$$\mathbf{a}' = \mathbf{a} + n\boldsymbol{\kappa},$$
$$\mathbf{J}' = \mathbf{J} - n\mathbf{K} - n^2\mathbf{I}. \tag{10.102}$$

A look at the two sets of Eqs. (10.45) and (10.101) reveals that they have the same structure in terms of the two sets of variables $\{\boldsymbol{\omega}, \boldsymbol{\gamma}\}$ and $\{\boldsymbol{\omega}', \boldsymbol{\gamma}\}$ and that they differ only in the values of parameters $\mathbf{a}, \mathbf{J}, \mathbf{K}$, which are transformed to $\mathbf{a}', \mathbf{J}', \mathbf{K}'$, respectively, containing the extra-parameter n. When one sets $n = 0$, $\boldsymbol{\omega}' = \boldsymbol{\omega}$ and the three primed matrix-parameters reduce to their original (unprimed) values. It is an easy exercise to show that the consecutive application of two transformations with parameters n_1, n_2 is equivalent to the application of one transformation with the parameter $n_1 + n_2$.

10.11.2 Lagrangian Derivation

Consider the problem described by the equations of motion (10.45) derived from the Lagrangian (10.49). Let us affect the transformation. It can be readily checked that this transformation changes the Lagrangian (10.49) to the similar form

$$L' = \frac{1}{2}\boldsymbol{\omega}'\mathbf{I}\cdot\boldsymbol{\omega}' + \boldsymbol{l}'.\boldsymbol{\omega}' - V', \tag{10.103}$$

where

$$V' = V + n\boldsymbol{l}\cdot\boldsymbol{\gamma} - \frac{1}{2}n^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}, V = \mathbf{a}\cdot\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma},$$
$$\boldsymbol{l}' = \mathbf{l} - n\boldsymbol{\gamma}\mathbf{I}, \boldsymbol{l} = \boldsymbol{\kappa} - \frac{1}{2}\boldsymbol{\gamma}\mathbf{K}. \tag{10.104}$$

and renders the equations of motion (10.45) to

$$\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}'\times(\boldsymbol{\omega}'\mathbf{I} + {}^{-\prime}) = \boldsymbol{\gamma}\times\frac{\partial V'}{\partial\boldsymbol{\gamma}},$$
$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}'\times\boldsymbol{\gamma} = \mathbf{0} \tag{10.105}$$

where

$$\boldsymbol{\mu}' = \boldsymbol{\mu} + 2n\boldsymbol{\gamma}\bar{\mathbf{I}}$$
$$= \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}} + 2n\boldsymbol{\gamma}\bar{\mathbf{I}}.$$

Equation (10.105) admit the integrals

$$I_1 \equiv \frac{1}{2}\boldsymbol{\omega}'\mathbf{I}\cdot\boldsymbol{\omega}' + \mathbf{a}'\cdot\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\gamma}\mathbf{J}'\cdot\boldsymbol{\gamma} = h',$$
$$I_2 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\boldsymbol{\gamma}\bar{\mathbf{K}}')\cdot\boldsymbol{\gamma} = f',$$
$$I_3 = \boldsymbol{\gamma}^2 = 1, \tag{10.106}$$

with the constants

$$h' = h + nf,\ f' = f. \tag{10.107}$$

10.11.3 Physical and Mechanical Significance of the Transformation

The transformation (10.97) was used by Tisserand in [353] (See also [354]) to illustrate the effect of Coriolis and centrifugal forces on the motion of a rigid body with one point fixed on the rotating earth. It was implicitly used by other authors (e.g. [21, 51]) while studying the stability of relative equilibria in the problem of motion of a satellite in a circular orbit. It was applied only to the integrals of motion, but the transformed equations were not obtained and the full significance of the transformation was not revealed.

The transformation was applied for the first time to the full equations of motion of a charged and magnetized body in [378], where all its properties were revealed. It was also applied to the problem of motion of a satellite in a circular orbit to obtain its equations of motion relative to the orbital frame [382], in a form that resembles equations of motion of a rigid body about a fixed point under the action of given potential and gyroscopic forces. The invariance of the equations of motion of the body in a liquid under this transformation was first recognized in our work [383].

The presence of the parameter n in the transformed Lagrangian and the transformed equations of motion, in the framework of the equivalent physical problem, turns on a simultaneous combination of three physical effects:

(1) The effect of displacing the centre of mass of the body by an amount $n\boldsymbol{\kappa}$, proportional to the gyrostatic moment.

    When $\mathbf{a}$ is proportional to $\boldsymbol{\kappa}$, say, $\mathbf{a} = m\boldsymbol{\kappa}$, then one can choose $n = -m$, so that $\mathbf{a}' = \mathbf{0}$ and thus getting rid of the uniform gravity field in the transformed problem.

(2) The matrix $\mathbf{K}$ is changed by the amount $n\mathbf{I}$. This can be interpreted as the matrix of coefficients in the vector potential of a static, on the body, charge distribution

whose inertia matrix is proportional to the inertia matrix of the distribution of mass in the body and subject to the Lorentz forces due to a uniform magnetic field of intensity $\mathcal{B} = -2n$ in the direction of $\boldsymbol{\gamma}$.

If the matrix $\mathbf{K}$ is proportional to $\mathbf{I}$, say, $\mathbf{K} = m\mathbf{I}$, then the regular precession transformation can be used to make $\mathbf{K}' = \mathbf{0}$ by taking $n = m$.

(3) The matrix $\mathbf{J}$ of coefficients of the quadratic part of the potential is modified by adding two terms: $-n\mathbf{K} - n^2\mathbf{I}$. The potential resulting from those terms can be interpreted as due to magnetized (electrically charged) parts of the body influenced by a magnetic (electric) field with second-harmonic potential.

In the next sections, we shall use the uniform precession transformation in the two ways: to construct more general solutions containing the parameter n from known simpler ones and to simplify some other cases of motion by using that parameter to reduce the number of physical constants in them, in order to facilitate obtaining their solutions.

As a quick illustrative example, we work out an explicit solution of Euler's case generalized by the uniform precession transformation. For the transformed motion

$$V = -\frac{1}{2}n^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}, \qquad \boldsymbol{\mu} = 2n\boldsymbol{\gamma}\bar{\mathbf{I}}. \tag{10.108}$$

The equations of motion for this case take the form

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I} + 2n\boldsymbol{\gamma}\bar{\mathbf{I}}) &= -n^2\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.109}$$

They admit the complementary integral

$$A^2(p + n\gamma_1)^2 + B^2(q + n\gamma_2)^2 + C^2(r + n\gamma_3)^2 = G^2. \tag{10.110}$$

On one hand, Eq. (10.109) characterizes an integrable case of motion of the body in a liquid, which lies on the intersection of the cases of Clebsch and Steklov (See Table 10.2 below). On the other hand, in the framework of the equivalent problem, they describe the motion of a body under the influence of potential and Lorentz' forces. A family of explicit solutions of this case[2] can be written down immediately by transforming the solution constructed for Euler's case in Chap. 4 Sect. 4.1.

---

[2] In Euler's case we have solved only the dynamical equations of motion and adopted a very special solution of Poisson's equations in which the vectors $\boldsymbol{\gamma}$ and $\mathbf{G}$ are parallel. Hoever, the transformation applies equally well to the general solution of thewhole Euler-Poisson system.

$$p = \pm(\mu - \frac{n}{D}A)\sqrt{\frac{D(D-C)}{A(A-C)}}\mathrm{cn}\lambda(t-t_0),$$
$$q = (\mu - \frac{n}{D}B)\sqrt{\frac{D(D-C)}{B(B-C)}}\mathrm{sn}\lambda(t-t_0),$$
$$r = \pm(\mu - \frac{n}{D}C)\sqrt{\frac{D(A-D)}{C(A-C)}}\mathrm{dn}\lambda(t-t_0), \tag{10.111}$$

$$\gamma_1 = \pm\sqrt{\frac{A(D-C)}{D(A-C)}}\mathrm{cn}\lambda(t-t_0),$$
$$\gamma_2 = \sqrt{\frac{B(D-C)}{D(B-C)}}\mathrm{sn}\lambda(t-t_0),$$
$$\gamma_3 = \pm\sqrt{\frac{C(A-D)}{D(A-C)}}\mathrm{dn}\lambda(t-t_0), \tag{10.112}$$

where λ, D, μ and k, the modulus of elliptic functions, are the same as in Chap. 4 Sect. 4.1. The motion of the body is quite different from that in Euler's case. For example, choosing $n = \frac{\mu D}{A}$ we make $p \equiv 0$. The angular velocity lies permanently in the yz-plane, and it is still expressed in elliptic functions of time.

10.11.4 Uniform Precession Transformation in Hamiltonian Formalism

The expression (10.82) gives the Hamiltonian corresponding to the Lagrangian (10.49). Let H' be the Hamiltonian corresponding to the Lagrangian L' in (11.6), i.e. the Lagrangian obtained from (10.49) by the replacement $\boldsymbol{\omega} \to \boldsymbol{\omega}' = \boldsymbol{\omega} + n\boldsymbol{\gamma}$. It can be shown by direct calculation that

$$H' = H + n\mathbf{M}\cdot\boldsymbol{\gamma}. \tag{10.113}$$

The precession transformation is equivalent to adding the term $n\mathbf{M}\cdot\boldsymbol{\gamma}$ to the Hamiltonian, which is the precession parameter n multiplied by the areas integral (the second integral in (10.85)). The transformed Hamiltonian is a constant of motion

$$H' = h' = h + nf, \tag{10.114}$$

in agreement with (10.107). We also have

$$\begin{aligned}\boldsymbol{\omega}' &= \frac{\partial H'}{\partial \mathbf{M}} \\ &= \frac{\partial H}{\partial \mathbf{M}} + n\boldsymbol{\gamma} \\ &= \boldsymbol{\omega} + n\boldsymbol{\gamma}.\end{aligned} \tag{10.115}$$

Moreover, it is easy to show that the transformed Hamiltonian H' in (10.113) produces the same equations of motion as the original Hamiltonian H. In fact, using H' in (10.83), we obtain the equations

$$\begin{aligned}\dot{\mathbf{M}} &= \mathbf{M}\times\frac{\partial H'}{\partial \mathbf{M}} + \boldsymbol{\gamma}\times\frac{\partial H'}{\partial \boldsymbol{\gamma}} \\ &= \mathbf{M}\times(\frac{\partial H}{\partial \mathbf{M}} + n\boldsymbol{\gamma}) + \boldsymbol{\gamma}\times(\frac{\partial H}{\partial \boldsymbol{\gamma}} + n\mathbf{M}) \\ &= \mathbf{M}\times\frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma}\times\frac{\partial H}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma}\times\frac{\partial H'}{\partial \mathbf{M}} \\ &= \boldsymbol{\gamma}\times(\frac{\partial H}{\partial \mathbf{M}} + n\boldsymbol{\gamma}) \\ &= \boldsymbol{\gamma}\times\frac{\partial H}{\partial \mathbf{M}},\end{aligned} \tag{10.116}$$

which are identical to the original Eq. (10.83).

In contrast to the transformed Lagrangian (10.103), the transformed Hamiltonian (10.113) does not reveal any of the physical effects of the uniform precession transformation, which we listed in the last subsection. Thus, *the part of the physical effects induced by the uniform precession transformation in the problem is completely hidden by the Hamiltonian form of the equations of motion.* The Hamiltonian formalism identifies the whole family of mechanical systems depending on the arbitrary parameter n into a single Hamiltonian system. The Hamiltonian flow on the integral manifold of that system is the same for all physical problems, which differ only in the value of n. In the problem of motion of a body in a liquid that is a family of bodies with differing shape characteristics, but in the alternative problem it means a body subject to a family of potential and gyroscopic forces depending on n. As usual in the search for integrable cases, one assumes only the Hamiltonian form of the equations of motion and tries to determine the coefficients in a general form (ansatz) of the Hamiltonian and the complementary integral. In Hamiltonians constructed in this way, a term of the form $n\mathbf{M}\cdot\boldsymbol{\gamma}$ is missing and the dynamical behaviour of the original physical system will be determined up to a precessional motion with a constant rate.

10.12 Generalization of General Integrable Cases

The most important consequence of the form-invariance of equations of motion under the transformation (10.97) is the possibility it opens to generalize general (unconditional) and conditional integrable cases and also particular solutions through adding the precession parameter n into their structure, and thus enriching the physical problem by adding new physical effects. In the present section, we formulate this result for general integrable cases and give concrete examples of its applications. Conditional and particular cases are discussed in the next sections.

Theorem 10.2 *Let for some set of parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}', \mathbf{a}', \mathbf{J}'$, *Eq.* (10.105) *admit a complementary integral* $I_4 = I_4(\omega', \gamma)$, *so that they become integrable for arbitrary initial conditions, and let their solution be* $\{\boldsymbol{\omega} = \Omega(t), \gamma = \Gamma(t)\}$. *Then Eq.* (10.54) *are also integrable for arbitrary initial conditions, for the set of values of the parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}, \mathbf{a}, \mathbf{J}$:

$$\begin{aligned} \tilde{\mathbf{K}} &= \tilde{\mathbf{K}}' - 2n\mathbf{I}, \\ \mathbf{a} &= \mathbf{a}' - n\boldsymbol{\kappa}, \\ \mathbf{J} &= \mathbf{J}' + n\mathbf{K} + n^2\mathbf{I}, \end{aligned} \tag{10.117}$$

their complementary integral is

$$I_4 = I_4(\boldsymbol{\omega} + n\boldsymbol{\gamma}, \boldsymbol{\gamma}), \tag{10.118}$$

and their general solution is $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t) + n\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$. *It contains the additional arbitrary real parameter* n. *When* $n = 0$, *the generalized solution renders to the original solution.*

Any one of the hierarchy of integrable cases provided in the previous chapters admits a generalization as a case of the motion of a rigid body in a liquid. Transformed cases are of the same type (general or conditional) as the original ones. Examples are given in the next subsections: In Sokolov's case, the introduction of the parameter n results in a new integrable case. Even when the parameter n already enters in the structure of a known system, like in the case due to Rubanovsky [317], the regular precession transformation can be used to simplify the process of construction of an explicit solution of the equations of motion in terms of time. In such cases, the solution can be found first for the simpler case $n = 0$ and then generalized by that transformation for arbitrary n by the formulas in the last theorem.

10.12.1 Generalization of the Integrable Case Found by Sokolov

In 2002, Sokolov [336] obtained an integrable case of the rigid body in a liquid, which adds a parameter c to a former case by Yehia [380] (1986). The body has the Kowalevski configuration $A = B = 2C$. The centre of mass lies in the equatorial plane. The functions $V, \mathbf{l}, \boldsymbol{\mu}$ and the integrals I_3 and I_4 are given, according to the order followed in this book, as

$$
\begin{aligned}
V &= C[kc\gamma_1 + a_2\gamma_2 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2)], \\
\mathbf{l} &= C(2c\gamma_3, 0, k - c\gamma_1), \; \boldsymbol{\mu} = C(c\gamma_3, 0, k + c\gamma_1), \qquad (10.119) \\
I_3 &= 2(p\gamma_1 + q\gamma_2) + (r + k + c\gamma_1)\gamma_3, \\
I_4 &= \left[p^2 - q^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r - k)\right]^2 \\
&\quad + \left[2pq - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r - k)\right]^2 \\
&\quad + 2k(r - k + c\gamma_1)\left[p^2 + q^2 + 2cp\gamma_3\right] \\
&\quad - 2kc^2\{2\gamma_3[2p\gamma_1 + c\gamma_1\gamma_3 + 2q\gamma_2 + r\gamma_3] \\
&\quad + k\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + c\gamma_1)\} - 4a_2kq\gamma_3. \qquad (10.120)
\end{aligned}
$$

It was pointed out in [411] (2003) that the parameter n can be added to this case to produce a non-trivial generalization, represented by the formulas

$$
\begin{aligned}
V &= C[kc\gamma_1 + a_2\gamma_2 - nk\gamma_3 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2) \\
&\quad - nc\gamma_1\gamma_3 - \frac{n^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)], \\
\mathbf{l} &= C(2c\gamma_3 + 2n\gamma_1, 2n\gamma_2, k - c\gamma_1 + n\gamma_3), \\
\boldsymbol{\mu} &= C(c\gamma_3 - n\gamma_1, -n\gamma_2, k + c\gamma_1 - 3n\gamma_3), \qquad (10.121)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= 2(p\gamma_1 + q\gamma_2) + (r + k + c\gamma_1)\gamma_3 + n(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2), \\
I_4 &= \left[(p + n\gamma_1)^2 - (q + n\gamma_2)^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r + n\gamma_3 - k)\right]^2 \\
&\quad + \left[2(p + n\gamma_1)(q + n\gamma_2) - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r + n\gamma_3 - k)\right]^2 \\
&\quad + 2k(r + n\gamma_3 - k + c\gamma_1)\left[(p + n\gamma_1)^2 + (q + n\gamma_2)^2 + 2c(p + n\gamma_1)\gamma_3\right] \\
&\quad - 2kc^2\{2\gamma_3[2(p + n\gamma_1)\gamma_1 + c\gamma_1\gamma_3 + 2(q + n\gamma_2)\gamma_2 + (r + n\gamma_3)\gamma_3] \\
&\quad + k\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + n\gamma_3 + c\gamma_1)\} - 4a_2k(q + n\gamma_2)\gamma_3. \qquad (10.122)
\end{aligned}
$$

Comparing (10.121), (10.122), (10.119), (10.120) we note that, unlike in Sokolov's case, the centre of mass in the generalized case does not lie in the equatorial plane since it has three non-zero coordinates. Also, the vector potential **l** and the gyroscopic vector $\boldsymbol{\mu}$ do not lie in a meridional plane as in Sokolov's case.

Remark. On the other hand, the regular precession transformation can be used in the reverse direction. For example, to seek the explicit solution of the equations of motion or to study the stability of a given motion, it suffices to study equations of motion for the Sokolov case. The solution may be extended to the generalized system with the extra-parameter n and the conclusion about stability will be the same before and after adding this parameter to the system.

10.12.2 Steklov's Case and Its Generalizations

One of the first known integrable cases of Kirchhoff equations (The version of (10.45) with $\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$)

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\gamma}\bar{\mathbf{K}}) = \boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{J}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}. \end{aligned} \tag{10.123}$$

describing the motion of a rigid body with a singly connected surface in a liquid was discovered in 1895 by Steklov [345]. It corresponds to the choice

$$\mathbf{J} = \mathbf{0}. \tag{10.124}$$

Using (10.123), (10.124) we calculate the derivative

$$\frac{d}{dt}\{\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2 + \boldsymbol{\gamma}\bar{\mathbf{K}} \cdot (\boldsymbol{\omega}\mathbf{I})\} = -\boldsymbol{\omega} \cdot [(\boldsymbol{\omega} \times \boldsymbol{\gamma})\bar{\mathbf{K}}\mathbf{I}].$$

Under the condition

$$\bar{\mathbf{K}}\mathbf{I} = \varepsilon\boldsymbol{\delta},$$

where $\boldsymbol{\delta}$ is the unit matrix, or, equivalently,

$$\bar{\mathbf{K}} = \varepsilon\mathbf{I}^{-1} \tag{10.125}$$

the right-hand side of the last equation vanishes, which leads to Steklov's complementary integral of motion

$$\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2 + \varepsilon\boldsymbol{\omega} \cdot \boldsymbol{\gamma} = \text{const}\,. \tag{10.126}$$

This is the classical case of Steklov. When $\varepsilon = 0$, it turns into Euler's case.

Kharlamov [192, 197] investigated the full equations of motion which describe the problem of motion of a body with a multi-connected surface equivalent to (10.45) but in certain modified Clebsch variables. We write them in our form (10.45)

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}}) &= \boldsymbol{\gamma}\times(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J}), \\ \dot{\boldsymbol{\gamma}}+\boldsymbol{\omega}\times\boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.127}$$

In those equations, the analog of the gyrostatic momentum and the centre of mass are present. Using what can be called a "brute force" method, Kharlamov found a generalization of Steklov's result under the conditions

$$a_1 + n\kappa_1 = a_3 + n\kappa_3 = 0, a_2 = \kappa_2 = 0 \tag{10.128}$$

and expressed the complementary integral as a quadratic polynomial in the variables [197]. Somewhat later, Rubanovsky [320] by a similar method replaced Kharlamov's non-symmetrical conditions $a_2 = \kappa_2 = 0$ by the less restrictive and more symmetric condition

$$a_2 + n\kappa_2 = 0, \tag{10.129}$$

so that the two vectors $\mathbf{a}$ and $\boldsymbol{\kappa}$ are now proportional, i.e.

$$\mathbf{a} = -n\boldsymbol{\kappa}. \tag{10.130}$$

Let us now consider a system of equations of motion containing only the arbitrary gyrostatic vector $\boldsymbol{\kappa}$, in addition to Eq. (10.123) and take Steklov's condition (10.128) into account. We write them as

$$\begin{aligned} \dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}'\times(\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} + \varepsilon\mathbf{I}^{-1}) &= \mathbf{0}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.131}$$

It can be easily verified that this system admits the following complementary integral, which generalizes (10.126):

$$\frac{1}{2}|\boldsymbol{\omega}'\mathbf{I}+\boldsymbol{\kappa}|^2 + \varepsilon\boldsymbol{\omega}' \cdot \boldsymbol{\gamma} = \text{const}\,. \tag{10.132}$$

Now we apply the regular precession transformation to the system (10.131) and its integral (10.132). Equation (10.131) transforms to (10.127), in which

$$\bar{\mathbf{K}} = -\frac{1}{2}\varepsilon\mathbf{I}^{-1} + n\bar{\mathbf{I}}, \mathbf{J} = -n^2\mathbf{I}, a = -n\boldsymbol{\kappa}. \tag{10.133}$$

This gives at once Rubanovsky's generalization of Kharlamov's result. It also enables to write the complementary integral in the very simple and transparent way:

$$I_4 = \frac{1}{2}[\boldsymbol{\omega}\mathbf{I} + n\boldsymbol{\gamma}\mathbf{I}+\boldsymbol{\kappa}]^2 + \varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\gamma} = \text{const}\,. \tag{10.134}$$

Although this case was noted by Rubanovsky, the method used here helped write the integral I_4 in this simple form. A notable advantage of the regular precession transformation is that one can use it here in the reverse way. To construct the explicit solution in terms of functions in time, it is sufficient to do that for the case $n = 0$. This means to construct the solution of the system (10.131) with the fourth integral (10.132). Having completed this task, i.e. having found $\boldsymbol{\omega}' = \Omega(t), \;= \Gamma(t)$, one can just write down the solution for the generalized case

$$\boldsymbol{\omega} = \Omega(t) - n\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t). \tag{10.135}$$

Explicit time solution of the full case (10.131) is not constructed to the present moment. This was achieved only in two particular cases:

(1) In Steklov's case $\boldsymbol{\kappa} = \mathbf{0}$, by Kötter in terms of theta functions of two variables [235].
(2) In Joukovsky's case $\varepsilon = 0$, the solution was obtained by Volterra [366] in terms of Weierstrass functions, which are complex functions in t. An alternative solution in terms of *real* Jacobi's elliptic functions was constructed by Wittenburg [369].

Despite the interest in applying methods of modern algebraic geometry (e.g. [71]), the general solution for the full basic case $\varepsilon|\boldsymbol{\kappa}|\neq 0$ was not considered.

10.13 Generalization of Conditional Integrable Cases

Theorem 10.3 *Let for some set of parameters* $\mathbf{I}, \boldsymbol{\kappa}, \bar{\mathbf{K}}', \mathbf{a}', \mathbf{J}'$, *Eq.* (10.105) *be integrable on the integral level* $I_2 = f_0$ *with the complementary integral* $I_4 = I_4(\omega', \gamma)$, *and let their solution be* $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$. *Then Eq.* (10.54) *is also integrable on the same integral level* $I_2 = f_0$, *for the set of values of the parameters* $\mathbf{I}, \boldsymbol{\kappa}, \tilde{\mathbf{K}}, \mathbf{a}, \mathbf{J}$,

$$\mathbf{K} = \mathbf{K}' - 2n\mathbf{I},$$
$$\mathbf{a} = \mathbf{a}' - n\boldsymbol{\kappa},$$
$$\mathbf{J} = \mathbf{J}' + n\mathbf{K} + n^2\mathbf{I},$$

their complementary integral is

$$I_4 = I_4(\boldsymbol{\omega} + n\boldsymbol{\gamma}, \boldsymbol{\gamma}),$$

and their general solution on that level is $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t) + n\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$. *It contains the additional arbitrary real parameter* n. *When* $n = 0$, *the generalized solution renders to the original solution.*

There are two conditionally integrable cases known presently in the problem of motion of a body in a liquid. We now demonstrate how the uniform precession transformation works on one of them, namely, the Goryachev–Chaplygin hierarchy of cases. The second hierarchy is based on a conditional subcase of Kowalevski's case and the integrable problem of a body in a liquid found by Chaplygin. The last problem will be treated in more detail later in this chapter.

10.13.1 Generalization of Goryachev–Chaplygin's, Sretensky's and Sokolov–Tsiganov Cases

The first and most famous conditional integrable case of Goryachev and Chaplygin of the classical problem (See Chap. 4 Sect. 4.4) was built in 1900–1901 for a body satisfying the conditions $A = B = 4C$ and $z_0 = 0$. This case was generalized through the addition of a gyroscope along the axis of dynamical symmetry by Sretensky in 1963 [341]. For more details, see Chap. 5 Sect. 5.3. Sokolov and Tsiganov [337] (2002) added two more parameters. In our way of writing, the last case corresponds to the choice

$$V = C[a_1\gamma_1 + a_2\gamma_2 + \frac{1}{2}(c_2\gamma_1 - c_1\gamma_2)^2], \tag{10.136}$$

and

$$\boldsymbol{\mu} = C(c_1\gamma_3, c_2\gamma_3, \kappa + c_1\gamma_1 + c_2\gamma_2). \tag{10.137}$$

This case can be readily generalized by the regular precession transformation to include the parameter n as follows:

$$\begin{aligned} V &= C[a_1\gamma_1 + a_2\gamma_2 - n\kappa\gamma_3 + \frac{1}{2}(c_2\gamma_1 - c_1\gamma_2)^2 \\ &\quad - n\gamma_3(c_1\gamma_1 + c_2\gamma_2) - \frac{n^2}{2}(4\gamma_1^2 + 4\gamma_2^2 + \gamma_3^2)], \\ \boldsymbol{\mu} &= C(c_1\gamma_3 - n\gamma_1, c_2\gamma_3 - n\gamma_2, \kappa + c_1\gamma_1 + c_2\gamma_2 - 7n\gamma_3). \end{aligned} \tag{10.138}$$

The transformation adds several terms to the potential, including linear and quadratic terms, and some linear terms to $\boldsymbol{\mu}$. The complementary integral for the generalized case is

$$\begin{aligned} I_4 = (r - \kappa + c_1\gamma_1 + c_2\gamma_2 + n\gamma_3)[(p + n\gamma_1 + \frac{1}{2}c_1\gamma_3)^2 + (q + n\gamma_2 + \frac{1}{2}c_2\gamma_3)^2] \\ + \gamma_3[(\kappa c_1 - a_1)(p + n\gamma_1) + (\kappa c_2 - a_2)(q + n\gamma_2)] \\ + \frac{1}{2}\gamma_3^2[\kappa(c_1^2 + c_2^2) - c_1a_1 - c_2a_2]. \end{aligned} \tag{10.139}$$

Explicit time solution for this case is not found yet. But to find this solution, it suffices to express solution for the special case $n = 0$, i.e. for the case found by Sokolov and Tsiganov [337].

10.14 Generalizations of Particular Solvable Cases

All the twelve particular solvable cases of the classical problem presented in Chap. 8 can be immediately generalized by the regular precession transformation. All those cases produce new cases in the problem of motion of a rigid body in liquid. The same remark fully applies for all the known solvable cases of the motion of a gyrostat (See Chaps. 13 and 14) and also any particular solution known in the problem of motion of a body in a liquid. The parameter n can be added to all those cases, with all possible implications on the nature of forces acting on the body.

We shall not make a complete list of those generalized cases, but we shall provide some of the most illustrative examples. We first formulate the following theorems:

Theorem 10.4 *Let for some set of parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}', \mathbf{a}', \mathbf{J}'$ *and initial conditions* $\boldsymbol{\omega} = \boldsymbol{\Omega}_0, \ = \Gamma_0,$ *the Eq. (10.105) admit a particular solution* $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(\mathbf{t}), \gamma = \boldsymbol{\Gamma}(\mathbf{t})\}$, *then for the set of values of the parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}, a, J$ *and for the initial conditions* $\boldsymbol{\omega} = \boldsymbol{\Omega}_0 + n\boldsymbol{\Gamma}_0, \gamma = \boldsymbol{\Gamma}_0$ *Eq. (10.54) admit a particular solution* $\{\boldsymbol{\omega} = \boldsymbol{\omega}(t) + n\boldsymbol{\Gamma}(t), \gamma = \boldsymbol{\Gamma}(t)\}$ *containing the additional arbitrary real parameter* $n. When n = 0$, *the generalized particular solution renders to the original particular solution.*

Corollary. *Any motion of a body in a liquid whose angular velocity has the form* $\boldsymbol{\omega} = \omega_1\gamma + \boldsymbol{\omega}_2$, *i.e. involves a component* ω_1 *in the direction of* γ, *can be reduced by using the transformation (10.97) with* $n = -\omega_1$ *to a motion with angular velocity* $\omega = \omega_2$ *and vise versa. In particular,*

1- A uniform (permanent) rotation about a vertical axis can be reduced to a position of equilibrium.

2- A regular precession with a vertical precession axis can be reduced to a permanent rotation about the configuration axis (fixed in the body), which becomes fixed in space.

3- The so-called semi-regular precession (composed of pendulum-like motion of the rigid body about a horizontal axis and a uniform rotation about the vertical) can be reduced to pendulum-like motion, parallel to a fixed plane.

Theorem 10.5 *All properties of the first solution in the last theorem, like stability in the sense of Lyapunov, stability in (or by) the first approximation, instability and periodicity*[3] *are passed to the second solution.*

[3] Here, periodicity relates only to the Euler-Poisson variables $\boldsymbol{\omega}, \boldsymbol{\gamma}$. The motion is periodic relative to the body system of axes. The motion can be periodic in space only under commensurability condition between the periods of the relative and the precessional motions.

The proof follows immediately from the fact that the stability, instability or periodicity of one of the pairs of solutions $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t), \gamma = \boldsymbol{\Gamma}(t)\}$ and $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t) + n\boldsymbol{\Gamma}(t), \gamma = \boldsymbol{\Gamma}(t)\}$ implies the same to the other pair.

This theorem allows a great simplification to the study of properties of motion, as in the last corollary, a permanent rotation reduces to an equilibrium. Also, a regular precession (a periodic motion) reduces to a uniform rotation. Note that the equations of variation for the precession are periodic in time, while those for uniform rotation are of constant coefficients and hence their analysis is much simpler.

We give here only a few illustrative examples to show how the transformation can be used in the direct or in the reverse directions, to generalize a given case or to simplify it. We present results partly in the framework of the problem of motion of a rigid body about a fixed point and partly in the equivalent problem, according to our analogy described earlier in this chapter, of motion of a body in a liquid.

10.14.1 Example 1. Equilibria and Permanent Rotations About a Vertical Axis

It is evident that a position of equilibrium of the body governed by Eq. (10.45) can be transformed by the regular precession transformation. The image for a given finite real n is a permanent rotation about an axis fixed in the body and taking a vertical position. Conversely, a permanent rotation can always be reduced to a relative equilibrium in a coordinate system moving with the same precession speed as the body.

Equilibria. Consider an equilibrium position of the system (10.45). Those are the solutions $\{\boldsymbol{\omega} = \mathbf{0}, \gamma = \gamma_0\}$ where γ_0 satisfies

$$\gamma_0 \times (\mathbf{a} + \gamma_0 \mathbf{J}) = \mathbf{0}. \tag{10.140}$$

1. For the classical problem, when $\mathbf{J} = \mathbf{0}, \mathbf{a} \neq \mathbf{0}$, there are two equilibria: the upper and lower equilibria of the centre of mass above or below the fixed point.

2. When $\mathbf{a} = \mathbf{0}, \mathbf{J} \neq \mathbf{0}$, there are six equilibria, in which one of the principal axes of the matrix $\mathbf{J}$ is directed along or against the vector $\boldsymbol{\gamma}$.

3. In the generic case γ_0 satisfies a relation $\mathbf{a} + \gamma_0 \mathbf{J} = \breve{}\gamma_0$, so that

$$\gamma_0 = -\mathbf{a}(\mathbf{J} - \lambda\boldsymbol{\delta})^{-1}, \tag{10.141}$$

where λ is a root of the sixth-degree equation

$$\left|\mathbf{a}(\mathbf{J} - \lambda\boldsymbol{\delta})^{-1}\right|^2 = 1. \tag{10.142}$$

In this case also we have a maximal number of six positions of equilibrium and minimum number of two.

Permanent rotations. Permanent rotations were discussed in detail in Sect. 10.7. In a coordinate system moving with the same precession speed as the body, the permanent rotation looks like an equilibrium position, which is determined from the Eq. (10.140) but with the transformed parameters $\mathbf{a}'$ and $\mathbf{J}'$, *i.e*.

$$\gamma_0 \times (\mathbf{a}' + \gamma_0 \mathbf{J}') = \mathbf{0}. \tag{10.143}$$

Substituting those parameters from (10.102) into the last relation, we get the condition for a permanent rotation as

$$\gamma_0 \times [\mathbf{a} + n\boldsymbol{\kappa} + \gamma_0(\mathbf{J} - n\mathbf{K} - n^2\mathbf{I})] = \mathbf{0}. \tag{10.144}$$

This equation can now be compared with the condition for the permanent rotation (10.66). They become identical, provided we take $n = -\omega_0$.

10.14.2 Example 2. Permanent Rotations About a Tilted Axis and Precessional Motions About the Vertical

Consider a precessional motion, in which the angular velocity of the body is given by

$$\boldsymbol{\omega} = \Omega_0 \mathbf{e} + \Omega_1 \gamma, \tag{10.145}$$

where Ω_0, Ω_1 are constants and $\mathbf{e}$ is a unit vector fixed in the body at the fixed point O. The body rotates about $\mathbf{e}$ with angular velocity Ω_0, while this axis precesses about the vertical with angular velocity Ω_1. Using the transformation (10.97) with the choice $n = -\Omega_1$, we have

$$\boldsymbol{\omega}' = \Omega_0 \mathbf{e}. \tag{10.146}$$

The precessional motion is reduced by this transformation to a uniform rotation about an axis fixed in the body and in space and inclined to the vertical at a fixed angle. That is the permanent rotational motion described in Sect. 10.8. Similarly, a permanent rotational motion with angular velocity (10.146) can be transformed by the inverse transformation $n = \Omega_1$ to the precessional motion.

Solutions of the equations of motion corresponding to regular precessions were investigated in [123] (See also [125]). The conditions for existence of such precession are obtained in a quite complicated form (conditions (18) in [123]). Those conditions can be easily shown to be equivalent to conditions (10.75) followed by the rotation transformation (10.97). According to the said above, one could consider only uniform rotational motions about an axis fixed in space and inclined to the vertical. The whole class of precessional motions generated by transforming uniform rotations about an inclined axis using (10.97) with the parameter n taking all real values are equivalent to that rotation and, moreover, have the same properties, for example, as concerns stability of the motions.

10.14.3 Example 3. generalization of grioli's precession [402, 405]

On a dynamical basis, Grioli established the possibility of a regular precession of the heavy rigid body about a non-vertical axis under certain conditions on the parameters of the body [138]. Guliaev derived the full explicit solution of this case [141] (see also [256]). We present the necessary details in brief. The solution differs from that of Guliaev only in that we have assigned a certain value for the initial time moment, so that the solution becomes more transparent.

Let the axes be arranged such that $A \geq B \geq C$. For

$$\mathbf{a}' = (a, 0, c), \boldsymbol{\kappa} = \mathbf{0}, \mathbf{K}' = \mathbf{J}' = \mathbf{0}, \tag{10.147}$$

where $a\sqrt{B-C} = c\sqrt{A-B}$, the system of Eq. (10.105) admits a particular solution (See 8.10)

$$\begin{aligned} p' &= \frac{\Omega}{s}(a - c\cos(\Omega t)),\, q' = \Omega\sin(\Omega t),\, r' = \frac{\Omega}{s}(c + a\cos(\Omega t)),\\ \gamma_1 &= -\frac{\Omega^2}{s^2}[Cc\cos(\Omega t) + (B-C)a\sin^2(\Omega t)],\\ \gamma_2 &= \frac{\Omega^2}{s^3}\sin(\Omega t)[(Aa^2 + Cc^2) - (A-C)ac\cos(\Omega t)],\\ \gamma_3 &= \frac{\Omega^2}{s^2}[Aa\cos(\Omega t) + (A-B)c\sin^2(\Omega t)], \end{aligned} \tag{10.148}$$

where $s = \sqrt{a^2 + c^2}$, $\Omega^2 = \frac{s}{\sqrt{(A-B+C)^2 + (A-B)(B-C)}}$. This solution corresponds to a regular precession of the body. The angular velocity $\boldsymbol{\omega}'$ can be written as the sum of two terms

$$\boldsymbol{\omega}' = \Omega\boldsymbol{\zeta} + \Omega\boldsymbol{\alpha}, \tag{10.149}$$

where $\boldsymbol{\zeta}$, $\boldsymbol{\alpha}$ are two unit vectors: the first fixed in the body (orthogonal to a circular section of the inertia ellipsoid) and the second fixed in space [141], so that in the body system

$$\boldsymbol{\zeta} = (\frac{a}{s}, 0, \frac{c}{s}),\, \boldsymbol{\alpha} = (-\frac{c}{s}\cos(\Omega t), \sin(\Omega t), \frac{a}{s}\cos(\Omega t)). \tag{10.150}$$

Note that $\boldsymbol{\zeta}$ is orthogonal to $\boldsymbol{\alpha}$ and that $\boldsymbol{\alpha}$ is inclined to the upward vertical vector $\boldsymbol{\gamma}$ at a fixed angle δ,

$$\cos\delta = \frac{A - B + C}{\sqrt{(A - B + C)^2 + (A - B)(B - C)}}. \tag{10.151}$$

The body rotates with the uniform velocity Ω around the vector $\boldsymbol{\zeta}$ fixed in it, while that vector rotates with the same velocity Ω about the direction $\boldsymbol{\alpha}$ fixed in space.

We now consider another case of motion of the same body as above, but we will replace V', $\boldsymbol{\mu}'$ by

$$\begin{aligned}
&\mathbf{a}' = (a, 0, c), \boldsymbol{\kappa} = \mathbf{0}, \mathbf{K}' = \mathbf{J}' = \mathbf{0},\\
&V = a\gamma_1 + c\gamma_3 - \frac{1}{2}n^2(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\\
&\boldsymbol{\mu} = n((B + C - A)\gamma_1, (C + A - B)\gamma_2, (A + B - C)\gamma_3)
\end{aligned} \tag{10.152}$$

where, for simplicity, n is taken as a constant. It is easy to verify that applying the substitution $\boldsymbol{\omega} = \boldsymbol{\omega}' + n\boldsymbol{\gamma}$ transforms (10.152) into (10.147). Thus, the system with (10.152) admits a particular solution representing Grioli's precession uniformly rotated with speed n about the vertical. In this solution $\gamma_1, \gamma_2, \gamma_3$ are the same as in (10.148), while

$$\begin{aligned}
p &= \frac{\Omega}{s}(a - c\cos(\Omega t)) - \frac{n\Omega^2}{s^2}[Cc\cos(\Omega t) + (B - C)a\sin^2(\Omega t)],\\
q &= \Omega\sin(\Omega t) + \frac{n\Omega^2}{s^3}\sin(\Omega t)[(Aa^2 + Cc^2) - (A - C)ac\cos(\Omega t)],\\
r &= \frac{\Omega}{s}(c + a\cos(\Omega t)) + \frac{n\Omega^2}{s^2}[Aa\cos(\Omega t) + (A - B)c\sin^2(\Omega t)].
\end{aligned} \tag{10.153}$$

This case is a non-trivial generalization of Grioli's result [138]. The whole picture of Grioli's precession about the inclined axis precesses about the vertical at an arbitrary angular speed n. The resulting motion admits two interpretations as a motion of a body in liquid [383] or a motion of a charged body under potential and Lorentz forces as described in Sect. 2.2 above. It is noteworthy that this gives a new result in both interpretations.

The angular velocity $\boldsymbol{\omega} = \Omega(\boldsymbol{\zeta} + \boldsymbol{\alpha}) + n\boldsymbol{\gamma}$ no longer has constant magnitude as was the case in Grioli's precession. The resulting motion is not a regular precession. Although $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are periodic functions of time, the motion is not in general periodic in space for arbitrary values of n. However, if $\frac{n}{\Omega}$ is rational the body returns periodically to its initial position. As far as we know, such motions have not been considered previously. This solution can be generalized by adding a rotor to the body along the normal to a circular cross-section of the ellipsoid of inertia of the body. The solution for the last case was found by Keis [167] and rediscovered by Kharlamova. The resulting case of motion of a heavy gyrostat can be transformed using the regular precession transformation to a case of motion of another body by inertia in a liquid or a case of motion of an electrically charged body in gravity and magnetic fields. Formulas received will generalize (10.152), (10.153).

10.14.4 *Example 4. Regularly Precessing Pendulum*

By this motion, we mean a generalization of the motion of a physical pendulum, such that the axis of the pendulum rotation performs regular precession about the vertical. A near, but different, term "semi-regular precession" was coined by Grioli [139] in certain problems lower in the hierarchy than that of the present chapter. The same name was used later by many authors, e.g. [357]. (See also Gorr [119] and for more detail [126]). This name refers to the motion in which the body rotates with a time-dependent (i.e. non-uniform) angular velocity about an axis fixed in it, while this axis makes regular precession about an axis fixed in space. Thus, the regularly precessing pendulum motion is a type of semi-regular precession, but the last may comprise motions that do not fit in the pendulum type in addition to a regular precession.

Conditions for existence of semi-regular precessions of a rigid body in a liquid involving a pendulum-like motion about an axis fixed in the body and regular precession about the (virtual) vertical were first found in [247] where solutions of the equations of motion were sought such that the angular velocity has the form

$$\dot{\varphi}\mathbf{e} + n\boldsymbol{\gamma}, \tag{10.154}$$

where n is a constant, $\boldsymbol{\gamma}$ is the unit vector along the (virtual) vertical and $\mathbf{e}$ is a unit vector constant in the body. This formula is substituted into the equations of motion, compatibility conditions are found and then a differential equation is obtained for the determination of φ.

Independently, and slightly later, of [247] the same motions were considered more comprehensively in the Ph.D. Thesis [148]. In this work not only conditions for the existence of pendulum motions and their transformed version (the semi-regular precession) are obtained, but also a detailed study was made on the orbital stability of certain special cases of those motions.

In our presentation of the precessing pendulum motion, we shall use the method used in [148]. The study of the motion is made in two steps:

A) The motion is studied in a rotating reference frame in which the body performs a pendulum motion about a horizontal axis fixed in this system. Conditions necessary for performing this motion are found on the transformed parameters of the body $\mathbf{I}$, $\mathbf{a}'$, $\mathbf{J}'$, $\boldsymbol{\kappa}$, $\bar{\mathbf{K}}'$.

B) The motion is transformed back to the inertial frame, the regular precession will be added. The relevant conditions on the original parameters of the body are obtained from

$$\boldsymbol{\omega}' = \boldsymbol{\omega} + n\boldsymbol{\gamma}. \tag{10.155}$$

10.14.4.1 Pendulum Motion

Consider the motion of the body as a physical pendulum, taking place around a principal axis of the inertia matrix of the body-liquid system, while this axis keeps a permanent horizontal position (Fig. 10.2).

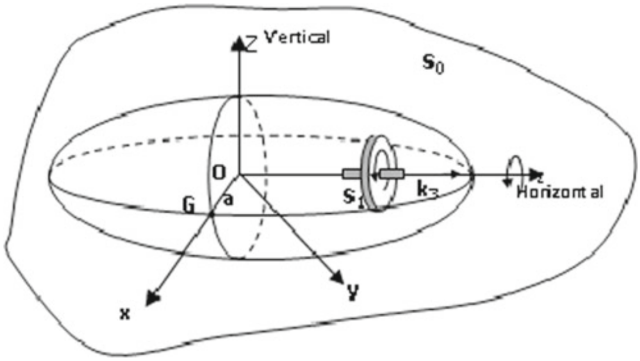


Fig. 10.2 The body configuration of pendulum motion. G the centre of mass, S the rotor with gyrostatic moment k3

Let us choose the third principal axis to be the axis of proper rotation with variable angular velocity $\dot{\varphi}$. The solution corresponding to this motion with $\theta = \frac{\pi}{2}$ is

$$\boldsymbol{\omega} = (p, q, r) = (0, 0, \dot{\varphi}),$$
$$\gamma = (\sin\varphi, \cos\varphi, 0). \tag{10.156}$$

We shall find conditions on the matrices $\bar{\mathbf{K}}$, $\mathbf{J}$ and the vectors $\boldsymbol{\kappa} = (k_1, k_2, k_3)$ and $\mathbf{a} = (a_{1,}a_2, a_3)$ that allow the body to perform pendulum motion. Substituting into Eq. (10.45), the Poisson equations are identically satisfied and the first two dynamical equations give:

$$-\dot{\varphi}[\bar{K}_{12}\sin\varphi - 2\bar{K}_{22}\cos\varphi + k_2] - \cos\varphi[J_{13}\sin\varphi + J_{23}\cos\varphi + a_3] = 0,$$
$$\dot{\varphi}[\bar{K}_{11}\sin\varphi - 2\bar{K}_{12}\cos\varphi + k_1] + \sin(\varphi)[J_{13}\sin\varphi + J_{23}\cos\varphi + a_3] = 0, \tag{10.157}$$

while the third dynamical equation is replaced by the energy integral

$$\frac{1}{2}C\dot{\varphi}^2 + \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma = E, \tag{10.158}$$

where E is the energy constant of the motion.

A combination of (10.157) gives

$$\dot{\varphi}[k_1\sin\varphi - k_2\cos\varphi - (\bar{K}_{11} - \bar{K}_{22})\sin 2\varphi - 2\bar{K}_{12}\cos 2\varphi] = 0,$$

which leads to the conditions

$$k_1 = k_2 = 0,$$
$$\bar{K}_{12} = 0, \bar{K}_{22} = \bar{K}_{11}. \tag{10.159}$$

Now, getting back to (10.157) we obtain one equation

$$\bar{K}_{11}\dot{\varphi} + J_{13}\sin\varphi + J_{23}\cos\varphi + a_3 = 0. \tag{10.160}$$

This equation in φ gives a law of motion contradicting the pendulum law, and hence should be satisfied as an identity. We obtain, in addition to (10.159), the conditions

$$\bar{K}_{11} = 0, a_3 = 0, J_{13} = J_{23} = 0. \tag{10.161}$$

Summing up, for this version one can write the parameters of the problem in the form:

$$\mathbf{a} = (a_1, a_2, 0), \boldsymbol{\kappa} = (0, 0, k_3),$$
$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{12} & J_{22} & 0 \\ 0 & 0 & J_{33} \end{pmatrix}, \tag{10.162}$$
$$\bar{\mathbf{K}} = \begin{pmatrix} 0 & 0 & \bar{K}_{13} \\ 0 & 0 & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33} \end{pmatrix}, \mathbf{K} = \begin{pmatrix} \bar{K}_{33} & 0 & -\bar{K}_{13} \\ 0 & \bar{K}_{33} & -\bar{K}_{23} \\ -\bar{K}_{13} & -\bar{K}_{23} & 0 \end{pmatrix}. \tag{10.163}$$

Those conditions mean that the centre of mass of the body lies in the xy-plane, perpendicular to the axis of pendulum rotation (the z-axis), which is a principal axis also of the matrix $\mathbf{J}$. The angle of proper rotation φ can be determined as an elliptic function of time by inverting the integral, obtained by separating variables in (10.158),

$$t = \int \frac{d\varphi}{\sqrt{2(E - a_1\sin\varphi - a_2\cos\varphi - J_{12}\sin\varphi\cos\varphi) - J_{11}\sin^2\varphi - J_{22}\cos^2\varphi}}. \tag{10.164}$$

This formula contains the energy constant E, which takes all real values on the interval $[V_-, \infty)$, V_- being the minimum of the potential V on the Poisson sphere. Pendulum motions constitute a family of periodic motions of two types: vibrational motions reversing their direction every half-period time and complete rotational motions going on in one direction all the time.

10.14.4.2 The Precessing Pendulum

Conditions (10.163) for existence of pendulum-like motion of the body in a liquid (or in the equivalent generalized problem) can now be generalized to generate conditions for the semi-regular precession. One can now transform the pendulum-like motion about its axis fixed in space to add the parameter n to the solution. Every pendulum-like motion generates a family of semi-regular precessions, with n taking all real values. The parameter n enters in the transformed conditions (10.154) according to the transformation formulas (10.102). Finally, we can write the parameters of the body, in order that the body can perform a semi-regular motion composed of a pendulum motion and a precession with angular velocity n :

$$\mathbf{a}' = (a_1, a_2, nk_3), \boldsymbol{\kappa} = (0, 0, k_3),$$
$$\bar{\mathbf{K}}' = \begin{pmatrix} 2n(B+C-A) & 0 & \bar{K}_{13} \\ 0 & 2n(C+A-B) & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33} + 2n(A+B-C) \end{pmatrix}$$
$$\mathbf{J}' = \begin{pmatrix} J_{11} - n^2 A - n\bar{K}_{33} & J_{12} & n\bar{K}_{13} \\ J_{12} & J_{22} - n^2 B - n\bar{K}_{33} & n\bar{K}_{23} \\ n\bar{K}_{13} & n\bar{K}_{23} & J_{33} - n^2 C \end{pmatrix}. \tag{10.165}$$

In the last formula, we used the relation $\mathbf{K} = \mathrm{tr}(\bar{\mathbf{K}}) - \bar{\mathbf{K}}$ to obtain the transformed parameters from (10.102).

Conditions (10.165) are well-ordered and much transparent than conditions in [247], where conditions for the semi-regular precession are given just as relations between the parameters.

10.14.4.3 The Space Picture of the Motion

The integral (10.164) is elliptic and can be evaluated using formulas from [130]. When $J_{12} = 0$ and $a_2 = 0$, i.e. when the vector $\mathbf{a}$ (the centre of mass) lies on a principal axis of inertia and $\mathbf{J}$, $\mathbf{I}$ have common principal axes, the integral becomes simpler and $\gamma_1 = \sin\varphi$ can be determined in terms of Jacobian elliptic function in time. This was done in [148], where also the translational motion was studied. It turned out that the central point can draw several types of trajectories in space.

In the special case when the parameters of the body satisfy the conditions

$$A = B,$$
$$J_{11} = J_{22} = \varepsilon A, J_{33} = \varepsilon C,$$
$$K_{13} = K_{23} = 0, K_{33} = -nC,$$

one finds

$$\gamma_1 = -1 + 2\,\mathrm{sn}^2\, v, \gamma_2 = 2\,\mathrm{sn}\, v\,\mathrm{cn}\, v,$$
$$p = -n\gamma_1, q = -n\gamma_2, r = \frac{2}{k}\sqrt{\frac{a_1}{C}}\,\mathrm{dn}\, v, \tag{10.166}$$

where

$$v = \sqrt{\frac{a_1}{C}}\frac{t}{k} \tag{10.167}$$

and the modulus of the elliptic functions

$$k = \sqrt{\frac{4a_1}{2h + 2a_1 - A(\varepsilon + n^2)}}. \tag{10.168}$$

Note that we have chosen the case of fast pendulum (in which the pendulum rotates with variable angular velocity in one direction).

From (10.166), we also get

$$\dot{\psi} = \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2} = -n,$$

so that we can write

$$\psi = -nt,$$

and thus we arrive at the following expressions for the base vectors in space

$$\begin{aligned} \boldsymbol{\alpha} &= (\gamma_2 \cos nt, -\gamma_1 \cos nt, -\sin nt), \\ \boldsymbol{\beta} &= (-\gamma_2 \sin nt, \gamma_1 \sin nt, -\cos nt). \end{aligned} \tag{10.169}$$

The velocity of the central point of the body can now be written from (10.60) as

$$\mathbf{u} = \mathbf{a} + \gamma \mathbf{J} - \frac{1}{2}\omega\, \mathbf{K}.$$

The space components of the velocity with respect to some inertial system of axes $\xi\eta\zeta$ are

$$\dot{\xi} = \mathbf{u} \cdot \boldsymbol{\alpha}, \dot{\eta} = \mathbf{u} \cdot \boldsymbol{\beta}, \dot{\zeta} = \mathbf{u} \cdot \boldsymbol{\gamma},$$

and can now be evaluated:

$$\begin{aligned} \frac{d\xi}{dt} &= F(t) \sin nt + 2a_1 \operatorname{sn} v \operatorname{cn} v \cos nt \\ \frac{d\eta}{dt} &= F(t) \cos nt - 2a_1 \operatorname{sn} v \operatorname{cn} v \sin nt \\ \frac{d\zeta}{dt} &= (\varepsilon + n^2)A - a_1 + nC + 2a_1 \operatorname{sn}^2 v, \end{aligned} \tag{10.170}$$

where

$$F(t) = n(k_3 + \frac{2}{k}\sqrt{Ca_1} \operatorname{dn} v).$$

By integrating (10.170) with respect to time, we obtain

$$\xi = -C(\frac{k_3}{C} + \frac{2}{k}\sqrt{\frac{a_1}{C}}\,\mathrm{dn}(\sqrt{\frac{a_1}{C}}\frac{t}{k}))\cos nt,$$
$$\eta = C(\frac{k_3}{C} + \frac{2}{k}\sqrt{\frac{a_1}{C}}\,\mathrm{dn}(\sqrt{\frac{a_1}{C}}\frac{t}{k}))\sin nt,$$
$$\zeta = [(\varepsilon + n^2)A - a_1 + nC + \frac{2a_1}{k^2}(1 - \frac{E(k)}{K(k)})]t - \frac{2}{k}\sqrt{\frac{a_1}{C}}Z(\sqrt{\frac{a_1}{C}}\frac{t}{k}) \quad (10.171)$$

where $K(k)$, $E(k)$ are complete elliptic integrals and Z is Jacobi's Zeta function of the same modulus k.

The functions dn, Z have period

$$T_1 = 2kK(k)\sqrt{\frac{C}{a_1}}, \quad (10.172)$$

while the trigonometric terms have period

$$T_2 = \frac{2\pi}{n}. \quad (10.173)$$

The position vector of the central point of the body P (say) is not periodic in time in general, but its projection on the $\xi\eta$-plane can be a closed curve if the ratio $\frac{T_1}{T_2}$ is a rational number.

Now we are ready to describe the space picture of the motion of the body. The body performs the periodic pendulum motion about its horizontal z-axis while this axis precesses with a uniform angular speed n in the (virtual) horizontal plane. The motion of the central point P of the body traces a space curve of helicoidal type about a (virtual) vertical axis. The radial distance ρ of P from the ζ-axis of the curve

$$\rho = k_3 + \frac{2}{k}\sqrt{Ca_1}\,\mathrm{dn}\,v \quad (10.174)$$

changes periodically, while rotating about the vertical ζ-axis with the same angular speed n of the body about the vertical Z-axis. As to its horizontal motion, the body moves around the ζ-axis and rotates in such a way that one face of the body is always directed to that axis. In celestial mechanics, this regime of motion is called $1-1$ rotation.

From the expression (10.174), we note that the effect of the gyrostatic moment k_3 appears in the motion of the central point as an increase (decrease) of the radial distance between that point and the ζ-axis. This means widening or narrowing the lateral dimensions of the helicoidal trajectory, according to the sign of k_3.

The motion of the central point in the ζ-direction is not periodic in general, due to the presence of a secular term (linear in t). After each (orbital) revolution about the ζ-axis the central point elevates (or descends) a certain distance above (or below) the horizontal plane that passed through P at the initial moment $t = 0$. The space path of P is helicoidal-like.

However, if the coefficient of t in the secular term vanishes, i.e. if

$$(\varepsilon + n^2)A - a_1 + nC + \frac{2a_1}{k^2}(1 - \frac{E(k)}{K(k)}) = 0, \tag{10.175}$$

then the vertical motion of P is periodic with period T_1. If, moreover, T_1 and T_2 are commensurable, say, $\frac{T_1}{N_1} = \frac{T_2}{N_2}$, then the space trajectory of P closes after a number N of revolutions $N = \text{LCM}(N_1, N_2)$ (The least common multiple of the two numbers).

The following figures illustrate the shapes of some space trajectories of the central point. In all of them we suppose that (10.175) is satisfied and set $C = 1$ and substitute$\sqrt{a_1} = 2kK(k)/T1$, so that the equation of the space curve becomes

$$\begin{aligned} \xi &= -[k_3 + \frac{2}{k}\sqrt{a_1}\,\text{dn}(2K(k)\frac{t}{T_1})]\cos(2\pi\frac{t}{T_2}), \\ \eta &= [k_3 + \frac{2}{k}\sqrt{a_1}\,\text{dn}(2K(k)\frac{t}{T_1})]\sin(2\pi\frac{t}{T_2}), \\ \zeta &= -\frac{2}{k}\sqrt{a_1}Z(2K(k)\frac{t}{T_1}). \end{aligned} \tag{10.176}$$

The following Fig. 10.3 shows the space orbit of the central point of the body for values of the parameters:

$$k_3 = 1, k = 0.9, n = 1, a_1 = k^2 = 0.81.$$

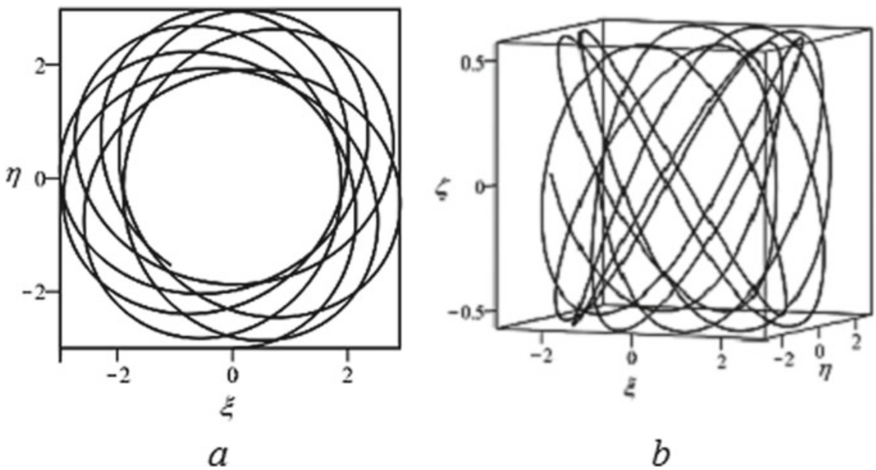


Fig. 10.3 Space trajectory of pendulum. **a** An upper view **b** A side view

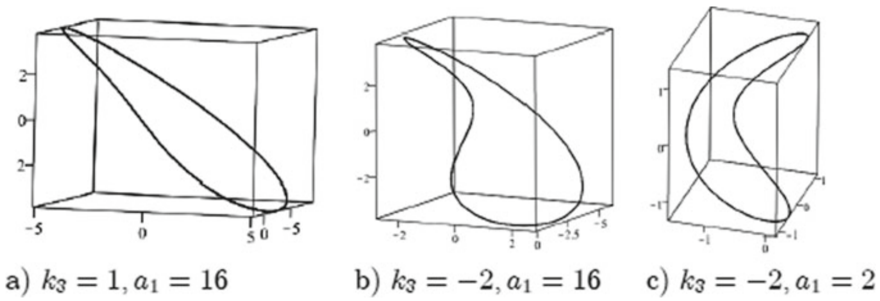


Fig. 10.4 $T_1 = T_2$

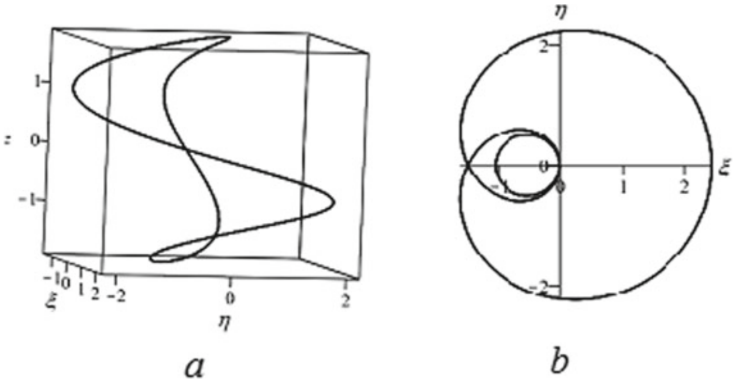


Fig. 10.5 $k_3 = -3, a_1 = 4, T_1 = 2T_2(N_1 = 2, N_2 = 1)$. **a** Side view. **b** Projection on $\xi\eta$-plane

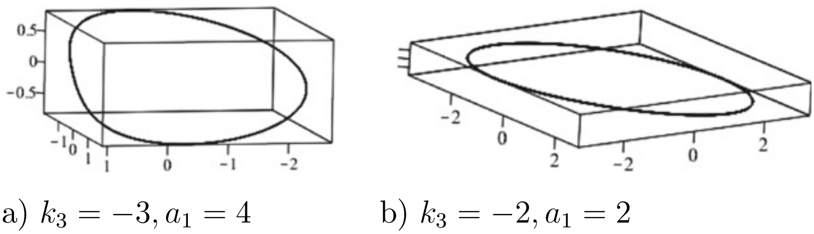


Fig. 10.6 Side view for different values of k_3 and a_1

The shapes of some closed space curves are shown for different values of the parameters k_3, k and the integers N_1 and N_2. They show the diversity of the forms of trajectories, even for a very limited set of initial conditions (Figs. 10.4, 10.5, 10.6, 10.7, 10.8).

Fast pendulum rotations $k = 0.99$,

Slower rotation $k = 0.5$ $N_1 = 1, N_2 = 1$

$N_1 = 2, N_2 = 1$

The case of a simple body $k_3 = 0$:

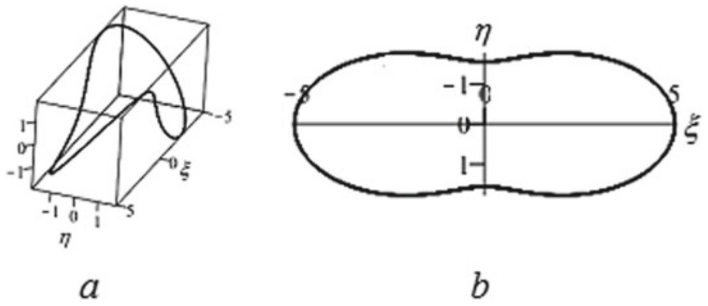


Fig. 10.7 $k_3 = -3, a_1 = 4, T_2 = 2T_1(N_1 = 1, N_2 = 2)$. **a** side view **b** Projection on $\xi\eta$-plane

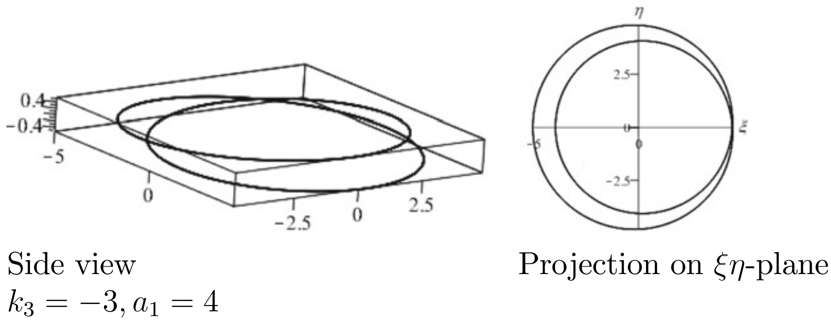


Fig. 10.8 $k_3 = -3, a_1 = 4$ **a** Side view **b** Projection on $\xi\eta$-plane

It has been noticed before that the presence of the gyrostatic momentum k_3 affects only the lateral dimensions of the trajectory of the body. We now examine some periodic motions of the body in the absence of k_3. To keep the variation of the radial distance ρ somewhat large, we give the modulus of elliptic functions k the value 0.99. The following figures are obtained by taking $a_1 = 4$. The values of N_1 and N_2 are given for each figure. The motion of the body consists of

- Complete rotations of the body as a physical pendulum, with period of rotation T_1, about its z-axis, which is always horizontal (orthogonal to the virtual vertical γ) and directed to the ζ-axis.
- A 1-1 regime of rotation about the ζ-axis (One side of the body always faces that axis) of periodic time $T_2 = \frac{2\pi}{n}$.
- Radial displacement of the central point from the ζ-axis of periodic time T_1 (The same as the periodic time of the pendulum).
- Oscillations of the central point in the direction of the ζ-axis (the virtual vertical) of periodic time T_1.

Figure 10.9 depicts the case of equal T_1 and T_2. The space trajectory of the central point of the body closes after a single rotation about the ζ-axis. The central point ascends from the lowest point to the highest point on the part of the trajectory near to the ζ-axis and then descends on the farther part to the first point. At the same time,

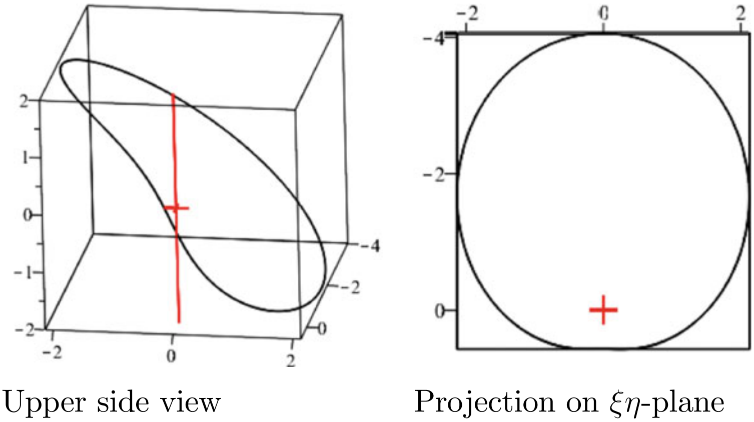


Fig. 10.9 $T_1 = T_2$

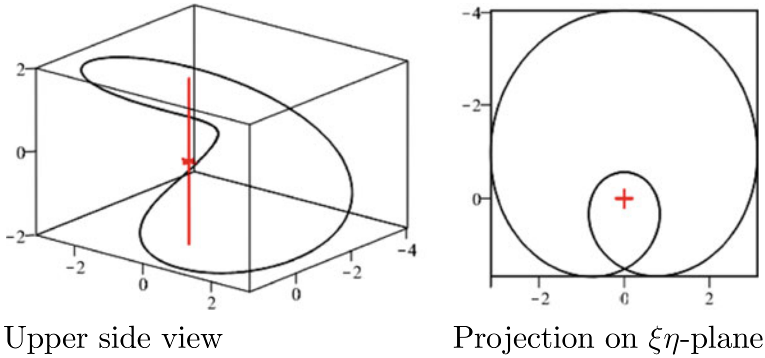


Fig. 10.10 $T_1 = 2T_2$

the body completes a pendulum rotation cycle in its vertical plane which rotates, in turn, so that the axis of the pendulum motion remains all the time directed to the ζ-axis. In Fig. 10.10, at the time of a complete pendulum rotation cycle the body completes two rotations about the ζ-axis, giving always the same face to that axis. The central point of the body ascends along the narrower loop and descends along the wider one.

In Fig. 10.11, each pendulum rotation cycle corresponds to one vertical oscillation but corresponds to ten precession cycles associated with ten loops around the ζ-axis forming a ten-loop solenoid. The central point of the body ascends along the narrow part of the solenoid and descends along the wider part to the lowest point. Figure 10.12 shows the reverse case $T_2 = 10T_1$. The time of one complete precession cycle of the body and rotation of its central point is enough for ten cycles of the vertical and lateral vibrations of the central point and ten complete cycles of the pendulum vibration.

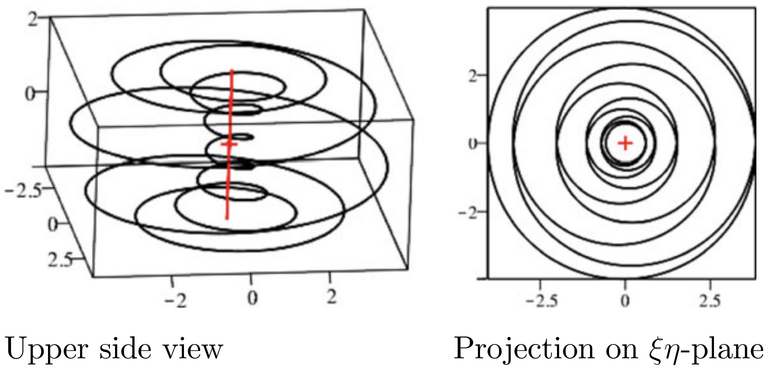


Fig. 10.11 $T_1 = 10T_2$

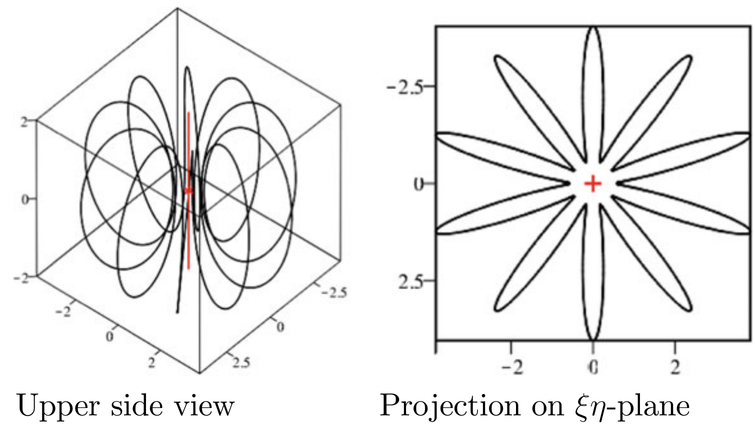


Fig. 10.12 $T_2 = 10T_1$

Remark: It is now time to repeat our previous assertion, that all regular pendulum precessions generated by a certain pendulum motion share all qualitative properties with that motion. For example, the condition for the (orbital) stability of the horizontal axis of semi-regular precession is the same as the condition of stability of the pendulum-like motion generating it, for $n = 0$. Conditions for stability of precessional motion are obtained from the former conditions by replacing the original parameters by the primed ones, which involve the precession parameter n. Some information about orbital stability of pendulum-like motions in the two problems of this chapter will be included in the exercises.

10.15 Tables of Integrable Cases of Motion of a Rigid Body in a Liquid

In this section, we provide tables of general and conditional integrable cases of motion of a rigid body by inertia in an ideal incompressible liquid, infinitely extending in all directions and resting at infinity. results are displayed for the case of a body bounded by a multi-connected surface, i.e. for a perforated body. This case is characterized by the presence of the two vectors **a** and $\boldsymbol{\kappa}$ in our equations in the framework of the equivalent problem of motion about a fixed point of a rigid body acted upon by potential and gyroscopic forces. To follow the same pattern as in previous and coming chapters, we have chosen to put the problem of the present chapter in the context of the second problem. In the tables, all integrable cases of the problem of motion of a body in a liquid are presented in their most general form, as cases of integrability of our new Eq. 10.45 and in terms of the parameter matrices and vectors **I**, **J**, **K**,$\boldsymbol{\kappa}$ and **a**. To express the integrability conditions and the integrals of motion in terms of the original (Kirchhoff-Lamb) parameters, one should use (10.43) or (10.30) as needed.

As to the classification of cases in each table, we organized cases according to the degree of the complementary integral as a function in the components of the angular velocity.

For each case, we provide

(1) The full hierarchy of earlier cases, to which the given case reduces under relevant conditions on the parameters.
(2) The conditions on the parameters on the body and fields, under which the case is valid.
(3) The potential function V.
(4) The vector functions $\boldsymbol{l}$ and $\boldsymbol{\mu}$, which describe the gyroscopic forces acting on the body: The first enters in the Lagrangian and the second in the equations of motion.
(5) The explicit forms of the first integrals I_3 and I_4 in the Euler-Poisson variables.
(6) A Hamiltonian function H is given for each case, together with the corresponding form of the complementary integral in terms of the variables ($\mathbf{M}$,γ) (See Sect. 10.10).

Remark 16 Regarding the fact that most integrable cases are obtained by using inverse method, different hamiltonians may be constructed for one and the same case.

10.15.1 General Integrable Cases

The number of known general integrable cases in the two equivalent problems of the present chapter is seven. Most of them are solutions of Thomson-Lamb equations. They developed historically from solutions of the simpler cases of integrability of the Kirchhoff equations and, in cases, from cases of integrability of problems lower in the hierarchy, which are presented in the previous chapters.

Remark: The regular precession transformation parameter n figures in five of the seven integrable cases, namely, cases 2,3,5,6 and 7. If n is introduced in cases 1 and 4, it can be absorbed in other parameters of the problem and can give no new effects. In cases 2,3 and 5, the parameter n appeared at some stage in their development and not necessarily from the first discovery of the case. In the remaining cases (number 6,7), the introduction of that parameter in [411] was a significant generalization of the case found by Sokolov [336].

Table 10.1 General integrable cases

1	The case of axi-symmetric body Generalization of Lagrange's case Kirchhoff [219] (1870) (see also [220]) $a_3 = 0$, $\kappa_3 = 0$
	$A = B,$ $V = a_3\gamma_3 + \frac{1}{2}[b_1(\gamma_1^2 + \gamma_2^2) + b_3\gamma_3^2],$ $\mathbf{l} = (K_1\gamma_1, K_1\gamma_2, K_3\gamma_3 + \kappa),$ $\boldsymbol{\mu} = (-K_3\gamma_1, -K_3\gamma_2, (K_3 - 2K_1)\gamma_3 + \kappa),$ $I_3 = A(p\gamma_1 + q\gamma_2) + (Cr + \kappa)\gamma_3 + K_1(\gamma_1^2 + \gamma_2^2) + K_3\gamma_3^2),$ $I_4 = Cr + \kappa + K_3\gamma_3$
	$H = \frac{M_1^2+M_2^2}{2A} + \frac{M_3^2}{2C} - \frac{K_1}{A}(M_1\gamma_1 + M_2\gamma_2) - \frac{M_3}{C}(K_3\gamma_3 + \kappa)$ $+a_3^*\gamma_3 + b_1^*(\gamma_1^2 + \gamma_2^2) + b_3^*\gamma_3^2,$
	$I_4 = M_3$

where a_3^*, b_1^* and b_3^* are constants

2	Clebsch [55] (1870). Euler [79] (1758). $n = b = 0$
	$V = (b - \frac{1}{2}n^2)(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$ $\mathbf{l} = n(A\gamma_1, B\gamma_2, C\gamma_3)$ $\boldsymbol{\mu} = n((A - B - C)\gamma_1, (B - C - A)\gamma_2, (C - A - B)\gamma_3)$ $I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 + n(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),$ $I_4 = \frac{1}{2}[A^2(p + n\gamma_1)^2 + B^2(q + n\gamma_2)^2 + C^2(r + n\gamma_3)^2]$ $-b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2).$
	$H = \frac{1}{2}(M_1^2/A + M_2^2/B + M_3^2/C) + b(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$ $-n(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3),$
	$I_4 = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) - b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2).$

Somewhat later, after [55], the special version ($n = 0$) of this case case was found, apparently independently, by Tisserand [354] (1891) and Brun [47] (1893) in the context of the motion of a body acted upon by approximate Newtonian gravitational forces (See Chap. 6). At that time the Steklov analogy, described above in this chapter, between the two problems was still unknown.

	3	Clebsch [55] (1870) -
		$A = B = C$ $V = \frac{1}{2}A(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2)$ $\mathbf{l} = nA\boldsymbol{\gamma}, \boldsymbol{\mu} = -nA\boldsymbol{\gamma}$ $I_3 = A(p\gamma_1 + q\gamma_2 + r\gamma_3),$ $I_4 = A[c_1(p + n\gamma_1)^2 + c_2(q +$ $n\gamma_2)^2 + c_3(r + n\gamma_3)^2$ $-(c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2)]$
		$H = \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2)$ $-n\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,,$
		$I_4 = c_1M_1^2 + c_2M_2^2 + c_3M_3^2$ $-A[c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2]$

In cases 2 and 3, the parameter n invokes gyroscopic terms due to circulation of the liquid through perforations. Setting $n = 0$ makes the body simply connected.

	4	Rubanovsky [317] (1968), Kharlamov ($\kappa_2 = 0$) [192] (1963), Steklov ($\boldsymbol{\kappa}= \mathbf{0}$) [344] (1893), Joukowsky ($a = 0$) [163] (1885), Euler ($\kappa= \mathbf{0}$, $a = 0$) [79] (175	
		$\mathbf{I} = \text{diag}(A, B, C), \bar{\mathbf{I}} = \frac{1}{2}tr(\mathbf{I}) - \mathbf{I},$ $\mathbf{J} = [tr(\mathbf{I}^{-1})\boldsymbol{\delta}-\mathbf{I}^{-1}]$ $= \text{diag}(\frac{1}{B} + \frac{1}{C}, \frac{1}{A} + \frac{1}{C}, \frac{1}{A} + \frac{1}{B})$ $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3),$ $V = -n(\boldsymbol{\kappa} \cdot \boldsymbol{\gamma} - a\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot \boldsymbol{\gamma}) - \frac{1}{2}n^2\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma},$ $\boldsymbol{l} = \boldsymbol{\kappa} + a\boldsymbol{\gamma}\mathbf{J} + n\boldsymbol{\gamma}\mathbf{I},$ $\boldsymbol{\mu} = \boldsymbol{\kappa} + 2a\boldsymbol{\gamma}\mathbf{I}^{-1} - 2n\boldsymbol{\gamma}\bar{\mathbf{I}},$ $I_3 = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l}) \cdot ,$ $I_4 = \frac{1}{2} \mid (\boldsymbol{\omega} + n\boldsymbol{\gamma})\mathbf{I} + \boldsymbol{\kappa} \mid^2 - 2a\boldsymbol{\omega} \cdot \boldsymbol{\gamma}$	
		$H = \frac{1}{2}(\mathbf{M} - \check{} - a\boldsymbol{\gamma}\mathbf{J})\mathbf{I}^{-1} \cdot (\mathbf{M} - \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J})$ $-n\mathbf{M} \cdot ,$	
		$I_4 = \frac{1}{2} \mid \mathbf{M} - a\boldsymbol{\gamma}\mathbf{J} \mid^2 - a\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot (\mathbf{M} - \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J}).$	

5	Rubanovsky [317] (1968) Lyapunov [267] ($a_1 = a_2 = a_3 = 0$) (1893)
	$A = B = C$, $V = C\{a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3$ $\quad -\frac{1}{2}[(bc + b_0)\gamma_1^2 + (ca + b_0)\gamma_2^2 + (ab + b_0)\gamma_3^2]\}$, $\mathbf{l} = -\frac{1}{2}C((b + c)\gamma_1, (c + a)\gamma_2, (a + b)\gamma_3)$, $\boldsymbol{\mu} = C(a\gamma_1, b\gamma_2, c\gamma_3)$, $I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 + \frac{1}{2}(a\gamma_1^2 + b\gamma_2^2 + c\gamma_3^2)$, $I_4 = \frac{1}{2}[(b + c)\, p^2 + (c + a)\, q^2 + (a + b)\, r^2] - abc(p\frac{\gamma_1}{a} + q\frac{\gamma_2}{b} + r\frac{\gamma_3}{c})$ $\quad + a_1(p + a\gamma_1) + a_2(q + b\gamma_2) + a_3(r + c\gamma_3)$.
	$H = \frac{1}{2C}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}[(b + c)M_1\gamma_1 + (c + a)M_2\gamma_2 + (a + b)M_3\gamma_3]$ $\quad + C(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3)$ $\quad - \frac{C}{8}[(a^2 + 2bc)\gamma_1^2 + (b^2 + 2ac)\gamma_2^2 + (c^2 + 2ab)\gamma_3^2]$,
	$I_4 = (b + c)M_1^2 + (c + a)M_2^2 + (a + b)M_3^2$ $\quad + C\{[(b^2 + c^2)\gamma_1 + 2a_1]M_1 + [(a^2 + c^2)\gamma_2 + 2a_2]M_2 + [(a^2 + b^2)\gamma_3 + 2a_3]M_3\}$ $\quad + \frac{C^2}{4}[(b + c)(b - c)^2\gamma_1^2 + (c + a)(c - a)^2\gamma_2^2 + (a + b)(a - b)^2\gamma_3^2]$ $\quad + C^2[(a + b + c)(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3) + 2(a_1 a\gamma_1 + a_2 b\gamma_2 + a_3 c\gamma_3)]$.

The parameter b_0 has meaning in the problem of motion of a body in a liquid. In the alternative problem it is immaterial.

6	Yehia [411] (2003) Sokolov [336] $n = 0$ (2002) Yehia [380] $n = c = 0$ (1986) Kowalevski [238] $n = c = \kappa = 0$ (1889)
	$A = B = 2C$, $\boldsymbol{\kappa} = C(0, 0, \kappa)$, $V = C[\kappa c\gamma_1 + a_2\gamma_2 - n\kappa\gamma_3 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2) - nc\gamma_1\gamma_3$ $\quad - \frac{n^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)]$, $\mathbf{l} = C(2c\gamma_3 + 2n\gamma_1, 2n\gamma_2, \kappa - c\gamma_1 + n\gamma_3)$, $\boldsymbol{\mu} = C(c\gamma_3 - n\gamma_1, -n\gamma_2, \kappa + c\gamma_1 - 3n\gamma_3)$,
	$I_3 = 2(p\gamma_1 + q\gamma_2) + (r + \kappa + c\gamma_1)\gamma_3 + n(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)$,
	$I_4 = \left[(p + n\gamma_1)^2 - (q + n\gamma_2)^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r + n\gamma_3 - \kappa)\right]^2$ $\quad + \left[2(p + n\gamma_1)(q + n\gamma_2) - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r + n\gamma_3 - \kappa)\right]^2$ $\quad + 2\kappa(r + n\gamma_3 - \kappa + c\gamma_1)\left[(p + n\gamma_1)^2 + (q + n\gamma_2)^2 + 2c(p + n\gamma_1)\gamma_3\right]$ $\quad - 2\kappa c^2\{2\gamma_3[2(p + n\gamma_1)\gamma_1 + c\gamma_1\gamma_3 + 2(q + n\gamma_2)\gamma_2 + (r + n\gamma_3)\gamma_3]$ $\quad + \kappa\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + n\gamma_3 + c\gamma_1)\} - 4a_2\kappa(q + n\gamma_2)\gamma_3$.
	$H = \frac{1}{2C}(\frac{M_1^2}{2} + \frac{M_2^2}{2} + M_3^2) - c\gamma_3 M_1 + (c\gamma_1 - \kappa)M_3 + Ca_2\gamma_2$ $\quad - n(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$
	$I_4 = [\frac{M_1^2 - M_2^2}{4} + Cc\,(-M_1\, z + M_3\, x) + \left(a_2\, y + c^2\right) C^2]^2$ $\quad + [\frac{M_1 M_2}{2} + Cc\,(-M_2\, z + M_3 y) - C^2 a_2\, x]^2 + \frac{1}{2}Ck\,(M_3 - 2Ck)\left(M_1^2 + M_2^2\right)$ $\quad + C^2 k[-2Ca_2 M_2\; z - 2M_2\; (M_1\, y - M_2\, x)\, c - 2Cc^2 M_3]$

	7	The parameter n is added here to B-M-S result. Borisov, Mamaev and Sokolov $n=s=0$ [39] (2001) Sokolov [336] $n=0$ (2001) Kowalevski $n=m=0$ (1888)	
		$A=B=2C$, $V=C\{s(c_1\gamma_1+c_2\gamma_2)+\frac{1}{2}m^2[(c_1\gamma_1+c_2\gamma_2)^2-(c_1^2+c_2^2)\gamma_3^2]$ $+nm\gamma_3(c_2\gamma_1-c_1\gamma_2)\gamma_3-\frac{n^2}{2}(2\gamma_1^2+2\gamma_2^2+\gamma_3^2)\}$, $\mathbf{l}=C(2n\gamma_1,2n\gamma_2,m(c_2\gamma_1-c_1\gamma_2)+n\gamma_3)$, $\boldsymbol{\mu}=C(-mc_2\gamma_3-n\gamma_1,-mc_1\gamma_3-n\gamma_2,m(c_2\gamma_1-c_1\gamma_2)-3n\gamma_3)$,	
		$I_3=2(p\gamma_1+q\gamma_2)+[r+m(c_2\gamma_1-c_1\gamma_2)]\gamma_3+n(2\gamma_1^2+2\gamma_2^2+\gamma_3^2)$,	
		$I_4=A_4^2+B_4^2$, where $A_4=(p+n\gamma_1)^2-(q+n\gamma_2)^2+s(c_2\gamma_2-c_1\gamma_1)$ $+m(r+n\gamma_3)(c_2\gamma_1+c_1\gamma_2)+m^2(c_2^2\gamma_2^2-c_1^2\gamma_1^2)$, $B_4=2(p+n\gamma_1)(q+n\gamma_2)-s(c_1\gamma_2+c_2\gamma_1)$ $-m(r+n\gamma_3)(c_1\gamma_1-c_2\gamma_2)-m^2(c_1\gamma_2+c_2\gamma_1)(c_1\gamma_1+c_2\gamma_2)]^2$.	
		$H=\frac{1}{2C}[\frac{M_1^2+M_2^2}{2}+M_3^2]+m(c_1\gamma_2-c_2\gamma_1)M_3+Cs(c_1\gamma_1+c_2\gamma_2)$ $-Cm^2(c_1^2+c_2^2)\gamma_3^2-n\,(M_1\,\gamma_1+M_2\,\gamma_2+M_3\,\gamma_3)$,	
		$I_4=[\frac{M_1^2-M_2^2}{4C^2}+\frac{m(c_1\gamma_2+c_2\gamma_1)}{C}M_3+s(-c_1\gamma_1+c_2\gamma_2)-m^2(c_1^2+c_2^2)(\gamma_1^2-\gamma_2^2)]^2$ $+[\frac{M_1M_2}{2C^2}-\frac{m(c_1\gamma_1-c_2\gamma_2)}{C}M_3-s(c_1\gamma_2+c_2\gamma_1)-2m^2(c_1^2+c_2^2)\gamma_1\gamma_2]^2$.	

Strictly speaking, case 7 is related to case 6 and can be considered as its special case. We prefer, for future uses (See Chap. 12), to consider case 7 as independent case in its most general form containing maximum number of parameters.

In case 7, the integral I_4 is the sum of two squares. It is the squared modulus of the complex quantity

$$A_4+iB_4=[p+iq+n(\gamma_1+i\gamma_2)]^2 \\ -(c_1+ic_2)(\gamma_1+i\gamma_2)[s+im(r+n\gamma_3)+m^2(c_1\gamma_1+c_2\gamma_2)]. \tag{10.177}$$

The quantities A_4, B_4 satisfy the relations

$$\dot{A}_4=[r+n\gamma_3+m(c_1\gamma_2-c_2\gamma_1)]B_4,\ \dot{B}_4=-[r+n\gamma_3+m(c_1\gamma_2-c_2\gamma_1)]A_4, \tag{10.178}$$

so that the set of conditions

$$\{A_4=0,\ B_4=0\}$$

define an invariant manifold.

When $s=0$, case 7 renders to the case discussed in Sect. 10.10 and the integral becomes expressible, as in (10.91), in the form of the product of two functions one linear and the other cubic in velocities.

10.15.2 Conditional Integrable Cases on the Level $f = 0$

Two conditional integrable cases are known. Those cases can be generalized, as will be shown later, using an arbitrary function $\nu(\boldsymbol{\gamma})$ instead of n. Nevertheless, we write them down here with that parameter, as it adds physically significant terms to both problems considered in this chapter: the problem of motion of a body in a liquid and the alternative problem of motion under the action of potential and gyroscopic forces.

Table 10.2 Cases integrable on the level

1	Parameter n is added here to the result of Sokolov and Tsiganov $n = 0$ Sokolov-Tsiganov [338], 2002, $n = c_1 = c_2 = 0$ Sretensky [341], 1963, $n = c_1 = c_2 = \kappa = 0$ Goryachev-Chaplygin
	$A = B = 4C$, $V = C[a_1\gamma_1 + a_2\gamma_2 - n\kappa\gamma_3 + \frac{1}{2}(c_2\gamma_1 - c_1\gamma_2)^2$ $\quad -n\gamma_3(c_1\gamma_1 + c_2\gamma_2) - \frac{n^2}{2}(4\gamma_1^2 + 4\gamma_2^2 + \gamma_3^2)]$, $\mathbf{l} = C(\frac{c_1}{2}\gamma_3 + 4n\gamma_1, \frac{c_2}{2}\gamma_3 + 4n\gamma_2, \frac{1}{2}(c_1\gamma_1 + c_2\gamma_2) + n\gamma_3)$, $\boldsymbol{\mu} = C(c_1\gamma_3 - n\gamma_1, c_2\gamma_3 - n\gamma_2, \kappa + c_1\gamma_1 + c_2\gamma_2 - 7n\gamma_3)$, $I_3 = 4p\gamma_1 + 4q\gamma_2 + [r + \kappa + c_1\gamma_1 + c_2\gamma_2]\gamma_3 + n(4\gamma_1^2 + 4\gamma_2^2 + \gamma_3^2)$, $I_4 = (r - \kappa + c_1\gamma_1 + c_2\gamma_2 + n\gamma_3)\times$ $\quad \times[(p + n\gamma_1 + \frac{1}{2}c_1\gamma_3)^2 + (q + n\gamma_2 + \frac{1}{2}c_2\gamma_3)^2]$ $\quad +\gamma_3[(\kappa c_1 - a_1)(p + n\gamma_1) + (\kappa c_2 - a_2)(q + n\gamma_2)]$ $\quad +\frac{1}{2}\gamma_3^2[\kappa(c_1^2 + c_2^2) - c_1a_1 - c_2a_2]$
	$H = \frac{1}{2C}(\frac{M_1^2}{4} + \frac{M_2^2}{4} + M_3^2) + (-\kappa + 2c_1\gamma_1 + 2c_2\gamma_2)M_3$ $\quad -\gamma_3(c_1M_1 + c_2M_2) + C(a_1\gamma_1 + a_2\gamma_2)$ $\quad -n(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$,
	$I_4 = [(M_3 - 2C\kappa + 4C(c_1\gamma_1 + c_2\gamma_2)](M_1^2 + M_2^2)$ $\quad -4C^2\gamma_3(a_1M_1 + a_2M_2)$

$I_3 = f = 0$

In [337, 338], Sokolov and Tsiganov did not give the complementary integral for this full case. The above formulas are adjusted from [41] (English edition 2017).

	2	Yehia [386] 1987, $n=\kappa=0$ Chaplygin [53] 1903, $n=\kappa=b_1=b_2=0$ Kowalevski [238] 1888 (Special case $f=0$),	
		$A=B=2C$, $V=C[a_1\gamma_1+a_2\gamma_2-n\kappa\gamma_3+b_1(\gamma_1^2-\gamma_2^2)+2b_2\gamma_1\gamma_2$ $-\frac{n^2}{2}(2\gamma_1^2+2\gamma_2^2+\gamma_3^2)]$, $\boldsymbol{l}=C(2n\gamma_1,2n\gamma_2,\kappa+n\gamma_3)$, $\boldsymbol{\mu}=C(-n\gamma_1,-n\gamma_2,\kappa-3n\gamma_3)$,	
		$I_3=2p\gamma_1+2q\gamma_2+(r+\kappa)\gamma_3+n(2\gamma_1^2+2\gamma_2^2+\gamma_3^2)$, $I_4=[(p+n\gamma_1)^2-(q+n\gamma_2)^2-a_1\gamma_1+a_2\gamma_2+b_1\gamma_3^2]^2$ $+[2(p+n\gamma_1)(q+n\gamma_2)-a_1\gamma_2-a_2\gamma_1+b_2\gamma_3^2]^2$ $+2\kappa(r+n\gamma_3-\kappa)[(p+n\gamma_1)^2+(q+n\gamma_2)^2]$ $-4\kappa\gamma_3[(p+n\gamma_1)(a_1+b_1\gamma_1+b_2\gamma_2)$ $+(q+n\gamma_2)(a_2+b_2\gamma_1-b_1\gamma_2)]$.	
		$H=\frac{1}{2C}(\frac{M_1^2}{2}+\frac{M_2^2}{2}+M_3^2)-\kappa M_3$ $+C[a_1\gamma_1+a_2\gamma_2+b_1(\gamma_1^2-\gamma_2^2)+2b_2\gamma_1\gamma_2]$ $-n(M_1\gamma_1+M_2\gamma_2+M_3\gamma_3)$, $I_4=[\frac{M_1^2-M_2^2}{4C^2}-a_1\gamma_1+a_2\gamma_2+b_1\gamma_3^2]^2$ $+[\frac{M_1M_2}{2C^2}-a_1\gamma_2-a_2\gamma_1+b_2\gamma_3^2]^2$ $-\frac{\kappa}{2C^3}(2C\kappa-M_3)(M_1^2+M_2^2)$ $-\frac{2\kappa}{C}\gamma_3[M_1(a_1+b_1\gamma_1+b_2\gamma_2)+M_2(a_2+b_2\gamma_1-b_1\gamma_2)]$.	

10.16 Further Studies on Integrable Cases

In the set of integrable cases in the dynamics of a body in a liquid, the presence of the complementary (fourth) integral makes it possible to perform several analytical and qualitative investigations on each one of the integrable cases, something we are not able to do in the generic case missing the fourth integral. Those investigations include the final explicit solution of the equations of motion in terms of time, bifurcation and topological classification of orbits of the integrable system on its integral manifold and the stability analysis of certain motions like stationary and periodic motions.

In this section, we try to give a quick review of some of those investigations performed for the problem of motion of a body in a liquid.

10.16.1 Separation of Variables, Explicit Solutions and

Separation of the variables was performed most easily in the general integrable cases of Euler and Lagrange of the classical problem. Both cases were reduced to elliptic quadratures and hence the explicit solution was expressed in terms of elliptic functions of time. The general integrable case of Kowalevski and the conditional case of Goryachev and Chaplygin were reduced to hyper-elliptic quadratures (For more information see Chap. 4). Explicit time solution of various integrable cases of motion of a body in a liquid was investigated by several authors. The present status of this aspect is summarized in the following:

General integrable cases:

(1) Kirchhoff reduced the case of simply connected body ($a_3 = \kappa = 0$) to an elliptic quadrature and expressed some particular motions in terms of elliptic functions [219]. Detailed analysis of the general solution in elliptic functions was performed by Halphen [146] and Greenhill [135, 136]. The full general case 1 of Table 10.1 can be easily solved also in terms of elliptic functions of time. This is shown in Chap. 12 Sect. 12.1 as a special version of a more general separable case of Lagrange's type (Case 7 of Table 10.1).

(2) The two cases 2 and 3 discovered by Clebsch were shown by Kötter [233] in 1892 to have their general solution in terms of Theta functions in two variables. The special version $f = 0$ of the asymmetric case of Clebsch was solved, in the same set of functions by Weber, somewhat earlier using separation of variables in Hamilton–Jacobi equation [367]. The version $f = 0$ of the spherical Clebsch case is equivalent to Neumann's problem solved in Theta functions of two variables (See Chap. 9 Sect. 9.7.3). Equations of motion in the Lax pair form and generalization to n-dimensional space are briefly discussed in [306]. Some later trials led to separation of variables in a much more complicated form, e.g. [260, 273]. Recently, the full version and Weber's one have been reconsidered in [95, 271].

(3) Steklov and Lyapunov subcases of cases 4 and 5 are conjugate in the same sense as the two cases of Clebsch. A solution of those subcases proposed by Kötter in Theta functions of two variables [235] was presented in a very compact and complicated way, which led to some controversies between his contemporaries. Tsiganov [360] reconsidered the separation problem for Steklov and Lyapunov subcases and recently, in [363], the full versions of cases 4, 5 due to Rubanovsky. However, no explicit formulas are given. Thus, the separation of variables for the Rubanovsky cases cannot be considered complete, except for the Steklov and Lyapunov subcases separated by Kötter.

(4) As concerns case 6, only the lower level of this hierarchy (Kowalevski's case) is separated by Kötter [232, 234]. The status of the second level (Yehia's gyrostat) is described in Chap. 5. A successful procedure like that followed by Kötter has not been found. Separation of variables is not yet achieved for that level and for the next one (Sokolov's generalization of Yehia's gyrostat). Note that if the solution of the Sokolov case ($n = 0$) is constructed, the solution of the last level with $n \neq 0$ is the same, as concerns the vector $\boldsymbol{\gamma}(t)$. The vector $\boldsymbol{\omega}(t)$ is then readily obtained by applying the regular precession transformation (See Sect. 10.11).

Topological classification of the case of Yehia's gyrostat in the uniform gravity field (See Chap. 5) is discussed in detail in the book of M. Kharlamov et al. [184] (See also [185]). The generalized version when $n \neq 0$ and for non-zero Sokolov parameter c, was not investigated until now.

(5) Variable separation for the Sokolov case in the hierarchy 7 (without a gyrostatic momentum) was proposed in [227]. It generalizes the one used for Kowalevski's case by Kowalevski and Kötter. Explicit separation and expressions for dynamical variables were given in [70] , in terms of two intermediate variables, which are expressed in genus-2 Theta functions. In the last level of the hierarchy, after the introduction of the parameter n, the solution is obtained by applying the uniform precession transformation. The same separation variables of [227] were used in [186] for detailed investigation of the integral manifolds and their bifurcation and also complete description of the phase topology of this case.

Conditional integrable cases

(1) Separation of variables is known for the first two levels of the hierarchy. For Goryachev–Chaplygin's see Chap. 4 Sect. 4.4 and for Sretensky's level see Chap. 5. Explicit time solution for the full case (1) of Table 10.2 is not found yet.

(2) The second case of Table 10.2 involves 6 significant parameters $a_1, a_2, b_1, b_2, \varkappa, n$, of which the last one can be set equal to zero for variable separation and it can be restored in the system by the uniform precession transformation. Separation of variables and explicit expressions for the dynamical variables are known in the following subcases:

a- The special version $f = 0$ of Kowalevski's case ($n = \varkappa = a_2 = b_1 = b_2 = 0$). By a rotation of the x, y axes fixed in the body by a constant arbitrary angle in their plane, one can construct a solution in which both coefficients a_1, a_2 are present.

b- Chaplygin [53] first established the integrability, on the level $f = 0$, of the case $n = \varkappa = a_1 = a_2 = b_2 = 0$, that describes the motion of a simply connected body in a liquid (with only one parameter b_1 present in the potential). Then he achieved a separation of variables for this case and expressed the dynamical variables in terms of two parameters s_1, s_2, each of which can be expressed as an elliptic function of t. This solution is presented in detail in the next section. By a rotation of the x, y axes fixed in the body at a constant arbitrary angle in their plane, one can construct a solution in which both coefficients b_1, b_2 are present.

c- From the results of [416], it follows that the problem of motion of a rigid body with $A = 2C$ and arbitrary B, subject to forces with potential

$$V = a_1\gamma_1 + b_1(\gamma_1^2 - \gamma_2^2),$$

under the additional restrictions

$$q = 0, f = 0,$$

is solvable in elliptic functions of time.

10.16.2 Topological Classification of Integrable Cases

The classical integrable cases of the problem of motion of a rigid body in a liquid served as a fertile land for the application of methods of algebraic geometry and topology, created specially for the study of the phase space of integrable systems. Topological classification offers an alternative, which determines the picture of the foliation of the Liouville tori and hence sheds some light on the general (qualitative) features of motion that can hardly be obtained from the explicit solution of complicated problems. General methods of the study of bifurcation of integral manifolds and phase topology of the integrable cases of the classical problem and gyrostat motion were developed by M. Kharlamov (See e.g. [183, 185]). Steklov's case was investigated by Bogoyavlensky and Ivakh [33].

Fomenko constructed what may be called "Morse theory of integrable Hamiltonian systems" [87–90], building on previous results of many authors including, in particular, works of Smale. This theory was further developed by Fomenko, his colleagues and coworkers (e.g. [34, 35, 92, 303]). Topological classification is made for several two- and multi-dimensional integrable Hamiltonian systems. Most interesting, in particular, are Hamiltonian systems with two degrees of freedom. Those include reductions of higher dimensional systems with cyclic coordinates. A theory of topological invariants of such systems was developed, which gives their classification up to Liouville equivalence, i.e. up to deformation of Liouville tori. For basic information about the theory and some applications to rigid body dynamics, the reader is referred to papers in [90], the works cited above and references therein. In this subsection, some results about topological classification are pointed out parallel to information about explicit solution for each integrable case of the dynamics of a rigid body in a liquid. It has to be said here that topological classification is not a characteristic property of an integrable system. Chaplygin's case of rigid body in a liquid, discussed in the next section, is an example.

10.17 Chaplygin's Case of Integrability

In [53], Chaplygin discovered the conditional case, integrable on the zero level ($f = 0$) of the areas integral and like Kowalevski's case valid under the condition $A = B = 2C$ and characterized by the choice

$$V = Cc(\gamma_1^2 - \gamma_2^2),\ \boldsymbol{\mu} = \mathbf{0}, \tag{10.179}$$

in the equations of motion (10.54), which, in this case take the form

$$\begin{aligned} 2\dot{p} - qr &= 2c\gamma_2\gamma_3, \\ 2\dot{q} + pr &= 2c\gamma_1\gamma_3, \\ \dot{r} &= -4c\gamma_1\gamma_2, \end{aligned}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{10.180}$$

The four integrals of motion are

$$\begin{aligned} p^2+q^2+\frac{1}{2}r^2+c(\gamma_1^2-\gamma_2^2) &= h,\\ 2p\gamma_1+2q\gamma_2+r\gamma_3 &= 0,\\ \gamma_1^2+\gamma_2^2+\gamma_3^2 &= 1,\\ (p^2-q^2+c\gamma_3^2)^2+4p^2q^2 &= K^2. \end{aligned} \tag{10.181}$$

They involve two arbitrary constants h and K. The sign of K is immaterial and, without loss of generality, we can assume that $K \geq 0$.

Chaplygin's case is highly interesting for many reasons. In particular,

(1) It was the second conditional case in rigid body dynamics, after the Goryachev–Chaplygin case of the classical problem (See Chap. 3 Sect. 3.4).
(2) It turned out that separation of variables is much simpler than in the former case and leads to explicit expressions of the Euler-Poisson in terms of elliptic functions. In fact, it is a rare example of dynamical problem, with a relatively simple solution that can be explicitly written in terms of two sets of elliptic functions, which have two independent moduli.
(3) Because, as will be seen later in Chap. 2, it can be brought to equivalence with a completely different problem. Namely, it is that of motion of a body with the Kowalevski configuration about a fixed point, while acted upon by two irreducible uniform fields.
(4) It was the subject of many later generalizations, as will be seen in Chap. 13.

Chaplygin's case was a favourite subject for topological analysis by many authors. Topology of the iso-energy surfaces, bifurcation diagrams in the plane $\{K^2, h\}$ and topological classification of the Liouville tori are studied in [300, 322]. More detailed topological analysis can be found in [295]. In [91], topological equivalence of Chaplygin's case is established with two other problems, the Euler case of rigid body dynamics and Jacobi's problem of geodesics on an ellipsoid. Nevertheless, it seems that not much is done in the literature dealing with the explicit analytical forms of the solution or the qualitative properties of motion. For all those reasons, we now give a somewhat detailed description of the solution and possible types of motion of the body.

10.17.1 Separation of Variables

We give here the expressions for the Euler–Poisson variables in terms of Chaplygin's separation variables. Some more details on the separation process can be found in [53].

$$p = 1/2\,\frac{\sqrt{2}\sqrt{K}\sqrt{s_1-1}\sqrt{1-s_2}}{\sqrt{s_1-s_2}},$$
$$q = 1/2\,\frac{\sqrt{2}\sqrt{K}\sqrt{s_1+1}\sqrt{1+s_2}}{\sqrt{s_1-s_2}},$$
$$r = \frac{\sqrt{2}[\sqrt{\left(s_1^2-1\right)(\beta-cs_2)}+\sqrt{\left(1-s_2^2\right)(cs_1-\alpha)}]}{s_1-s_2}), \tag{10.182}$$

$$\gamma_1 = \frac{[\sqrt{(s_1+1)(1-s_2)(\beta-cs_2)}-\sqrt{(1+s_2)(s_1-1)(cs_1-\alpha)}]}{\sqrt{2c}\,(s_1-s_2)},$$
$$\gamma_2 = \frac{[\sqrt{(s_1-1)(1+s_2)(\beta-cs_2)}+\sqrt{(1-s_2)(s_1+1)(cs_1-\alpha)}]}{\sqrt{2c}\,(s_1-s_2)},$$
$$\gamma_3 = -\frac{\sqrt{2K}}{\sqrt{c(s_1-s_2)}}, \tag{10.183}$$

where $\alpha = h + K$, $\beta = h - K$. Note that $\alpha \geq \beta$ and equality occurs when $K = 0$. The two variables s_1, s_2 are solutions of the equations

$$\dot{s}_1 = -\sqrt{2\left(s_1^2-1\right)(cs_1-\alpha)},$$
$$\dot{s}_2 = -\sqrt{2(1-s_2^2)(\beta-cs_2)}. \tag{10.184}$$

It is not hard to see that for a real solution of those equations $s_1 \in [\max(\alpha, 1), \infty]$, while $s_2 \in [-1, \min(\beta, 1)]$.

Now, for more visibility of the results, one can choose the units of measuring time, so that the constant $c = 1$. It is essential to find the conditions of repeated roots in the under-root polynomials in (10.184). Those are, respectively,

$$h = -K \pm 1, h = K \pm 1. \tag{10.185}$$

For simplicity, we introduce the parameters

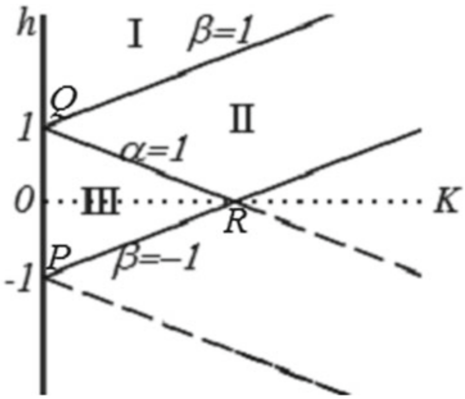
$$\alpha = h + K, \beta = h - K. \tag{10.186}$$

The bifurcation diagram in the Kh-plane is shown in Fig. 10.13.

There are three admissible regions I, II and III, inscribed by solid lines. In those regions we have

$$-1 < 1 < \beta < \alpha \text{ in region I},$$
$$-1 < \beta < 1 < \alpha \text{ in region II},$$
$$-1 < \beta < \alpha < 1 \text{ in region III}.$$

Fig. 10.13 Bifurcation diagram for Chaplygin's case



Inside each region the analytical form of the solution of (10.184) and its qualitative properties do not change, and so does the topological type of the invariant two-dimensional manifold which consists of tori on which the trajectories are wind in the phase space. Only crossing the boundaries between those regions, those properties can change.

By integrating (10.184), it is not hard to obtain the following formulas for s_1 and s_2 in terms of time:

$$
\begin{aligned}
s_1 &= \alpha + (\alpha - 1)\frac{\mathrm{sn}^2(\sqrt{\frac{\alpha+1}{2}}t, k_1)}{\mathrm{cn}^2(\sqrt{\frac{\alpha+1}{2}}t, k_1)}, \qquad k_1 = \sqrt{\frac{2}{\alpha+1}}, \quad \alpha > 1, \\
&= 1 + (1-\alpha)\frac{\mathrm{sn}^2(t,\ \nu_1)}{\mathrm{cn}^2(t,\ \nu_1)}, \qquad \nu_1 = \sqrt{\frac{\alpha+1}{2}}, \qquad \alpha < 1,
\end{aligned} \tag{10.187}
$$

and

$$
\begin{aligned}
s_2 &= -1 + 2\,\mathrm{sn}^2(\sqrt{\frac{\beta+1}{2}}\tau, k_2), \qquad k_2 = \sqrt{\frac{2}{\beta+1}}, \quad \beta > 1, \\
&= -1 + (\beta+1)\,\mathrm{sn}^2(\tau,\ \nu_2), \qquad \nu_2 = \sqrt{\frac{\beta+1}{2}}, \quad \beta < 1.
\end{aligned} \tag{10.188}
$$

Here $\tau = t - t_0$, t_0 is an arbitrary constant. Note that $0 \le t_0 < T$, where T is the period of the Jacobi elliptic functions of moduli k_2. A similar constant appears in the first two formulas is set equal to zero without loss of generality by choosing the initial time moment. The generic motion is quasi-periodic, but becomes periodic when the periods T_1 and T_2 of the two sets of Jacobi's functions are commensurable.

According to the general Liouville-Arnold theorem for completely integrable hamiltonian systems, the integral manifold of the Chaplygin system, corresponding to a fixed pair of the parameters $\{K, h\}$, is a 2-torus or a union of such tori, each

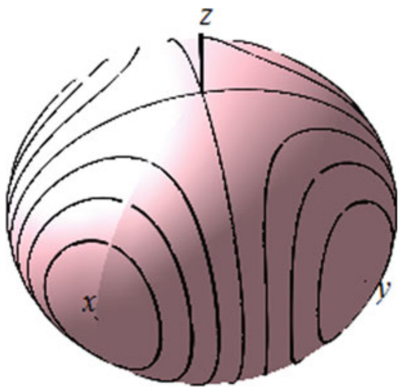


Fig. 10.14 Iso-potentials on the Poisson sphere, the same as zero-velocity curves

of which is filled (winded) by quasi-periodic phase trajectories. On each torus, a trajectory is singled out by the value of the parameter t_0. The projection of a trajectory of the system on the Poisson sphere is the trajectory of the apex of the vector $\boldsymbol{\gamma}$, during the motion of the body, on that sphere. That is what we try to clarify in the following subsections.

10.17.2 Forms of Motion on the Poisson Sphere

A look at Eq. (10.180) reveals that they have six equilibrium positions, in which an end of one of the principal axes of inertia is directed vertically upwards. Two positions correspond to potential minima $V = -1$ at the points $\boldsymbol{\gamma} = (0, \pm 1, 0)$, two correspond to potential saddle point $V = 0$ at $\boldsymbol{\gamma} = (0, 0, \pm 1)$ and the last two correspond to potential maxima $V = 1$ at $\boldsymbol{\gamma} = (\pm 1, 0, 0)$.

From the energy integral in (10.181), one can see that any real possible motion or equilibrium must satisfy the condition

$$V = \gamma_1^2 - \gamma_2^2 \le h. \tag{10.189}$$

The region determined by this condition on the Poisson sphere is called *the region of possible motions.* On its boundary $\gamma_1^2 - \gamma_2^2 = h$, the angular velocity of the body vanishes. If exists, this boundary is named *the zero-velocity curve ZVC*.

Figure 10.14 depicts iso-potential lines on the Poisson sphere. At the minimum value of the energy parameter $h = -1$, the ZVC is composed of two opposite points, corresponding to two stable equilibrium positions[4] of the body with either ends of

[4] Here we mean the alternative problem of motion about a fixed point. In the Chaplygin problem, it corresponds to a steady translational motion of the body in the liquid.

the y-axis directed along the upward vertical (The vector γ). As h increases, namely, for $h \in (-1, 0)$ ZVC consists of two components, each of which is closed around one of the ends of the y-axis. The region of possible motion is composed of the two areas inside the two components of the ZVC. The value $h = 0$ is a critical one. At this value, the ZVC renders to a pair of great circles intersecting on the z-axis and the two regions meet at the two ends of the z-axis. For greater values of $h \in (0, 1)$, the two components of the ZVC become closed around the tips of the x-axis and the region of possible motion is the whole sphere with the exception of the two regions inscribed by the ZVC. For $h = 1$, the region of possible motion is the whole sphere and the equilibrium is possible with the x-axis in vertical position. Finally, for $h > 1$, the region of possible motion is the whole sphere and no ZVC exists.

In the bifurcation diagram Fig. 10.13, one can readily see that the least value of the energy parameter for a possible motion is $h = -1$ at P and at this point $K = 0$. Those values correspond to two equilibrium positions $\gamma = (0, \pm 1, 0)$ at two potential minima. The y-axis is then directed up or down the positive Z-axis fixed in space. The trajectory of the apex of y consists of two points. If we move in the bifurcation diagram on the boundary PQ, the trajectory becomes an arc of the great circle $\gamma_3 = 0$, corresponding to a periodic pendulum-like motion about the z-axis. As we approach the point Q, the motion becomes asymptotic to one of the two equilibrium positions at the two potential maxima at $\gamma = (\pm 1, 0, 0)$. At all points beyond Q, the trajectory is the whole circle, corresponding to complete uni-directional plane (pendulum-like) rotations about the z-axis.

A similar pattern is noted also on the line PR. The motion begins as pendulum-like vibration about the x-axis with increasing amplitude that reaches $\pi/2$ at R, where the motion becomes asymptotic to the equilibrium positions at $\gamma = (0, 0, \pm 1)$, the saddle points of the potential. Beyond R, the motion is a pendulum complete rotation about the x-axis.

An exceptional family of motions corresponds to parameters on the segment RQ. The motion begins as a pendulum-like vibration about the y-axis with increasing amplitude that reaches $\pi/2$ at Q, where it becomes asymptotic to the equilibrium positions at two potential maxima at $\gamma = (\pm 1, 0, 0)$.

Finally, on the critical line $h = K + 1$, the motion is asymptotic to pendulum-like complete rotations about the y-axis.

10.17.3 Explicit Solution

Now, substituing relevant expressions for s_1 and s_2 from (10.187), (10.188) into (10.182), (10.183), one can write down all the Euler-Poisson variables as functions of time. Doing that, one has to choose the signs of the radicals $\sqrt{s_1 - 1}$, $\sqrt{1 - s_2}$, $\cdots$ and$\sqrt{s_1 - s_2}$. However, one has to take only the combinations of signs which are compatible with the areas integral, the second one in (10.181). To make it more definite, we first note that the equations of motion and the integrals of motion enjoy the property of being invariant under each of the simultaneous changes of signs of the tuples

$$\begin{aligned}&1)\{\gamma_1,\gamma_2,\gamma_3\},\\&2)\{p,q,\gamma_3\},\\&3)\{r,\gamma_1,\gamma_2,t\}.\end{aligned}\tag{10.190}$$

Some more changes can be obtained as products of those three. Example is the change $\{\boldsymbol{\omega}(t),\boldsymbol{\gamma}(t)\}\rightarrow\{-\boldsymbol{\omega}(-t),\boldsymbol{\gamma}(t)\}$, obtainable as the product of the three changes. Alternatively, the last change follows from a general principle, the time-reversibility of the motion of natural mechanical systems, acted upon by purely potential forces.

We now write down the final forms in the three zones of the primary solution, obtained by giving all radicals in (10.182), (10.183) a positive sign. In all illustrating examples in the accompanying figures below, we have set $t_0=0$, i.e. we have projected one trajectory of the infinite number on an integral torus of the problem. Note that those figures were plotted on different time intervals, sufficient for suitable visualization (Figs. 10.15, 10.16, 10.17).

10.17.3.1 In Zone I ($1\leq\beta\leq\alpha<\infty$,)

$$\begin{aligned}p&=\frac{\sqrt{K(\alpha-1)}\,\mathrm{cn}\,(\nu_2\tau,k_2)}{\sqrt{\Delta_1}},\\q&=\frac{\sqrt{K(\alpha+1)}\,\mathrm{dn}\,(\nu_1t,k_1)\,\mathrm{sn}\,(\nu_2\tau,k_2)}{\sqrt{\Delta_1}},\\r&=\frac{\sqrt{2(\alpha-1)}}{\Delta_1}[2\mathrm{sn}(\nu_1t,k_1)\,\mathrm{cn}(\nu_1t,k_1)\,\mathrm{sn}\,(\nu_2\tau,k_2)\,\mathrm{cn}\,(\nu_2\tau,k_2)\\&\quad+\sqrt{(\alpha+1)(\beta+1)}\,\mathrm{dn}\,(\nu_2\tau,k_2)\,\mathrm{dn}(\nu_1t,k_1)],\end{aligned}\tag{10.191}$$

$$\begin{aligned}\gamma_1&=\frac{1}{\Delta_1}[\sqrt{(\alpha+1)(\beta+1)}\,\mathrm{cn}(\nu_1t,k_1)\,\mathrm{dn}(\nu_1t,k_1)\,\mathrm{cn}\,(\nu_2\tau,k_2)\;\mathrm{dn}\,(\nu_2\tau,k_2)\\&\quad-(\alpha-1)\,\mathrm{sn}\,(\nu_2\tau,k_2)\;\mathrm{sn}(\nu_1t,k_1)],\\\gamma_2&=\frac{\sqrt{\alpha-1}}{\Delta_1}[\sqrt{(\alpha+1)}\,\mathrm{sn}(\nu_1t,k_1)\,\mathrm{dn}(\nu_1t,k_1)\;\mathrm{cn}\,(\nu_2\tau,k_2)\\&\quad+\sqrt{(\beta+1)}\,\mathrm{sn}\,(\nu_2\tau,k_2)\;\mathrm{dn}\,(\nu_2\tau,k_2)\;\mathrm{cn}(\nu_1t,k_1)\,],\\\gamma_3&=-\frac{\sqrt{2}\sqrt{K}\,\mathrm{cn}(\nu_1t,k_1)}{\sqrt{\Delta_1}},\end{aligned}\tag{10.192}$$

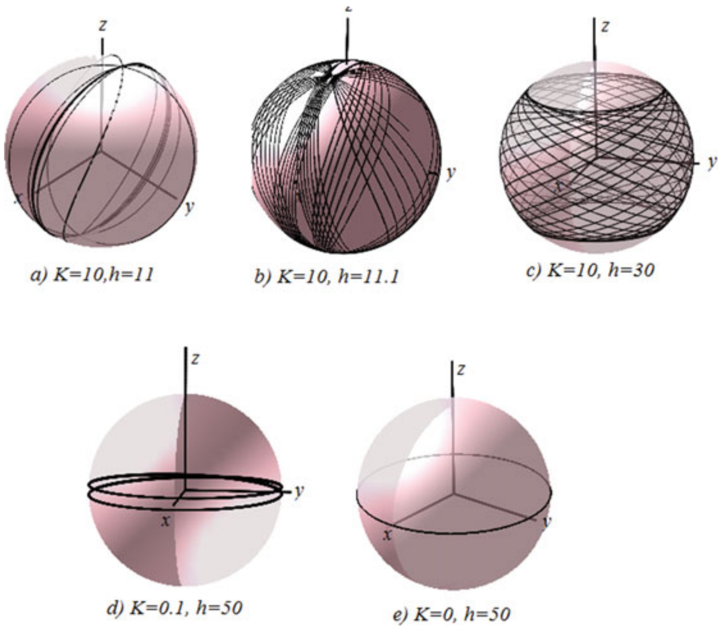


Fig. 10.15 Examples of trajectories from zone I

where

$$\Delta_1 = (\alpha + 1)\,\mathrm{dn}^2\,(\nu_1 t, k_1) - 2\,\mathrm{cn}^2\,(\nu_1 t, k_1)\,\mathrm{sn}^2\,(\nu_2 \tau, k_2)\,. \tag{10.193}$$

10.17.3.2 In Zone II ($-1 \le \beta \le 1 \le \alpha < \infty$)

$$p = \frac{\sqrt{K(\alpha - 1)}\,\mathrm{dn}\,(\tau, \nu_2)}{\sqrt{\Delta_2}},$$

$$q = \frac{\sqrt{K(\alpha + 1)(\beta + 1)}\,\mathrm{dn}\,(\nu_1 t, k_1)\,\mathrm{sn}\,(\tau, \nu_2)}{\sqrt{2\Delta_2}},$$

$$r = \frac{\sqrt{2(\alpha - 1)(\beta + 1)}}{\Delta_2}$$

$$[\sqrt{2}\,\mathrm{sn}(\nu_1 t, k_1)\,\mathrm{cn}(\nu_1 t, k_1)\,\mathrm{sn}\,(\tau, \nu_2)\,\mathrm{dn}\,(\tau, \nu_2) + \sqrt{\alpha + 1}\,\mathrm{cn}\,(\tau, \nu_2)\,\mathrm{dn}(\nu_1 t, k_1)], \tag{10.194}$$

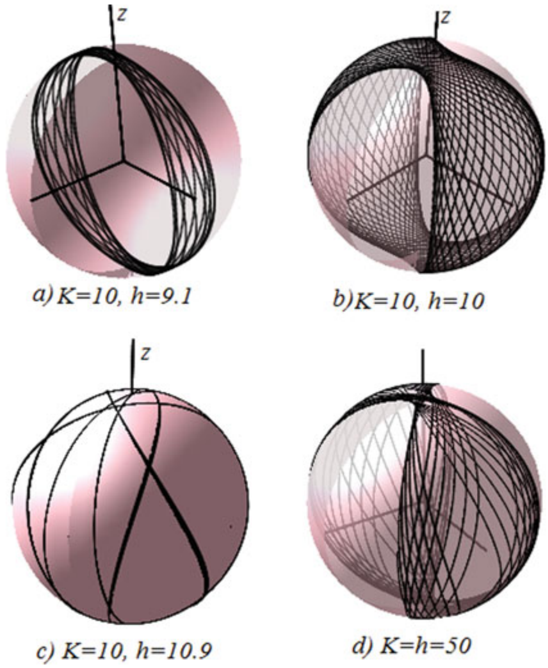


Fig. 10.16 Examples of zone II

$$\gamma_1 = -\frac{\sqrt{\beta+1}}{2\Delta_2}[-2\sqrt{\alpha+1}\,\mathrm{dn}(\nu_1 t, k_1)\ \mathrm{dn}\,(\tau, \nu_2)\ \mathrm{cn}\,(\tau, \nu_2)\ \mathrm{cn}(\nu_1 t, k_1)$$

$$+ \sqrt{2}\ (\alpha - 1)\,\mathrm{sn}\,(\tau, \nu_2)\ \mathrm{sn}(\nu_1 t, k_1)],$$

$$\gamma_2 = \frac{\sqrt{2(\alpha-1)}}{2\Delta_2}[\sqrt{2(\alpha+1)}\,\mathrm{sn}(\nu_1 t, k_1)\,\mathrm{dn}(\nu_1 t, k_1)\ \mathrm{dn}\,(\tau, \nu_2)$$

$$+ (\beta+1)\,\mathrm{sn}\,(\tau, \nu_2)\ \mathrm{cn}\,(\tau, \nu_2)\ \mathrm{cn}(\nu_1 t, k_1)\,],$$

$$\gamma_3 = -\frac{\sqrt{2}\sqrt{K}\,\mathrm{cn}(\nu_1 t, k_1)}{\sqrt{\Delta_2}}, \tag{10.195}$$

where

$$\Delta_2 = (\alpha+1)\,\mathrm{dn}^2\,(\nu_1 t, k_1) - (\beta+1)\,\mathrm{cn}^2\,(\nu_1 t, k_1)\,\mathrm{sn}^2\,(\tau, \nu_2)\,. \tag{10.196}$$

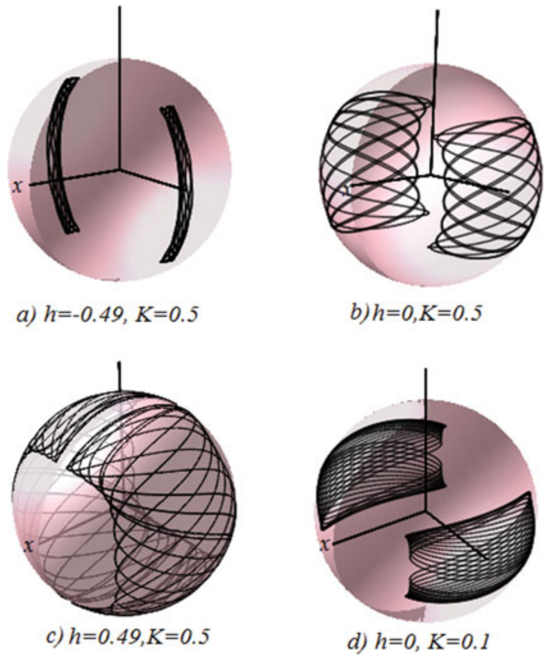


Fig. 10.17 Examples of trajectories in zone III

10.17.3.3 In Zone III ($-1 \leq \beta \leq \alpha \leq 1 < \infty$)

$$
\begin{aligned}
p &= \frac{\sqrt{K(1-a)}\,\mathrm{sn}\,(t,\nu_1)\,\mathrm{dn}\,(\tau,\nu_2)}{\sqrt{\Delta_3}}, \\
q &= \frac{\sqrt{K(\beta+1)}\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2)}{\sqrt{2\Delta_3}}, \\
r &= \frac{\sqrt{2(1-\alpha)(\beta+1)}}{\Delta_3}[\mathrm{sn}\,(t,\nu_1)\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{cn}\,(\tau,\nu_2) \\
&\quad + \mathrm{cn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2)\,\mathrm{dn}\,(\tau,\nu_2)],
\end{aligned} \tag{10.197}
$$

$$
\begin{aligned}
\gamma_1 &= \frac{\nu_2}{\Delta_3}[(a-1)\,\mathrm{sn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2) + 2\,\mathrm{cn}\,(t,\nu_1)\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{cn}\,(\tau,\nu_2)\,\mathrm{dn}\,(\tau,\nu_2)] \\
\gamma_2 &= \frac{\sqrt{2(1-a)}}{2\Delta_3}[2\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{dn}\,(\tau,\nu_2) \\
&\quad + (\beta+1)\,\mathrm{sn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2)\,\mathrm{cn}\,(t,\nu_1)\,\mathrm{cn}\,(\tau,\nu_2)] \\
\gamma_3 &= -\frac{\sqrt{2K}\,\mathrm{cn}(t,\nu_1)}{\sqrt{\Delta_3}},
\end{aligned} \tag{10.198}
$$

where

$$\Delta_3 = 2\,\mathrm{dn}^2(t,\nu_1) - (\beta+1)\,\mathrm{sn}^2(\tau,\nu_2)\,\mathrm{cn}^2(t,\nu_1)\,. \tag{10.199}$$

10.18 Integrability Issues

To begin with, let us note that just as in the problems considered in the previous chapters, the equations of motion (10.45) satisfy Jacobi's divergence condition, which may be written as

$$\frac{\partial}{\partial \boldsymbol{\omega}} \cdot \dot{\boldsymbol{\omega}} + \frac{\partial}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} = 0$$

and thus we need only one general integral (involving a new arbitrary constant) to complete the integration of the problem of motion.

The problem of motion of a body by inertia in an ideal fluid is described by Lagrangian and Hamiltonian equations, isomorphic by the analogy introduced in this chapter to the equations describing the motion of a rigid body about a fixed point under the action of an axi-symmetric combination of three classical fields. The last problem has three degrees of freedom and thus requires for complete integrability the existence of a fourth integral independent of the three known ones. In any set of generalized coordinates, say, Euler's angles, the geometric integral degenerates into an identity and we are left with three integrals, the number of integrals required for complete integrability in the sense of Liouville. Thus, Jacobi's and Liouville's approaches lead to the same requirement.

Neither Eqs. (10.45) and (10.41) nor the equivalent Thomson-Lamb equations were investigated in their full form for the existence of a fourth (complementary) integral. The situation is somewhat better for Kirchhoff's equations, which describe the motion of a body bounded by a simply connected surface. We give here only brief account of the various research on this matter.

10.18.1 Results Concerning Kirchhoff's Equations

10.18.1.1 The Case of Tri-Axial Ellipsoid of the Matrix $\bar{\mathbf{a}}$

Existence of a real-analytic fourth integral: One of the notable results is due to Kozlov and Onishchenko [246] (See also [41]), who used Eq. (10.12) to establish that when the matrices $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}$ are simultaneously diagonal, i.e.

$$\bar{\mathbf{a}} = \mathrm{diag}(a_1, a_2, a_3),$$
$$\bar{\mathbf{b}} = \mathrm{diag}(b_1, b_2, b_3),$$
$$\bar{\mathbf{c}} = \mathrm{diag}(c_1, c_2, c_3),$$

and under the condition that $a_1 \neq a_2 \neq a_3 \neq a_1$, there exists no real-analytic complementary integral of (10.12) independent of the three known general integrals (10.13), except in the two cases when the following necessary relations hold between the matrices:

$$\text{A)}\ \frac{c_1 - c_3}{a_2} + \frac{c_2 - c_1}{a_3} + \frac{c_3 - c_2}{a_1} = 0, \bar{\mathbf{b}} = \mathbf{0}, \tag{10.200}$$

Conditions (A) are also sufficient for integrability. They correspond to Clebsch's integrable case (Case 2 of Table 10.1 above) under the restriction $n = 0$, n the regular precession transformation parameter.[5]

$$\text{B)}\ \frac{b_1 - b_3}{a_2} + \frac{b_2 - b_1}{a_3} + \frac{b_3 - b_2}{a_1} = 0. \tag{10.201}$$

This condition is necessary but not sufficient. The classical case of Steklov (Case 5 of Table 10.1 above, with $n = 0$, $\boldsymbol{\kappa} = \mathbf{0}$) satisfies this condition and existence of the fourth integral is secured by the additional restriction $\bar{\mathbf{c}} = \mathbf{0}$.

Branching of solution: The large number of parameters involved in the Thomson-Lamb equations of motion of a body in a liquid has become an obstacle for further analytical studies of those equations. In spite of its huge success in the classical problem, the approach used by Kowalevski [238] to isolate possible cases in which the solution of equations of motion has only poles as critical points in the complex t-plane does not seem efficient in the problem of motion of a body in a liquid. However, analogous result was established for Kirchhoff's equations:

Under the condition that $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ are simultaneously diagonal and $a_1 \neq a_2 \neq a_3 \neq a_1$, *the general solution of* (10.12) *is meromorphic only for the cases of Clebsch and Steklov* [316].

In both cases, the complementary integral is known and the explicit time solution is expressed in terms of Theta functions.

Existence of a single-valued or algebraic fourth integral: The investigation of existence of a single-valued complementary integral was performed in [36]. It turned out that when $a_1 \neq a_2 \neq a_3 \neq a_1$, branching of solutions is an obstacle for existence of single-valued integrals. It is shown that

Under the condition that $\bar{a}$, $\bar{b}$, $\bar{c}$ *are simultaneously diagonal and* $a_1 \neq a_2 \neq a_3 \neq a_1$, *the cases of Clebsch and Steklov are the only cases, when Eq.* (10.12) *admit a single-valued fourth integral.*

[5] In fact, the condition $\bar{\mathbf{b}} = \mathbf{0}$, is over-restrictive. The result holds when $\bar{\mathbf{b}}$ is propertional to $\bar{\mathbf{I}} = \frac{1}{2}\,\mathrm{tr}(\mathbf{I})\delta - \mathbf{I} = \frac{1}{2}\,\mathrm{tr}(\bar{\mathbf{a}}^{-1})\delta - \bar{\mathbf{a}}^{-1}$. Compare with Case 2 of Table 10.1. The full form, consistent with that in Table 10.1, was given in [36].

The existence of a polynomial integral of Kirchhoff was also considered in some recent works [263, 428], following a line due to Darboux.

10.18.1.2 Case when $\bar{a}$ Has an Ellipsoid of Revolution

Preserving the assumption of diagonal three matrices and adding the restriction $a_1 = a_2$, Sadetov [327, 328] has shown that a complementary algebraic integral of the equations of motion does not exist, except in the Kirchhoff case (Case 1 of Table 10.1) and the special versions of the cases of Clebsch under the extra-condition $a_1 = a_2$.

Remark: As we have seen, in all methods used to investigate integrability of the problem, diagonality of all matrices was a common assumption. This situation greatly reduced the efficiency of those methods. None of them has pointed out a new integrable case. Ironically, in the case which generalizes the classical case of Kowalevski, found later by Sokolov, the matrix K has off-diagonal elements. It was not predicted by any method, but came as a result of the application of a brute force method. An ansatz of an integral of degree 4 was used and a symbolic program was used to solve the resulting conditions on the coefficients and on the system parameters.

10.19 Remark Concerning Particular Solutions of the Problem

The above tables of general and conditional integrable cases of Thomson-Lamb and Kirchhoff equations give a complete up-to-date list and full identification of those cases. Although we also know a large number of particular exact solutions, we have not tried to make a complete list of them. At present, some of those solutions are scattered in journal papers. We have described the most important of those solutions in the present chapter, as examples on solutions of various forms of the equations of motion and also in examples of application of the regular precession transformation.

The largest collection of particular exact solutions of problems of motion of a body in a liquid may be found in books of Gorr and co-authors [121, 125, 126]. Cases are classified by the nature of motion: permanent rotations, regular precessions, semi-regular precessions and so on. However, those books concentrate more on the research of the Donetsk group and in general on results published in Russian. Some results may have been disguised by the use of various sets of variables and may need careful revision. In general, further effort is needed to compare, complete, classify and tabulate all existing results.

10.20 The Donetsk School of Mechanics and Its Attitude to Competing Works

Be not proud because thou art learned; but discourse with the ignorant man as with the sage. For no limit can be set to skill, neither is there any craftsman that possesseth full advantages
Ptah-Hotep (2880 BC) [72]

Although founded by Euler and developed by the basic works of D'Alembert, Poisson and Lagrange, the field of dynamics of a rigid body acted upon by various forces suffered from stagnation for almost a century. Over that period, the search for integrable cases or particular solutions didn't lead to any notable results, even in the simplest problem, the classical problem of motion of a body about a fixed point under its own weight, more than Euler's and Lagrange's cases.

The first breakthrough in the classical problem was made, in 1888, by Kowalevski, who discovered the third integrable case. To that she was not led by a physical or mechanical conservation law, as in the previous two cases, but was led only by a purely mathematical property of the solution of the equations of motion. Kowalevski's success encouraged a number of several of the classics in mathematics and mechanics to invest huge efforts in the same problem. Over the next two decades, the search of such eminent scientists as Joukovsky, Lyapunov, Steklov, Chaplygin and Goryachev produced several integrable cases and particular solutions not only in the classical problem, but also in the gyrostat problem and the problem of motion of a body in a liquid. We have listed those results in relevant chapters of this book. It can be noted that of the eight solutions known up to the first decades of the twentieth century, four cases were found by Russian authors. The next four decades have brought no significant changes in the status of the field, but in 1948, Grioli announced the discovery of a regular precession about an axis inclined to the vertical.

Donetsk school headed by P.V. Kharlamov has made a significant advance in the subject of rigid body dynamics in the period extending from the mid-fifties to the late eighties. For most of this period the Donetsk school comprised a large number of coworkers who worked on all aspects of the classical problem and its generalizations, and especially, the problem of motion of the gyrostat. The group made several notable achievements: three new particular solutions of the classical problem raised the ratio of exact particular solutions constructed by Russian-writing authors to seven cases out of a total of twelve known at the present time. Donetsk school's success was exclusive in the problem of motion of a heavy gyrostat about a fixed point. As pointed out in Sect. 15 of Chap. 5, a considerable part of our present knowledge of exact particular solutions of the equations of motion of a gyrostat belongs to that school. Those are mostly cases generalizing known ones of the classical problem, but a few ones have no analogs in the classical problem. However, the Donetsk school did not find any general integrable cases of the gyrostat problem. This is a key remark to which we shall return later.

The group used a "brute force" policy in the search for exact solutions. Problems are scanned for the possibility of admitting a solution of a prescribed form. Each of the resulting cases certainly required a high cost of manual calculations. The view of an expression or a solution with coefficients written in one or two pages was normal and mostly expected. In the classical field of rigid body dynamics no easy results are left. The group earned credibility and authority in the area of constructing exact solutions and renewed the spirit that prevailed at the turn of the 19th to the 20th century, when the field of rigid body dynamics was, almost completely, a Russian-language science.

Researchers from the Donetsk school have shown that the few results announced in the thirties by Field, Corliss and Fabbri mainly repeat or are special cases of the former results of Russian authors. Also, a result of Mertsalov (1946) was shown to be in error. The overall performance of the research group was more than successful. This gave the group a sense of responsibility to the Russian heritage: they kept its competence among other schools of mechanics and turned into custodian not only of the Russian contributions but also of the whole subject of rigid body dynamics.

In the mid-eighties, the author introduced the simple transformation discussed in detail in Sect. 10.11, which led to an automatic generalization of all general and conditional integrable cases as well as particular solutions of the classical problem and its generalizations by inserting an additional parameter n that invokes a simultaneous combination of potential and gyroscopic forces. Results have interpretations as new integrable and solvable cases in the problem of motion of a body in a liquid. Nearly at the same time, the author devised a method for constructing two-dimensional integrable systems that admit a complementary integral, polynomial in velocity. This method had two main advantages. Firstly, it produced systems living on Riemannian manifolds and not only on flat spaces. Secondly, those systems are time-irreversible, and thus accommodate reductions of 3D systems with a cyclic integral. Those two advantages made the method able to obtain a new integrable system that needed some restrictions to produce a case of motion of the gyrostat, which turned out to be the long-waited generalization of the historical Kowalevski's case, by adding a rotor to the body along its axis of dynamical symmetry (For details, see Chap. 5 Sect. 5.6). There were some other new results, like the new form of the equations of motion of a body in a liquid, which we presented in detail in Sect. 10.4.

The new results have shocked the Donetsk school in more than one way. On one hand, a significant contribution came from outside the Donetsk school. On the other hand, no brute force was used, nor needed, in obtaining those results. The Donetsk school behaved in reaction to the appearance of the new results in a strange way. We give here few brief quotations from the publications of members of the Donetsk school of mechanics, to show to what extent some scientific criticism can go when a strongly overconfident group of researchers have full control over a well-known scientific journal. A rebuttal of some of those criticizing publications was published in 2001 in [405], too late, after some comments were included in Mathematical Reviews [281] and Zentralblatt (Zbl 1025.70007). After the publication of our article [405], it seems that the Donetsk group, at last, realized that they were in error, nevertheless, no one of the authors of the aggressive publications came out to declare that. The direct aggressive series of criticism was stopped. Only in few occasions they were resumed, and mainly indirectly (e.g. [203]).

After the detailed presentation of the problem of motion of a body in a liquid in this chapter, one can hardly need any comments on the claims in the papers of the Donetsk group. However, we find it necessary to pick up some of the most offensive claims. In fact, there are some lessons here to be learned from them.

10.20.1 The Attitude to the Uniform Precession Transformation

In a series of publications, which were brought to our attention only in the fall of 1996, the authors claimed that this method of generalization is void, meaningless and leads to nothing new [200, 211, 216, 217, 292, 350]. We give few quotations, referring the reader to original sources.

(A) In a rare example of unjustified criticism, one of those authors (Kharlamov P.V. [200]) stated, after appraising the criticism in [216, 217, 350], that this criticism:

> "*restores the truth in the most difficult problems of the dynamics of a rigid body, and cleans the field of study from rubbish introduced by faulty and illiterate papers of H. M. Yehia ...* " [200].

The same quotation was included in the review (MR 93g: 01040), written for Mathematical Reviews by Konosevich, the colleague of the authors in the same institute.

(B) Another author writes [350]:

> *"H. Yehia has announced so significant results, that, in case they were true, the state of the classical problems of rigid body dynamics could have radically changed. ...*
> *Astonishing is the lightness with which the achievements of the greatest scientists including prominent nationals were "generalized" in a single stroke by means of a trivial change of variables and introducing a nonsignificant parameter to the system. But neither V. A. Steklov, N. E. Joukovsky, S. A. Chaplygin nor G. V. Kolossov can defend themselves against Yehia's generalizations"* .

(C) As was explained by Kharlamov in [200] (The same reasoning also in [216, 217, 350]), the main point of their criticism is the following:

"Let the system of differential equations

$$\dot{x}_i = X_i(x_1, x_2, ..., x_n), i = 1, ..., n \qquad (10.202)$$

be transformed by means of the invertible substitution

$$y_i = y_i(x_1, x_2, ..., x_n; \nu), i = 1, ..., n \qquad (10.203)$$

to the form

$$\dot{y}_i = Y_i(y_1, y_2, ..., y_n; \nu), i = 1, ..., n. \qquad (10.204)$$

If (10.202) admits an integral

$$I(x_1, x_2, ..., x_n) = const. \tag{10.205}$$

then (10.204) has the integral

$$J(y_1, y_2, ..., y_n; \nu) = const. \tag{10.206}$$

obtained from (10.205) by the substitution (10.203). Even a beginner in Mathematics can realize that the factiously introduced parameter ν in (10.204) and (10.206) can have no significant meaning, since it can be eliminated again by the use of the inverse transformation of (10.203). Thus Yehia's idea to generalize in this way all the known results in the dynamics of rigid bodies (belonging to Joukovsky, Kowalevski and others) is empty and meaningless".

(1) The first lesson here to learn is that a scientific journal not owned and edited by that research group, could not allow the use of words like "*rubbish, faulty and illiterate*" to describe the publications of a competing author.
(2) The second is that over-confidence caused the whole group to deny or disbelieve scientific achievements of others.
(3) The third lesson is that the review data bases Mathematical Reviews and Zentralblatt sometimes adopt the easy solution: to assign each of the members of a certain scientific institution to review the publications of other members, and thus allowing less probability of fair reviews and objective evaluations. In our case, each of the members of the IAMM (Institute of Applied Mathematics and Mechanics) reviewed other members' works. The circle is thus closed: The Authors are the Editors of the Journal (Mekh. Tverd. Tela) and reviewers of their articles and, at last, the reviewers of their publications for the MR and Zbl bases.[6]

10.20.2 The Attitude to the Equations of Motion in the Form (10.45)

In 2001, Kharlamov P.V., Mozalevskaya G.V. and Lesina M.E. published the paper [203], in which the equations of motion of a body in a liquid are observed to be written in four different forms, as per the choice of the principal variables in the equations. The first is the classical Tomson-Lamb Eq. (10.16) using the variables $\boldsymbol{\omega}, \mathbf{u}$. The second is (10.23) using $\mathbf{M}, \mathbf{p}$. In fact, they use a slightly modified form due to Kharlamov [192], praising this form as being chosen by Chaplygin and Kharlamov in their research and giving it the term "principal (main) representation" of the equations of motion. The third form uses $\boldsymbol{\omega}$ and $\mathbf{p}(\gamma)$, which are in fact our equations presented in Sect. 10.4 and deduced originally in 1986 [383], but they are not presented in [203] in full form. The fourth form uses $\mathbf{M}, \mathbf{u}$ and is termed as the worst choice. As the authors tried to give references and names for the first two forms, they pass by the third form of the equations without giving any references nor referring to any

[6] In fact, articles were rejected from publication in the Russian journal PMM J. Appl. Math. Mech. (See [200]). Namely, this rejection evoked the publication of the whole series of papers in "Mekh. Tverd. Tela".

authority in the field. In particular, our 1986 paper [383], which is most relevant to this context was not mentioned nor cited in [203].

It may be interesting in this context to recall the next quotation from the Zentralblatt review (Zbl 1025.70007) concerning the above paper [203] and just repeating all the claims advanced in that paper:

"*It is noticed, that the objective factors, that characterize the given mechanical object, should be separated from the subjective factors, brought by the investigator into the mathematical model of this object. In this connection it is shown, that the equations used by H.M. Yehia in some of his papers are not new, but they are partial cases of Kirchhoff's equations. It is also noticed that some of the generalizations of known integrable cases given by H.M. Yehia are not new too, but they can be obtained from the initial integrable cases by coordinate transformation. To avoid such mistakes the authors suggest that all results in this area should be compared with the corresponding results for the main form*[7] *of Kirchhoff's equations.*"

Reviewer: Boris Ivanovich Konosevich (Donetsk)

It is notable here that the reviewer and the authors of the article [203] are members of the same institute.

10.21 Exercises

(1) A solid of revolution moves through a liquid and its kinetic energy T is given by

$$T = \frac{1}{2}[A(p^2 + q^2) + Cr^2 + A'(u_1^2 + u_2^2) + C'u_3^2].$$

Prove that the steady motion given by

$$p = q = 0, r = \Omega, u_1 = u_2 = 0, u_3 = v$$

is stable in the linear approximation, provided

$$\Omega^2 = 4v^2 \frac{AC'(A' - C')}{A'C^2}.$$

[Lamb]

(2) Show that in the classical problem of motion of a heavy rigid body fixed from on point the permanent rotations around a tilted axis (Sect. 10.8) is possible, only when the body is fixed from its centre of mass, and the axis of rotation is a principal axis of inertia of the body at the fixed point.

[7] The "main form" means the second representation, i.e. the one used by Kharlamov (See the last paragraph).

(3) A body fixed from its centre of mass moves under the action of forces with potential $V = \frac{1}{2}\sum J_{ij}\gamma_i\gamma_j$, γ_i are the direction cosines of a certain line fixed in space. Show that a uniform rotation of the body about an axis inclined to that line is possible only when the axis of rotation is a common principal axis of the matrices **J**, **I** and this axis takes a horizontal position.

(4) A body bounded by a simply connected surface is moving in an ideal incompressible fluid, infinitely extending in all directions and at rest at infinity. The equations of motion have the form (10.45), with $\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$. Show that uniform rotation about the z-axis of the body is possible only if the three matrices have the form:

$$\mathbf{J} = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{11} & 0 \\ 0 & 0 & J_{33} \end{bmatrix},$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{11} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix},$$

$$\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & \frac{K_{11}}{2(J_{33}-J_{11})}K_{13} \\ I_{12} & I_{22} & \frac{K_{11}}{2(J_{33}-J_{11})}K_{23} \\ \frac{K_{11}}{2(J_{33}-J_{11})}K_{13} & \frac{K_{11}}{2(J_{33}-J_{11})}K_{23} & I_{33} \end{bmatrix},$$

provided $J_{33} \neq J_{11}$, $K_{11} \neq 0$, the angle θ_0 is chosen arbitrarily and the angular speed of rotation

$$\Omega = -\frac{2(J_{33} - J_{11})\cos\theta_0}{K_{11}}.$$

(5) Consider the critical cases of exercise 4: $K_{11} = 0$, $J_{33} \neq J_{11}$ and $K_{11} = 0$, $J_{33} = J_{11}$.

(6) Show that the consecutive application of two transformations with parameters n_1, n_2 is equivalent to the application of one transformation with the parameter $n_1 + n_2$.

(7) In Sect. 10.14.4, when $\bar{K}_{11} \neq 0$ use Eq. (10.160) to show that the relation between the rotation angle φ and time is determined from the equation

$$t = -\bar{K}_{11} \int \frac{d\varphi}{J_{13}\sin\varphi + J_{23}\cos\varphi + a_3} \tag{10.207}$$

under the condition that the parameters of the body are given by

$$\mathbf{a} = a_3\left(\frac{-CJ_{13}}{K_{11}^2}, \frac{-CJ_{23}}{K_{11}^2}, 1\right), \boldsymbol{\kappa} = (0, 0, \kappa_3),$$

$$\mathbf{J} = \begin{pmatrix} J_{11} & -\frac{CJ_{13}J_{23}}{K_{11}^2} & J_{13} \\ -\frac{CJ_{13}J_{23}}{K_{11}^2} & J_{11} + \frac{C}{K_{11}^2}(J_{13}^2 - J_{23}^2) & J_{23} \\ J_{13} & J_{23} & J_{33} \end{pmatrix},$$

$$\bar{\mathbf{K}} = \begin{pmatrix} \bar{K}_{11} & 0 & \bar{K}_{13} \\ 0 & \bar{K}_{11} & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33} \end{pmatrix}. \tag{10.208}$$

Show that in contrast to the case of pendulum motion, this motion has a definite energy value, which depends on the parameters of the body

$$E = \frac{1}{2}[J_{11} + \frac{C}{K_{11}^2}(J_{13}^2 + a_3^2)]. \tag{10.209}$$

(8) Starting from the Lagrangian (10.49) of the generalized problem (10.45) (the Routhian of the problem of motion of a body in liquid after ignoring the cyclic translational coordinates):

(a) Ignore the angle of precession retaining the Poisson variables (the components of $\boldsymbol{\gamma}$) as redundant configurational variables.
(b) Apply Hamilton's principle in the form of Jacobi to the reduced time-irreversible Routhian system. Equations of motion are deduced from a variational problem of the type $\delta \int R dt = 0$. Applying Maupertuis' principle to eliminate the time differential from the variational problem.
(c) Use γ_1 as the independent variable and obtain the following second-order differential in γ_3, to which the equations of motion of the body in a liquid are reduced on the integral level $\{I_1 = h, I_2 = f\}$ [384]:

$$\begin{aligned} & D(1-\gamma_1^2-\gamma_3^2)\gamma_3'' + C\gamma_3(1-\gamma_3^2) \\ & - \gamma_1[A-(A+2C)\gamma_3^2]\gamma_3' + \gamma_3[C-(C+2A)\gamma_1^2]\gamma_3'^2 \\ & - A\gamma_1(1-\gamma_1^2)\gamma_3'^3 \\ & - \frac{\rho}{ABCD}\{C\gamma_3[(A-B)(A+B-C)\gamma_1^2 + B(B-C)(1-\gamma_3^2)] \\ & \qquad + A\gamma_1[(B-C)(B+C-A)\gamma_3^2 + B(A-B)(1-\gamma_1^2)]\gamma_3'\} \\ & + \frac{\rho}{2ABC(h-V_1)}[\frac{\partial V_1}{\partial \gamma_3}(\lambda+\mu\gamma_3') - \frac{\partial V_1}{\partial \gamma_1}(\mu+\nu\gamma_3')] \\ & + \frac{\rho^{3/2}}{ABC\sqrt{aD^3(h-V_1)}} \\ & \times \{f[(A-B)(A+B-C)\gamma_1^2 - B(A-B+C) + (C-B)(B+C-A)\gamma_3^2] \\ & \quad + \Lambda\} \\ & = 0, \end{aligned} \tag{10.210}$$

where

$$\rho = \lambda + 2\mu\gamma_3' + \nu\gamma_3'^2,$$
$$\lambda = C[B(1-\gamma_3^2) + (A-B)\gamma_1^2],$$
$$\mu = AC\gamma_1\gamma_3$$
$$\nu = A[B(1-\gamma_1^2) + (C-B)\gamma_3^2].$$

and

$$V_1 = \mathbf{a}\cdot\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma} + \frac{1}{2D}[f - \boldsymbol{\kappa}\cdot\boldsymbol{\gamma} + \frac{1}{2}\mathbf{K}\cdot\boldsymbol{\gamma}]^2,$$
$$\Lambda = D\boldsymbol{\kappa}\cdot\boldsymbol{\gamma}\mathbf{I} + \boldsymbol{\kappa}\cdot\boldsymbol{\gamma}[D\,\mathrm{tr}(\mathbf{I}) - 2|\boldsymbol{\gamma}\boldsymbol{I}|^2]$$
$$+ |\boldsymbol{\gamma}\mathbf{I}|^2\boldsymbol{\gamma}\mathbf{K}\cdot\boldsymbol{\gamma} + D[\mathrm{tr}(\mathbf{K})D - \boldsymbol{\gamma}\mathbf{IK}\cdot\boldsymbol{\gamma} - \mathrm{tr}(\mathbf{I})\boldsymbol{\gamma}\mathbf{K}\cdot\boldsymbol{\gamma}].$$

(9) Under conditions (10.163) a solution of the orbital equation in the previous exercise is possible in the form $\gamma_3 = 0$.

[This characterizes the precessing pendulum motion, including the pendulum motion about a fixed axis.]

(10) Let the particle of unit mass and unit electric charge moving on the fixed smooth ellipsoid

$$Ax^2 + By^2 + Cz^2 = 1$$

be acted upon by forces with potential

$$V = \frac{1}{2}[\frac{k}{A^2x^2 + B^2y^2 + C^2z^2} + \frac{J^2}{(A^2x^2 + B^2y^2 + C^2z^2)^2}]$$

where k, J are constants, and effective magnetic field H whose component H_n orthogonal to the surface is given by

$$H_n = J\frac{[A^2(B+C-A)x^2 + B^2(C+A-B)y^2 + C^2(A+B-C)z^2]}{[A^2x^2 + B^2y^2 + C^2z^2]^{5/2}}.$$

Show that this system admits in addition to Jacobi's integral, the quadratic integral

$$I = (A^2x^2 + B^2y^2 + C^2z^2)(A\dot{x}^2 + B\dot{y}^2 + C\dot{z}^2) - k\frac{A^3x^2 + B^3y^2 + C^3z^2}{A^2x^2 + B^2y^2 + C^2z^2}$$
$$+ 2J\frac{[BC(B-C)yz\dot{x} + CA(C-A)zx\dot{y} + AB(A-B)xy\dot{z}]}{A^2x^2 + B^2y^2 + C^2z^2}$$
$$+ J^2\frac{[A^2(B+C-A)x^2 + B^2(C+A-B)y^2 + C^2(A+B-C)z^2]}{(A^2x^2 + B^2y^2 + C^2z^2)^2},$$

and is consequently integrable.

[Use the Lagrangian (10.53) with the choice

$$V = \frac{1}{2} b(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),$$
$$\mathbf{l} = n(A\gamma_1, B\gamma_2, C\gamma_3),$$

which characterize Clebsch's case of tri-axial body (Case 2 of Table 10.1). After Routhian reduction by the cyclic variable ψ perform Minkowsky change of variables. For detailed solution see [412].

11. A pendulum of unit length whose bulb has unit mass and carries a unit electric charge is moving under the influence of forces whose potential is $V(\mathbf{r})$ and a magnetic field $\mathbf{H}(\mathbf{r})$. Show that the equations of motion on the unit sphere can be written in the form [404]:

$$\mathbf{r}\times\ddot{\mathbf{r}} = H_r\dot{\mathbf{r}} - \mathbf{r} \times \frac{\partial V}{\partial \mathbf{r}}, \tag{10.211}$$

$\mathbf{r} = (x, y, z)$ is the position vector of the bulb, $H_r = \mathbf{H} \cdot \mathbf{r}$ is the radial component of the magnetic field. The motion is completely determined by the two scalar functions V and H_r. The two cases of motion of a dynamically spherical body in a liquid generate the following two cases of motion of a particle on the sphere:

(1) The case corresponding to Clebsch's case.

It is characterized by the pair of functions

$$\begin{aligned} V &= ax^2 + by^2 + cz^2, \\ H_r &= f. \end{aligned} \tag{10.212}$$

The second integral of motion for this case can be obtained from Clebsch's integral substiuting $\boldsymbol{\omega} \to f\mathbf{r} - \mathbf{r}\times\dot{\mathbf{r}}$ (compare to (2.33)).

$$\begin{aligned} I = {} & a(y\dot{z} - z\dot{y} - fx)^2 + b(z\dot{x} - x\dot{z} - fy)^2 + c(x\dot{y} - y\dot{x} - fz)^2 \\ & - (bcx^2 + cay^2 + abz^2). \end{aligned} \tag{10.213}$$

This case is a non-separable generalization of the well-known separable Neumann integrable problem [294] by the presence of the gyroscopic forces and reduces to it when $f = 0$.

(2) The case corresponding to the Rubanovsky–Lyapunov case

$$V = s_1 x + s_2 y + s_3 z - \frac{abc}{2}(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c})$$
$$+ \frac{1}{8}[2f + (b+c)x^2 + (c+a)y^2 + (a+b)z^2]^2,$$
$$H_r = f + \frac{1}{2}[a + b + c - 3(ax^2 + by^2 + cz^2)]. \tag{10.214}$$

The second integral of motion is

$$I = (b+c)(y\dot{z} - z\dot{y} - Nx)^2 + (c+a)(z\dot{x} - x\dot{z} - Ny)^2$$
$$+ (a+b)(x\dot{y} - y\dot{x} - Nz)^2 + s_1[(N+a)x + z\dot{y} - y\dot{z}]$$
$$+ s_2[(N+b)y + x\dot{z} - z\dot{x}] + s_3[(N+c)z + y\dot{x} - x\dot{y}]$$
$$- (bcx^2 + cay^2 + abz^2) \tag{10.215}$$

where $N = f + \frac{1}{2}[(b+c)x^2 + (c+a)y^2 + (a+b)z^2]$.