Advances in Mechanics and Mathematics
45

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A Lagrangian Approach

Birkhäuser

Advances in Mechanics and Mathematics

Volume 45

Birkhäuser

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ISSN 1571-8689 ISSN 1876-9896 (electronic)
Advances in Mechanics and Mathematics
ISBN 978-3-030-96335-4 ISBN 978-3-030-96336-1 (eBook)
https://doi.org/10.1007/978-3-030-96336-1

Mathematics Subject Classification: 70-XX, 70Exx, 70Hxx

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

This book is dedicated to my family:
Nagwa, Hani and Magid
and to the memory of
my teacher and friend Vladimir G. Dumin

Introduction

Rigid body dynamics is one of the oldest and most challenging subjects in classical mechanics. It was initiated by Leonhard Euler, who formulated the equations of motion for a general (asymmetric) torque-free body and obtained the first integrable, in quadratures, case known after his name. With the efforts of D'Alembert, Poinsot, Lagrange and Poisson, the equations of motion of a body about a fixed point under the action of forces were written in their present form known as the Euler–Poisson equations: a system of six first-order differential equations for which three integrals are known. Lagrange found the second integrable case, the case of a heavy axi-symmetric body, usually named as Lagrange's top. In both cases of Euler and Lagrange, an integral of motion followed from general principles of mechanics, constancy of the angular momentum in the first and due to the cyclic angle of rotation about the axis of symmetry in the second. An important moment was that in both cases, the equations of motion were solved to the end and the solution expressed through elliptic functions, invented by Jacobi, and certain integrals involving them.

The search for integrable cases continued, but, although the problem attracted the attention of several eminent mathematicians, the search did not lead to any other cases. A whole century later, Sofia Kowalevski found a new integrable case of the heavy rigid body. That was not in virtue of a physical conservation principle, but using a purely mathematical condition: all solutions of the equations of motion should have only poles as their singularities as functions of time in the complex t-plane. This property is satisfied by the solutions in the two known integrable cases of Euler and Lagrange, being expressible in terms of elliptic functions of time. Having isolated three cases of this type, two cases of Euler and Lagrange and a new third one, Kowalevski tried and found the complementary integral in the third case. That integral turned out to be the first instance ever of a polynomial integral of degree four in the dynamical variables in a dynamical problem. Kowalevski also reduced the problem to quadratures and expressed all the dynamical variables in terms of hyper-elliptic functions of time, which are far more complicated than elliptic functions, but share with them the property of having only poles as singular points in the complex plane.

From this point on, instead of searching for new integrable cases, researchers directed their efforts to find methods to prove non-integrability, non-existence of a fourth integral of motion, in addition to the elementary three general ones, which is algebraic (Liouville, Husson and Burgatti) or even single-valued (Poincaré). This trend reached its perfection after the KAM (Kolmogorov–Arnold–Mozer) theory for integrable Hamiltonian systems was established. In the last few decades, new concepts were applied to rigid body dynamics in the works of several authors, among which those of Kozlov and Ziglin play a distinguished role.

The last quarter of the nineteenth century was a golden era for rigid body dynamics. More complicated problems including certain generalizations of the classical problem were investigated. The integrable case of the motion of a body carrying a symmetric rotor (the gyrostat) was found by Joukovsky and Volterra. Brun found integrals of motion for the motion of a body acted upon by asymmetric Newtonian force of attraction. The equations of motion of a body in a liquid were constructed by Kirchhoff and integrable cases of that problem are associated with the names of Clebsch, Steklov and Lyapunov. The same period is also characterized by the appearance of several cases of exact particular solutions of the classical problem, which were obtained by Staude, Hess, Goryachev, Chaplygin, Kowalewsky, Bobylev and Steklov. Those cases constitute more than half of the exact solutions of the classical problem, known to us to date.

The next half-century or so has elapsed without significant advancements in problems of rigid body dynamics as concerns integrable cases. The second half of the twentieth century, on the other hand, witnessed a renaissance of the subject. Interest has grown in integrable problems in general and, in particular, in those of rigid body dynamics. Several important results were obtained, including new exact solutions of the classical problem and the problem of motion of a heavy gyrostat. New problems emerged and underwent intensive investigation. One of them was that of motion of a body acted upon by more general conservative potential and gyroscopic forces. The last four chapters of this book present mostly innovations brought into the subject in the past few decades, in which the author had some substantial contributions.

The idea of writing this book emerged more than a decade ago. It was delayed so long due to the social and political upheaval that arose in Egypt at the time and had a direct impact on every aspect of life. The original motivation was two-fold, **first and foremost**, there was a need for a new survey on the subject of rigid body dynamics. The core of such a survey should be classification and a complete up-to-date account of all the known but scattered in the literature integrable cases and particular solutions of the diverse problems treated within this subject.

Only in integrable cases can one study the motion in the whole phase space and draw conclusions about the behaviour of the mechanical system over an infinite interval of time. It will be evident as we go through the book that integrable cases are a rarity in problems of rigid body dynamics, and in some problems, there are even proofs that no more integrable cases are there to be found in the future. This situation ensures the high importance of every integrable case and justifies that each one is recorded under the name of its discoverer. Also of great importance are particular exact solutions, obtained only under certain conditions on the initial state of motion of the body. Those constitute the only window through which one can visualize or study a little fragment of the solution in non-integrable problems over an infinite time interval.

Direct methods already used in the construction of new integrable problems since the late nineteenth century have been exhausted and are no longer capable of identifying integrable cases in more complicated problems. Completely new ways of thinking were required to establish new methods and to make substantial advancements in the subject. One of those ways was writing equations of motion in the form of the Lax pairs and using advanced tools of algebraic geometry. In this way, many integrable systems were constructed in the dynamics of particles and some in rigid body dynamics. Later, those methods helped to construct the formal exact solution of some problems. An example is the integrable case of motion of a body acted upon by two skew uniform fields, which generalizes Kowalevski's case of a single field due to Reyman and Semenov-Tian-Shansky. Two integrable versions of this problem were known when the problem admits a linear integral, but using the above method made it possible to obtain the case, with a quadratic integral, unifying those versions. The solution of the complexified equations was also pointed out in terms of Theta functions.

Another method of finding integrable problems in rigid body dynamics was an inverse one. The author of the present book developed a method for the construction of integrable generalized natural systems of two degrees of freedom, which admit integrals of motion in the form of polynomial in the velocities with coefficients depending on the position. This method led to the appearance of vast families of such systems, living on two-dimensional Riemannian manifolds. Designating special values for the parameters, it was possible to construct integrable cases on some known manifolds. In this way, a comparatively large collection of general and conditional integrable problems in rigid body dynamics was constructed over the past decades. Examples are the case of a gyrostat, which generalizes the classical case of Kowalevski, and a large set of conditional cases. Every one of the integrable cases poses new mathematical challenges: to investigate qualitative properties using integrals of motion, to achieve separation of variables and study topological properties of integral manifolds in the phase space. Another possible task is the construction of explicit time solution of the equations of motion, usually by inverting quadratures in case of separation of variables or by using Lax pair representation of a certain type.

Moreover, the application of certain transformations to known integrable cases with cyclic coordinates has led to the construction of much more general integrable cases. This produced new general integrable cases depending on several extra parameters and added physical effects to all the integrable cases known earlier.

Thus, the state of the subject has radically changed since the time of the well-known monographs of Leimanis and Magnus. In 2005, Borisov and Mamaev published their book "Rigid Body Dynamics" and an English translation appeared in 2017. This marvellous book lays emphasis on mathematical structures and generalizations of integrable cases to higher dimensions. Several important topics are not covered in it. Moreover, reading this book requires professional mathematical knowledge.

The second motivation for writing the book was to showcase some novel methods developed by the author that have led to substantial results and completely new collection of integrable cases in rigid body dynamics. Although almost all these cases have definite physical interpretation, they have not been given due attention. In the mean time, as we essentially use the Lagrangian approach, against a currently prevailing trend that the Hamiltonian (or Poisson bracket) approach is the one that should be followed in all problems of mechanics whenever it is applicable. We had to explain why we intentionally use the Lagrangian formalism to present some features of mechanical systems that are vividly seen in that formalism, while disguised in the Hamiltonian formalism. Our standpoint is that the use of this or that approach is not an arguable issue. None of them can be said to be absolutely better than the other. Each approach must be applied, with no prejudice, to problems for which it is most suited. The problem of classifying and tabulating integrable cases of motion of a rigid body subject to potential and gyroscopic forces, which occupies most of the book, is an example. Those cases are time-irreversible, i.e. equations of motion are not invariant under the change of sign of the time variable. As will be established later in the book, every such case is completely determined by two (scalar and vector) functions V and μ. The scalar is the potential and the vector uniquely determines the moment exerted by gyroscopic forces. The pair (V, μ) uniquely characterizes the physics of the problem and its equations of motion and thus can be used as a basis for the classification and tabulation of integrable cases. On the other hand, in inverse methods used to construct integrable systems, gauge terms that arise as a part of the solution of partial and ordinary differential equations enter in the definition of momenta. The Hamiltonian function and the Hamiltonian equations of motion depend on those terms and obscure the necessary terms that determine potential and gyroscopic forces acting on the body together with the physics of the problem. Hamiltonian equations of motion of an irreversible mechanical system can be written in an infinite number of equivalent forms. In fact, to determine whether two Hamiltonians are equivalent, one has to do some steps that are equivalent to finding the equations of motion in the Lagrangian form. Concrete examples are given in the last chapters of the book, beginning with Chap. 10. Nevertheless, in Chaps. 10 and 12, after classification of integrable cases on the Lagrangian basis, to conform with the reference character of the book, we also give Hamiltonians and complementary integrals in terms of momenta for all integrable cases. These cases involve gyroscopic moments, which depend on the position in a complicated manner. For such systems, the Lagrangian approach is not just an awkward presentation, but it faithfully and uniquely presents the physics of the problem under consideration. On the other hand, a system of Hamiltonian equations of motion can represent a whole class of physically different mechanical systems on that level of complication. Nevertheless, the whole integrability theory is most easily and clearly in terms of the Hamiltonian approach.

In the plan of this book, it soon became clear that the original motivation to include all new changes in the field of rigid body dynamics to produce something similar to Routh's tractate of the late nineteenth century is too ambitious and rather impractical. The changes in the subject in the preceding half-century are far more extensive, to be included in one volume. A narrower line had to be set as an aim for the book. We have, thus, made the decision to make a survey of known integrable cases in various problems in the dynamics of a rigid body moving about a fixed point under given forces, including certain problems of motion of a body with no fixed point, but which reduce, after some transformation or reduction, to the first type of problems (with a fixed point). Examples of such problems are the problem of motion of a body in a liquid and that of motion of a satellite in a circular orbit about a spherical planet.

Thus, no place was left for some important problems. The problem of motion of a rigid body subject to a non-holonomic constraint, rolling on a plane or a surface, is an example. Other examples are the motion of a gyroscope in gimbals, problems involving motion of a system of connected bodies in a general state of motion and the motion of a body with a cavity, completely filled with a liquid in a state of vortex flow. The last problem is described by Poincaré–Joukovsky's equations of motion. We have also excluded a large set of existing solutions, which are not in finite exact form, like asymptotic solutions and series solutions or perturbations of exact solutions in power series of a small parameter. On the other hand, problems of rigid body dynamics in which integrability is not a principal issue are not considered. An example is the controlled rotational motion of the rigid body.

Even in the main course of the problem considered in the book, we had to make some definite selections of the material to be included. In the first place, we intended to make a complete up-to-date account of all known integrable cases in the subject. A considerable part of such content is scattered in the literature and would be presented in the form of a book for the first time. The information about integrable cases should contain conditions for their existence, full historical context of their discovery or development from former cases and sufficiently detailed forms of the first integrals in each case. This covers all general integrable cases, i.e. cases integrable in the whole phase space (for arbitrary initial conditions) as well as conditional integrable cases, i.e. integrable on a fixed level of the relevant linear integral of the motion (the areas integral). In most elementary cases, we tried to illuminate as much as possible the process of obtaining the explicit solution of the equations of motion. By this, we mean the expression of all the physical phase variables in terms of time. As will be seen in most of the integrable problems considered in this book, the separation of variables, inverting quadratures and constructing explicit solutions have turned into a separate art and in their majority still represent open mathematical challenges. Even in solved cases, a frequently met drawback is that some explicit solutions are expressed using complex functions of time, a situation that obstructs their use in numerical calculations or simulation. For this reason, greater importance is devoted to the construction of some particular solutions expressible in terms of elliptic, trigonometric or simpler functions of time. Apart from the integrable cases, complete, and somewhat detailed, account of all the twelve known exact particular solutions of the classical problem of motion of a heavy rigid body is given. But this could not be pursued in other higher problems of the hierarchy. That could simply double the size of the book and also the time to compile the existing information.

The book is divided into two parts:

The first part, the elementary part, grew mainly as a course on rigid body dynamics delivered over years to undergraduate mathematics students of the Faculty of Science and it can be used for this purpose. This part includes Chaps. 1–7, covering the material necessary for a mathematics or physics student to get acquainted with the subject of rigid body dynamics, its main problems, techniques and historical development. A few sections of Chap. 8 can be selected to augment that content with some examples of a particular solution in rigid body dynamics.

We begin in Chap. 1 with a study of the characteristics of mass distributions: the centre of mass and the inertia matrix. Even in this classical material, some innovative element was introduced. A new theorem is given, determining natural bounds on the location of the centre of mass of a body with given moments of inertia. Chapter 2 is devoted to different ways of the description of finite rotations, infinitesimal rotations and the angular velocity vector. Introduced here are Euler's angles, the rotation matrix and quaternions for describing the orientation of the body.

Chapter 3 includes a brief study of the classical problem of motion of a rigid body about a fixed point under the action of its own weight. Different forms of the equations of motion and their integrals are derived in different reference frames fixed in the body and moving with it. Equations of motion are obtained in Lagrangian, Routhian and Hamiltonian forms.

In Chap. 4, the three general integrable cases of the classical problem known after the names of Euler, Lagrange and Kowalevski are presented in some detail. Explicit time solution of the equations of motion is given in terms of elliptic functions of time for Euler's case of a torque-free body. The solution of Lagrange's case is reduced to an elliptic quadrature, which may be used to express it in elliptic functions as well. However, we relied, following Poisson, on the use of integrals of motion to establish certain qualitative aspects of the motion, without referring to explicit time solution. Kowalevski's case is formally reduced to hyper-elliptic quadratures. Some degenerate cases are solved in elliptic or simpler functions in Appendix B. The conditional case of integrability bearing the names of Goryachev and Chaplygin is also presented with its separation of variables belonging to Chaplygin. Degenerations of hyper-elliptic quadratures are presented in some detail in Appendix C.

Chapter 5 is devoted to the study of the problem of motion of a heavy gyrostat. In its simplest form, the gyrostat is a rigid body in which a symmetric rotor is placed with its axis fixed in the carrier body by cylindrical smooth joint(s) and given a constant angular speed with respect to the main body. The gyrostatic effects are in wide use in several problems of science and technology. Equations of motion were formulated in the last decades of the nineteenth century. An integrable case is readily recognized, which is a trivial generalization of Lagrange's case, when the main body is axially symmetric and the rotor is aligned along its axis of symmetry. The second case generalizes Euler's case in the classical problem by adding a rotor in an arbitrary direction fixed in the body. This case was found by Joukovsky and shortly later by Volterra. The third general integrable case, Yehia's case, was found as a generalization of Kowalevski's case in the classical problem by adding a gyrostatic momentum. The conditional case of Goryachev and Chaplygin that was generalized to the gyrostat by Sretensky is also presented. The chapter concludes with some applications of the gyrostat dynamics to stabilize certain motions.

In Chap. 6, the problem of motion of a gyrostat about a fixed point while acted upon by the force of a Newtonian centre of attraction is presented. Especially interesting is the case when the attraction centre is far from the fixed point. This case is treated in some detail, for it has several applications in certain problems in astronomy and physics. Two integrable cases are known. They generalize Euler's and Lagrange's cases. In this chapter, we also present what is called Brun's problem, which is equivalent to a special version of the former problem. A quite interesting property is proved that Brun's potential is the only one that admits an integral of motion, quadratic in the velocities.

Chapter 7 contains a brief account of the problem of motion of a body having no fixed point. Equations of motion will be useful in certain topics later to be exposed in this book. A quite interesting example, the motion of a top on a smooth plane (called Poisson's top), is considered at least for its educational importance.

The second part of the book contains mostly new research material that was not compiled before in book form.

The original course included a few examples of particular solutions of the equations of motion. In the final plan of the book, I found it necessary to make separate Chap. 8 collecting the basic information and results about all the twelve known exact particular solutions of the equations of motion in the classical problem. This information, which accumulated in the period from the 1890s to 1970s, to the date the last case was found, has never been presented in a source in the English language. Just a few cases are pointed out in Leimanis' book [256], not all of them are correct. The same situation applies to Magnus' monograph [270]. Borisov and Mamaev mention only half of these cases, with somewhat detailed analysis of the earlier results of Staude, Hess, Bobylev, Steklov and Grioli. In our presentation, some new features were added. In most cases of a particular solution, we show the curve drawn during the motion by the apex of the vertical unit vector γ on the unit sphere fixed in the body. This graph gives a full idea on how the body moves relative to the vertical through the fixed point, i.e. up to a rotation about it.

Exact particular solutions of the gyrostat problem are considered only in one brief section of Chap. 8. Those solutions were intensively studied almost exclusively by the school of Mechanics in Donetsk. All those cases are listed with the essential information on each case. Those are nine cases generalizing their counterparts of the classical problem and four new cases with no classical analogs. For each case, we provided necessary information and references that would help the interested reader to track every case in original works.

In Chap. 9, we consider the analogy between the motion of a rigid body about a fixed point and the problem of motion of a particle on a smooth ellipsoid. This analogy, noted first by Minkowski, furnishes several easy ways for the reduction of the order of equations of motion, using the known integrals of motion. The climax in this direction is the maximal reduction to a single differential equation of the second order named as the "orbital equation". This equation settles once for all the question of maximal reduction of order raised in rigid body dynamics the since 1890s and discussed by a long list of authors.

Chapter 10 is devoted to the problem of motion of a body by inertia in an ideal incompressible fluid extending to infinity in all directions. This problem originally belongs to the field of hydrodynamics, which is described by boundary-value problems on partial differential equations. Nevertheless, the efforts of several authors led to the result that the pressure of the fluid on the body can be completely avoided, leaving us with a mechanical system of six degrees of freedom. Equations of motion of that system were given by Kirchhoff and Clebsch for a simply connected body and by Lamb for a multi-connected (perforated) body.

Also in this chapter, a new form of the equations of motion and a transformation have changed the way of presentation so deeply, gave a new insight into the problem and revealed its inherent relation to other physical problems that were treated before as completely separate from each other. An analogy has been established between this problem and a special form of the problem of motion about a fixed point of a heavy and magnetized body, which carries immovable in it electric charges under the action of an axi-symmetric combination of gravitational, electric, magnetic and Lorentz forces. This opened the way to study systematically, for the first time, the motion of a heavy, magnetized and electrically charged body or gyrostat. The full list of known integrable cases, seven general and two conditional, valid for the two equivalent problems, is given in a unified form. For each case, we give relevant historical information and essential contributions to its study. For completeness, we also provide the Hamiltonian and the complementary integral for every integrable case in the tables, beginning from Chap. 10, where gyroscopic forces will play a more prominent role and Hamiltonians found by inverse methods are usually obscured by gauge terms.

The analogy just described above of the problem of motion of a body in a liquid and the alternative problem has placed the last problem on the top of a hierarchy of the problems considered in all previous chapters and paved the way to create a higher and richer level of that hierarchy that was never treated before. It may have been considered as hopelessly complicated to yield significant results.

In Chap. 11, the use of the Lagrangian approach and certain peculiarities of the equations of motion has pushed the whole subject beyond its common limits. Equations of motion are given in their historical context. They formally generalize the new alternative form of equations of motion of a body in a liquid and go much further from the physical point of view. Transformations are given, which generate new integrable cases of the most complicated nature from the ones in the lower hierarchies, by adding more parameters into their structures.

In Chap. 12, unprecedented and quite complicated integrable cases involving large numbers of parameters were constructed in an exotic, but effortless, way. In fact, we have used certain tricky properties of the Lagrangian formalism to add extra parameters of physical significance to the structures of the known general integrable cases of a body in a liquid. The number of those additional parameters depends on the structure of the potential part of the Lagrangian of the integrable problem. The new cases are the only known examples in our days of integrable cases of motion of a rigid body acted upon by potential and gyroscopic forces of the most complicated structure. The variety of those cases may shed some light on some of the most intractable problems of mechanics concerning the motion of natural and artificial bodies in extreme conditions when the fields applied to the body have comparable effects and none of them can be treated as negligible.

In this chapter, we also introduce a new type of generalization of each general case of integrability of motion of a body in a liquid a conditional integrable case involving an arbitrary function. This type of generalization is valid on a fixed level of the cyclic (areas) integral.

It was argued by some authors that those two types of generalization are trivial and lose their value if the problem under consideration is written in Hamiltonian formalism. We use the Hamiltonian formalism to show that this view of the subject is not factual.

In Chap. 13, we give a full list of the known up-to-date conditional integrable cases of the problem of motion of a rigid body. The tables for those cases are given here in their last and most general form, with no regard to their physical interpretation. Cases in those tables are ordered according to the degree of the complementary integral as a function of the components of the angular velocity of the body. Some of them acquire a physical meaning only for certain values of the parameters present in them. Other ones do not seem presently to have physical meaning at all, mostly because their potentials involve singular terms of certain types, not usually attainable by natural fields. The 22 cases known at present of this class were obtained mainly in the works of the author and some with his coworkers. Most of those cases have resulted as special cases of certain generalized natural multi-parameter integrable systems that were constructed by the author over the last few decades. Some of those cases have even stimulated research to find new ways for the separation of variables and other mathematical topics.

Chapter 14 is devoted to a systematic presentation of the present status of the problem of motion of a rigid body about a fixed point under the action of an asymmetric combination of potential and gyroscopic forces (crossed fields). Equations of motion are derived in the Euler–Poisson variables. Known integrable cases are collected and classified. First presented are integrable cases of a body acted upon by two and three skew uniform fields, then cases with a potential that is quadratic in the direction cosines. Apart from some special cases, in both types of problems, the mechanical system has strictly three degrees of freedom, i.e. does not admit a cyclic integral. In the last two sections of this chapter, we present two classes of problems admitting a cyclic coordinate. The problem of motion of a (physically) axi-symmetric body under the action of asymmetric forces admits the Eulerian proper rotation angle as a cyclic coordinate. The symmetry leading to a cyclic integral in the second problem is not about a fixed axis, neither in space nor in the body. It may be interpreted as axial symmetry in the quaternion space. The cyclic coordinate is the sum (or difference) of the two Eulerian angles of precession and proper rotation. For the last two classes, the method described in Chap. 11 and applied in Chap. 12 gives some exotic generalizations of the well-known cases.

Exercises constitute an essential component of the book. A large number of them provide some supplemental information to the main text or introduce in a brief way additional topics of special interest that could not be presented in more detail.

The elementary part of the present book should be easily readable by anyone who has completed courses on calculus, differential equations and analytical dynamics. Some parts require some knowledge of elliptic integrals and Jacobi's elliptic functions. The advanced part provides mostly new material but it is written in the most elementary way. It will be tractable for readers of different mathematical backgrounds: students, Mathematicians, Physicists and Engineers. I hope that it will stimulate research in the field. Many of the new cases may be investigated for explicit solution either by separation of variables or by the use of Lax pairs. Topological classification and qualitative properties of motion can be studied for every case, for which separation of variables is achieved. Many cases are waiting for appropriate concrete physical interpretation.

This book is intended to be a reference book for integrable cases and exact solutions. I have taken possible care of checking the large number of formulas involved, mostly by using computer packages of symbolic computation.

Throughout the book, I used the usual notations for mathematical terms and operations. Vectors and matrices are denoted by bold symbols, and scalar and vector products by dot and cross, respectively. We have found it much easier and more consistent to use for multiplication of a vector $\mathbf{v}$ by a matrix $\mathbf{M}$ the usual matrix form $\mathbf{vM}$, instead of the mostly used operator form $\mathbf{Mv}$. The first produces vectors in the usual row form and brings some advantage in avoiding the need in many sources to switch between row and column forms of vectors.

During work on this book, I enjoyed generous help from many friends, to whom I express my sincere gratitude. A conversation with David Gao at a conference in Poland was encouraging and inspiring. Michael and Irina Kharlamov and Pavel Ryabov provided me with some old papers in Russian. Gennady Gorr secured for me issues of MTT and longtime chats discussing many details of the subject. Our long-years friendship was in no way affected by our differences on some of the content of books coauthored by him. Alexey Borisov made some publications of RCD available to me, in addition to his books coauthored by Ivan Mamaev, including their marvellous book on rigid body dynamics (2005) and its recent English translation (2017). On the other hand, Ahmed Ghaleb has read extensive parts of the manuscript and made many suggestions to improve the English text. Adel Elmandouh and Ashraf Hussein helped me by checking some mathematical calculations and resolving some issues concerning LaTeX editing of the manuscript. Hani Yehia and Ashraf Hussein helped me to improve the quality of some graphics.

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Contents

Part I
The Elementary Part

Chapter 1
Distribution of Mass

In the study of the dynamics of a rigid body, we deal in a natural way with certain quantities which are determined by the distribution of mass in that body. In this chapter, we introduce those quantities and study their properties and the relations between them.

1.1 The Moment of Mass—The Centre of Mass

A rigid body is defined as a finite distribution of mass in which the relative positions of all its mass elements do not change with time, regardless of the position of the body in space and the external forces exerted on it. We do not assume any conditions on the structure or shape of the body, which may be composed of invariable rigidly connected parts that can comprise in any way discrete point masses and continuous line, surface or volume distributions of mass. The term "element of mass" we use below should be interpreted in each case accordingly.

1.1.1 Moments of a Mass Distribution

The moment of mass of a given rigid body is a vector defined by the integral

$$\boldsymbol{\sigma} = \int \mathbf{r} dm, \tag{1.1}$$

where dm is an infinitesimal mass element, $\mathbf{r}$ is the position vector of that element and the integral is taken over all mass elements of the body. In a given system of axes $Oxyz$, the components are

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_1

$$(\int xdm, \int ydm, \int zdm), \tag{1.2}$$

Those are the first moments of the mass distribution with respect to the given coordinate system.

Generally, moments of arbitrary degree n for a mass distribution are also defined

$$\boldsymbol{\sigma}(n_1, n_2, n_3) = \int x^{n_1} y^{n_2} z^{n_3} dm, \tag{1.3}$$

where n_1, n_2, n_3 are non-negative integers and $n = n_1 + n_2 + n_3$. In dynamics of rigid bodies, zeroth-, first- and second-degree moments ($n = 0, 1, 2$) appear naturally in the equations of motion. Higher moments are also met when the potential of a rigid body is calculated in certain models of gravitational potential in the field of attraction of other bodies. The simplest case is that of approximating the potential of the body in the Newtonian field of a far centre of attraction. We shall return to this point later with more detail.

1.1.2 Centre of Mass

Let M be the total mass of the body

$$M = \int dm. \tag{1.4}$$

Obviously, M is positive and finite. The vector

$$\mathbf{r}_0 = \boldsymbol{\sigma}/M = (\frac{1}{M}\int xdm, \frac{1}{M}\int ydm, \frac{1}{M}\int zdm) \tag{1.5}$$

defines a unique point in the body, called the centre of mass. This point has the fundamental property that the resultant of forces exerted on the body by an arbitrary uniform gravity field always passes through it. However, this property can be lost for any non-uniform gravitational field.

1.2 Second Moments and Inertia Matrix of a Mass Distribution

In the course of our study of rigid body dynamics, we deal with two related matrices (in fact, tensors):

1.2.1 Second Moments Matrix of Mass Distribution

Define the symmetric matrix $\bar{\mathbf{I}} = (\bar{I}_{i,j})_{i,j=1}^{3}$

$$\bar{I}_{i,j} = \int r_i r_j dm, \qquad i, j = 1...3, \tag{1.6}$$

where r_i stands for the i-th component of the position vector $\mathbf{r} = (x, y, z)$ of the mass element dm of the body, i.e.

$$\bar{\mathbf{I}} = \begin{pmatrix} \int x^2 dm & \int xydm & \int xzdm \\ \int xydm & \int y^2 dm & \int yzdm \\ \int xzdm & \int yzdm & \int z^2 dm \end{pmatrix}. \tag{1.7}$$

Diagonal elements $\int x^2 dm$, $\int y^2 dm$ and $\int z^2 dm$ are called moments of inertia of the body with respect to the planes yz, zx and xy, respectively.

1.2.2 Inertia Matrix of Mass Distribution

In most dynamical considerations, we more frequently meet the inertia matrix defined as

$$\mathbf{I} = tr(\bar{\mathbf{I}})\delta - \bar{\mathbf{I}}, \tag{1.8}$$

where δ is the unit matrix. This makes

$$\mathbf{I} = (I_{i,j})_{i,j=1}^{3} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (z^2 + x^2) dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2 + y^2) dm \end{pmatrix}. \tag{1.9}$$

The diagonal elements of the inertia matrix I_{11}, I_{22} and I_{33} are called moments of inertia of the mass distribution with respect to the axes x, y, z, respectively, while the off-diagonal ones are termed the products of inertia with respect to the three coordinate planes. As the mass element is always positive, the moments of inertia are non-negative. Moreover, a moment of inertia of a body about an axis vanishes only if the body mass is distributed (continuously or discretely) on that axis.

Note that

$$tr(\mathbf{I}) = 2tr(\bar{\mathbf{I}}), \tag{1.10}$$

so that the inverse of the relation (1.8) can be written as

$$\bar{\mathbf{I}} = \frac{1}{2} tr(\mathbf{I})\delta - \mathbf{I}. \tag{1.11}$$

1.3 Properties of the Inertia Matrix

1.3.1 The Triangle Inequalities

Although the moments of inertia are always positive, not every three positive quantities can represent moments of inertia of some body about three perpendicular axes. In fact, moments of inertia of a body satisfy the inequalities

$$I_{11} + I_{22} - I_{33} \geqslant 0, I_{22} + I_{33} - I_{11} \geqslant 0, I_{11} + I_{33} - I_{22} \geqslant 0. \tag{1.12}$$

The equality holds only for plane mass distributions. Those inequalities suggest that the three moments of inertia about three perpendicular axes can be represented by lengths of three sides of a triangle. We shall return to this point with more detail later in this chapter.

To prove those inequalities, we notice from (1.7) that the diagonal elements of $\bar{\mathbf{I}}$ are non-negative, for being quadratic moments of mass with respect to the coordinate planes. The three inequalities follow from the relation (1.11). Equality holds only for bodies whose mass is distributed in one of the coordinate planes.

1.3.2 Theorem of Parallel Axes

Let $\mathbf{I}$ be the inertia matrix of a given body of total mass M with respect to some Cartesian frame $Oxyz$ with origin O at the centre of mass of the body. We shall calculate the inertia matrix $\mathbf{I}'$ with respect to another Cartesian frame $O'x'y'z'$ parallel to the first, so that O' has relative to O the position vector $\mathbf{r}_1 = (x_1, y_2, z_1)$. The first of those elements is

$$\begin{aligned} I'_{11} &= \int (y'^2 + z'^2) dm \\ &= \int [(y - y_1)^2 + (z - z_1)^2] dm \\ &= \int (y^2 + z^2) dm + (y_1^2 + z_1^2) \int dm - 2y_1 \int y dm - 2z_1 \int z dm. \end{aligned}$$

Since $\int y dm = \int z dm = 0$, we get

$$I'_{11} = I_{11} + M(y_1^2 + z_1^2). \tag{1.13}$$

In a similar way, we can show that

$$\mathbf{I}' = \mathbf{I} + \begin{pmatrix} M(y_1^2 + z_1^2) & -Mx_1y_1 & -Mx_1z_1 \\ -Mx_1y_1 & M(z_1^2 + x_1^2) & -My_1z_1 \\ -Mx_1z_1 & -My_1z_1 & M(x_1^2 + y_1^2) \end{pmatrix}, \tag{1.14}$$

which can be stated as follows:

Theorem 1.1 *The inertia matrix of a body about a system of axes is equal to its inertia matrix relative to a parallel system with the origin at the centre of mass of the body plus inertia matrix of a point mass equal to the mass of the body placed at the centre of mass of the body relative to the first system.*

From (1.13), we deduce that *among all parallel axes, the axis around which the moment of inertia of a given body is minimal is the one passing through centre of mass of the body.*

1.3.3 Ellipsoid of Inertia

Let I be the inertia matrix of a given body with respect to a Cartesian reference frame $Oxyz$. Denote by $\mathbf{r}$ the position vector of the current mass element dm. The moment of inertia of that body about an arbitrary straight line OP, passing through O, whose direction cosines are given by the vector $\mathbf{t} = (\alpha, \beta, \gamma)$ can be written in the form (Fig. 1.1)

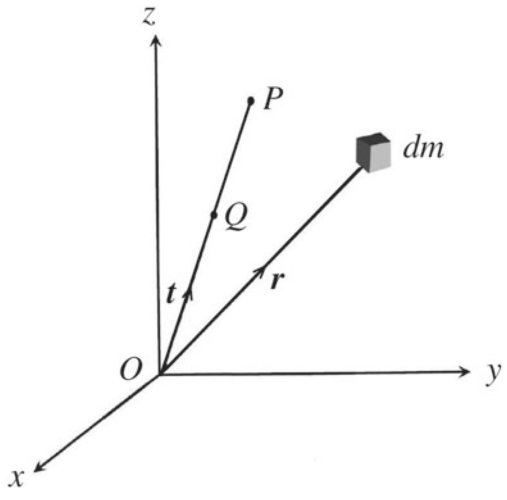

Fig. 1.1 Body-element reference frame

$$
\begin{aligned}
I_{OP} &= \int |\mathbf{r} \times \mathbf{t}| dm \\
&= \int [|\mathbf{r}|^2 |\mathbf{t}|^2 - (\mathbf{r} \cdot \mathbf{t})^2] dm \\
&= \int [(x^2 + y^2 + z^2)(\alpha^2 + \beta^2 + \gamma^2) - (\alpha x + \beta y + \gamma z)^2] dm \\
&= I_{11}\alpha^2 + I_{22}\beta^2 + I_{33}\gamma^2 + 2I_{12}\alpha\beta + 2I_{23}\beta\gamma + 2I_{13}\alpha\gamma. \qquad (1.15)
\end{aligned}
$$

We now take a point Q on OP such that I_{OP} is inversely proportional to $\overline{OQ}^2$ with a proportionality constant of dimensions ML^4 and magnitude equal to 1, i.e. $I_{OP} = \frac{1.}{\overline{OQ}^2}$. Combining this with (1.15), we get

$$
\overline{OQ}^2 (I_{11}\alpha^2 + I_{22}\beta^2 + I_{33}\gamma^2 - 2I_{12}\alpha\beta - 2I_{23}\beta\gamma - 2I_{13}\alpha\gamma) = 1,
$$

and denoting by (X, Y, Z) the coordinates of Q, we can put the last equation in the form

$$
I_{11}X^2 + I_{22}Y^2 + I_{33}Z^2 + 2I_{12}XY + 2I_{23}YZ + 2I_{13}XZ = 1. \qquad (1.16)
$$

This means that the point Q lies on a quadratic surface with centre at the origin. Since the moment of inertia of the body around any line is always positive and finite, the surface (1.16) is an ellipsoid, usually termed “inertia ellipsoid” or “Cauchy’s ellipsoid”. Equation (1.16) can be written in short form as

$$
\mathbf{r}\mathbf{I} \cdot \mathbf{r} = 1, \qquad (1.17)
$$

where $\mathbf{r} = (X, Y, Z)$.

With the inertia ellipsoid, we gain a clear geometric image of the distribution of moments of inertia about axes meeting at one point. We know from geometry that the axes can be rotated to a position $O\xi\eta\zeta$, say, so as to eliminate the products of inertia and hence all mixed terms in (1.16) and write the ellipsoid equation in the standard form

$$
A\xi^2 + B\eta^2 + C\zeta^2 = 1, \qquad (1.18)
$$

where A, B, C are called the principal moments of inertia. They are the eigenvalues of the inertia matrix I, i.e. roots of the equation

$$
\begin{vmatrix}
I_{11} - \lambda & I_{12} & I_{13} \\
I_{12} & I_{22} - \lambda & I_{23} \\
I_{13} & I_{23} & I_{33} - \lambda
\end{vmatrix} = 0 \qquad (1.19)
$$

or, in expanded form,

$$
\begin{aligned}
&\lambda^3 - (I_{11} + I_{22} + I_{33})\lambda^2 \\
&+(I_{11}I_{22} + I_{22}I_{33} + I_{33}I_{11} - I_{12}^2 - I_{23}^2 - I_{13}^2)\lambda \\
&-I_{11}I_{22}I_{33} + 2I_{12}I_{23}I_{13} + I_{11}I_{23}^2 + I_{22}I_{13}^2 + I_{33}I_{12}^2 \\
&= 0.
\end{aligned} \tag{1.20}
$$

One should notice here that the coefficients of the last equation, which defines a unique set of principal moments of inertia, must be invariant with respect to the rotation of the axes, i.e. they must be independent of the current set of axes at O. This implies the three relations between the elements of the current inertia matrix at O and the principal moments of inertia at the same point:

$$
\begin{aligned}
I_{11} + I_{22} + I_{33} &= A + B + C, \\
I_{11}I_{22} + I_{22}I_{33} + I_{33}I_{11} - I_{12}^2 - I_{23}^2 - I_{13}^2 &= AB + BC + AC, \\
I_{11}I_{22}I_{33} - 2I_{12}I_{23}I_{13} - I_{11}I_{23}^2 - I_{22}I_{13}^2 - I_{33}I_{12}^2 &= ABC.
\end{aligned} \tag{1.21}
$$

The axes ξ, η, ζ of the ellipsoid (in which I is diagonal) are called the principal axes of inertia and they are in the directions of the eigenvectors of I, i.e. the solutions of the linear system

$$
\begin{pmatrix} I_{11} - \lambda & I_{12} & I_{13} \\ I_{12} & I_{22} - \lambda & I_{23} \\ I_{13} & I_{23} & I_{33} - \lambda \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = 0 \tag{1.22}
$$

corresponding to the three eigenvalues $\lambda = A, B, C$. The planes $\xi = 0, \eta = 0, \zeta = 0$ are called principal planes.

Remark 1 As clear from (1.18), the semi-axes of the ellipsoid of inertia are

$$
a = \frac{1}{\sqrt{A}}, b = \frac{1}{\sqrt{B}}, c = \frac{1}{\sqrt{C}},
$$

and thus the semi-minor axis of the ellipsoid of inertia is the axis of maximal moment of inertia and the semi-major axis of the ellipsoid of inertia is the axis of minimal moment of inertia.

Remark 2 An arbitrary ellipsoid with semi-axes a, b, c cannot be the ellipsoid of inertia of a real body unless those semi-axes satisfy the three inequalities $\frac{1}{a^2} + \frac{1}{b^2} \geqq \frac{1}{c^2}$, $\frac{1}{b^2} + \frac{1}{c^2} \geqq \frac{1}{a^2}$, $\frac{1}{c^2} + \frac{1}{a^2} \geqq \frac{1}{b^2}$.

Remark 3 When two of the principal moments of inertia are equal, $A = B$, say, the ellipsoid of inertia becomes an ellipsoid of revolution with axis of symmetry ζ. Then all axes passing through O in the equatorial plane $\xi\eta$ of the ellipsoid are principal axes.

Remark 4 When $A = B = C$, the point O is called a spherical point. All axes through it are principal axes.

Remark 5 The matrices $\bar{I}, I$ have the same principal axes and in those axes, in virtue of (1.11),

$$\bar{I} = \mathrm{diag}(\bar{A}, \bar{B}, \bar{C}) = \mathrm{diag}\left(\frac{B+C-A}{2}, \frac{C+A-B}{2}, \frac{A+B-C}{2}\right), \quad (1.23)$$

and regarding the inverse relation (1.8), we also have

$$I = \mathrm{diag}(\bar{B}+\bar{C}, \bar{C}+\bar{A}, \bar{A}+\bar{B}). \quad (1.24)$$

As we shall see in several occasions in the sequel, it is advantageous to use the second moments matrix instead of the inertia matrix. Several relations take a simpler form when expressed in terms of $\bar{\mathbf{I}}$. Moreover,

Remark 6 Contrary to moments of inertia, any given three positive quantities can serve as the principal second moments of a mass distribution with respect to three orthogonal axes.

Remark 7 If the axes are arranged so that $A \geqslant B \geqslant C$, then $\bar{A} \leq \bar{B} \leq \bar{C}$.

Remark 8 The condition of axial dynamical symmetry $A = B$ is equivalent to $\bar{A} = \bar{B}$.

Remark 9 The diagonal elements of the inertia matrix defined in (1.8) (expressed in terms of $\bar{\mathbf{I}}$) automatically satisfy the triangle inequalities

$$B+C-A = 2\bar{A} \geqslant 0, C+A-B = 2\bar{B} \geqslant 0, A+B-C = 2\bar{C} \geqslant 0. \quad (1.25)$$

1.3.4 The Gyration Ellipsoid

The inverse I^{-1} of the inertia matrix is called the gyration matrix. The gyration ellipsoid can be defined, in a way analogous to inertia ellipsoid, by the equation

$$\mathbf{r}\mathbf{I}^{-1}\cdot\mathbf{r} = 1, \quad (1.26)$$

where $\mathbf{r} = (x, y, z)$ is a current point on that ellipsoid. Thus, inertia and gyration ellipsoids have the same principal axes and their semi-axes are reciprocal to each other. In the principal axes of inertia, the gyration ellipsoid is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1. \quad (1.27)$$

1.3.5 Representation of Principal Moments of Inertia

In many problems involving rigid body dynamics, the results depend on the values of the moments of inertia. This can be represented in several ways. Here, we describe two of the most frequently used methods:

1.3.5.1 The First Method

In most problems, only the ratios between moments of inertia and not the moments themselves are significant to the results. The simplest method is to represent moments of inertia in the plane of their ratios $(\xi, \eta) = (\frac{B}{A}, \frac{C}{A})$. The set of admissible points, i.e. points which correspond to real bodies are those inside or on the border of the semi-infinite strip shown in Fig. 1.2. This strip is divided into zones with different relative order of the moments by lines representing cases of two equal moments and cases when the body is a plane disc. We refer to those zones by Roman numerals.

I) $A > B > C$,
II) $B > A > C$,
III) $B > C > A$,
IV) $A > C > B$,
V) $C > A > B$,
VI) $C > B > A$.

1.3.5.2 The Second Method

Ratios of moments of inertia are represented by points in the plane of the two quantities

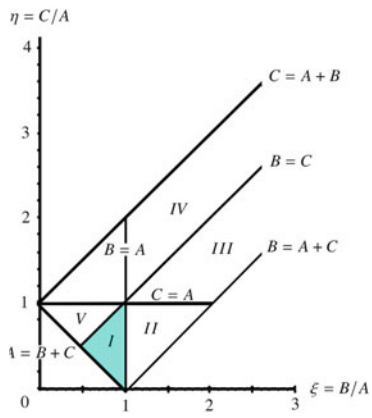

Fig. 1.2 Representing moments of inertia in the plane of their ratios

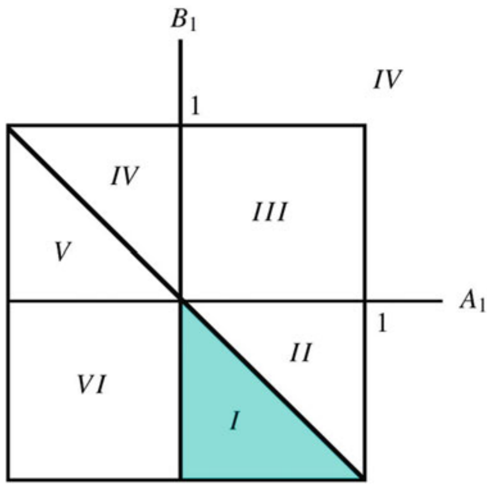

Fig. 1.3 Representing ratios of moments in the plane A_1B_1

$$A_1 = \frac{B - C}{A},\ B_1 = \frac{C - A}{B}. \tag{1.28}$$

Recalling that the largest value of B is $A + C$, we note that the maximum value of A_1 is 1. Also, the largest value of C is $A + B$ and thus A_1 has a minimum of -1. Similarly, we can show that $-1 \leq B_1 \leq 1$. The admissible area in the given plane is the square depicted in Fig. 1.3.

Comparing the three methods, we note that the circular boundaries between zones are not comfortable in use. Two of the zones extend to infinity, in which the moment A approaches zero. In the second method, the boundaries are straight lines, but we also have two infinite zones. In the third method, all zones are finite and boundaries are straight, but the quantities defining the plane are not as simple as in the second method. Thus, we shall prefer to use one of the last two methods when we need to illustrate the dependence of certain results on moments of inertia of the body.

1.4 Relations Between the Centre of Mass and the Inertia Ellipsoid

In several circumstances, we deal with a rigid body and results of the study depend on the parameters: moments of inertia of that body, its mass and the position of its centre of mass. In the analysis of such results, one has to know the relative order of parameters. Also, in numerical simulations of motion, one needs to choose definite numerical values of all the parameters, corresponding to a realistic body.

We know that three moments of inertia A, B and C are subject only to triangle inequalities. When one of the triangle inequalities for moments of inertia approaches equality, say, $C = A + B$, the body turns into a distribution of mass in the plane $z = 0$. Obviously, some condition (or conditions) must hold in the general case and turn to $z = 0$ in the case of equality.

This question is answered by the following:

Theorem 1.2 [389] *Let $Oxyz$ be the Cartesian coordinate system coinciding with the principal axes of inertia at O of a body of mass M and given principal moments of inertia A, B and C, respectively. The centre of mass of the body lies inside or on the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1.29}$$

where

$$a^2 = \frac{\bar{A}}{M} = \frac{B + C - A}{2M}, b^2 == \frac{\bar{B}}{M} = \frac{C + A - B}{2M}, c^2 = \frac{\bar{C}}{M} = \frac{A + B - C}{2M}. \tag{1.30}$$

Remark 10 Note that if $C \to A + B$, then $c \to 0$ and the centre of mass lies in the ellipse

$$\frac{x^2}{B/M} + \frac{y^2}{A/M} = 1 \tag{1.31}$$

of the plane $z = 0$, where $A = \int y^2 dm$, $B = \int x^2 dm$.

This theorem can be proved[1] by applying the conditions that the quadratic form

$$F(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \int (\alpha_0 + x_1\alpha_1 + y_1\alpha_2 + z_1\alpha_3)^2 dm$$

is non-negative for all real $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ where dm is the mass element at (x_1, y_1, z_1) and the integral is taken over the whole mass of the body. This gives

$$\begin{aligned} F(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = {} & \alpha_0^2 \int dm \\ & +2\alpha_0(\alpha_1 \int x_1 dm + \alpha_2 \int y_1 dm + \alpha_3 \int z_1 dm) \\ & +\alpha_1^2 \int x_1^2 dm + \alpha_2^2 \int y_1^2 dm + \alpha_3^2 \int z_1^2 dm \\ = {} & M[\alpha_0^2 + 2\alpha_0(\alpha_1 x + \alpha_2 y + \alpha_3 z)] \\ & +\bar{A}\alpha_1^2 + \bar{B}\alpha_2^2 + \bar{C}\alpha_3^2. \end{aligned}$$

[1] For detailed proof, generalization to arbitrary dimension and degenerate cases, see [408].

For a mass distribution $M > 0$, and the required condition is that all top left determinants of the matrix

$$\begin{pmatrix} 1 & x & y & z \\ x & \frac{\bar{A}}{M} & 0 & 0 \\ y & 0 & \frac{\bar{B}}{M} & 0 \\ z & 0 & 0 & \frac{\bar{C}}{M} \end{pmatrix}$$

should be non-negative. It turns out that, unless the mass distribution lies in a plane, those conditions reduce to three ones of the form

$$x^2 \le a^2, y^2 \le b^2, z^2 \le c^2, \tag{1.32}$$

where

$$a^2 = \frac{\bar{A}}{M}, b^2 = \frac{\bar{B}}{M}, c^2 = \frac{\bar{C}}{M} \tag{1.33}$$

and the last one, resulting from the whole determinant, which we put in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1. \tag{1.34}$$

Conditions (1.32) restrict the centre of mass of the mass distribution to lie inside or on the cuboid centred at the origin and with sides $2a$, $2b$, $2c$. The condition (1.34) is stronger than the three previous conditions. It tells that the centre of mass lies inside or on the ellipsoid with semi-axes a, b, c inscribed in this cuboid.

The semi-axes of the ellipsoid in the condition (1.34) have the simplest form (1.33) in terms of the matrix of second moments. Written in terms of moments of inertia, they become

$$a^2 = \frac{B + C - A}{2M}, b^2 = \frac{C + A - B}{2M}, c^2 = \frac{A + B - C}{2M}. \tag{1.35}$$

Remark 11 Theorem 1.2 was firstly used in [388], where it helped to establish that the number of zones of stability and instability of pendulum-like motion cannot exceed a given finite number. Disregarding such result led to assuming countable number of zones in some works, e.g. [17]. In some engineering works in order to simulate the motion of a rigid body, authors have to give the parameters of the problem certain numerical values. Ignoring the above conditions can lead to the use of impossible combination of parameters like the location of the centre of mass that does not belong to any physical body, e.g. [160]. Recently, this result was applied in [315].

1.5 Solved Examples

Example 1

Find the loci of the axes at O about which the moments of inertia are equal.

Solution

Let (ξ, η, ζ) be a point on one of those axes, about which the moment of inertia is I. The point of intersection of this axis with the inertia ellipsoid

$$A\xi^2 + B\eta^2 + C\zeta^2 = 1$$

lies on the sphere

$$I(\xi^2 + \eta^2 + \zeta^2) = 1.$$

Subtracting the two equations, we obtain

$$(A - I)\xi^2 + (B - I)\eta^2 + (C - I)\zeta^2 = 0. \tag{1.36}$$

This homogeneous quadratic equation determines the required locus, which is a cone with vertex at O. This cone is called equimomental cone (Fig. 1.4).

For determinacy, we assume the moments of inertia to be organized so that $A > B > C$. Equation (1.36) is meaningful only for values of I satisfying $C \leq I \leq A$ (Fig. 1.5).

The shape of the cone depends on the value of I as follows:

(1) For $I = C$, the equimomental cone degenerates into the line $\xi = \eta = 0$, i.e. into the $z-$axis.
(2) For values of I on the interval $C < I < B$, Eq. (1.36) takes the form

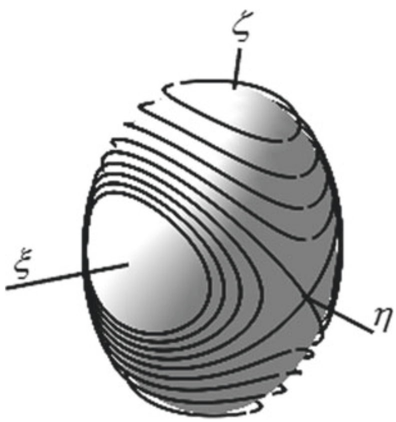

Fig. 1.4 Iso-momental lines on the ellipsoid of inertia

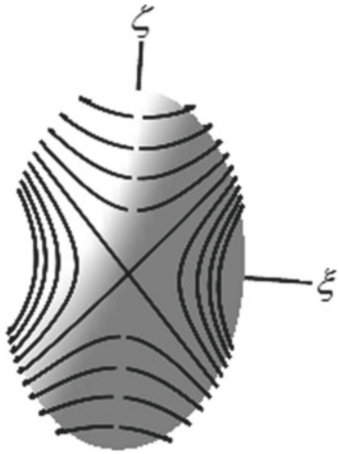

Fig. 1.5 Projections of iso-momental lines on the xz-plane

$$(A - I)\xi^2 + (B - I)\eta^2 = (I - C)\zeta^2.$$

The axis of the cone is the $z-$axis.

(3) When $I = B$, the middle moment of inertia, the cone degenerates into the two planes

$$\sqrt{(A - B)}\xi = \pm\sqrt{(B - C)}\zeta$$

which intersect at the middle axis of the inertia ellipsoid.

(4) For values of I on the interval $B < I < A$, Eq. (1.36) can be written in the form

$$(A - I)\xi^2 = (I - B)\eta^2 + (I - C)\zeta^2.$$

The axis of the cone is the $x-$axis.

(5) Finally, when $I = A$, the cone degenerates into $x-$axis.

Example 2

A_0, B_0 and C_0 are the central principal moments of inertia of a given body (the principal moments of inertia at its centre of mass). Show that no point exists, at which the three principal moments of inertia A, B, C are equal (spherical point), unless the smaller two of the central moments are equal.

Solution

Let the point $P(x, y, z)$ be a spherical point referred to axes at the centre of mass O of the body. All axes passing through P are principal axes. Choose the axis OP to be the $z-$axis. The inertia matrix in parallel axes at P is

$$\begin{pmatrix} A + Mz^2 & 0 & 0 \\ 0 & B + Mz^2 & 0 \\ 0 & 0 & C \end{pmatrix},$$

and its diagonal elements are the principal moments at P. Equating those, we get

$$A + Mz^2 = B + Mz^2 = C.$$

This is possible only when $A = B < C$, i.e. the z-axis is the axis of dynamical symmetry and the moment around it is the maximal moment at O. In that case, there are two spherical points $(0, 0, \pm\sqrt{\frac{C-A}{M}})$, both lying on that axis at equal distances from the centre of mass.

Example 3

(a) Find the cone formed by all axes passing through some point O of a rigid body, so that each of them is a principal axis of inertia at one of its points.

Solution

(a) Let xyz be a system of axes at the given point O and $\mathbf{r} = (x, y, z)$ be the position vector of the current mass element in that system. Take an axis z' passing through another point O' at a distance s from O and two orthogonal axes to complete a new Cartesian system $x'y'z'$ at O'. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma$ be three orthogonal unit vectors in the directions of the new axes. We have

$$x' = \mathbf{r} \cdot \boldsymbol{\alpha},\ y' = \mathbf{r} \cdot \boldsymbol{\beta},\ z' = \mathbf{r} \cdot \gamma - s.$$

The condition that Oz' be a principal axis of inertia of the given body at O' gives two equations

$$\int x'z'dm = 0,\ \int y'z'dm = 0,$$

which can be written in the form

$$\boldsymbol{\alpha} \cdot (\gamma\bar{\mathbf{I}} - Ms\mathbf{r}_0) = 0,\ \boldsymbol{\beta} \cdot (\gamma\bar{\mathbf{I}} - Ms\mathbf{r}_0) = 0$$

in which $\mathbf{r}_0$ is the centre of mass and $\bar{\mathbf{I}}$ is the second moments matrix. The vector in brackets is thus orthogonal to both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, i.e. parallel to γ. That is

$$\gamma \times (\gamma\bar{\mathbf{I}} - Ms\mathbf{r}_0) = 0. \tag{1.37}$$

Eliminating s by multiplying scalarly by $\mathbf{r}_0$, we get the equation of the required locus as

$$\mathbf{r}_0 \cdot (\gamma \times \gamma\bar{\mathbf{I}}) = \mathbf{0}. \tag{1.38}$$

In this form, this equation can be expressed in terms of components with respect to any axes at O. In particular, if we take xyz as the principal axes of $\bar{\mathbf{I}}$ (and also of the inertia matrix **I**) at O, then (1.38) takes the form

$$\begin{vmatrix} x_0 & y_0 & z_0 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \bar{A}\gamma_1 & \bar{B}\gamma_2 & \bar{C}\gamma_3 \end{vmatrix} = 0 \tag{1.39}$$

or, equivalently,

$$x_0(\bar{C} - \bar{B})\gamma_2\gamma_3 + y_0(\bar{A} - \bar{C})\gamma_3\gamma_1 + z_0(\bar{B} - \bar{A})\gamma_2\gamma_1 = 0. \tag{1.40}$$

This homogeneous quadratic equation represents a cone with vertex at O, called Ampère's cone. Recalling the relation (1.11), one can write the relations (1.38–1.40) in terms of the inertia matrix by simply dropping bars from $\bar{\mathbf{I}}(\bar{A}, \bar{B}, \bar{C})$. In particular, the last equation becomes

$$x_0(C - B)\gamma_2\gamma_3 + y_0(A - C)\gamma_3\gamma_1 + z_0(B - A)\gamma_2\gamma_1 = 0. \tag{1.41}$$

The three principal axes are generators of that cone. Other generators are passing through the points $\mathbf{r}_0, \mathbf{r}_0\bar{\mathbf{I}}^{-1}, \mathbf{r}_0\mathbf{I}^{-1}$. The general view of Ampère's cone is shown in Fig. 1.6. We shall meet this cone in the sequel, in connection with the stationary rotations of a rigid body known as Staude's rotations (see Chap. 8 Sect. 8.3).

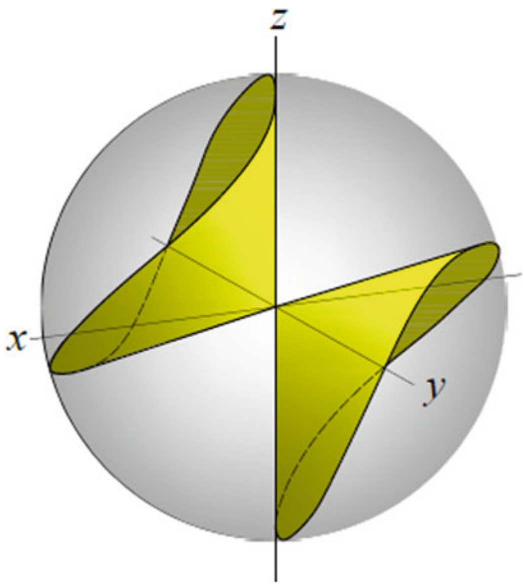

Fig. 1.6 The intersection of Ampere's cone with the unit sphere, for $A : B : C :: 2 : 3 : 4$ and $x_0 : y_0 : z_0 :: 2 : 3 : 7$

1.6 Exercises

1. Show that for an axis to be a principal axis of inertia of a given rigid body at two points on it, this axis should pass through the centre of mass of the body, and in that case, it is a principal axis for all its points.
2. Given a rigid body and a point O on an axis Oz fixed in it, show that one can always find two orthogonal pairs of axes x, y and x_1, y_1 at O orthogonal also to z, such that
   (a) the product of inertia with respect to x, y vanishes;
   (b) the moments of inertia about x_1, y_1 are equal, and find the angle between the two pairs.
3. Given a body with central principal moments of inertia A, B, C, show that its matrix of inertia can be obtained at an arbitrary point of space by replacing the body with an ellipsoid of the same mass M, uniform density and semi-axes $\left(\sqrt{\frac{5(B+C-A)}{2M}}, \sqrt{\frac{5(C+A-B)}{2M}}, \sqrt{\frac{5(A+B-C)}{2M}}\right)$, respectively, in the directions of the central principal axes.
4. Show that the two circular cross-sections of the gyration ellipsoid (1.27) relative to principal axes at O lie in the two planes

$$x\sqrt{C(A-B)} \pm z\sqrt{A(B-C)} = 0.$$

   In Hess' case, the centre of mass (x_0, y_0, z_0) lies on the line drawn from O perpendicular to one of those cross-sections, show that the following conditions are satisfied:

$$y_0 = 0, x_0^2 A(B-C) = z_0^2 C(A-B).$$

5. Show that the equation of the gyration ellipsoid (1.26), in arbitrary Cartesian axes at the same origin, can be put in the form

$$\begin{vmatrix} 1 & x & y & z \\ x & I_{11} & I_{12} & I_{13} \\ y & I_{12} & I_{22} & I_{23} \\ z & I_{13} & I_{23} & I_{33} \end{vmatrix} = 0.$$

6. Show that the equation of the ellipsoid (1.29), inside or on which lies the centre of mass (x, y, z) of a body in an arbitrary system of axes $Oxyz$, can be written in the form

$$\begin{vmatrix} M & Mx & My & Mz \\ Mx & \bar{I}_{11} & \bar{I}_{12} & \bar{I}_{13} \\ My & \bar{I}_{12} & \bar{I}_{22} & \bar{I}_{23} \\ Mz & \bar{I}_{13} & \bar{I}_{23} & \bar{I}_{33} \end{vmatrix} = 0,$$

   $\bar{\mathbf{I}}$ being the second moments matrix of the body in those axes.

Chapter 2
Description of Rotation of a Rigid Body About a Fixed Point

Although in motion of a rigid body about a fixed point, we deal with continuous change of position, i.e. with a sequence of infinitesimal rotations, it turns out that the study of finite rotations is essential in understanding several concepts concerning infinitesimal ones. In this chapter, we present a brief theory of finite rotations to elucidate their properties and relation to infinitesimal rotations. We shall concentrate on the use of Euler's angles and the Euler–Rodrigues or quaternion coordinates as the most relevant to the Lagrangian approach. For space considerations, some other alternative descriptions, like Cayley–Klein, are not considered. We also use the most common notation. For a useful historical survey, the reader may consult [332] and several references therein.

2.1 The Position of a Rigid Body. Euler's Angles

The position of a rigid body moving about a fixed point O is completely determined by the position of a Cartesian coordinate system $Oxyz$ fixed in the body and moving with it with respect to the system $OXYZ$ fixed in space. The number of parameters necessary for the description must be three, the number of degrees of freedom of the rotational motion of the body. Several types of angles are used to this end, and their choice depends mainly on the suitability for the concrete problem of motion under consideration. For example, some angles are most suitable to use in the study of motion of a ship and others are suitable for describing the flight of a plane. Although, of course, different sets of angles must be equivalent for arbitrary position of the body, each of them can have specific privileges or drawbacks in regard to certain application.

One of the most frequently used sets of angles in many applications and especially for theoretical purposes are Euler's angles ψ, θ and φ. They are defined as follows (see Fig. 2.1):

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_2

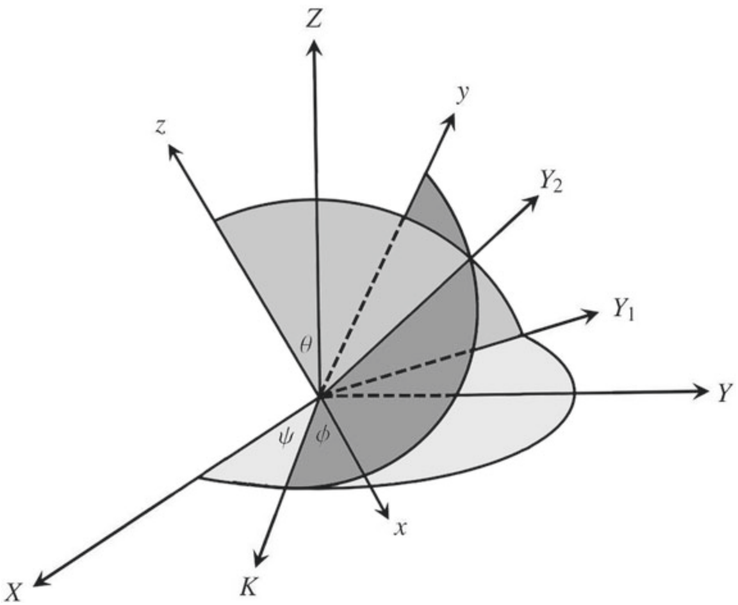

Fig. 2.1 Euler's angles

We begin by the body axes $Oxyz$ coinciding with the space-fixed axes $OXYZ$, and from that position, we give the body system a rotation by an angle ψ (the *precession* angle) about the Z-axis, so that the body system takes the position OKY_1Z. Then, the last system is given a rotation by an angle θ (the angle of *nutation*) about OK. This brings the body system to the position OKY_2z. We now fix the z-axis in the body and give the body system a rotation by an angle φ (the angle of *proper rotation*) about the z-axis to reach its final position $Oxyz$, fixed in the body. In this way, ψ is the angle of rotation of the body about the space axis Z, θ is the angle between z and Z and φ is the angle of rotation about z. The line OK is the intersection of the two planes Oxy and OXY. It is called the *line of nodes*.

Now, let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the space axes XYZ and the body axes xyz, respectively. Let also $\mathbf{n}$ be a unit vector along the nodal line OK and $\mathbf{j}_1, \mathbf{j}_2$ be unit vectors along OY_1 and OY_2, respectively. One can express the components of the fixed unit vectors with respect to the moving axes. For example,

$$
\begin{aligned}
\boldsymbol{\alpha} &= \cos\psi\mathbf{n} - \sin\psi\mathbf{j}_1 \\
&= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, \sin\theta\sin\psi), \\
\boldsymbol{\beta} &= (\sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, -\sin\theta\cos\psi), \\
\boldsymbol{\gamma} &= (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta).
\end{aligned} \tag{2.1}
$$

We can also write the unit vector $\mathbf{n}$ in the body and space axes, respectively, as

$$\mathbf{n} = \cos\varphi\mathbf{i} - \sin\varphi\mathbf{j} = \cos\psi\boldsymbol{\alpha} + \sin\psi\boldsymbol{\beta}. \tag{2.2}$$

Conversely, we can express the moving unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the fixed (space) basis. They have the form

$$\begin{aligned}
\mathbf{i} &= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, \sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, \sin\theta\sin\varphi),\\
\mathbf{j} &= (-\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, \sin\theta\cos\varphi),\\
\mathbf{k} &= (\sin\theta\sin\psi, -\sin\theta\cos\psi, \cos\theta).
\end{aligned} \tag{2.3}$$

2.2 The Rotation Matrix

Let (X, Y, Z) be the coordinates of a point P in the system of axes XYZ. When the system XYZ is rotated by an angle Φ around Z-axis, the point P is displaced to the new point $P'(X', Y', Z')$. One can easily write

$$\begin{aligned}
X' &= X\cos\Phi + Y\sin\Phi,\\
Y' &= -X\sin\Phi + Y\cos\Phi,\\
Z' &= Z,
\end{aligned}$$

which can be put in the matrix form

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \mathbf{R}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

where

$$\mathbf{R} = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{R}$ is called the rotation matrix. Note that $\mathbf{R}$ is an orthogonal matrix (all rows are orthogonal and also columns), its inverse is its transpose ($\mathbf{R}^{-1} = \mathbf{R}^T$) and its determinant is 1.

Now we consider the characteristic equation of $\mathbf{R}$. That is

$$\begin{vmatrix} \cos\Phi - \lambda & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0.$$

In the final form, this becomes

$$(1-\lambda)(\lambda^2 - 2\lambda\cos\Phi + 1) = 0. \tag{2.4}$$

As we see, one of the characteristic roots equals 1. It corresponds to the eigenvector $(0, 0, 1)$, which coincides with the axis of the rotation. The other two roots are complex

$$\cos\Phi \pm \sqrt{\cos^2\Phi - 1} = \cos\Phi \pm i\sin\Phi = e^{\pm i\Phi}.$$

Thus, the eigenvector vector corresponding to the unit eigenvalue of the rotation matrix coincides with the axis of the rotation and the argument of the complex pair of eigenvalues directly expresses the angle of rotation.

Now we apply the same conception to an arbitrary rotation. Let $\mathbf{r}$ be a vector whose components are (X, Y, Z) in the space-fixed axes and (x, y, z) in the body-fixed axes. We can find the relations between the components of the vector in the two systems as follows:

The components of $\mathbf{r}$ in the system OKY_1Z after a rotation by an angle ψ around the Z-axis

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \mathbf{R}_\psi \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2.5}$$

and in the system OKY_2Z after a rotation by an angle θ around OK

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \mathbf{R}_\theta \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}. \tag{2.6}$$

Finally, after a rotation by an angle φ around the z-axis, we find the components of $\mathbf{r}$ in the body system

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathbf{R}_\varphi \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\ &= \mathbf{R} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \mathbf{R} &= \mathbf{R}_\varphi \mathbf{R}_\theta \mathbf{R}_\psi \\ &= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi & \sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi & \sin\theta\sin\varphi \\ -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi & -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi & \sin\theta\cos\varphi \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}. \tag{2.8}$$

Comparing with (2.1), we conclude that

$$\mathbf{R} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

and again we note that $\mathbf{R}$ has all rows and columns orthogonal unit vectors. Its inverse is its transpose $\mathbf{R}^{-1} = \mathbf{R}^T$ (check that $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \boldsymbol{\delta}$) and its determinant equals 1. Note also that the columns of are the components of the fixed unit vectors referred to the moving (body) axes while its rows are the components of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, in the directions of the movable axes xyz, referred to the fixed axes XYZ.

2.2.1 The Angle of Rotation

We now proceed to form the characteristic equation of the rotation matrix (2.8). After some manipulations and factorization, we get

$$\begin{aligned} |\mathbf{R} - \lambda\boldsymbol{\delta}| &= (1-\lambda)\{\lambda^2 + [(1-\cos\theta) - (1+\cos\theta)\cos(\psi+\varphi)]\lambda + 1\} \\ &= (1-\lambda)\{\lambda^2 + 2[1 - 2\cos^2\frac{\theta}{2}\cos^2\frac{\psi+\varphi}{2}]\lambda + 1\} = 0. \end{aligned} \tag{2.9}$$

From here, we see that one of the characteristic roots of the rotation matrix is 1. The eigenvector $\mathbf{v}$ corresponding to that root satisfies the equation

$$\mathbf{R}\mathbf{a} = \mathbf{a},$$

i.e. the rotation represented by the matrix $\mathbf{R}$ leaves that vector unchanged. This vector coincides with the axis of the rotation. On the other hand, to obtain the rotation angle of the rotation Φ, we compare the quadratic factors in (2.9) and (2.4). We get

$$\cos\Phi = 2\cos^2\frac{\theta}{2}\cos^2\frac{\psi+\varphi}{2} - 1. \tag{2.10}$$

This can be also written in the form

$$\cos\frac{\Phi}{2} = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2} \tag{2.11}$$

in which we have chosen positive sign.

Remark: In spite of their simplicity, Euler's angles suffer from the defect that the two angles ψ and φ lose their independence when the third angle θ takes one of the values 0 or π. In those cases, the rotation matrix takes the form

$$\begin{pmatrix} \cos(\psi \pm \varphi) & \sin(\psi \pm \varphi) & 0 \\ -\sin(\psi \pm \varphi) & \cos(\psi \pm \varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which involves not the two angles, but their sum when $\theta = 0$ and difference when $\theta = \pi$.

2.3 Description of Finite Rotation

We have just seen that the finite rotation can be always and completely represented by a rotation matrix. On the other hand, it is evident that such rotation can be completely determined by giving the axis of rotation and the angle of rotation about that axis, i.e. one can say the rotation is determined by a scalar quantity and a direction. Nevertheless, it cannot be represented in the full sense by a vector, since an essential rule of vector algebra, the commutation rule, is not followed by matrices. This means that performing two consequent rotations R_1 and then R_2 gives different resultant from that of reverse order R_2 and then R_1. This can be clearly illustrated by the following

Example: Let us perform to the body in Fig. 2.2a two consecutive rotations, each by a right angle,[1] the first about the x-axis and the second about the y-axis. Figure 2.2b shows what we get in this case, but Fig. 2.2c shows the completely different result of performing the rotations in the reverse order.

Analytically, R_1 and R_2 can be represented by the matrices

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \mathbf{R}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Resultant of the first sequence is

$$\mathbf{S} = \mathbf{R}_2\mathbf{R}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the second sequence

[1] We mean rotation by an angle described in the positive sense about an axis, i.e. counterclockwise as viewed from the positive end of that axis.

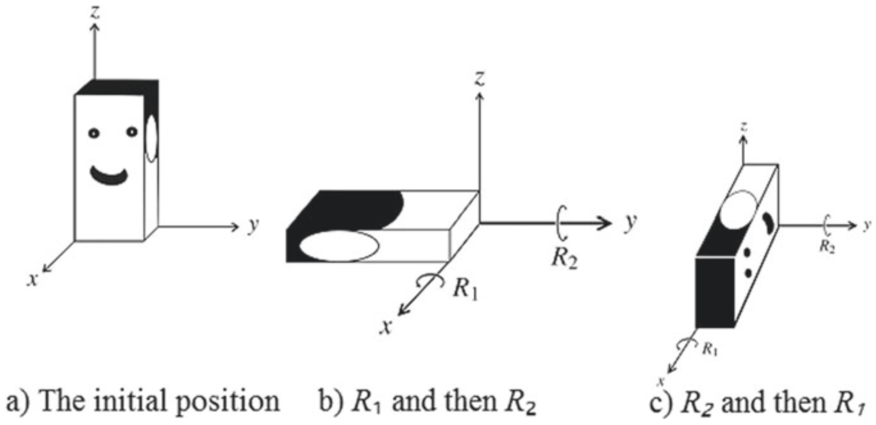

Fig. 2.2 Finite rotations are not commutative

$$\mathbf{S}' = \mathbf{R}_1\mathbf{R}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

The two matrices **S** and **S**′ are obviously different. We can go a little further and construct the characteristic equation for **S**. That is

$$1 - \lambda^3 = (1 - \lambda)(\lambda^2 + \lambda + 1) = 0,$$

and its roots are $1, e^{\pm i\frac{2\pi}{3}}$. This means that the angle of the rotation **S** is equal to $\frac{2\pi}{3}$. To find the axis of the rotation, we solve the equations

$$\mathbf{S}\mathbf{v}' = \mathbf{v}',$$

which gives the column vector

$$v' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, **S** is a rotation by angle $\frac{2\pi}{3}$ around the axis in the direction parallel to the vector $\mathbf{v} = (1, 1, 1)$.

Similarly, one can show that **S**′ is a rotation by angle $\frac{2\pi}{3}$ around the vector $\mathbf{v} = (1, 1, -1)$.

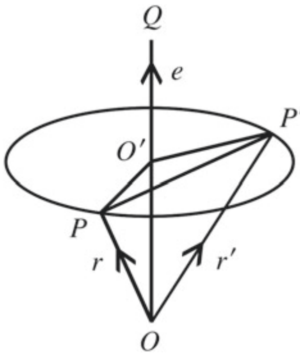

Fig. 2.3 Geometry of a rotation vector

2.4 Representation of Finite Rotation by Means of a Vector

The above example shows that finite rotation cannot be completely represented by a vector as we represent the position vector or the velocity of a particle. Thus, we shall try now to find the formula that expresses the finite rotation as a vector quantity and to find a suitable rule for the resultant of two rotations, a rule that must account for the non-commutation of rotations.

2.4.1 The Rotation Vector

Let us begin with some vector $\mathbf{r} = \overrightarrow{OP}$ and an axis $\overline{OQ}$ with a unit vector $\mathbf{e}$ in its direction. The rotation of $\mathbf{r}$ by an angle Φ around OQ in the positive direction carries $\mathbf{r}$ to its new position $\mathbf{r}' = \overrightarrow{OP'}$ and the point P along the circular arc $\widehat{PP'}$ to P' (4.1). Let also O' be the centre of PP'. Our aim now is to express $\mathbf{r}'$ in terms of $\mathbf{r}$ and the angle and direction of the rotation (see Fig. 2.3).

The plane Fig. 2.4 shows the circle $O'PP'$. $O'R$ is orthogonal to PP' and RS is orthogonal to $O'P$. Note that $\mathbf{e}$ is the outward unit vector normal to the plane of the figure and $SR = O'P \sin\frac{\Phi}{2}\cos\frac{\Phi}{2}$, so that $SR = \mathbf{e} \times O'P \sin\frac{\Phi}{2}\cos\frac{\Phi}{2}$.

From simple geometry, we find that

$$PP' = 2PR = 2(PS + SR) = 2[-O'P \sin^2\frac{\Phi}{2} + \mathbf{e} \times O'P \sin\frac{\Phi}{2}\cos\frac{\Phi}{2}].$$

But since

$$O'P = \mathbf{r} - (\mathbf{r}\cdot\mathbf{e})\mathbf{e} = (\mathbf{e}\cdot\mathbf{e})\mathbf{r} - (\mathbf{e}\cdot\mathbf{r})\mathbf{e} = -\mathbf{e} \times (\mathbf{e} \times \mathbf{r}),$$

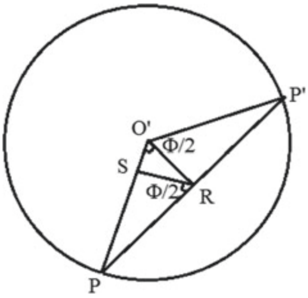

Fig. 2.4 Rotation angle

then we obtain

$$PP' = 2[\sin^2\frac{\Phi}{2}\mathbf{e}\times(\mathbf{e}\times\mathbf{r})+\sin\frac{\Phi}{2}\cos\frac{\Phi}{2}\mathbf{e}\times\mathbf{r}],$$

so that we can finally write

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + PP' \\ &= \mathbf{r} + 2[\sin\frac{\Phi}{2}\cos\frac{\Phi}{2}\mathbf{e}\times\mathbf{r}+\sin^2\frac{\Phi}{2}\mathbf{e}\times(\mathbf{e}\times\mathbf{r})].\end{aligned} \tag{2.12}$$

This is the Rodrigues formula, which expresses the rotated vector $\mathbf{r}'$ in terms of the initial vector $\mathbf{r}$, the angle of rotation Φ and the direction $\mathbf{e}$ of the rotation axis. It can be written also in the form

$$\mathbf{r}' = \mathbf{r} + (\sin\Phi)\mathbf{e}\times\mathbf{r}+(1-\cos\Phi)\mathbf{e}\times(\mathbf{e}\times\mathbf{r}). \tag{2.13}$$

It is valid for arbitrary angle and arbitrary direction of the rotation. As expected, for a point on the axis of rotation, $\mathbf{r} = \mathbf{e}$ and $\mathbf{r}' = \mathbf{r}$. Also, a rotation with an angle 2π brings all points of space to their initial positions.

To push forward the concept of a vector representing a finite rotation, we assume that the rotation angle $\Phi \neq \pi$, i.e. $\cos\frac{\Phi}{2} \neq 0$. Then we can write (2.12) in the form

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + 2\cos^2\frac{\Phi}{2}[\tan\frac{\Phi}{2}\mathbf{e}\times\mathbf{r}+\tan^2\frac{\Phi}{2}\mathbf{e}\times(\mathbf{e}\times\mathbf{r})] \\ &= \mathbf{r} + \frac{2}{1+\tan^2\frac{\Phi}{2}}[\tan\frac{\Phi}{2}\mathbf{e}\times\mathbf{r}+\tan^2\frac{\Phi}{2}\mathbf{e}\times(\mathbf{e}\times\mathbf{r})].\end{aligned}$$

Introducing the notation

$$\boldsymbol{\rho} = \tan\frac{\Phi}{2}\mathbf{e}, \tag{2.14}$$

then we rewrite the last formula as

$$\mathbf{r}' = \mathbf{r} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times [\mathbf{r} + \boldsymbol{\rho} \times \mathbf{r}]. \tag{2.15}$$

Using (2.14), one can verify that

(1) $\boldsymbol{\rho}(\Phi + 2\pi, \mathbf{e}) = \boldsymbol{\rho}(\Phi, \mathbf{e})$.
(2) $\boldsymbol{\rho}(-\Phi, -\mathbf{e}) = \boldsymbol{\rho}(\Phi, \mathbf{e})$.
Those properties are geometrically obvious.
(3) Formula (2.14) is not suitable for expressing any rotation with an angle π about any axis. That is the singular point of the function tan . This is not related to the rotation itself, but due to the way of representing the rotation as a vector in (2.14). The previous formulas (2.12), (2.13) are still valid for the angle $\Phi = \pi$.
(4) Finite rotation is thus represented by a vector, which may be written in terms of its components as $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$ or $\boldsymbol{\rho} = \rho_1\mathbf{i} + \rho_2\mathbf{j} + \rho_3\mathbf{k}$, but this determines the magnitude and direction of the rotation and does not mean at all that the rotation is the resultant of its parts, or equivalent to any sequence of those parts.
(5) It is evident that rotation vectors do not commute and cannot be summed according to rules of vector algebra. However, it can be easily shown that infinitesimally small rotations do commute and obey the rule of summation of vectors.
Let $\boldsymbol{\rho}_1$ be a rotation by a small angle Φ_1 the rotation vector $\boldsymbol{\rho}_1 = \tan\frac{\Phi_1}{2}\mathbf{e}_1 = \frac{\Phi_1}{2}\mathbf{e}_1$. After neglecting nonlinear terms in the rotation vector, formula (2.15) takes the form

$$\mathbf{r}' = \mathbf{r} + 2\boldsymbol{\rho}_1 \times \mathbf{r}. \tag{2.16}$$

If $\boldsymbol{\rho}_1$ is followed by another small rotation $\boldsymbol{\rho}_2 = \frac{\Phi_2}{2}\mathbf{e}_2$, the vector $\mathbf{r}'$ is transformed to

$$\begin{aligned} \mathbf{r}'' &= \mathbf{r}' + 2\boldsymbol{\rho}_2 \times \mathbf{r}' \\ &= \mathbf{r} + 2\boldsymbol{\rho}_1 \times \mathbf{r} + 2\boldsymbol{\rho}_2 \times (\mathbf{r} + 2\boldsymbol{\rho}_1 \times \mathbf{r}). \end{aligned} \tag{2.17}$$

Neglecting the nonlinear term, we get

$$\mathbf{r}'' = \mathbf{r} + 2(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2) \times \mathbf{r}. \tag{2.18}$$

Small rotations are summed according to vector addition rule, and their sum does not depend on the order of the rotations.

Now we return to formula (2.14) to see how a rotation $\boldsymbol{\rho} = \tan\frac{\Phi}{2}\mathbf{e}$ acts on the unit vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ fixed in the directions of XYZ and bring them to be coincident with the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively. According to (2.15), we have

$$
\begin{aligned}
\mathbf{i} &= \boldsymbol{\alpha} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times (\boldsymbol{\alpha} + \boldsymbol{\rho} \times \boldsymbol{\alpha}), \\
\mathbf{j} &= \boldsymbol{\beta} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times (\boldsymbol{\beta} + \boldsymbol{\rho} \times \boldsymbol{\beta}), \\
\mathbf{k} &= \boldsymbol{\gamma} + \frac{2}{1+\boldsymbol{\rho}^2}\boldsymbol{\rho} \times (\boldsymbol{\gamma} + \boldsymbol{\rho} \times \boldsymbol{\gamma}).
\end{aligned} \tag{2.19}
$$

One immediately notices that

$$\boldsymbol{\rho} \cdot \mathbf{i} = \boldsymbol{\rho} \cdot \boldsymbol{\alpha}, \boldsymbol{\rho} \cdot \mathbf{j} = \boldsymbol{\rho} \cdot \boldsymbol{\beta}, \boldsymbol{\rho} \cdot \mathbf{k} = \boldsymbol{\rho} \cdot \boldsymbol{\gamma},$$

i.e. the components of the rotation vector are the same in the directions of the initial and final axes. We shall denote those components by $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$.

We now express the rotation matrix in terms of the components of the rotation vector

$$
\begin{aligned}
R &= \begin{pmatrix} \mathbf{i}\cdot\boldsymbol{\alpha} & \mathbf{i}\cdot\boldsymbol{\beta} & \mathbf{i}\cdot\boldsymbol{\gamma} \\ \mathbf{j}\cdot\boldsymbol{\alpha} & \mathbf{j}\cdot\boldsymbol{\beta} & \mathbf{j}\cdot\boldsymbol{\gamma} \\ \mathbf{k}\cdot\boldsymbol{\alpha} & \mathbf{k}\cdot\boldsymbol{\beta} & \mathbf{k}\cdot\boldsymbol{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} 1 - 2\frac{\rho_2^2+\rho_3^2}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_1\rho_2+\rho_3}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_1\rho_3-\rho_2}{1+\boldsymbol{\rho}^2} \\ 2\frac{\rho_1\rho_2-\rho_3}{1+\boldsymbol{\rho}^2} & 1 - 2\frac{\rho_1^2+\rho_3^2}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_2\rho_3+\rho_1}{1+\boldsymbol{\rho}^2} \\ 2\frac{\rho_1\rho_3+\rho_2}{1+\boldsymbol{\rho}^2} & 2\frac{\rho_2\rho_3-\rho_1}{1+\boldsymbol{\rho}^2} & 1 - 2\frac{\rho_1^2+\rho_2^2}{1+\boldsymbol{\rho}^2} \end{pmatrix}.
\end{aligned} \tag{2.20}
$$

2.5 Hamilton–Rodrigues' Parameters

We now introduce the four quantities $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to express the three components of the rotation vector, such that

$$(\rho_1, \rho_2, \rho_3) = \frac{(\lambda_1, \lambda_2, \lambda_3)}{\lambda_0}. \tag{2.21}$$

As we have one redundant parameter, we assume that the new parameters satisfy the condition

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1, \tag{2.22}$$

so that the end of the four-dimensional vector $\boldsymbol{\Lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ lies on a three-dimensional sphere of unit radius. This implies the relation

$$\lambda_0\dot{\lambda}_0 + \lambda_1\dot{\lambda}_1 + \lambda_2\dot{\lambda}_2 + \lambda_3\dot{\lambda}_3 = 0. \tag{2.23}$$

From (2.21), we calculate

$$1 + \rho^2 = 1 + \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{\lambda_0^2} = \frac{1}{\lambda_0^2}, \tag{2.24}$$

and substituting (2.21) in (2.20) and using the last relation, we obtain the expression of the rotation matrix

$$\mathbf{R} = \begin{pmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_0\lambda_3 + \lambda_1\lambda_2) & 2(-\lambda_0\lambda_2 + \lambda_1\lambda_3) \\ 2(-\lambda_0\lambda_3 + \lambda_1\lambda_2) & \lambda_0^2 + \lambda_2^2 - \lambda_1^2 - \lambda_3^2 & 2(\lambda_0\lambda_1 + \lambda_2\lambda_3) \\ 2(\lambda_0\lambda_2 + \lambda_1\lambda_3) & 2(-\lambda_0\lambda_1 + \lambda_2\lambda_3) & \lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2 \end{pmatrix}. \tag{2.25}$$

This form of the rotation matrix is more symmetric than that in terms of the rotation vector or in terms of Euler's angles. Moreover, it does not have the problem of degeneration of Euler's angles at $\theta = 0$ or π, nor the singularity of the rotation vector corresponding to a rotation by an angle π. This makes Euler–Rodrigues' parameters in certain problems appropriate for use as variables describing motion and finite rotations of the rigid body.

On the other hand, one can readily notice that the two sets of the Hamilton–Rodrigues parameters $\pm\Lambda$ correspond to the same rotation matrix. Thus, any expression designating a quantity of physical meaning should contain only even terms in Λ, otherwise it will be double-valued on the group of rotations $SO3$. This remark will have some implications in later chapters.

Remark 12 The expression (2.25) for the rotation matrix can be decomposed into three parts of simpler structure (two symmetric and one antisymmetric):

$$\begin{aligned} \mathbf{R} = (2\lambda_0^2 - 1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{pmatrix} \\ +2\lambda_0\begin{pmatrix} 0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & 0 \end{pmatrix} \end{aligned} \tag{2.26}$$

or in the shorter tensor form

$$R_{ij} = (2\lambda_0^2 - 1)\delta_{ij} + 2\lambda_i\lambda_j + 2\lambda_0\epsilon_{ijk}\lambda_k, \tag{2.27}$$

where δ is the Kronecker delta and ϵ is the Levi-Civita tensor.

Remark 13 It is clear from (2.25) that the points $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ and $(-\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3)$ represent the same rotation matrix. The sphere (2.22) covers the configuration space of the rotating body twice. The configuration space can, thus, be represented by one half of that sphere, say, the half on which $\lambda_0 \geqq 0$.

Remark 14 From (2.24) and (2.14), we have

$$\frac{1}{\lambda_0^2} = 1 + \rho^2 = 1 + \tan^2\frac{\Phi}{2} = \sec^2\frac{\Phi}{2},$$

so that

$$\lambda_0 = \cos\frac{\Phi}{2}, \tag{2.28}$$

and hence

$$\begin{aligned}(\lambda_1, \lambda_2, \lambda_3) &= \lambda_0 \boldsymbol{\rho}\\ &= \cos\frac{\Phi}{2}\tan\frac{\Phi}{2}\mathbf{e}\\ &= \sin\frac{\Phi}{2}\mathbf{e}.\end{aligned}$$

Thus, we can write the following expression for the Euler–Rodriguez parameters

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (\cos\frac{\Phi}{2}, \sin\frac{\Phi}{2}\mathbf{e}). \tag{2.29}$$

2.6 The Angular Velocity Vector

During motion of the body, Euler's angles change with time t, and also the rotation matrix. To express the time derivatives of an arbitrary quantity, one may first check the following relations using expressions (2.1)

$$\begin{array}{lll}\frac{\partial\boldsymbol{\alpha}}{\partial\psi} = \boldsymbol{\alpha}\times\boldsymbol{\gamma}, & \frac{\partial\boldsymbol{\beta}}{\partial\psi} = \boldsymbol{\beta}\times\boldsymbol{\gamma}, & \frac{\partial\boldsymbol{\gamma}}{\partial\psi} = 0,\\ \frac{\partial\boldsymbol{\alpha}}{\partial\theta} = \boldsymbol{\alpha}\times\mathbf{n}, & \frac{\partial\boldsymbol{\beta}}{\partial\theta} = \boldsymbol{\beta}\times\mathbf{n}, & \frac{\partial\boldsymbol{\gamma}}{\partial\theta} = \boldsymbol{\gamma}\times\mathbf{n},\\ \frac{\partial\boldsymbol{\alpha}}{\partial\varphi} = \boldsymbol{\alpha}\times\mathbf{k}, & \frac{\partial\boldsymbol{\beta}}{\partial\varphi} = \boldsymbol{\beta}\times\mathbf{k}, & \frac{\partial\boldsymbol{\gamma}}{\partial\varphi} = \boldsymbol{\gamma}\times\mathbf{k}.\end{array} \tag{2.30}$$

From those, we get

$$\dot{\boldsymbol{\alpha}} = \frac{\partial\boldsymbol{\alpha}}{\partial\psi}\dot{\psi} + \frac{\partial\boldsymbol{\alpha}}{\partial\theta}\dot{\theta} + \frac{\partial\boldsymbol{\alpha}}{\partial\varphi}\dot{\varphi} \tag{2.31}$$

$$\begin{aligned}&= \boldsymbol{\alpha}\times\boldsymbol{\gamma}\dot{\psi} + \boldsymbol{\alpha}\times\mathbf{n}\dot{\theta} + \boldsymbol{\alpha}\times\mathbf{k}\dot{\varphi}\\ &= \boldsymbol{\alpha}\times(\dot{\psi}\boldsymbol{\gamma}+\dot{\theta}\mathbf{n}+\dot{\varphi}\mathbf{k}).\end{aligned} \tag{2.32}$$

Let us now introduce the notation

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{\gamma}+\dot{\theta}\mathbf{n}+\dot{\varphi}\mathbf{k}. \tag{2.33}$$

The vector $\boldsymbol{\omega}$ is called the angular velocity of the body and it is, in fact, the usual vector sum of the three vectors $\dot{\psi}\boldsymbol{\gamma}, \dot{\theta}\mathbf{n}, \dot{\varphi}\mathbf{k}$, which represent the angular velocities $\dot{\psi}$, $\dot{\theta}$ and $\dot{\varphi}$ about the axes Z, K and z, respectively. In a similar way, we can get two expressions for the derivatives of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, so that we can write

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}. \tag{2.34}$$

Those equations satisfied by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are called Poisson's equations and they express the constancy of those vectors in space, as we shall see soon. They play an important role in the dynamics of rigid body as will be seen in due course.

2.7 Space and Relative Time Rates of Change of a Vector

During the motion of the body, the three unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ fixed in the body along the axes x, y, z change with time. The rate of change of one of them, $\mathbf{i}$, say, is the velocity $\frac{d\mathbf{i}}{dt}$ of its end point in space. Hence, we have

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i}, \ \frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j}, \ \frac{d\mathbf{k}}{dt} = \boldsymbol{\omega} \times \mathbf{k}, \tag{2.35}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the body.

Now, let $\mathbf{u}$ be a vector given by its components (u_1, u_2, u_3) in the body system of axes, so that we write

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}. \tag{2.36}$$

The time derivative of this vector is

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k} + u_1\frac{d\mathbf{i}}{dt} + u_2\frac{d\mathbf{j}}{dt} + u_3\frac{d\mathbf{k}}{dt}.$$

Using (2.35) in the last three terms, we get

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k} + u_1\boldsymbol{\omega} \times \mathbf{i} + u_2\boldsymbol{\omega} \times \mathbf{j} + u_3\boldsymbol{\omega} \times \mathbf{k} \\ &= \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k} + \boldsymbol{\omega} \times (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}). \end{aligned}$$

Now we introduce the notation $\dot{\mathbf{u}} = \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k}$, i.e. $\dot{\mathbf{u}}$ is the time derivative of the vector $\mathbf{u}$ as if the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ were constant vectors, or as seen by an observer fixed in the body and moving with it. This derivative will be called the relative derivative or the relative rate of change of $\mathbf{u}$. The last relation becomes

$$\frac{d\mathbf{u}}{dt} = \dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}. \tag{2.37}$$

Thus, we have split the space derivative into two terms: the relative derivative $\dot{\mathbf{u}}$ in the body system and the term $\boldsymbol{\omega} \times \mathbf{u}$ resulting from the rotation of the body system. When $\boldsymbol{\omega} = \mathbf{0}$ the two derivatives coincide.

As an example, we apply this rule to the three vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ fixed in space. As $\frac{d\boldsymbol{\alpha}}{dt} = \frac{d\boldsymbol{\beta}}{dt} = \frac{d\boldsymbol{\gamma}}{dt} = 0$, we have

$$\dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}, \dot{\boldsymbol{\beta}} + \boldsymbol{\omega} \times \boldsymbol{\beta} = \mathbf{0}, \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{2.38}$$

so that we again obtain Poisson's equations (2.34).

2.7.1 Components of the Angular Velocity in the Body Axes and Space Axes

The direction of the angular velocity at the fixed point determines a line called the instantaneous axis of rotation. Points of the body lying on that line at any moment of time are instantaneously at rest. The magnitude of $\boldsymbol{\omega}$ is a measure of the angular speed of rotation of the body. In case of rotation about a fixed axis, the angular velocity is the time rate of change of the angle of rotation about that axis, but it is not possible in general to write a rotation angle such that the angular velocity is represented as its rate of change.

We shall denote by p, q, r and p', q', r' the components of the angular velocity $\boldsymbol{\omega}$ in the moving and in the fixed axes, respectively. Using (2.2) and (2.3) together with (2.33), we have

$$\begin{aligned} p &= \dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi, \\ q &= \dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi, \\ r &= \dot{\psi} \cos\theta + \dot{\varphi}, \end{aligned} \tag{2.39}$$

and similarly, we write

$$\begin{aligned} p' &= \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \\ q' &= -\dot{\varphi} \sin\theta \cos\psi + \dot{\theta} \sin\psi, \\ r' &= \dot{\varphi} \cos\theta + \dot{\psi}. \end{aligned} \tag{2.40}$$

2.7.2 The Use of the Euler–Rodrigues Parameters

From (2.11) and (2.28), we have $\lambda_0 = \cos\frac{\theta}{2} \cos\frac{\psi+\varphi}{2}$. One can easily obtain expressions for the other three parameters by comparing corresponding elements of the rotation matrix $\mathbf{R}$ in formulas (2.8) and (2.25). It is even easier to compare the antisymmetric parts, e.g. one can see from (2.26) and (2.8) that

$$\lambda_1 = \frac{R_{23} - R_{32}}{4\lambda_0} = \frac{\sin\theta(\cos\varphi + \cos\psi)}{4\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}}$$
$$= \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}$$

and so on, so that we get the expressions

$$\lambda_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2},\ \lambda_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2},$$
$$\lambda_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2},\ \lambda_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}. \tag{2.41}$$

The angular velocity has the expression

$$p = 2(\lambda_0\dot{\lambda}_1 - \lambda_1\dot{\lambda}_0 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3),$$
$$q = 2(\lambda_0\dot{\lambda}_2 - \lambda_2\dot{\lambda}_0 + \lambda_1\dot{\lambda}_3 - \lambda_3\dot{\lambda}_1),$$
$$r = 2(\lambda_0\dot{\lambda}_3 - \lambda_3\dot{\lambda}_0 + \lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2). \tag{2.42}$$

This may be written in the vector form

$$\boldsymbol{\omega} = 2[\lambda_0\dot{\boldsymbol{\lambda}} - \dot{\lambda}_0\boldsymbol{\lambda} - \boldsymbol{\lambda}\times\dot{\boldsymbol{\lambda}}], \tag{2.43}$$

where $\boldsymbol{\lambda}$ denotes the three-dimensional vector $(\lambda_1, \lambda_2, \lambda_3)$. In this notation, (2.22) and (2.23) take the form

$$\lambda_0^2 + \boldsymbol{\lambda}^2 = 1, \tag{2.44}$$
$$\lambda_0\dot{\lambda}_0 + \boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}} = 0. \tag{2.45}$$

The formula (2.43) together with (2.44), (2.45) can be used to obtain a remarkable expression for the square of the angular velocity. Squaring both sides of (2.43) and noting that the third term on the right-hand side is orthogonal to the other two, we write

$$\omega^2 = 4[\lambda_0^2\dot{\boldsymbol{\lambda}}^2 - 2\lambda_0\dot{\lambda}_0\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}}\dot{\boldsymbol{\lambda}} + \dot{\lambda}_0^2\boldsymbol{\lambda}^2 + |\boldsymbol{\lambda}\times\dot{\boldsymbol{\lambda}}|^2]$$
$$= 4[\lambda_0^2\dot{\boldsymbol{\lambda}}^2 - 2\lambda_0\dot{\lambda}_0\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}} + \dot{\lambda}_0^2\boldsymbol{\lambda}^2 + \boldsymbol{\lambda}^2\dot{\boldsymbol{\lambda}}^2 - (\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}})^2]$$
$$= 4[(\lambda_0^2 + \boldsymbol{\lambda}^2)\dot{\boldsymbol{\lambda}}^2 + 2\lambda_0^2\dot{\lambda}_0^2 + \dot{\lambda}_0^2\boldsymbol{\lambda}^2 - (\lambda_0\dot{\lambda}_0)^2$$
$$= 4[(\lambda_0^2 + \boldsymbol{\lambda}^2)\dot{\boldsymbol{\lambda}}^2 + (\lambda_0^2 + \boldsymbol{\lambda}^2)\dot{\lambda}_0^2]$$
$$= 4(\lambda_0^2 + \boldsymbol{\lambda}^2)(\dot{\lambda}_0^2 + \dot{\boldsymbol{\lambda}}^2)$$
$$= 4(\dot{\lambda}_0^2 + \dot{\boldsymbol{\lambda}}^2),$$

so that, finally, we have

$$\omega^2 = p^2 + q^2 + r^2 = 4(\dot{\lambda}_1^2 + \dot{\lambda}_2^2 + \dot{\lambda}_3^2 + \dot{\lambda}_0^2) = 4\dot{\mathbf{\Lambda}}^2. \tag{2.46}$$

That is four times the square of the speed of the point $\mathbf{\Lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ moving on the unit sphere (2.22).

2.8 Quaternions and Representation of Finite Rotation

Quaternions, or hypercomplex numbers, discovered by Hamilton, are a generalization of the ordinary complex number system. A quaternion is composed of one real component and three imaginary ones. A general quaternion can be written in the form

$$Q = (a, A_1, A_2, A_3) = a + A_1 i + A_2 j + A_3 k, \tag{2.47}$$

where i, j, k are imaginary units satisfying the multiplication rules

$$\begin{aligned} &i^2 = j^2 = k^2 = 1, \\ &ij = -ji = k, jk = -kj = i, ki = -ik = j. \end{aligned} \tag{2.48}$$

In (2.47), the first part a is an ordinary real part and the remaining parts can be viewed as a vector $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, and thus, we write

$$Q = a + \mathbf{A}. \tag{2.49}$$

Now it is easy to check that the product of the quaternions Q and $Q' = a' + \mathbf{A}'$ according to the rules (2.48) can be put in the usual form using scalar and vector products of vectors as

$$QQ' = aa' - \mathbf{A} \cdot \mathbf{A}' + a\mathbf{A}' + a'\mathbf{A} + \mathbf{A} \times \mathbf{A}', \tag{2.50}$$

and if we define the conjugate quaternion $\bar{Q} = a - \mathbf{A}$, we easily note that the quantity

$$Q\bar{Q} = \bar{Q}Q = a^2 + \mathbf{A} \cdot \mathbf{A} \tag{2.51}$$

is a positive real number which we adopt as the squared magnitude of the quaternion

$$|Q| = \sqrt{Q\bar{Q}} = \sqrt{a^2 + A_1^2 + A_2^2 + A_3^2}. \tag{2.52}$$

From the last, we get that for a non-zero quaternion Q

$$Q\frac{\bar{Q}}{|Q|^2} = 1. \tag{2.53}$$

That is the multiplicative inverse of Q (which satisfies $QQ^{-1} = Q^{-1}Q = 1$) is $Q^{-1} = \frac{\bar{Q}}{|Q|^2}$.

Consider now the quaternion formed by Euler–Rodrigues' parameters $Q = \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k$. It can be put in the form

$$Q = \lambda_0 + \lambda_0 \boldsymbol{\rho} = \cos\frac{\Phi}{2}(1 + \boldsymbol{\rho}). \tag{2.54}$$

Note that

$$|Q| = 1,\ Q^{-1} = \cos\frac{\Phi}{2}(1 - \boldsymbol{\rho}). \tag{2.55}$$

Let $\mathbf{r}$ be a vector, i.e. a quaternion with zero real part. Applying (2.50), we calculate the product

$$\begin{aligned}
Q\mathbf{r}Q^{-1} &= \cos^2\frac{\Phi}{2}(1 + \boldsymbol{\rho})[(0 + \mathbf{r})(1 - \boldsymbol{\rho})] \\
&= \cos^2\frac{\Phi}{2}(1 + \boldsymbol{\rho})(\boldsymbol{\rho}\cdot\mathbf{r} + \mathbf{r} + \boldsymbol{\rho}\times\mathbf{r}) \\
&= \cos^2\frac{\Phi}{2}[\boldsymbol{\rho}\cdot\mathbf{r} + \mathbf{r} + \boldsymbol{\rho}\times\mathbf{r} - \boldsymbol{\rho}\cdot(\mathbf{r} + \boldsymbol{\rho}\times\mathbf{r}) \\
&\quad +(\boldsymbol{\rho}\cdot\mathbf{r})\boldsymbol{\rho} + \boldsymbol{\rho}\times(\mathbf{r}+\boldsymbol{\rho}\times\mathbf{r})] \\
&= \cos^2\frac{\Phi}{2}[\mathbf{r} + 2\boldsymbol{\rho}\times\mathbf{r} + (\boldsymbol{\rho}\cdot\mathbf{r})\boldsymbol{\rho} + \boldsymbol{\rho}\times(\boldsymbol{\rho}\times\mathbf{r})] \\
&= \cos^2\frac{\Phi}{2}[\mathbf{r} + 2\boldsymbol{\rho}\times\mathbf{r} + 2\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\mathbf{r}) + \boldsymbol{\rho}^2\mathbf{r}] \\
&= \cos^2\frac{\Phi}{2}[(1 + \boldsymbol{\rho}^2)\mathbf{r} + 2\boldsymbol{\rho}\times\mathbf{r} + 2\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\mathbf{r})] \\
&= \cos^2\frac{\Phi}{2}(1 + \boldsymbol{\rho}^2)[],
\end{aligned}$$

and using (2.14), we finally get

$$Q\mathbf{r}Q^{-1} = \mathbf{r} + \frac{2\boldsymbol{\rho}}{1 + \boldsymbol{\rho}^2}\times(\mathbf{r} + \boldsymbol{\rho}\times\mathbf{r}). \tag{2.56}$$

Comparing this formula with (2.15), we note that the rotation $\boldsymbol{\rho}$ transfers the vector $\mathbf{r}$ to

$$\mathbf{r}' = Q\mathbf{r}Q^{-1}, \tag{2.57}$$

so that the rotation $\boldsymbol{\rho}$ is completely determined by the quaternion Q of unit magnitude.

Also, we have

$$
\begin{aligned}
Q^{-1}\mathbf{r}'Q &= Q^{-1}(Q\mathbf{r}Q^{-1})Q \\
&= (Q^{-1}Q)\mathbf{r}(Q^{-1}Q) \\
&= \mathbf{r},
\end{aligned}
$$

so that the inverse of the rotation is given by the quaternion $Q^{-1} = \bar{Q}$.

2.9 Composition of Two Rotations

Formula (2.57) is due to Cayley. Although equivalent to (2.15), it is much simpler in dealing with finite rotations. We use it now to obtain a formula for the composition of two rotations.

Consider a rotation through an angle Φ_1 around the axis in the direction $\mathbf{e}_1$. This rotation is completely described either by the rotation vector $\boldsymbol{\rho}_1 = \tan\frac{\Phi_1}{2}\mathbf{e}_1$ or by the quaternion $q_1 = \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k$. Let the rotation vector $\boldsymbol{\rho}_2 = \tan\frac{\Phi_2}{2}\mathbf{e}_2$ and the quaternion $q_2 = \mu_0 + \mu_1 i + \mu_2 j + \mu_3 k$ correspond to another rotation through an angle Φ_2 around $\mathbf{e}_2$. We have

$$
q_1 = \cos\frac{\Phi_1}{2}(1 + \boldsymbol{\rho}_1),\, q_2 = \cos\frac{\Phi_2}{2}(1 + \boldsymbol{\rho}_2). \tag{2.58}
$$

The vector $\mathbf{r}$ is transformed by the first rotation to

$$
\mathbf{r}' = q_1\mathbf{r}q_1^{-1} \tag{2.59}
$$

and then by the second rotation to

$$
\mathbf{r}'' = q_2\mathbf{r}'q_2^{-1} = q_2q_1\mathbf{r}q_1^{-1}q_2^{-1} = (q_2q_1)\mathbf{r}(q_2q_1)^{-1}. \tag{2.60}
$$

Thus, the resultant rotation corresponds to the quaternion

$$
\begin{aligned}
Q &= q_2q_1 \\
&= (Q_0, Q_1, Q_2, Q_3),
\end{aligned} \tag{2.61}
$$

where

$$
\begin{aligned}
Q_0 &= \lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3, \\
Q_1 &= \lambda_1\mu_0 + \lambda_0\mu_1 + \lambda_3\mu_2 - \lambda_2\mu_3, \\
Q_2 &= \lambda_2\mu_0 + \lambda_0\mu_2 + \lambda_1\mu_3 - \lambda_3\mu_1, \\
Q_3 &= \lambda_3\mu_0 + \lambda_0\mu_3 + \lambda_2\mu_1 - \lambda_1\mu_2.
\end{aligned} \tag{2.62}
$$

If we like to express the resultant rotation in terms of the rotation vectors, we use (2.58) to write

$$
\begin{aligned}
Q &= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}(1+\boldsymbol{\rho}_2)(1+\boldsymbol{\rho}_1) \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}[(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)+\boldsymbol{\rho}_1+\boldsymbol{\rho}_2+\boldsymbol{\rho}_2\times\boldsymbol{\rho}_1] \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)[1+\frac{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2+\boldsymbol{\rho}_2\times\boldsymbol{\rho}_1}{(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)}].
\end{aligned}
\tag{2.63}
$$

Comparing this with (2.58), we can write the resultant rotation quaternion in the form

$$
Q = \cos\frac{\Phi}{2}(1+\boldsymbol{\rho}). \tag{2.64}
$$

We find

$$
\boldsymbol{\rho} = \frac{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2+\boldsymbol{\rho}_2\times\boldsymbol{\rho}_1}{(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)} \tag{2.65}
$$

and

$$
\begin{aligned}
\cos\frac{\Phi}{2} &= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2}(1-\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2) \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\mathbf{e}_1\cdot\mathbf{e}_2 \\
&= \cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\cos\chi.
\end{aligned}
\tag{2.66}
$$

Rodrigues's formula (2.65) gives the resultant rotation vector and (2.66) gives the angle of the resultant rotation in terms of the two rotation angles and the angle χ between the two axes of the rotations.

Note that Rodrigues's formula (2.65) is not valid when $\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2 = 1$. In that case, from (2.66), we see that the rotation angle $\Phi = \pi$. This is expected whenever we deal with vectors of rotation.

2.10 Exercises

1- Show that the resultant of two half turns around different axes intersecting at an angle θ is equivalent to a rotation at an angle 2θ around the axis orthogonal to the two axes.

2- Put the formula (2.42) for the angular velocity in the quaternion form

$$
\boldsymbol{\omega} = 2q^{-1}\dot{q}, \tag{2.67}
$$

and hence prove the formula (2.46).

Chapter 3
The Classical Problem: The Motion of a Heavy Rigid Body About a Fixed Point

In the present chapter, we present detailed analysis of the classical problem of motion of a rigid body about a fixed point under the action of its own weight. This problem has a long history that began with the work of Euler and continued to the present day. Various powerful methods belonging to eminent specialists in mechanics and mathematics were applied to this problem without stopping, sometimes successfully and sometimes with less success. The list of basic contributors to this problem from our perspective, the construction of integrable cases, will be clearly presented as we proceed through this preliminary chapter and the following two chapters, after we make clear the meanings of general integrable, conditional integrable and particular solvable cases of the classical problem. This chapter is mainly concerned with basic concepts and various forms of the equations of motion, each of which would be more suited for use in certain investigations of the classical problem.

3.1 Equations of Motion

In this section, we derive the equation of rotational motion of the rigid body about a fixed point, under the action of arbitrary forces, which are not necessarily conservative or even having a potential. For such general setting, the Lagrangian approach is not a suitable choice and it is preferable to use ordinary vector mechanics. Denote by $\mathbf{r}$ the position vector of a mass element dm and by $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ its velocity. The angular momentum of the body, denoted here by $\mathbf{G}$, is the sum of moments of momenta of the elements about the origin O, the fixed point of the body,

$$\mathbf{G} = \int \mathbf{r} \times (\mathbf{v} dm), \tag{3.1}$$

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_3

where the integral is taken over the whole mass of the body. Differentiating the last relation, we get

$$\frac{d\mathbf{G}}{dt} = \int \mathbf{v} \times \mathbf{v} dm + \int \mathbf{r} \times \frac{d\mathbf{v}}{dt} dm. \tag{3.2}$$

The first integral vanishes, and from the equation of motion of the mass element dm, we have

$$dm \frac{d\mathbf{v}}{dt} = d\mathbf{F} + d\mathbf{F}',$$

in which $d\mathbf{F}$, $d\mathbf{F}'$ are, respectively, the resultant external and internal forces exerted on that element. Inserting this into (3.2), we write

$$\frac{d\mathbf{G}}{dt} = \int \mathbf{r} \times (d\mathbf{F} + d\mathbf{F}'). \tag{3.3}$$

Since the internal forces appear only in equal and opposite pairs, their overall moment vanishes, i.e. $\int \mathbf{r} \times d\mathbf{F}' = 0$. Thus, we finally have

$$\frac{d\mathbf{G}}{dt} = \mathbf{L}, \tag{3.4}$$

where $\mathbf{L} = \int \mathbf{r} \times d\mathbf{F}$ is the resultant moment of all the external forces acting on the body about the fixed point. This is the equation of rotational motion of the rigid body about a fixed point, under the action of arbitrary forces with moment $\mathbf{L}$. It is curious that this equation is similar to the equation of motion of a particle $\frac{d\mathbf{P}}{dt} = \mathbf{F}$, but replacing the Linear momentum $\mathbf{P}$ and the force $\mathbf{F}$ by the angular momentum and the moment of forces about the fixed point.

3.2 The Heavy Rigid Body

Equation (3.4) is quite general. It is valid for an arbitrary rigid body subject to arbitrary system of forces. In this chapter, we are concerned with the simplest case of motion of a body subject only to its own weight. For such a body, let $\mathbf{g}$ be the intensity of the gravity field directed vertically downwards. We have

$$\begin{aligned}\mathbf{L} &= \int \mathbf{r} \times (\mathbf{g} dm) \\ &= \int \mathbf{r} dm \times \mathbf{g}.\end{aligned}$$

Recalling the definition of the centre of gravity (the centre of mass) of the body in Chap. 1, we write the last relation in the form

$$\mathbf{L} = M\mathbf{r}_0 \times \mathbf{g}. \tag{3.5}$$

Without loss of generality, for most applications, one can take the Z-axis in the vertical direction upwards, so that the gravity field becomes

$$\mathbf{g} = -g\boldsymbol{\gamma}. \tag{3.6}$$

Thus, in the case of a heavy rigid body, the equation of motion takes the form

$$\frac{d\mathbf{G}}{dt} = -Mg\mathbf{r}_0 \times \boldsymbol{\gamma}. \tag{3.7}$$

3.3 The Angular Momentum of a Rigid Body

The mass element at the point $\mathbf{r}$ has velocity

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}. \tag{3.8}$$

Recalling the definition (3.1), we write

$$\begin{aligned}\mathbf{G} &= \int \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r})dm\\ &= \int [\mathbf{r}^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r})\mathbf{r}]dm.\end{aligned}$$

The i-th component is

$$\begin{aligned}G_i &= \omega_i \int \mathbf{r}^2 dm - \int \sum_{j=1}^{3} \omega_j r_j r_i dm\\ &= \sum_{j=1}^{3} \omega_j \int [\mathbf{r}^2\delta_{ij} - r_i r_j]dm\\ &= \sum_{j=1}^{3} \omega_j I_{ij},\end{aligned}$$

where $\mathbf{I} = (I_{ij})_{i,j=1}^3$ is the inertia matrix in the system of axes at the fixed point O. Making use of the symmetry of the inertia matrix, we write the last expressions in the form

$$\mathbf{G} = \boldsymbol{\omega}\mathbf{I}. \tag{3.9}$$

Remark 15 In most textbooks, the last relation is usually written as

$$\mathbf{G}' = \mathbf{I}\boldsymbol{\omega},$$

where $\mathbf{G}'$ is a column vector. We use the notation (3.9) to express $\mathbf{G}$ as a normal row vector. This is especially convenient in applying rules of vector algebra to that vector.

The relation (3.9) can be written in the expanded form as

$$\begin{aligned}\mathbf{G} &= (p, q, r)\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} \\ &= (I_{11}p + I_{12}q + I_{13}r, I_{12}p + I_{22}q + I_{23}r, I_{13}p + I_{23}q + I_{33}r). \quad (3.10)\end{aligned}$$

In the inertial system of axes at O, the body moves and its orientation changes with time. Equation (3.7) then involves 12 variable quantities: six moments and products of inertia, three coordinates of the mass centre and three components of the angular velocity. This makes the equations of motion quite complicated and impractical to use.

The system of axes fixed in the body with origin at the fixed point O enjoys the advantage that the inertia matrix is constant and also the position vector of the centre of mass. This makes it plausible to use a coordinate system fixed in the body to express Eq. (3.7) in it. A question arises, how to express that equation which is derived in the inertial system of axes in the body system? The answer will be given soon.

3.4 Kinetic Energy of a Moving Body

Summing the kinetic energy of mass elements and making use of the formulas of the last subsection, we get

$$\begin{aligned}T &= \int \frac{1}{2}\mathbf{v}^2 dm \\ &= \frac{1}{2}\int \mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r})dm \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot\int [\mathbf{r}\times\mathbf{v}]dm \\ &\equiv \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega}. \quad (3.11)\end{aligned}$$

In expanded form, this means

$$T = \frac{1}{2}(I_{11}p^2 + I_{22}q^2 + I_{33}r^2 + 2I_{12}pq + 2I_{23}qr + 2I_{13}pr). \tag{3.12}$$

3.5 Equations of Motion in the Moving Coordinate System

Let us now write Eq. (3.7) in the body system. It takes the form, called Euler's equation,

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0. \tag{3.13}$$

In addition to the vector $\boldsymbol{\omega}$, this equation involves the vertical unit vector $\boldsymbol{\gamma}$, which has variable components in the body system $Oxyz$. Being constant in space, the vector $\boldsymbol{\gamma}$ satisfies $\frac{d\boldsymbol{\gamma}}{dt} = 0$ in the inertial frame. In the body system, this is equivalent to

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{3.14}$$

which bears the name of Poisson's equation.

The pair of vector Eqs. (3.13) and (3.14), known as the *Euler–Poisson* equations, constitute a closed system of six scalar first-order differential equations in six variables, which can be chosen as either $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$ or $\mathbf{G}$, $\boldsymbol{\gamma}$.

3.5.1 The Use of the Variables $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$. Special Axes Related to the Inertia Matrix

In that case, using (3.9), we write the Euler–Poisson equation as

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0, \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{3.15}$$

which is the most commonly used form of those equations. For arbitrary choice of the body axes, they have the expanded form

$$\begin{aligned}
I_{11}\dot{p} + I_{12}\dot{q} + I_{13}\dot{r} + (I_{33} - I_{22})qr + I_{23}(q^2 - r^2) + p\,(I_{13}q - I_{12}r) &= Mg(z_0\gamma_2 - y_0\gamma_3),\\
I_{12}\dot{p} + I_{22}\dot{q} + I_{23}\dot{r} + (I_{11} - I_{33})\,pr + I_{13}\left(r^2 - p^2\right) + q\,(I_{12}r - I_{23}p) &= Mg(x_0\gamma_3 - z_0\gamma_1),\\
I_{13}\dot{p} + I_{23}\dot{q} + I_{33}\dot{r} + (I_{22} - I_{11})pq + I_{12}\left(p^2 - q^2\right) + r\,(I_{23}p - I_{13}q) &= Mg(y_0\gamma_1 - x_0\gamma_2),
\end{aligned} \tag{3.16}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{3.17}$$

In this form of equations, the inertia matrix has six elements. The system is not solved for the derivatives, a situation that is not in favour of a process of solution.

3.6 Integrals of Motion

Equations of motion (3.13), (3.14) or in expanded form (3.16), (3.17) are essentially nonlinear. For their solution, in the sense of reduction to quadratures, the application of Jacobi's theorem about the last integrating multiplier (See, e.g. [305]) requires the knowledge of four integrals of motion.

The first step is to see how much general integrals the above system admits in its most general form.

3.6.1 *The Energy Integral*

The rigid body is assumed to be smoothly fixed at O and moving in the uniform field of gravity, whose potential is

$$V = Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma}. \tag{3.18}$$

Regarding expressions (3.11), (3.12) and (3.18), one can immediately write the energy integral as

$$\begin{aligned} I_1 &\equiv T + V \\ &\equiv \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{G} + Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} \\ &\equiv \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} + Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} = h, \end{aligned} \tag{3.19}$$

h being the arbitrary constant of conserved total energy of the motion. In the expanded form, we can write

$$\begin{aligned} I_1 \equiv\ & \frac{1}{2}(I_{11}p^2 + I_{22}q^2 + I_{33}r^2 + 2I_{12}pq + 2I_{23}qr + 2I_{13}pr) \\ & + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) \\ =\ & h. \end{aligned} \tag{3.20}$$

3.6.2 *The Area's Integral*

Now we rewrite the equation of rotational motion (3.7)

$$\frac{d\mathbf{G}}{dt} = -Mg\mathbf{r}_0 \times \boldsymbol{\gamma},$$

and note that on multiplying scalarly by the vector $\boldsymbol{\gamma}$ on both sides, we get

$$\gamma \cdot \frac{d\mathbf{G}}{dt} = 0,$$

which may be now written as

$$\frac{d}{dt}(\mathbf{G} \cdot \gamma) = 0,$$

so that we obtain the second general integral of motion

$$I_2 \equiv \mathbf{G} \cdot \gamma = f, \tag{3.21}$$

f being an arbitrary parameter. In the moving axes, it has the form

$$I_2 \equiv \boldsymbol{\omega}\mathbf{I} \cdot \gamma = f. \tag{3.22}$$

In a general body system, it may be written as

$$I_2 = (I_{11}p + I_{12}q + I_{13}r)\gamma_1 + (I_{12}p + I_{22}q + I_{23}r)\gamma_2 + (I_{13}p + I_{23}q + I_{33}r)\gamma_3 = f.$$

The integral of motion (3.21) or (3.22) is linear in the components of the angular velocity. In accordance with the tradition prevailing in celestial mechanics, it is called the *areas integral*.

3.6.3 The Geometric Integral

The vector γ is defined as the unit vector directed vertically upwards. From this definition, it directly follows that its square is a constant of motion, with its constant value normalized to 1:

$$I_3 \equiv \gamma^2 \equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{3.23}$$

3.6.4 Exercise

Use Euler–Poisson's equations of motion in the vector form (3.15) to directly obtain the three general integrals of motion.

3.7 Special Axes Associated with the Inertia Matrix

Equation (3.16) can be somewhat simplified, by a suitable choice of the body axes. For example, we can take the z-axis as the one joining the fixed point with the centre of mass, so that $\mathbf{r}_0$ can be written as

$$\mathbf{r}_0 = (0, 0, z_0). \tag{3.24}$$

Moreover, we still have the freedom to rotate the x, y-axes in their plane to a position in which

$$I_{12} = 0. \tag{3.25}$$

We shall call the final set of axes *the special axes associated to the inertia matrix*. Those axes are most convenient in describing some particular solutions of the classical problem and other problems in rigid body dynamics, such as the regular precessions. This will be made clear later on.

The Euler equations now take the form

$$\begin{aligned} I_{11}\dot{p} + I_{13}\dot{r} + (I_{33} - I_{22})qr + I_{23}(q^2 - r^2) + I_{13}pq &= Mgz_0\gamma_2, \\ I_{22}\dot{q} + I_{23}\dot{r} + (I_{11} - I_{33})\,pr + I_{13}\left(r^2 - p^2\right) - I_{23}pq &= -Mgz_0\gamma_1, \\ I_{13}\dot{p} + I_{23}\dot{q} + I_{33}\dot{r} + (I_{22} - I_{11})pq + r\,(I_{23}p - I_{13}q) &= 0, \end{aligned} \tag{3.26}$$

while Poisson's equations still have the form (3.17), i.e.

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0.$$

The general integrals of motion can be written in the given coordinate system in the form

$$\begin{aligned} I_1 &\equiv \frac{1}{2}(I_{11}p^2 + I_{22}q^2 + I_{33}r^2 + 2I_{23}qr + 2I_{13}pr) \\ &\quad + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) \\ &= h, \\ I_2 &\equiv (I_{11}p + I_{13}r)\gamma_1 + (I_{22}q + I_{23}r)\gamma_2 + (I_{13}p + I_{23}q + I_{33}r)\gamma_3 = f, \\ I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{3.27}$$

3.8 The Use of Principal Axes of Inertia of the Body

In the special case, when the body axes are chosen to be the principal axes of the body at O, we have

$$\mathbf{I} = \mathrm{diag}(A, B, C), \mathbf{G} = (Ap, Bq, Cr), \mathbf{r}_0 = (x_0, y_0, z_0). \tag{3.28}$$

The equations of motion take the form

$$
\begin{aligned}
A\dot{p} + (C - B)qr &= Mg(z_0\gamma_2 - y_0\gamma_3),\\
B\dot{q} + (A - C)\,pr &= Mg(x_0\gamma_3 - z_0\gamma_1),\\
C\dot{r} + (B - A)pq &= Mg(y_0\gamma_1 - x_0\gamma_2),\\
\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 + r\gamma_1 &- p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0,
\end{aligned}
\tag{3.29}
$$

and the integrals of motion become

$$
\begin{aligned}
I_1 &\equiv \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h,\\
I_2 &\equiv Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f,\\
I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.
\end{aligned}
\tag{3.30}
$$

The equations of motion acquire in (3.29) their simplest and most symmetric form. The most favoured form is in scientific and technical literature. Those equations involve six parameters: three principal moments of inertia and three quantities formed by multiplying three coordinates of the centre of mass by the body weight. Unlike Eq. (3.16), Eq. (3.29) is readily solved in the derivatives $\dot{p}, \dot{q}, \dot{r}$, which is quite an advantage.

In the sequel, we shall mostly adhere to this form of the equations of motion. Only in exceptional occasions, we find other forms more appropriate or easier to use.

3.9 Determination of Euler's Angles

Solving the system of six equations of motion (3.29), we determine the vectors $\boldsymbol{\omega}(t)$ and $\boldsymbol{\gamma}(t)$as functions of the time t and only five arbitrary constants of integration, since the initial values of $\boldsymbol{\gamma}$ satisfy the geometric integral without arbitrary constant. This determines the Eulerian angles of nutation and proper rotation θ and φ as

$$
\theta = \cos^{-1}\gamma_3,\ \varphi = \tan^{-1}\frac{\gamma_1}{\gamma_2}. \tag{3.31}
$$

To complete the solution of the dynamical problem, i.e. to determine the orientation of the body in space, we should also determine the precession angle ψ. To this end, we use (2.39) of Chap. 2 to write

$$
\dot{\psi} = \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2}, \tag{3.32}
$$

so that we finally obtain

$$\psi = \psi_0 + \int_0^t \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2} dt, \tag{3.33}$$

ψ_0 is the sixth integration constant of the solution. This completes the solution of the problem of motion about a fixed point.

3.10 The Movable and Immovable Hodographs

Applied to the angular velocity vector the relation (2.37) gives

$$\frac{d\boldsymbol{\omega}}{dt} = \dot{\boldsymbol{\omega}}, \tag{3.34}$$

i.e. the angular velocity has the same rate of change in space as in the body. This formula, noted by Poisson, means that the infinitesimal change in the angular velocity at any moment of time is the same in both space and body reference frames. This has a very useful interpretation. Let the spatial curves Γ, named as the movable angular velocity hodograph, and Γ_0, the immovable angular velocity hodograph, be the loci of the angular velocity vector in the body and space system of axes, respectively. The two curves have the same tangent at every moment of time. The motion of the body in space can be represented as rolling the movable hodograph Γ without slipping on the immovable hodograph Γ_0 (fixed in space). The hodograph motion was studied as a way of geometric visualization of the motion in solvable cases. A voluminous literature exists on this topic. Interested readers may see, e.g. [108, 121] for several concrete examples.

3.11 The Use of the Variables G, γ. Special Axes Associated with the Gyration Ellipsoid

Let $\mathbf{G} = (P, Q, R)$ denote the angular momentum of the body and its components referred to the body axes. In that case, inverting the relation (3.9), we write

$$\boldsymbol{\omega} = \mathbf{GA}, \mathbf{A} = \mathbf{I}^{-1}, \tag{3.35}$$

so that (3.13) and (3.17) take the form

$$\dot{\mathbf{G}} + \mathbf{GA} \times \mathbf{G} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0, \dot{\boldsymbol{\gamma}} + \mathbf{GA} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{3.36}$$

and the integrals of motion become

$$\begin{aligned} \frac{1}{2}\mathbf{GA}\cdot\mathbf{G}+Mg\mathbf{r}_0\cdot\gamma &= h, \\ \mathbf{G}\cdot\gamma &= f, \\ \gamma^2 &= 1. \end{aligned} \tag{3.37}$$

The main advantage of Eq. (3.36) is that they are solved for the derivatives, in the sense that each of the six equations involves only one derivative of one component of $\mathbf{G}$ or γ. In this form, also the areas integral takes its simplest form. The situation can be made more advantageous by using the so-called "*Special axes associated with the gyration ellipsoid*", introduced and extensively used by Kharlamov [191]. They are formed in the following way: Choose the z-axis as the one joining the fixed point with the centre of mass, so that $\mathbf{r}_0$ can be written as

$$\mathbf{r}_0 = (0, 0, z_0), \tag{3.38}$$

and then rotate the x, y-axes in their plane to a position in which

$$A_{12} = 0. \tag{3.39}$$

In those special axes, the angular velocity

$$\boldsymbol{\omega} = (A_{11}P + A_{13}R,\ A_{22}Q + A_{23}R,\ A_{33}R + A_{13}P + A_{23}Q), \tag{3.40}$$

and Euler–Poisson's Eq. (3.36) become

$$\begin{aligned} \dot{P} + (A_{22} - A_{33})QR - A_{13}PQ + A_{23}(R^2 - Q^2) &= Mgz_0\gamma_2, \\ \dot{Q} + (A_{33} - A_{11})PR + A_{23}PQ + A_{13}(P^2 - R^2) &= -Mgz_0\gamma_1, \\ \dot{R} + (A_{11} - A_{22})PQ + (A_{13}Q - A_{23}P)R &= 0, \\ \dot{\gamma}_1 + (A_{22}Q + A_{23}R)\gamma_3 - (A_{33}R + A_{13}P + A_{23}Q)\gamma_2 &= 0, \\ \dot{\gamma}_2 + (A_{33}R + A_{13}P + A_{23}Q)\gamma_1 - (A_{11}P + A_{13}R)\gamma_3 &= 0, \\ \dot{\gamma}_3 + (A_{11}P + A_{13}R)\gamma_2 - (A_{22}Q + A_{23}R)\gamma_1 &= 0. \end{aligned} \tag{3.41}$$

As to the integrals of motion in the special axes, we note that the areas and geometric integrals still have the form as in (3.37), but the energy integral takes the form

$$I_1 = \frac{1}{2}(A_{11}P^2 + A_{22}Q^2 + A_{33}R^2 + 2A_{23}QR + 2A_{13}PR) + Mgz_0\gamma_3 = h. \tag{3.42}$$

3.12 Equations of Motion in Generalized Coordinates

The Euler–Poisson form is mostly preferred in the study of rigid body motion. Nevertheless, in certain situations, it is advantageous to write the Lagrangian form of the equations of motion, sometimes using the Eulerian angles as generalized coordinates and other times using different coordinates or some redundant coordinates, for example, the components of the vector $\boldsymbol{\gamma}$ or the quaternions. This formalism turns out to be most useful in the case of a dynamically symmetric body, but we shall not impose this condition for the time being.

The Lagrangian can be written in arbitrary coordinate system fixed in the body, but to obtain a more tractable form, we use the principal axes of inertia of the body at the fixed point as body axes. We write

$$L = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) - Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma}. \tag{3.43}$$

Using Eqs. (2.39) and (14.1), the Lagrangian takes the form

$$\begin{aligned} L = &\frac{1}{2}[A(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)^2 \\ &+B(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi)^2 + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\ &-Mg(x_0\sin\theta\sin\varphi + y_0\sin\theta\cos\varphi + z_0\cos\theta). \end{aligned} \tag{3.44}$$

We note at once two properties of the Lagrangian leading to two integrals:

(1) The system is conservative and hence admits the energy integral

$$\begin{aligned} I_1 \equiv &\frac{1}{2}[A(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)^2 \\ &+B(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi)^2 + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\ &+Mg(x_0\sin\theta\sin\varphi + y_0\sin\theta\cos\varphi + z_0\cos\theta) = h. \end{aligned} \tag{3.45}$$

(2) The angle of precession ψ is a cyclic coordinate and this leads to the cyclic integral

$$\begin{aligned} I_2 &\equiv \frac{\partial L}{\partial \dot{\psi}} \\ &= A\sin\theta\sin\varphi(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi) \\ &\quad +B\sin\theta\cos\varphi(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi) + C\cos\theta(\dot{\psi}\cos\theta + \dot{\varphi}) \\ &= (A\sin^2\theta\sin^2\varphi + B\sin^2\theta\cos^2\varphi + C\cos^2\theta)\dot{\psi} \\ &\quad +(A-B)\sin\theta\sin\varphi\cos\varphi\dot{\theta} \\ &= f. \end{aligned} \tag{3.46}$$

It is evident that those integrals are the same as the two in (3.30). Note that the geometric integral in (3.30) turns into an identity in the Euler angles as coordinates. In fact, $\gamma_1, \gamma_2, \gamma_3$ are redundant coordinates, i.e. they are dependent coordinates subject to the geometric integral as a constraint.

3.13 Canonical Equations of Motion in Euler's Angles

For certain important applications, such as different perturbation procedures, it may be advantageous to use the Hamiltonian formalism. We shall give now the Hamiltonian function and canonical equations of motion in Euler's angles and their conjugate momenta p_ψ, p_θ, p_φ. From (3.44), we get

$$\begin{aligned}
p_\psi &= \frac{\partial L}{\partial \dot\psi} = D\dot\psi + (A - B)\sin\theta\sin\varphi\cos\varphi\dot\theta + C\cos\theta\dot\varphi,\\
p_\theta &= \frac{\partial L}{\partial \dot\theta} = (A - B)\sin\theta\cos\theta\sin\varphi\dot\psi + (A\cos^2\varphi + B\sin^2\varphi)\dot\theta,\\
p_\varphi &= \frac{\partial L}{\partial \dot\varphi} = C(\dot\psi\cos\theta + \dot\varphi),
\end{aligned} \tag{3.47}$$

where

$$D = A\sin^2\theta\sin^2\varphi + B\sin^2\theta\cos^2\varphi + C\cos^2\theta. \tag{3.48}$$

Then, after solving (3.47) for $\dot\psi$, $\dot\theta$, $\dot\varphi$, we calculate the Hamiltonian

$$\begin{aligned}
H &= \dot\psi p_\psi + \dot\theta p_\theta + \dot\varphi p_\varphi - L\\
&= \frac{(A\cos^2\varphi + B\sin^2\varphi)}{2AB\sin^2\theta}(p_\psi - p_\varphi\cos\theta)^2 + \frac{(A\sin^2\varphi + B\cos^2\varphi)}{2AB}p_\theta^2\\
&\quad - \frac{(A - B)\sin\varphi\cos\varphi}{AB\sin\theta}(p_\psi - p_\varphi\cos\theta)p_\theta + \frac{p_\varphi^2}{2C}\\
&\quad + Mg(x_0\sin\theta\sin\varphi + y_0\sin\theta\cos\varphi + z_0\cos\theta).
\end{aligned} \tag{3.49}$$

The equations of motion can be written in the form

$$\begin{aligned}
\dot p_\psi &= -\frac{\partial H}{\partial \psi}, \dot\psi = \frac{\partial H}{\partial p_\psi},\\
\dot p_\theta &= -\frac{\partial H}{\partial \theta}, \dot\theta = \frac{\partial H}{\partial p_\theta},\\
\dot p_\varphi &= -\frac{\partial H}{\partial \varphi}, \dot\varphi = \frac{\partial H}{\partial p_\varphi}.
\end{aligned} \tag{3.50}$$

Since H does not depend on ψ, we have $\dot{p}_\psi = 0$, i.e. $p_\psi = \text{const}$. In conformity with (3.46), we take this constant to be f, so that

$$p_\psi = f. \tag{3.51}$$

The second and third pairs of equations give

$$\begin{aligned}
\dot{p}_\theta &= \frac{(B-A)\sin\varphi\cos\varphi}{AB\sin^2\theta}(f\cos\theta - p_\varphi)p_\theta \\
&\quad + \frac{(A\cos^2\varphi + B\sin^2\varphi)}{AB\sin^2\theta}(f - p_\varphi\cos\theta)(f\cos\theta - p_\varphi) \\
&\quad - Mg\,(x_0\cos\theta\sin\varphi + y_0\cos\theta\cos\varphi - z_0\sin\theta)\,, \\
\dot{\theta} &= \frac{(B-A)\sin\varphi\cos\varphi}{AB\sin\theta}(f - p_\varphi\cos\theta) + \frac{(A\sin^2\varphi + B\cos^2\varphi)}{AB}p_\theta, \\
\dot{p}_\varphi &= \frac{(A-B)}{AB}\Big[\frac{\sin\varphi}{\sin\theta}(f - p_\varphi\cos\theta) + p_\theta\cos\varphi\Big] \\
&\quad \times\Big[\frac{\cos\varphi}{\sin\theta}(f - p_\varphi\cos\theta) - p_\theta\sin\varphi\Big] \\
&\quad - Mg\sin\theta\,(x_0\cos\varphi - y_0\sin\varphi)\,, \\
\dot{\varphi} &= \frac{p_\varphi}{C} + \frac{(A-B)\sin\varphi\cos\varphi\cos\theta}{AB\sin\theta}p_\theta \\
&\quad - \frac{(A\cos^2\varphi + B\sin^2\varphi)\cos\theta}{AB\sin^2\theta}(f - p_\varphi\cos\theta).
\end{aligned} \tag{3.52}$$

If a solution is obtained for the last system giving θ, φ, $\dot{\theta}$, $\dot{\varphi}$ as functions of time, the precession angle ψ can be then determined by integrating the second equation in (3.50), which is now written as

$$\dot{\psi} = \frac{(A\cos^2\varphi + B\sin^2\varphi)}{AB\sin^2\theta}(f - p_\varphi\cos\theta) - \frac{(A-B)\sin\varphi\cos\varphi}{AB\sin\theta}p_\theta. \tag{3.53}$$

3.14 The Routhian Reduction

From (3.46), we find

$$\dot{\psi} = \frac{f - (A-B)\sin\theta\sin\varphi\cos\varphi\dot{\theta} - C\cos\theta\dot{\varphi}}{(A\sin^2\theta\sin^2\varphi + B\sin^2\theta\cos^2\varphi + C\cos^2\theta)}. \tag{3.54}$$

One can now use Routh's procedure to ignore the cyclic coordinate ψ and reduce the problem of motion to a system of two degrees of freedom. The Routhian of the system is

$$\begin{aligned} R &= L - f\dot{\psi} \\ &= R_2 + R_1 - V_1, \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} R_2 &= \frac{1}{2D}\{C\sin^2\theta(A\sin^2\varphi + B\cos^2\varphi)\dot{\varphi}^2 - \\ &-\frac{1}{2}C(A-B)\sin 2\theta\sin 2\varphi\dot{\theta}\dot{\varphi} \\ &+[D(A\cos^2\varphi + B\sin^2\varphi) - (A-B)^2\sin^2\theta\sin^2\varphi\cos^2\varphi]\dot{\theta}^2\}, \\ R_1 &= \frac{f}{D}[C\cos\theta\dot{\varphi} + (A-B)\sin\theta\sin\varphi\cos\varphi\dot{\theta}], \\ V_1 &= V + \frac{f^2}{2D}. \end{aligned} \tag{3.56}$$

The function V_1 is called the reduced potential, while V is the original potential of the problem. The equations of motion are

$$\frac{d}{dt}\frac{\partial R}{\partial\dot{\theta}} - \frac{\partial R}{\partial\theta} = 0, \frac{d}{dt}\frac{\partial R}{\partial\dot{\varphi}} - \frac{\partial R}{\partial\varphi} = 0. \tag{3.57}$$

We shall not write them down in the expanded form because they lack symmetry and they are not easy to use in general. However, they can be used much easily in case of a dynamically symmetric body. Such concrete applications are not in the focus of the present book and can be found in several books on perturbation problems. Those are two second-order equations in the two variables θ and φ. After solving those equations and expressing the two angles in terms of time, one can determine the ignored angle ψ by integrating (3.54) with respect to time.

Remark 1: The Lagrangian (3.44) (and equations of motion derived from it in any generalized coordinates) is time-reversible, i.e. the Lagrangian and equations remain invariant if the sign of time t is changed. On the contrary, the Routhian and Routhian equations of motion are not time-reversible. They are invariant only on the simultaneous change of signs of t and f.

When $f = 0$, the Routhian becomes

$$R = R_2 - V,$$

and the Routhian equations of motion are time-reversible.

Remark 2: In the case of axial dynamical symmetry $B = A$, a significant simplification occurs in the Routhian (3.55). It renders to the form

$$\begin{aligned} R &= \frac{A}{2D}(C\sin^2\theta\dot{\varphi}^2 + D\dot{\theta}^2) + \frac{f}{D}C\cos\theta\dot{\varphi} - V_1, \\ V_1 &= V + \frac{f^2}{2D}. \end{aligned} \tag{3.58}$$

Finally, when $A = C$, i.e. in the case of complete dynamical symmetry, $D = A$ and we have

$$R = \frac{A}{2}(\sin^2\theta\dot{\varphi}^2 + \dot{\theta}^2) + f\cos\theta\dot{\varphi} - V(\theta,\varphi). \tag{3.59}$$

This is the Lagrangian of a particle moving on a smooth sphere with θ and φ as polar coordinates on that sphere. The particle is subject to forces with potential V and gyroscopic forces represented by the term linear in $\dot{\varphi}$. Note that the last term is proportional to f and thus vanishes when $f = 0$.

3.15 Exercises

(1) A heavy rigid body is moving about a fixed point, which is not coincident with the centre of mass of the body. Use the energy integral and Euler's equations to express the vector $\boldsymbol{\gamma}$ in terms of the angular velocity $\boldsymbol{\omega}$ and its time derivative $\dot{\boldsymbol{\omega}}$ in the form

$$\boldsymbol{\gamma} = \frac{1}{Mg|\mathbf{r}_0|^2}[\mathbf{r}_0 \times (\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I}) + (h - \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega})\mathbf{r}_0], \tag{3.60}$$

where h is the energy constant and $\mathbf{r}_0 \neq \mathbf{0}$ is the position vector of the centre of mass of the body with respect to the fixed point.

(2) Use the last result to reduce the equations of motion of the classical problem to the form of three autonomous first-order differential equations in the components of the angular velocity with respect to an arbitrary system of axes fixed in the body in time as independent variable to the form[1]

$$\begin{aligned}(\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I}) \cdot \mathbf{r}_0 &= 0,\\ (h - \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega})^2 + (\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I})^2 &= M^2g^2|\mathbf{r}_0|^2,\\ \mathbf{r}_0 \cdot [(h - \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega})\boldsymbol{\omega}\mathbf{I} + \dot{\boldsymbol{\omega}}\mathbf{I} \times \boldsymbol{\omega}\mathbf{I} - |\boldsymbol{\omega}\mathbf{I}|^2\boldsymbol{\omega}] &= Mgf|\mathbf{r}_0|^2,\end{aligned} \tag{3.61}$$

where f is the areas constant and other parameters as defined above.

Hint: Use the following equations:

$$\begin{aligned}(\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I}) \cdot \mathbf{r}_0 &= 0,\\ |\boldsymbol{\gamma}|^2 &= 1,\\ \boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\gamma} &= f.\end{aligned}$$

[1] For this form, or (3.63), to be equivalent to the original Euler–Poisson system, a condition on the motion must be satisfied (See the two theorems in Sect. 8.1).

(3) Show that in terms of $\mathbf{G}$, the formula (3.60) and reduced Eq. (3.61) take the following form:

$$\gamma = \frac{1}{Mg|\mathbf{r}_0|^2}[\mathbf{r}_0 \times (\dot{\mathbf{G}} + \mathbf{G}\mathbf{A} \times \mathbf{G}) + (h - \frac{1}{2}\mathbf{G}\mathbf{A} \cdot \mathbf{G})\mathbf{r}_0], \tag{3.62}$$

and

$$\begin{aligned} (\dot{\mathbf{G}} + \mathbf{G}\mathbf{A} \times \mathbf{G}) \cdot \mathbf{r}_0 &= 0, \\ (h - \frac{1}{2}\mathbf{G}\mathbf{A} \cdot \mathbf{G})^2 + (\dot{\mathbf{G}} + \mathbf{G}\mathbf{A} \times \mathbf{G})^2 &= M^2 g^2 |\mathbf{r}_0|^2, \\ \mathbf{r}_0 \cdot [(h - \frac{1}{2}\mathbf{G}\mathbf{A} \cdot \mathbf{G})\mathbf{G} + \dot{\mathbf{G}} \times \mathbf{G} - |\mathbf{G}|^2 \mathbf{G}\mathbf{A}] &= Mgf|\mathbf{r}_0|^2, \end{aligned} \tag{3.63}$$

where $\mathbf{A} = \mathbf{I}^{-1}$.

Chapter 4
General and Conditional Integrable Cases of the Classical Problem

In the classical problem of motion of a rigid body about a fixed point, there are only three general and one conditional integrable cases. By a general integrable case, we mean that case in which the problem of motion is integrable for arbitrary initial conditions, while the term "conditionally integrable" is reserved for cases which are integrable only on a fixed level of the areas integral. In this chapter, we give a brief account of integrable cases of the classical problem. We first rewrite the Euler–Poisson equations of motion in their most used form in the system of principal axes of inertia of the body at the fixed point:

$$\begin{aligned} A\dot{p} + (C - B)qr &= Mg(z_0\gamma_2 - y_0\gamma_3), \\ B\dot{q} + (A - C)\,pr &= Mg(x_0\gamma_3 - z_0\gamma_1), \\ C\dot{r} + (B - A)pq &= Mg(y_0\gamma_1 - x_0\gamma_2), \\ \dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 &+ r\gamma_1 - p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0, \end{aligned} \tag{4.1}$$

and also their integrals of motion

$$\begin{aligned} I_1 &\equiv \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h, \\ I_2 &\equiv Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f, \\ I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{4.2}$$

The list of integrable cases is provided in the following Tables 4.1 and 4.2:

In both Tables 4.1 and 4.2 $a_1 = Mgx_0/C$, $a_2 = Mgy_0/C$. Although in the cases of Kowalevski and Goryachev–Chaplygin, one can always rotate the coordinate axes xy in their plane in order that the x-axis passes through the centre of mass, i.e. $y_0 = 0$, we here keep both coordinates in the general position, keeping in mind some future occasions, when the extra-parameter makes it possible to build integrable cases

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_4

Table 4.1 General (Unconditional) cases

	Author	Conditions
1	Euler [78] (1758). $I_4 = A^2p^2 + B^2q^2 + C^2r^2$.	$g\mathbf{r}_0 = \mathbf{0}$.
2	Lagrange's top [251] (1788). (Axially symmetric case) $I_4 = Cr$.	$B = A$, $x_0 = y_0 = 0$,
3	Kowalevski (1889) [238] $I_4 = (p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2)^2 + (2pq - a_1\gamma_2 - a_2\gamma_1)^2$.	$A = B = 2C$, $z_0 = 0$.

Table 4.2 Conditional case $f = 0$

1	Goryachev–Chaplygin [115] (1900) [52] (1901)	$A = B = 4C$, $z_0 = 0$
	$I_4 = r(p^2 + q^2) - \gamma_3(a_1 p + a_2 q)$	

containing additional parameters in a nonlinear way (See, e.g. case 6 of Table 12.1 in Chap. 12).

4.1 Euler's Case (1758). The Torque-Free Rigid Body

4.1.1 Explicit Time-Solution

In the same work [78], where he published the final form of the equations of motion of a rigid body about a fixed point, Euler noted that the case of a free rigid body (when the moment of forces applied to the body about the fixed point vanishes) can be solved and reduced that case to a quadrature expressing the relation of the components of angular velocity to time. In the present section, we shall solve the equations of motion in Euler's case explicitly using the elliptic functions invented by Jacobi after almost a century, in the middle of the nineteenth century.

The equations of motion are obtained from Eq. (3.29) of the previous chapter by setting either $g = 0$ (the case of absence of gravity force) or the equivalent case $\mathbf{r}_0 = \mathbf{0}$ (the case of a heavy body fixed from its centre of mass). Euler's equation can be written as

$$\frac{d\mathbf{G}}{dt} \equiv \dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = 0, \tag{4.3}$$

or, in components,

$$A\dot{p} - (B - C)qr = 0,$$

$$B\dot{q} - (C - A)pr = 0,$$
$$C\dot{r} - (A - B)pq = 0. \tag{4.4}$$

Apart from Poisson's equations, (4.4) form a closed system of three equations in three variables, and can be solved independently. One can also notice that the energy integral (3.30) for this case takes the form

$$I_1 \equiv \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) = h, \tag{4.5}$$

so that it gives an integral of the system (4.4).

We can easily construct another integral of that system. In fact, Eq. (4.3) expresses the constancy of the angular momentum vector $\mathbf{G}$ in the inertial space. The modulus of this vector will be constant in all frames and in particular in the body frame. We can write the integral

$$\mathbf{G}^2 = const. = G_0^2,$$

Alternatively, we can multiply (4.3) scalarly by $\mathbf{G}$ to obtain

$$\mathbf{G}\cdot\dot{\mathbf{G}} = 0,$$

which leads to the same result. Thus, the new integral can be written as

$$A^2p^2 + B^2q^2 + C^2r^2 = G_0^2. \tag{4.6}$$

Now, we construct the general solution for the equations of motion. First, we introduce the two constants D and μ through the relations

$$2h = \mu^2 D, G_0 = D\mu, \tag{4.7}$$

provided none of the constants G_0, h is zero, which can happen only on an equilibrium ($p = q = r = 0$). The two Eqs. (4.5, 4.6) become

$$Ap^2 + Bq^2 + Cr^2 = D\mu^2, \tag{4.8}$$
$$A^2p^2 + B^2q^2 + C^2r^2 = D^2\mu^2. \tag{4.9}$$

Note that D has the dimension of a moment of inertia, while μ has that of an angular velocity.

Eliminating r from the two Eqs. (4.8, 4.9), we have

$$p^2 = \frac{1}{A(A - C)}[\mu^2 D(D - C) - B(B - C)q^2]. \tag{4.10}$$

Also, eliminating p from the same equations

$$r^2 = \frac{1}{A(A-C)}[\mu^2 D(A-D) - (A-B)q^2]. \tag{4.11}$$

Without loss of generality, we assume $A > B > C$, it is easy to show that the variables p and r have real values only if the parameter D satisfies the condition

$$C \leqslant D \leqslant A. \tag{4.12}$$

Inserting the two expressions (4.10, 4.11) into the middle equation in (4.4), we obtain

$$B\dot{q} = \frac{\pm 1}{\sqrt{AC}}\sqrt{\mu^2 D(A-D) - B(A-B)q^2}\sqrt{\mu^2 D(D-C) - B(B-C)q^2}. \tag{4.13}$$

Noting that the right-hand side of the last equation is the square root of a 4th degree polynomial, we conclude that the separation of variables leads to an elliptic integral of the first kind. Since q^2 admits the values from zero to the minimum value of

$$q_1^2 = \frac{\mu^2 D(A-D)}{B(A-B)}, \qquad q_2^2 = \frac{\mu^2 D(D-C)}{B(B-C)}.$$

Thus, we have two cases:

4.1.1.1 The First Case

If $q_2^2 < q_1^2$ (this is hold only if $\frac{D-C}{B-C} < \frac{A-D}{A-B}$, i.e. $C < D < B$) then q^2 is changed from 0 to q_2^2. Setting

$$q = q_2 x,$$

where x is a new variable that varies from -1 to 1. Inserting the last expression in Eq. (4.13), we get

$$\dot{x} = \lambda\sqrt{1-x^2}\sqrt{1-k^2x^2}, \tag{4.14}$$

where

$$\lambda = \pm\mu\sqrt{\frac{D(A-D)(B-C)}{ABC}}, \quad k^2 = \frac{(A-B)(D-C)}{(A-D)(B-C)}. \tag{4.15}$$

Separating the variables in (4.14) and integrating, we get

$$\lambda(t-t_0) = \int_0^x \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \tag{4.16}$$

where t_0 is an arbitrary integration constant. Equation 4.16 can be solved to give

$$x = \mathrm{sn}\lambda(t-t_0). \tag{4.17}$$

Taking all obtained expressions into account, we can write

$$
\begin{aligned}
p &= \pm\mu\sqrt{\frac{D(D-C)}{A(A-C)}}\operatorname{cn}\lambda(t-t_0),\\
q &= \mu\sqrt{\frac{D(D-C)}{B(B-C)}}\operatorname{sn}\lambda(t-t_0),\\
r &= \pm\mu\sqrt{\frac{D(A-D)}{C(A-C)}}\operatorname{dn}\lambda(t-t_0).
\end{aligned}
\tag{4.18}
$$

The last expressions show that the components of angular velocity are periodic functions in the time t with period $\frac{4}{\lambda}K(k)$ for p, q and $\frac{2}{\lambda}K(k)$ for r. Furthermore, the sign of p and q change during the motion while r does not change its sign. Therefore, the rotation about the third axis that has the minimum value of inertia is always in one and the same direction.

4.1.1.2 The Second Case

In this case, $B < D < A$. The reader can easily check that the last formulas are replaced by

$$
\begin{aligned}
p &= \mu\sqrt{\frac{D(D-C)}{A(A-C)}}\operatorname{dn}\lambda(t-t_0),\\
q &= \mu\sqrt{\frac{D(A-D)}{B(A-B)}}\operatorname{sn}\lambda(t-t_0),\\
r &= \mu\sqrt{\frac{D(A-D)}{C(A-C)}}\operatorname{cn}\lambda(t-t_0),
\end{aligned}
\tag{4.19}
$$

where

$$
\lambda = \pm\mu\sqrt{\frac{D(D-C)(A-B)}{ABC}}, \quad k = \sqrt{\frac{(A-D)(B-C)}{(A-B)(D-C)}}.
$$

Note that this time the rotation about the first axis (x-axis, with the maximum moment of inertia) remains in one direction.

4.1.2 Permanent Rotations

Permanent rotations are motions with constant angular velocity in the body, i.e. motions satisfying $\dot{\boldsymbol{\omega}} = \mathbf{0}$. Since $\frac{d\omega}{dt} = \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$, those motions are are also time independent in space. For them Eqs. (4.4) become

$$(C - A)qr = (A - C)rp = (B - A)pq = 0.$$

They have three solutions: $\{p = \Omega, q = r = 0\}$, $\{q = \Omega, p = r = 0\}$ and $\{r = \Omega, p = q = 0\}$, each of which represent a permanent rotation with an arbitrary angular velocity Ω about one of the principal axes of inertia of the body which takes an arbitrary fixed position in the inertial space.

4.1.3 The Degeneracy of Elliptic Function

When $D = C$, we have $k = 0$ and Eq. (4.18) gives $p = q = 0$ and $r = \pm\mu$. This refers to the permanent rotation of the body around the third axis that has the minimum principal inertia. The second degeneration occurs when $D = B$. Then, we have $k = 1$ and the expressions (4.18) or (4.19) become

$$\begin{aligned} p &= \pm\mu\sqrt{\frac{B(B-C)}{A(A-C)}}\frac{1}{\cosh\lambda'(t-t_0)}, \\ q &= \mu\tanh\lambda'(t-t_0), \\ r &= \pm\mu\sqrt{\frac{B(A-B)}{C(A-C)}}\frac{1}{\cosh\lambda'(t-t_0)}, \end{aligned}$$

where $\lambda' = \mu\sqrt{\frac{(A-B)(B-C)}{AC}}$. This is an asymptotic motion. As the time $t \to \pm\infty$, this motion becomes a permanent rotation with angular velocity $\pm\mu$ around the middle axis.

4.1.4 The Case of Dynamically Axi-Symmetric Body

In this case, when $B = A(> C)$ (the case of a body with a prolate inertia spheroid), we can obtain relevant formulas of the solution by substituting $B = A$ in (4.18). We obtain

$$\begin{aligned} p &= \pm\mu\sqrt{\frac{D(D-C)}{A(A-C)}}\cos\lambda(t-t_0), \\ q &= \mu\sqrt{\frac{D(D-C)}{A(A-C)}}\sin\lambda(t-t_0), \end{aligned}$$

$$r = \pm\mu\sqrt{\frac{D(A-D)}{C(A-C)}} \tag{4.20}$$

where $\lambda = \pm\mu\sqrt{\frac{D(A-D)(A-C)}{A^2C}}$. It is easy to identify the two results.

On the other hand, when $B = C < A$, i.e. in the case of a body with an oblate ellipsoid of inertia, from (4.19) we get

$$\begin{aligned} p &= \mu\sqrt{\frac{D(D-C)}{A(A-C)}}, \\ q &= \mu\sqrt{\frac{D(A-D)}{C(A-C)}}\sin\lambda(t-t_0), \\ r &= \mu\sqrt{\frac{D(A-D)}{C(A-C)}}\cos\lambda(t-t_0), \end{aligned} \tag{4.21}$$

where

$$\lambda = \mu\sqrt{\frac{D(D-C)(A-C)}{AC^2}}.$$

4.1.5 *Euler's Angles in Terms of Time*

In Euler's case, there is no distinct direction in space to be taken as the direction of gravity. The orthonormal basis vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ can be chosen arbitrarily in space. There also exist three areas integrals

$$\mathbf{G}\cdot\boldsymbol{\gamma} = G_0, \mathbf{G}\cdot\boldsymbol{\alpha} = G_0', \mathbf{G}\cdot\boldsymbol{\beta} = G_0'' \text{ (say)}. \tag{4.22}$$

For determinacy, one can use the direction of the constant angular momentum to be that of $\boldsymbol{\gamma}$, i.e. one can take

$$\boldsymbol{\gamma} = \frac{\mathbf{G}}{G_0} = \frac{(Ap, Bq, Cr)}{G_0}. \tag{4.23}$$

Corresponding to this choice, one must take $G_0' = G_0'' = 0$ in (4.22). Using the expressions (4.18), we can write the Eulerian angles in terms of time. Thus, we have

$$\cos\theta = \frac{Cr}{G_0} = \frac{C}{D}\sqrt{\frac{D(A-D)}{C(A-C)}}\,\mathrm{dn}\lambda(t-t_0),$$

$$\tan\varphi = \frac{Ap}{Bq} = \sqrt{\frac{A(B-C)}{B(A-C)}}\frac{\mathrm{cn}\lambda(t-t_0)}{\mathrm{sn}\lambda(t-t_0)}, \tag{4.24}$$

as elliptic functions in the time t. The angle of precession ψ can be determined from

$$\dot{\psi} = \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2} = G\frac{Ap^2 + Bq^2}{A^2p^2 + B^2q^2} = \mu\frac{D}{C}\left[1 - \frac{\frac{A-C}{A}}{1 + \frac{C(A-B)}{A(B-C)}s^2}\right],$$

where $s = \mathrm{sn}\lambda(t-t_0)$. Thus, ψ is obtained as an elliptic integral of third kind in time

$$\psi = \psi_0 + \mu\frac{D}{C}\left[t - t_0 - \frac{A-C}{A}\int_{t_0}^{t}\frac{dt}{1 + \frac{C(A-B)}{A(B-C)}\,\mathrm{sn}^2\,\lambda(t-t_0)}\right]. \tag{4.25}$$

Further analytical evaluation of the last integral may be found in several treatizes, e.g. [9] and will not be pursued here.

In the case of dynamical symmetry $A = B$, we get

$$\begin{aligned}
\cos\theta &= \frac{Cr_0}{G},\\
\dot{\psi} &= \frac{G_0}{A},\\
\dot{\varphi} &= r_0 - \dot{\psi}\cos\theta = (1 - \frac{C}{A})r_0.
\end{aligned} \tag{4.26}$$

It is evident that this motion describes a uniform rotation around the axis of symmetry while this axis rotates around the vertical with a constant angular velocity. Euler's angles are then given by

$$\begin{aligned}
\theta &= \cos^{-1}\frac{Cr_0}{G},\\
\psi &= \psi_0 + \frac{G_0}{A}(t - t_0),\\
\varphi &= \varphi_0 + (1 - \frac{C}{A})r_0(t - t_0).
\end{aligned} \tag{4.27}$$

This motion is named regular precession. The axis of symmetry of the body is called the figure axis and the axis fixed in space in the direction of $\mathbf{G}$ and $\boldsymbol{\gamma}$ is called the precession axis. This type of motion will be met in several other situations later in this book.

4.1.6 Geometrical Interpretation of the Motion (Poinsot 1851)

The explicit analytical solution of Euler's case provided in Sect. 4.1.1 is not of much help in giving a clear geometric idea about how the body moves in space. An elegant geometric description of the motion constructed by Poinsot [308] has become popular in textbooks on the subject. We give a quick presentation of this construction here.

In the system of principal axes of inertia of the body $Oxyz$ at the fixed point O, the inertia ellipsoid of the body has the equation

$$Ax^2 + By^2 + Cz^2 = 1. \tag{4.28}$$

Let the body be in motion at certain moment of time with instantaneous angular velocity $\boldsymbol{\omega} = (p, q, r)$ referred to that system. Denote by $\mathbf{r} = (x, y, z)$ the position vector of the pole P (the point of intersection of the vector $\boldsymbol{\omega}$ with the inertia ellipsoid). Note that since P lies on the instantaneous axis of rotation of the body, it is always in instantaneous rest.

It is easy to deduce the following properties of the motion of the inertia ellipsoid.

Property 1: *The angular velocity of the body is proportional to the line segment* $\overline{OP}$, *cut of the rotation axis by the inertia ellipsoid.*

In fact, one can write

$$\boldsymbol{\omega} = \lambda \mathbf{r}, \tag{4.29}$$

where λ is a multiplier to be determined. Substituting this expression into (4.28), we obtain

$$\lambda^{-2}(Ap^2 + Bq^2 + Cr^2) = 1,$$

and in virtue of the energy integral this gives

$$\lambda^2 = 2h.$$

It follows that

$$\lambda = \frac{1}{\sqrt{2h}}.$$

so that the proportionality factor λ is a constant. We have here fixed the positive sign for λ, in order to have $\mathbf{r}$ in the same direction of $\boldsymbol{\omega}$, and thus from (4.29) we get

$$\boldsymbol{\omega} = \sqrt{2h}\mathbf{r}. \tag{4.30}$$

Property 2: *The tangent plane* Π *to the inertia ellipsoid at the pole* P *keeps a fixed direction in space for all the time of motion.*

The unit vector $\mathbf{n}$ in the direction orthogonal to the tangent plane at P can be written as

$$\mathbf{n} = \frac{\mathbf{rI}}{\sqrt{|\mathbf{rI}|^2}} = \frac{\omega\mathbf{I}}{\sqrt{|\omega\mathbf{I}|^2}} = \frac{\mathbf{G}}{G}, \tag{4.31}$$

where we have used (4.30). As **G** is a constant vector in space, property 2 is proved.

Property 3: *The tangent plane to the inertia ellipsoid at the pole P remains fixed in space for all the time of motion.*

To prove this property it is easy, using properties 1 and 2, to deduce that the orthogonal distance of the centre O of the ellipsoid from the tangent plane at P

$$\delta = \mathbf{r}\cdot\mathbf{n} = \frac{\mathbf{rI}\cdot\mathbf{r}}{\sqrt{|\mathbf{rI}|^2}} = \frac{1}{\sqrt{\mathbf{rI}^2\cdot\mathbf{r}}} = \frac{\sqrt{2h}}{G} \tag{4.32}$$

is really constant during motion.

Now we are ready to make the final conclusion about the motion of the body based on the motion of its ellipsoid of inertia (fixed in that body):

The motion of a rigid body by inertia about the fixed point O is realized by rolling the inertia ellipsoid, with centre at O, without slipping on the immovable plane Π (See Fig. 4.1).

At any time moment, the point of contact P is momentarily at rest. The curve traced out on the ellipsoid of inertia by the pole P is called *polhode*, and it is obviously a closed curve. The locus of P on the plane Π has the name *herpolhode.* It is not closed in general, but it may close under some conditions on the parameters of the body and of the motion. In the case of a dynamical axial symmetry, both the polhode and herpolhode become circular.

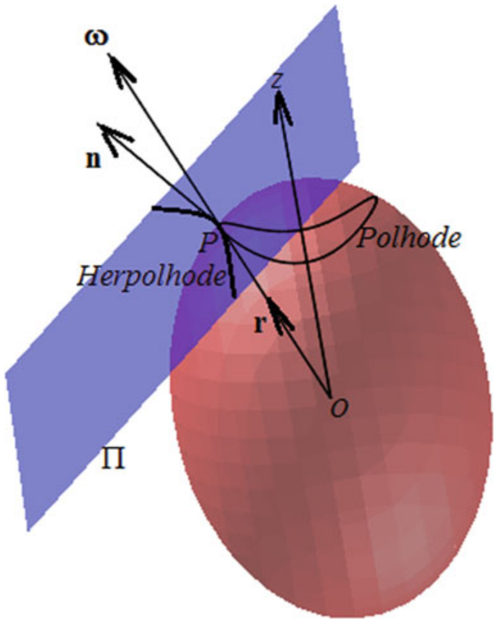

Fig. 4.1 Poinsot's construction

It should be noted that Poinsot's description is purely geometrical. It shows the consecutive positions occupied at different moments by the inertia ellipsoid of the body, but has nothing to do with the time sequence in which those positions occur.

Exercise 1 Show that in Euler's case the angular momentum of the body describes a cone, whose equation referred to the principal axes of inertia at the fixed point is

$$(1-\frac{D}{A})x^2+(1-\frac{D}{B})y^2+(1-\frac{D}{C})z^2=0.$$

4.2 Lagrange's Case (1788). The Top with a Fixed Point

Lagrange's top is characterized by axial symmetry, when two of the principal moments of inertia at O are equal and the centre of the mass of the body lies on the axis of dynamical symmetry, so that one can write

$$A=B, x_0=y_0=0. \tag{4.33}$$

This case was first described as a solvable problem by Lagrange in his historical book on analytical mechanics [251]. A significant contribution of Poisson [309] made a clear and elegant picture of the motion of the apex of the top. This picture will be briefly described below in this section.

The study of motion of Lagrange's top is most interesting as for theoretical understanding of gyroscopic effects of rotating bodies and from the point of view of scientific and technical applications. In fact, the approximate theory of gyroscopes relies on the fundamental properties of motion of Lagrange's heavy top and the axisymmetric version of the free top (in absence of gravity). That is why that top is a favourite subject in all books on analytical dynamics in general and, particularly, in books on gyroscopes. (See, e.g. [112] for precession of equinoxes and [132, 270] for gyroscopic effects of rotors). One of the most detailed descriptions of the dynamical behaviour of Lagrange's top can be found in [222], where expressions for the angular velocities and the geometric variables are also given in terms of theta functions (See also [269, 368]). Interest in the subject never faded away over years. Various aspects of Lagrange's top were studied extensively in the literature. The reader can see [34, 61, 62, 104, 106, 143, 262, 310, 362], just as examples. In this section, we give a brief account of the analytical solution and the most important qualitative properties of the motion.

4.2.1 The Solution

In the case of Lagrange, both coordinates ψ and φ are cyclic and one can use Routh's procedure to ignore them and reduce the problem of motion in the classical Lagrangian form to one degree of freedom. However, we shall follow here a slightly different approach, more consistent with that used in other cases throughout this book. In fact, under conditions (4.33) Euler–Poisson's equations reduce to

$$\begin{aligned} A\dot{p} - (A - C)qr &= Mgz_0\gamma_2, \\ A\dot{q} - (A - C)pr &= -Mgz_0\gamma_1, \\ C\dot{r} &= 0, \end{aligned} \tag{4.34}$$

$$\begin{aligned} \dot{\gamma}_1 + q\gamma_3 - r\gamma_2 &= 0, \\ \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 &= 0, \\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 &= 0. \end{aligned} \tag{4.35}$$

The last equation of (4.34) at once gives the fourth integral

$$r = r_0 \tag{4.36}$$

where r_0 is an arbitrary parameter. We also write the three general integrals:

The energy integral

$$A(p^2 + q^2) + Cr^2 + 2Mgz_0\gamma_3 = 2h, \tag{4.37}$$

the areas integral

$$A(p\gamma_1 + q\gamma_2) + Cr\gamma_3 = f, \tag{4.38}$$

and the geometric integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{4.39}$$

where h, f are arbitrary parameters. According to Jacobi's theorem, the equations of motion can be solved completely and the solution is expressed in terms of elliptic functions as in Euler's case. In fact, using Eq. 4.35 and the geometric integral, we obtain

$$\begin{aligned} \dot{\gamma}_3{}^2 &= (p\gamma_2 - q\gamma_1)^2 = p^2(1 - \gamma_3^2 - \gamma_1^2) + q^2(1 - \gamma_3^2 - \gamma_2^2) - 2pq\gamma_1\gamma_2 \\ &= (1 - \gamma_3^2)(p^2 + q^2) - (p\gamma_1 + q\gamma_2)^2. \end{aligned} \tag{4.40}$$

One can also use the four integrals of motion in the last equation to obtain the following equation for γ_3:

$$\dot{\gamma_3}^2 = (1-\gamma_3^2)(E - a\gamma_3) - (\frac{f - Cr_0\gamma_3}{A})^2, \tag{4.41}$$

where $E = \frac{1}{A}(2h - Cr_0^2)$, $a = \frac{2Mgz_0}{A}$ and, without loss of generality, we assume $a > 0$. In this equation, one can separate the variables and integrate to obtain the relation

$$t = \int \frac{d\gamma_3}{\sqrt{F(\gamma_3)}}, \tag{4.42}$$

where

$$F(\gamma_3) = (1-\gamma_3^2)(E - a\gamma_3) - \frac{1}{A^2}(f - Cr_0\gamma_3)^2. \tag{4.43}$$

The function $F(\gamma_3)$ is a cubic polynomial in the variable γ_3 and thus, the integral in (4.42) is an elliptic integral of the first kind and its inverse $\gamma_3(t)$ is an elliptic function. The three roots of F, which we denote by $u_1, u_2, u_3 (u_1 \le u_2 \le u_3)$, play a decisive role in determining the character of the motion. We shall now closely investigate the function F on the real γ_3 line. First we note that

$$F(\mp 1) = -\frac{1}{A^2}(f \pm Cr_0)^2,$$
$$F(+\infty) = +\infty. \tag{4.44}$$

That is, apart from the very special cases $f \pm Cr_0 = 0$, F is negative at the terminal points of the interval $[-1, 1]$. One of the roots, namely u_3, lies to the right of $\gamma_3 = 1$ and for physically meaningful motion the other two roots must be real and must lie in the interval $[-1, 1]$. The point representing γ_3 moves on the interval $[u_1, u_2]$, where F is non-negative, (See Fig. 4.2), traversing this interval in one direction and then in the other with its velocity vanishing at the ends. In the generic case of three different roots of F, γ_3 is periodic in t with period $T = 2\int_{u_1}^{u_2} \frac{d\gamma_3}{\sqrt{F(\gamma_3)}}$. Explicit solution was given in [134] in terms of Jacobi's elliptic functions of time. The most detailed solution is given in the treatise of Klein and Sommerfeld [222] in terms of Theta functions. Alternative form in Weierstrass' functions, the most popular in

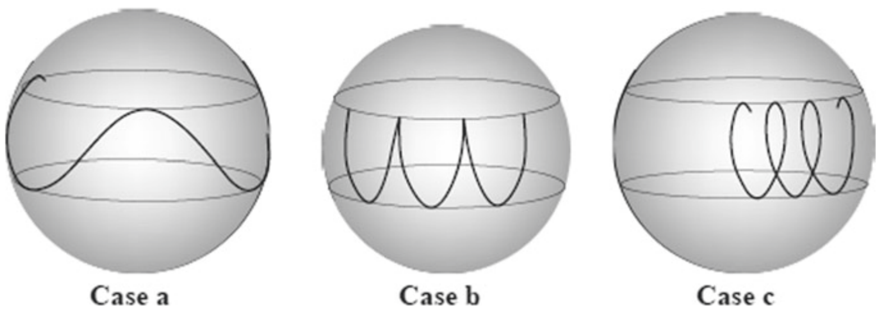

Fig. 4.2 The path of the apex on the fixed sphere

application at that time, can be found in Whittaker's book [368]. We shall not give explicit analytical formulas here, but we can draw, following Poisson, a very useful qualitative picture of the motion of the body.

Now, we turn to determine Euler's angles. The nutation angle θ is readily expressed as $\theta = \cos^{-1}\gamma_3$. The two relations

$$p\gamma_1 + q\gamma_2 = \dot{\psi}\sin^2\theta, \quad r = \dot{\psi}\cos\theta + \dot{\varphi}$$

can be used, together with the areas integral, to obtain expressions

$$\dot{\psi} = \frac{f - Cr_0\gamma_3}{A(1-\gamma_3^2)}, \tag{4.45}$$

$$\dot{\varphi} = r_0 - \frac{\gamma_3(f - Cr_0\gamma_3)}{A(1-\gamma_3^2)}. \tag{4.46}$$

One can obtain ψ, φ by integrating the last two expressions with respect to time. The general solution of the problem is thus constructed. It contains six arbitrary constants: h, f, r_0 and the integration constants that appear in the last three integration processes.

4.2.2 *The Study of the Motion*

Lagrange's top is assumed to have only dynamical symmetry about the z−axis. It is possible to think of it as an ordinary top having complete axial symmetry, geometric and physical. From this perspective, the way in which the proper rotation angle φ changes does not make difference to the observer. The significant part of the motion is that of the figure axis. This is determined by the two spherical coordinates $\theta = \cos^{-1}\gamma_3$ and ψ, governed by Eqs. (4.42) and (4.45), respectively.

4.2.2.1 Motion of the Apex

During the motion of the body, the apex of the figure axis traces a curve on the unit sphere fixed in space and with centre at the fixed point. The figure axis nutates, ascending from the angle $\theta_1 = \cos^{-1}u_1$ to $\theta_2 = \cos^{-1}u_2$ and then descending to θ_1, in a periodic manner, while precessing about the vertical. The precession velocity $\dot{\psi}$, as given by (4.45), takes its minimum value at the upper-most position θ_2. This value may be positive, zero or negative, depending on the quantity $f - Cr_0u_2$. Accordingly, the spherical curve traced by the apex touches the upper circle $u = u_2$ in the same direction as it makes with the lower one, has a cusp on that circle or touches it in the opposite direction. The trace of the apex takes one of the forms a, b or c, depicted in Fig. 4.2.

4.2.2.2 Regular Precession of Lagrange's Top

Generic motion of Lagrange's top corresponds to motion of the figure axis between the two circles u_1 and u_2. In Fig. 4.3a, u varies on the interval $[u_1, u_2]$. Regular precession corresponds to the nutation angle θ taking a constant value θ^* (say), and then from (4.45, 4.46) we find that the other two Eulerian angles ψ, φ change with time in constant rates. This occurs in two qualitatively different ways:

1- When $u_1 = u_2 = u^*$, $-1 \le u^* \le 1$. This happens at inclination u^*, provided two conditions $F(u^*) = F'(u^*) = 0$ are satisfied. These conditions give, respectively,

$$E = au^* + \frac{(Cr_0u^* - f)^2}{A(1 - u^{*2})},$$
$$f = \frac{1}{2u^*}[Cr_0(1 + u^{*2}) \pm (1 - u^{*2})\sqrt{C^2r_0^2 - 2Aau^*}]. \tag{4.47}$$

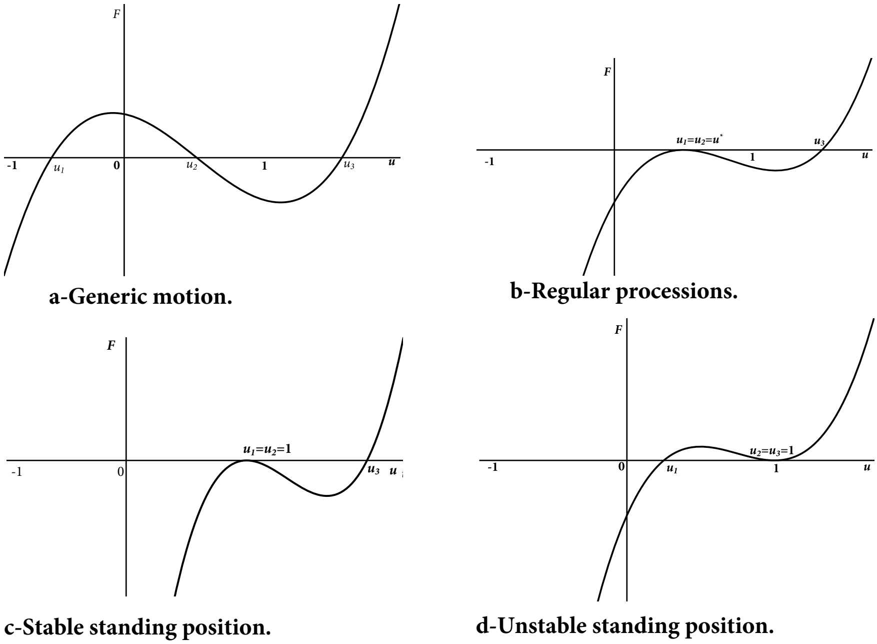

Fig. 4.3 Possible cases of motion of Lagrange's top

When the figure axis makes with the vertical upwards an angle $\theta^* = \cos^{-1} u^*(\frac{\pi}{2} < \theta^* \leq \pi)$, there are two different real values of f corresponding to two precessional motions that may be named as fast and slow precessions. On the other hand, when $0 \leq \theta^* < \frac{\pi}{2}$, the two precessions are possible only when $|r_0|$ exceeds a minimal value $\frac{\sqrt{2Aau^*}}{C}$ (Fig. 4.3b). This family of precessional motions include the hanging gyroscope position $u^* = -1$ and the standing gyroscope position $u^* = 1$, termed also as the sleeping top position (Fig. 4.3c). Regular precessions of this family are all stable, since a slight perturbation of the motion causes splitting of the two roots in a small neighbourhood of u^*. This leads to a small periodic change in the nutation angle θ and consequently small wobbling in the rates $\dot{\psi}$ and $\dot{\varphi}$.

2- When $u_1 < 1, u_3 = u_2 = 1$. This gives a different standing position, corresponding to Fig. 4.3d. On perturbation, the equal roots split into $u_3 > 1, u_2 < 1$. The figure axis begins a finite periodic motion, in which it goes near to position u_1 before it returns near to u_2. This standing position is unstable.

Comparing Fig. 4.3c, d, one concludes that the standing gyroscope position is stable when F has a maximum at its double root $u = 1$ and unstable when it has a minimum at that position. Analytically, the condition for a stable upright spinning position is

$$F(1) = F'(1) = 0, \; F''(1) < 0. \tag{4.48}$$

This finally gives the condition

$$r_0^2 > 2a\frac{A^2}{C^2} = \frac{4MgAz_0}{C^2}. \tag{4.49}$$

There is a critical value $r_0^* = 2\frac{\sqrt{MgAz_0}}{C}$ of the angular speed $|r_0|$ of the sleeping top, under which the upward position is unstable and above which that position becomes stable. The same stabilizing effect can be achieved even when the body stands at rest in its upper equilibrium position, with the help of a symmetric rotor whose axis is fixed in the body in alignment with the body axis. We shall return to this point in the next chapter.

4.3 Kowalevski's Case (1888)

In both cases of Euler and Lagrange, an integral of motion followed from general principles of mechanics, constancy of the angular momentum in the first and cyclicity of the angle of rotation about the axis of symmetry in the second. An important moment was that in both cases the equations of motion were solved to the end and the solution expressed through Jacobi's elliptic functions and certain integrals involving them.

The search for integrable cases continued, but, although the problem attracted attention of many eminent mathematicians, the search didn't lead to any other cases.

A whole century later, a new integrable case of the heavy rigid body was found by Sofia Kowalevski. That was not in virtue of a physical conservation principle, but using a purely mathematical condition: all solutions of the equations of motion (4.1) should have only poles as their singularities as functions of time in the complex t-plane. This property is satisfied by the solutions in the two known integrable cases of Euler and Lagrange, being expressible in terms of elliptic functions of time. It implies that the Euler–Poisson variables $\boldsymbol{\omega}, \boldsymbol{\gamma}$ can be represented in the vicinity of a pole at t_0 (say) by a Laurent series of the form

$$\omega_i = \frac{1}{(t-t_0)^{\alpha_i}} \sum_{k=0}^{\infty} \omega_i^{(k)} (t-t_0)^k, \gamma_i = \frac{1}{(t-t_0)^{\beta_i}} \sum_{k=0}^{\infty} \gamma_i^{(k)} (t-t_0)^k, \tag{4.50}$$

in which α_i, β_i are positive integers and the coefficients $\omega_i^{(k)}, \gamma_i^{(k)}$ are (in general complex) constants to be determined. In order for the series (4.50) to represent the general solution of the six-ordered system (4.1), five constants must remain arbitrary in it. The sixth one, as explained in Sect. 3.9, is associated with the integration process that determines the angle of precession.

Kowalevski used the values $\alpha_i = 1, \beta_i = 2$ and analyzed the successive systems of equations in the coefficients resulting from substituting the expansion in the Euler–Poisson equations of motion and their integrals. As a result, she isolated three cases of the required type, the two cases of Euler and Lagrange and a new third one, characterized by the conditions $A = B = 2C, z_0 = 0$. Thus, the centre of mass of the body lies in the plane of equal moments of inertia, i.e. $z_0 = 0$. By virtue of the condition $A = B$, we can assume that the centre of the mass lies on x-axis, so that we finally write Kowalevski's conditions for the new third case[1]

$$A = B = 2C, y_0 = z_0 = 0. \tag{4.51}$$

Note that those conditions guarantee only that property of the solution in the complex plane, but with no definite conclusion about integrability of the third case, i.e. about the existence of a complementary fourth integral, necessary for the integration of the equations of motion. Kowalevski tried and found the complementary integral in the third case, thus proving the integrability of that case. She also reduced problem to quadratures and expressed the solution in terms of hyper-elliptic functions of time, which are more complicated than elliptic functions, but they share with them the property of having only poles as singular points in the complex plane.

In this section, we throw some light on the Kowalevski case. In our presentation, more attention is paid to available principal results. Detailed citations to relevant sources covering the abridged material are given.

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[1] Some authors argue that the condition $y_0 = 0$ is a restriction on the physical parameters of the problem and call for correcting this error [98, 371]. For a discussion see Chap. 8 Sect. 8.14.

4.3.1 Integration of the Equations of Motion

Under Kowalevski's conditions (4.51), the Euler–Poisson equations take the form

$$\begin{aligned} 2\dot{p} - qr &= 0, \\ 2\dot{q} + pr &= a\gamma_3, \\ \dot{r} + a\gamma_2 &= 0, \end{aligned} \tag{4.52}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0, \tag{4.53}$$

in which $a = \frac{Mgx_0}{C}$. The three general integrals of motion may be written as

$$\begin{aligned} I_1 &= 2(p^2 + q^2) + r^2 + 2a\gamma_3 = 2h, \\ I_2 &= 2(p\gamma_1 + q\gamma_2) + r\gamma_3 = f, \\ I_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{4.54}$$

From the first two equations of (4.52), we get

$$2\frac{d}{dt}(p + iq) = -ir(p + iq) + ia\gamma_3, \tag{4.55}$$

and from the first two equations of (4.53)

$$\frac{d}{dt}(\gamma_1 + i\gamma_2) = -ir(\gamma_1 + i\gamma_2) + i\gamma_3(p + iq). \tag{4.56}$$

Multiplying (4.55) by $p + iq$ and (4.56) by a and subtracting, we obtain

$$\frac{d}{dt}\left[(p + iq)^2 - a(\gamma_1 + i\gamma_2)\right] = -ir\left[(p + iq)^2 - a(\gamma_1 + i\gamma_2)\right].$$

The last equation can be written in the equivalent form

$$\frac{d}{dt}\ln\left[(p + iq)^2 - a(\gamma_1 + i\gamma_2)\right] = -ir.$$

Taking the conjugate of the last equation, noting that all the variables are real, we get

$$\frac{d}{dt}\ln\overline{\left[(p + iq)^2 - a(\gamma_1 + i\gamma_2)\right]} = ir,$$

and adding the last two equations gives

$$\frac{d}{dt}\ln|(p + iq)^2 - a(\gamma_1 + i\gamma_2)|^2 = 0.$$

Thus, we obtain the fourth first integral in the form

$$|(p+iq)^2 - a(\gamma_1 + i\gamma_2)|^2 = k^2,$$

where k is an arbitrary integration constant. Its final form takes the form

$$I_4 = (p^2 - q^2 - a\gamma_1)^2 + (2pq - a\gamma_2)^2 = k^2. \tag{4.57}$$

According to Jacobi's theorem, the general solution of Euler–Poisson equations in this case can be constructed. To find it, Kowalevski introduced two new variables s_1, s_2 and showed that they satisfy the following two ordinary differential equations

$$\begin{aligned} \frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} &= 0, \\ \frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} &= \frac{dt}{2}, \end{aligned} \tag{4.58}$$

where

$$\Phi(s) = -(s-h-k)(s-h+k)F(s), \; F(s) = s[(s-h)^2 + a^2 - k^2] - \frac{a^2 f^2}{2}, \tag{4.59}$$

i.e. $\Phi(s)$ is a polynomial of degree five in the variable s. Those equations can be put in the equivalent form

$$\begin{aligned} \frac{\dot{s}_1}{\sqrt{\Phi(s_1)}} &= \frac{1}{2(s_1 - s_2)}, \\ \frac{\dot{s}_2}{\sqrt{\Phi(s_2)}} &= \frac{-1}{2(s_1 - s_2)}. \end{aligned} \tag{4.60}$$

Equations (4.58) are termed Abel–Jacobi equations. On integration, they give

$$\begin{aligned} \int_{s_1^0}^{s_1} \frac{ds_1}{\sqrt{\Phi(s_1)}} + \int_{s_2^0}^{s_2} \frac{ds_2}{\sqrt{\Phi(s_2)}} &= 0, \\ \int_{s_1^0}^{s_1} \frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \int_{s_2^0}^{s_2} \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} &= \frac{1}{2}(t - t_0), \end{aligned} \tag{4.61}$$

where s_1^0, s_2^0 are the initial values at $t = t_0$. The inversion of those ultra-elliptic quadrature is a classical solved problem. It gives the two variables s_1, s_2 in terms of ultra-elliptic functions of time, depending on the parameters h, f, k and two more integration constants. Alternatively, they can be expressed in terms of Theta functions in two variables. A detailed account of the inversion problem can be found in [113].

Kowalevski was also able to express all the dynamical variables in terms of s_1, s_2 [238, 256]. To this end she widely used the properties of elliptic functions, which were so popular at the time. Kötter [234] simplified and systematized Kowalevski's method and made it somewhat more transparent (See also [259] and references therein). Several later trials to simplify Kowalevski's and Kötter's derivations by using Lax representation of the equations of motion and methods of algebraic geometry have led to more complicated solutions in terms of Theta functions of three variables, e.g. [1, 26] (See also comments in [307]).

The motion of the body in Kowalevski's case takes its simplest form in the plane of the Kowalevski separation variables $s_{1,2}$. Let e_i, $i = 1, ..., 3$, be the roots of the cubic polynomial $F(s)$. Two options are possible, either e_1 is real and e_2, e_3 are complex conjugate, or the three roots are real and $e_1 \geq e_2 \geq e_3$. We shall introduce also the notation $e_4 = h - k$, $e_5 = h + k$. Note that, for determinacy, k can always be considered a non-negative constant, while a is made positive by a suitable choice of the coordinate x axis.

Each of the variables s_1, s_2 takes its values on one of the admissible intervals on which $\Phi(s)$ takes non-negative values. But, in view of Eq. (4.60), both variables cannot vary simultaneously in one and the same interval. After the works of Appelrot [9] (See also [108]), it turned out that in the generic case, when Φ has no equal roots, then real motions in Kowalevski's case correspond to the intervals of variation of Kowalevski's variables shown in the following table (Table 4.3).

Kowalevski's expressions for the phase variables in terms of the auxilliary variables s_1, s_2 can be written as

$$\begin{aligned}
p &= -\frac{\beta_1 P_1 + \beta_2 P_2 + \beta_3 P_3}{\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3}, \\
q &= \frac{1}{\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3}, \\
r &= \sqrt{2}\frac{\alpha_1 P_{23} + \alpha_2 P_{31} + \alpha_3 P_{12}}{\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3}, \\
\gamma_3 &= -\sqrt{2}\frac{\beta_1 P_{23} + \beta_2 P_{31} + \beta_3 P_{12}}{\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3}, \\
\gamma_1 \pm i\gamma_2 &= -\frac{\sum_{n=1}^{3}(e_n - e_4)\alpha_n P_{n4} \mp \sum_{n=1}^{3}(e_n - e_5)\alpha_n P_{n5}}{\sum_{n=1}^{3}\alpha_n P_{n4} \mp \sum_{n=1}^{3}\alpha_n P_{n5}},
\end{aligned} \tag{4.62}$$

Table 4.3 Admissible intervals of Kowalevski's variables s_1, s_2 when Φ has all roots distinct

1	$-\infty < s_2 \leq e_4 < e_1 \leq s_1 \leq e_5$	F has one real root e_1
2	$-\infty < s_2 \leq e_1 < e_4 \leq s_1 \leq e_5$	
3	$-\infty < s_2 \leq e_3 < e_2 < e_1 \leq s_1 \leq e_5$	$e_3 < e_4 < e_1$
4	$-\infty < s_2 \leq e_4 < e_3 < e_2 < e_1 \leq s_1 \leq e_5$	
5	$-\infty < s_2 \leq e_3 < e_2 < e_1 < e_4 \leq s_1 \leq e_5$	

where

$$P_n = \sqrt{(s_1 - e_n)(s_2 - e_n)},$$
$$P_{nm} = \frac{P_n P_m}{(s_1 - s_2)} \left\{ \frac{\sqrt{\Phi(s_1)}}{(s_1 - e_n)(s_1 - e_m)} - \frac{\sqrt{\Phi(s_2)}}{(s_2 - e_n)(s_2 - e_m)} \right\},$$

and the constant coefficients are given by

$$\alpha_n = \frac{\sqrt{2e_n}}{F'(e_n)},$$
$$\beta_1 = \frac{\sqrt{e_2 e_3}}{F'(e_1)}, \beta_2 = \frac{\sqrt{e_3 e_1}}{F'(e_2)}, \beta_3 = \frac{\sqrt{e_1 e_2}}{F'(e_3)}.$$

However, those formulas serve to show how complicated the solution is. The generic analytical solution is almost useless in the study of qualitative properties of motion. Even in some cases, when the solution takes simpler form for certain values of the parameters, it turns out to be more practical to return to direct solution of the equations of motion under given conditions, instead of substituting in the generic solution.

In real motion, the point (s_1, s_2) moves either in a rectangular area or in a semi-infinite strip. In this area, the s_1, s_2 are related to time through the ultra-elliptic integrals. For such motions, the general behaviour of the angles of precession and proper rotation was briefly investigated by Kozlov [240]. However, as the expressions of the dynamical variables given by Kowalevski and Kötter are too complicated to help drawing more qualitative conclusions about the motion, it acquires great importance to point out some special classes of motions for which the ultra-elliptic quadratures degenerate into elliptic or even into simpler ones. Those classes correspond to cases when the polynomial $\Phi(s)$ has multiple roots. They were first studied by Appelrot [11] and investigated later in [67, 158] (See also [183] or [108]). An account of those cases will be presented in Appendix B.

4.4 The Goryachev–Chaplygin Case: A Conditional Integrable Case

Here, we are going to deal with a new type of integrable cases in rigid body dynamics that we shall meet frequently later in this book, a conditional case. That is a case when the complementary fourth integral exists only under one condition: the initial motion $(\boldsymbol{\omega}_0, \boldsymbol{\gamma}_0)$ satisfies the restriction

$$\boldsymbol{\omega}_0 \mathbf{I} \cdot \boldsymbol{\gamma}_0 = 0,$$

and hence, from the integral of areas, we shall have at all subsequent times

$$\boldsymbol{\omega}\mathbf{I}\cdot\gamma = f = 0.$$

The angular momentum vector is horizontal at the initial moment and will stay so all the time.

4.4.1 The Fourth Integral

When the distribution of mass in the body satisfies the conditions

$$A = B = 4C, \; y_0 = z_0 = 0, \tag{4.63}$$

the equations of motion (3.29) take the form

$$\begin{aligned} 4\dot{p} &= 3qr, \\ 4\dot{q} &= -3pr + a\gamma_3, \\ \dot{r} &= -a\gamma_2, \\ \dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \; &\dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \; \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0 \end{aligned} \tag{4.64}$$

where $a = Mgx_0$. As was shown first by Goryachev, it can be easily verified that

$$\frac{d}{dt}[r(p^2 + q^2) - ap\gamma_3] = \frac{aq}{4C} f, \tag{4.65}$$

where f is the constant of areas. Now, we note, as was first done by Chaplygin, that when

$$f = \mathbf{G}\cdot\boldsymbol{\gamma} = 0, \tag{4.66}$$

i.e. when the vertical component f of the kinetic moment of the body vanishes, the quantity in the square bracket becomes an integral of motion

$$r(p^2 + q^2) - ap\gamma_3 = G \tag{4.67}$$

where G is an arbitrary constant. Note that, as in the three general cases of integrability, this integral is a polynomial, cubic in the Euler–Poisson variables. The integral (4.67) was obtained by Chaplygin and will be named Chaplygin's integral. The particular case found slightly earlier by Goryachev is subject to the additional restriction $G = 0$ and an invariant relation $\sqrt[3]{p}\gamma_3 = br$, b is an arbitrary constant.

In the three general cases of integrability, the problem of motion was integrable for all initial conditions, which is not the case here. The integral (4.67) is valid only for motions satisfying the condition (4.66), i.e.

$$4p\gamma_1 + 4q\gamma_2 + r\gamma_3 = 0. \tag{4.68}$$

Of course, it suffices to have this condition satisfied at the beginning of motion, and it will be satisfied at all times since the quantity in hand is an integral of motion. This condition means that the kinetic moment of the body should lie at the initial moment of motion in a horizontal plane passing through the fixed point, and it will continue to be in the same plane at all times.

4.4.2 Separation of Variables. Solution of the Equations of Motion

As we have mentioned earlier, the mere presence of the four integrals of motion does not always mean that a known procedure can be utilized to reduce equations of motion to a separation of variables and, consequently, to complete the process of explicit expression of all the phase variables of the problem in terms of certain functions of time. In most cases, one has to design for a new case a new method, almost independent of methods applied in the solution of other cases.

Here, we present for Goryachev–Chaplygin case the method which Chaplygin was able to devise for separation of variables. We first write all the integrals of motion

$$\begin{aligned}
\frac{1}{2}[4(p^2 + q^2) + r^2] + a\gamma_1 &= \frac{h}{C} = E, \\
4p\gamma_1 + 4q\gamma_2 + r\gamma_3 &= f/C = 0, \\
r(p^2 + q^2) - ap\gamma_3 &= G, \\
\gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1.
\end{aligned} \tag{4.69}$$

Following Chaplygin, we introduce new variables u and v by the relations

$$\begin{aligned}
r &= u - v, \\
4(p^2 + q^2) &= uv.
\end{aligned} \tag{4.70}$$

Substituting in (4.69) and solving the first three equations, which are linear in $\gamma_1, \gamma_2, \gamma_3$, and inserting the solution in the fourth equation, we determine p and q. After some manipulations we write down the solution of the algebraic system (4.69)–(4.70) is

$$\begin{aligned}
p &= \frac{U_1 V_2 - U_2 V_1}{8a}, \\
q &= \frac{U_1 V_1 + U_2 V_2}{8a}, \\
r &= u - v,
\end{aligned}$$

$$\begin{aligned}\gamma_1 &= -\frac{U+V}{2a(u+v)},\\ \gamma_2 &= -\frac{U_1U_2 - V_1V_2}{2a(u+v)},\\ \gamma_3 &= -\frac{U_1V_2 + V_1U_2}{2a(u+v)},\end{aligned} \tag{4.71}$$

where

$$\begin{aligned} U &= u^3 - 2Eu - 4G, & V &= v^3 - 2Ev + 4G,\\ U_1^2 &= U - 2au, & V_1^2 &= V - 2av,\\ U_2^2 &= -U - 2au, & V_2^2 &= -V - 2av. \end{aligned} \tag{4.72}$$

Thus, we have expressed the six Euler–Poisson variables in terms of the two variables u and v. It remains to determine the dependence of u, v on time. Using the first equation of (4.70), the third of (4.64) and (4.71), we get

$$\frac{du}{dt} - \frac{dv}{dt} = -a\gamma_2 = \frac{U_1U_2 - V_1V_2}{2(u+v)}. \tag{4.73}$$

Also, using the second equation of (4.70), the first and second of (4.64) and (4.71), we find

$$\begin{aligned} v\frac{du}{dt} + u\frac{dv}{dt} &= 8(p\dot{p} + q\dot{q})\\ &= 2aq\gamma_3\\ &= -\frac{U_1V_1 + U_2V_2}{8a}\frac{U_1V_2 + V_1U_2}{(u+v)}, \end{aligned} \tag{4.74}$$

so that we finally obtain two differential equations in u and v

$$2(u+v)\frac{du}{dt} = U_1U_2, \quad 2(u+v)\frac{dv}{dt} = V_1V_2. \tag{4.75}$$

The last equations can be put in the form

$$\frac{du}{U_1U_2} - \frac{dv}{V_1V_2} = 0, \quad \frac{udu}{U_1U_2} + \frac{vdv}{V_1V_2} = \frac{dt}{2}. \tag{4.76}$$

Integrating from the initial position u_0, v_0 at t_0, we can now write

$$\begin{aligned} &\int_{u_0}^{u} \frac{du}{\sqrt{4a^2u^2 - (u^3 - 2Eu - 4G)^2}} - \int_{v_0}^{v} \frac{dv}{\sqrt{4a^2v^2 - (v^3 - 2Ev + 4G)^2}} = 0,\\ &\int_{u_0}^{u} \frac{udu}{\sqrt{4a^2u^2 - (u^3 - 2Eu - 4G)^2}} + \int_{v_0}^{v} \frac{vdv}{\sqrt{4a^2v^2 - (v^3 - 2Ev + 4G)^2}} = \frac{1}{2}(t - t_0). \end{aligned} \tag{4.77}$$

With this step, we have completed the process of separation of variables as was proposed by Chaplygin. Integrals in (4.77) are in the standard form of hyper-elliptic integrals, where the denominators of terms are square roots of polynomials of the sixth degree. The inverse functions, i.e. the solution of the system (4.77) in the variables u, v as functions of time can be expressed in terms of Theta functions of two variables, each of which is a linear function of time. This step was completed for the present problem by Marcolongo and Olsson and the reader may be referred to their original works [272, 298]. The behaviour of the variables u, v depending on the number and multiplicity of real roots was studied by Dokshevich [67]. The analysis is much simpler than in Kowalevski's case, since here we deal with a system with only two parameters.

It is noteworthy that the separation of variables in Goryachev–Chaplygin's case was attained in [240] by an elegant approach, using canonical Andoyer–Deprit variables. Also, certain qualitative properties of motion were studied, e.g. the long time behaviour of the angles of precession and proper rotations. Particularly simple solutions of Goryachev–Chaplygin case are discussed in Appendix C.

4.5 Integrability and Nonintegrability Issues

As mentioned above, Kowalevski used the analytical theory of differential equations to isolate all possible combinations of the physical parameters A, B, C, x_0, y_0 and z_0, for which the general solution of the Euler-Poisson system of equations of motion, for arbitrary initial conditions, can be expressed as single-valued functions that have only poles as their singularities in the complex plane of time t. Her approach led to the third integrable case in the classical problem of motion of a rigid body about a fixed point. Kowalevski's result was continued and supplemented by Appelrot [10] and Lyapunov [266]. Lyapunov's method is characterized by the introduction of a small parameter and involves the analysis of the monodromy of variational equations for some particular solutions of Euler–Poisson's equations. It showed that, except for the cases of Euler, Lagrange and Kowalevski, the general solution of those equations branches in the complex time plane in all cases.

The subject of integrability was a favourite field for several mathematicians over a large period of time.

As we have noticed, the fourth integral in the three general integrable cases has polynomial form in the Euler–Poisson variables. Husson investigated the possibility that there exists an additional general integral algebraic in those variables [154, 155]. It turned out that this happens only in the three known integrable cases. Ziglin [435] has also shown that a meromorphic complementary first integral of the Euler–Poisson equations exists only in the cases of Euler, Lagrange and Kowalevski and also in the conditional ($f = 0$) Goryachev–Chaplygin case. We note also the recent proof of the non-existence of meromorphic integrals [268], obtained by means of a technique based on differential Galois theory.

One has to mention also the contributions and different approaches of Poincaré, Kozlov [240, 245] and Ziglin [434, 435] and others. For a more detailed presentation see, e.g. [41]. Detailed analysis of the complete algebraic integrability and the anaytical structure of the solution in the complex plane for the integrable cases can be found in [2]. However, we note also that although methods of proving integrability or non-integrability had succeeded in the study of the classical problem, they become much less effective in more general problems involving the effects of additional potential and gyroscopic forces on the body.

Chapter 5
The Motion of a Heavy Gyrostat

Originally, the gyrostat, as the terminology was coined by Lord Kelvin, is a heavy rigid body with a rotor or a fly-wheel spinning with a constant angular speed about its axis of symmetry. The subject gained a great interest at the first two decades of the twentieth century. Examples are the two books by Crabtree [59] (1909) and Gray [133] (1918), devoted exclusively to describing gyroscopic phenomena, specially the stabilizing effects of rotors, and the ways to make use of them in warfare of World War I. Today, gyroscopic apparatuses are indispensable in cell phones, in so many applications in terrestrial and cosmic navigation and in technology. Most useful is the stabilizing effect of fast rotors on normally unstable motions and equilibria.

In this chapter, different types of mechanical systems having the same equations of motion as the gyrostat are presented. General and conditional integrable cases of motion are presented. In fact, these are generalizations of the relevant integrable cases in the classical problem, and reduce to them when the gyrostatic momentum vanishes.

At present, several particular solutions to the problem of motion of a gyrostat are known, namely, 13 solutions. Some of them are generalizations of classical counterparts by adding a gyrostatic momentum. Other cases lose their meaning when the gyrostatic moment vanishes.

5.1 Models of the Gyrostat

5.1.1 The Classical Model

Consider a system S, composed of two joint rigid bodies. The first, S_0, which we shall call the carrier (or the main) body, is fixed in the inertial space from its point

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_5

O. The second body, the rotor S_1, is an axially symmetric body fixed from its axis of symmetry in the main body. Its centre of mass, O_1, lies on its axis of symmetry. Usually, such a symmetric body is called gyroscope. Because of the symmetry of the rotor, its rotation does not change the distribution of mass in the system. Let $\mathbf{I}$ and $\mathbf{r}_0$ be the inertia matrix and the position vector of the centre of mass of the system, referred to the system of axes $Oxyz$ fixed in the main body. Let also $\mathbf{J}$ be the inertia matrix of the rotor with respect to a system of axes $O_1x_1y_1z_1$ fixed in it with z_1 along its axis of symmetry. From symmetry, it is clear that $O_1x_1y_1z_1$ is a system of principal axes of the rotor and hence we can write $\mathbf{J} = \mathrm{diag}(J_1, J_1, J)$.

Let the rotor be set and kept in motion about its axis with a constant angular velocity Ω, by means of some device. Let $\mathbf{r}_1 = \overrightarrow{OO_1}$ and denote by $\mathbf{r}$ the position vector of a mass element dm of the system with respect to O. The velocity of that element is $\boldsymbol{\omega} \times \mathbf{r}$ if it belongs to S_0 and $\boldsymbol{\omega}\times\mathbf{r}+ \Omega\mathbf{e} \times \mathbf{r}'$ for elements of S_1, where $\mathbf{r}'$ is the position vector of the mass element of the rotor with respect to O_1. The angular momentum of the system can be written as

$$\begin{aligned}
\mathbf{G} &= \int_{S_0} \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r})dm + \int_{S_1} \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r}+\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \int_{S_0} \mathbf{r} \times (\boldsymbol{\omega}\times\mathbf{r})dm + \int_{S_1} \mathbf{r} \times (\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \boldsymbol{\omega}\mathbf{I}+ \int_{S_1} (\mathbf{r}_1 + \mathbf{r}')\times(\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \boldsymbol{\omega}\mathbf{I} + \mathbf{r}_1\times(\Omega\mathbf{e} \times \int_{S_1} \mathbf{r}'dm) + \int_{S_1} \mathbf{r}'\times(\Omega\mathbf{e} \times \mathbf{r}')dm \\
&= \boldsymbol{\omega}\mathbf{I} + \mathbf{0}+\Omega\mathbf{e}\mathbf{J} \\
&= \boldsymbol{\omega}\mathbf{I}+\Omega J\mathbf{e}.
\end{aligned} \tag{5.1}$$

Here we have used $\int_{S_1} \mathbf{r}'dm = \mathbf{0}$. The last expression will be written as

$$\mathbf{G} = \boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}, \tag{5.2}$$

where $\boldsymbol{\kappa} = \Omega J\mathbf{e}$ is the gyrostatic momentum, the angular momentum of the rotor relative to the carrier body. It is directed along the axis of symmetry of the rotor.

Now we write down the equation of motion of the system. The mutual forces between the main body and the rotor are internal forces in the system and do not appear in this equation. One has

$$\dot{\boldsymbol{G}} + \boldsymbol{\omega} \times \boldsymbol{G} = Mg\boldsymbol{\gamma}\times\mathbf{r}_0.$$

Since $\boldsymbol{\kappa}$ is kept constant in the body, $\dot{\boldsymbol{\kappa}} = \mathbf{0}$, and the last equation reduces to

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa}) = Mg\boldsymbol{\gamma}\times\mathbf{r}_0. \tag{5.3}$$

This is the final form of the dynamical equation of motion of the gyrostat. Together with Poisson's equation

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{5.4}$$

one obtains a closed system which we now write in the following scalar form of six first-order differential equations:

$$\begin{aligned} A\dot{p} + (C - B)qr + \kappa_3 q - \kappa_2 r &= Mg(z_0\gamma_2 - y_0\gamma_3), \\ B\dot{q} + (A - C)pr + \kappa_1 r - \kappa_3 p &= Mg(x_0\gamma_3 - z_0\gamma_1), \\ C\dot{r} + (B - A)pq + \kappa_2 p - \kappa_1 q &= Mg(y_0\gamma_1 - x_0\gamma_2), \end{aligned} \tag{5.5}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{5.6}$$

This system admits the general integrals:

$$\begin{aligned} &I_1 \equiv Ap^2 + Bq^2 + Cr^2 + Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h, \\ &I_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ &I_3 = (Ap + \kappa_1)\gamma_1 + (Bq + \kappa_2)\gamma_2 + (Cr + \kappa_3)\gamma_3 = f. \end{aligned} \tag{5.7}$$

The first integral is usually termed Jacobi's integral for the system, since it is different from the total energy of the system, which contains terms linear in the components of $\boldsymbol{\omega}$.

When the angular speed Ω of the rotor vanishes, gyrostatic momentum $\boldsymbol{\kappa} = \mathbf{0}$, and equations (5.5) and the integrals (5.7) reduce to their counterparts of the classical problem.

5.1.2 *The Free Rotor Model*

In the previous model, the angular velocity of the rotor was kept constant relative to the carrier body. In an interesting alternative, due to Levi-Civita [261], the rotor is left to move freely around its axis of symmetry fixed in the body, so that the system will have an additional rotational degree of freedom. Let χ be the angle of rotation of the rotor relative to the body. Using the same symbols as in the previous subsection, the kinetic energy of the system is expressed as the sum of two parts

$$\begin{aligned} T &= \frac{1}{2}\int_{S_0} (\boldsymbol{\omega}\times\mathbf{r})^2 dm + \frac{1}{2}\int_{S_1} (\boldsymbol{\omega}\times\mathbf{r}+\dot{\chi}\mathbf{e}\times\mathbf{r}')^2 dm \\ &= \frac{1}{2}\int_{S_0} (\boldsymbol{\omega}\times\mathbf{r})^2 dm + \dot{\chi}\int_{S_1} (\boldsymbol{\omega}\times\mathbf{r})\cdot(\mathbf{e}\times\mathbf{r}')dm + \frac{1}{2}\dot{\chi}^2\int_{S_1} (\mathbf{e}\times\mathbf{r}')^2 dm \end{aligned}$$

$$= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \dot{\chi}\int_{S_1}(\mathbf{r}_1+\mathbf{r}')\times(\mathbf{e}\times\mathbf{r}')dm\cdot\boldsymbol{\omega} + \frac{1}{2}\dot{\chi}^2\int_{S_1}\mathbf{r}'\times(\mathbf{e}\times\mathbf{r}')dm\cdot\mathbf{e}.$$

Noting that

$$\mathbf{r}_1\times(\mathbf{e}\times\int_{S_1}\mathbf{r}'dm) = 0, \int_{S_1}\mathbf{r}'\times(\mathbf{e}\times\mathbf{r}')dm = \mathbf{eJ} = J\mathbf{e},$$

we obtain

$$T = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + J\dot{\chi}\mathbf{e}\cdot\boldsymbol{\omega} + \frac{1}{2}J\dot{\chi}^2$$

and hence the Lagrangian of the system may be written as

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + J\dot{\chi}\mathbf{e}\cdot\boldsymbol{\omega} + \frac{1}{2}J\dot{\chi}^2 - \mathbf{a}\cdot\boldsymbol{\gamma} \tag{5.8}$$

where $\mathbf{a} = Mg\mathbf{r}_0$. Obviously, the angle χ is a cyclic variable. The corresponding cyclic integral is

$$\frac{\partial L}{\partial \dot{\chi}} = J(\mathbf{e}\cdot\boldsymbol{\omega} + \dot{\chi}) = \kappa, \tag{5.9}$$

κ is an integration constant. Note that this integral means that the component of the total angular velocity of the rotor along its axis of symmetry remains constant during motion, i.e.

$$\mathbf{e}\cdot\boldsymbol{\omega} + \dot{\chi} = \frac{\kappa}{J}.$$

Now, ignoring the cyclic coordinate, we obtain the Routhian

$$\begin{aligned} R &= L - \kappa\dot{\chi} \\ &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} - \frac{1}{2}J(\boldsymbol{\omega}\cdot\mathbf{e})^2 + \kappa\mathbf{e}\cdot\boldsymbol{\omega} - \mathbf{a}\cdot\boldsymbol{\gamma} - \frac{\kappa^2}{2J} \\ &= \frac{1}{2}\boldsymbol{\omega}\tilde{\mathbf{I}}\cdot\boldsymbol{\omega} + \kappa\mathbf{e}\cdot\boldsymbol{\omega} - \mathbf{a}\cdot\boldsymbol{\gamma} \end{aligned} \tag{5.10}$$

where

$$\tilde{I}_{ij} = I_{ij} - Je_ie_j \tag{5.11}$$

and a constant $\frac{\kappa^2}{2J}$ has been ignored. In the way described in Chap. 3, the Euler dynamical equation derived from this Routhian are

$$\dot{\boldsymbol{\omega}}\tilde{\mathbf{I}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\tilde{\mathbf{I}} + \boldsymbol{\kappa}) = \boldsymbol{\gamma}\times\mathbf{a}, \tag{5.12}$$

where $\boldsymbol{\kappa} = \kappa\mathbf{e}$.

This equation has the same structure as (5.3), but the matrix $\tilde{\mathbf{I}}$ is not simply the matrix of inertia of the system, but depends on the axial moment of inertia of the rotor and on the orientation of its axis relative to the main body. The meaning of the vector $\boldsymbol{\kappa}$, the gyrostatic momentum is different in the two equations. Moreover, it should be noted that, in view of (5.11), the matrix $\tilde{\mathbf{I}}$ may not satisfy conditions, normal to ordinary inertia matrix of the simple body, like positivity of diagonal elements, triangle inequalities, etc.

5.1.3 Joukovsky's Model

Generalizing previous particular cases considered by Stokes and Neumann, Joukovsky established that "a fluid mass with an initial velocity in a multiply-connected cavity in the rigid body performs an action that is similar to the action of some rotor attached to the rigid body" [163] (see also [41, 286]).

The gyrostatic moment can also be due to internal cyclic degrees of freedom such as circulation of fluid in tubes inside the body or to forced stationary motions as motors, whose axes are fixed in the body.

As will be seen in a later chapter, terms in the equations of motion similar to gyrostatic momentum appear in problems of motion of a perforated rigid body (a body bounded by a multi-connected surface) in a liquid, as a result of the presence of circulations through perforations.

5.2 Equations of Motion in Hamiltonian Form

As in the classical problem (Chap. 3), one may use some generalized coordinates like Euler's angles, to construct the Hamiltonian function and the canonical equations of motion, involving those coordinates and momenta conjugate to them. This form of the equations of motion is rarely used in applications and is left as an exercise. Non-canonical equations

$$\mathbf{M} = \frac{\partial R}{\partial \boldsymbol{\omega}} = \boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa}, \tag{5.13}$$

so that

$$\boldsymbol{\omega} = (\mathbf{M} - \boldsymbol{\kappa})\mathbf{I}^{-1}. \tag{5.14}$$

Also, the Hamiltonian corresponding to the same Routhian as a function in $\mathbf{M}$ and $\boldsymbol{\gamma}$ is

$$H = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{a}\cdot\boldsymbol{\gamma}$$

$$
\begin{aligned}
&= \frac{1}{2}(\mathbf{M} - \boldsymbol{\kappa})\mathbf{I}^{-1} \cdot (\mathbf{M} - \boldsymbol{\kappa}) + \mathbf{a} \cdot \boldsymbol{\gamma} \\
&= \frac{1}{2}\mathbf{M}\mathbf{I}^{-1} \cdot \mathbf{M} - \boldsymbol{\kappa}\mathbf{I}^{\mathbf{-1}} \cdot \mathbf{M} + \mathbf{a} \cdot \boldsymbol{\gamma},
\end{aligned}
\tag{5.15}
$$

so that the equations of motion can be written as

$$
\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \\
\dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}}.
\end{aligned}
\tag{5.16}
$$

or, in the expanded form,

$$
\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M} \times (\mathbf{M} - \check{\ })\mathbf{I}^{-1} + \boldsymbol{\gamma} \times \mathbf{a}, \\
\dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times (\mathbf{M} - \boldsymbol{\kappa})\mathbf{I}^{-1}.
\end{aligned}
\tag{5.17}
$$

5.3 Tables of Integrable Cases

Equations (5.5 and 5.6) have three known general and one conditional integrable cases, which generalize the four cases of a simple heavy body. Those are listed in the following Tables 5.1 and 5.2.

Table 5.1 Unconditional cases

	Author	Conditions
1	Joukovsky 1885 [163] and Volterra 1899 [366] Euler $\kappa_1 = \kappa_2 = \kappa_3 = 0$. $I_4 = (Ap + \kappa_1)^2 + (Bq + \kappa_2)^2 + (Cr + \kappa_3)^2$.	$g\mathbf{r}_0 = \mathbf{0}$.
2	Axially symmetric case (Generalization of Lagrange's top) $I_4 = Cr + \kappa_3$.	$B = A$, $x_0 = y_0 = 0$, $\kappa_1 = \kappa_2 = 0$.
3	Yehia 1986 [380], [383]* Kowalevski 1889 ($\kappa = 0$) [238] $I_4 = (p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2)^2 + (2pq - a_1\gamma_2 - a_2\gamma_1)^2$ $+2\kappa\,(r - \kappa)\,(p^2 + q^2) - 4\kappa\gamma_3(a_1 p + a_2 q)$,	$A = B = 2C$, $z_0 = 0$, $\kappa_1 = \kappa_2 = 0$.

where $\kappa = \kappa_3/C$, $a_1 = Mgx_0/C$, $a_2 = Mgy_0/C$

*The case (3) was rediscovered in 1987 by Komarov [225] and Gavrilov [109]. In the monograph [41], it is attributed to Yehia, Komarov and Gavrilov, but in the Russian literature it is mostly called *Kowalevski–Yehia's* case

Table 5.2 Conditional cases $f = 0$

1	Sretensky (1963) [341]. Goryachev–Chaplygin 1900–1901 ($\kappa = 0$).	$A = B = 4C$, $z_0 = 0$, $\kappa_1 = \kappa_2 = 0$, $\kappa_3 = C\kappa$.
	$I_4 = (r - \kappa)(p^2 + q^2) - \gamma_3(a_1 p + a_2 q)$.	

where $\kappa = \kappa_3/C$, $a_1 = Mgx_0/C$, $a_2 = Mgy_0/C$

5.4 The Case of Joukovsky and Volterra

The first integrable case, which generalizes Euler's case, i.e. a balanced gyrostat or a gyrostat under no external torques, was noted in 1885 by Joukovsky in his study of the motion by inertia of a body containing liquid-filled cavities [163]. He also devised a geometric-mechanical interpretation of the motion in that case. Independently, in a trial to explain the displacement of Earth's poles by adding a rotor to the model of rigid Earth, Volterra gave in 1899 the full solution of the equations of motion in terms of Weierstrass' elliptic functions σ_i of time [366]. Those functions are complex in general. An alternative but real solution in terms of Jacobi's elliptic functions was constructed by Wittenburg [369]. Volterra's solution and stability analysis of the permanent rotations were reconsidered in [18].

5.5 The Case of Axially Symmetric Gyrostat

The axi-symmetric gyrostat is a trivial generalization of Lagrange's top and the solution of the equations of motion for it is practically the same as that of Lagrange's case. In a later Chap. 12, we will show a much richer generalization of this case.

We now prove the following

Theorem 5.1 *Any integrable case of an axi-symmetric body in a potential field, in which both ϕ and ψ are cyclic variables, can be generalized by the addition of a rotor aligned with the axis of symmetry.*

Theorem 5.2 *Consider the motion of an axi-symmetric gyrostat, with a gyrostatic momentum κ aligned along the axis of symmetry of the carrier body. The motion of the axis of the gyrostat is identical with the motion of a simple body with the same moments of inertia of the carrier body and moving in the same potential. The gyrostatic momentum is compensated by an additional angular speed κ/C given to the body about its axis, C being the axial moment of inertia of the body.*

Conversely, the motion of the axis of a simple axi-symmetric body, which given an additional angular speed Ω around that axis is identical to the motion the axis of a similar body, which carries a rotor with gyrostatic momentum $\kappa = C\Omega$.

Those two theorems can be proved by writing the Lagrangian of the simple body in a field with potential $V(\theta)$. Let the moments of inertia of the body be C about its z-axis of symmetry and A about any axis orthogonal to it. For such body, we have from (3.44)

$$\begin{aligned} L &= \frac{1}{2}[A(p^2+q^2)+Cr^2] - V(\theta) \\ &= \frac{1}{2}[A(\dot\theta^2+\sin^2\theta\dot\psi^2)+C(\dot\psi\cos\theta+\dot\varphi^2) - V(\theta). \end{aligned} \tag{5.18}$$

We now study the motion in another reference frame, which is rotating abut the z-axis with a constant angular rate Ω. This can be achieved by a substitution

$$\varphi = \varphi' + \Omega t,$$

which preserves the holonomicity of the system. The Lagrangian transforms to

$$\begin{aligned} L &= \frac{1}{2}\{A(\dot\theta^2+\sin^2\theta\dot\psi^2)+C[\dot\psi\cos\theta+(\dot\varphi'+\Omega)^2)]\} - V(\theta) \\ &= \frac{1}{2}\{A(\dot\theta^2+\sin^2\theta\dot\psi^2)+C(\dot\psi\cos\theta+\dot\varphi')^2\} \\ &\quad + C\Omega(\dot\psi\cos\theta+\dot\varphi') - V(\theta) + \frac{1}{2}C\Omega^2. \end{aligned}$$

Ignoring the last constant term, this can be regarded as describing the motion of the body referred to fixed axes in it, but with the coordinate φ' instead of φ and with the same potential V and additional gyroscopic term $C\Omega(\dot\psi\cos\theta+\dot\varphi') = C\Omega r'$. The last term is the contribution of a gyrostatic momentum $\boldsymbol{\kappa}$ directed along the z-axis

$$\boldsymbol{\kappa} = (\mathbf{0}, \mathbf{0}, C\Omega).$$

This proves Theorem 5.1 and the first part of Theorem 5.2. The second part of Theorem 5.2 follows naturally.

Those results may be used to express the quadrature resulting from separation of variables in the case 2 of Table 5.1, i.e. the gyrostatic generalization of Lagrange's case from the quadrature (4.42) by replacing r_0 by $r_0 + \frac{\kappa_3}{C}$, so that it becomes

$$t = \int \frac{d\gamma_3}{\sqrt{(1-\gamma_3^2)(E-a\gamma_3) - \frac{1}{A^2}[f-(Cr_0+\kappa_3)\gamma_3]^2}}.$$

5.6 Yehia's Case

The history of the third case in Table 5.1 has experienced some confusion and misunderstandings. It was direct and easy, guided by the same principle of conservation of angular momentum, to obtain the integrable case of Joukovsky, as a generalization of

Euler's case in the classical problem, by adding a constant gyrostatic momentum. The generalization of Lagrange's case of a symmetric body was even easier. Nevertheless, the search of a gyrostatic generalization of Kowalevski's case was so futile, that it was generally believed that, unlike Euler's and Lagrange's cases, Kowalevski's case does not admit generalization by the addition of a gyrostatic momentum. This trend may have been augmented by three contradicting results published in the mid-sixties by Keis:

(1) In the first of those works [167] (1963), Keis claimed having obtained generalizations to the gyrostat problem for four known integrable and particular cases of the classical problem, namely, Lagrange's, Hess', Bobylev–Steklov's and Delone's cases. The first case, the generalization of Lagrange's top, is trivially simple and the next two cases will be commented on in the relevant section on particular solutions. The last case (Delone's) is a special case of Kowalevski's, when Kowalevski's integral takes a zero value and splits into two invariant relations (See Sect. 4.3). Keis added to the body a constant gyrostatic momentum, aligned with the centre of mass in the equatorial plane of the inertia spheroid and claimed that the resulting system admits two invariant relations generalizing those of Delone's case. This claim was cited as being true in the review book [256].

(2) In the second paper [168] (1964), may be after realizing the flaw in his 1963 paper (cited in [168]), Keis used the method of Husson [154, 155] to give another theorem asserting that the equations of motion of the heavy gyrostat with Kowalevski's configuration $A = B = 2C$ admit an algebraic complementary integral only when the gyrostatic momentum vanishes ($\kappa_1 = \kappa_2 = \kappa_3 = 0$). He then formulated it as[1] "If $x_0^2 + y_0^2 + z_0^2 \neq 0$ and $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \neq 0$ a fourth algebraic integral is possible only when $A = B, x_0 = y_0 = 0, \ \kappa_1 = \kappa_2 = 0$", i.e. only in the case of Lagrange when both the centre of mass and the gyrostatic momentum are directed along the axis of dynamical symmetry of the body. This meant that Kowalevski's case has no extension to the gyrostat problem.

(3) In the third paper [170] (1965), Keis used Golubev's method [113] (In fact, Poincaré's method of small parameter) to establish a new result. The search for all cases, when all the solutions of the equations of motion of a heavy gyrostat are single-valued, reduces to investigation of the solution in three cases:

    (a) The torque-free gyrostat, or gyrostat fixed from its centre of mass, ($x_0 = y_0 = z_0 = 0$).
    (b) The axi-symmetric gyrostat ($A = B, x_0 = y_0 = 0, \ \kappa_1 = \kappa_2 = 0$).
    (c) The "Kowalevski gyrostat" ($A = B = 2C, y_0 = z_0 = 0, \kappa_1 = \kappa_2 = 0$).

    In this paper, the presence of the third component of the gyrostatic momentum does not give rise to multi-valued solutions in any of the three cases, up to the second degree of a small parameter. However, the conditions for case c are considered as necessary. The only way to ensure integrability is to find the

[1] Here we use the notation adopted in the present book.

complementary fourth integral of motion, a step which was not considered by Keis. Strangely, in the third paper the author does not refer to any of the other two papers, each of which announces a conflicting result.

Probably, influenced by the result in the second paper of Keis, published in the most influential Russian mechanics journal PMM, Kharlamov and coworkers concentrated on the search for particular solutions of the equations of motion. In this respect, they have succeeded in constructing the most part of the cases of that type known up to date. To this end, they used equations of motion in the form of Euler–Poisson and various modified forms. Kharlamov [198] obtained a particular solution of the heavy gyrostat with the Kowalevski configuration $A = B = 2C$, $y_0 = z_0 = 0$, involving a gyrostatic moment along the axis of dynamical symmetry of the body under certain restrictions on the initial motion. His case characterized by the existence of an invariant relation quadratic in the angular velocities fits as a special case of the third general integrable case in Table 5.1.

The full generalization of Kowalevski's case by the addition of a rotor to the body came out, in our work [380], in almost a century (exactly 98 years) after the publication of Kowalevski's case (See also [383]). Actually, it was not found as a solution of Euler–Poisson equations or any of their modifications. It was one of the first results obtained by the completely new method devised by the author of the present book to construct integrable 2D conservative mechanical systems, which admit a complementary integral polynomial in the velocities. After constructing a several-parameter integrable time-irreversible system of the above type, the parameters of the system are given certain values, such that the metric of the system could be identified with that of the Routhian reduction of the rigid body dynamics and then potential and gyroscopic forces could be identified and only then the appropriate Euler–Poisson equations are verified and the presentation of the new case in [380] was made in the last context.[2] The details of the method will not be presented here for space considerations, but the reader can get some acquaintance with it from the early papers [381, 419]. This method has proved fruitful and still gives new integrable cases of much more complicated problems in particle and rigid body dynamics (see, e.g. [411, 413, 422, 423]). All cases pertaining to rigid body dynamics obtained in this way will be described later in this book. They form the most part of the list of conditional integrable cases in Chap. 13.

The question of integrability of Eqs. (5.5 and 5.6) did not attract as much interest as the problem of a simple heavy body. Only one result on this aspect is known. It generalizes the above-mentioned theorem of Husson to the present problem of motion of a gyrostat.

Theorem 5.3 (Gavrilov [110, 111]) *The equations of motion of heavy gyrostat (5.5 and 5.6) possess an additional algebraic first integral only in the three cases of Joukowsky, Lagrange and Yehia.*

[2] This result was announced at the International Conference on Mechanics held in Moscow University, January 1986.

5.6.1 Separation of Variables

For greater clearness we first write down the equations and integrals of motion in the present case, after adding a simplifying condition $y_0 = 0$, which can be attained by a coordinate rotation. The equations have the form

$$\begin{aligned} 2\dot{p} - q(r-\kappa) &= 0, \\ 2\dot{q} + p(r-\kappa) &= a\gamma_3, \\ \dot{r} + a\gamma_2 &= 0, \end{aligned}$$

$$\dot{\gamma_1} + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma_2} + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma_3} + p\gamma_2 - q\gamma_1 = 0, \tag{5.19}$$

in which $a = \frac{Mgx_0}{C}$, $\kappa = \frac{\kappa_3}{C}$. The three general integrals of motion may be written as

$$\begin{aligned} I_1 &= 2(p^2+q^2) + r^2 + 2a\gamma_3 = 2h, \\ I_2 &= 2(p\gamma_1 + q\gamma_2) + (r+\kappa)\gamma_3 = f, \\ I_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ I_4 &= (p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2)^2 + (2pq - a_1\gamma_2 - a_2\gamma_1)^2 \\ &\quad + 2\kappa\,(r-\kappa)\,(p^2+q^2) - 4\kappa\gamma_3(a_1 p + a_2 q) \\ &= K \end{aligned} \tag{5.20}$$

and here we retained the same names for h and f after dividing by C and K is an arbitrary constant.

Unlike the case of Kowalevski's top (with $\kappa = 0$), the explicit solution of the equations of motion in terms of time in Yehia's case is still unsuccessful. Separation coordinates analogous to Kowalevski's s_1, s_2 (Chap. 4 Sect. 4.3) were not found, even on the zero level of the areas integral. However, there is an indirect indication about the class of functions needed to describe this solution. An idea of special interest was presented in [145], which relates Kowalevski's case to a special version ($f = 0$) of an integrable case of the problem of motion of a rigid body in a liquid, known as Clebsch's first case (Case 2 of Table 10.1. Chap. 10) A bi-rational complex transformation was found relating the two sets of variables describing the two integrable problems, so that explicit solution for one of the cases can be obtained from that of the other. In the meantime, Clebsch's case is known to be solvable in terms of Theta functions with two arguments.

As was established by Gavrilov [110]: "The gyrostat of Yehia can be realized in a similar way as (the full $f \neq 0$) Clebsch's geodesic motion on E_3. This leads, in particular, to formulas for its explicit solution in terms of genus-two hyper-elliptic Theta functions [233]". Of course, this construction is not practical as a method of solution and there must be another direct way to obtain the solution. This way has not been found yet.

Komarov and Tsiganov [229] (See also [228]) considered the trajectory isomorphism of what they call "the Kowalevski gyrostat" and the Clebsch problem.

Although appeared fifteen years later, the last works do not contain any reference to Gavrilov's work.

In a recent work [326], Ryabov shows that the separated equations of the Yehia (Kowalevski–Yehia) case, *on its zero level of area's integral,* can be formally written in the Abel–Jacobi form analogous to (4.58) with $\Phi(s)$ as a polynomial of degree five in the variable s. However, the relation of the original variables of the problem to separated ones are not obtained, so that the problem of separation of variables cannot be considered complete yet, even on the level $f = 0$.

An earlier work relying on the Lax pair representation of the equations of motion constructs separation variables that are simultaneously suitable, *on the zero level of the areas integral*, for what the author names as gyrostatic generalizations of Kowalevski's and Goryachev–Chaplygin's cases [249]. It is claimed there that the given separation is "much simpler than the Kowalevski separation". However, the impact of that situation on the solution of the equations of motion was not considered.

Further (unpublished) results are announced by Fedorov et al. concerning the case of a gyrostat in two constant uniform fields, which includes the present case as a special version. This will be commented later in Chap. 14 Sect. 14.2.1.1.

The gyrostatic generalizations of Appelrot's classes of motions are discussed in Appendix D.

5.7 The Conditional Case of Sretensky

In [341], Sretensky found the modification of the complementary integral in the generalization of Goryachev–Chaplygin's case. He also generalized the procedure due to Chaplygin for explicit solution (See Chap. 4 Sect. 4.4) by changing the definitions of the three quantities r, U and V in (4.70)–(4.72) to be

$$\begin{aligned} r &= u - v - \kappa, \\ U &= u(u-\kappa)^2 - 2Eu - 4G, \\ V &= v(v+\kappa)^2 - 2Ev + 4G. \end{aligned} \tag{5.21}$$

The ultra-elliptic quadratures remain the same as in (4.77). The investigation of the critical sets and bifurcation diagrams in Sretensky's case is performed in [172, 174] (See also [183]). The results generalize relevant ones for Goryachev–Chaplygin's case, but they are much more complicated in view of the presence of three significant parameters. For more detail about Sretensky's case see Appendix E.

5.8 Some Applications of the Gyrostat Motion

We have seen that the presence of the gyrostatic momentum leads to appearance of a gyroscopic moment $\kappa \times \omega$ in the equations of motion.

5.9 Exercises

1- Show that the Lagrangian

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \boldsymbol{\kappa}\cdot\boldsymbol{\omega} - V(\boldsymbol{\gamma}),$$

describes the motion of a gyrostat with gyrostatic momentum $\boldsymbol{\kappa}$ about a fixed point, while acted upon by axially symmetric forces with potential $V(\boldsymbol{\gamma})$ ($\boldsymbol{\gamma}$ is the unit vector along the axis of symmetry of the forces). Deduce the equations of motion in the form

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}) &= \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} &= \mathbf{0}. \end{aligned}$$

2- In the previous problem, show that all possible axes of stationary motions lie on the cone, with vertex at the origin and generators passing through the points of the spherical curve

$$\begin{aligned} [\boldsymbol{\gamma} \cdot (\boldsymbol{\gamma}\mathbf{I}\times\frac{\partial V}{\partial \boldsymbol{\gamma}})]^2 - [\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{I})][\boldsymbol{\kappa} \cdot (\boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}})] &= 0, \\ \boldsymbol{\gamma}^2 &= 1, \end{aligned}$$

and the angular velocity of the gyrostat about the axis in the direction of $\boldsymbol{\gamma}$ is given by any of the expressions

$$\omega = \frac{\boldsymbol{\gamma}\cdot(\boldsymbol{\gamma}\mathbf{I} \times \frac{\partial V}{\partial \boldsymbol{\gamma}})}{\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{I})} = \frac{\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}})}{\boldsymbol{\gamma}\cdot(\boldsymbol{\gamma}\mathbf{I}\times\frac{\partial V}{\partial \boldsymbol{\gamma}})}.$$

3- An axially symmetric gyrostat, moving about a fixed point under its own weight, has its centre of mass on its axis of symmetry and the gyrostatic momentum is collinear with that axis. Show that the upper equilibrium position of the gyrostat, which is unstable in the absence of gyrostatic moment, can always be stabilized and find the minimum angular velocity necessary for that effect.

4- Write down the Hamiltonian and Hamiltonian equations of motion of a heavy gyrostat moving about a fixed point, using Euler's angles as generalized coordinates.

5- A system composed of a main body S_0 fixed from one point O, while carrying another body S_1 whose axis $O'P$ is fixed in S_0 by means of a smooth cylindrical hinge and freely rotates about this axis. Let $OO' = \mathbf{r}_1$ and $\mathbf{e}$ is the unit vector in the direction of $O'P$. Show that the kinetic energy of the system is

$$T = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \dot{\chi}\mathbf{e}\mathbf{J}\cdot\boldsymbol{\omega} + \frac{1}{2}\dot{\chi}^2\mathbf{e}\mathbf{J}\cdot\mathbf{e}+M_1\dot{\chi}\boldsymbol{\omega}\cdot[\mathbf{r}_1\times(\mathbf{e} \times \mathbf{r}_0')],$$

where **I** is the inertia matrix of the system at the fixed point, **J** is the inertia matrix of the second body at O', M_1 is the mass of S_1 and $\mathbf{r}'_0$ is the position vector of its centre of mass.

Chapter 6
Motion of a Rigid Body About a Fixed Point in the Field of a Distant Newtonian Centre and Brun's Problem

In the preceding chapters, we concentrated on the study of motion of a rigid body about a fixed point in the uniform gravity field. This field serves also as a good approximation for the forces acting on the body in most purposes. However, for certain applications, like precise surveying instruments, the slight variations in the Earth's gravitational field from point to another must be taken into consideration. These variations play a decisive role in the determination of rotational motion of artificial satellites of Earth, especially those whose tasks demand high precision of orientation, for terrestrial, cosmic or astronomical purposes.

In the present chapter, we study in detail another approximate model for one rigid body moving in the gravitational field of a fixed body. In this model, we make two assumptions:

(a) The fixed body is spherically symmetric. This enables us to treat it as a point mass concentrated at the centre of the body.

(b) The diameter of the moving body is very small compared to the distance between the fixed point of the moving body and the centre of the fixed body.

Let a body of mass M be in motion about its fixed point O at a distance R from a fixed Newtonian attraction centre of mass M' at O'. Denote by $\boldsymbol{\gamma}$ the unit vector in the direction $\overrightarrow{O'O}$ and by V the potential of the body in the field of the centre. To calculate the moment of gravitational forces about O, we take a mass element dm of the body whose position vector referred to O is $\mathbf{r}$:

The potential of this element will be $dV = -\frac{\mu dm}{|R\boldsymbol{\gamma}+\mathbf{r}|}$, the force exerted on it by the attraction centre is $d\mathbf{F} = -\frac{\mu(R\boldsymbol{\gamma}+\mathbf{r})dm}{|R\boldsymbol{\gamma}+\mathbf{r}|^3}$ and its moment about O is $d\mathbf{L} = -\frac{\mu R\boldsymbol{\gamma}\times\mathbf{r}dm}{|R\boldsymbol{\gamma}+\mathbf{r}|^3}$, where μ is Gauss' constant of the centre ($\mu = M' \times$Newton's gravitational constant). Integrating over the whole mass of the body, we get

$$V = -\mu \int \frac{dm}{|R\boldsymbol{\gamma}+\mathbf{r}|}, \tag{6.1}$$

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_6

$$\mathbf{L} = -\mu R\boldsymbol{\gamma}\times \int \frac{\mathbf{r}dm}{|R\boldsymbol{\gamma}+\mathbf{r}|^3} \tag{6.2}$$

and recalling that $|R\boldsymbol{\gamma}+\mathbf{r}| = \sqrt{R^2 + 2R\boldsymbol{\gamma}\cdot\mathbf{r}+r^2}$, one can write (2) in the form

$$\mathbf{L} = \gamma\times\frac{\partial V}{\partial\gamma}. \tag{6.3}$$

6.1 Approximate Form of the Potential

It is evident from (6.3) that the knowledge of the potential V is sufficient for complete determination of the moment $\mathbf{L}$. However, in most cases it is difficult to calculate V from (6.1) due to the complexity of the shape of the body, its mass density or because the mass distribution in the body is unknown. In those cases, an efficient solution is to expand the potential in powers of $\frac{1}{R}$ and keeping terms up to the third degree in this parameter. First we write the expansion of the function

$$\begin{aligned}
\frac{1}{\sqrt{R^2 + 2R\gamma\cdot\mathbf{r}+r^2}} &= \frac{1}{R}[1 + \frac{2\gamma\cdot\mathbf{r}}{R} + \frac{r^2}{R^2}]^{-\frac{1}{2}} \\
&= \frac{1}{R}[1 - \frac{1}{2}(\frac{2\gamma\cdot\mathbf{r}}{R} + \frac{r^2}{R^2}) + \frac{3}{8}(\frac{2\gamma\cdot\mathbf{r}}{R} + \frac{r^2}{R^2})^2 + o(\frac{1}{R^3})] \\
&= \frac{1}{R} - \frac{\gamma\cdot\mathbf{r}}{R^2} - \frac{r^2}{2R^3} + \frac{3}{2R^3}(\gamma\cdot\mathbf{r})^2 + o(\frac{1}{R^4}).
\end{aligned}$$

Inserting this expression into (6.1), and neglecting terms of degrees higher than the third, we obtain

$$\begin{aligned}
V &= -\mu[\frac{1}{R}\int dm - \frac{1}{R^2}\gamma\cdot\int \mathbf{r}dm - \frac{1}{2R^3}\int r^2 dm + \frac{3}{2R^3}\int (\gamma\cdot\mathbf{r})^2 dm] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(3\sum\gamma_i\gamma_j\int x_i x_j dm - \frac{1}{2}tr(\mathbf{I}))] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(3\gamma\bar{\mathbf{I}}\cdot\gamma - \frac{1}{2}tr(\mathbf{I}))] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(3\gamma(\frac{1}{2}tr(\mathbf{I})\boldsymbol{\delta} - \mathbf{I})\cdot\gamma - \frac{1}{2}tr(\mathbf{I}))] \\
&= -\mu[\frac{M}{R} - \frac{M}{R^2}\gamma\cdot\mathbf{r}_0 + \frac{1}{2R^3}(tr(\mathbf{I}) - 3\gamma\mathbf{I}\cdot\gamma)].
\end{aligned} \tag{6.4}$$

In expanded form that is

$$V = -\frac{\mu M}{R} + \frac{\mu M}{R^2}(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3)$$

$$-\frac{\mu}{2R^3}tr(\mathbf{I}) + \frac{3\mu}{2R^3}(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2). \tag{6.5}$$

As from (6.3) only derivatives of V with respect to components of $\boldsymbol{\gamma}$, the constant terms may be discarded and thus the potential can be written as

$$\begin{aligned} V &= Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) + \frac{1}{2}\lambda(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) \\ &= Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} + \frac{1}{2}\lambda\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma}, \end{aligned} \tag{6.6}$$

where g is the gravity field intensity of the centre at the fixed point O of the body and $\lambda = \frac{3g}{R}$. Formula (6.6) gives the approximate form of the potential of the body by the knowledge of its centre of mass and moments of inertia, without need to completely specify the distribution of mass in the body.

Remark: Note that for a Newtonian centre of attraction both parameters g and λ are positive. In certain physical problems, the Newtonian attraction is replaced by Coulomb's electric interaction (e.g. [58]). In that case g and λ can be either positive or negative and $\mathbf{I}, Mg\mathbf{r}_0$ are replaced, respectively, by the inertia matrix of the electric charges and the moment of those charges multiplied by the intensity of the electric field of the central charge at O.

6.2 Brun's Problem

Brun considered the motion of the following model [47]:

Let a rigid body be in motion about a fixed point O, while each of its mass elements is influenced by a force proportional to its mass and its distance from a fixed plane at O in the direction perpendicular to that plane. In the usual notation, the axes $Oxyz$ are taken as the system of principal axes of the body at the fixed point and $OXYZ$ as the inertial frame, with the Z-axis orthogonal to the fixed plane and the unit vector γ along the Z direction.

The potential of the body

$$\begin{aligned} V &= \frac{1}{2}N\int Z^2 dm = \frac{1}{2}N\int (\mathbf{r}\cdot\boldsymbol{\gamma})^2 dm \\ &= \frac{1}{2}N\boldsymbol{\gamma}\bar{\mathbf{I}}\cdot\boldsymbol{\gamma} = -\frac{1}{2}N\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma} + const \end{aligned} \tag{6.7}$$

where N is proportionality constant. This potential is just the quadratic part of the potential (6.6), with λ replaced by $-N$.

Thus, Brun's problem is a special version of the general problem of motion of a rigid body in an approximate Newtonian field, namely, the case when the body is fixed from its centre of mass.

6.3 Equations of Motion and Integrals of Motion

Taking (6.6) into account, one can write the equations of motion of the body in the approximate field of a Newtonian centre of attraction in the form

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I} = \boldsymbol{\gamma}\times(Mg\mathbf{r}_0+\lambda\boldsymbol{\gamma}\mathbf{I}),\\
&\dot{\boldsymbol{\gamma}}+\boldsymbol{\omega}\times\boldsymbol{\gamma}= \mathbf{0}.
\end{aligned}\tag{6.8}$$

Or in expanded form

$$\begin{aligned}
A\dot{p} + (C-B)(qr-\lambda\gamma_2\gamma_3) &= Mg(z_0\gamma_2 - y_0\gamma_3),\\
B\dot{q} + (A-C)(pr-\lambda\gamma_1\gamma_3) &= Mg(x_0\gamma_3 - z_0\gamma_1),\\
C\dot{r} + (B-A)(pq-\lambda\gamma_1\gamma_2) &= Mg(y_0\gamma_1 - x_0\gamma_2),
\end{aligned}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{6.9}$$

This is a closed system of six first-order differential equations. For this system, we have three obvious general integrals of motion. They are the energy, areas and geometric integrals

$$\begin{aligned}
&I_1 = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega}+Mg\mathbf{r}_0\cdot\boldsymbol{\gamma}+\frac{1}{2}\lambda\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}= h,\\
&I_2 = \boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\gamma} = f,\\
&I_3 = \boldsymbol{\gamma}\cdot\boldsymbol{\gamma}= \mathbf{1}.
\end{aligned}\tag{6.10}$$

6.4 Integrable Cases

As will be seen later in this book, the problem under consideration is in fact a special version of the much more general problem considered in Chap. 10, dealing with the motion of a body in a liquid or a magnetized and electrically charged body about a fixed point. However, in virtue of the importance of this version in applications, we present here its integrable cases. For the system (6.9) to be integrable, just as in the problem of motion in the uniform gravity field, we have to find a fourth integral independent of those three. It turned out that this can be achieved in two cases.

6.4.1 Brun's Case [47] (Analog of Euler's Case)

In this case $\mathbf{r}_0 = 0$, i.e. the body is fixed from its centre of mass. To obtain the complementary integral, we write the Euler–Poisson equation

$$\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I}=\lambda\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I}$$

multiplying scalarly by $\boldsymbol{\omega}\mathbf{I}$ to get

$$\boldsymbol{\omega}\mathbf{I}\cdot\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I})=\lambda\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I}).$$

That is

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2&=-\lambda\boldsymbol{\gamma}\cdot(\boldsymbol{\omega}\mathbf{I}\times\boldsymbol{\gamma}\mathbf{I})\\&=-\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot(\boldsymbol{\omega}\times\boldsymbol{\gamma}).\end{aligned}\tag{6.11}$$

The last relation can be verified very easily by writing the triple scalar product in the form of a determinant and using the determinant's properties. Using Poisson's equation in this relation, we write

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2&=\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\dot{\boldsymbol{\gamma}}\\&=\frac{d}{dt}\frac{1}{2}\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}\end{aligned}$$

and hence finally we get

$$|\boldsymbol{\omega}\mathbf{I}|^2-\lambda ABC\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}=c,\tag{6.12}$$

or in expanded form

$$I_4=A^2p^2+B^2q^2+C^2r^2-\lambda ABC(\frac{\gamma_1^2}{A}+\frac{\gamma_2^2}{B}+\frac{\gamma_3^2}{C})=c.\tag{6.13}$$

This integral is an obvious generalization of the fourth integral in Euler's case (the square of the modulus of the angular momentum). Nevertheless, complete solution of the equations of motion (6.9) for Brun's case with the integrals (6.10) and (6.13) is much more complicated than in Euler's case. Kobb [223] expressed the Hamiltonian of the problem using Euler's angles as generalized coordinates and the momenta conjugate to them. By writing Hamilton–Jacobi equation and constructing a complete solution for it, he reduced the solution to certain quadratures, but he has not tried to solve those quadratures explicitly for the coordinates in terms of time. A similar approach was adopted by Kharlamova in [204] using different coordinates, the sphero-conic coordinates on the Poisson sphere (See Chap. 9).

6.4.2 *The Generalization of Lagrange's Case*

Let $A = B$ and $x_0 = y_0 = 0$, i.e. the body admits axial dynamical symmetry about the z-axis passing through the fixed point and the centre of mass of the body lies on the axis of dynamical symmetry. Under those conditions, it is easy to write the third equation of (6.9) in the form

$$C\dot{r} = 0$$

and thus the integral is the same as in Lagrange's case

$$I_4 = r = r_0. \tag{6.14}$$

To obtain the solution of the equations of motion, one can proceed exactly as in Lagrange's case in Chap. 4 (Sect. 4.2). The relation between γ_3 and time t has the same form as in (4.42)

$$t = \int \frac{d\gamma_3}{\sqrt{F(\gamma_3)}}, \tag{6.15}$$

but with

$$F(\gamma_3) = (1 - \gamma_3^2)(E - a\gamma_3 - a_1\gamma_3^2) - (b - cr_0\gamma_3)^2, \tag{6.16}$$

i.e. $F(\gamma_3)$ is here a polynomial of the fourth degree. Thus, γ_3 is expressible in terms of elliptic functions of time. This procedure was pointed out in [250], where the Eulerian angles are given expressions in terms of Weierstrass' elliptic functions.

Arkhangelsky [14] proved that the equations of motion (6.8) do not admit a complementary single-valued integral in any more cases than the above two. Note also that the case of axially symmetric body can be readily generalized by adding a gyrostatic moment, i.e. a rotor along the axis of symmetry of the body. The integral I_4 becomes $Cr + k_3$, i.e. $I_4 = r$ is still a constant of the motion.

6.4.3 *The Place of Brun's Potential*

At present we know very little about the answer to an important question: for which potentials are the Euler–Poisson equations

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I} = \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0} \tag{6.17}$$

integrable? Even for the simplest case of a quadratic integral we know a single partial result, obtained in [381] using reduced equations in isometric coordinates on the inertia ellipsoid. We deduce it here in a direct way from Euler–Poisson's equations. We formulate it as

Theorem 6.1 *Equation (6.17) is integrable with a quadratic complementary integral for arbitrary A, B, C, and for all initial conditions only for the potential*

$$V = \frac{1}{2}\lambda(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\ \lambda \textit{ arbitrary constant} \tag{6.18}$$

and their integral is

$$I_4 = A^2p^2 + B^2q^2 + C^2r^2 - \lambda(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2). \tag{6.19}$$

Proof Let for a general potential $V(\gamma)$ and $\boldsymbol{\mu} = \mathbf{0}$ the equations of motion have an integral quadratic in the angular velocities. This integral must have the form

$$I_4 = A^2p^2 + B^2q^2 + C^2r^2 + F(\gamma_1, \gamma_2, \gamma_3), \tag{6.20}$$

so that $F \equiv 0$ when $V \equiv 0$. Differentiating this integral we get

$$2\boldsymbol{\omega}\mathbf{I}\cdot\dot{\boldsymbol{\omega}}\mathbf{I} + \frac{\partial F}{\partial \boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} = \mathbf{0},$$

and using (6.17) this becomes

$$2\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\mathbf{I}) + \frac{\partial F}{\partial \boldsymbol{\gamma}} \cdot (-\boldsymbol{\omega}\times\boldsymbol{\gamma}) = \mathbf{0},$$

and finally we obtain the relation

$$2\boldsymbol{\omega}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}}) - \boldsymbol{\omega}\cdot(\boldsymbol{\gamma}\times\frac{\partial F}{\partial \boldsymbol{\gamma}}) = \mathbf{0}, \tag{6.21}$$

which is satisfied for arbitrary vector $\boldsymbol{\omega}$.

Now we replace $\boldsymbol{\omega}$ by $\boldsymbol{\gamma}$ in Eq. (6.21), to obtain an equation for the potential V

$$2\boldsymbol{\gamma}\mathbf{I}\cdot(\boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}}) = 0,$$

whose solution is readily

$$V = V(\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}, \boldsymbol{\gamma}^2).$$

Since we have $\boldsymbol{\gamma}^2 = 1$, the last expression can be written as

$$V = V(\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}). \tag{6.22}$$

In a similar way, we replace $\boldsymbol{\omega}$ by $\boldsymbol{\gamma}\mathbf{I}^{-1}$ in Eq. (6.21), to obtain the equation for F

$$\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot(\boldsymbol{\gamma}\times\frac{\partial F}{\partial \boldsymbol{\gamma}}) = \mathbf{0}.$$

It follows that

$$F = F(\gamma \mathbf{I}^{-1} \cdot \gamma). \tag{6.23}$$

Now, inserting (6.22) and (6.23) into (6.21), one gets

$$4V'(\gamma\mathbf{I}\cdot\gamma)\boldsymbol{\omega}\mathbf{I}\cdot(\gamma \times \gamma\mathbf{I}) - 2F'(\gamma\mathbf{I}^{-1}\cdot\gamma)\boldsymbol{\omega}\cdot(\gamma \times \gamma\mathbf{I}^{-1}) = \mathbf{0}. \tag{6.24}$$

Comparing the structures of the two terms in the last equation and noting that $\gamma\mathbf{I}\cdot\gamma$ and $\gamma\mathbf{I}^{-1}\cdot\gamma$ are independent functions for a tri-axial body ($A \neq B \neq C$), The two derivatives V' and F' must be constants. Neglecting an insignificant constant in each of V and F, we write

$$V = \frac{1}{2}\lambda(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\; F = N_1(\gamma_1^2/A + \gamma_2^2/B + \gamma_3^2/C). \tag{6.25}$$

Again, inserting the last expressions into (6.24), we obtain

$$\lambda\gamma\cdot(\boldsymbol{\omega}\mathbf{I}\times\gamma\mathbf{I}) + N_1\gamma\mathbf{I}^{-1}\cdot(\boldsymbol{\omega}\times\gamma) = \mathbf{0}.$$

Applying the identity (A.2) in Appendix A, we rewrite the last relation as

$$(\lambda ABC + N_1)\gamma\mathbf{I}^{-1}\cdot(\boldsymbol{\omega}\times\gamma) = \mathbf{0}$$

and thus we obtain

$$N_1 = -\lambda ABC.$$

This completes the determination of the integral as in (6.13). □

6.5 Exercises

1- In the classical problem of motion of a rigid body about a fixed point in the constant uniform gravity field two equilibrium positions are possible. In the problem of motion in approximate Newtonian field, show that in any equilibrium position one of the generators of Ampère's cone (Staude's cone) must be vertical (passes through the fixed point and the centre of attraction).

2- Noting that λ is positive in (6.8) for a centre of attraction, show that the vertical generator of Staude's cone in an equilibrium position must be one of the axes inadmissible for Staude's rotation in the uniform gravity field. [Compare the equations of equilibrium to equations of Staude's rotation and note that ω^2 in Staude's rotation is replaced by $-\lambda$]

3- For a body fixed from its centre of mass in a uniform gravity field, any position is a possible equilibrium position. Show that for a body fixed from its centre of mass in the approximate field of a Newtonian centre there are only six equilibrium positions and find them.

4- For a body fixed from its centre of mass, show that a uniform rotation is possible only in two cases:

(a) The rotation about a principal axis with arbitrary angular velocity.
(b) The rotation with angular velocity $\pm\sqrt{\lambda}$ about an arbitrary axis.

5- Show that all the axes of uniform rotation of the body in approximate field of a Newtonian centre are generators in Ampère's cone and that possible axes of Staude's rotation constitute a subset of possible axes of uniform rotation in the Newtonian field.

6- Using the terminology of the present chapter, **F** is the exact resultant force exerted on the body by the centre of attraction. Show that

(a) $\mathbf{F} = -\mu \int \frac{(R\gamma+\mathbf{r})}{|R\gamma+\mathbf{r}|^3} dm$,

(b) The component of this force in the direction of γ is $\mathbf{F}\cdot\gamma = -\frac{\partial V}{\partial R}$, V being the exact potential ($V = -\mu \int \frac{dm}{|R\gamma+\mathbf{r}|}$).

(c) The resultant force **F** can be written in terms of the potential in the form

$$\mathbf{F} = -\frac{\partial V}{\partial R}\gamma + \frac{1}{R}\gamma \times (\gamma \times \frac{\partial V}{\partial \gamma}).$$

Note that $\mathbf{F} \cdot \mathbf{L} = 0$, which agrees with the fact that the resultant attraction must be a single force passing through the centre of attraction whatever be the position of the body.

(d) The magnitude of the resultant force **F** is

$$|\mathbf{F}| = \sqrt{(\frac{\partial V}{\partial R})^2 + \frac{1}{R^2}|\gamma \times \frac{\partial V}{\partial \gamma}|^2}.$$

(e) The reaction of the fixed point on the body at any equilibrium position must be vertical.

7- A rigid body is fixed from a point and its centre of mass lies on a principal plane for that point, say, the xy-plane. The body is acted upon by the gravitational force due to a distant Newtonian attraction. Show that the body can perform a plane pendulum-like motion about a horizontal axis, described by the conditions $p = q = \gamma_3 = 0$, and determine the variables r, γ_1, γ_2 as functions of time.

8- Brun's model problem is modified, so that each mass element of the body is influenced by a force proportional to its mass and its distance from an arbitrary fixed plane and in the direction perpendicular to that plane. Show that the potential is equivalent to the general potential (6.6).

9- Investigate the stability of the uniform rotation of a body fixed from its centre of mass O in the approximate Newtonian field of a far centre at O', determined in Exercise 4-a. Show that the rotation with angular speed r_0 about the z-axis, which is directed to pass through the centre of attraction and the fixed point of the body, is

(a) unstable when the z-axis is the middle principal axis of inertia of the body at O,

(b) stable for all values of r_0, when C is the least principal moment of inertia, and
(c) when C is the largest principal moment of inertia, the rotation is unstable if

$$r_0 < \sqrt{\frac{3g}{R}} \frac{\sqrt{a}+\sqrt{b}}{1+\sqrt{ab}},$$

and stable if

$$r_0 > \sqrt{\frac{3g}{R}} \frac{\sqrt{a}+\sqrt{b}}{1+\sqrt{ab}},$$

where $a = \frac{C-B}{A}$, $b = \frac{C-A}{B}$, g is the acceleration of gravity at O and $R = |\overrightarrow{O'O}|$.

[Beletsky [20]. See also [256], Sect. 18.4]

Chapter 7
The Motion of a Body with No Fixed Point

7.1 General Considerations

In previous chapters, we studied problems of motion of a rigid body with one point fixed in space, i.e. in an inertial reference frame. In the present chapter, we study certain problems of motion when the body is not fixed from any point. For the moment, we shall not begin with constructing a Lagrangian for the motion. To keep the applicability of the equations of motion as wide as possible, we assume that the body is subject to a set of forces, which are not necessarily time independent and which may not have a potential.

As in Chap. 3, we write the equation of motion of an element dm of the body mass

$$dm\frac{d^2\mathbf{r}}{dt^2} = d\mathbf{F} + d\mathbf{F}' \tag{7.1}$$

summing over the mass of the body one gets

$$M\frac{d^2\mathbf{r}_0}{dt^2} = \mathbf{F}. \tag{7.2}$$

The centre of mass of the body moves as a particle acted upon by a force equal to the resultant of all external forces acting on the body.

On the other hand, let $\boldsymbol{\rho}$ be the position vector of the mass element dm with respect to the centre of mass, so that

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\rho}. \tag{7.3}$$

Multiplying (7.1) vectorially by $\boldsymbol{\rho}$ and integrating over the body, we obtain the equation for the rotational motion in the form

$$\frac{d\mathbf{G}_c}{dt} = \mathbf{L}_c, \tag{7.4}$$

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_7

where $\mathbf{G}_c = \int \boldsymbol{\rho} \times \frac{d\boldsymbol{\rho}}{dt} dm$ and $\mathbf{L}_c = \int \boldsymbol{\rho} \times d\mathbf{F}$ are, respectively, the angular momentum and the resultant moment of external forces about the centre of mass. The law of rotational motion about the centre of mass of the body resembles that of motion relative to the inertial frame.

Example 1 A rigid body is free to move in vacuo under the action of a uniform gravity field. Describe the motion.

The centre of mass of the body moves as a projectile, while the body performs a torque-free motion about the centre of mass as in Euler's case.

In particular, if the body begins with zero angular velocity, it will continue translational motion of its centre of mass without change in its orientation. If the body begins with a rotation about one of its principal axes of inertia, it will continue rotation with the same angular velocity about the same axis, which keeps fixed orientation in space.

Kinetic energy of the rigid body: To construct the Lagrangian of the motion, one needs to have an expression for the kinetic energy. That is

$$\begin{aligned} T &= \frac{1}{2} \int (\mathbf{v}_c + \frac{d\boldsymbol{\rho}}{dt})^2 dm \\ &= \frac{1}{2} M \mathbf{v}_c^2 + \frac{1}{2} \int (\frac{d\boldsymbol{\rho}}{dt})^2 dm. \end{aligned} \tag{7.5}$$

The kinetic energy of the body in a general translational and rotational motion is the sum of two terms: the kinetic energy of a mass M, equal to the mass of the body and moving with its centre of mass, and the kinetic energy of the rotation of the body about its centre of mass. The last formula gives a simple expression of the kinetic energy, very useful in application to many problems of rigid body dynamics. In the next section, we study an example of such application.

Example 2 A heavy magnetized rigid body is free to move in vacuo under the action of two skew uniform gravity and magnetic fields. Describe the motion.

Let (X, Y, Z) be the coordinates of the centre of mass P of the body relative to an inertial frame $OXYZ$ with axis Z directed vertically upwards and let $\mathbf{H} = H\mathbf{e}$ be the magnetic field of fixed magnitude and fixed direction in space determined by the unit vector $\mathbf{e}$. The Lagrangian of the problem can be written as

$$\begin{aligned} L = &\frac{1}{2} M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - MgZ \\ &+ \frac{1}{2}(A_c p^2 + B_c q^2 + C_c r^2) - \mathbf{m} \cdot \mathbf{H}, \end{aligned}$$

where M, $\mathbf{m}$ are the mass of the body and its magnetic moment, respectively, and A_c, B_c, C_c are the central principal moments of inertia. One can see at once that the translational motion of the centre of mass and the rotational motion are completely independent. The centre of mass P moves as a projectile on a parabola with its axis vertical and vertex upwards. The equations of the rotational motion are obtained from the Lagrangian

$$L' = \frac{1}{2}(A_c p^2 + B_c q^2 + C_c r^2) - H\mathbf{m} \cdot \mathbf{e}. \tag{7.6}$$

This Lagrangian is form-identical with (3.43) of the classical problem of motion of a heavy body about a fixed point and the problem can be put in Lagrangian, Hamiltonian or in the Euler–Poisson form. It follows immediately that the present problem is generally integrable only in the following three cases:

(1) The analog of Euler's case. The case of no magnetic effect $H = 0$ (no magnetic field) or $\mathbf{m} = \mathbf{0}$ (no magnetization in the body)
(2) The analog of Lagrange's case. The central inertia ellipsoid is a spheroid and the magnetic moment is parallel to the symmetry axis.
(3) The analog of Kowalevski's case. $A_c = B_c = 2C_c$ and the magnetic moment lies in the equatorial plane ($m_3 = 0$).

The conditional integrable case of Goryachev and Chaplygin has its analog valid on the level $f = 0$, f being the component of the angular momentum in the direction of the magnetic field ($\boldsymbol{\omega}\mathbf{I}_c \cdot \mathbf{e} = f$). The same applies to all particular solutions of the classical problem, a complete list of which will be provided in Chap. 8.

7.2 Poisson's Top. A Top on a Smooth Horizontal Plane

In our previous study of Lagrange's top (the axi-symmetric top), the assumption was made that the pin of the top serves as a fixed point in the inertial space. A related model was first considered by Poisson [309], but much rarely mentioned in textbooks on the subject. For a somewhat detailed treatment see [221, 222].

In this model, a symmetrical top moves with its apex Q constrained to move without friction on a horizontal plane, so that the reaction R of the plane on the body remains in the vertical direction. Denote by P the centre of mass of the top. From symmetry, P lies at a point on its axis of the top at a distance a (say) from Q. Let (x, y, z) be the coordinates of P relative to an inertial frame $Oxyz$ with axis z directed vertically upwards and the xy-plane is the plane of motion of the apex. In this system, the apex Q has coordinates $(X, Y, 0)$. The rotational motion of the top will be described by the Eulerian angles: ψ between the vertical plane containing QP and the xz-plane, θ between QP and the vertical upwards and finally φ the angle of proper rotation of the top about its axis (See Fig. 7.1).

To derive equations of motion, one may use (7.5) to write down the Lagrangian of the motion as

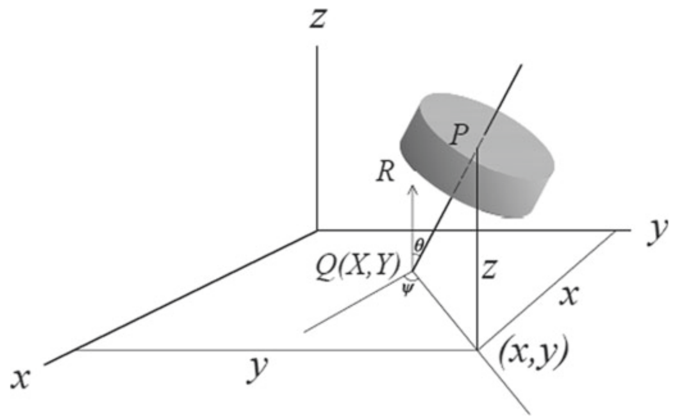


Fig. 7.1 Poisson's top

$$
\begin{aligned}
L = &\frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
&+\frac{1}{2}[A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\
&-Mgz
\end{aligned}
$$

where M is the mass of the top, C is its axial moment of inertia, A is the moment of inertia about any axis passing through P and orthogonal to QP and g is the acceleration of gravity. Recalling that $z = a\cos\theta$, we can rewrite the Lagrangian in the form

$$
\begin{aligned}
L = &\frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + a^2\sin^2\theta\dot{\theta}^2) \\
&+\frac{1}{2}[A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\
&-Mgz.
\end{aligned} \tag{7.7}
$$

The mechanical system under consideration has five degrees of freedom. It also admits five integrals of motion: the energy integral and four integrals corresponding to the four cyclic coordinates x, y, ψ and φ, so that there is no need to write any of the Lagrange's equations of motion, but only to write the integrals

$$
\begin{aligned}
\dot{x} &= U, \\
\dot{y} &= V, \\
\dot{\psi}\cos\theta + \dot{\varphi} &= r_0, \\
A\sin^2\theta\dot{\psi} + Cr_0\cos\theta &= n, \\
\frac{1}{2}M(U^2 + V^2 + a^2\sin^2\theta\dot{\theta}^2) + \frac{1}{2}[A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + Cr_0^2] + Mgz &= h,
\end{aligned} \tag{7.8}
$$

where U, V, r_0, n, h are arbitrary constants of motion.

From the first two integrals, it turns out that the centre of mass of the top moves with uniform horizontal velocity. This agrees with the fact that the body moves under the action of only two vertical forces: its weight and the reaction of the plane. From the last three integrals, one gets

$$\dot{\psi} = \frac{n - Cr_0 \cos\theta}{A \sin^2\theta},$$
$$\dot{\varphi} = r_0 - \frac{\cos\theta(n - Cr_0\cos\theta)}{A\sin^2\theta} \tag{7.9}$$

and the final equation for the angle θ

$$(A + Ma^2 \sin^2\theta)\dot{\theta}^2 = 2(E - Mga\cos\theta) - \frac{(n - Cr_0\cos\theta)^2}{A\sin^2\theta}, \tag{7.10}$$

where $E = h - \frac{1}{2}M(U^2 + V^2) - \frac{1}{2}Cr_0^2$.

From the analytical point of view, setting $\cos\theta = u$, we transform (7.10) to the form

$$[A + Ma^2(1 - u^2)]\dot{u}^2 = 2(E - Mgau)(1 - u^2) - \frac{1}{A}(n - Cr_0 u)^2, \tag{7.11}$$

which leads through separation of variables to the relation

$$t = \int^u \sqrt{\frac{A + Ma^2(1 - u^2)}{2(E - Mgau)(1 - u^2) - \frac{1}{A}(n - Cr_0u)^2}}\,du$$
$$= \sqrt{\frac{a}{2g}} \int^u \frac{du}{\sqrt{F(u)}}, \tag{7.12}$$

where

$$F(u) = \sqrt{\frac{(u - u_1)(u_2 - u)(u_3 - u)}{u_4^2 - u^2}},$$

and $0 \le u_1 \le u_2 \le 1 \le u_3$, $u_4 = \sqrt{1 + \frac{A}{Ma^2}} > 1$.

Now, we note that Eq. (7.9) is identical with the ones considered for Lagrange's top in Chap. 3. Also, (7.11) is similar to its corresponding Eq. (4.41) in Lagrange's case, and differs from it only by presence of the term $Ma^2(1 - u^2)$ in the coefficient of $\dot{u}^2$ on its left-hand side. However, this term does not change the sign of the coefficient. Thus, the general qualitative character of the motion of the Poisson top is almost the same as in Lagrange's top and it will not be repeated here.

In general, the last integral is hyper-elliptic, compared to the elliptic integral (4.42) for Lagrange's top. Inverting this relation we express $u = \cos\theta$ in terms of time, and then integrating (7.9) with respect to time we obtain ψ and φ. In Klein's work [221], where the top under consideration is termed the "toy top", explicit expressions of the Cayley–Klein parameters describing the motion as hyper-elliptic integrals in u were given, so that together with (7.12) this gives a parametric representation of the solution. Those results were detailed and refined in [339], where also degenerate cases when hyper-elliptic integral reduce to elliptic were singled out.

To find the equation of the trajectory of the tip of the top on the plane, we write

$$\begin{aligned} X &= x - a\sin\theta\cos\psi = x_0 + Ut - a\sin\theta\cos\psi, \\ Y &= y - a\sin\theta\sin\psi = y_0 + Vt - a\sin\theta\sin\psi. \end{aligned} \tag{7.13}$$

The reaction of the plane on the apex can be found from the equation of motion of the centre of mass in the z direction

$$M\ddot{z} = R - Mg,$$

so that

$$\begin{aligned} R &= M(g + \ddot{z}) \\ &= M(g + a\ddot{u}). \end{aligned} \tag{7.14}$$

Remark: Without any effect on the rotational motion of the top, it can be assumed that $U = V = x_0 = y_0 = 0$, so that $x = y = 0$. This fixes the choice of the inertial frame as the one whose Z-axis passes through the initial position of the centre of mass P of the top. In this frame, P moves only vertically up and down the Z-axis.

7.2.1 Regular Precession of Poisson's Top

As it was in the case of Lagrange's top, regular precession corresponds to the nutation angle θ taking a constant value θ^* (say), and then from (7.9) we find that the other two Eulerian angles φ, ψ change with time in constant rates. This occurs in two qualitatively different ways:

1- When $u_1 = u_2 = u^*$, $0 < u^* \leq 1$. This can happen at arbitrary inclination of the body axis to the vertical, including the standing gyroscope positions ($u = 1$). Regular precessions of this type correspond to Fig. 7.2a for the function F. They are all stable, since a slight perturbation of the motion causes splitting of the two roots in a small neighbourhood of u^*. This leads to a small periodic change in the nutation angle θ and consequently small wobbling in the rates $\dot{\psi}$ and $\dot{\varphi}$.

2- When $u_1 < 1$, $u_3 = u_2 = 1$. This gives a different standing position, corresponding to Fig. 7.2b. On perturbation, the equal roots split into $u_3 > 1$, $u_2 < 1$. The figure axis begins a finite periodic motion, in which it goes to position u_1 before it returns to u_2. This standing position is unstable.

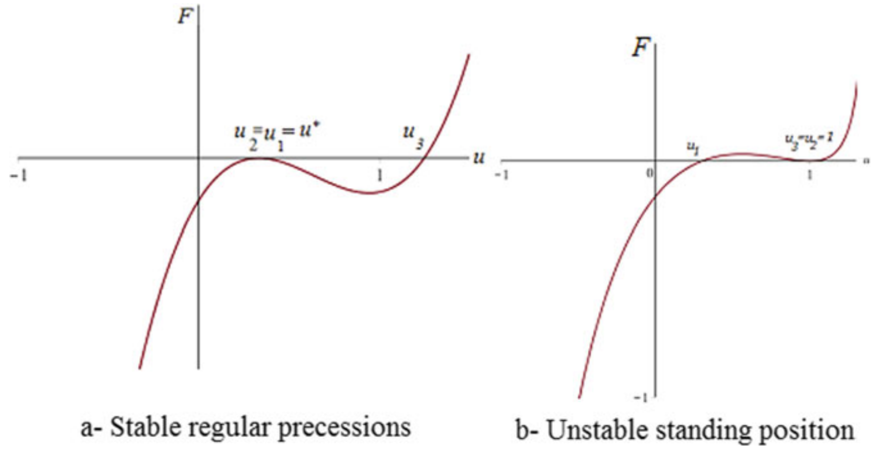


Fig. 7.2 Stability of regular precession for Poisson's top

7.3 Exercises

1. Show that the resulting problem of example 2 in Sect. 7.1 above can be generalized to the case of a body-gyrostat.

2. The Poisson top is given the upper equilibrium position. Show that this position, which is unstable when $r_0 = 0$, can be stabilized by giving the top an angular speed r_0 about its axis of symmetry provided

$$|r_0| > 2\frac{\sqrt{MgaA}}{C}.$$

3. A rigid body moves freely in a gravitational field with homogeneous quadratic potential

$$V_g = \frac{1}{2}(a\xi^2 + b\eta^2 + c\zeta^2)$$

in the inertial frame. Show that the equations of translational motion of the centre of mass and the rotational motion about the centre of mass are completely separate. For more details see Chap. 14 Sect. 14.3.1.

Part II
The Advanced Part

Chapter 8
Particular Solutions of the Classical Problem and Its Generalizations

8.1 The Notion of a Particular Solution

By particular solutions, we mean cases that are solvable under conditions on the initial state of motion, which cannot be stated as a condition only on the value f of the areas integral. After the great success of Sofia Kowalevski (Kovalevskaya) in isolating a new general integrable case, several Russian authors, including such celebrated names as Joukovsky, Lyapunov, Steklov and Chaplygin, concentrated on research on various aspects of rigid body dynamics. Their contributions have built up a large part of the information available today about various problems in the dynamics of rigid body.

In this chapter, we present known particular solutions of the Euler–Poisson equations describing motion in the classical problem. These are twelve solutions, valid under various conditions on the initial motion and on the distribution of mass in the body. This number does not include other particular solutions, which are special cases of general and conditional cases. For example, four particular solutions of the classical problem considered by Appelrot, who coined for them the name "Classes of the simplest motions", fit under the integrable case of Kowalevski [11].

In our presentation, we adhere to chronological order and also to referring each case to the author who was first to publish it, in case of repetition. After providing conditions and detailed solution in each particular case, we draw the orbit (or family of orbits) of motion on the Poisson sphere fixed in the body. That is the trajectory traced by the apex of the vertical unit vector $\gamma(t)$ on that sphere as the body moves. This simulates the motion of the body relative to the vertical and in fact determines the motion of the body in the inertial space up to the angle of precession.

Section 8.14 of the present chapter is devoted to a brief discussion of certain particular solutions, which were announced and turned out later to be either in error or repeating known solutions. At the end of this chapter, we have also inserted a brief survey of all the known up-to-date particular solutions of the problem of motion of a heavy gyrostat. In most cases, we aimed at giving minimal information that would help the interested reader to track any of the cases in original works.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_8

In constructing particular solutions of the classical problem, we deal with six first-order differential equations with three first integrals of motion, namely energy, areas and geometric integrals. A question arises, to what extent one can rely on using integrals of motion and ignore some of the six differential equations? This question is first answered in the following theorem:

Theorem 8.1 Let $\boldsymbol{\omega}(t)$ and $\boldsymbol{\gamma}(t)$ be two vectors satisfying Euler's dynamical equation

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0, \tag{8.1}$$

and the integrals of motion

$$\begin{aligned} I_1 &\equiv \frac{1}{2}\mathbf{G} \cdot \boldsymbol{\omega} + Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} = h, \\ I_2 &\equiv \mathbf{G} \cdot \boldsymbol{\gamma} = f, \\ I_3 &\equiv \boldsymbol{\gamma}^2 = 1, \end{aligned} \tag{8.2}$$

where $\mathbf{G} = \boldsymbol{\omega}\mathbf{I}$. Then, these vectors satisfy Poisson's equation

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{8.3}$$

provided the equation

$$\mathbf{G} \cdot (\mathbf{r_0} \times \boldsymbol{\gamma}) = \mathbf{0} \tag{8.4}$$

is not satisfied identically for all t.

Proof Differentiating I_1 with respect to time and using (8.1), we obtain

$$\mathbf{r}_0 \cdot (\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma}) = \mathbf{0}. \tag{8.5}$$

In a similar way, differentiating I_2, we get

$$\mathbf{G} \cdot (\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma}) = \mathbf{0}, \tag{8.6}$$

while differentiating I_3 yields

$$\boldsymbol{\gamma} \cdot \dot{\boldsymbol{\gamma}} = 0. \tag{8.7}$$

The general solution of (8.7) is

$$\dot{\boldsymbol{\gamma}} = -\boldsymbol{\Omega} \times \boldsymbol{\gamma}, \tag{8.8}$$

where $\boldsymbol{\Omega}$ is a vector to be determined. Inserting this into (8.5) gives

$$(\boldsymbol{\Omega} - \boldsymbol{\omega}) \cdot (\mathbf{r_0} \times \boldsymbol{\gamma}) = \mathbf{0},$$

so that one can write, since a component of $\boldsymbol{\Omega}$ parallel to $\boldsymbol{\gamma}$ is insignificant,

$$\boldsymbol{\Omega} = \boldsymbol{\omega} + \lambda \mathbf{r_0},$$

where λ is a still undetermined multiplier.

Now, only Eq. (8.6) remains to be satisfied. Inserting the last expression into (8.8) and then $\dot{\gamma}$ into (8.6), we obtain

$$\mathbf{G} \cdot [-(\boldsymbol{\omega} + \lambda \mathbf{r_0}) \times \boldsymbol{\gamma} + \boldsymbol{\omega} \times \boldsymbol{\gamma}] = 0,$$

and this finally becomes

$$\lambda[\mathbf{G} \cdot (\mathbf{r_0} \times \boldsymbol{\gamma})] = 0.$$

If the triple scalar product is not identically zero during motion, then $\lambda = 0$, $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and (8.8) becomes identical with Poisson's equation.

Thus, in constructing particular solutions of the Euler–Poisson system, it is sufficient in most cases to take into account Euler's dynamical equation, along with the three first integrals of motion. The resulting solution will also satisfy Poisson's equation, unless it comes out that the three vectors $\mathbf{G}$, $\mathbf{r_0}$,$\boldsymbol{\gamma}$ remain coplanar all the time.

We now note the following: Multiplying the dynamical Eq. (8.1) scalarly by $\mathbf{G}$, we get

$$\mathbf{G} \cdot \dot{\mathbf{G}} = \mathbf{MgG} \cdot (\boldsymbol{\gamma} \times \mathbf{r}_0),$$

so that if condition (8.4) is satisfied, then

$$\mathbf{G}^2 = G_0^2(\text{const}.).$$

The converse is also true. Thus, the last theorem can be reformulated as follows:

Theorem 8.2 If $\omega(t)$ and $\gamma(t)$ satisfy Euler's dynamical equation and the three relations $I_1 = h$, $I_2 = f$ and $I_3 = 1$ for some pair $\{h, f\}$, and if the magnitude of the angular momentum of the body is not identically a constant, then $\omega(t)$ and $\gamma(t)$ satisfy also Poisson's equation and constitute a solution of the classical problem.

This result will be used frequently in the present chapter. In most cases, it is comfortable to use the ansatz in the three integrals and to satisfy Euler's dynamical equations, with no need to verify Poisson's equations.

It should be noted that a solution satisfying (8.1)–(8.2) can be erroneous as a solution of the Euler–Poisson equations. An example is the recently published result of Ershkov [76] claiming a new particular solution in which the centre of mass lies either on a principal axis of inertia or in a principal plane. To this end, the author uses only the dynamical equation and the three integrals. As in these solutions $\mathbf{G} = C_0\boldsymbol{\gamma}$, i.e. $\mathbf{G}^2 = const.$, three Poisson's equations should have been checked, which was not done by the author of [76]. For more details, see Sect. 8.14 of the present chapter.

8.2 Planar Motion (Motion of the Body as a Physical Pendulum)

This is the motion of the body about the fixed point as a physical pendulum, i.e. motion about a horizontal axis which should be fixed in the body and space. This motion is also called planar motion, as all body elements move parallel to a fixed vertical plane. Let us take the axis of rotation to be the z-axis. The conditions for such a motion are

$$p = q = \gamma_3 = 0. \tag{8.9}$$

Substituting into the equations of motion (3.29), one obtains

$$\begin{aligned} 0 &= Mgz_0\gamma_2, \\ 0 &= -Mgz_0\gamma_1, \\ C\dot{r} &= Mg(y_0\gamma_1 - x_0\gamma_2), \\ \dot{\gamma}_1 - r\gamma_2 &= 0, \dot{\gamma}_2 + r\gamma_1 = 0. \end{aligned} \tag{8.10}$$

This means $z_0 = 0$, i.e. the centre of mass of the body lies in the principal xy-plane. Measuring Euler's angle φ from the downward vertical, one sees that conditions (8.9) reduce to the conditions

$$\theta = \frac{\pi}{2}, \dot{\psi} = 0, r = -\dot{\varphi}, \gamma_1 = -\cos\varphi, \gamma_2 = -\sin\varphi, \tag{8.11}$$

so that only one equation remains. Instead of integrating this equation, we use the energy integral to write

$$\frac{1}{2}Cr^2 + Mg(x_0\gamma_1 + y_0\gamma_2) = h, \tag{8.12}$$

where h is the total energy of the motion. Denoting by α the angle between the position vector of the centre of mass and the y-axis and by s the distance from the fixed point to the centre of mass, so that

$$x_0 = s\cos\alpha, y_0 = s\sin\alpha, \tag{8.13}$$

and completing substitution from (8.11), we get

$$\frac{1}{2}C\dot{\varphi}^2 - Mgs(\cos\alpha\cos\varphi + \sin\alpha\sin\varphi) = h, \tag{8.14}$$

or, equivalently,

$$\frac{1}{2}C\dot{\varphi}^2 = h + a\cos(\varphi - \alpha), \tag{8.15}$$

where $a = Mgs$. Now, separating variables in the last equation, we find

$$\pm\sqrt{\frac{2}{C}}dt = \frac{d\varphi}{\sqrt{h + a\cos(\varphi - \alpha)}}.$$

Introducing a new angle Φ by the relation $\varphi = \alpha + 2\Phi$ and integrating the last relation yield

$$\pm\sqrt{\frac{1}{2C}}(t - t_0) = \int_0^{\Phi} \frac{d\Phi}{\sqrt{h + a - 2a\sin^2\Phi}}, \tag{8.16}$$

where t_0 is the time at which $\Phi = 0$, i.e. when the centre of mass passes at the vertical position below the fixed point (the stable equilibrium position). The minimum energy for possible motions is $h = -a$ occurs at this position. Three qualitatively different types of motions are possible:

8.2.1 Rotational Motion

When $h > a$ we can write (8.16) in the form

$$\pm\sqrt{\frac{h+a}{2C}}(t - t_0) = \int_0^{\Phi} \frac{d\Phi}{\sqrt{1 - k^2\sin^2\Phi}}, \tag{8.17}$$

where

$$k^2 = \frac{2a}{h+a} < 1.$$

The angle Φ is readily expressed as

$$\begin{aligned} \Phi = \operatorname{am}(u, k),\ u = \pm\sqrt{\frac{h+a}{2C}}(t - t_0),\\ \sin\Phi = \operatorname{sn} u,\ \cos\Phi = \operatorname{cn} u, \end{aligned} \tag{8.18}$$

and hence

$$\begin{aligned} \gamma_1 &= -\cos\varphi = -\cos(\alpha + 2\Phi)\\ &= -\cos\alpha(1 - 2\sin^2\Phi) + 2\sin\alpha\sin\Phi\cos\Phi\\ &= -\cos\alpha(1 - 2\operatorname{sn}^2 u) + 2\sin\alpha\operatorname{sn} u\operatorname{cn} u,\\ \gamma_2 &= -\sin(\alpha + 2\Phi)\\ &= -\sin\alpha(1 - 2\operatorname{sn}^2 u) - 2\cos\alpha\operatorname{sn} u\operatorname{cn} u, \end{aligned} \tag{8.19}$$

and the angular velocity is given by

$$r = -2\frac{d\Phi}{du}\frac{du}{dt} = \pm 2\sqrt{\frac{h+a}{2C}}\,\mathrm{dn}\,u. \tag{8.20}$$

The body rotates with periodic angular velocity and makes complete revolution in one and the same direction all the time. The three variables γ_1, γ_2, r are periodic functions in t with period

$$T = 2K(k)\sqrt{\frac{2C}{h+a}}. \tag{8.21}$$

As the energy of rotation h increases to infinity, $k \to 0$, $K(k) \to \frac{\pi}{2}$, $\mathrm{dn}\,u \to 1$ and $r \to \sqrt{\frac{2h}{C}}$ and the periodic time for fast rotations becomes $\frac{2\pi}{r}$.

8.2.2 Vibrational Motion

When $-a \le h < a$, the energy of motion is not sufficient to let the centre of mass of the body go all the way to the highest point vertically above the fixed point. The motion is a vibration such that $\varphi - \alpha$ varies between $\pm\delta$, $\delta = \cos^{-1}\frac{-h}{a}$. The modulus k of the elliptic functions becomes >1 and using elliptic functions with modulus

$$\nu = \frac{1}{k} = \sin\frac{\delta}{2} = \sqrt{\frac{h+a}{2a}}, \tag{8.22}$$

the solution (8.19), (8.20) is replaced by

$$\begin{aligned}
\gamma_1 &= -\cos\alpha(1 - 2\nu^2\,\mathrm{sn}^2\,w) + 2\nu\sin\alpha\,\mathrm{sn}\,w\,\mathrm{dn}\,w,\\
\gamma_2 &= -\sin\alpha(1 - 2\nu^2\,\mathrm{sn}^2\,w) - 2\nu\cos\alpha\,\mathrm{sn}\,w\,\mathrm{dn}\,w,\\
r &= 2\sqrt{\frac{h+a}{2C}}\,\mathrm{cn}\,w,
\end{aligned} \tag{8.23}$$

where

$$w = \pm\sqrt{\frac{a}{C}}(t - t_0). \tag{8.24}$$

The periodic time of motion is

$$T = 4K(\nu)\sqrt{\frac{C}{a}}. \tag{8.25}$$

For small vibrations about the lower equilibrium position, $T = 2\pi\sqrt{\frac{C}{a}}$.

8.2.3 *The Limiting Motion*

Now we turn to the critical case between the above two classes of motions. When $h \to a$, both k and $\nu \to 1$ and the period becomes infinitely large. The solutions (8.19), (8.20) and (8.23) both reduce to

$$\begin{aligned}
\gamma_1 &= -\cos\alpha(1 - 2\tanh^2 w) + 2\nu \sin\alpha \tanh w \operatorname{sech} w, \\
\gamma_2 &= -\sin\alpha(1 - 2\tanh^2 w) - 2\nu \cos\alpha \tanh w \operatorname{sech} w, \\
r &= 2\sqrt{\frac{a}{C}} \operatorname{sech} w.
\end{aligned} \tag{8.26}$$

As $w \to \pm\infty$, $(\gamma_1, \gamma_2) \to (\cos\alpha, \sin\alpha)$ and $r \to 0$. The motion becomes asymptotic to or from the unstable (upper) equilibrium position.

8.2.4 *Orbital Stability*

The trajectory traced by the plane motion on the Poisson sphere is a great circle $\gamma_3 = 0$ for the case of rotations and an arc of such circle in the case of vibration. By the orbital stability of the motion, we mean the stability of the solution $\gamma_3 = 0$ with respect to lateral perturbations preserving the values of the total energy h and the areas integral $f = 0$. The study of orbital stability of pendulum motions in the linear setting was initiated in [379], continued in [375, 376, 388] and generalized for the gyrostat in [388, 420].

8.3 Permanent Rotation of a Heavy Rigid Body About a Fixed Point [343] (1894)

8.3.1 *Possible Axes of Permanent Rotation*

Following Staude [343], we now look for possible rotations of the body with a uniform angular velocity about an axis, which remains immovable both in space and in the body. Setting $\dot{\boldsymbol{\omega}} = \mathbf{0}$, $\dot{\boldsymbol{\gamma}} = \mathbf{0}$, in (3.15), we get

$$\boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I} = Mg\boldsymbol{\gamma} \times \mathbf{r}_0, \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}. \tag{8.27}$$

The second equation has the solution $\boldsymbol{\omega} = \omega_0 \boldsymbol{\gamma}$, ω_0 is a constant, so that the motion is possible only about a vertical axis. Substituting $\boldsymbol{\omega}$ in the dynamical equation gives

$$\boldsymbol{\gamma} \times (\omega_0^2 \boldsymbol{\gamma}\mathbf{I} - Mg\mathbf{r_0}) = \mathbf{0}. \tag{8.28}$$

Multiplying this equation scalarly by $\mathbf{r_0}$, we obtain the equation of the locus of the axes of permanent rotations

$$\mathbf{r_0} \cdot (\gamma \times \gamma \mathbf{I}) = 0. \tag{8.29}$$

In the system of principal axes of the body, this equation can be written in the determinantal form

$$\begin{vmatrix} x_0 & y_0 & z_0 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ A\gamma_1 & B\gamma_2 & C\gamma_3 \end{vmatrix} = 0 \tag{8.30}$$

or, in the expanded form,

$$x_0(B - C)\gamma_2\gamma_3 + y_0(C - A)\gamma_3\gamma_1 + z_0(A - B)\gamma_1\gamma_2 = 0. \tag{8.31}$$

This is a homogeneous quadratic equation. It represents a cone, fixed in the body and whose vertex is at the fixed point. This cone is called Staude's cone. It is identical with Ampére's cone mentioned in Chap. 1 in a completely different context as the cone formed by lines passing through the fixed point, each of which is a principal axis with respect to one of its points. The last property of Staude's cone seems to be first proved by Lecornu [255], but is usually ascribed to van der Woude [370].

There are three cases when Staude's cone becomes undetermined. Those are when

(1) the body is fixed from its mass centre ($x_0 = y_0 = z_0 = 0$). That is Euler's case;
(2) the body has spherical dynamical symmetry at the fixed point ($A = B = C$);
(3) the body is dynamically axi-symmetric and the centre of mass lies on the symmetry axis. This is Lagrange's case.

Staude's cone degenerates into two planes in three cases:

(1) When the body is dynamically axi-symmetric.
(2) When the centre of mass lies in a principal plane or on a principal axis of inertia.
(3) To those cases one has to add the case when a triangle inequality holds ($B + C = A, C + A = B, A + B = C$), in which Staude's cone formally persists to be non-degenerate, but loses its meaning because the body renders to a plane disc with centre of mass lying outside the plane. According to the theorem of Sect. 1.4, we have $z_0 = 0$.

8.3.2 Description of the Motion

We now try a more detailed description of the permanent rotations. Let a given vector γ determine a direction in the main body of an axis of permanent rotations, i.e. a generator of Staude's cone. This generator is directed vertically upwards and the body moves about it with a constant angular speed ω_0. During this motion the centre of mass (and also any other mass element) of the body uniformly traces a circular path in a horizontal plane. In the given motion not only the total energy of motion is conserved, but also both kinetic and potential energies remain constant.

The square of the angular velocity about this axis may be determined from any of the equations

$$\begin{aligned} \omega_0^2(B-C)\gamma_2\gamma_3 &= Mg(y_0\gamma_3 - z_0\gamma_2),\\ \omega_0^2(C-A)\gamma_3\gamma_1 &= Mg(z_0\gamma_1 - x_0\gamma_3),\\ \omega_0^2(A-B)\gamma_1\gamma_2 &= Mg(x_0\gamma_2 - y_0\gamma_1), \end{aligned} \tag{8.32}$$

obtained by writing (8.28) in components. Note that those equations are compatible in virtue of (8.29). This means that if a permanent rotation is possible (ω_0^2 positive) about an axis with one of its ends directed vertically upwards, then this motion can have one of the two angular velocities $\pm\sqrt{\omega_0^2}$, i.e. equal and opposite angular velocities. From the last equations, we notice that ω_0^2 changes sign when the direction of $\boldsymbol{\gamma}$ is reversed, so that the other end of the axis of permanent rotation cannot be itself directed upwards for another possible permanent rotation. In other words, every generator of Staude's cone is a possible axis of permanent rotation with one and only one end of it directed vertically upwards. Obvious exception is the generator that passes through the centre of mass. This generator can be directed vertically up or down, rendering the body to its two equilibrium positions (since $\omega_0^2 = 0$) with the centre of mass above or below the fixed point.

We now proceed to clarify the picture of Staude's cone and the distribution of the axes of permanent rotations on that cone. We assume that the centre of mass lies at a point in the positive octant. This can be always achieved, excluding for the moment some degenerate cases, when the centre of mass of the body lies on one of its principal axes of inertia. From the determinant Eq. (8.30), we notice that Staude's cone passes through the three principal axes of inertia of the body and also through the centre of mass $\mathbf{r}_0$ and its diametrically opposite point with respect to the origin $(-\mathbf{r}_0)$. Denote by P and Q, the points of intersection of the generator passing through $\mathbf{r}_0$ and $(-\mathbf{r}_0)$, respectively, with the Poisson sphere.

Staude's cone has vertex at the origin, and it intersects the Poisson sphere fixed in the body in two closed spherical curves, one of which passes through P, the positive ends of x, z-axes and the negative end of y. The other spherical curve is the reverse of the first with respect to the origin, i.e. passes through $Q, -x, -z, y$ (See Fig. 8.1).

For further determinacy, we shall assume that $A > B > C$. From Eqs. (8.32), it follows that ω_0 is real only on the axes intersecting the unit sphere in the four arcs (P, z), $(x, -y)$, $(Q, -x)$ and $(y, -z)$, shown as thick lines in Fig. 8.2. Each axis should be directed vertically upwards and the body must be given the appropriate (positive or negative) angular velocity about that axis. Note that the angular velocity corresponding to the permanent rotation about OP is $\omega_0 = 0$, i.e. this is the upper equilibrium position of the body. Similarly, the axis OQ corresponds to the lower equilibrium position. Note also that ω_0 tends to infinity whenever the axis of rotation approaches one of the six ends of the principal axes of inertia.

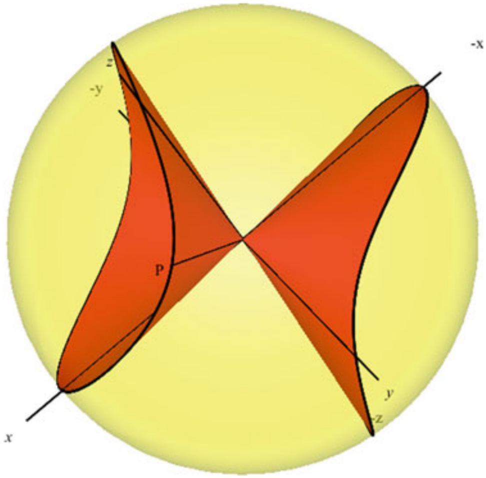


Fig. 8.1 Staude's cone and its intersection with the sphere

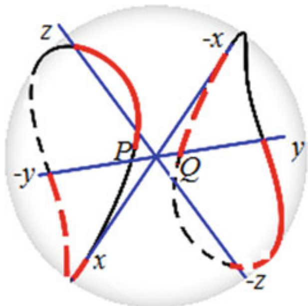


Fig. 8.2 Admissible arcs for Staude's permanent rotations are represented by thick (red) lines (Hidden lines are dashed)

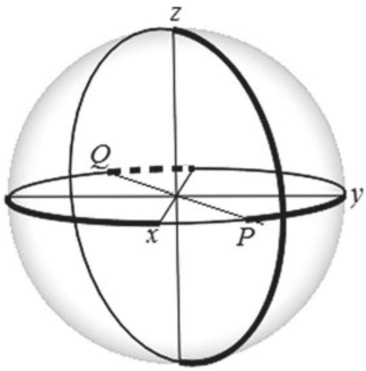


Fig. 8.3 The centre of mass lies in a principal plane (xy-plane)

Figure 8.3 depicts the admissible arcs on the sphere, corresponding to a centre of mass lying in the principal plane xy. For more detailed presentation of degenerate cases, see [321] (and also [256]).

8.3.3 Further Studies

Permanent rotations have attracted researchers for a long time and from different points of view. Linear stability analysis was performed for some special cases in [131], (See also [132]), [43] and others. More systematic results were obtained by Rumiantsev [321], who relied on wide use of linearized equations in the variations and construction of Lyapunov functions.

Staude's rotation is characterized by the constancy of the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$ in the body system (as well as in space). In such motion, only the Eulerian angle of precession ψ varies with time, while θ and φ take constant values. Noting that ψ is a cyclic coordinate, it becomes evident that Staude's rotations can be characterized as the equilibrium positions of the reduced system. This means permanent rotations are located at the extremal points of the reduced potential V_1 of the problem on the Poisson sphere (See (3.56)):

$$V_1 = Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} + \frac{f^2}{2(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)}. \tag{8.33}$$

Let P be one of those points. The permanent rotation about OP is stable if V_1 attains a minimum at P, unstable if it has a saddle point. If V_1 has a maximum at P, then one cannot draw any conclusion about the stability of the permanent rotation based on V_1 alone. In that case gyroscopic stabilization is possible and the equations of perturbed motion should be considered in the nonlinear setting. For more details and results in this direction, see [105, 351] and references therein. Detailed stability analysis of permanent rotations with special emphasis on those corresponding to the three integrable cases can be found in [330].

8.3.4 Excercises

(1) Show that the reaction of the fixed point on the body in the state of permanent rotation is given by the formula

$$\mathbf{R} = Mg\boldsymbol{\gamma} - M\omega_0^2\boldsymbol{\rho},$$

where ω_0 is determined from (8.32) and $\boldsymbol{\rho}$ is the current radius vector of the centre of mass of the body relative to the axis of rotation.

(2) Determine all possible uniform rotations of a rigid body about a fixed point O of it, when acted upon by a single constant force applied to a point distinct from both the fixed point and the centre of mass [Alfieri [6]].

8.4 Hess' Case (1890)

One of the earliest particular solutions in the dynamics of a rigid body was discovered by Hess in 1890 [150] and actually rediscovered by some other authors, e.g. Appelrot [10] and Shiff [331]. It differs from all other particular cases in an essential aspect, its solution is not completely reduced to quadratures. One may say it is a case of partial solution of the Euler–Poisson equations.

8.4.1 Equations of Motion

Let the principal axes of inertia of the body be arranged so that $A > B > C$, and assume that the centre of mass C_g lies in the xz-plane containing the major and minor axes of the inertia ellipsoid $y_0 = 0$. Equations (3.29) take the form

$$\begin{aligned} A\dot{p} + (C - B)qr &= Mgz_0\gamma_2, \\ B\dot{q} + (A - C)\,pr &= Mg(x_0\gamma_3 - z_0\gamma_1), \\ C\dot{r} + (B - A)pq &= -Mgx_0\gamma_2, \\ \dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 &+ r\gamma_1 - p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \end{aligned} \tag{8.34}$$

Eliminating γ_2 between the first and third equations, we get

$$Ax_0\dot{p} + Cz_0\dot{r} = q[(B - C)x_0r + (A - B)z_0p]. \tag{8.35}$$

Thus, if the distribution of mass in the body satisfies the condition

$$\frac{(A - B)z_0}{Ax_0} = \frac{(B - C)x_0}{Cz_0} = s \text{ (say)},$$

i.e. if

$$x_0\sqrt{A(B - C)} = \pm z_0\sqrt{C(A - B)}, \tag{8.36}$$

then Eq. (8.35) can be written as

$$\frac{d}{dt}(Ax_0p + Cz_0r) = sq(Ax_0p + Cz_0r). \tag{8.37}$$

If the motion at a certain time moment satisfies the condition that

$$Ax_0p + Cz_0r = 0, \tag{8.38}$$

the time derivative of this expression will also vanish, and hence it will be a constant of motion that vanishes at all times. This gives what we call an invariant relation. Note that the invariant relation (8.38) has a dynamical meaning. It signifies that the component of the angular momentum of the body vanishes in the direction from the fixed point to the centre of mass of the body.

Remark: Condition (8.36) together with $y_0 = 0$ mean that the centre of mass C_g lies on the line drawn from the origin perpendicular to one of the two circular cross-sections of the gyration ellipsoid (See Exercise 4 of Chap. 1).

Taking this remark into account, one can see that the angular momentum of the body in Hess' case lies always in one of the circular cross-sections of the gyration ellipsoid.

8.4.2 Solution

Let δ be the angle between OC_g and the z-axis. Regarding (8.36), we have

$$\cos\delta = \sqrt{\frac{A(B-C)}{B(A-C)}},\ \sin\delta = \pm\sqrt{\frac{C(A-B)}{B(A-C)}}. \tag{8.39}$$

Denote by Γ_1 and Γ_3 the projections of the vector $\boldsymbol{\gamma}$ on OC_g and the orthogonal to it. Then one can write

$$\begin{aligned}\Gamma_3 &= \cos\delta\gamma_1 + \sin\delta\gamma_3,\\ \Gamma_1 &= -\sin\delta\gamma_1 + \cos\delta\gamma_3.\end{aligned} \tag{8.40}$$

Those variables satisfy the relation

$$\Gamma_1^2 + \Gamma_3^2 = \gamma_1^2 + \gamma_3^2 = 1 - \gamma_2^2. \tag{8.41}$$

We now show that the variable Γ_3 can be determined by a quadrature. In fact, using the integrals of motion

$$\begin{aligned}Ap^2 + Bq^2 + Cr^2 + 2Mgs\Gamma_3 &= 2h,\\ Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 &= f,\end{aligned}$$

together with (8.38), one can find

$$\begin{aligned}A(1-\Gamma_3^2)p &= -\sin\delta(\sqrt{S}\gamma_2 + f\Gamma_1),\\ B(1-\Gamma_3^2)q &= -\sqrt{S}\Gamma_1 + f\gamma_2,\\ C(1-\Gamma_3^2)r &= \cos\delta(\sqrt{S}\gamma_2 + f\Gamma_1),\end{aligned} \tag{8.42}$$

where $S = S(\Gamma_3)$ is the cubic polynomial

$$S = 2B(1 - \Gamma_3^2)(h - s\Gamma_3) - f^2. \tag{8.43}$$

From Poisson's equations and using the last expressions (8.42), we readily obtain for Γ_3 the equation

$$B\dot{\Gamma}_3 = \sqrt{S(\Gamma_3)}. \tag{8.44}$$

Thus, Γ_3 can be expressed as an elliptic function of time by inverting the relation

$$t = \int^{\Gamma_3} \frac{du}{\sqrt{2B(1-u^2)(h-su) - f^2}}. \tag{8.45}$$

The last formula can be compared to formulas (4.42) of Sect. 4.2, concerning Lagrange's top. Equation (8.45) has the same structure as in Sect. 4.2 provided in the last we put $c = C/A = 0$, i.e. the Lagrange top has all its mass concentrated on the Γ_3-axis. This means that the centre of mass of the body moves with respect to the vertical as a spherical pendulum. This property was noted by Joukovsky [165].

8.4.3 The Use of Special Axes

Hess' case takes a simpler form if we choose the body axes to be associated with the gyration ellipsoid. The equations of motion take the form (3.41). Let the x, y-axes be in the plane of one of the circular cross-sections of that ellipsoid and the z-axis orthogonal to it. As the middle principal axis of inertia of the body is the intersection of the two circular cross-sections, this axis can always be chosen as the y-axis, so that $A_{23} = 0$. The gyration matrix becomes

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{11} & 0 \\ A_{13} & 0 & A_{33} \end{pmatrix}. \tag{8.46}$$

In those axes, the centre of mass lies on the z-axis at the point $(0, 0, z_0)$ (say), and Hess' integral becomes $R = 0$. The equations of motion (3.41) become

$$\begin{aligned} \dot{P} - A_{13}PQ &= c\gamma_2, \\ \dot{Q} + A_{13}P^2 &= -c\gamma_1, \end{aligned} \tag{8.47}$$

$$\begin{aligned}
\dot{\gamma}_1 - P\gamma_2 A_{13} + A_{11}Q\gamma_3 &= 0,\\
\dot{\gamma}_2 + P\gamma_1 A_{13} - A_{11}P\gamma_3 &= 0,\\
\dot{\gamma}_3 + A_{11}\,(P\gamma_2 - Q\gamma_1) &= 0,
\end{aligned}\tag{8.48}$$

where $c = Mgz_0$, and the integrals of motion

$$\begin{aligned}
\frac{1}{2}A_{11}(P^2 + Q^2) + c\gamma_3 &= h,\\
P\gamma_1 + Q\gamma_2 &= f,\\
\gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1.
\end{aligned}\tag{8.49}$$

It will be more convenient to introduce new variables by the relations

$$P = \rho\cos\chi,\ Q = \rho\sin\chi. \tag{8.50}$$

Those variables are related to the Eulerian angles θ, φ. Recalling the expression of $\boldsymbol{\gamma}$, one can use the integrals (8.49) to obtain the relations

$$\begin{aligned}
\gamma_3 = \cos\theta &= \frac{1}{c}(h - \frac{1}{2}A_{11}\rho^2),\\
\rho\sin\theta\sin(\chi + \varphi) &= f.
\end{aligned}\tag{8.51}$$

In the same special axes, the angular velocity of the body can be expressed, using (8.46), as

$$\begin{aligned}
\boldsymbol{\omega} \equiv (p, q, r) &= (A_{11}P,\ A_{11}Q,\ A_{13}P)\\
&= (A_{11}\rho\cos\chi,\ A_{11}\rho\sin\chi,\ A_{13}\rho\cos\chi).
\end{aligned}\tag{8.52}$$

Now, comparing this expression with the formula (2.39) and using the areas integral in (8.49), one can find the rates of change of the Eulerian angles

$$\begin{aligned}
\dot{\psi} &= \frac{A_{11}f}{\sin^2\theta},\\
\dot{\theta} &= A_{11}\rho\cos(\chi + \varphi),\\
\dot{\varphi} &= A_{13}\rho\cos\chi - \frac{A_{11}f\cos\theta}{\sin^2\theta}.
\end{aligned}\tag{8.53}$$

Using (8.50) and (8.47), we find for ρ and χ the equations

$$\rho\frac{d\rho}{dt} = -c\sqrt{F}, \tag{8.54}$$

where

$$F = A_{11}\rho^2[1 - (h - \frac{1}{2}A_{11}\rho^2)^2/c^2] - f^2,$$

and

$$\dot{\chi} = -[A_{13}\rho\cos\chi + \frac{cf}{\rho^2}]. \tag{8.55}$$

The first equation determines ρ^2 as an elliptic function of time t (the equivalent of (8.45)). Inserting this function into (8.55) we obtain for χ a nonlinear first-order differential equation with elliptic coefficients. Substituting $\tan\frac{\chi}{2} = \eta$, the last equation can be put in the form of Riccati equation and if further one uses the substitution $\eta = \dot{\zeta}/\zeta$, it can be reduced to a linear second-order equation with elliptic coefficients [293]. An equivalent form was obtained by Golubev using a somewhat different method in [113]. A detailed analysis of Hess' case was performed by Kovalev in [236, 237], aiming mainly at constructing the angular velocity hodograph for Hess' case. A linear equation for $\zeta(\rho)$, equivalent to Nekrassov's equation, was derived [237] (See also [108]). However, none of those forms helped to complete the solution in the general case.

8.4.4 Solution of the Case $f = 0$

The simplest case and the only one when the solution of the equations of motion is completed is that when the areas constant f vanishes and then Eq. (8.55) becomes separable. In that case, the second equation in (8.51) determines the angle χ in the form

$$\chi = -\varphi, \tag{8.56}$$

and, hence, Eqs. (8.53) become

$$\begin{aligned} \dot{\psi} &= 0, \\ \dot{\theta} &= A_{11}\rho, \\ \dot{\varphi} &= A_{13}\rho\cos\varphi. \end{aligned} \tag{8.57}$$

The first equation means that the Eulerian angle ψ during motion preserves a constant value. Without loss of generality, this value can be taken as $\psi = 0$. The motion of the body can be interpreted as follows:

The body rotates with angular velocity $\dot{\varphi}$ about its barycentric axis (the z-axis, carrying the centre of mass), while that axis performs a periodic pendulum-like motion in a fixed vertical plane passing through the fixed point. That is a type of precession with a variable angular speed $\dot{\theta}$ about a horizontal axis, namely the type pointed out by Bressan [46] (1957) for the rigid body in the case of Hess.

Explicit expressions of the variables in terms of time differ for the three cases of pendulum motions: rotational, vibrational and asymptotic. We write down here the full expressions for the case of complete pendulum rotations, characterized by the condition $h > c$. Detailed qualitative analysis of possible types of motion can be found in [108].

In that case from the above formulas we can find

$$\cos\theta = -1 + 2\,\mathrm{sn}^2\,u, \tag{8.58}$$

$$\rho = \sqrt{\frac{2(h+c)}{A_{11}}}\,\mathrm{dn}\,u, \tag{8.59}$$

where

$$u = A_{11}\sqrt{\frac{h+c}{2}}t,$$

and the modulus of Jacobi's elliptic functions

$$k = \sqrt{\frac{2c}{h+c}}.$$

On the other hand, dividing the last two equations in (8.57) and separating variables, we get

$$\frac{d\varphi}{\cos\varphi} = \frac{A_{13}}{A_{11}}d\theta.$$

And on integration, this gives

$$\varphi = \tan^{-1}(\sinh(\frac{A_{13}}{A_{11}}\theta)), \tag{8.60}$$

where an insignificant integration constant is taken zero. The trajectory (8.60) on the Poisson sphere is the locus of the end of the vertical unit vector $\boldsymbol{\gamma}$

$$\boldsymbol{\gamma} = (\sin\theta\tanh(\frac{A_{13}}{A_{11}}\theta), \sin\theta\,\mathrm{sech}(\frac{A_{13}}{A_{11}}\theta), \cos\theta)$$

on that sphere. As θ increases indefinitely with the pendulum rotation of the z-axis in the fixed vertical plane, the angle of proper rotation φ about that axis tends to the value $\pi/2$, as depicted in Fig. 8.4. The larger the quantity $\frac{A_{13}}{A_{11}}$, the faster φ approaches $\pi/2$. We conclude that the motion of the body asymptotically approaches the motion as a physical pendulum about the medium principal axis of inertia, which takes a permanent horizontal position.

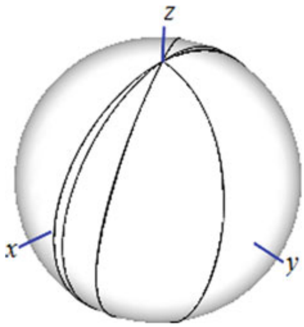


Fig. 8.4 Orbit of apex of γ on the Poisson sphere

8.5 The Case of Bobylev and Steklov (1896)

Almost simultaneously, in 1896, two communications of Bobylev [27] and Steklov [346] appeared announcing one and the same solvable case of the equations of motion of a heavy rigid body fixed from one point.

The key condition in this case is that one of the components of the angular velocity permanently vanishes, say, $q = 0$, i.e. the angular velocity lies permanently in the principal xz-plane. A great simplification occurs in the equations of motion (3.29) if one adds the assumption that the centre of mass of the body lies on the z-axis, i.e. $x_0 = y_0 = 0$. Then, under the compatibility condition $C = 2A$, we will have

$$\begin{aligned} A\dot{p} &= a\gamma_2, \\ -Apr &= -a\gamma_1, \\ C\dot{r} &= 0, \\ \dot{\gamma}_1 - r\gamma_2 = 0, \dot{\gamma}_2 &+ r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 = 0, \end{aligned} \tag{8.61}$$

where $a = Mgz_0$. From those equations, we immediately get

$$\begin{aligned} r &= r_0 = const, \\ p &= \frac{a}{Ar_0}\gamma_1, \qquad r_0 \neq 0, \\ \gamma_2 &= \frac{\dot{\gamma}_1}{r_0}. \end{aligned} \tag{8.62}$$

Now, using the areas integral, we obtain

$$\gamma_3 = \frac{1}{2Ar_0}\left(f - \frac{a}{r_0}\gamma_1^2\right), \tag{8.63}$$

so that all variables are expressed in term of γ_1 and it remains to determine γ_1 using the geometric integral, which leads to the first-order separable equation

$$\dot{\gamma}_1 = \pm r_0 \sqrt{1 - \gamma_1^2 - \frac{1}{4A^2 r_0^2}\left(f - \frac{a}{r_0}\gamma_1^2\right)^2}. \tag{8.64}$$

Note that one can verify that all the Euler–Poisson equations (8.61) are all satisfied, in virtue of (8.62)–(8.64).

To simplify the classification of the possible motions, we introduce the new dimensionless parameters ρ, j by the relations

$$r_0 = \sqrt{\frac{a}{A}}\,\rho \quad , \quad f = \sqrt{aA}\,j. \tag{8.65}$$

Thus, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ will be expressed as follows:

$$\begin{aligned} \boldsymbol{\omega} &= \sqrt{\frac{a}{A}}\left(\frac{\gamma_1}{\rho}, 0, \rho\right), \\ \boldsymbol{\gamma} &= \left(\gamma_1, \sqrt{\frac{A}{a}}\frac{\dot{\gamma}_1}{\rho}, \frac{j}{2\rho} - \frac{\gamma_1^2}{2\rho^2}\right). \end{aligned} \tag{8.66}$$

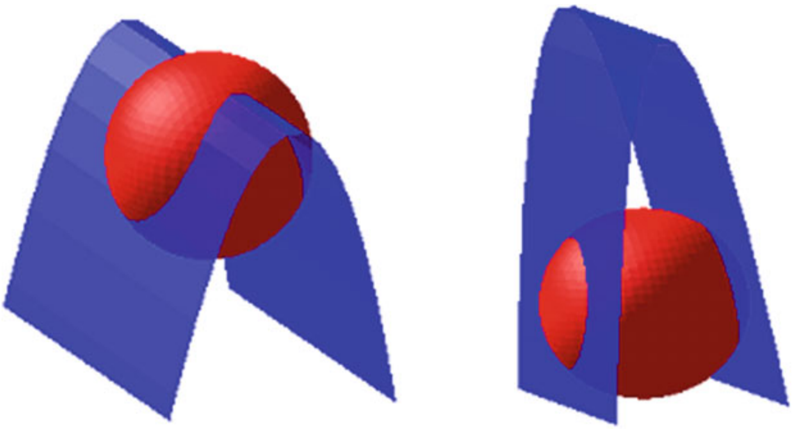
Note that a simultaneous change of the signs of parameters r_0, f (ρ, j) results in a change of signs of p, $\dot{\gamma}_1$ and leaves unchanged the vector $\boldsymbol{\gamma}$ as function of time. On the other hand, formulas (8.66) are not valid for $\rho = 0$. Thus, it suffices to consider motions with parameters ρ, j in the open right half-plane $\rho > 0$. Motions in the other half-plane are the same but traversed in the reverse direction.

Now, integrating Eq. (8.64) gives

$$\pm\sqrt{\frac{a}{A}}\frac{(t - t_0)}{2\rho} = \int \frac{d\gamma_1}{\sqrt{F(\gamma_1)}}, \tag{8.67}$$

where $F(\gamma_1) = \sqrt{\left(4\rho^4 - j^2\rho^2\right) + \left(2j\rho - 4\rho^4\right)\gamma_1^2 - \gamma_1^4}$ and t_0 is an arbitrary constant. The integral in the right-hand side is an elliptic integral of the first kind. Inverting this integral, we express γ_1 as an elliptic function in the time t. Depending on the two parameters ρ and j, the fourth-degree polynomial $F(\gamma_1)$ may have either two or four real roots, so that in the ρj-plane we have two different regions corresponding to two qualitatively distinct classes of motions.

A clear geometrical picture of the two classes is obtained by examining the trajectories drawn during the motion of the body by the tip of the vector $\boldsymbol{\gamma}$ on the Poisson sphere. That is the curve of intersection of the sphere $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ and the parabolic cylinder $\gamma_3 = \frac{j}{2\rho} - \frac{\gamma_1^2}{2\rho^2}$. It can be readily seen that when $-1 \le \frac{j}{2\rho} < 1$ the intersection is a single simple closed curve symmetric with respect to the yz-plane, beginning at $j = -2\rho$ with the single point $(0, 0, -1)$. Example of this type



a) A single spherical curve. b) Two spherical curves.

Fig. 8.5 Path of the vector γ on the unit sphere

is illustrated in Fig. 8.5a. For $j > 2\rho$ if $j \leq \rho^3 + \frac{1}{\rho}$, the intersection consists of two similar closed curves symmetrically situated with respect to the yz-plane (Fig. 8.5b). On the boundary $j = \rho^3 + \frac{1}{\rho}$, the two curves shrink to two points. We now describe the analytical solution of the equations of motion in the two regions.

8.5.1 Region I: The First Class of Motions

In the region I, $-2\rho \leq \ j \leq 2\rho$, (see Fig. 5.1) the polynomial $F(\gamma_1)$ has only two real roots. Equation (8.67) will have a *cn* solution and the Euler–Poisson variables take the form

$$\gamma = (M\,\mathrm{cn}(u,k)\,,\, -\frac{M}{\rho}\sqrt{d}\,\mathrm{sn}(u,k)\,\mathrm{dn}(u,k), \frac{j}{2\rho} - \frac{M^2}{2\rho^2}\,\mathrm{cn}^2(u,k)),$$
$$\omega = \sqrt{\frac{a}{A}}(\frac{M}{\rho}\,\mathrm{cn}(u,k), 0, \rho), \tag{8.68}$$

where

$$M = \sqrt{\rho[j - 2\rho^3 + 2\rho d]}, d = \sqrt{\rho^4 - j\rho + 1},$$
$$k^2 = \frac{1}{2}(1 + \frac{j - 2\rho^3}{2\rho d}),$$
$$u = \pm\sqrt{\frac{ad}{A}}\,(t - t_0)\,. \tag{8.69}$$

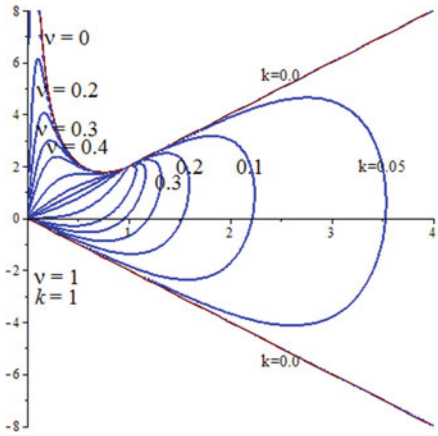
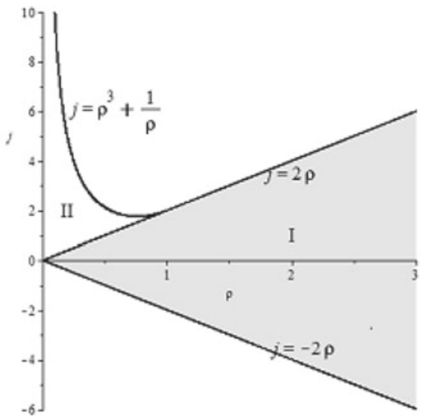


Fig. 8.6 Contour lines of k, ν

The expressions (8.68) are periodic functions in time with period $\frac{4K(k)}{\sqrt[4]{\rho^4 - j\rho + 1}}\sqrt{\frac{A}{a}}$. Contours of k in the ρj-plane are illustrated in Fig. 8.6. The motion is composed of a uniform rotation about the z-axis and a vibration about the x-axis.



Zones I and II

On the two rays $j = -2\rho$ and $j = 2\rho > 2$, the modulus $k = 0$ (see Fig. 8.7) and the motion reduces to Staude's rotation with the uniform angular velocity $r_0 = \sqrt{\frac{a}{A}}\rho$ around the z-axis, whose positive half bearing the centre of mass is directed vertically downwards on the first ray and upwards on the second.

On the line $j = 2\rho$, $0 < \rho \leq 1$, (8.68) reduces to

$$\gamma = (\frac{\pm 2\rho\sqrt{1-\rho^2}}{\cosh(w)}, \frac{2\left(1-\rho^2\right)\sinh(w)}{\cosh^2(w)}, 1 - \frac{2\left(1-\rho^2\right)}{\cosh^2(w)}),$$
$$\boldsymbol{\omega} = \sqrt{\frac{a}{A}}(\frac{\pm 2\rho\sqrt{1-\rho^2}}{\cosh w}, 0, \rho), \tag{8.70}$$

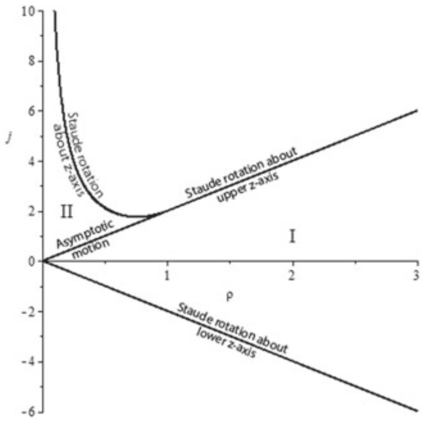


Fig. 8.7 Limiting cases

where $w = \sqrt{\frac{a}{A}(1-\rho^2)}(t-t_0)$. Those solutions describe asymptotic motions and tend to Staude's rotation with velocity $\omega = \sqrt{\frac{a}{A}}\rho$ around z-axis directed upwards.

8.5.2 Region II. The Second Class of Motions

In the region II (see Fig. 1)

$$j \le \rho^3 + \frac{1}{\rho}, \;\; j \ge 2\rho, \tag{8.71}$$

the polynomial $F(\gamma_1)$ has four real roots and Eq. (8.67) will take the form

$$\gamma_1 = \pm N \,\mathrm{dn}(w, \nu), \tag{8.72}$$

where

$$w = \sqrt{\frac{a}{A\rho}}\frac{\sqrt{j - 2\rho^3 + 2\rho d}}{2}(t - t_0),$$
$$N = \sqrt{\rho(j - 2\rho^3 + 2\rho d)}, \nu^2 = \frac{4\rho d}{j - 2\rho^3 + 2\rho d}. \tag{8.73}$$

Contours of ν are shown in Fig. 2. The vectors $\boldsymbol{\gamma}, \boldsymbol{\omega}$ will have the form

$$\boldsymbol{\gamma} = (\pm N \,\mathrm{dn}(w, \nu), \frac{2d}{N}\,\mathrm{sn}(w, \nu)\,\mathrm{cn}(w, \nu), \frac{j}{2\rho} - \frac{N^2}{2\rho^2}\,\mathrm{dn}^2(w, \nu)),$$
$$\boldsymbol{\omega} = \sqrt{\frac{a}{A}}(\pm\frac{N}{\rho}\,\mathrm{dn}(w, \nu), 0, \rho). \tag{8.74}$$

On the boundary curve $j = \rho^3 + \frac{1}{\rho}$, $\rho \le 1$, the motion degenerates into Staude's regular rotation with angular velocity $\omega = \sqrt{\frac{a}{A}}\frac{1}{\rho}$ around one of the directions $\boldsymbol{\gamma} = (\pm\sqrt{1-\rho^2}, 0, \rho)$ (see Fig. 3). On the line $j = 2\rho$, $0 < \rho \le 1$, we get the pair of solutions (8.70) describing motions asymptotic to Staude's rotations around z-axis.

Remark: The, so-called, orbital stability of the Bobylev–Steklov motion was recently investigated in [426]. Some results will be presented in due course.

8.6 Steklov's Case (1899)

8.6.1 Conditions and Solution

Suppose that the centre of mass lies on the first principal axis, i.e. $y_0 = z_0 = 0$. Equations of motion (3.29) take the form

$$
\begin{aligned}
A\dot{p} + (C - B)qr &= 0,\\
B\dot{q} + (A - C)\,pr &= a\gamma_3,\\
C\dot{r} + (B - A)pq &= -a\gamma_2,
\end{aligned}
\tag{8.75}
$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0, \tag{8.76}$$

where we have put $Mgx_0 = a$. Without loss of generality we assume $a > 0$. The general first integrals of motion become

$$
\begin{aligned}
I_1 &\equiv \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + a\gamma_1 = h,\\
I_2 &\equiv Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f,\\
I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,
\end{aligned}
\tag{8.77}
$$

where h, f are certain parameters, determined by initial state of motion.

When $B = C$, the equations describe Lagrange's integrable case discussed in Chap. 2. Thus, without loss of generality, one can choose the y-axis to be the one of greater moment of inertia, i.e. assume that $B > C$. Steklov assumes a solution of (8.75), (8.76), such that

$$\gamma_2 = n_2pq,\ \gamma_3 = n_3pr, \tag{8.78}$$

without indicating any line of thought leading to that assumption. It is not hard to reveal that under this assumption (8.75) can be cast in the form of equations of motion of a heavy body about its centre of mass, i.e. can be rendered to the integrable case of Euler and Poinsot, see Sect. 4.1, thus guaranteeing integrability and solution in terms of elliptic functions of time. Equations (8.76) will serve as compatibility conditions, may be, leading to some restrictions on the parameters of the problem.

Now, substituting (8.78) into (8.75)–(8.77) and using again the resulting dynamical equations, we determine the values of the parameters:

$$\begin{aligned} f &= 0, \\ h &= \epsilon a[1 - \frac{A^2}{2(A-B)(A-C)}], \\ n_2 &= \frac{(A-B)\,(A-C)}{a\,(A-2\,C)}, \\ n_3 &= \frac{(A-B)\,(A-C)}{a\,(A-2\,B)}, \end{aligned} \tag{8.79}$$

where $\epsilon = \pm 1$ and the relations between the variables

$$\begin{aligned} q^2 &= \frac{A}{B-C}[\frac{A-C}{A-2\,B}p^2 - \frac{a\epsilon\,(A-2\,C)}{(A-B)\,(A-C)}], \\ r^2 &= \frac{A}{B-C}[-\frac{A-B}{A-2\,C}p^2 + \frac{a\epsilon\,(A-2\,B)}{(A-B)\,(A-C)}]. \end{aligned} \tag{8.80}$$

It remains to find expression for p in terms of time. To this end, we use (8.80) in the first of Eqs. (8.75). We get

$$\dot{p} = \pm\sqrt{[\frac{A-C}{A-2\,B}p^2 - \frac{a\epsilon\,(A-2\,C)}{(A-B)\,(A-C)}][-\frac{A-B}{A-2\,C}p^2 + \frac{a\epsilon\,(A-2\,B)}{(A-B)\,(A-C)}]}, \tag{8.81}$$

so that p and hence q, r are elliptic functions of time. However, we deal here with real motion, where all variables are real, and this leads only to two classes of motion, characterized by the two different choices $\epsilon = \pm 1$. For those classes we have some conditions on the moments of inertia satisfied in addition to the triangle inequalities:

$$\begin{aligned} &B > A > 2C && \text{in the first case } (\epsilon = 1), \\ &A > B > C,\ A > 2C && \text{in the second case } (\epsilon = -1). \end{aligned}$$

In his original 1899 paper [347], Steklov considered only the first class of motions, but formulas derived there cover also the second class, which was noted and considered in detail by De Angeli [8] in 1934. The latter class was also rediscovered by Kuzmin in 1952 [248]). It is also noteworthy that the method devised by Steklov to obtain his solution was used and generalized in later works of other authors as Goryachev [114], Chaplygin [54], Kowalewski [239] and Kharlamov [196]. Solutions obtained in this way will be presented below.

The regions where the conditions on moments of inertia are satisfied are shown in Fig. 8.8 in the plane of $(B/A, C/A)$. They are represented in Fig. 8.8, respectively,

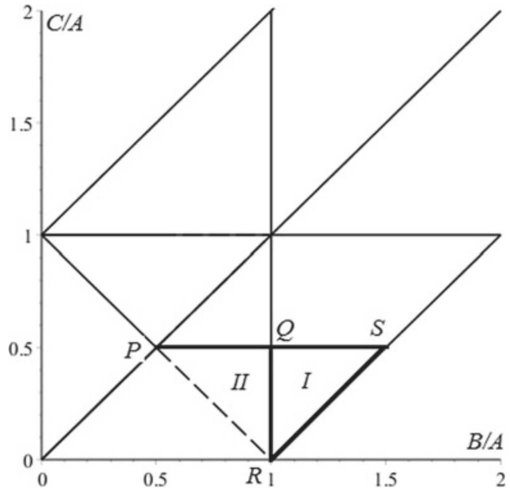


Fig. 8.8 Regions of reality of Steklov's solution

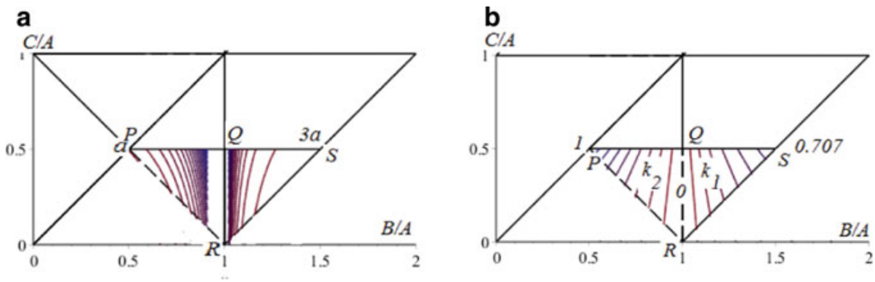


Fig. 8.9 **a** Contours of h. **b** Contours of k_1, k_2

by the two triangular regions $I(PQR)$ and $II(QRS)$. Note that on the line PR in region II we have $B + C = A$ and the body becomes a distribution of mass in the yz-plane, while its centre of mass lies on the x-axis outside that plane. The line PR is thus excluded from region II.

The energy constant h in the first region takes a value ranging from $3a$ at S to infinity on QR. In the second region it ranges from a at P to infinity on QR. Contour lines of h are shown in Fig. 8.9a. Figure 8.9b shows contours of $k_1 = \sqrt{\frac{B-A}{B-C}}$ and $k_2 = \sqrt{\frac{A-B}{A-C}}$, the moduli of elliptic functions in regions I and II, respectively. Note that k_1 changes from 0 on QR to $1/\sqrt{2}$ at S, while k_2 changes from 0 on QR to 1 at P.

We now write the solution of the equations of motion in the final explicit form.

8.6.2 The First Class

In this case the centre of mass lies on the middle axis of inertia. The components of the angular velocity are expressed as

$$
\begin{aligned}
p &= \sqrt{\frac{(2B-A)(A-2C)a}{(B-A)(A-C)^2}}\,\mathrm{cn}(u_1),\\
q &= \sqrt{\frac{A(A-2C)a}{(B-A)(B-C)(A-C)}}\,\mathrm{sn}(u_1),\\
r &= \sqrt{\frac{(2B-A)Aa}{(B-A)(A-C)^2}}\,\mathrm{dn}(u_1),
\end{aligned}
\tag{8.82}
$$

where $u_1 = \Omega_1(t - t_0)$, t_0 is an arbitrary constant and $\Omega_1 = \sqrt{\frac{(B-C)a}{(B-A)(A-C)}}$. The vector γ is given by

$$
\begin{aligned}
\gamma_1 &= \frac{A\,\mathrm{sn}^2(u_1) - C}{A - C} = 1 - \frac{A}{A-C}\,\mathrm{cn}^2(u_1),\\
\gamma_2 &= \sqrt{\frac{A(2B-A)}{(B-C)(A-C)}}\,\mathrm{sn}(u_1)\,\mathrm{cn}(u_1),\\
\gamma_3 &= \frac{\sqrt{A(A-2C)}}{A-C}\,\mathrm{cn}(u_1)\,\mathrm{dn}(u_1).
\end{aligned}
\tag{8.83}
$$

8.6.3 The Second Class

In this case the centre of mass of the body lies on the axis of largest moment of inertia.

$$
\begin{aligned}
p &= \sqrt{\frac{(2B-A)(A-2C)a}{(A-B)(A-C)^2}}\,\mathrm{sn}(u_2),\\
q &= \sqrt{\frac{A(A-2C)a}{(A-B)(B-C)(A-C)}}\,\mathrm{cn}(u_2),\\
r &= \sqrt{\frac{(2B-A)Aa}{(A-B)(B-C)(A-C)}}\,\mathrm{dn}(u_2),
\end{aligned}
\tag{8.84}
$$

where $u_2 = \Omega_2(t - t_0)$, $\Omega_2 = \sqrt{\frac{a}{A-B}}$ and γ is given by

$$\gamma_1 = -1 + \frac{A\,\mathrm{sn}^2(u_2)}{A - C},$$
$$\gamma_2 = -\sqrt{\frac{A(2B - A)}{(B - C)(A - C)}}\,\mathrm{sn}(u_2)\,\mathrm{cn}(u_2),$$
$$\gamma_3 = \sqrt{\frac{A(A - 2C)}{(A - C)(B - C)}}\,\mathrm{sn}(u_2)\,\mathrm{dn}(u_2). \tag{8.85}$$

8.6.4 Some Properties of the Motion

8.6.4.1 1

We have suppressed some different combinations of signs of the square roots in (8.82)–(8.85) which are equally possible. Those may be obtained by applying the transformations $u \to u + 2K$ or $(t, \boldsymbol{\omega}, \boldsymbol{\gamma}) \to (-t, -\boldsymbol{\omega}, \boldsymbol{\gamma})$.

8.6.4.2 2

Both classes of motion are periodic. The periodic times for them are

$$T_i = \frac{4K(k_i)}{\Omega_i}, i = 1, 2. \tag{8.86}$$

The motions consist of vibrations with period T about x- and y-axes and unidirectional rotation with variable angular speed about the z-axis with period $T/2$. The two periods become zero on the line QR $(B = A)$ corresponding to the limiting case of infinitely large angular velocities.

8.6.4.3 3

One way to understand the geometry of motion, as was done in the Bobylev–Steklov case, is to examine what we call the "trajectory" or "orbit" of motion. That is the curve traced during the motion of the body by the tip of the vector $\boldsymbol{\gamma}$ on the Poisson sphere $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$. For the first class of motions this curve can be described as the intersection of the sphere with the elliptic cylinder

$$(2B - A)\,(\gamma_1 - 1)\,[(A - C)\,\gamma_1 + C] + A(B - C)\gamma_2^2 = 0. \tag{8.87}$$

Noting that for all values of the parameters the point $P_1(1, 0, 0)$ lies on that intersection, so that this point is common between all orbits of the first class. Moreover, the

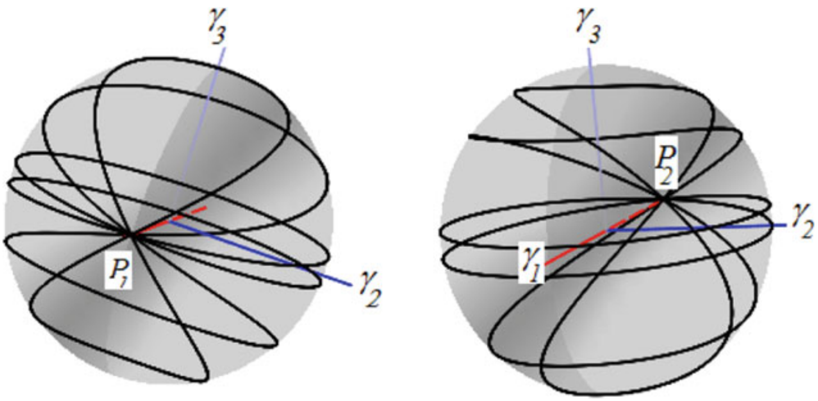


Fig. 8.10 **a** Three class 1 orbits passing through P_1. **b** Three class 2 orbits passing through P_2

projection of the orbit on the yz-plane in the vicinity of the point P_1 is described by the equation

$$\gamma_3^2 - \frac{A-2C}{2B-A}\gamma_2^2 = 0 \tag{8.88}$$

representing two line segments intersecting at that point. The generic orbits of motions of the first class on the Poisson sphere are Fig. 8-shaped curves, all with node at P_1 (See Fig. 8.10a). It is easy to show that orbits of the second class of motions are also Fig. 8 curves, but with their node at $P_2(-1, 0, 0)$.

8.6.4.4 4

On the border line PS both classes of motion degenerate into plane (pendulum-like) motions with $p = q = 0$ about the z-axis, which is taking a horizontal position. Those are rotations with a variable but unidirectional angular velocity

$$r = 2\sqrt{\frac{(B-C)a}{C(B-2C)}}\,\mathrm{dn}(u_1, k_1) \text{ for the first class of motions },$$
$$r = 2\sqrt{\frac{a}{2C-B}}\,\mathrm{dn}(u_2, k_2) \quad \text{for the second class of motions },$$

where $\Omega_1, \Omega_2, k_1, k_2$ are calculated for $A = 2C$. The vertical unit vector is given by

$$\gamma_1 = -1 + 2\,\mathrm{sn}^2(u_2),\ \gamma_2 = -2\,\mathrm{sn}(u_2)\,\mathrm{cn}(u_2),\ \gamma_3 = 0.$$

The energy constant for the first class of motions varies on QS from $3a$ at S to infinity at Q, while for the second class of motions it varies on PQ from a at P to infinity at Q. Note that on the energy level $h = a$, the plane motion becomes asymptotic to the upper (unstable) equilibrium position.

On the line PS in the plane of inertia parameters, the two halves of Fig. 8 orbits coincide and degenerate into circular orbits $\gamma_3 = 0$, corresponding to pendulum-like motion. Figure 8.10a, b shows two sequences of three Fig. 8 orbits of classes 1 and 2, respectively, with the last one close to the circular section $\gamma_3 = 0$.

8.6.5 Exercises

1- Substituting γ_2, γ_3 from (8.78) with n_2, n_3 as given in (8.79) in Eqs. (8.75), the last equations take the form

$$\begin{aligned} A\dot{p} + \frac{1}{2}(C' - B')qr &= 0, \\ B'\dot{q} + \frac{1}{2}\left(A - C'\right)pr &= 0, \\ C'\dot{r} + \frac{1}{2}(B' - A)pq &= 0, \end{aligned} \tag{8.89}$$

in which $B' = 2B - A$, $C' = 2C - A$. Those equations can be written in the vector form

$$\dot{\mathbf{G}}' + \frac{1}{2}\boldsymbol{\omega} \times \mathbf{G}' = 0, \tag{8.90}$$

where $\mathbf{G}' = (Ap, B'q, C'r)$. This form shows clearly that the equations are identical with the equations of motion about a fixed point by inertia of a hypothetical body with inertia matrix $\mathbf{I}' = \text{diag}(A, B', C')$ and angular velocity $\boldsymbol{\omega}$ referred to a system of axes moving with angular velocity $\frac{1}{2}\boldsymbol{\omega}$. However, this analogy should be taken with caution. The constants of motion of the hypothetical body are not arbitrary:

(a) Show that the analog of the energy integral corresponding to Steklov's solution has the value zero.

$$\frac{1}{2}(Ap^2 + B'q^2 + C'r^2) = 0.$$

This makes no contradiction, since the hypothetical body has one negative moment of inertia $C' = 2C - A < 0$ for both of Steklov's families of solutions.

(b) The angular momentum vector $\mathbf{G}'$ of the hypothetical body is a constant vector in the coordinate system moving with angular velocity $\frac{1}{2}\boldsymbol{\omega}$ about O. Then its square modulus $\mathbf{G}'^2$ is constant in all coordinate systems. Show also that the analog of the integral of the angular momentum corresponding to Steklov's solution has the value

$$\mathbf{G}'^2 = A^2p^2 + B'^2q^2 + C'^2r^2 = 2\epsilon Aa\frac{(A-2C)(2B-A)}{(A-C)(B-A)}.$$

The right-hand side is positive for both classes of Steklov's motions.

8.7 Goryachev's Case [114] (1899)

8.7.1 Conditions and Solution

Like in the Bobylev–Steklov and Steklov cases, the centre of mass of the body is assumed on one of the principal axes of inertia at the fixed point. Without loss of generality we take this axis to be the x-axis, so that the motion is described by the same equations of the previous section (8.75), (8.76) and the same integrals (8.77). For a solution, Goryachev tries two relations of the form

$$\gamma_2 = pq(n_1 + n_2p^2), \gamma_3 = n_3pr \tag{8.91}$$

and finds that this is possible only when the principal moments of inertia of the body are subject to the restriction

$$A(9B - 8C) = 16C(B - C). \tag{8.92}$$

In the plane of inertia parameters, the condition (8.92) is satisfied on the curve $PQNR$ (Fig. 8.11), so that the character of motion of a body is determined by the point representing the body on that curve. Denoting the ratio C/A by c and solving (8.92) for B one can write

$$C = Ac, B = A\frac{8c(2c-1)}{16c-9}. \tag{8.93}$$

Regarding triangle inequalities, the parameter c ranges from $\frac{3}{5}$ at P to ∞ at R (the point at infinity).

From the equations of motion, it follows also that the areas and energy parameters of the motion must take the values

$$\begin{aligned} f &= 0, \\ h &= 2\varepsilon a\frac{(2c-1)(9-56c+64c^2)}{(3-4c)(3-8c)(3-4c)}, \end{aligned} \tag{8.94}$$

where

$$\varepsilon = \pm 1.$$

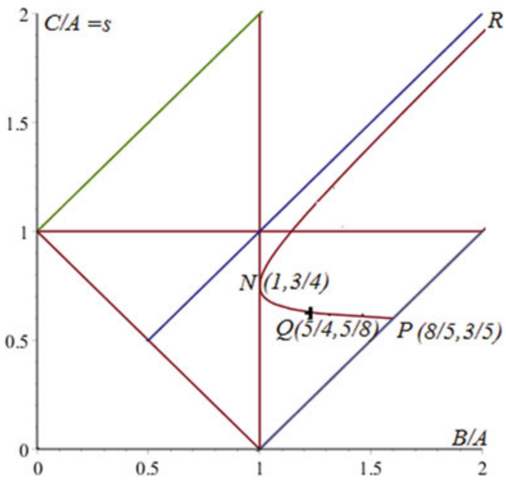


Fig. 8.11 Goryachev's condition (8.92) is satisfied on the hyperbolic branch $PNQR$

Under the above conditions, five of Euler–Poisson's variables can be expressed in terms of p as

$$\begin{aligned}
q &= \frac{16c-9}{4}\sqrt{\frac{32\varepsilon(2c-1)a}{c(4c-3)(8c-5)(8c-3)A} - \frac{p^2}{c^2}},\\
r &= \frac{1}{8}\{-\frac{128\varepsilon(16c-9)a}{c(4c-3)A} + 32\frac{(4c-3)(48c^2-44c+9)}{c^2(2c-1)}p^2\\
&\quad -\frac{\varepsilon(8c-5)(8c-3)(4c-1)(4c-3)^3A}{c^3(2c-1)^2a}p^4\}^{1/2},\\
\gamma_1 &= \varepsilon - \frac{(4c-3)(16c^2-15c+3)A}{2c(2c-1)a}p^2\\
&\quad -\frac{(8c-3)(8c-5)(4c-1)(4c-3)^3A^2}{128c^2(2c-1)^2a^2}p^4,\\
\gamma_2 &= \frac{(4c-3)(8c-3)(8c-5)A}{2(2c-1)(16c-9)a}pq[-1+\varepsilon\frac{(4c-1)(4c-3)^2A}{16c(2c-1)a}p^2],\\
\gamma_3 &= \frac{(3-4c)A}{2a}pr,
\end{aligned} \tag{8.95}$$

while p is determined as a function of time from the relation

$$\int^p \frac{dp}{\sqrt{F(p)}} = \frac{B-C}{A}t, \tag{8.96}$$

where $F(p) = q^2(p)r^2(p)$ is a polynomial of the sixth degree in p.

It remains to ensure that the two square roots figuring in the expressions for q and r in (8.95) take only real values, depending on the ratio c of the principal moments of inertia of the body and the sign of ε. Regarding the first expression in (8.95), the arc PR is divided into three parts:

(1) PQ, on which $c\epsilon i_0 = [\frac{3}{5}, \frac{5}{8})$. On this arc, one should choose $\varepsilon = 1$ and then $h\epsilon[\frac{26}{9}a, \infty)$.
(2) QN, on which $c\epsilon(\frac{5}{8}, \frac{3}{4})$. On this arc, if one chooses $\varepsilon = -1$, q can take real values. However, in that case, all the three coefficients under the square root sign in the expression for r become negative, so that r takes only imaginary values.
(3) NR, with $c\epsilon(\frac{3}{4}, \infty)$ and $\varepsilon = 1$. On this arc, it can be shown that $h < -a$, so that the kinetic energy of the body is negative.

Thus, the motion is possible only for bodies corresponding to points of the arc PQ with the choice $\varepsilon = 1$.

In fact, with the use of the change of variable

$$p = 4\sqrt{\frac{2c(2c-1)a}{(3-4c)(8c-3)(5-8c)A}}\,v, \tag{8.97}$$

the relation (8.96) is rendered to

$$2\sqrt{\frac{(4c-1)(3-4c)a}{c(8c-3)(5-8c)A}}\,t = \int \frac{dv}{\sqrt{(v-v_0)v(v_1-v)(1-v)}}, \tag{8.98}$$

where

$$v_{0,1} = -1/2\,\frac{48\,c^2 - 44\,c + 9}{(4\,c-1)\,(3-4c)} \mp 1/4\,\frac{\sqrt{2\left(9 - 64c + 160\,c^2 - 128\,c^3\right)}}{(4\,c-1)\sqrt{(3-4\,c)}}. \tag{8.99}$$

We first note that, as shown in Fig. 8.12, on the interval $i_0 = [\frac{3}{5}, \frac{5}{8})$, the roots of the fourth-degree polynomial under the square root sign in (8.98) have the order $v_0 < 0 < v_1 < 1$.

The polynomial under the quadratic root sign in (8.98) has degree four, and thus, the integral is elliptic of the first kind. From tables of integrals, e.g. [130], we evaluate this integral and hence obtain the inversion formula

$$v = \frac{n\,\mathrm{sn}^2(u,k)}{1 - m\,\mathrm{sn}^2(u,k)}, \tag{8.100}$$

where

$$u = \mu t + u_0,$$

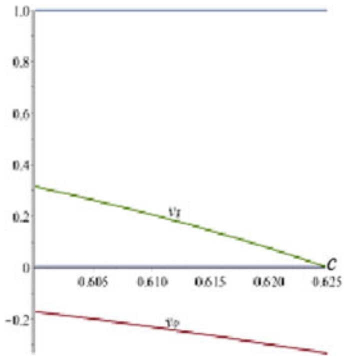


Fig. 8.12 The order of the roots of the fourth-degree polynomial in (8.98)

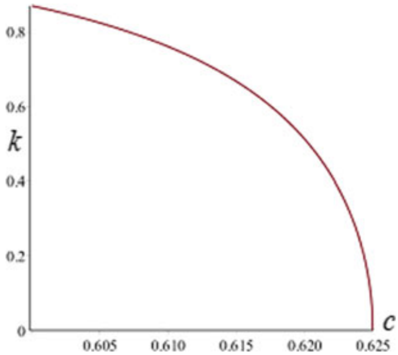


Fig. 8.13 k v. c

$$
\begin{aligned}
\mu &= \frac{\sqrt[4]{(4c-3)(128c^3-160c^2+64c-9)}\sqrt{a}}{\sqrt{2c(8c-3)(5-8c)A}},\\
k &= \sqrt{\frac{v_1(1-v_0)}{v_1-v_0}}\\
&= \sqrt{\frac{1}{2}+\frac{\sqrt{2}(256c^3-320c^2+144c-27)}{8(4c-3)^{3/2}\sqrt{128c^3-160c^2+64c-9}}},\\
n &= \frac{2^{3/2}}{16}\frac{(8c-5)(8c-3)(16c-9)}{\sqrt{(4c-3)(128c^3-160c^2+64c-9)}},\\
m &= \frac{1}{2}[1-\frac{\sqrt{2}(48c^2-44c+9)}{\sqrt{(4c-3)(128c^3-160c^2+64c-9)}}],
\end{aligned}
\tag{8.101}
$$

and u_0 is an arbitrary constant. Figure 8.13 shows the graph for $k(c)$ as drawn from the last expression. It ranges from 0.87064540 at $c=\frac{3}{5}$ to 0 at $c=\frac{5}{8}$.

Using (8.100), we express the components of the angular velocity as follows:

$$p = p_1\sqrt{\frac{a}{A}}\frac{\operatorname{sn}(u,k)}{\sqrt{1-m\operatorname{sn}^2(u,k)}},$$
$$q = q_1\sqrt{\frac{a}{A}}\frac{\operatorname{dn}(u,k)}{\sqrt{1-m\operatorname{sn}^2(u,k)}},$$
$$r = r_1\sqrt{\frac{a}{A}}\frac{\operatorname{cn}(u,k)}{1-m\operatorname{sn}^2(u,k)}, \tag{8.102}$$

in which

$$p_1 = \frac{2\sqrt{\sqrt{2}c(2c-1)(16c-9)}}{\sqrt[4]{(4c-3)(128c^3-160c^2+64c-9)}},$$
$$q_1 = (16c-9)\sqrt{\frac{2(2c-1)}{c(4c-3)(8c-3)(8c-5)}},$$
$$r_1 = \sqrt{\frac{2(16c-9)}{c(3-4c)}}. \tag{8.103}$$

The components of the vertical unit vector $\boldsymbol{\gamma}$ take the following form:

$$\gamma_1 = 1-2\sqrt{2}\frac{(16c-9)(16c^2-15c+3)}{\sqrt{(4c-3)^3(128c^3-160c^2+64c-9)}}\frac{\operatorname{sn}^2(u,k)}{1-m\operatorname{sn}^2(u,k)}$$
$$-\frac{(4c-1)(8c-3)(5-8c)(16c-9)^2}{4(4c-3)^2(128c^3-160c^2+64c-9)}\frac{\operatorname{sn}^4(u,k)}{(1-m\operatorname{sn}^2(u,k))^2},$$
$$\gamma_2 = \frac{(8c-3)(8c-5)(4c-3)}{8((2c-1)(16c-9))}p_1q_1\frac{\operatorname{sn}(u,k)\operatorname{dn}(u,k)}{(1-m\operatorname{sn}^2(u,k))}$$
$$\times[\frac{\sqrt{2}(4c-1)(16c-9)}{\sqrt{(4c-3)\left(128c^3-160c^2+64c-9\right)}}\frac{\operatorname{sn}^2(u,k)}{1-m\operatorname{sn}^2(u,k)}-4],$$
$$\gamma_3 = \frac{3-4c}{2}p_1r_1\frac{\operatorname{sn}(u,k)\operatorname{cn}(u,k)}{(1-m\operatorname{sn}^2(u,k))^{3/2}}. \tag{8.104}$$

8.7.2 *Properties of the Motion*

8.7.2.1 The Initial Motion

To obtain symmetric views in the graphics, we give the constant u_0 the value 0. This corresponds to the choice of the initial moment $t = 0$ as the one at which $p = 0$. At this same moment, from (8.104), we have $\boldsymbol{\gamma} = (1, 0, 0)$. The x-axis carrying the centre of mass begins motion from a position vertically above the fixed point. The

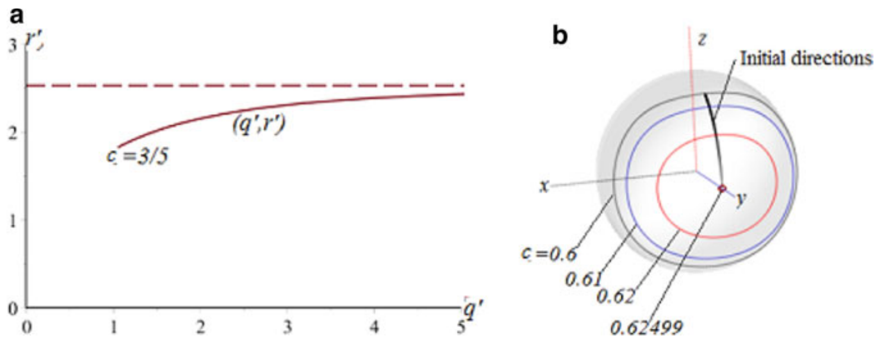


Fig. 8.14 **a** The initial angular velocity (q', r') as c varies. **b** Directions of angular velocity during motion as c varies

initial angular velocity that should be given to the body to commence a motion of the Goryachev type lies in the yz-plane and its direction depends on the parameter c. Figure 8.14 shows the variation of the dimensionless quantities $(q' = \sqrt{\frac{A}{a}}q, r' = \sqrt{\frac{A}{a}}r)$ as c varies on the interval i_0. From this figure, it is obvious that as c tends to $\frac{5}{8}$, q tends to infinity while r tends to a finite limit $\frac{4\sqrt{10}}{5}\sqrt{\frac{a}{A}} \simeq 2.5298\sqrt{\frac{a}{A}}$.

8.7.2.2 Periodicity of the Motion

We now clarify the general character of motion after this initial setting. We readily note that the second component q of the angular velocity does not change sign. In a time period $\frac{2K(k)}{\mu}$, it ranges from a minimum q_0 to a maximum $\frac{q_0}{\sqrt{1-v_1}}$ and then to q_0 again. The motion of the body is composed of a rotation around the y-axis and vibrations around the x, z-axes. The whole motion is periodic with period

$$T = \frac{4K(k)}{\mu} = 4K(k)\frac{\sqrt{2c(8c-3)(5-8c)A/a}}{\sqrt[4]{(3-4c)(9-64c+160c^2-128c^3)}}. \tag{8.105}$$

The period of motion decreases monotonically from its value at $c = 3/5$ to zero at $c = 5/8$, corresponding to very fast uniform rotation about the y-axis.

In Fig. 8.14b, we show four closed curves representing the traces on the unit sphere fixed in the body by the unit vector $\frac{\omega}{\omega}$ in the direction of the angular velocity along a period of the motion. All curves are closed around the y-axis. As c approaches $5/8$, the curve becomes a very small one shrinking to that axis.

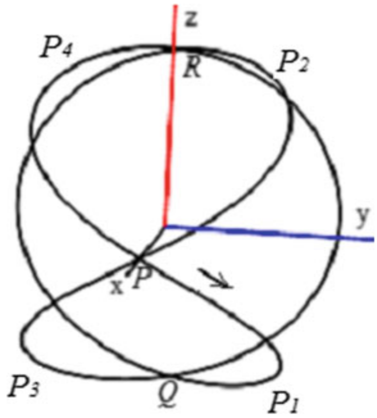


Fig. 8.15 Space view of the trajectory for $c = 0.6$ on a transparent Poisson sphere

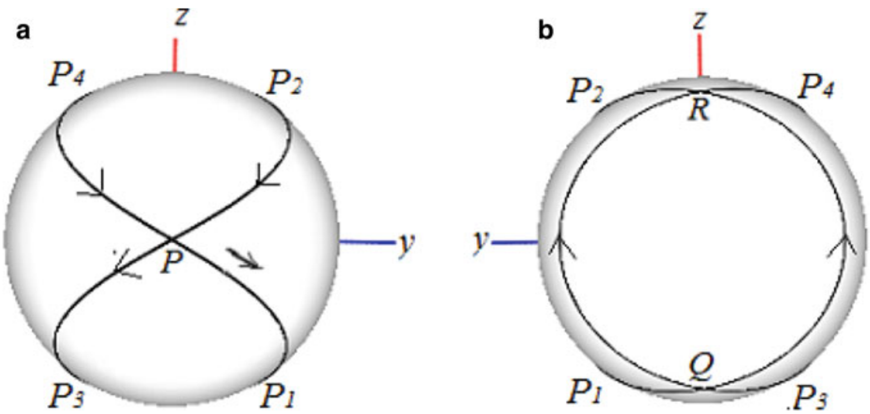


Fig. 8.16 Motion direction on two halves of an opaque sphere

8.7.2.3 Orbits on the Poisson Sphere

Now, we turn to the picture of the trajectories of the motion, the trace of the vertical vector γ on the Poisson sphere. As we have seen above, the point $P(1, 0, 0)$ is common between all orbits. Figure 8.15 shows the space view of the orbit corresponding to the value $c = 3/5$, as an example of the generic orbits. It begins from the point P on the x-axis and goes through the points P_1, Q, R, P_2, P, P_3, Q, R, P_4 and then closes at P. The orbit has three self-intersection points P, Q and R, so that it makes three loops, two small and one larger. The direction of motion is shown on all arcs of the orbit on the two halves of an opaque sphere (Fig. 8.16a, b).

Although those orbits are not simple in the space view, they have two planes of symmetry: xz and xy. The projections of four orbits on the xz-plane are shown in Fig. 8.17 for values of c ranging from the beginning to near the end of the interval

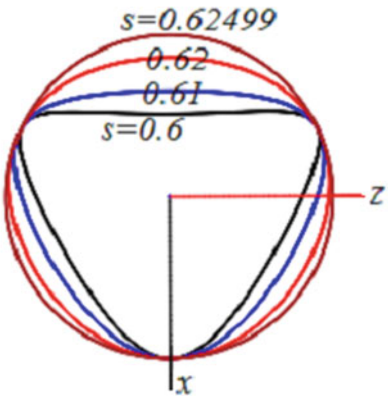


Fig. 8.17 Projections of trajectories on the xz-plane corresponding to four values of c (View from the top of y-axis)

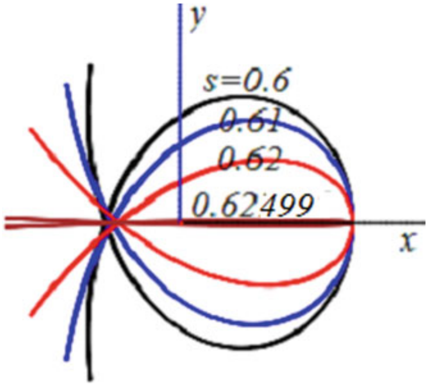


Fig. 8.18 Projections of trajectories on the xy-plane corresponding to four values of c (View from the top of z-axis)

$[\frac{3}{5}, \frac{5}{8})$. Each orbit can be seen as the intersection of the Poisson sphere with a cylindrical surface, which is parallel to the y-axis and touches the unit sphere at three points. With increasing c, the projection becomes wider and encloses the orbits with smaller values of c. As $c \to 5/8$, the projection of the orbit approaches a circle of unit radius. In fact, the limiting orbit at this value of c is the circle $\gamma_2 = 0$, corresponding to very fast rotations about the y-axis, which takes a horizontal position.

Figure 8.18 depicts the projections of the same orbits on the xy-plane. This projection begins wider at $c = 3/5$ and becomes narrower with increasing c to coincide in the limit, as $c \to 5/8$, with the diameter of the sphere on the x-axis. Figure 8.19 shows the same orbits on the front and the rear halves of the sphere.

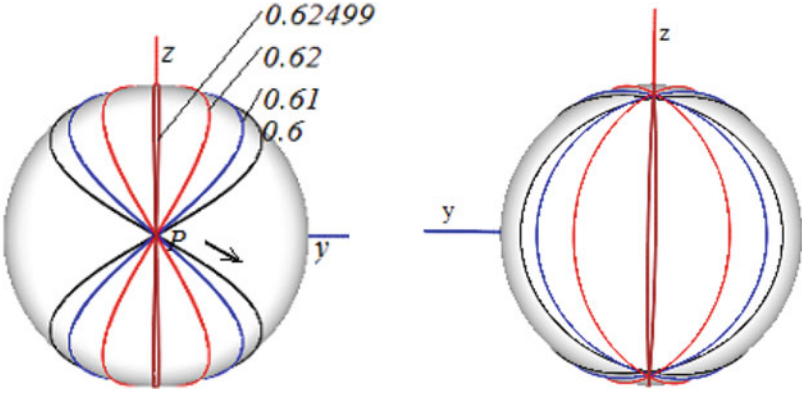


Fig. 8.19 Front and rear views of the four trajectories on the Poisson sphere

8.7.3 The History of Goryachev's Case

(1) In his 1899 work [114], Goryachev obtained the condition (8.92) and the expressions (8.95), but he erroneously concluded that the integral (8.96) is hyper-elliptic.
(2) In 1965, Kharlamov [196] derived a relation equivalent to (8.98) and concluded that the motion is periodic. He also evaluated the time period of the motion, but he did not proceed to invert the elliptic integral.
(3) In 1974, Stepanova [349] gave an explicit inversion formula of the integral (8.96) in the form

$$p = p_0\sqrt{\frac{1 - \mathrm{sn}(\chi t, k_0)}{1 + n_0\,\mathrm{sn}(\chi t, k_0)}}$$

and expressed the six Euler–Poisson variables as functions of time. Unfortunately, the solution given by Stepanova is not adequate for describing the motion, since it involves odd powers of the auxiliary quantity p, which is not differentiable at all its zeros and takes the wrong sign on half of each period. As a result, expressions for p, r and γ_2 are not analytic functions of time. Moreover, expressions for the parameters m_0, n_0, χ and the modulus k_0 of the Jacobi elliptic functions are extremely complicated. In fact, we have established that elliptic functions used in (8.102)–(8.104) and the elliptic functions used in Stepanova's solution are connected by second-order transformation of the Landen type (See, e.g. [75]).

The expression for γ_2 in [349] is written erroneously. A corrected expression for γ_2 is given in [108] (Sect. 8.3). The authors of [108] remarked also that Stepanova's solution is two-valued and that its period is only half the period found earlier by

Kharlamov based on the analysis of the quadrature in (8.98), but they did not realize the need to modify the way of inversion of the integral used by Stepanova.

8.8 Chaplygin's Case (1904)

As in the previous two cases, the centre of mass of the body is assumed on the x-axis, so that the motion is described by same Eqs. (8.75), (8.76) and integrals (8.77). In Chaplygin's own words in his original paper [54], one reaches the conclusion that the present case was found by certain extension of the method used in Steklov's and Goryachev's cases. Generalizing (8.78) and (8.91), Chaplygin tries two relations of the form

$$a\gamma_2 = q(\alpha p + \lambda p^n), a\gamma_3 = r(\beta p + \mu p^n), \tag{8.106}$$

where n, α, β, λ and μ are constants to be determined. Equations (8.75) give

$$\begin{aligned} \frac{B}{A}(B-C)q^2 &= (C-A-\beta)p^2 - \frac{2\mu}{n+1}p^{n+1} + Bk, \\ \frac{C}{A}(B-C)r^2 &= (A-B+\alpha)p^2 + \frac{2\lambda}{n+1}p^{n+1} + Cl, \end{aligned} \tag{8.107}$$

in which k, l are integration constants. Proceeding to satisfy Eqs. (8.76), Chaplygin reaches the conclusion that three choices are possible:

(1) $\lambda = \mu = 0$, which leads to the case of Steklov.
(2) $\lambda = 0, n = 3$, which leads to Goryachev's case.
(3) $n = -\frac{1}{3}, k = l = 0$, which gives the new case known now under his name.

To make the nature much clearer, we shall now describe this case in a slightly modified way than that of Chaplygin [54] and several authors who mostly repeated Chaplygin's approach [108, 195].

Substituting expressions (8.106) into the integrals of motion and Euler's equations, we obtain the values of the parameters

$$\begin{aligned} h &= 0, & f &= 0, \\ \alpha &= \frac{(B-A)\,(A-C)}{(A-2\,C)}, & \beta &= \frac{(B-A)\,(A-C)}{(A-2\,B)}, \\ \lambda &= \frac{3A-2B}{2C-A}Cs, & \mu &= \frac{3A-2C}{2B-A}Bs, \end{aligned} \tag{8.108}$$

s being a constant, determined from the equation

$$9A^3(2B+2C-3A)\nu^3 = 4a^2\frac{(2B-A)^2(2C-A)^2}{(3A-2B)(3A-2C)}, \tag{8.109}$$

and the moments of inertia are subject to the restriction

$$9(2B - A)(2C - A) = 4BC. \tag{8.110}$$

The last condition determines the curve PQR in the plane $(B/A, C/A)$ (Fig. 8.20). This curve can be parametrized by a parameter c so that $C = Ac$, $B = \frac{9A(2c-1)}{2(16c-9)}$. The edge point P, at which the equality $B = A + C$ is satisfied, corresponds to $c = \frac{1+\sqrt{73}}{16} = 0.5965$.

On the other hand, to avoid the appearance of cubic roots, we introduce an intermediate variable u by the relation $p = u^3$. The Euler–Poisson variables can be written in the following form:

$$\begin{aligned}
p &= u^3,\\
q &= -\frac{(9-16c)\sqrt{3s(3-2c)-(1-c)u^4}\,u}{\sqrt{c(8c-3)(3-4c)}},\\
r &= \frac{\sqrt{36s(3-5c)-(9-14c)u^4}\,u}{\sqrt{(8c-3)(3-4c)(2c-1)}},\\
\gamma_1 &= -1/4\,\frac{Au^2\left(-(14c-9)(c-1)u^4+9\left(12c^2-22c+9\right)s\right)}{ca(2c-1)},\\
\gamma_2 &= -\frac{A\sqrt{3s(3-2c)-(1-c)u^4}\left(12cs(3-5c)-u^4(9-14c)(1-c)\right)}{2a\sqrt{c}\sqrt{8c-3}\sqrt{3-4c}\,(2c-1)},\\
\gamma_3 &= -\frac{A}{4a}\,\frac{\left(-(14c-9)(c-1)u^4+9(2c-1)s(2c-3)\right)}{c\sqrt{8c-3}\sqrt{3-4c}\sqrt{2c-1}}\times\\
&\quad\times\sqrt{36(3-5c)s-(9-14c)u^4},
\end{aligned} \tag{8.111}$$

where s is now given from

$$s^3 = \frac{4a^2c^2(2c-1)^2}{27A^2(3-4c)^2(3-5c)(3-2c)}. \tag{8.112}$$

In order that all the variables be real, the parameter c must be restricted to the interval $[c_0, c^*] \equiv [\frac{1+\sqrt{73}}{16}, \frac{3}{5}]$ corresponding to the arc PQ in Fig. 8.20.

The change of the variable u with time is determined from the equation

$$\dot{u} = \frac{\sqrt{36s(3-5c)-(9-14c)u^4}\sqrt{3s(3-2c)-(1-c)u^4}}{6\sqrt{c(2c-1)}}, \tag{8.113}$$

so that the time can be evaluated by the expression

$$t = 6\sqrt{\frac{c(2c-1)A}{a}}\int\frac{du}{\sqrt{36s(3-5c)-(9-14c)u^4}\sqrt{3s(3-2c)-(1-c)u^4}}. \tag{8.114}$$

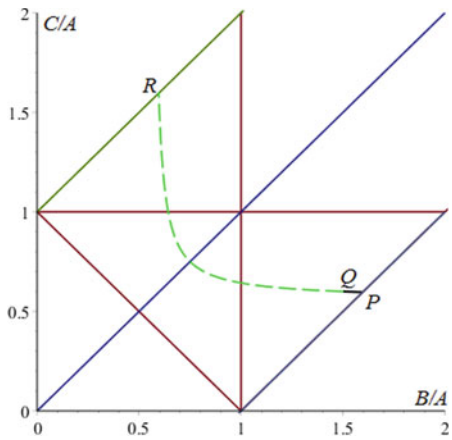


Fig. 8.20 Physically admissible moments of inertia lie on the arc PQ

Remark. An equation equivalent to this equation was obtained by Chaplygin [54], who proceeded to express the last integral as the sum of two elliptic integrals (see exercises). He also obtained the equation of the moving hodograph of the angular velocity as a cone of the second degree and the fixed hodograph as a surface of revolution. The motion of the body should be represented geometrically as rolling the first hodograph on the second. This description, however, turns out to be inaccurate, because, as we shall see below, the actual hodograph is only a part of the movable cone. The full quadratic cone never makes any complete revolutions.

Remark. Kharlamov [196] introduced instead of u in (8.111) a variable σ by the relation $p = \sigma^{3/2}$. This led to some complications in his reasoning, but he was able to show that only a part of the movable hodograph (cone) rolls over an equal arc of the immovable hodograph in one direction and the body reaches an instant state of rest and then immediately begins rolling on the same arc in the opposite direction.

Here we shall go the same way as in the last few cases. Let us write (8.114) in the form

$$t = \frac{1}{s}\sqrt{\frac{c(2c-1)A}{(3-5c)\,(3-2c)\,a}}\int \frac{du}{\sqrt{1-\frac{u^4}{u_1^4}}\sqrt{1-\frac{u^4}{u_2^4}}}, \tag{8.115}$$

where

$$\begin{aligned} u_1 &= \frac{2^{2/3}3^{1/4}[c(2c-1)(3-5c)]^{1/6}}{(9-14c)^{1/4}(3-2c)^{1/12}(3-4c)^{1/6}}, \\ u_2 &= \frac{[2c(2c-1)(3-2c)]^{1/6}}{(1-c)^{1/4}(3-5c)^{1/12}(3-4c)^{1/6}}. \end{aligned} \tag{8.116}$$

Note that on the interval of change of c, we have $u_2 > u_1 \geq 0$. The variable u then takes its values on the interval $[-u_1, u_1]$. Thus, we can use the substitution

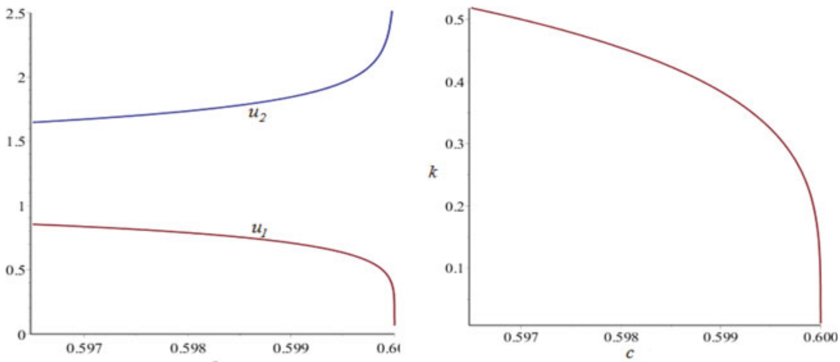
$$u = u_1 \sin\phi, \tag{8.117}$$

so that (8.115) becomes

$$t = \frac{3\sqrt{(3-4c)A/a}}{[3(3-2c)(9-14c)]^{1/4}} \int_0^{\phi} \frac{d\phi}{\sqrt{1+\sin^2\phi}\sqrt{1-k^4\sin^4\phi}}, \tag{8.118}$$

where $k = u_1/u_2$. We have chosen $\phi = 0$ at $t = 0$, so that we suppressed an immaterial arbitrary integration constant. Thus, ϕ monotonically increases with t and the Euler–Poisson variables take their final parametric form in terms of ϕ

$$\begin{aligned}
p &= \sqrt{\frac{a}{A}}\,\frac{4\cdot 3^{3/4}\sqrt{c}\sqrt{2\,c-1}\sqrt{3-5\,c}}{(9-14\,c)^{3/4}\sqrt[4]{3-2\,c}\sqrt{3-4\,c}}\sin^3\phi,\\
q &= \sqrt{\frac{a}{A}}\,\frac{2\sqrt[4]{3}\sqrt{2\,c-1}\sqrt[4]{3-2\,c}\,(16\,c-9)}{(9-14\,c)^{1/4}\,(4\,c-3)\,\sqrt{8\,c-3}}\sin(\phi)\sqrt{1-k^4\sin^4\phi},\\
r &= \sqrt{\frac{a}{A}}\,\frac{4\cdot 3^{3/4}\sqrt{c}\sqrt{3-5\,c}\sqrt{(\sin(\phi))^2+1}\,\sin(\phi)\cos(\phi)}{\sqrt[4]{3-2\,c}\,(4\,c-3)\sqrt[4]{9-14\,c}\sqrt{8\,c-3}},\\
\gamma_1 &= \frac{3\sqrt{3}\sin^2\phi[12\,c^2-22\,c+9-4\,(3-5\,c)\,(1-c)\sin^4\phi]}{\sqrt{3-2\,c}\,(3-4\,c)\,\sqrt{9-14\,c}},\\
\gamma_2 &= \frac{4\sqrt{c}\sqrt{3-5\,c}[c-3\,(1-c)\sin^4\phi]}{(3-4\,c)^{3/2}\,\sqrt{8\,c-3}}\sqrt{1-k^4\sin^4\phi},\\
\gamma_3 &= \frac{3\sqrt{3}\sqrt{2\,c-1}\cos\phi\sqrt{\sin^2\phi+1}}{\sqrt{2\,c-3}\,(4\,c-3)^{3/2}\,\sqrt{8\,c-3}}\times\\
&\quad\times[4\,(5\,c-3)\,(c-1)\sin^4\phi-(2\,c-1)\,(2\,c-3)].
\end{aligned} \tag{8.119}$$



k

8.8.1 *Properties of the Motion*

8.8.1.1 The Periodicity

The motion is periodic in time with period T

$$T = \frac{3\sqrt{(3-4c)A/a}}{[3(3-2c)(9-14c)]^{1/4}} \int_0^{2\pi} \frac{d\phi}{\sqrt{1+\sin^2\phi}\sqrt{1-k^4\sin^4\phi}}$$
$$= \frac{12\sqrt{(3-4c)A/a}}{[3(3-2c)(9-14c)]^{1/4}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1+\sin^2\phi}\sqrt{1-k^4\sin^4\phi}}. \quad (8.120)$$

8.8.1.2 The Hanging Centre of Mass

From (8.108) we have $h = 0$. The energy integral gives

$$\gamma_1 = -\frac{1}{2a}(Ap^2 + Bq^2 + Cr^2). \quad (8.121)$$

This means that the x-axis, which carries the centre of mass of the body, never rises over the horizontal plane passing through the fixed point. When the centre of mass touches that plane, the angular velocity of the body vanishes.

8.8.1.3 The Initial Motions

From the qualitative point of view, the motion varies slightly as c varies on the admissible period $[c_0, c^*]$. Equations (8.119) tell that at $t = 0 (\phi = 0)$ for all values of c the initial angular velocity vanishes $p = q = r = 0$. The body begins from a rest position. At this moment, the x-axis carrying the centre of mass occupies a horizontal position $\gamma_1 = 0$, and the positive y-axis makes with the horizontal an angle

$$\delta = \arctan \frac{4\sqrt{3}}{9} \frac{c^{3/2}\sqrt{3-5c}}{(2c-1)^{3/2}\sqrt{3-2c}}. \quad (8.122)$$

8.8.1.4 A Border Case—The Physical Pendulum Motion

For the border value c^* we have $\delta = 0$, i.e. the body begins motion with x, y horizontal and z vertical upwards. On the following motion, we have

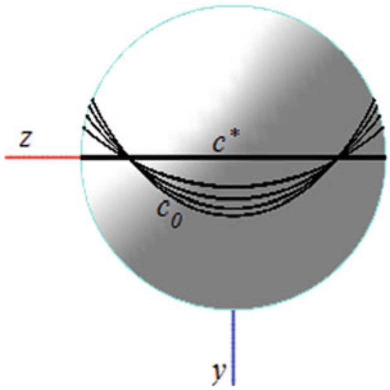


Fig. 8.21 Traces of γ on the Poisson sphere from the negative side of the x-axis. View from top of -ve x-axis

$$\omega = (0, -\frac{2}{\sqrt{3}} \sin\phi, 0),$$
$$\gamma = (-\sin^2\phi, 0, \cos\phi\sqrt{1+\sin^2\phi}).$$

The centre of mass begins to drop down, moving on a vertical circle, until it reaches the horizontal position opposite to its initial position and then reverses the direction of motion returning again to that position. The whole motion is just a planar vibration of the body as a pendulum about the y-axis which keeps a horizontal position. In this case, $k = 0$. The integral in the relation (8.118) turns into an elliptic integral of the first kind and, as expected, all variables in (8.119) render to elliptic functions of time.

8.8.1.5 The Orbits on the Poisson Sphere

Figures 8.21 and 8.22 depict the trajectories of the point of intersection of the vertical upward with the Poisson sphere for five equidistant values of c on the interval $[c_0, c^*]$. For $c = c^*$, the trajectory is the semi circle in the xz-plane. As c decreases to c_0, the trajectories become slightly deformed from that trajectory as seen from the two figures.

8.9 Kowalewski's Case [239] (1908)

Kowalewski assumed, as in the previous three cases, that the centre of mass of the body lies on the first principal axis of inertia. Thus, the problem is described, as in those cases, by the same equations of motion (8.75)–(8.76) and admit the same integrals (8.77).

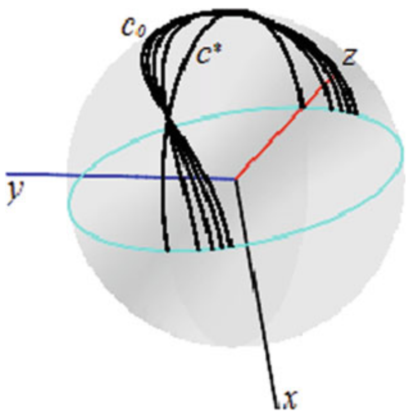


Fig. 8.22 The same trajectories on a transparent Poisson sphere

After the success of Goryachev and Chaplygin in finding new solvable cases using slight variations of Steklov's method, Kowalewski developed an extension of that method. With this method, he restored the three previous cases and found one more. For a detailed presentation of that method, the reader is referred to Kowalewski's original paper [239] or Leimanis' book. We shall directly use his ansatz for the new solution of the Euler–Poisson system of equations in the same way as in the last three cases. This expresses four of the variables in the form

$$\begin{aligned} \gamma_2 &= q(n_0 + n_1 p + n_2 p^2), \gamma_3 = r(m_0 + m_1 p), \\ q^2 &= q_0 + q_1 p + q_2 p^2, \\ r^2 &= r_0 + r_1 p + r_2 p^2 + r_3 p^3, \end{aligned} \tag{8.123}$$

and the fifth is obtained from the energy integral as

$$\gamma_1 = 1/a\{h - \frac{1}{2}[Ap^2 + Bq^2(p) + Cr^2(p)]\},$$

so that five variables are expressed in terms of p. Inserting this choice into the remaining two integrals of motion and the dynamical equations, we find that the following condition on the moments of inertia must be satisfied:

$$A = 18C\frac{B - C}{10B - 9C}. \tag{8.124}$$

In the plane of ratios of moments of inertia, this condition is satisfied on the hyperbolic branch PQR. The point $P(\frac{27}{17}, \frac{10}{17}) = P(1.5882, 0.58824)$ is determined by the triangle inequality $B = C + A$.

It will be more comfortable to use a parametric representation of the ratios of the moments of inertia (Fig. 8.23)

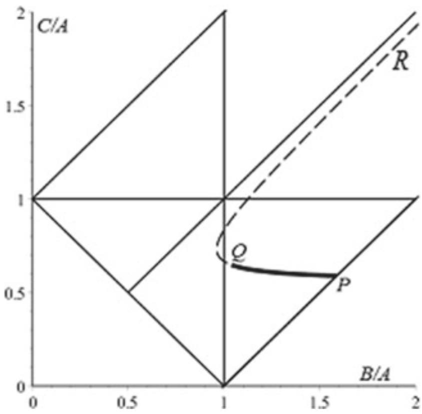


Fig. 8.23 Kowalewski's condition (8.124) satisfied on the arc PQR

$$C/A = c,\ B/A = \frac{9}{2}c\frac{2c-1}{9c-5}. \tag{8.125}$$

This will help writing formulas in a unique way using two parameters A and c. The point P corresponds to the minimum value of c, say, $c_0 = 10/17$.

For h, f, we get the values

$$h = -\frac{(2c-1)(3c-1)P_3(c)a}{\sqrt{(3\,c-2)\,P_9(c)}},$$
$$f = \frac{\sqrt{2Aa}\,(3-4\,c)\,P_3(c)\sqrt{c(2c-1)^3(3c-1)}}{[(3\,c-2)\,P_9^3(c)]^{1/4}}, \tag{8.126}$$

where

$$P_3(c) = 384\,c^3 - 624\,c^2 + 305\,c - 40,$$
$$P_9(c) = 1769472\,c^9 - 8257536\,c^8 + 16831488\,c^7 - 19642368\,c^6 + 14443596\,c^5$$
$$-6930624\,c^4 + 2167313\,c^3 - 425427\,c^2 + 47520\,c - 2300. \tag{8.127}$$

The Euler–Poisson variables can be written in terms of an auxiliary non-dimensional variable v as follows:

$$p = N\sqrt{\frac{a}{A}}v,$$
$$q = 1/3\,\frac{(9\,c-5)\,N}{c\,(2\,c-1)}\sqrt{\frac{a}{A}}\sqrt{\frac{R_2(v)}{(3\,c-1)\,(2-3\,c)}},$$
$$r = 1/2\,\frac{N}{c}\sqrt{\frac{a}{A}}\sqrt{R_3(v)},$$

$$\gamma_1 = \frac{N^2}{c}[2\ (3\,c-1)\,(-2+3\,c)^2\,v^3 - \frac{(-2+3\,c)\left(72\,c^2-71\,c+16\right)v^2}{2\,c-1}$$
$$+1/4\,\frac{\left(864\,c^3-1424\,c^2+767\,c-134\right)v}{(2\,c-1)^2}$$
$$-1/4\,\frac{384\,c^4-752\,c^3+561\,c^2-189\,c+24}{(2\,c-1)^3\,(3\,c-1)}], \tag{8.128}$$
$$\gamma_2 = \frac{N^2}{c\,(2\,c-1)}\sqrt{\frac{R_2(v)}{(3\,c-1)\,(2-3\,c)}}[\,(3\,c-1)\,(-2+3\,c)^2\,v^2$$
$$-\frac{(-2+3\,c)\left(30\,c^2-30\,c+7\right)v}{2\,c-1} + 1/8\,\frac{384\,c^3-624\,c^2+333\,c-58}{(2\,c-1)^2}],$$
$$\gamma_3 = 1/2\,\frac{N\,((2-3\,c)\,Nv+N)}{c}\sqrt{R_3(v)}, \tag{8.129}$$

where

$$N = (64\,\frac{c^2\,(-2+3\,c)\,(3\,c-1)^2\,(2\,c-1)^6}{P_9(c)})^{1/4},$$
$$R_2(v) = 1-4\,(3c-1)\,(2c-1)\,(2-3c)\,[(2c-1)\,v^2-2v],$$
$$R_3(v) = -16\ (3c-1)\,(3c-2)^2\,v^3 + 4\,\frac{(3c-2)\left(168c^2-164c+37\right)v^2}{2\,c-1}$$
$$-6\,\frac{\left(384\,c^3-624\,c^2+333\,c-58\right)v}{(2c-1)^2}$$
$$-\frac{384\,c^4-1112\,c^3+1101\,c^2-453\,c+66}{(3c-1)\,(3c-2)\,(2c-1)^3}. \tag{8.130}$$

We note first that the polynomial $R_2(v)$ figuring under square root in the expressions for q and γ_2 in (8.129) has a negative leading coefficient. If we denote by $v1$, $v2$ the roots of R_2, we can write

$$q = \frac{2(9c-5)}{3c}\sqrt{(v-v_1)(v_2-v)}, \tag{8.131}$$

where

$$v_1 = \frac{1}{(2c-1)}\left(1-\frac{1}{2}\sqrt{\frac{\left(-36c+36c^2+7\right)}{(3c-1)\,(3c-2)}}\right),$$
$$v_2 = \frac{1}{(2c-1)}\left(1+\frac{1}{2}\sqrt{\frac{\left(-36c+36c^2+7\right)}{(3c-1)\,(3c-2)}}\right). \tag{8.132}$$

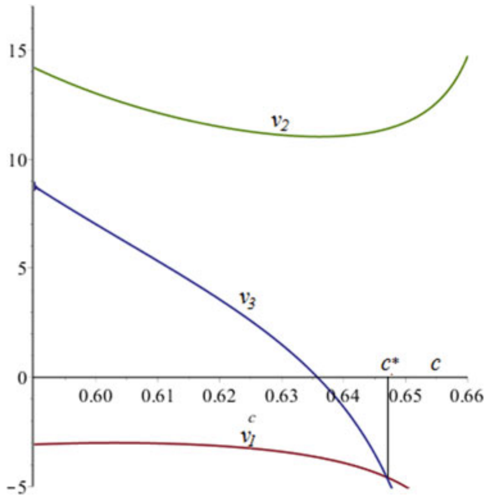


Fig. 8.24 The roots of P_2 and P_3

As $v2 > v1$, the variable v can take its values only in the interval $[v_1, v_2]$.

The cubic R_3 figuring under root in expressions for r, γ_3 has negative discriminant on $[v_1, v_2]$, so that it has one real root, say, v_3 and two complex conjugate roots v_4 and $\bar{v}_4$. We can write

$$r = -\frac{2(2-3c)\sqrt{3c-1}}{c}\sqrt{(v_3 - v)(v^2 - 2\,\mathrm{Re}(v_4) + |v_4|^2)}. \tag{8.133}$$

It follows that $v \leq v_3$, and thus v changes on the interval $[v_1, \min(v_2, v_3)]$. But comparing the graphs of v_1, v_2, v_3 in Fig. 8.24, we finally have

$$v1 \leq v \leq v_3 < v2. \tag{8.134}$$

From the same figure, we see that with growing c the interval $[v1, v_3]$ shrinks to zero at some value c^*, say, of c. This value at which the root v_3 of R_3 equals the root v_1 of R_2 is a root of the resultant of R_2 and R_2. This can be easily seen to be a root of the eighth-degree polynomial

$$\begin{aligned} P_8 = {} & 147456\,c^8 - 563712\,c^7 + 920032\,c^6 - 836796\,c^5 + 463806\,c^4 \\ & -160441\,c^3 + 33838\,c^2 - 3980\,c + 200. \end{aligned} \tag{8.135}$$

Numerical solution gives the value

$$c^* = 0.6469380234. \tag{8.136}$$

The value c^* corresponds to the point $Q(1.040241051, 0.6469380234)$ in Fig. 8.23, so that the feasible points constitute the arc PQ.

The relation of the variable v with time is obtained by substituting p, q, r into the first equation of (8.75). We obtain

$$\begin{aligned}\dot{v} &= \frac{c}{2(9c-5)}\sqrt{\frac{A}{a}}qr \\ &= \frac{2(2-3c)\sqrt{3c-1}}{3c}\sqrt{\frac{a}{A}}\sqrt{(v-v_1)(v_2-v)}\sqrt{(v_3-v)|v-v_4|^2)},\end{aligned} \quad (8.137)$$

so that, after separating variables and integration, we have t given by a hyper-elliptic integral

$$t = \frac{3c\sqrt{A/a}}{2(2-3c)\sqrt{3c-1}}\int\frac{dv}{\sqrt{(v-v_1)(v_2-v)}\sqrt{(v_3-v)}|v-v_4|}. \quad (8.138)$$

In this integral if v starts from v_1, it will increase up to v_3, decreases to v_1 and then increases again. The motion is periodic.

We now introduce a new variable, the angle ϕ, by the relation

$$v = v_1 + (v_3 - v_1)\sin^2\phi, \quad (8.139)$$

so that $\phi = 0$ at the initial time $t = 0$. The expression (8.138) becomes

$$\begin{aligned}t = &\frac{3c\sqrt{A/a}}{(2-3c)\sqrt{3c-1}}\times \\ &\times\int_0^\phi \frac{d\phi}{\sqrt{[v_2-v_1-(v_3-v_1)\sin^2\phi]}|v_1-v_4+(v_3-v_1)\sin^2\phi|}.\end{aligned} \quad (8.140)$$

The variable ϕ increases monotonically with time. Components of the angular velocity become

$$\begin{aligned}p &= N\sqrt{\frac{a}{A}}[v_1 + (v_3-v_1)\sin^2\phi], \\ q &= \frac{2}{3c}(9c-5)\sqrt{v_3-v_1}\sqrt{v_2-v_1-(v_3-v_1)\sin^2\phi}\sin\phi, \\ r &= -\frac{2}{c}(2-3c)\sqrt{(3c-1)(v_3-v_1)}|v_1-v_4+(v_3-v_1)\sin^2\phi|\cos\phi.\end{aligned} \quad (8.141)$$

The components of $\boldsymbol{\gamma}$ can be written accordingly. The overall period of the motion is 2π in ϕ. The time period T of the motion is given by

$$
\begin{aligned}
T &= \frac{3c\sqrt{A/a}}{(2-3c)\sqrt{3c-1}} \times \\
&\quad \times \int_0^{2\pi} \frac{d\phi}{\sqrt{[v_2 - v_1 - (v_3 - v_1)\sin^2\phi]|v_1 - v_4 + (v_3 - v_1)\sin^2\phi|}} \\
&= \frac{12c\sqrt{A/a}}{(2-3c)\sqrt{3c-1}} \times \\
&\quad \times \int_0^{\pi/2} \frac{d\phi}{\sqrt{[v_2 - v_1 - (v_3 - v_1)\sin^2\phi]|v_1 - v_4 + (v_3 - v_1)\sin^2\phi|}}.
\end{aligned}
\tag{8.142}
$$

We note that in the limiting case $c = c^*$, from (8.141), we get

$$p = -1.922773239\sqrt{\frac{a}{A}}, q = r = 0.$$

We can also show that for the same value of c

$$\gamma_1 = -1, \gamma_2 = \gamma_3 = 0.$$

Thus, at $c = c^*$, the motion of the body in the case of Kowalewski renders to a rotation with uniform angular speed about the first principal axis of inertia, while the positive half of that axis, carrying the centre of mass points, vertically downward.

Figures 8.25, 8.26, 8.27, 8.28 and 8.29 depict trajectories of the motion on the Poisson sphere fixed in the body for some values ranging from c_0 to near to c^*. The trajectories have two planes of symmetry xy and xz. Projections on both planes are shown in the first four cases.

In Fig. 8.25, we note a trajectory with four loops, large enough to intersect pairwise.

In Fig. 8.26, the loops become smaller and do not intersect.

In Fig. 8.27, the four loops become very small.

In Fig. 8.28, the loops disappear and the trajectory becomes a simple closed curve. In Fig. 8.29, we show trajectories for two values of c near to c^*. Those trajectories are closed around the negative end of the x-axis. As c approaches c^*, the trajectory shrinks and the motion tends to uniform rotation about that axis, which then occupies a vertical position.

8.10 Grioli's Case (1947): The Regular Precession About a Tilted Axis

The regular precession is that in which the body rotates uniformly about an axis fixed in it (the figure axis), while that axis precesses also uniformly about an axis fixed in space (the precession axis), keeping with it a fixed angle. As we have seen

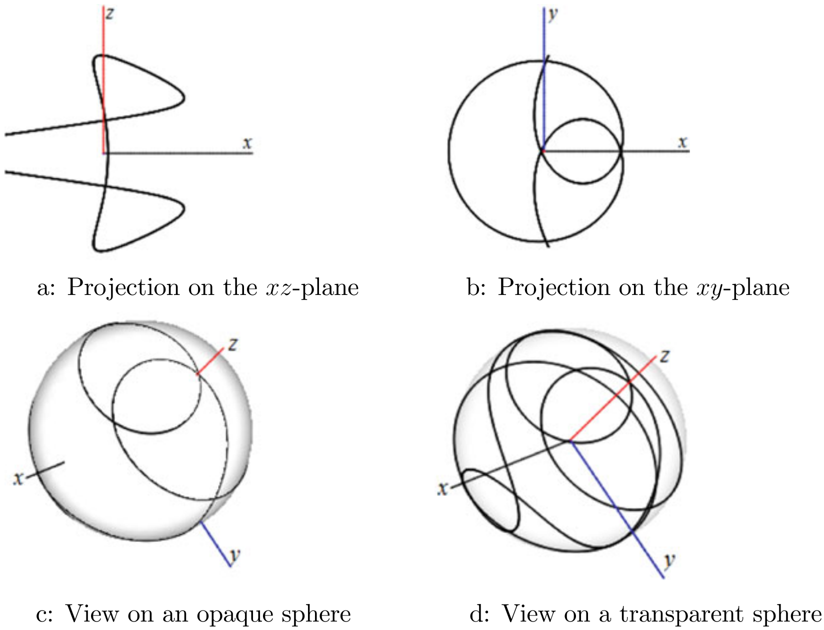


Fig. 8.25 Views of the trajectory for the minimal value $c_0 = 10/17$

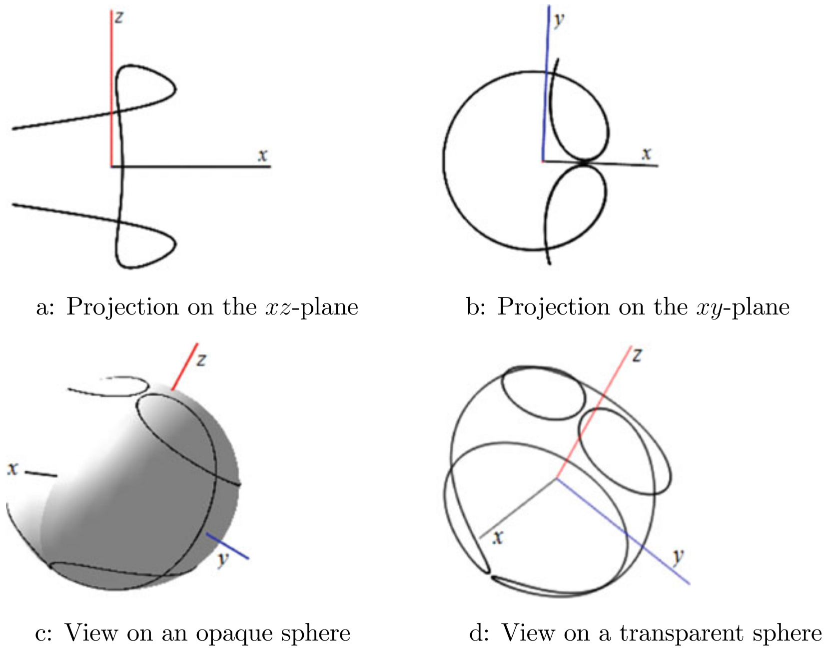
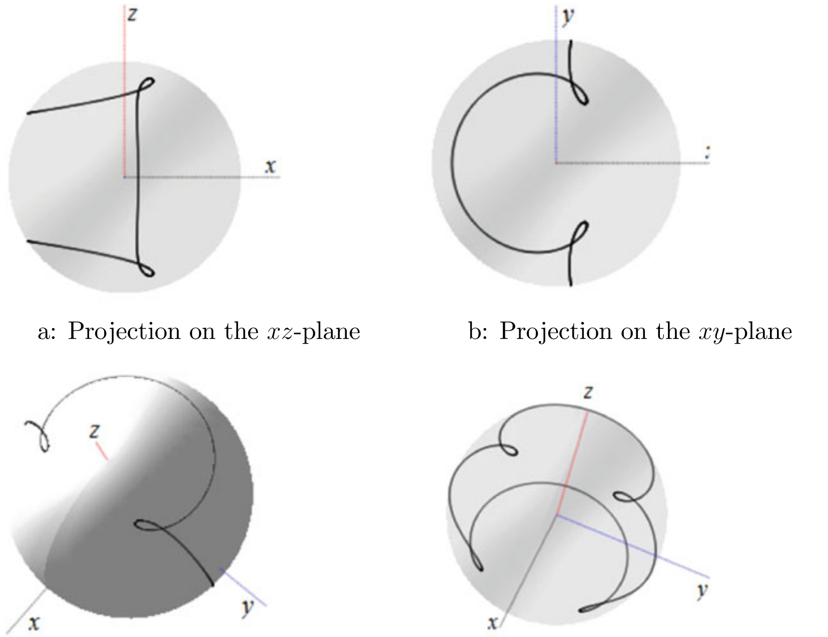


Fig. 8.26 Views of the trajectory for the value $c = 3/5$



a: Projection on the xz-plane b: Projection on the xy-plane

c: View on an opaque sphere d: View on a transparent sphere

Fig. 8.27 Views of the trajectory for the value $c = 0.614$

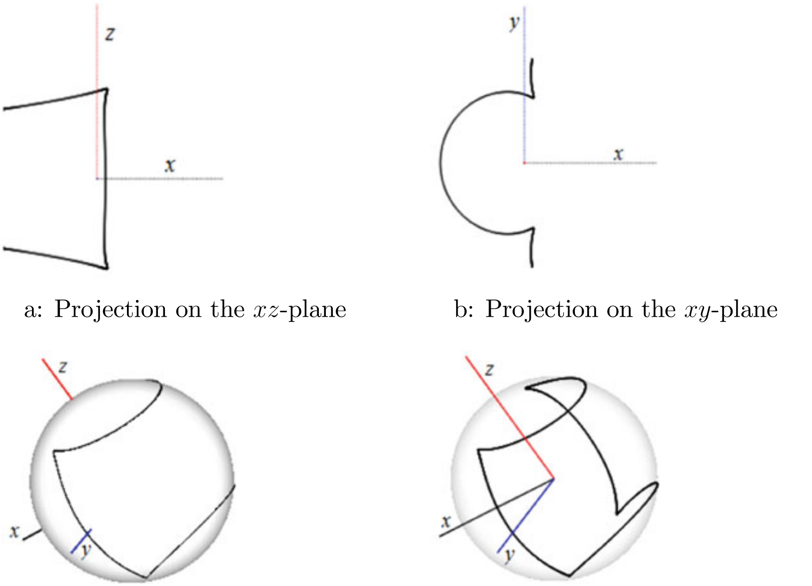
in Sect. 4.2, a wide class of regular precessional motions can be performed by an axially symmetric body (Lagrange's top). In all those motions, the figure axis is the axis of symmetry and the precession axis is vertical. Initial conditions can even be chosen so as to give the angle between the two axes any preassigned value.

As was shown long ago by Routh [312], an asymmetric body, in which no two of the three principal moments of inertia are equal, is not capable of performing a regular precession, having for the axis of precession the vertical and for the figure axis one of the three principal axes of inertia.

Grioli established, on a purely dynamical basis, the possibility of a regular precession of the heavy rigid body about a non-vertical axis under certain conditions on the parameters of the body [138]. Gulyaev derived the full explicit solution of this case [141] (see, e.g. [256]). We present the necessary details in Gulyaev's direct and transparent derivation, which uses principal axes of inertia as the body system. Another derivation, using general axes, will be given later in dealing with motion of the rigid body under the action of two or three skew fields.

Let the axes be arranged such that $A \geq B \geq C$, so that we deal with a body for which the ratios of inertia are confined to the triangle PQR in Fig. 8.30. Moreover, assume that the conditions

$$x_0\sqrt{B-C} = z_0\sqrt{A-B},\ y_0 = 0 \tag{8.143}$$



a: Projection on the xz-plane   b: Projection on the xy-plane

c: View on an opaque sphere   d: View on a transparent sphere

Fig. 8.28 Views of the trajectory for the value $c = 0.62$

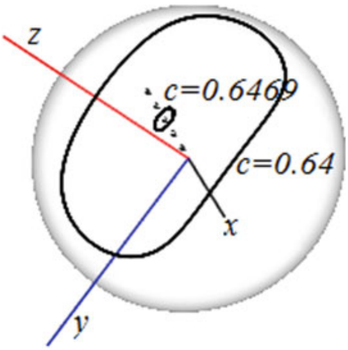


Fig. 8.29 Trajectories shrink as c approaches $c^* = 0.6469380234$

hold for the centre of mass and the principal moments of inertia. These conditions mean that the centre of mass of the body lies on the perpendicular from the fixed point O to the plane of one of the circular cross-sections of the ellipsoid of inertia. Note also that on the side PQ, $A = B$ and from (8.143) $x_0 = 0$. Similarly, on QR, we have $B = C$, $z_0 = 0$. On both sides, the body becomes axially symmetric with the centre of mass lying on the axis of symmetry, i.e. the body turns into Lagrange's top. On the third side PR of the triangle, $B + C = A$ and the body becomes a disc in the yz-plane and thus $x_0 = 0$, which contradicts (8.143). The side PR should be excluded, and thus an asymmetric body satisfying (8.143) corresponds to points of the interior of the triangle PQR in Fig. 8.30.

As was shown by Gulyaev in [141], under conditions (8.143), the system of Eqs. (3.29) admits the particular solution:

$$\begin{aligned} p &= \frac{\Omega}{s}(a - c\cos(\Omega t)), q = \Omega \sin(\Omega t), r = \frac{\Omega}{s}(c + a\cos(\Omega t)), \\ \gamma_1 &= -\frac{\Omega^2}{s^2}[Cc\cos(\Omega t) + (B - C)a\sin^2(\Omega t)], \\ \gamma_2 &= \frac{\Omega^2}{s^3}\sin(\Omega t)[(Aa^2 + Cc^2) - (A - C)ac\cos(\Omega t)], \\ \gamma_3 &= \frac{\Omega^2}{s^2}[Aa\cos(\Omega t) + (A - B)c\sin^2(\Omega t)], \end{aligned} \tag{8.144}$$

where $a = Mgx_0, b = Mgy_0, s = \sqrt{a^2 + c^2}, \Omega^2 = \frac{s}{\sqrt{(A-B+C)^2+(A-B)(B-C)}}$. This solution corresponds to a uniform precession of the body. The angular velocity $\boldsymbol{\omega}$ can be written as the sum of two terms

$$\boldsymbol{\omega} = \Omega\boldsymbol{\zeta} + \Omega\boldsymbol{\alpha}, \tag{8.145}$$

where

$$\boldsymbol{\zeta} = (\frac{a}{s}, 0, \frac{c}{s}), \boldsymbol{\alpha} = (-\frac{c}{s}\cos(\Omega t), \sin(\Omega t), \frac{a}{s}\cos(\Omega t)). \tag{8.146}$$

The first vector $\boldsymbol{\zeta}$ is fixed in the body along the configuration axis. This axis carries the centre of mass of the body and is orthogonal to a circular section of the inertia ellipsoid. The second vector, $\boldsymbol{\alpha}$, can be easily shown to have the following properties:

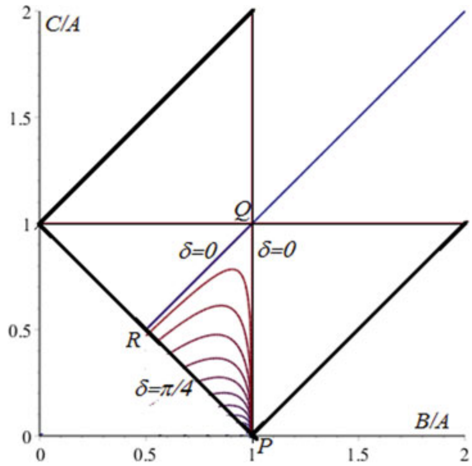
(1) $\boldsymbol{\alpha}$ is orthogonal to $\boldsymbol{\zeta}$. In fact $\boldsymbol{\alpha} \cdot \boldsymbol{\zeta} = 0$.
(2) $\boldsymbol{\alpha}$ is fixed in space. In virtue of (8.146) and (8.145), it satisfies the equation $\dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}$.

Thus, $\boldsymbol{\alpha}$ is the unit vector along the precession axis. Note that the angular velocities of the body around the configuration and precession axes are equal (each equals Ω).

Now, we determine the angle δ of the inclination of the axis of precession to the upward vertical, which must be a fixed angle. Using the relation $\cos\delta = \boldsymbol{\alpha} \cdot \boldsymbol{\gamma}$, we find

$$\begin{aligned} \delta &= \arccos\frac{A - B + C}{\sqrt{(A - B + C)^2 + (A - B)(B - C)}} \\ &= \arctan\frac{\sqrt{(A - B)(B - C)}}{A - B + C}. \end{aligned} \tag{8.147}$$

Fig. 8.30 Contours of the angle δ



Thus, in the final general picture of the motion, the body rotates with the uniform velocity Ω around the vector $\boldsymbol{\zeta}$ fixed in it, while that vector rotates with the same angular velocity Ω about the direction $\boldsymbol{\alpha}$ orthogonal to $\boldsymbol{\zeta}$ but fixed in space and making with the vertical upwards a fixed angle δ. On the two sides PQ, QR of the triangle PQR in Fig. 8.30, we have $\delta = 0$. As we have seen above, on both sides, the body becomes Lagrange's top. As in Lagrange's case, the figure axis is the axis of symmetry of the body and the axis of precession is the vertical at the fixed point, but here we have two additional conditions: the configuration and precession axes are orthogonal and the velocities of rotation and precession are equal. We recall that for Lagrange's top precession, one or the two conditions may not be satisfied.

Figure 8.30 shows the contour lines of equal δ. As seen from it, in the interior of the triangle PQR, the inclination angle δ increases downwards and approaches a limit $\frac{\pi}{2}$ as the current point approaches P. This means that for asymmetric bodies, for which $A - B$ and C take very small positive values, the precession axis occupies an almost horizontal position and the configuration axis precesses around it in a nearly vertical plane.

8.10.1 Motion of the Centre of Mass

From (8.144) and (8.146), we find the angle, Δ (say), between the vector $\boldsymbol{\zeta}$ directed to the centre of mass and the vertical. We get

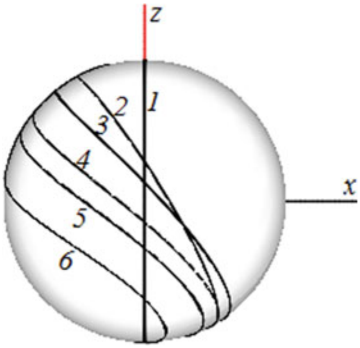


Fig. 8.31 Projection of orbits on the xz-plane of symmetry

$$
\begin{aligned}
\cos\Delta &= \boldsymbol{\zeta}\cdot\boldsymbol{\gamma} \\
&= \sqrt{\frac{(A-B)(B-C)}{(A-B+C)^2+(A-B)(B-C)}}\cos(\Omega t) \\
&= \sin\delta\cos(\Omega t).
\end{aligned} \tag{8.148}
$$

Thus, the angle Δ varies with time from $\pi/2-\delta$ to $\pi/2+\delta$. The centre of mass spends equal time intervals above and below the horizontal plane through the fixed point.

8.10.2 Orbits of Motion on the Poisson Sphere

The picture of the motion can be most clearly interpreted as the motion of a right circular cone with vertical angle $\pi/2$ rolling without slipping on a similar cone fixed in space. This will be presented later in this book, based on the use of non-principal system of axes. But it is of interest here to depict some orbits of the motion on the Poisson sphere and show how the orbit changes with the change of moments of inertia.

(1) From (8.144), γ_1, γ_3 are even functions in t, while γ_2 is odd. The orbits are symmetric with respect to the xz-plane.

(2) At $\delta = 0$, the orbit is a great circle $\gamma_1 = 0$. For increasing values of δ (see Fig. 8.30), the orbit remains simple and smooth. When δ reaches $\frac{\pi}{4}$, a cusp appears on the orbit on the plane of symmetry xz. The cusp turns into a small second loop, which grows with increasing δ.

Figures 8.31 and 8.32 show sample orbits corresponding to moments of inertia represented by six points in Fig. 8.30 lying on the line drawn from P and bisecting QR. The orbits are numbered from 1 to 6 as δ varies from 0 to 79°.

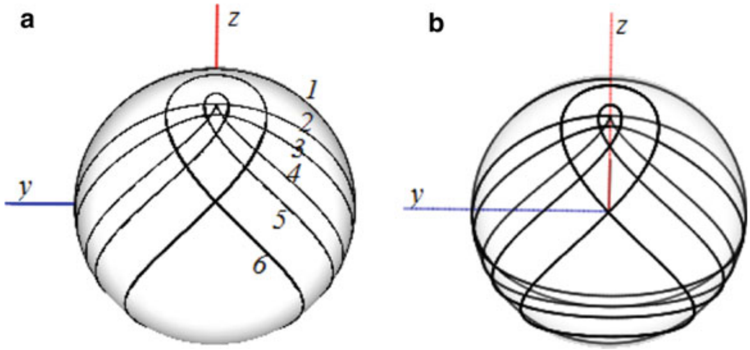


Fig. 8.32 Orbits of motion on the Poisson sphere **a** View on opaque sphere **b** View on transparent sphere

8.11 Dokshevich's First Case [63] (1966)

This case bears some resemblance to the case of Hess described in Sect. 8.4. The centre of mass lies on a circular section of the gyration ellipsoid. It is possible to investigate this case using the principal set of axes at the fixed point, but, as presented in Sect. 7.9 of [108], expressions of the Euler–Poisson variables in terms of a regularizing variable become too complicated.

8.11.1 Use of Special System of Axes

We shall use here the special system of axes fixed in the body and associated with the gyration ellipsoid. In this system expressions are less complicated, but, most importantly, the process of separation of variables and the explicit expressions of the variables in terms of time is more straightforward. Our presentation is a slight modification of those given in [63, 108].

Let, for determinacy, the centre of mass of the body lie on the x-axis. Let also the gyration matrix at the fixed point in the chosen axes be written as

$$\mathbf{J} \equiv \mathbf{I}^{-1} = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{8.149}$$

so that the equation of the gyration ellipsoid becomes

$$ax^2 + 2bxy + c(y^2 + z^2) = 1, \tag{8.150}$$

and the plane $x = 0$ contains the circular cross-section of the ellipsoid. The components of the angular velocity are related to the components of the vector of angular momentum, which we denote by

$$\mathbf{G} = (P, Q, R), \tag{8.151}$$

by the relations $\boldsymbol{\omega} = \mathbf{GJ}$ or in expanded form

$$\begin{aligned} p &= aP + bQ, \\ q &= bP + cQ, \\ r &= cR. \end{aligned} \tag{8.152}$$

Equations of the motion take the form

$$\begin{aligned} \dot{P} &= -bPR, \\ \dot{Q} &= [(a-c)P + bQ]R + s\gamma_3, \\ \dot{R} &= b(P^2 - Q^2) + (c-a)PQ - s\gamma_2, \\ \dot{\gamma}_1 + (bP + cQ)\gamma_3 - cR\gamma_2 &= 0, \\ \dot{\gamma}_2 + cR\gamma_1 - (aP + bQ)\gamma_3 &= 0, \\ \dot{\gamma}_3 + (aP + bQ)\gamma_2 - (bP + cQ)\gamma_1 &= 0, \end{aligned} \tag{8.153}$$

where $s = Mgx_0$ and the first integrals of motion become

$$\begin{aligned} I_1 &\equiv \frac{1}{2}[aP^2 + 2bPQ + c(Q^2 + R^2)] + s\gamma_1 = h, \\ I_2 &\equiv P\gamma_1 + Q\gamma_2 + R\gamma_3 = f, \\ I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{8.154}$$

Dokshevich [63] used the ansatz

$$\begin{aligned} \gamma_2 &= g_0 + g_1 P^2, \\ Q &= Q_1 P + \frac{Q_2}{P}, \\ R &= \frac{1}{P}\sqrt{R_0 + R_1 P^2 + R_2 P^4}, \end{aligned} \tag{8.155}$$

and γ_1, γ_3 are expressed from the first two integrals of motion in (8.154). The third integral and the equations of motion are used to obtain conditions on the parameters of the problem. It can be found that

$$h = -2\,\frac{s\left((a-7/2\,c)\sqrt{a^2-ac+3\,b^2+c^2}+a^2-4\,ac+6\,b^2+c^2\right)}{(2\,a-4\,c)\sqrt{a^2-ac+3\,b^2+c^2}+2\,a^2-5\,ac+12\,b^2+5\,c^2},$$
$$f = 0,$$

$$\begin{aligned}
Q_1 &= \frac{1}{3b}(c-2a+\sqrt{a^2-ac+3b^2+c^2}),\\
Q_2 &= \frac{cs}{b(2bQ_1+a-c)},\\
R_0 &= -[\frac{cs}{b(2bQ_1+a-c)}]^2,\\
R_1 &= -\frac{cs}{b^2}-\frac{cs}{a(2bQ_1+2a-c)}+\frac{(a-c)s(ac-b^2)}{ab^2(2bQ_1+a-c)},\\
R_2 &= -Q_1^2-\frac{a}{2bQ_1+2a-c},\\
g_0 &= -\frac{c}{b},\\
g_1 &= -\frac{(2bQ_1+a-c)[2bQ_1^2+(2a-c)Q_1-b]}{s(2bQ_1+2a-c)}.
\end{aligned} \tag{8.156}$$

$$\begin{aligned}
\gamma_1 &= -1-\frac{(bQ_1+a)(2bQ_1+a-c)}{s(2bQ_1+2a-c)}P^2,\\
\gamma_2 &= -\frac{c}{b}-\frac{(2bQ_1+a-c)[2bQ_1^2+(2a-c)Q_1-b]}{s(2bQ_1+2a-c)}P^2,\\
\gamma_3 &= \sqrt{1-\gamma_1^2-\gamma_2^2}.
\end{aligned} \tag{8.157}$$

From (8.153) and (8.155), we obtain the relation of P and time

$$\dot{P} = -b\sqrt{R_0+R_1P^2+R_2P^4}, \tag{8.158}$$

so that P can be expressed as an elliptic function of time. In fact, it can be shown that [63]

$$P = P_0\,\mathrm{dn}(\rho t, k), \tag{8.159}$$

where P_0, ρ, k are determined through coefficients R_0, R_1, R_2.

8.11.2 *Orbits on the Poisson Sphere*

Recently, the first solution of Dokshevich was isolated on the basis of a property of the orbits of motion on the Poisson sphere [427]. To this end, we use the orbital equation, which will be introduced in a later chapter of this book Chap. 9. This equation, resulting from eliminating dynamical variables using integrals of motion, is a second-order differential equation connecting two of the components of the vector γ. From a known solution of the orbital equation, one can construct the corresponding full solution of the Euler–Poisson equations.

It was shown in [427] that the only solution of the classical problem when the orbit of motion on the Poisson sphere is a circular section of that sphere is Dokshevich's first solution. This agrees with (8.157), from which we see that eliminating P we obtain a linear relation between γ_1 and γ_2.

8.12 The Case of Konosevich and Pozdnyakovich [230, 231] (1968)

The solution in this case was found in a trial to express it in the form of truncated trigonometric series in an intermediate variable σ. In the system of principal axes of inertia at the fixed point, let the centre of mass lies on the x-axis, so that the equations of motion have the form (8.75) and let a and u denote Mgx_0 and $\frac{C}{A}$, respectively. The Euler–Poisson variables have the following expressions:

$$
\begin{aligned}
p &= \sqrt{\frac{a}{A\Gamma}}u[2R_1\cos(2\sigma) + 2\cos(\sigma) \\
&\quad -\frac{3R_1}{2P_1}(3u-2)(9u-4)(11u-4)(u-1)], \\
q &= -2\sqrt{\frac{aC}{\Gamma}\frac{s_0}{Bu}}(1-u)(2u-1)[2R_1\sin(2\sigma)+\sin(\sigma)], \\
r^2 &= -2/3\,\frac{au^2}{C\Gamma\,(2u-1)(3u-2)^2(9u-4)\left(17u^2-16u+4\right)}\times \\
&\quad \times\{P_1\cos(4\sigma) \\
&\quad +6\,(2u-1)(3u-2)(9u-4)(11u-6)R_1\cos(3\sigma) \\
&\quad +3\,(2u-1)(3u-2)\left(92u^2-89u+22\right)\cos(2\sigma) \\
&\quad -\frac{(2u-1)\left(1329u^4-2915u^3+2290u^2-766u+92\right)\cos(\sigma)}{(9u-4)R_1} \\
&\quad +1/8\,\frac{(2u-1)P_3}{(3u-1)(3u-2)(9u-4)P_1}\},
\end{aligned}
\tag{8.160}
$$

$$\begin{aligned}
\gamma_1 = {} & \frac{2}{3\Gamma}\,\frac{1-u}{(3\,u-2)^2\,(9\,u-4)}\{\frac{P_1\,\cos(4\,\sigma)}{2\,u-1}\\
& +3\,(3\,u-2)\,(9\,u-4)\,(15\,u-8)\,R_1\cos(3\,\sigma)\\
& +3\,(3\,u-2)\left(35\,u^2-33\,u+8\right)\cos(2\,\sigma)\\
& -1/2\,\frac{\left(1473\,u^4-3235\,u^3+2582\,u^2-890\,u+112\right)}{(9\,u-4)\,R_1}\cos(\sigma)\\
& +1/8\,\frac{P_2}{P_1}\},\\
\gamma_2 = {} & -\frac{s_0}{3\Gamma}\,\frac{(1-u)^2\,(2\,u-1)}{(3\,u-2)^2\,(9\,u-4)}\{\frac{P_1\,\sin(4\,\sigma)}{(2\,u-1)^2}\\
& +36\,(3\,u-2)\,(9\,u-4)\,R_1\sin(3\,\sigma)\\
& +3\,(3\,u-2)\,(13\,u-8)\sin(2\,\sigma)\\
& -2\,\frac{\left(309\,u^3-464\,u^2+222\,u-34\right)\sin(\sigma)}{(9\,u-4)\,R_1}\},\\
\gamma_3 = {} & -\frac{2A(1-u)}{\Gamma c_1}r[(9u-4)R_1\cos(2\sigma)+(3u-1)\cos(\sigma)\\
& +\frac{u(1-u)(11u-4)}{4(3u-2)(9u-4)R_1}],
\end{aligned} \tag{8.161}$$

in which

$$\begin{aligned}
P_1 = {} & 97\,u^4-271\,u^3+258\,u^2-101\,u+14,\\
P_2 = {} & 729185\,u^8-3592714\,u^7+7498373\,u^6-8683880\,u^5\\
& +6116476\,u^4-2687840\,u^3+720872\,u^2-108064\,u+6944,\\
P_3 = {} & 334695915\,u^{11}-2239025631\,u^{10}+6709632635\,u^9\\
& -11904287977\,u^8+13908377054\,u^7-11245436972\,u^6\\
& +6425316760\,u^5-2596052504\,u^4+727305680\,u^3\\
& -134634208\,u^2+14828480\,u-736512,
\end{aligned} \tag{8.162}$$

$$\begin{aligned}
s_0^2 &= \frac{4u}{(1-u)(17u^2-16u+4)},\\
R_1^2 &= \frac{P_1}{3(3u-2)^2(9u-4)^2},\\
c_1^2 &= \frac{Ca}{\Gamma},
\end{aligned} \tag{8.163}$$

and

$$\Gamma^2 = \frac{(5\,u-2)\,(u-1)^3}{144\,(3\,u-2)^5\,(9\,u-4)^2\,(2\,u-1)^2\left(97\,u^4-271\,u^3+258\,u^2-101\,u+14\right)^2(3\,u-1)}\times$$
$$\times(3828317502420\,u^{18}-45928438243128\,u^{17}+256746575766313\,u^{16}$$
$$-888956089381273\,u^{15}+2137100780408016\,u^{14}-3789579706038612\,u^{13}$$
$$+5138532374554515\,u^{12}-5449736202969735\,u^{11}+4584313633529960\,u^{10}$$
$$-3082573845813260\,u^{9}+1661356742951664\,u^{8}-716191818324608\,u^{7}$$
$$+245195273873024\,u^{6}-65783181186816\,u^{5}+13530394373376\,u^{4}$$
$$-2059297120512\,u^{3}+218459750400\,u^{2}-14419084288\,u+445763584). \tag{8.164}$$

Formulas (8.160)–(8.164) are slight modifications of the presentation in [108].

Using the first of dynamical Eqs. (8.75), the following equation is obtained for the intermediate variable

$$\dot{\sigma} = \frac{1}{2}\sqrt{\frac{(1-u)(17u^2-16u+4)}{u}}r. \tag{8.165}$$

The moments of inertia of the body are subject to two conditions: a simple one

$$B = 4A\frac{2C-A}{17C-8A}, \tag{8.166}$$

and the second is that u must be a root of the seventh-degree polynomial equation

$$\begin{aligned} P_7 &= 437511u^7 - 1822945u^6 + 3227896u^5 - 3146990u^4 \\ &\quad +1823596u^3 - 627920u^2 + 118960u - 9568 \\ &= 0. \end{aligned} \tag{8.167}$$

Although this equation has three real roots, only two roots lead to real expressions for r and Γ, and hence for all (8.160)–(8.161), provided that the variable σ is restricted for each root to certain interval $[\sigma_1, \sigma_2]$.

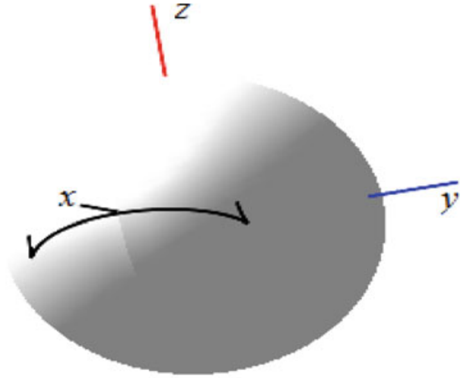
However, we cannot go further with the exact solution and must use approximate values of the roots $u_1 = 0.4119045665$, $u_2 = 0.7081863612$, and hence replace the whole solution by a one with approximate numerical coefficients. From now on, the case under consideration splits into two different subcases, which differ not only in the value of u, but also in the description of motion accordingly.

It is notable here that only the factor $\sqrt{\frac{a}{A}}$ figures in the components of $\boldsymbol{\omega}$, while γ contains no parameters. This means we are dealing with a single orbit of the apex on the Poisson sphere corresponding to each of the roots u_1, u_2. Orbits are shown for the two subcases, together with the necessary values of the parameters.

8.12.1 The First Subcase

$$
\begin{aligned}
u &= u_1 = 0.4119045665, \\
A &: B : C :: 1 : 0.7064431289 : 0.4119045665, \\
\Gamma &= 0.029056173, \\
\sigma_1 &= 2.7431083805, \\
\sigma_2 &= 2\pi - \sigma_1.
\end{aligned} \tag{8.168}
$$

At the ends of this interval, r and γ_3 change their signs and hence the motion is periodic time.



The orbit

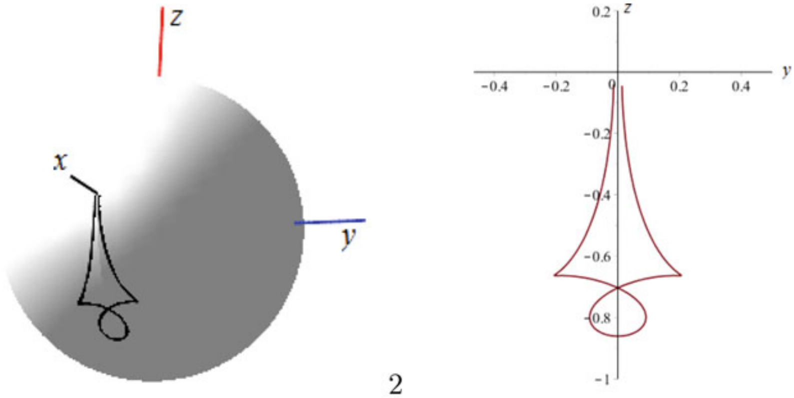
The orbit is symmetric with respect to the xz-plane and intersects it at $\sigma = \pi$. The projection shows clearly that the motion is a vibration, in which the apex of the vertical draws the orbit from one end to the other and then $\dot{\sigma}$ changes its sign, and the orbit is described in the reverse direction. The motion is periodic in time. The periodic time T is obtained from (8.165) as

$$
\begin{aligned}
T &= 4\sqrt{\frac{u}{(1-u)(17u^2-16u+4)}} \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{r} \\
&= 8\sqrt{\frac{u}{(1-u)(17u^2-16u+4)}} \int_{\sigma_1}^{\pi} \frac{d\sigma}{r}.
\end{aligned} \tag{8.169}
$$

8.12.2 The Second Subcase

$$
\begin{aligned}
u &= u_2 = 0.7081863612, \\
A &: B : C :: 1 : 0.4123351224 : 0.7081863612,
\end{aligned}
$$

$$\Gamma = 14.9952481138,$$
$$\sigma_1 = 1.0117323905,$$
$$\sigma_2 = 2\pi - \sigma_1.$$



2

The orbit of motion on the P. sphere. Orbit projected on the yz-pl.

The orbit is symmetric with respect to the xz-plane and intersects it at $\sigma = \pi$. The projection shows clearly that the motion is also a vibration, in which the apex of the vertical draws the orbit from one end to the other and then $\dot{\sigma}$ changes its sign, and the orbit is described in the reverse direction. The motion is periodic and the period is given by the same formula as (8.169).

8.13 Dokshevich's Second Case [64] (1970)

This is the last case discovered until now in the classical problem of motion of a heavy rigid body about a fixed point. As all the cases discovered are not obtained on an exhaustive basis, it is not known whether some new cases of solvability of the Euler–Poisson equations for the classical problem in the near (or even distant) future to be found.

Under the condition assumed in most of the particular cases treated above, the centre of mass of the body is assumed to lie on one of the principal axes of inertia of the body at the fixed body. The equations of motion are the same as (8.75)–(8.76) and their first integrals have the form (8.77).

To obtain this solution, assume the components of the angular velocity in the form

$$p = \sqrt{a_2(u^2 - u_1^2)},$$
$$q = \sqrt{b_2(u^2 + b_1 u + b_0)},$$
$$r = u\sqrt{c_1(u - u_3)}. \tag{8.170}$$

The first component of the vector $\boldsymbol{\gamma}$ is determined from the energy integral in (8.77)

$$\gamma_1 = \frac{1}{a}[h - \frac{1}{2}(Ap^2 + Bq^2 + Cr^2)], \tag{8.171}$$

and the other two from third and second dynamical Eqs. (8.75)

$$\begin{aligned}\gamma_2 &= -\frac{1}{a}[C\dot{r} + (B - A)pq],\\ \gamma_3 &= \frac{1}{a}[B\dot{q} + (A - C)\, pr].\end{aligned} \tag{8.172}$$

The first dynamical equation is used to express $\dot{u}$ as

$$\begin{aligned}\dot{u} &= \frac{(B - C)}{Aa_2u}pqr\\ &= \frac{(B - C)}{Aa_2}\sqrt{a_2b_2c_1(u^2 - u_1^2)(u^2 + b_1u + b_0)(u - u_3)}.\end{aligned} \tag{8.173}$$

Substituting the above expressions into the second and third integrals of motion

$$\begin{aligned}I_2 &\equiv Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f,\\ I_3 &\equiv \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,\end{aligned} \tag{8.174}$$

we obtain two polynomials in u whose coefficients must be all zeros. From those equations, it turns out that the moments of inertia are related by the same relation as in the case of Kowalewski

$$A(10B - 9C) = 18C(B - C),$$

and one more relation

$$192(\frac{C}{A})^2 - 184\frac{C}{A} + 41 = 0.$$

Together, those relations give

$$\begin{aligned}B &= \frac{649 + 71\sqrt{37}}{848}A = 1.274\,6A,\\ C &= \frac{23 + \sqrt{37}}{48}A = 0.605\,89A.\end{aligned} \tag{8.175}$$

Without loss of generality, we can normalize the moments of inertia so that $A = 1$, and no free parameters remain in the solution. We also find the parameters

$$f = 0, a = \frac{7 + 11\sqrt{37}}{256},$$
$$h = \frac{\sqrt{(2816\sqrt{37} - 4411)(67801 + 11975\sqrt{37})}}{5366592} = 2.955473898. \quad (8.176)$$

The final expressions for the Euler–Poisson variables can be written as follows:

$$p = \frac{N\sqrt{\left(9327212 + 1782163\sqrt{37}\right)}}{13\,068\sqrt{231}}\sqrt{\left(9u^2 - 17 - 8\sqrt{37}\right)}$$
$$= \frac{N\sqrt{\left(9327212 + 1782163\sqrt{37}\right)}}{4356\sqrt{231}}\sqrt{u^2 - u_1^2},$$
$$q = \frac{N\sqrt{\left(1635292 - 474787\sqrt{37}\right)}}{13\,068\sqrt{231}}\sqrt{36u^2 - 3\left(25 + 7\sqrt{37}\right)u - 352 - 64\sqrt{37}}$$
$$= \frac{N\sqrt{\left(1635292 - 474787\sqrt{37}\right)}}{2178\sqrt{231}}\sqrt{(u - u_4)(u - u_5)},$$
$$r = \frac{N\sqrt{\left(1853 - 4463\sqrt{37}\right)}}{1452\sqrt{21}}u\sqrt{4u - 7 - \sqrt{37}}$$
$$= \frac{N\sqrt{\left(1853 - 4463\sqrt{37}\right)}}{1452\sqrt{21}}u\sqrt{u - u_3}, \quad (8.177)$$

and

$$\gamma_1 = \frac{9[24u^3 - 12(5 + \sqrt{37})u^2 - 3(17 + 5\sqrt{37})u + 4(127 + 19\sqrt{37})]}{(1 + \sqrt{37})\sqrt{2816\sqrt{37} - 4411}},$$
$$\gamma_2 = \frac{176(12u - 31 - \sqrt{37})}{1649 + 61\sqrt{37}}pq,$$
$$\gamma_3 = \frac{2(53\sqrt{37} - 235)(4u + 7 + \sqrt{37})}{1107}p\frac{r}{u}, \quad (8.178)$$

where

$$N = \sqrt[4]{11(256\sqrt{37} - 401)},$$
$$u_1 = \frac{1}{3}\sqrt{17 + 8\sqrt{37}} = 2.701\,1,$$

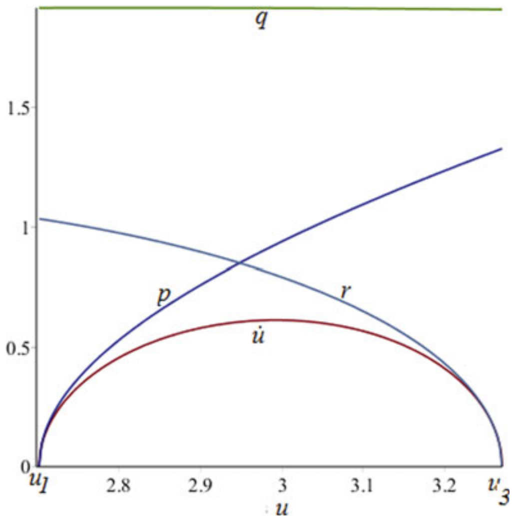


Fig. 8.33 Graphs of $\dot{u}, p, q, r$

$$
\begin{aligned}
u_2 &= -\frac{1}{3}\sqrt{17+8\sqrt{37}} = -2.7011,\\
u_3 &= \frac{1}{4}(7+\sqrt{37}) = 3.2707,\\
u_4 &= \frac{1}{24}(25+7\sqrt{37}+\sqrt{8070+1374\sqrt{37}}) = 8.1562,\\
u_5 &= \frac{1}{24}(25+7\sqrt{37}-\sqrt{8070+1374\sqrt{37}}) = -2.5246.
\end{aligned}
\tag{8.179}
$$

From Eq. (8.173), the time can be expressed as a hyper-elliptic integral. Comparing numerical values of the roots u_i, we easily see that they have the order

$$u_4 > u_3 > u_1 > u_5 > u_2. \tag{8.180}$$

Graphical check (Fig. 8.33) shows that $p, q, r, \dot{u}$ are all real only on the interval $u_1 \le u \le u_3$. Moreover, the q component of the angular velocity does not change its sign on the whole motion.

To complete the description of motion, we introduce an auxiliary angle ϕ by the substitution

$$u = u_1 + (u_3 - u_1)\sin\phi. \tag{8.181}$$

Now, the angular velocities p, q, r are expressed as analytical functions of ϕ.

$$p = \frac{11^{3/4}\sqrt{21}\sqrt{2}\sqrt{9327212 + 1782163\sqrt{37}}\sqrt[4]{256\sqrt{37} - 401}}{12074832} \times$$
$$\times \sqrt{251 - \sqrt{37} - [523 + 127\sqrt{37} - 12(7 + \sqrt{37})\sqrt{8\sqrt{37} + 17}]\cos^2\phi}\,\sin\phi,$$
$$q = \frac{\sqrt{21 \cdot 22(474787\sqrt{37} - 1635292)}}{6037416}\sqrt{\delta_1\cos^4\phi + \delta_2\cos^2\phi + 176\sqrt{37} + 968},$$
$$r = \frac{N\sqrt{7}\sqrt{-1853 + 4463\sqrt{37}}(21 + 3\sqrt{37} - 4\sqrt{8\sqrt{37} + 17})^{3/2}}{639966096} \times$$
$$\times [2763 + 450\sqrt{37} + (43\sqrt{37} + 299)\sqrt{8\sqrt{37} + 17} - 1749\cos^2\phi]\cos\phi, \tag{8.182}$$

where

$$\delta_1 = (12\sqrt{37} + 84)\sqrt{8\sqrt{37} + 17} - 523 - 127\sqrt{37},$$
$$\delta_2 = 2(\sqrt{37} - 17)\sqrt{8\sqrt{37} + 17} + 123 + 15\sqrt{37},$$

and the components of $\boldsymbol{\gamma}$ are changed accordingly.

8.13.1 Periodicity of the Motion

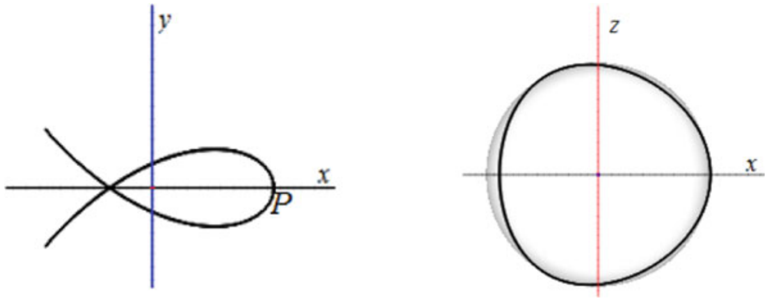
From (8.173), we see that the time can be expressed as a hyper-elliptic integral in the auxiliary variable u. However, it turns out that the angle ϕ is more suited to complete the description of motion. In fact, using (8.173) and (8.181), one can find

$$\dot{\phi} = K\sqrt{P_2 P_4},$$

where

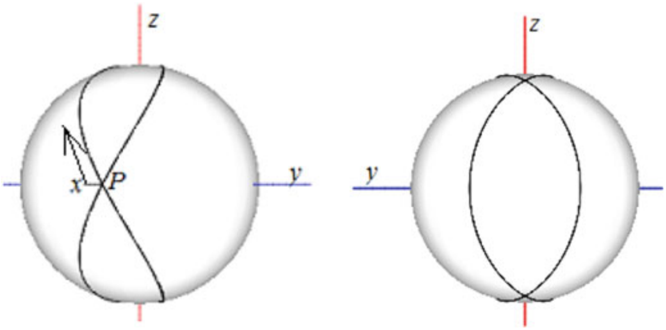
$$K = \frac{\sqrt{\left(25549\sqrt{37} + 116399\right)\sqrt{8\sqrt{37} + 17} + 84(23\sqrt{37} + 21514)}}{14\,256\sqrt{8162}},$$
$$P_2 = 251 - \sqrt{37} - [523 + 127\sqrt{37} - 12(7 + \sqrt{37})\sqrt{8\sqrt{37} + 17}]\cos^2\phi,$$
$$P_4 = \delta_1\cos^4\phi + \delta_2\cos^2\phi + 176\sqrt{37} + 968. \tag{8.183}$$

It is evident that P_2 and P_4 have no real zeros. Thus, $\dot{\phi}$ is positive for all ϕ, so that ϕ monotonically increases with time. The motion of the body is periodic with period 2π in ϕ. The time elapsed in motion from the initial point P ($\phi = 0$) to a general position ϕ is



a) Projection of orbit on xy-pl. b) Projection of orbit on xz-pl.

Fig. 8.34 Planes of symmetry



c) The two small loops. b) The larger loop.

Fig. 8.35 Two sides of the sphere

$$t = \frac{1}{K}\int_0^{\phi} \frac{d\phi}{\sqrt{P_2 P_4}},$$

and the time period is

$$T = \frac{1}{K}\int_0^{2\pi} \frac{d\phi}{\sqrt{P_2 P_4}}.$$

8.13.2 Orbits on the Poisson Sphere

The present case bears some similarity to Goryachev's case above. Graphics reveal that the orbit of motion on the Poisson sphere is three-loop curve with the two planes xy and xz as planes of symmetry. Two smaller loops have a common vertex P at the point $(1, 0, 0)$ attained periodically at $\phi = 0, \pi, 2\pi$ and so on. At $\phi = 2\pi$, after

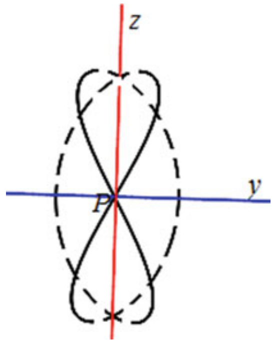


Fig. 8.36 The whole orbit. Hidden lines dashed

making two complete rotations in the same direction about the y-axis, the point returns to P and the orbit closes (Figs. 8.34, 8.35 and 8.36).

8.14 Unsuccessful Cases and Incorrect Claims

Most of the particular solutions of the problem of motion of a heavy rigid body about a fixed point were constructed in the last decade of the nineteenth century and the first decade of the twentieth century. The first successful trial to build the next particular solution was that made by Grioli in 1947. Along the period between those times, several publications appeared announcing new solutions. However, it turned out later that those publications carried wrong claims. Some of them are repeated or were special cases of known results discovered earlier by other authors. Other ones were erroneous and contained no solution at all. Erroneous solutions continued to appear in some publications until the present time, e.g. the very recent publications [76, 98].

In view of the fact that some of those solutions were presented in books on the subject as valid solutions, e.g. [256, 270], we shall give here a brief account of the ones that were published in English or used and cited in works in English. We comment on their shortcomings and give the references, where the reader can find more details about them. Some more incorrect results appeared in Russian, but had no further implications. Those are not listed here. An example is Arjanikh's claim that the classical problem of heavy rigid body dynamics in all its generality admits an integral of motion linear in velocities different from the areas integral [13].

8.14.1 Shiff's Work [331] (1903)

Shiff's short article [331] first introduces a transformation of the equations of motion of a heavy rigid body about a fixed point to a certain form, which can be easily identified with that obtained much earlier by Hess [150]. Based on those equations, he introduces a particular solution of those equations, claiming it to be a new solution. Hamel [147] noted that the investigations of Shiff, Stäckel and Hess, who used equations of Hess' type, are defective. They ignored Poisson's equations, while the question about equivalence of the equations they used and the original Euler–Poisson equations was not addressed.

On the other hand, the solution introduced by Shiff satisfies the equality $\mathbf{G}^2 =$ const, and, in the light of Sect. 8, the check of Poisson's equations reveals that two of them are not satisfied.

8.14.2 Field's Works [85, 86] (1934)

The main results of those two articles are not erroneous, but the case pointed out in them is a special case of Steklov's case presented in Sect. 8.6, under the additional restriction on the moments of inertia $A^2 = 2BC$. On the other hand, Field pointed out an interesting property of that special version, namely the equality of the angles of precession and proper rotation.

8.14.3 Corliss' Works [56, 57] (1932–1934)

In the first article, Corliss announces two particular solutions of the classical problem. The first reproduces one version of Steklov's solution, although Steklov's paper [347] is cited in it. The second is a special case of Kowalevski's general integrable case. This also is not a new result, as it describes the class of motions of Kowalevski's gyroscope described much earlier by Appelrot under the name of "The second class of the simplest motions of Kowalevski's gyroscope" [11]. In his second article, Corliss introduces one more particular solution, this time repeating Goryachev's result [114].

8.14.4 Fabbri's Works [80, 81] (1934)

Fabbri's result in [80] also repeats Steklov's. In the case introduced in [81], some of the variables are not real-valued [120].

8.14.5 Concerning Mertsalov's Work [283] (1946)

A case is included in Leimanis' book [256] (Sect. 8.7) and listed in the table of exactly solvable cases in Magnus' book [270] (Sect. 3.3.1), under the name "Case of Mertsalov". This is a particular solution that could generalize in some sense the conditional integrable case of Goryachev and Chaplygin $A = B = 4C$ (Sect. 4.4) by removing the restriction on the areas integral and imposing another condition on the initial velocity. An integral supposed to generalize (4.68) is provided. However, no exact solution is included in Mertsalov's paper, which is devoted to building an approximate solution. In fact, Mertsalov did not claim obtaining an exact solution in his work.

8.14.6 Gao's Work [98] (2003)

The purpose of this paper is to present two results that extend the famous integrable case due to Kowalevski of motion of a heavy rigid body about a fixed point. Unfortunately, one of these results is correct but not new. The other one is new but not correct [282].

Case 1. As per her original work, Kowalevski's case is characterized by the conditions $A = B = 2C$, $z_0 = 0$, i.e. the centre of mass lies in the equatorial plane of the inertia ellipsoid of revolution. As pointed out by Kowalevski, one can rotate the x, y-axes in this plane, and they will remain as principal axes, so as to have the centre of mass of the body on the x-axis and, equivalently, make $y_0 = 0$. This is done in most works on the subject, including Kowalevski's original work [238] of 1888 and classical textbooks like the ones by Whittaker and Leimanis. In the case introduced here y_0 is not necessarily zero. It is in no way an "extension" or "generalization" of Kowalevski's result. In fact, It is just an unneeded complication of the choice of the body reference system.

Case 2. This case, characterized by the conditions $A = B = 2C$, $y_0 = 0$, $x_0 z_0 \neq 0$, is in conflict with the necessary conditions obtained by Husson [154] in 1906 for the Euler–Poisson equations to have a fourth algebraic integral and cannot be true in principle. Moreover, it is easy to find that the expression H_4 given by the author can be an integral in only two cases:

(a) $z_0 = 0$, and this gives Kowalevski's case;

(b) $x_0 = 0$, and this case can be easily identified as a special case of Lagrange's top.

It is noteworthy that the same incorrect generalization as Case 2 is met much earlier in the classical monograph of Hagihara on celestial mechanics [144].

8.14.7 The Work of Yanxia and Keying [371] (2005)

Here is another claim to a generalization of Kowalevski's case by assuming $y_0 \neq 0$, which is merely referring the centre of mass to another coordinate system, rotated about the z-axis by some fixed angle. That is the same as Case 1 of Gao's work.

8.14.8 Ershkov's Work [76] (2014)

In his article, Ershkov announced a new exact solution of the classical problem, which, as the author states, generalizes the famous case of motion by inertia, found originally by Euler. Some details were further treated in [77]. In their comment [329], Sanduleanu and Petrov showed inconsistency of this result. To satisfy all the equations of motion, it reduces to a subcase of Euler's case.

The source of error in Ershkov's work is that he relied on the dynamical equations with the three integrals of motion and completely ignored Poisson's equations. He also assumed that $\mathbf{G}^2 = G_0^2$ (const). As shown in Sect. 8.1, under this condition, the three integrals of motion cannot replace Poisson's equation. It is necessary to check directly Poisson's equation on the given solution. Doing that, we find that on the Ershkov solution, only one of three Poisson's equations is satisfied.

8.15 Particular Solutions in the Problem of Motion of a Heavy Gyrostat

At present, fourteen particular solutions of the equations of motion of a heavy gyrostat are known, nine of which are generalizations of their counterparts in the classical problem of motion of a heavy simple rigid body. The other five have no analog as the gyrostatic momentum vanishes. However, the presence of the gyrostatic terms brings significant complications to the solution, especially to the process of separation of variables. Therefore, cases that were expressed in terms of elliptic functions are not brought to the same final state of the solution, due to complicated quadratures.

Here we give a brief note about each of the known cases. For clarity of information and for the sake of comparing the original and generalized cases, we preserve the order of those cases as in the classical problem. For each case, we give authors' names, references and the conditions on the gyrostatic momentum $\boldsymbol{\kappa}$. For full information about those cases, the reader is referred to the original works or one of the books, which bear review character, devoted to the motion of the gyrostat [121, 125].

8.15.1 *Generalizations of Particular Cases Known in the Classical Problem*

8.15.1.1 Generalization of Planar (Pendulum) Motion

The generalization is quite simple by adding a gyrostatic moment along the z-axis, the axis of rotation of the pendulum. Conditions on the system parameters are

$$\kappa_1 = \kappa_2 = z_0 = 0. \tag{8.184}$$

The solution is not affected by the presence of the gyrostatic moment. It is still given by formulas of Sect. 8.2. The motion is described and its orbital stability is investigated in [388, 420]. It turns out that the gyrostatic momentum plays a decisive effect on the stability of the motion.

8.15.1.2 Stationary Motion (Permanent Rotations) of the Gyrostat

(A) Permanent rotations of the gyrostat in Joukovsky–Volterra's case (The balanced gyrostat). In the case of absence of the gravity field (or in case of a gyrostat fixed from its centre of mass), permanent rotations satisfy the equations

$$\boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa}) = \mathbf{0}, \tag{8.185}$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}. \tag{8.186}$$

The first equation is completely independent of $\boldsymbol{\gamma}$ and determines possible values of the vector $\boldsymbol{\omega}$, constant in the body and in the space systems. Having obtained $\boldsymbol{\omega}$, the second equation can be solved for $\boldsymbol{\gamma}$.

Let us first multiply (8.185) scalarly by $\boldsymbol{\kappa}$. We get

$$\boldsymbol{\kappa} \cdot (\boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I}) = \mathbf{0}. \tag{8.187}$$

This means that in all possible permanent rotations, the angular velocity lies along a generator of a quadratic cone

$$\kappa_1(C - B)qr + \kappa_2(A - C)pr + \kappa_3(B - A)pq = 0. \tag{8.188}$$

This cone passes through the points $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}\mathbf{I}^{-1}$ and the three principal axes of inertia of the gyrostat. It is analogous to Staude's cone (8.31) for the permanent rotations of a simple heavy body with the gyrostatic momentum $\boldsymbol{\kappa}$ playing the role of the position vector of the centre of mass and $\boldsymbol{\omega}$ the role of the vertical unit vector $\boldsymbol{\gamma}$.

Dynamical Eq. (8.185) can be put in the parametric form

$$\frac{Ap+\kappa_1}{p}=\frac{Bq+\kappa_2}{q}=\frac{Cr+\kappa_3}{r}=s,\ \text{(say)}$$

so that one can write

$$p=\frac{\kappa_1}{s-A},\, q=\frac{\kappa_2}{s-B},\, r=\frac{\kappa_3}{s-C}. \tag{8.189}$$

This parametrization plays important role in the explicit time solution of the equations of motion of the balanced gyrostat [369].

We note also that, as in case of Staude's rotation, the angular velocity about an axis of permanent rotation grows indefinitely as this axis approaches any one of the principal axes of the gyrostat. Stability analysis of permanent rotations of the free gyrostat can be found in [369].

(B) Permanent rotations of the heavy gyrostat. This motion is characterized by the constancy of the two vectors $\boldsymbol{\omega}, \boldsymbol{\gamma}$ in the body axes. Setting $\dot{\boldsymbol{\omega}}=\mathbf{0}, \dot{\boldsymbol{\gamma}}=\mathbf{0}$, in (5.3), (5.4), we get

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa})=Mg\boldsymbol{\gamma}\times\mathbf{r}_0,\, \boldsymbol{\omega}\times\boldsymbol{\gamma}=\mathbf{0}. \tag{8.190}$$

From the second equation, we can write, as in the case of Staude's motion (See Sect. 8.3), $\boldsymbol{\omega}=\omega_0\boldsymbol{\gamma}$, so that the dynamical equation gives

$$\boldsymbol{\gamma}\times(\omega_0^2\boldsymbol{\gamma}\mathbf{I}+\omega_0\boldsymbol{\kappa}-Mg\mathbf{r_0})=\mathbf{0}. \tag{8.191}$$

This is the condition that the line along the vector $\boldsymbol{\gamma}$ intersects at right angle the ellipsoid

$$\Phi=\frac{1}{2}\omega_0^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}+(\omega_0\boldsymbol{\kappa}-\mathbf{M}g\mathbf{r_0})\cdot\boldsymbol{\gamma}=const. \tag{8.192}$$

For generic values of the parameters, the centre of this ellipsoid is displaced from the origin and only two solutions of (8.191) are guaranteed for arbitrary ω_0.

Let a given vector $\boldsymbol{\gamma}$ determine a direction in the main body of an axis of permanent rotations. Multiplying (8.191) scalarly by $\boldsymbol{\kappa}$ and by $\mathbf{r}_0$, we obtain two expressions for ω_0:

$$\omega_0=Mg\frac{\mathbf{r_0}\cdot(\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I})}{\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I})}=\frac{\boldsymbol{\gamma}\cdot(\mathbf{r}_0\times\boldsymbol{\kappa})}{\mathbf{r_0}\cdot(\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I})}. \tag{8.193}$$

Those expressions are compatible only on the fourth-degree surface

$$Mg[\mathbf{r_0}\cdot(\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I})]^2-[\boldsymbol{\gamma}\cdot(\mathbf{r}_0\times\boldsymbol{\kappa})][\boldsymbol{\kappa}\cdot(\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I})]=0. \tag{8.194}$$

This surface intersects the Poisson sphere $\boldsymbol{\gamma}^2=1$ in a curve. The generators of the cone formed by the axes of permanent rotations are drawn by connecting the origin to the points of that curve.

A notable degeneration of (8.193) and (8.194) occurs when $\boldsymbol{\kappa}$ is collinear with $\mathbf{r_0}$. In that case, if we choose the uniform angular velocity ω_0 such that Γ

$$\omega_0 \boldsymbol{\kappa} = Mg\mathbf{r_0}, \tag{8.195}$$

then (8.191) reduces to

$$\gamma \times \gamma \mathbf{I} = \mathbf{0}. \tag{8.196}$$

Stationary rotation is possible about any of the three principal axes of inertia, which would be directed vertically upwards. The angular velocity ω_0 has positive or negative sign, according to whether $\boldsymbol{\kappa}$ and $\mathbf{r_0}$ are parallel or anti-parallel, so that only three different motions of this type are possible.

8.15.1.3 Generalization of Hess' Case (Sretensky [340, 341] 1963)

When $y_0 = 0$, $\kappa_2 = 0$, under Hess' condition (8.36)

$$x_0\sqrt{A(B-C)} = \pm z_0\sqrt{C(A-B)},$$

equations of motion (5.5)–(5.6) admit the invariant relation

$$[(A-B)p + \kappa_1]x_0 + [(B-C)r - \kappa_3]z_0 = 0$$

which generalizes that of Hess by the presence of the components of the gyrostatic momentum. It is possible in that case to use the variables Γ_1, Γ_3 from (8.40) and obtain instead of (8.45) the relation

$$t = \int^{\Gamma_3} \frac{du}{\sqrt{2B(1-u^2)(h-su) - [f + (\frac{\kappa_1 \cos\delta}{A-B} - \frac{\kappa_2 \sin\delta}{B-C})u]^2}},$$

so that the centre of mass of the body performs with respect to vertical a motion like a spherical pendulum. A separable case analogous to the one of Hess' case and detailed analysis of the motion are given in [121] using the special axes attached to the gyration ellipsoid.

8.15.1.4 Generalization of the Bobylev–Steklov Case (Kharlamov [193] 1964)

This case is characterized by the conditions

$$\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3),$$

$$(2C - A)\kappa_2 x_0 = (2C - B)\kappa_1 y_0, z_0 = 0, \tag{8.197}$$

and the invariant relations

$$p = \lambda x_0, q = \lambda y_0. \tag{8.198}$$

As in the Bobylev–Steklov case, the solution is expressible in elliptic functions of time.

In a slightly earlier paper [167], Keis found a gyrostat generalization of the Bobylev–Steklov case. His result may be obtained from (8.197) by putting $\kappa_2 = y_0 = 0$.

8.15.1.5 Generalization of Steklov's and Kowalewski's Cases (Kharlamov [196] 1965)

In [196], Kharlamov investigated the possibility to generalize the invariant relations (8.80) and (8.123), which express q and r in terms of p, under the simplifying restriction that the gyrostatic moment is directed along the first coordinate axis x carrying the centre of mass of the body. In that case, one can write

$$\boldsymbol{\kappa} = (\kappa, 0, 0). \tag{8.199}$$

It turned out that both cases accommodate such gyrostatic moment. From formulas provided in [195, 196], it follows that the Euler–Poisson variables can still be expressed in elliptic functions of time in the generalization of Steklov's solution. However, this step was not performed due to the complication due to the presence of terms linear in p in the two quadratic factors generalizing (8.81).

In the generalization of Kowalewski's case, the relation of p and time is still determined by a hyper-elliptic quadrature [195] (See also [125]).

8.15.1.6 Generalizations of Grioli's Case (Keis 1965, Kharlamova 1969)

Keis (1963) [167] (see also [169]) discussed the existence of regular precession in the motion of a gyrostat about a fixed point under the action of gravity. It turned out that Grioli's case admits generalization, in which the gyrostatic momentum is directed parallel to the line from the fixed point to the centre of mass of the body. Kharlamova obtained the same generalization of Grioli's case in non-principal axes [208] in a search for linear invariant relation. Detailed explicit analysis of the regular precessional motion of a heavy gyrostat was again performed by Gulyaev using principal axes of inertia in [142].

8.15.1.7 Generalization of Dokshevich's Second Case (Kharlamova 1971)

In [209], Kharlamova constructed a new exact particular solution under the same condition on the distribution of mass that characterizes the body in Hess' case, i.e. the centre of mass lies on the orthogonal to the circular cross-section of the gyration ellipsoid. This case differs from that of Sretensky in that the gyrostatic momentum lies in the principal plane of inertia containing the major and minor axes. The Euler–Poisson variables were expressed in terms of an auxiliary variable, which may be expressed as an elliptic function of time. Formulas provided in [209] are quite complicated. The same case was later reconsidered by Kharlamov, who made some simplification of the conditions on the parameters and established that the solution has four different versions [199].

8.15.1.8 Generalization of Konosevich–Pozdnyakovich's Case (Kharlamova and Mozalevskaya [212] 1986)

In the book [212], Kharlamova and Mozalevskaya give three particular solutions constructed using invariant relations of the type used in the case of Konosevich–Pozdnyakovich. One of those cases contains Konosevich–Pozdnyakovich's case as a special case and the other two are different and have no analog in the classical problem. However, those cases are quite complicated and as presented in [212] are not given in the complete and verifiable form. The same content is copied in the two recent books [121, 125] without further clarification. The reader may consult the mentioned books for more information.

8.15.2 Solvable Cases of the Gyrostat, Having No Classical Analog

8.15.2.1 Case of Kharlamov and Kovalev [202] (1970) and of Kharlamova and Kharlamov [210] (1969)

This case is valid under the restriction on the principal moments of inertia

$$A = 18C\frac{B-C}{10B-9C}, \tag{8.200}$$

i.e. under the same condition as Kowalewski's case in the classical problem and its generalization Sect. 8.15.1.4. The centre of mass lies on the x-axis and the gyrostatic momentum is directed along the y-axis, i.e.

$$\mathbf{r}_0 = (x_0, 0, 0), \boldsymbol{\kappa} = (0, \kappa, 0). \tag{8.201}$$

The invariant relations are taken in the form $p^2 = P_2(q), r^2 = P_3(q)$, where P_i is a polynomial of degree i. The variables of the problem are expressed in terms of q, while q is related to time by an elliptic quadrature. A somewhat simpler parametrization is given but explicit inversion of this quadrature is not performed. More detailed presentation is given in [202].

It is notable that in the present solution the gyrostatic momentum κ figures in denominators in most formulas, so that the solution loses its meaning as κ tends to zero.

A generalized version of this solution due to Kharlamov and Kharlamova was published in [210] (See also [212]). In that case, one more component of the gyrostatic moment is added to the body along the z-direction.

8.15.2.2 Case of Mozalevskaya [291] (1970)

This case is valid for a body whose moments of inertia do not satisfy one of the triangle inequality. Thus, it is of no physical significance neither for the problem of motion of a gyrostat nor for the classical problem. It is noted that such a case has its physical interpretation in the frame of the problem of motion of a rigid body in a liquid, in which the triangle inequalities are not necessary.

8.15.2.3 Case of Dokshevich [65] (1970)

This case is valid under the conditions

$$\begin{aligned} B &= C, \\ y_0 &= z_0 = 0, \\ \boldsymbol{\kappa} &= (\kappa_1, \kappa_2, 0), \end{aligned} \tag{8.202}$$

where $\kappa_2 \neq 0$. Invariant relations are sought in the form $q = P_2, r^2 = P_4$, P_i is a polynomial of degree i. It was shown in [65] that the solution can be expressed in elliptic functions of time. The parameter κ_2 occurs in denominators in several formulas, so that the solution loses its meaning when $\kappa_2 = 0$.

8.15.2.4 Two Cases of Kharlamova and Mozalevskaya [212] 1986

Together with the generalization of Konosevich–Pozdnyakovich's case (item No. 8 in the previous subsection), the authors found two similar other cases. The solution is given by relations of a higher level of complexity. Moreover, they lose their meaning when the gyrostatic moment vanishes.

8.15.3 Known Cases of the Classical Problem, Which Are not Presently Generalized to the Gyrostat Problem

Three cases are left at present without generalization by adding a gyrostatic moment. Those are cases due to Goryachev, Chaplygin and Dokshevich's first case.

Chapter 9
The Rigid Body in a Potential Field

In this chapter, we present some results which can not only be applied to the classical problem of motion of a heavy rigid body about a fixed point, but can also be easily extended to cases of motion of a rigid body acted upon by general forces which admit symmetry about an axis fixed in space and passing through the fixed point. For such fields, the potential is a function of the Eulerian angles θ, φ, and the precession angle is a cyclic coordinate. The areas integral is preserved. The study of the Routhian reduction of the problem opens some possibilities for deeper analysis of properties of motion.

9.1 The Routhian in $\gamma_1, \gamma_2, \gamma_3$ as Redundant Variables

Reduced equations of motion in their form Chap. 3 (3.57) are not usually convenient in use because of lacking symmetry. It is possible to give the Routhian a more symmetric form that allows to go further in applications, by using the components of the vector $\boldsymbol{\gamma}$. To this end, one can use a direct substitution

$$\theta = \cos^{-1} \gamma_3, \varphi = \tan^{-1} \frac{\gamma_1}{\gamma_2} \tag{9.1}$$

in the formulas Chap. 3 (3.55 and 3.56) and making use of the constraint

$$\gamma_1 \dot{\gamma}_1 + \gamma_2 \dot{\gamma}_2 + \gamma_3 \dot{\gamma}_3 = 0,$$

resulting from the differentiation of the geometric integral. However, we shall use a less direct, but more effective procedure as follows.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_9

9.1.1 Expression of ω in Terms of γ and $\dot{\gamma}$

Multiplying Poisson's equation (3.14) vectorially by $\boldsymbol{\gamma}\mathbf{I}$ and using the areas integral (3.22), one gets

$$\boldsymbol{\omega} = \frac{f\gamma + \dot{\gamma} \times \gamma\mathbf{I}}{D}, \tag{9.2}$$

where D is given by, the same as in (3.48),

$$D = \boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma} = A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2. \tag{9.3}$$

Kinetic energy in the redundant coordinates:
Using (9.2), one can write

$$\begin{aligned} T &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} \\ &= \frac{1}{2D}\boldsymbol{\omega}\mathbf{I} \cdot [f\gamma + \dot{\gamma} \times \gamma\mathbf{I}] \\ &= \frac{1}{2D}[f^2 + \dot{\gamma} \cdot (\gamma\mathbf{I} \times \boldsymbol{\omega}\mathbf{I})], \end{aligned}$$

and using an identity (A) from appendix, we rewrite the last expression as

$$\begin{aligned} T &= \frac{1}{2D}[f^2 + \dot{\gamma}(adj(\mathbf{I})) \cdot (\gamma \times !)] \\ &= \frac{1}{2D}[f^2 + \dot{\gamma}(adj(\mathbf{I})) \cdot \dot{\gamma}] \\ &= \frac{1}{2D}[f^2 + ABC\dot{\gamma}\mathbf{I}^{-1} \cdot \dot{\gamma}]. \end{aligned} \tag{9.4}$$

The Routhian in the redundant coordinates $\gamma_1, \gamma_2, \gamma_3$
Writing from (3.55) and using (9.4) and (9.1), one obtains

$$\begin{aligned} R &= L - f\dot{\psi} \\ &= T - Mg\mathbf{r}_0 \cdot \gamma - f\dot{\psi} \\ &= \frac{1}{2D}[f^2 + ABC\dot{\gamma}\mathbf{I}^{-1} \cdot \dot{\gamma}] - Mg\mathbf{r}_0 \cdot \gamma \\ &\quad - f.\frac{1}{D}[f - (A - B)\sin\theta\sin\varphi\cos\varphi\dot{\theta} - C\cos\theta\dot{\varphi}] \\ &= \frac{ABC}{2D}\dot{\gamma}\mathbf{I}^{-1} \cdot \dot{\gamma} + f[(A - B)\sin\theta\sin\varphi\cos\varphi\dot{\theta} + C\cos\theta\dot{\varphi}] \\ &\quad -[V + \frac{f^2}{2D}] \end{aligned}$$

and we finally have, in the principal axes of inertia of the body at the fixed point,

$$
\begin{aligned}
R = {} & \frac{ABC}{2D}\left(\frac{\dot\gamma_1^2}{A} + \frac{\dot\gamma_2^2}{B} + \frac{\dot\gamma_3^2}{C}\right) \\
& + \frac{f}{D(1-\gamma_3^2)}[C\gamma_3(\gamma_2\dot\gamma_1 - \gamma_1\dot\gamma_2) - (A-B)\gamma_1\gamma_2\dot\gamma_3] \\
& - V_1,
\end{aligned}
\tag{9.5}
$$

where $V_1 = [V + \frac{f^2}{2D}]$ and D is given by (9.3).

Equations of motion can now be deduced in the form of three second-order equations involving a Lagrangian multiplier $\lambda(t)$—say, resulting from the holonomic constraint

$$\gamma^2 - 1 = 0, \tag{9.6}$$

which should be added to the Routhian R. Solving the resulting equations is equivalent to the determination of the Eulerian angles θ, φ, and then one can substitute this solution into (9.2) to get an expression for the angular velocity $\boldsymbol{\omega}$. One can also substitute the solution into (3.54) and integrate it with respect to time to complete the determination of the precession angle ψ.

However, we shall not write these equations down, since they will not play an important role in the sequel. The real significance of the Routhian (9.5) (or, in other words, the Lagrangian of the reduced system) will be elucidated in the following sections, where it will be transformed into various more flexible forms that lead to significant consequences.

Remark: It should be noted that after solving the Routhian equations of motion, one can obtain the cyclic angle ψ by integrating the formula

$$
\begin{aligned}
\dot\psi &= -\frac{\partial R}{\partial f} \\
&= \frac{1}{A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2}\{f + \frac{1}{1-\gamma_3^2}[C\gamma_3(\gamma_1\dot\gamma_2 - \gamma_2\dot\gamma_1) + (A-B)\gamma_1\gamma_2\dot\gamma_3]\}.
\end{aligned}
\tag{9.7}
$$

9.1.2 The Case of Complete Dynamical Symmetry of the Body

When $B = C = A$, the Routhian (9.5) takes the form

$$
\begin{aligned}
R = {} & \frac{1}{2}A(\dot\gamma_1^2 + \dot\gamma_2^2 + \dot\gamma_3^2) \\
& + \frac{fA\gamma_3}{1-\gamma_3^2}(\gamma_2\dot\gamma_1 - \gamma_1\dot\gamma_2) - V - \frac{f^2}{2A}.
\end{aligned}
\tag{9.8}
$$

The last term in this Routhian is now a constant and can be discarded from the potential. But one has to remember that $\dot\psi$ is still determined by (9.7), which for the case of spherical dynamical symmetry takes the form

$$\dot{\psi} = \frac{A\gamma_3}{1-\gamma_3^2}(\gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2) - \frac{f}{A}. \tag{9.9}$$

The Routhian (9.8) may be interpreted as describing the problem of motion of a particle on the Poisson sphere

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

under the action of forces with potential V and certain gyroscopic forces. This analogy will be used more than once later in this book.

9.2 Maximal Reduction of Order of the Differential Equations of Motion of a Rigid Body About a Fixed Point

9.2.1 Reduction

The six Euler–Poisson first-order differential equations describing the classical problem of motion of a rigid body about a fixed point admit three first integrals. All the equations and integrals are time-independent. This situation lead several authors, along nearly a whole century since 1890, to consider using the three integrals to eliminate three of the Euler–Poisson variables and thus reduce the system to three first-order autonomic equations. Eliminating time (differential) between those equations, we are then left with a system of two equations of the first order.

The history of the reduction of order began with Hess' work [150] (1890), in which he used the integrals of motion to express $\boldsymbol{\gamma}$ in terms of $\boldsymbol{\omega}$ and eliminated $\boldsymbol{\gamma}$ from the dynamical equations of motion of a heavy rigid body. A system of three autonomous first-order equations is obtained. The question of equivalence of those equations to the original problem has led Hess to the known solvable case under his name (see Sect. 8.4). However, the reduced equations were not used in any application. Relying also on algebraic elimination of the vector $\boldsymbol{\gamma}$, Shiff [331] and Stäckel [342] practically repeated Hess' main result in their works. Kharlamov [195] made the most use of this direction for the motion of a heavy gyrostat, referring the components of the angular velocity to the special axes of the gyration ellipsoid. He wrote the three autonomous equations in a more compact form, which was later used by the Donetsk School of Mechanics in the search for particular solvable cases. The ultimate reduction to a single second-order equation was tried by Kharlamova in [206], but the equation was so complicated that it was not written explicitly in final form and was not used in any application.

Bilimovich [23] went another way. He used Euler's angles as generalized coordinates to write equations of motion in Hamiltonian form and the integrals of areas and energy to eliminate the variables ψ, t. Bilimovich suggested the use of variables $x = \tan\frac{\theta}{2}$, $y = \tan\frac{\varphi}{2}$ and pointed out a sequence of operations which could lead to a single second-order equation of the structure

$$y'' = P_3 + V\sqrt{V_1 P_2^3}, \tag{9.10}$$

where $y' = \frac{dy}{dx}$, V, V_1 are rational functions of x, y and P_2, P_3 are polynomials of degrees 2 and 3 in y' with coefficients rational in x, y. Equation (9.10) was not explicitly written down in final form.

A similar situation was reached by Kharlamova in [207], using Poincaré–Cartan integral invariant. The variables ψ, t are eliminated and a procedure is pointed out to obtain an equation of second order, equivalent to (9.10), with the structure

$$\frac{d^2\phi}{d\theta^2} = F(\theta, \varphi, \frac{d\phi}{d\theta}),$$

but the final explicit form was not given. Such complicated equation was not used in any application.

The ultimate solution of the reduction problem in the geometric direction initiated by Bilimovich turned out to be using the components of the vector $\boldsymbol{\gamma}$ as coordinates describing the motion of the reduced system, after eliminating the cyclic coordinate ψ by Routh's procedure. The final form of the equation produced by this method is relatively simple and has certain symmetry properties. It turned out to be useful in deriving simple solutions and in the study of the stability of certain known solutions. Examples of its uses are given in the relevant chapters. A notable advantage of the method is that it is equally applicable to obtain a reduction to a second-order equation in the most general problem of motion of a rigid body under the action of conservative position-dependent potential and gyroscopic forces.

The result we are going to establish here was reached by elementary, but more lengthy steps in our 1983 and 1986 works. Here, we shall first apply Hamilton's principle in the form of Jacobi (see, for example, [305]) to the time-irreversible system with the Routhian (9.5). Equations of motion are deduced from the variational problem

$$\delta\int R dt = 0, \tag{9.11}$$

and applying Maupertuis' principle to eliminate the time differential from the variation, we arrive at the final variational problem

$$\delta \int \sqrt{\frac{2ABC(h-V_1)}{D}(\frac{d\gamma_1^2}{A}+\frac{d\gamma_2^2}{B}+\frac{d\gamma_3^2}{C})}$$
$$+\frac{f}{D(1-\gamma_3^2)}[C\gamma_3(\gamma_2 d\gamma_1-\gamma_1 d\gamma_2)-(A-B)\gamma_1\gamma_2 d\gamma_3]$$
$$=0, \tag{9.12}$$

where $V_1 = V+\frac{f^2}{2D}$ and variations of trajectories are made among the ones with total energy h of the original system or Jacobi's constant h of the Routhian system, the areas integral f as well as the condition that the point $\boldsymbol{\gamma}$ lies on the Poisson unit sphere.

Now we use the last condition to eliminate γ_2. Moreover, we choose γ_1 (say) to be the independent variable and γ_3 as the dependent one. We reduce the (9.12) to the variational problem

$$\delta \int S d\gamma_1 = 0, \tag{9.13}$$

where

$$S = \sqrt{\frac{2ABC(h-V_1)}{D(1-\gamma_1^2-\gamma_3^2)}(\boldsymbol{\lambda}+2\mu\gamma_3'+\nu\gamma_3'^2)}$$
$$+\frac{f}{D\sqrt{1-\gamma_1^2-\gamma_3^2}}\{C\gamma_3+\gamma_1[C\gamma_3^2-(A-B)(1-\gamma_1^2-\gamma_3^2)]\gamma_3'\}, \tag{9.14}$$

$$\begin{aligned} D &= B+(A-B)\gamma_1^2+(C-B)\gamma_3^2,\\ \lambda &= C[A\gamma_1^2+B(1-\gamma_1^2-\gamma_3^2)],\\ \mu &= AC\gamma_1\gamma_3,\\ \nu &= A[C\gamma_3^2+B(1-\gamma_1^2-\gamma_3^2)]. \end{aligned} \tag{9.15}$$

The prime denotes differentiation with respect to γ_1 and in V_1 the variable γ_2 is eliminated by the geometric constraint.

The function S is a Lagrangian function which describes a one-degree-of-freedom system. The equation of motion of that system can be written as

$$\frac{d}{d\gamma_1}\frac{\partial S}{\partial \gamma_3'}-\frac{\partial S}{\partial \gamma_3}=0.$$

After some tedious calculations, we finally obtain the equation

$$\begin{aligned}
&D(1-\gamma_1^2-\gamma_3^2)\gamma_3''+C\gamma_3(1-\gamma_3^2)\\
&-\gamma_1[A-(A+2C)\gamma_3^2]\gamma_3'+\gamma_3[C-(C+2A)\gamma_1^2]\gamma_3'^2\\
&-A\gamma_1(1-\gamma_1^2)\gamma_3'^3\\
&-\frac{\rho}{ABCD}\{C\gamma_3[(A-B)(A+B-C)\gamma_1^2+B(B-C)(1-\gamma_3^2)]\\
&\qquad+A\gamma_1[(B-C)(B+C-A)\gamma_3^2+B(A-B)(1-\gamma_1^2)]\gamma_3'\}\\
&+\frac{\rho}{2ABC(h-V_1)}[\frac{\partial V_1}{\partial\gamma_3}(\lambda+\mu\gamma_3')-\frac{\partial V_1}{\partial\gamma_1}(\mu+\nu\gamma_3')]\\
&+\frac{f\rho^{3/2}}{ABC\sqrt{2D^3(h-V_1)}}\times\\
&\times[(A-B)(A+B-C)\gamma_1^2-B(A-B+C)+(C-B)(B+C-A)\gamma_3^2]\\
=\;&0,
\end{aligned}\tag{9.16}$$

where

$$\begin{aligned}
\rho &= \lambda+2\mu\gamma_3'+\nu\gamma_3'^2,\\
\lambda &= C[B(1-\gamma_3^2)+(A-B)\gamma_1^2],\\
\mu &= AC\gamma_1\gamma_3\\
\nu &= A[B(1-\gamma_1^2)+(C-B)\gamma_3^2].
\end{aligned}$$

This equation is satisfied by some orbits on the Poisson sphere, drawn by the apex of the vector $\boldsymbol{\gamma}$ on the surface during motion of the body, and hence we shall call it "the orbital equation". It involves the two parameters h (the total energy) and f (the areas constant of the motion). If one has a solution of this equation in the form

$$\gamma_3=\gamma_3(\gamma_1),$$

then one can establish the relation of variables with time. From the energy integral, eliminating γ_2, we write

$$dt^2=\frac{\lambda d\gamma_1^2+2\mu d\gamma_1 d\gamma_3+\nu d\gamma_3^2}{2D(1-\gamma_1^2-\gamma_3^2)(h-V_1)},$$

so that the relation between γ_1 and time is given by

$$t=\int\sqrt{\frac{\lambda+2\mu\gamma_3'+\nu\gamma_3'^2}{2D(1-\gamma_1^2-\gamma_3^2)(h-V_1)}}d\gamma_1.\tag{9.17}$$

Inverting this integral, we obtain $\gamma_1(t)$ and immediately we also have $\gamma_3=\gamma_3(\gamma_1(t))$. The component γ_2 is obtained from the geometric constraint. This completes the determination of $\boldsymbol{\gamma}$ as a function of t. Returning to the formula (9.2), we substitute $\boldsymbol{\gamma}$

and its derivative $\dot{\gamma}$ to construct an expression for the angular velocity $\boldsymbol{\omega}$ in terms of time. We have thus completed the reduction of the problem of motion of a rigid body about a fixed point in a potential field, in all its generality, to the single second-order ODE (9.16).

9.2.2 Applications

Although the orbital equation (9.16) is not a simple one, it is compact and, to some extent, symmetric. It turned out to be so effective for some purposes.

9.2.2.1 Orbital Stability of Known Exact Solutions

For example, under some conditions on the potential V and under the condition $f = 0$, Eq. (9.16) admits the simplest solution

$$\gamma_3 = 0. \tag{9.18}$$

This solution corresponds in the original problem the planar motion of the body (pendulum motion) about the third principal axis of inertia, which takes a constant horizontal position during motion.

These motions are either oscillation or rotation, according to the total energy of the body, and they are periodic in time. The equations in the variations of periodic motions are linear equations with periodic coefficients. Thus, for the study of the stability of a periodic motion of the rigid body, we deal with six equations with periodic coefficients, which admit three integrals. The study of such system is quite difficult and can be performed mainly numerically. The study of the stability of such motion seemed intractable and was never considered in the literature. The orbital equation opened up a way to tackle this problem. The study was initiated in [374] using the reduced second-order equation of the orbit, and thus called orbital stability. It means that after the perturbations preserving the energy h and the areas integral $f = 0$, the orbit of motion on the Poisson sphere remains near to the circle (9.18). The equation in the variation for this periodic motion reduces in general to Hill's linear differential equation with a periodic coefficient, which depends on the parameters of the body and on h.

The simplest case occurs when the body is dynamically symmetric and the centre of mass lies on a principal axis x. The equation of the variation is transformed into Lame's equation:

$$\frac{d^2\gamma_3}{du^2} + [\alpha(\alpha\nu^2 + \frac{1}{2}) - \alpha(\alpha + 1)\nu^2 \operatorname{sn}^2 u]\gamma_3 = 0, \tag{9.19}$$

for vibrational motions, where $\alpha = 2C/A$, $\nu = \sqrt{\frac{h+a}{2a}} < 1$ (the modulus of elliptic functions), $a = Mgx_0$ and u is the independent variable, linearly depending on time. In virtue of the triangle inequalities for the moments of inertia A, A, C, it follows that α is restricted to the interval $0 < \alpha \leq 4$.

For rotational pendulum motions, the equation in the variation takes the form

$$\frac{d^2\gamma_3}{dv^2} + [\alpha(\alpha + \frac{1}{2}k^2) - \alpha(\alpha+1)k^2 \operatorname{sn}^2 v]\gamma_3 = 0, \tag{9.20}$$

$k = \sqrt{\frac{2a}{h+a}} < 1$ (the modulus of elliptic functions) and v linearly depends on t.

The zones of stability and instability of the orbit of motion in the plane of the parameters α, ν are separated by curves in that plane, which carry primary periodic solutions. Explicit analytical expressions for those curves were given in [374], mainly with the help of some results of Ince [156, 157] concerning periodic solutions of the Laméequation. The study was extended to include pendulum motions of a rigid body in an approximate Newtonian field of a far centre of attraction [376], the rigid body carrying a rotor (gyrostat) [377] and the case of general (not dynamically symmetric) body [388].

Plane motions and their stability were considered in some later papers on the basis of Euler–Poisson equations. Tkhai and Schvigin analysed numerically the case of vibration for three values of energy parameter [355]. Dovbysh investigated some qualitative properties of rotational motions [68]. Markeev and coworkers studied the stability of pendulum motions not only in the linear approximation but also in the nonlinear setting, using reduction of the Hamiltonian to the Birkhoff normal form [274]. However, this was done mainly for bodies, whose distribution of mass satisfies conditions of integrable cases, for example, Kowalevski's case [275] and the Goryachev–Chaplygin case [15, 279]. Unfortunately, the rather recent works [15–17, 279] do not mention the results of quite earlier works [375–377, 388], concerning the stability of pendulum motions in the linear setting, although some comparison of the results could have been useful. Note that the assertion in [17], that a countable number of stability and instability zones appear in the plane of parameters, is dubious, since it is incompatible with our earlier results.

Stability analysis was performed also for some other known periodic solutions of the Euler–Poisson equations of motion. Grioli's regular precessional motion is one of the most famous periodic solutions Chap. 8 (8.10). Its stability was considered in Grioli's original 1947 paper [138]. Tkhai studied the full problem numerically and obtained partial results [356]. The most exhaustive analysis of the stability of Grioli's motion was performed by Markeev in two papers [276, 277]. Zones of stability and instability were presented in the plane of the two parameters that determine the state of the system. Markeev also studied the stability of the periodic solution of Euler–Poisson equations known after Steklov [33]. The problem also involves two parameters, and results are presented in their plane.

In [420], the stability of pendulum motion of a heavy body carrying a rotor is a generalization to the classical problem. The orbital equation was also successfully used to investigate the stability of the particular solution of the classical problem known after the names of Bobylev and Steklov (see Chap. 8 (8.5)). Zones of stability were drawn in 3D space [426]. Results obtained by the use of the orbital equation were not obtained earlier by any other method.

9.2.2.2 Search for New Exact Solutions of the Equations of Motion

In a recent paper [427], the orbital equation was used in a quite different application. An exact solution of (9.16) is sought in the form $\gamma_3 = A_0 + A_1\gamma_1$. The forces acting on the body are assumed to be a combination of the uniform gravity field and Newtonian field of attraction of a centre, which is sufficiently far from the body to be taken in the approximate form $Mg\mathbf{r}_0 \cdot \boldsymbol{\gamma} + \frac{1}{2}\lambda\boldsymbol{\gamma} I \cdot \boldsymbol{\gamma}$. A particular solution is constructed, which turned out to be a generalization of the first case of Dokshevich (see Chap. 8 (8.11)) and turns into it as $\lambda = 0$.

9.3 Reduction to the Motion of a Particle on an Ellipsoid

The form of the Routhian (9.5) suggests using the transformation of the coordinates $\gamma_1, \gamma_2, \gamma_3$ by the relations

$$\begin{aligned}\gamma_1 &= \sqrt{A}\xi,\\ \gamma_2 &= \sqrt{B}\eta,\\ \gamma_3 &= \sqrt{C}\zeta,\end{aligned} \tag{9.21}$$

so that the geometric integral becomes

$$A\xi^2 + B\eta^2 + C\zeta^2 = 1. \tag{9.22}$$

Clearly, ξ, η, ζ are Cartesian coordinates of a point on the inertia ellipsoid. The Routhian transforms to

$$\begin{aligned}R = &\frac{ABC}{2D}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\\ &+\frac{fABC}{1 - C\zeta^2}[C\zeta(\eta\dot{\xi} - \xi\dot{\eta}) - (A - B)\xi\eta\dot{\zeta}]\\ &-V_1,\end{aligned} \tag{9.23}$$

where

$$D = A^2\xi^2 + B^2\eta^2 + C^2\zeta^2, \tag{9.24}$$

$$V_1 = V + \frac{f^2}{2D}. \tag{9.25}$$

The last Routhian may be given the interpretation as describing the problem of motion of a particle on the smooth surface of ellipsoid (9.22), but instead of the standard Euclidean metric this ellipsoid is endowed by a Riemannian metric conformal to it that has the form

$$ds^2 = \frac{d\xi^2 + d\eta^2 + d\zeta^2}{A^2\xi^2 + B^2\eta^2 + C^2\zeta^2}. \tag{9.26}$$

This alternative description of the quadratic part of the Routhian of the reduced problem of rigid body dynamics appeared first in a paper of Minkowsky [284] in 1888, devoted to another problem, namely that of the motion of a rigid body in a fluid.

A usual procedure, frequently used to get out from such situation, is to make a change of the time variable t and use a new independent variable τ, related to it by the differential relation

$$dt = \frac{ABC}{D}d\tau. \tag{9.27}$$

This procedure is applicable only to iso-energetic motions, i.e. motions on which the energy constant of the original system with Lagrangian Chap. 3 (3.44) and Jacobi's constant for the Routhian (9.23) keep a constant value h. The transformation renders the Routhian to a new one given by

$$\begin{aligned} R^* = {} & \frac{1}{2}(\xi'^2 + \eta'^2 + \zeta'^2) \\ & + \frac{fABC}{1 - C\zeta^2}[C\zeta(\eta\xi' - \xi\eta') - (A - B)\xi\eta\zeta'] \\ & - V^*, \end{aligned} \tag{9.28}$$

where $V^* = \frac{ABC}{D}(V_1 - h)$. For a detailed account of the time transformation, see, for example, the book of Pars [305]. Note that Jacobi's integral for the new Routhian R^* must be taken to be zero,

$$\frac{1}{2}(\xi'^2 + \eta'^2 + \zeta'^2) + V^* = 0, \tag{9.29}$$

while Jacobi's constant h for the pre-transformation Routhian R now enters as a parameter in the reduced potential.

The advantage of the new Routhian R^* is that it describes the motion of a particle of unit mass on the ellipsoid of inertia (9.22) of the body with the standard Euclidean metric. The motion is driven by two types of conservative forces: potential forces with the potential function V^* and gyroscopic forces characterized by terms linear in velocities in (9.28).

9.4 The Use of Elliptic Coordinates on the Inertia Ellipsoid [224]

The idea to use Jacobi's coordinates on the inertia ellipsoid as generalized coordinates in the description of rigid body dynamics appeared in Minkowsky's work [284], devoted to the study of the problem of motion of a rigid body in a liquid in the integrable case due to Clebsch. In that work, Minkowsky suggested using the same idea in the general problem of motion of a rigid body about a fixed point. Building on Minkowsky's result, Kolossov [224] realized this idea and used Routh's procedure to ignore the cyclic coordinate, the angle of precession, and thus simplifying the calculations.

Suppose that $A \neq B \neq C$. The point on the ellipsoid of inertia (9.22) is parametrized in elliptic coordinates by the following relations:

$$\begin{aligned}
\xi^2 &= \frac{1}{A}\frac{(u-\frac{1}{A})(v-\frac{1}{A})}{(\frac{1}{B}-\frac{1}{A})(\frac{1}{C}-\frac{1}{A})},\\
\eta^2 &= \frac{1}{B}\frac{(\frac{1}{B}-u)(v-\frac{1}{B})}{(\frac{1}{B}-\frac{1}{A})(\frac{1}{C}-\frac{1}{B})},\\
\zeta^2 &= \frac{1}{C}\frac{(\frac{1}{C}-u)(\frac{1}{C}-v)}{(\frac{1}{C}-\frac{1}{B})(\frac{1}{C}-\frac{1}{A})}.
\end{aligned} \tag{9.30}$$

This parametrization is valid, provided B is the middle moment of inertia. For determinacy, we assume that $C \leq B \leq A$ and $\frac{1}{A} \leq u \leq \frac{1}{B} \leq v \leq \frac{1}{C}$. In those coordinates $D = ABCuv$.

Substituting (9.30) in the Routhian (3.57), we obtain

$$\begin{aligned}
R^* = \frac{1}{2}\frac{v-u}{4}&\Big[\frac{u}{(u-\frac{1}{A})(u-\frac{1}{B})(u-\frac{1}{C})}u'^2 + \frac{v}{(v-\frac{1}{A})(v-\frac{1}{B})(\frac{1}{C}-v)}v'^2\Big]\\
&+lu' + mv' - V^*,
\end{aligned} \tag{9.31}$$

where

$$\begin{aligned}
l &= \frac{fABC}{1-C\zeta^2}\Big[C\zeta\Big(\eta\frac{\partial \xi}{\partial u} - \xi\frac{\partial \eta}{\partial u}\Big) - (A-B)\xi\eta\frac{\partial \zeta}{\partial u}\Big],\\
m &= \frac{fABC}{1-C\zeta^2}\Big[C\zeta\Big(\eta\frac{\partial \xi}{\partial v} - \xi\frac{\partial \eta}{\partial v}\Big) - (A-B)\xi\eta\frac{\partial \zeta}{\partial v}\Big],\\
V^* &= \frac{1}{uv}\Big(V + \frac{f^2}{2ABCuv} - h\Big).
\end{aligned} \tag{9.32}$$

Practically, this form of the Routhian was used by Kolossov [224], but only giving zero value to the areas constant f and thus reducing the Routhian to a time-reversible one.

Kolossov proceeded to consider certain potentials which lead to cases of separation of variables in the Hamilton–Jacobi equation for the transformed problem at $f = 0$, finally reaching cases solved earlier in terms of time in the works of Weber [367], Kobb [223] and others.

9.5 The Use of Isometric Coordinates on the Inertia Ellipsoid

A formal, but important, step was taken by Arjanykh [13], without referring to Minkowsky or Kolossov. He replaced Jacobi's coordinates with isometric coordinates on the inertia ellipsoid. In those coordinates, the equations of motion of a heavy rigid body about a fixed point are transformed into a problem of motion of a particle in the plane under the action of certain potential and gyroscopic forces.

Let us define two new coordinates

$$\begin{aligned} X &= \int \frac{\sqrt{u}du}{\sqrt{(u-\frac{1}{A})(\frac{1}{B}-u)(\frac{1}{C}-u)}}, \\ Y &= \int \frac{\sqrt{v}dv}{\sqrt{(v-\frac{1}{A})(v-\frac{1}{B})(\frac{1}{C}-v)}}. \end{aligned} \tag{9.33}$$

In those coordinates, the quadratic part of the Routhian becomes $\frac{1}{2}\frac{v-u}{4}(X'^2+Y'^2)$. If, further, we modify the change of the time variable (9.27) to be

$$\begin{aligned} dt &= \frac{ABC}{D}.\frac{(v-u)}{4}d\tau \\ &= \frac{(v-u)}{4uv}d\tau, \end{aligned} \tag{9.34}$$

the Routhian is transformed to

$$\begin{aligned} R^* = \frac{1}{2}&(X'^2+Y'^2) \\ &+l_1X'+m_1Y'-V^{**}, \end{aligned} \tag{9.35}$$

in which

$$l_1 = l\frac{du}{dX}, m_1 = m\frac{dv}{dY},$$
$$V^{**} = \frac{(v-u)}{4uv}(V + \frac{f^2}{2ABCuv} - h). \tag{9.36}$$

The equations of motion now take the form

$$X'' + \Omega Y' = -\frac{\partial V^{**}}{\partial X}, Y'' - \Omega X' = -\frac{\partial V^{**}}{\partial Y}, \tag{9.37}$$

where

$$\Omega = \frac{\partial l_1}{\partial Y} - \frac{\partial m_1}{\partial X} \tag{9.38}$$
$$= \frac{du}{dX}\frac{dv}{dY}(\frac{\partial l}{\partial v} - \frac{\partial m}{\partial u})$$
$$= \frac{(v-u)}{4ABCuv\sqrt{uv}}(A + B + C - 2\frac{u+v}{uv}). \tag{9.39}$$

Those equations describe the motion of a particle in the plane XY, while acted upon by forces with potential V^{**} and a gyroscopic force

$$\Omega(-Y', X'),$$

which may be interpreted as due to Lorentz's force exerted on a unit electric charge by a magnetic field Ω orthogonal to the XY plane.

The equations of motion in the form (9.37) were written in 1954 by Arjanykh [13] but with the function Ω as in (9.38), unevaluated. The final form (9.39) was published first by M. Kharlamov [171] in 1976.

9.6 Reduction to the Simplest Form of Orbital Differential Equation

In case any two of the principal moments of inertia are equal, the transformation of coordinates (9.30) used on the ellipsoid of inertia by Kolossov becomes singular and the equations of motion based on that transformation become invalid. The idea to modify this transformation to make it applicable for an arbitrary body appeared in [378]. An examination of the integrals in (9.33) reveals that they both are elliptic integrals of the third kind. The two variables, which we denote by σ and ρ, are chosen so that those third-type elliptic integrals defining the isometric coordinates take their normal (Legendre) form.

9.6.1 Sphero-Conic Coordinates on the Poisson Sphere [378]

For determinacy, we arrange the principal axes of inertia so that the inequalities

$$A \geqslant B \geqslant C \tag{9.40}$$

are satisfied. We use new sphero-elliptic coordinates σ, ρ on the Poisson sphere and the new isometric coordinates x, y defined by the formulas

$$\begin{aligned}
\gamma_1 &= \sqrt{\frac{B}{A}}\frac{\sigma\sqrt{1-k'^2\rho^2}}{\sqrt{(1-n\sigma^2)(1+m\rho^2)}},\\
\gamma_2 &= \frac{\sqrt{(1-\sigma^2)(1-\rho^2)}}{\sqrt{(1-n\sigma^2)(1+m\rho^2)}},\\
\gamma_3 &= \sqrt{\frac{B}{C}}\frac{\rho\sqrt{1-k^2\sigma^2}}{\sqrt{(1-n\sigma^2)(1+m\rho^2)}},
\end{aligned}\tag{9.41}$$

and

$$\begin{aligned}
x &= \sqrt{\frac{C}{A}}\int_0^{\sigma}\frac{d\sigma}{(1-n\sigma^2)\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}},\\
y &= \sqrt{\frac{A}{C}}\int_0^{\rho}\frac{d\rho}{(1+m\rho^2)\sqrt{(1-\rho^2)(1-k'^2\rho^2)}},
\end{aligned}\tag{9.42}$$

where

$$\begin{aligned}
k^2 &= \frac{A-B}{A-C}, k'^2 = \frac{B-C}{A-C},\\
n &= \frac{A-B}{A}, m = \frac{B-C}{C}.
\end{aligned}$$

Note that $0 \leq k \leq 1$, $0 \leq k' \leq 1$, $0 \leq n \leq 1$, $0 \leq m \leq 1$, and the intermediate variables satisfy the inequalities

$$-1 \leq \sigma \leq 1, -1 \leq \rho \leq 1. \tag{9.43}$$

Note that the transformation (9.41), unlike Kolossov's transformation (9.30), remains valid when $B = A$ or $B = C$. This privilege is much helpful in several applications and we shall return to this point later on.

Substituting (9.41), (9.42) into (9.5), we obtain

$$R = \frac{1}{2}B\chi(\dot{x}^2 + \dot{y}^2) + f(\frac{P}{M}\dot{x} - \frac{Q}{M}\dot{y}) - V_1, \tag{9.44}$$

where

$$\chi = B(1 - k^2\sigma^2 - k'^2\rho^2)$$
$$P = \rho\sqrt{(1-\rho^2)(1+k'^2\rho^2)}\sqrt{(1+m\rho^2)(1-n\sigma^2)}(1 - \frac{A+B-C}{A}k^2\sigma^2),$$
$$Q = n\sigma\sqrt{(1-\sigma^2)(1-k^2\sigma^2)}\sqrt{(1+m\rho^2)(1-n\sigma^2)}(1 - \frac{A+B-C}{C}\rho^2),$$
$$M = 1 - n\sigma^2 - \rho^2 + \frac{(A-B)(A+B-C)}{A(A-C)}\sigma^2\rho^2, \tag{9.45}$$

and V_1 now takes the form

$$V_1 = V + \frac{f^2}{2B}(1-n\sigma^2)(1+m\rho^2). \tag{9.46}$$

Now, we change the time variable t by the relation

$$dt = B\chi d\tau. \tag{9.47}$$

The Routhian (9.44) becomes

$$R^* = \frac{1}{2}(x'^2 + y'^2) + f(\frac{P}{M}x' - \frac{Q}{M}y') + U, \tag{9.48}$$

in which

$$\begin{aligned} U &= B\chi(h - V_1) \\ &= B(1 - k^2\sigma^2 - k'^2\rho^2)[h - V - \frac{f^2}{2B}(1-n\sigma^2)(1+m\rho^2)] \end{aligned} \tag{9.49}$$

and primes denote derivatives with respect to τ.

The equations of motion of the rigid body in the potential V on the level $\{h, f\}$ of the total energy and area's integrals are reduced in the xy-plane and in the fictitious time τ, to the form

$$x'' + \Omega y' = \frac{\partial U}{\partial x},\ y'' - \Omega x' = \frac{\partial U}{\partial y}, \tag{9.50}$$

in which

$$\begin{aligned} \Omega &= f[\frac{\partial}{\partial y}(\frac{P}{M}) + \frac{\partial}{\partial x}(\frac{Q}{M})] \\ &= f\frac{\chi}{\sqrt{AC}}\sqrt{(1-n\sigma^2)(1+m\rho^2)} \\ &\quad \times[A - B + C - 2(A-B)\sigma^2 + 2(B-C)\rho^2]. \end{aligned} \tag{9.51}$$

Equations (9.50), obtained in [372], serve as equations of motion of a mechanical system, comprised of a particle moving under the influence of certain potential and gyroscopic forces in the xy-plane. Note that the motion of this system is subject to the condition that the Jacobi integral is zero,

$$\frac{1}{2}(x'^2 + y'^2) + V_1 = 0. \tag{9.52}$$

The energy constant h of the original motion of the rigid body enters as a parameter in the potential term V_1 of the reduced problem (9.50).

Equations of motion in the form (9.50) turned out to be most relevant in working with the method developed by the author in [381] and are used in numerous works for the construction of integrable 2D natural and generalized natural mechanical systems that admit an integral polynomial in velocities.

Because of their symmetric form, Eq. (9.50) was easily applied in [373] to construct periodic solutions of the problem of the motion of a rigid body about a fixed point in a potential field, near to equilibrium points of (9.50), which correspond to stationary solutions of Euler–Poisson equations, usually called "uniform" or "permanent" rotations. In the classical problem, i.e. when the body moves under the action of the uniform gravity field, this is the Staude motion (see Chap. 8, Sect. 8.3). Equations similar to (9.50) but in a less finished form were used in [364, 365].

The variables x, y are uniformizing variables on the Poisson sphere. One can easily show that the function Ω is an analytical function in the whole xy-plane. If V is analytic in the components of γ, then it is also analytic in the xy-plane. On the other hand, from (9.42) one can deduce that σ and ρ are periodic in x and y, respectively. In fact, substituting $\sigma = \sin(\phi)$, the expression for x in (9.42) gives

$$x = \sqrt{\frac{C}{A}} \int_0^{\phi} \frac{d\phi}{(1 - n\sin^2(\phi))\sqrt{(1 - k^2\sin^2(\phi))}}.$$

Thus, x is monotone increasing in ϕ on $\mathbb{R}$. Also, a change 2π in ϕ adds to x a constant

$$\begin{aligned} X &= 4\sqrt{\frac{C}{A}} \int_0^{\pi/2} \frac{d\phi}{(1 - n\sin^2(\phi))\sqrt{(1 - k^2\sin^2(\phi))}} \\ &= 4\sqrt{\frac{C}{A}} \int_0^{1} \frac{d\sigma}{(1 - n\sigma^2)\sqrt{(1 - \sigma^2)(1 - k^2\sigma^2)}}. \end{aligned} \tag{9.53}$$

The inverse function $\varphi(x)$ is thus defined and single-valued (in the real sense) on $\mathbb{R}$. It is also quasi-periodic. It increases by 2π when x is increased by X. Finally, we see that σ is an analytic and periodic function in x. The same can be said about ρ as a function in y. It is an analytic and periodic function in y with period

$$Y = 4\sqrt{\frac{A}{C}}\int_0^1 \frac{d\rho}{(1+m\rho^2)\sqrt{(1-\rho^2)(1-k'^2\rho^2)}}. \quad (9.54)$$

Thus, a real algebraic function of σ, ρ is periodic in both directions on the xy-plane with periods X and Y.

However, it can be shown that the configuration space of the reduced problem $\{\varphi \in [0, 2\pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$ is covered twice by a rectangle of periods in the xy-plane. The variables x, y can be considered as new angular coordinates, which enter in the equations of motion in a more symmetric way than the Eulerian angles. In fact, it suffices to study motion in half of the square of periods, i.e. on one of the rectangles S_1, S_2, where

$$S_1 = \{x \in [-\frac{X}{2}, \frac{X}{2}), y \in [-\frac{Y}{4}, \frac{Y}{4})\},$$
$$S_2 = \{x \in [-\frac{X}{4}, \frac{X}{4}), y \in [-\frac{Y}{2}, \frac{Y}{2})\}. \quad (9.55)$$

9.6.2 Reduction to a Single Differential Equation [378]

Now, we proceed to obtain the orbital equation of the system described by the Routhian (9.48), i.e. the equation satisfied by the trajectories of that system on the Poisson sphere. From (9.52), we get

$$d\tau = \sqrt{\frac{1+(\frac{dy}{dx})^2}{2U}}dx, U = B\chi[h - V - \frac{f^2}{2B}(1-n\sigma^2)(1+m\rho^2)]. \quad (9.56)$$

Eliminating $d\tau$ from the second equation of motion, we arrive at the second-order differential equation

$$\frac{d^2y}{dx^2} = \frac{\Omega}{\sqrt{2U}}[1+(\frac{dy}{dx})^2]^{3/2} + \frac{1}{2U}(\frac{\partial U}{\partial y} - \frac{\partial U}{\partial x}\frac{dy}{dx})[1+(\frac{dy}{dx})^2], \quad (9.57)$$

in which Ω, U are given in (9.51) and (9.49).

Equation (9.56) is formally the simplest possible form of the orbital equation (9.16) under transformations of coordinates. This form was obtained in [378], prior to Eq. (9.16), which is more transparent, but of a much more complicated structure. It was used in [374] to obtain the equation in the variation to study the stability of the orbit $y = 0$, corresponding to pendulum (plane parallel) motion.

9.6.3 Special Cases

In the above equations, two limiting cases are easily separated:

(1) The first case when $B = A > C$. Then we have $k = n = 0$, $k' = 1$, $m = \frac{A}{C} - 1$, $X = 2\pi\sqrt{\frac{C}{A}}$, $\sigma = \sin(\sqrt{\frac{A}{C}}x)$, $Y = \infty$ and the relevant rectangle S_1 becomes an infinite strip. The transformation (9.41) becomes

$$\gamma_1 = \frac{\sin(\sqrt{\frac{A}{C}}x)\sqrt{1-\rho^2}}{\sqrt{(1+m\rho^2)}},$$
$$\gamma_2 = \frac{\cos(\sqrt{\frac{A}{C}}x)\sqrt{(1-\rho^2)}}{\sqrt{(1+m\rho^2)}},$$
$$\gamma_3 = \sqrt{\frac{A}{C}}\frac{\rho}{\sqrt{(1+m\rho^2)}}, \tag{9.58}$$

while ρ satisfies

$$y = \sqrt{\frac{A}{C}}\int_0^\rho \frac{d\rho}{(1+m\rho^2)(1-\rho^2)}$$
$$= \sqrt{\frac{C}{A}}[\frac{1}{2}\ln\frac{1+\rho}{1-\rho} + \sqrt{\frac{A}{C}-1}\tan^{-1}(\sqrt{\frac{A}{C}-1}\rho)]. \tag{9.59}$$

(2) The second case $A > B = C$. In that case, the transformation (9.41) becomes

$$\gamma_1 = \sqrt{\frac{C}{A}}\frac{\sigma}{\sqrt{(1-n\sigma^2)}},$$
$$\gamma_2 = \frac{\sqrt{(1-\sigma^2)}\cos(\sqrt{\frac{C}{A}}y)}{\sqrt{(1-n\sigma^2)}},$$
$$\gamma_3 = \frac{\sin(\sqrt{\frac{C}{A}}y)\sqrt{1-\sigma^2}}{\sqrt{(1-n\sigma^2)}}, \tag{9.60}$$

where σ is determined from

$$x = \sqrt{\frac{C}{A}}\int_0^\sigma \frac{d\sigma}{(1-\frac{A-C}{A}\sigma^2)(1-\sigma^2)}$$
$$= \frac{1}{2}\sqrt{\frac{A}{C}}\ln\{\frac{1+\sigma}{1-\sigma}(\frac{1-\sqrt{1-\frac{C}{A}}\sigma}{1+\sqrt{1-\frac{C}{A}}\sigma})^{\sqrt{1-\frac{C}{A}}}\}, \tag{9.61}$$

i.e. from

$$\frac{1+\sigma}{1-\sigma}\left(\frac{1-\sqrt{1-\frac{C}{A}}\sigma}{1+\sqrt{1-\frac{C}{A}}\sigma}\right)^{\sqrt{1-\frac{C}{A}}} = e^{2\sqrt{\frac{C}{A}}x}. \tag{9.62}$$

It is quite interesting that in some cases, when $\sqrt{1-\frac{C}{A}}$ is a rational number $\frac{N}{N'}$, such that $N, N'(N < N')$ are relatively prime numbers, the last equation becomes

$$\left(\frac{1+\sigma}{1-\sigma}\right)^{N'}\left(\frac{N'-N\sigma}{N'+N\sigma}\right)^{N} = e^{2\sqrt{N'^2-N^2}x}. \tag{9.63}$$

This is a polynomial equation in σ and can be solved for low N, N' $(N+N' \leq 4)$. We have three cases:

(a) $N = 0$. That is the case of complete dynamical symmetry $A = B = C$. Then

$$\sigma = \tanh x, \rho = \sin y. \tag{9.64}$$

(b) $N = 1, N' = 2.(B = C = \frac{3}{4}A)$

$$\sigma = 2\cos\frac{1}{3}[\pi + \cos^{-1}(\tanh(\sqrt{3}x)], \rho = \sin(\frac{\sqrt{3}}{2}y). \tag{9.65}$$

(c) $N = 1, N' = 3.(B = C = \frac{8}{9}A)$

$$\sigma = -\sqrt{1+\mu} + \sqrt{2-\mu+2\sqrt{1-\mu+\mu^2}}, \mu = \sinh^{-\frac{2}{3}}(\frac{16}{9}\sqrt{2}x),$$
$$\rho = \sin(\frac{2\sqrt{2}}{3}y). \tag{9.66}$$

In those three cases, the functions V_1, Ω and then the equations of motion (9.50) can be expressed explicitly in terms of x, y, and the problem can be studied as a proper problem of motion of a particle in the plane. However, this only illustrates the relative degree of complication in the dynamics of a rigid body compared to the dynamics of a particle.

9.7 Separable Potentials in Rigid Body Dynamics (Conditional Integrable Problems)

The general reduced problem of the motion of the rigid body about a fixed point in a potential field was transformed in some of the past sections to other problems of motion on the surface of a sphere, ellipsoid or other surfaces. When the areas constant $f = 0$, the equations of motion become time-reversible and it becomes possible to use certain methods of separation of variables known in the analogous problem. Usually, those cases of separation are obtained by applying the Hamilton–Jacobi equation. They will be presented here in the form already used in the above sections, after using a time transformation to attain the Liouville separation.

9.7.1 Potentials Separable for Axi-Symmetric Body

Let the body with axial dynamical symmetry $B = A$ be in motion about the fixed point O, under the action of forces whose potential is a given function $V(\theta, \varphi)$. Let also the areas constant $f = 0$. Equations of motion are obtained from (3.58), which here becomes

$$R = \frac{A}{2}\left(\frac{C\sin^2\theta}{A\sin^2\theta + C\cos^2\theta}\dot{\varphi}^2 + \dot{\theta}^2\right) - V(\theta, \varphi). \tag{9.67}$$

Using the time transformation

$$dt = \frac{C\sin^2\theta}{A\sin^2\theta + C\cos^2\theta}d\tau, \tag{9.68}$$

on the total energy level h, we get

$$R^* = \frac{1}{2}A\left(\varphi'^2 + \frac{A\sin^2\theta + C\cos^2\theta}{C\sin^2\theta}\theta'^2\right) + \frac{C\sin^2\theta}{A\sin^2\theta + C\cos^2\theta}[h - V(\theta, \varphi)]. \tag{9.69}$$

This system described by a Lagrangian of the last form becomes separable when

$$V(\theta, \varphi) = \frac{A\sin^2\theta + C\cos^2\theta}{C\sin^2\theta}v_1(\varphi) + v_2(\theta). \tag{9.70}$$

Then the Lagrangian R^* splits into two independent Lagrangians, each of one degree of freedom:

$$R^* = \left[\frac{1}{2}A\varphi'^2 - v_1(\varphi)\right] + \left[\frac{1}{2}\frac{A\sin^2\theta + C\cos^2\theta}{C\sin^2\theta}(A\theta'^2 + h) - v_2(\theta)\right].$$

Each Lagrangian also has its own energy integral, but with the whole system's energy equal to zero, i.e. we have

$$\begin{aligned} \frac{1}{2}A\varphi'^2 + v_1(\varphi) &= h_1, \\ \frac{1}{2}\frac{A\sin^2\theta + C\cos^2\theta}{C\sin^2\theta}(A\theta'^2 - h) + v_2(\theta) &= -h_1, \end{aligned} \tag{9.71}$$

where h_1 is the separation constant. After solving the separated equations and expressing φ and θ in terms of τ, the relation with the original time variable is obtained by integrating (9.68).

Potentials of the structure (9.70), or in the more transparent form:

$$V(\theta,\varphi) = \frac{A\sin^2\theta + C\cos^2\theta}{C\sin^2\theta} v_1(\varphi) + \Theta(\theta), \tag{9.72}$$

suffer a serious drawback. They are always singular at the two poles $\theta = 0, \pi$. This singularity cannot be removed by any choice of $v_1(\varphi)$. It is not probable that this class of separable potential can play some role in physical applications, but, as we shall see later, they appear in some integrable cases obtained as generalizations of Kowalevski's classical case. In those integrable cases, the complementary integral has degree four in the components of the angular velocity, but it can be rewritten as the sum of two parts, one of which is the square of a quadratic integral resulting from separation.

In order to write the quadratic complementary integral for this separable case in the Euler–Poisson variables, we first note from (9.72) that

$$V(\theta,\varphi) = V(\boldsymbol{\gamma}) = \frac{A-(A-C)\gamma_3^2}{C(1-\gamma_3^2)} F_1\left(\frac{\gamma_1}{\gamma_2}\right) + F_2(\gamma_3), \tag{9.73}$$

and F_1, F_2 are functions of their arguments. It can be verified that I_4 can be written as

$$I_4 = \frac{1}{2}Cr^2 + F_1\left(\frac{\gamma_1}{\gamma_2}\right). \tag{9.74}$$

9.7.2 Potentials Separable for an Asymmetric Body

Instead of the Jacobian elliptic coordinates on the ellipsoid of inertia, one may use their reciprocals λ, μ in the substitution

$$\begin{aligned}
\gamma_1 &= \sqrt{\frac{BC(A-\lambda)(A-\mu)}{(A-B)(A-C)\lambda\mu}},\\
\gamma_2 &= \sqrt{\frac{CA(\lambda-B)(B-\mu)}{(A-B)(B-C)\lambda\mu}},\\
\gamma_3 &= \sqrt{\frac{AB(\lambda-C)(\mu-C)}{(A-C)(B-C)\lambda\mu}},
\end{aligned} \tag{9.75}$$

and define isometric coordinates on the ellipsoid by the relations

$$X = \sqrt{ABC}\int \frac{d\lambda}{\lambda\sqrt{(A-\lambda)(\lambda-B)(\lambda-C)}},$$
$$Y = \sqrt{ABC}\int \frac{d\mu}{\mu\sqrt{(A-\lambda)(B-\mu)(\mu-C)}}, \tag{9.76}$$

and the change of the independent variable

$$dt = \frac{1}{4}(\lambda-\mu)d\tau, \tag{9.77}$$

then one can reduce the equations of motion on the common level $\{h, f\}$ of the first integrals to the form

$$X'' + \Omega Y' = -\frac{\partial V_1}{\partial X}, Y'' - \Omega X' = -\frac{\partial V_1}{\partial Y},$$
$$X'^2 + Y'^2 + V_1 = 0, \tag{9.78}$$

equivalent to (9.37), where

$$V_1 = \frac{\lambda-\mu}{4}(V + \frac{f^2\lambda\mu}{2ABC} - h), \tag{9.79}$$

V being the original potential of the rigid body, and

$$\Omega = \frac{f(\lambda-\mu)}{4ABC}\sqrt{\lambda\mu}[A+B+C-2(\lambda+\mu)]. \tag{9.80}$$

Note that in virtue of (9.75),

$$D \equiv A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2 = \frac{ABC}{\lambda\mu},$$
$$\alpha \equiv ABC[\mathrm{tr}(\mathbf{I}^{-1}) - \boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}] = D(\lambda+\mu), \tag{9.81}$$

so that λ, μ are the two roots of the equation

$$D\lambda^2 - \alpha\lambda + ABC = 0, \tag{9.82}$$

and

$$\lambda - \mu = \frac{\sqrt{\beta}}{D}, \tag{9.83}$$

where

$$\beta = \alpha^2 - 4ABCD. \tag{9.84}$$

On the level $f = 0$, the equations of motion become separable if

$$V_1 = F_1(\lambda) + F_2(\mu),$$

i.e. the original potential has the form

$$\begin{aligned} V &= \frac{F_1(\lambda) + F_2(\mu)}{\lambda - \mu} \\ &= \frac{A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2}{\sqrt{\beta}}[u(\alpha + \sqrt{\beta}) + v(\alpha - \sqrt{\beta})]. \end{aligned} \tag{9.85}$$

It can be checked that the fourth integral (on the level $f = 0$) is

$$I_4 = A^2p^2 + B^2q^2 + C^2r^2 + \frac{1}{\sqrt{\beta}}[(\alpha - \sqrt{\beta})v(\alpha - \sqrt{\beta}) + (\alpha + \sqrt{\beta})u(\alpha + \sqrt{\beta})]. \tag{9.86}$$

The general form of the potentials was first given by Kolossov in terms of elliptic coordinates on the ellipsoid of inertia [224]. He also considered the two cases

(1) $u(x) = -v(x) = ax + bx^2$, leading to the potential

$$V = \boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}(a' + b'\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}), \tag{9.87}$$

where a, b, a', b' are arbitrary constants. The first term of this potential is Brun's potential.

(2) $u(x) = -v(x) = a/x^2$, leading to the rational potential

$$V = a'\frac{\alpha}{D} = a'\frac{tr(\mathbf{I}^{-1}) - \boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}}{\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}}. \tag{9.88}$$

Potentials polynomial (or entire functions when $N = \infty$) in α and D of the form

$$V = \frac{D}{\sqrt{\beta}}\sum_{n=1}^{N} a_n[(\alpha - \sqrt{\beta})^n - (\alpha + \sqrt{\beta})^n] \tag{9.89}$$

were considered by Bogoyavlensky [31], and families of rational potentials were given in [381].

Remark: It is interesting to note that of all potentials (9.85) allowing separation of variables on the level $f = 0$, only one potential

$$V = \text{const} \times \boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}$$

preserves integrability on arbitrary level $f \neq 0$ (see Chap. 6 Sect. 6.4.3). Of course, there are many other potentials which allow integrability for arbitrary f, but all with gyroscopic forces in addition to potential forces.

9.7.3 Potentials Separable for a Body of Spherical Dynamical Symmetry

In the case of a body of complete dynamical symmetry ($A = B = C$) and when $f = 0$, the Lagrangian (9.8) takes the form

$$R = \frac{1}{2}A(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2) - V. \tag{9.90}$$

That is a problem of motion of a particle on a sphere. In elliptic coordinates u, v, the sphere is parametrized as

$$\begin{aligned} x &= \sqrt{\frac{(a-u)(a-v)}{(a-b)(a-c)}}, \\ y &= \sqrt{\frac{(u-b)(b-v)}{(a-b)(b-c)}}, \\ z &= \sqrt{\frac{(u-c)(v-c)}{(a-c)(b-c)}}. \end{aligned} \tag{9.91}$$

This transforms the Lagrangian to

$$R = \frac{1}{8}A(u-v)[\frac{\dot{u}^2}{(a-u)(u-b)(u-c)} + \frac{\dot{v}^2}{(a-v)(b-v)(v-c)}] - V. \tag{9.92}$$

Noting that

$$\begin{aligned} a\gamma_1^2 + b\gamma_2^2 + c\gamma_3^2 &= a + b + c - (u + v), \\ abc(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}) &= uv, \end{aligned} \tag{9.93}$$

i.e. u, v are roots of the quadratic equation

$$u^2 - [a + b + c - (a\gamma_1^2 + b\gamma_2^2 + c\gamma_3^2)]u + abc(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}) = 0.$$

The separable potentials in elliptic coordinates on the Poisson sphere attached to the body have the form

$$V = \frac{[u_1(\alpha^{'} - \sqrt{\beta^{'}}) + u_2(\alpha^{'} + \sqrt{\beta^{'}})]}{\sqrt{\beta^{'}}},$$
$$\alpha^{'} = a + b + c - (a\gamma_1^2 + b\gamma_2^2 + c\gamma_3^2), \beta^{'} = \alpha^2 - 4abc(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}). \tag{9.94}$$

The corresponding quadratic integral is

$$I_4 = A(ap^2 + bq^2 + cr^2) + \frac{1}{\sqrt{\beta^{'}}}[(\alpha^{'} + \sqrt{\beta^{'}})u_1(\alpha^{'} - \sqrt{\beta^{'}}) + (\alpha^{'} - \sqrt{\beta^{'}})u_2(\alpha^{'} + \sqrt{\beta^{'}})]. \tag{9.95}$$

One of the simple choices

$$u_2(x) = -u_1(x) = \frac{x^2}{8}$$

leads to the potential

$$V = \frac{1}{2}(a\gamma_1^2 + b\gamma_2^2 + c\gamma_3^2), \tag{9.96}$$

and the integral

$$I_4 = A(ap^2 + bq^2 + cr^2) - abc(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}). \tag{9.97}$$

This example corresponds, on the one hand, to Neumann's problem of motion of a particle on a sphere acted upon by a quadratic potential in the Euclidian coordinates [294].[1] On the other hand, in rigid body dynamics it is the separable special case on the level $f = 0$, of an integrable problem of motion of a body in a liquid due to Clebsch, namely case number 3 in Table 10.1 of integrable cases in Chap. 10.

9.8 Exercises

1 - Permanent rotations of the rigid body correspond to stationary (time-independent) solutions of the Euler–Poisson equations. They correspond to equilibrium positions of the system described by the Routhian (9.5) and the condition (9.6). Show that the locus of those equilibrium positions satisfies the equation

[1] In that work, Neumann reduced the problem to hyper-elliptic quadratures and solved them in terms of Rosenhein Theta functions in two variables.

$$\frac{\partial V}{\partial \boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{I}) = 0. \tag{9.98}$$

For the classical problem, this equation becomes

$$\mathbf{r}_0 \cdot (\boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{I}) = 0,$$

which gives Staude's cone (compare with Chap. 8 (8.29)).

[**Hint**: Extremals of V_1 on the unit sphere are obtained from the vector equation

$$\frac{\partial V_1}{\partial \boldsymbol{\gamma}} = \lambda \boldsymbol{\gamma},$$

where $V_1 = V + \frac{f^2}{2D}$ and λ is an undetermined multiplier. That is

$$\frac{\partial V}{\partial \boldsymbol{\gamma}} - \frac{f^2}{D^2}\boldsymbol{\gamma}\mathbf{I} = \lambda\boldsymbol{\gamma}.$$

Eliminating f, λ, (9.98) follows.]

2 - A dynamically axi-symmetric body (with $B = A$) is in motion about a fixed point in the field of a distant Newtonian centre. It has potential (see Chap. 6)

$$V = a\gamma_1 + \frac{3g}{2R}[A(\gamma_1^2 + \gamma_2^2) + C\gamma_3^2].$$

Show that Eq. (9.16) for this body admits a solution $\gamma_3 = 0$, corresponding to pendulum-like motion about the z-axis, which occupies a horizontal position. Use (9.16) also to show that the equation in the variation about that solution can be reduced to the Lame equation

$$\frac{d^2\gamma_3}{du^2} + \{\alpha[\alpha\nu^2 + \frac{1}{2} + \beta(\alpha - 2)] - \alpha(\alpha + 1)\nu^2 \operatorname{sn}^2 u\}\gamma_3 = 0,$$

for vibrations and

$$\frac{d^2\gamma_3}{dv^2} + \{\alpha[\alpha + \frac{1}{2}k^2 + \beta(\alpha - 2)k^2] - \alpha(\alpha + 1)k^2 \operatorname{sn}^2 v\}\gamma_3 = 0,$$

for rotations, where $\nu = \frac{1}{k} = \sqrt{\frac{h+a}{2a}}$ and $\beta = \frac{3gA}{4Ra}$ is a parameter characterizing the approximate Newtonian part of the field and other parameters are as above in this chapter.

[For zones of stability and instability of this motion, see [376, 379].]

3 - A gyrostat with constant gyrostatic momentum $\boldsymbol{\kappa}$ moves about a fixed point while acted upon by potential forces with potential $V(\boldsymbol{\gamma})$. Show that the reduction of order as performed above in the present section leads to the equation [377]:

$$\begin{aligned}
&D(1-\gamma_1^2-\gamma_3^2)\gamma_3'' + C\gamma_3(1-\gamma_3^2)\\
&-\gamma_1[A-(A+2C)\gamma_3^2]\gamma_3' + \gamma_3[C-(C+2A)\gamma_1^2]\gamma_3'^2\\
&-A\gamma_1(1-\gamma_1^2)\gamma_3'^3\\
&-\frac{\rho}{ABCD}\{C\gamma_3[(A-B)(A+B-C)\gamma_1^2 + B(B-C)(1-\gamma_3^2)]\\
&\qquad +A\gamma_1[(B-C)(B+C-A)\gamma_3^2 + B(A-B)(1-\gamma_1^2)]\gamma_3'\}\\
&+\frac{\rho}{2ABC(h-V_1)}[\frac{\partial V_1}{\partial\gamma_3}(\lambda+\mu\gamma_3') - \frac{\partial V_1}{\partial\gamma_1}(\mu+\nu\gamma_3')]\\
&+\frac{\rho^{3/2}}{ABC\sqrt{aD^3(h-V_1)}}\\
&\times\{f[(A-B)(A+B-C)\gamma_1^2 - B(A-B+C) + (C-B)(B+C-A)\gamma_3^2]\\
&\quad +D[\kappa_1(2A+B+C)\gamma_1 + \kappa_2(A+2B+C)\gamma_2 + \kappa_3(A+B+2C)\gamma_3]\\
&\quad -(A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)(\kappa_1\gamma_1 + \kappa_2\gamma_2 + \kappa_3\gamma_3)\}\\
&= 0,
\end{aligned}$$

where

$$\begin{aligned}
\rho &= \lambda + 2\mu\gamma_3' + \nu\gamma_3'^2,\\
\lambda &= C[B(1-\gamma_3^2) + (A-B)\gamma_1^2],\\
\mu &= AC\gamma_1\gamma_3\\
\nu &= A[B(1-\gamma_1^2) + (C-B)\gamma_3^2],
\end{aligned}$$

and γ_2, still standing in few places is just abbreviation for $\sqrt{1-\gamma_1^2-\gamma_3^2}$ and $V_1 = V + \frac{(f-\mathbf{k}\cdot\gamma)^2}{2D}$ is calculated after eliminating γ_2.

4 - The particular solvable case due to Bobylev and Steklov in the classical problem is based on the condition $q \equiv 0$. Determine the general form of the potential $V(\gamma_1, \gamma_2, \gamma_3)$ of the problem of motion of a rigid body about a fixed point, which allows the angular velocity to remain permanently in a principal plane of inertia of the body. Find also the motion of the body.

[This problem is completely solved on the level $f = 0$ and unsolved yet for arbitrary f. It turns out that for a general tri-axial body under the conditions $q = 0$, $f = 0$, the general form of the potential is

$$V = (A\gamma_1^2 + C\gamma_3^2)\{\int^{A\gamma_1^2+C\gamma_3^2} F(\frac{[A(\gamma_1^2+\gamma_3^2)-u]^{A/C}}{C(\gamma_1^2+\gamma_3^2)-u})\frac{du}{u^2} + G(\gamma_2)\},$$

where F and G are arbitrary functions. With this form of the potential, we can write the expressions for the Euler–Poisson variables in their final form parametrized by γ_1

$$(p, q, r) = (-\sqrt{\frac{2C}{A}}\lambda\gamma_1^{C/A}, 0, \sqrt{\frac{2A}{C}}\gamma_1)\sqrt{\varpi},$$
$$(\gamma_1, \gamma_2, \gamma_3) = (\gamma_1, \sqrt{1-\gamma_1^2-\lambda^2\gamma_1^{\frac{2C}{A}}},\ \lambda\gamma_1^{C/A}),$$

where

$$\varpi = \int^{A\gamma_1^2+C\lambda^2\gamma_1^{2C/A}} F(\frac{[A(\gamma_1^2+\lambda^2\gamma_1^{2C/A})-u]^{A/C}}{C(\gamma_1^2+\lambda^2\gamma_1^{2C/A})-u})\frac{du}{u^2}$$
$$+\frac{F(-\lambda^{\frac{2A}{C}}(A-C)^{\frac{A}{C}-1})}{A\gamma_1^2+C\lambda^2\gamma_1^{2C/A}} + G(\sqrt{1-\gamma_1^2-\lambda^2\gamma_1^{\frac{2C}{A}}})$$

and γ_1 is determined in terms of time by inverting the integral

$$t = \int \frac{d\gamma_1}{\sqrt{g(\gamma_1)}}, \quad g(\gamma_1) = -2\frac{A\gamma_1^2}{C}(1-\gamma_1^2-\lambda^2\gamma_1^{\frac{2C}{A}})\varpi.$$

More details and special cases can be found in [416].]

5- A body moving by inertia about a fixed point O has an axis of symmetry Oz and a particle of mass m moves on that axis without friction, subject to a force with potential $V(z)$. The problem of motion is described by the Lagrangian

$$L = \frac{1}{2}[(A+mz^2)(\dot\theta^2+\sin^2\theta\dot\psi^2)+C(\dot\psi\cos\theta+\dot\varphi)^2+m\dot z^2]-V(z).$$

Show that the problem is integrable for arbitrary initial conditions by Liouville separation and reduce its solution to quadratures.

[For solution and a few more details, see [399].]

Chapter 10
The Problem of Motion of a Body in a Liquid

The present chapter is devoted to the investigation of the problem of motion of a rigid body by inertia in an ideal incompressible fluid, infinitely extending in all directions and at rest at infinity. Strictly speaking, this problem belongs to the field of fluid dynamics. The problem evolved namely in this way. The ordinary differential equations of motion of the solid are simultaneously solved with partial differential equation governing the motion of the liquid under boundary conditions satisfied on the surface of the moving solid. In this process, the pressure of the liquid had to be explicitly calculated at every point of the surface of the body. Nevertheless, after the study of some simple cases, and mainly in the works of Thomson and Tait [352] and of Kirchhoff [219], it became clear that the body and the liquid can be treated as forming together one dynamical system of six degrees of freedom, so that the detailed picture of the pressure of the fluid on the surface of the body is completely avoided. This system, composed of the body and liquid, was reduced to the motion of a rigid body with modified characteristics to compensate the motion of the liquid. When referred to a coordinate frame fixed in the body, the kinetic energy of this system is expressed as a quadratic form of the components of the angular and linear velocities of the body with constant coefficients. This step was decisive in the evolution of the subject along the next few decades.

In this setting, the present problem has six degrees of freedom: three for the rotational motion and three for the translation of a point of the body and is traditionally described for a simply connected body by Kirchhoff's equations [219] (see also [220]) or by their Hamiltonian form, mostly used by mathematicians, which are due in their final form to Clebsch [55]. For a perforated body (a body bounded by a multi-connected surface) the equations of motion are usually taken in the form due to Lamb [253], or in the equivalent Hamiltonian form (see e.g. [41]).

Research in the problem of motion of a body in a liquid passed through a period of vigorous activity in the last decades of the nineteenth century. After the formulation of the equations of motion in their final most general form by Kirchhoff, Clebsch and Lamb, a lot of significant results was obtained by several eminent, and mostly

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_10

Russian, scientists, including Minkowsky [284], Lyapunov [267], Chaplygin [53] and Steklov [344, 345, 348]. For almost half a century, the research in the problem entered a state of stagnation. As stated by Aref and Jones [12] "*The Kirchhoff equations present a most remarkable simplification of a problem that, in principle, involves an infinite number of degrees of freedom. Surprisingly, the literature exploring these equations from the point of view of dynamical systems theory is rather sparse*". Half a century later, the first significant results concerning the integrable cases were obtained by Rubanovsky [317–320] (See also books: [121, 125]) using a modified form, due to Kharlamov P., of Clebsch's equations of motion. In most works outside Russia, the form of Clebsch (also Hamiltonian) was mostly used for some qualitative studies of the motion, e.g. [151, 263] (See also references of the last paper).

It turned out that the form of equations of motion involving the variables of Euler–Poisson type, rather than those of Hamiltonian type, enjoy some privileges that will be explained later in this chapter. Those are equations formulated, for the first time, in their most general form in [383]. They are in fact a form of Lagrangian equations, using redundant non-Lagrangian variables. Such Lagrangian equations are not completely new. Similar equations were used by Minkowski, in the special case of Kirchhoff's equations, to establish his brilliant theorem about the isomorphism between the reduced problem of rigid body motion and the motion of a particle on a smooth ellipsoid through a time transformation [284].

In this chapter, equations of motion are presented in their original forms of Kirchhoff, Clebsch and Lamb. Our new equations of Lagrangian, in fact Routhian, form [383] are also presented. This form turned out to be so effective that they put the problem in a unified context with other problems considered in this book. Those problems form a hierarchy, ascending from the classical problem to the one of the present chapter. This hierarchy is extended in the next part of this book to include the most general problem of motion of a rigid body under the action of conservative potential and gyroscopic forces which have a common axis of symmetry through the fixed point. The last problem reduces under some restrictions on the forces, to the problem, equivalent to that of motion of a body in a liquid. Going lower in the hierarchy, we note that every problem in it contains all the problems considered before it as a special case. As a result of this representation of the equations of motion, a striking property of the equations of motion of a rigid body in a liquid is revealed. It is the first problem which is closed under the regular precession transformation. Referring the equations to a coordinate frame precessing with a uniform speed with respect to the inertial frame, results in the same equations, as if in the inertial frame, but with changed characteristics of the body. Thus, this transformation generates from any solution in the present problem or any problem lower in the hierarchy, a new solution that contains the precession speed as an extra-parameter. This situation helped to re-organize the known information about the subject and to fill gaps in it. Some recently discovered integrable cases are generalized. Tables are given for all integrable cases, general and conditional. The most important known families of particular solutions to the problem are discussed on different levels of detail.

In our presentation of the subject, the problem of motion of a body in a liquid plays a rather unusual role. Results obtained in this problem by various methods and accumulated along a century have grown into a core for the advancement of some other problems of motion of a rigid body under more sophisticated forces. In later chapters, we shall use some transformations to obtain new integrable extensions which are more general from the physical and mathematical aspects and which were not subjected to any studies before.

10.1 Equations of Motion

10.1.1 Kirchhoff's Equations

Consider a rigid body moving in an ideal incompressible liquid, extending to infinity in all directions and at rest at infinity. Assume that the body is bounded by a simply connected surface and is moving by inertia, i.e. under no forces, except those exerted on it by the pressure of the liquid on its surface. Let O' and O, respectively, be the origins of the inertial coordinate system $O'XYZ$ and another system $Oxyz$ fixed in the body and let $\mathbf{r} = O'O$. Denote by $\boldsymbol{\omega}$ the angular velocity of the body and by $\mathbf{u}$ the velocity of O with respect to O', so that $\mathbf{u} = \frac{d\mathbf{r}}{dt}$. The equations of motion were derived in Lagrangian form using the Lagrangian function L (kinetic energy T, since no external forces are present):

$$L = T = \frac{1}{2}(\omega\mathbf{A} \cdot \omega + \mathbf{2uB} \cdot \omega + \mathbf{uC} \cdot \mathbf{u}) \tag{10.1}$$

in which $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are constant 3×3 real matrices; $\mathbf{A}, \mathbf{C}$ symmetric and $\mathbf{B}$ is not necessarily symmetric. Here, the state variables $\boldsymbol{\omega}$ and $\mathbf{u}$ and all quantities (parameters of the problem) are referred to the body system. Of course, as a quadratic form, T must be positive definite in the six variables. For this the three matrices must satisfy certain inequalities.

We shall not go through the explicit derivation of the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ from the underlying hydrodynamical problem, because that would increase the size of this chapter beyond preassigned limits. All this material can be found in the treatise of Lamb [253]. It will be helpful in dealing with motion of bodies with certain symmetry properties to borrow the cases presented in the following table from that treatise. A similar table is presented in [41].

Table 0:

	Symmetry	Matrices	Examples
1	Plane of symmetry xy	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \begin{pmatrix} 0 & 0 & B_{13} \\ 0 & 0 & B_{23} \\ B_{13} & B_{23} & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}$.	
2	Two orthogonal planes of symmetry xy, xz	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B_{23} \\ 0 & B_{23} & 0 \end{pmatrix}$, $\mathbf{C} = \text{diag}(C_1, C_2, C_3)$	
3	Three orthogonal planes of symmetry xy, xz, yz	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = \text{diag}(C_1, C_2, C_3)$	Tri-axial ellipsoid, Parallele-piped.
4	Rotation through an angle π about axis Oz.	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}$.	Two-bladed ship screw.
5	Rotation through an angle $\pi/2$ about axis Oz.	$\mathbf{A} = \text{diag}(A_1, A_2, A_3)$, $\mathbf{B} = \text{diag}(B_1, B_2, B_3)$, $\mathbf{C} = \text{diag}(C_1, C_2, C_3)$.	Four-bladed ship screw.
6	Helicoidal symmetry about axis Oz.	$\mathbf{A} = \text{diag}(A, A, A_3)$, $\mathbf{B} = \text{diag}(B, B, A_3)$, $\mathbf{C} = \text{diag}(C, C, C_3)$.	Helicoid.
7	Oz is axis of symmetry (or rotation through an angle $\frac{2\pi}{n}$, $n \notin \{2, 4\}$ around z-axis).	$\mathbf{A} = \text{diag}(A, A, A_3)$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = \text{diag}(C, C, C_3)$.	Spheroid, Three-bladed ship screw.
8	The body is similarly related to each of the coordinate planes.	$\mathbf{A} = A\boldsymbol{\delta}$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = C\boldsymbol{\delta}$.	Cube, sphere.

It is usually argued that the origin of the movable coordinate system can always be shifted so that O coincides with a certain point of the body, called the central point, at which the matrix $\mathbf{B}$ becomes symmetric. It is also usually assumed that the axes of the body system are rotated to the principal axes of the matrix $\mathbf{A}$, so that the matrix $\mathbf{A}$ becomes diagonal. However, we shall see soon that there is no need for those steps for the time being, if one is not concerned in using the original variables $\boldsymbol{\omega}$ and $\mathbf{u}$.

The equations of motion are [219]

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} + \boldsymbol{\omega} \times \frac{\partial L}{\partial \boldsymbol{\omega}} + \mathbf{u}\times\frac{\partial L}{\partial \mathbf{u}} = 0,$$
$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{u}} + \boldsymbol{\omega} \times \frac{\partial L}{\partial \mathbf{u}} = 0. \tag{10.2}$$

Explicitly, Kirchhoff's equations can be written in vector form

$$\dot{\omega}\mathbf{A}+\dot{\mathbf{u}}\mathbf{B} + \omega \times (\omega\mathbf{A} + \mathbf{u}\mathbf{B}) + \mathbf{u} \times (\omega\mathbf{B}^T+\mathbf{u}\mathbf{C}) = 0,$$
$$\dot{\omega}\mathbf{B}^T+\dot{\mathbf{u}}\mathbf{C} + \omega \times (\omega\mathbf{B}^T+\mathbf{u}\mathbf{C}) = 0 \tag{10.3}$$

or, if one introduces the notation

$$\mathbf{M} = \frac{\partial L}{\partial \boldsymbol{\omega}} = \omega\mathbf{A} + \mathbf{u}\mathbf{B}, \tag{10.4}$$

and

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = \omega\mathbf{B}^T+\mathbf{u}\mathbf{C}, \tag{10.5}$$

in the alternative form

$$\dot{\mathbf{M}} + \omega \times \mathbf{M} + \mathbf{u} \times \mathbf{p} = 0,$$
$$\dot{\mathbf{p}} + \omega \times \mathbf{p} = 0. \tag{10.6}$$

Equation (10.3) are quite complicated. An obvious disadvantage is that they are not solved with respect to the derivatives. Every scalar equation of motion may contain the six components of the derivatives $\dot{\omega}$ and $\dot{\mathbf{u}}$. Following Kirchhoff, we also note that those equations admit three integrals of motion:

1. The energy integral, as the Lagrangian is a homogeneous quadratic polynomial of the velocities

$$I_1 = \frac{1}{2}(\omega\mathbf{A} \cdot \omega + \mathbf{2uB} \cdot \omega + \mathbf{uC} \cdot \mathbf{u}). \tag{10.7}$$

2. From the second equation in (10.6), it follows that the magnitude of the vector $\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}}$ is conserved.

$$I_2 = |\mathbf{p}|^2 = |\omega\mathbf{B}^T+\mathbf{u}\mathbf{C}|^2. \tag{10.8}$$

3. Also, using both Eq. (10.6), we get

$$I_3 = \mathbf{M} \cdot \mathbf{p} = (\omega\mathbf{A} + \mathbf{u}\mathbf{B}) \cdot (\omega\mathbf{B}^T+\mathbf{u}\mathbf{C}). \tag{10.9}$$

The system of Eq. (10.3) was used in the treatment of certain simple cases.

10.1.2 Example: Permanent Translational Motions

For example, Kirchhoff investigated the possibility that the body performs uniform translational motion without rotation. From (10.3), setting $\boldsymbol{\omega}=\mathbf{0}$, it turns out that the condition for this motion is

$$\mathbf{u} \times \mathbf{uC} = \mathbf{0}.$$

That is, the vector $\mathbf{u}$ must be directed along one of the principal axes of the matrix $\mathbf{C}$. Thus, a body of an arbitrary shape (with a tri-axial ellipsoid of the matrix $\mathbf{C}$) always has three mutually orthogonal axes such that if the body is set in motion parallel to one of them along any direction in space and then left to itself, it will permanently continue this motion with constant velocity. In case of two equal principal axes, all axes at the equatorial plane are possible axes of permanent translation and also the polar axis, and in case of spherical symmetry all directions in the body are possible for permanent translation. Note that actual spherical symmetry of the body is not necessary. It is a dynamical property that the matrix C has three equal eigenvalues. This condition is satisfied by cubes as well as by spheres [253].

10.1.3 Clebsch's Form of Kirchhoff's Equations

Clebsch [55] transformed Eq. (10.6) to Hamiltonian form using the variables $\mathbf{M}$, $\mathbf{p}$ and the Legendre transformation

$$\begin{aligned} H(\mathbf{M},\mathbf{p}) &= \mathbf{M}\cdot\boldsymbol{\omega} + \mathbf{p}\cdot\mathbf{u} - L \\ &= \frac{1}{2}(\mathbf{M}\tilde{\mathbf{a}}\cdot\mathbf{M} + 2\mathbf{M}\tilde{\mathbf{b}}\cdot\mathbf{p} + \mathbf{p}\tilde{\mathbf{c}}\cdot\mathbf{p}) \end{aligned} \tag{10.10}$$

where

$$\begin{aligned} \tilde{\mathbf{a}} &= (\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1}, \\ \tilde{\mathbf{b}} &= -\mathbf{C}^{-1}\mathbf{B}(\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1} \\ \tilde{\mathbf{c}} &= \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}(\mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{C}^{-1}. \end{aligned}$$

Note that $\tilde{\mathbf{a}}$, $\tilde{\mathbf{c}}$ are symmetric but $\tilde{\mathbf{b}}$ is not.

The equations of motion acquire the Hamiltonian form due to Clebsch

$$\dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}},\ \dot{\mathbf{p}} = \mathbf{p} \times \frac{\partial H}{\partial \mathbf{M}} \tag{10.11}$$

or, in expanded form,

$$\begin{aligned}\dot{\mathbf{M}} &= \mathbf{M}\times(\mathbf{M}\tilde{\mathbf{a}}+\mathbf{p}\tilde{\mathbf{b}}^T) + \mathbf{p}\times(\mathbf{M}\tilde{\mathbf{b}}+\mathbf{p}\tilde{\mathbf{c}}),\\ \dot{\mathbf{p}} &= \mathbf{p}\times(\mathbf{M}\tilde{\mathbf{a}}+\mathbf{p}\tilde{\mathbf{b}}^T),\end{aligned} \tag{10.12}$$

which is used, usually assuming symmetry of the matrix $\tilde{\mathbf{b}}$, until recently, e.g. [41, 246, 263, 327, 328]. For the Hamiltonian form of the Kirchhoff equations see also [12, 257, 258, 280].

The general integrals of motion take their simplest form in the variables $\mathbf{M}$, $\mathbf{p}$:

$$\begin{aligned} I_1 &= H,\\ I_2 &= \mathbf{M}\cdot\mathbf{p}\\ I_3 &= \mathbf{p}^2. \end{aligned} \tag{10.13}$$

10.2 Thomson-Lamb's Equations

By the words of the contemporary of Thomson and Lamb, A.B. Basset [19] "the general theory of motion of a ring in an infinite liquid, when there is cyclic irrotational motion through its aperture, was first given by Sir William Thomson in the *Philosophical Magazine* (1871), and his theory has been subsequently developed by Professor Lamb, in his *Treatise on the motion of fluids*" [252]. Hence, and although the equations of motion are direct generalization of Kirchhoff's equations, I will give the name "Thomson-Lamb's theory" to the theory of equations of motion of a multi-connected (perforated) rigid body in a liquid. This problem was not considered in its generality in the western literature for about a century. In fact, after the works of Basset and Fawcett on the motion of perforated bodies in liquid (e.g. [19, 83]) in the last two decades of the nineteenth century, no significant results are seen in this area until the equations of motion were reformed by Kharlamov in the sixties. The deduction of the Lagrangian (Euler–Poisson type) equations appeared in 1986, a whole century later. This may have been caused by the historical nature of that period at the beginning of the twentieth century. The period of birth of new physical theories: atomic physics, relativity, old quantum and then quantum theories. Research in branches of classical mechanics was significantly retarded.

It may be noted here that the generalization of Kirchhoff equations for perforated body was given by some authors the name "Kirchhoff–Poisson equations". As examples, see [121, 125]. This name seems to us irrational, since Poisson had no relation at all to the present circle of problems.

Let O' and O, respectively, be the origins of the inertial coordinate system and the system fixed in the body, and let $\mathbf{r} = \overrightarrow{O'O}$. Denote by $\boldsymbol{\omega}$ the angular velocity of the body and by $\mathbf{u}$ the velocity of O with respect to O', so that $\mathbf{u} = \frac{d\mathbf{r}}{dt}$. The equations as in [253] are derived from a Lagrangian function (kinetic energy, since no external forces are present):

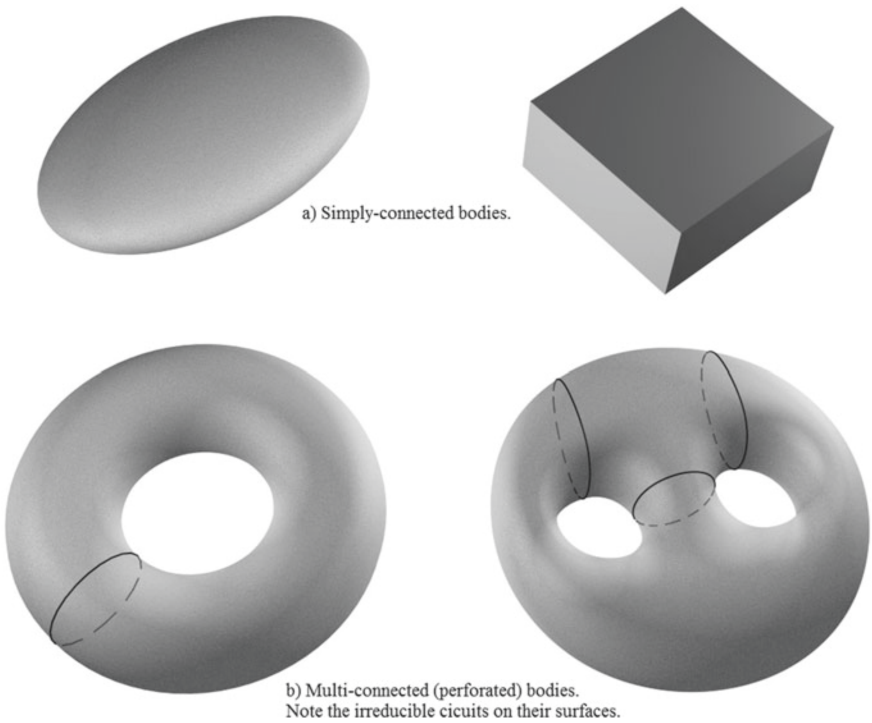


Fig. 10.1 Simple and perforated bodies

$$T = \frac{1}{2}(\omega\mathbf{A} \cdot \omega + \mathbf{2uB} \cdot \omega + \mathbf{uC} \cdot \mathbf{u}) + \bar{\alpha} \cdot \omega + \bar{\beta} \cdot \mathbf{u} \tag{10.14}$$

in which $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are constant 3×3 real matrices; $\mathbf{A}$,$\mathbf{C}$ symmetric and $\mathbf{B}$ is not necessarily symmetric and $\bar{\alpha}$, $\bar{\beta}$ are constant vectors, which characterize the multi-connectedness of the body and the circulations of the fluid on irreducible circuits drawn on its surface (Fig. 10.1b). Here, the state variables $\boldsymbol{\omega}$ and $\mathbf{u}$ and all quantities (parameters of the problem) are referred to the body system. For a body bounded by a simply connected surface the vectors $\bar{\alpha}$, $\bar{\beta}$ vanish and the Lagrangian turns into the one used by Kirchhoff and Clebsch.

It is usually argued that the origin of the movable coordinate system can always be shifted to a certain point of the body, called the central point, at which the matrix $\mathbf{B}$ becomes symmetric if it is not so at O, and hence it is also usually assumed that the axes of the system are rotated, so that the matrix $\mathbf{A}$ becomes diagonal. However, we shall see soon that there is no need for those steps for the time being.

The equations of motion are [253]

$$\frac{d}{dt}\frac{\partial T}{\partial \boldsymbol{\omega}} + \boldsymbol{\omega} \times \frac{\partial T}{\partial \boldsymbol{\omega}} + \mathbf{u}\times\frac{\partial T}{\partial \mathbf{u}} = 0,$$
$$\frac{d}{dt}\frac{\partial T}{\partial \mathbf{u}} + \boldsymbol{\omega} \times \frac{\partial T}{\partial \mathbf{u}} = 0. \quad (10.15)$$

Explicitly, Lamb's equations can be written in vector form

$$\dot{\boldsymbol{\omega}}\mathbf{A}+\dot{\mathbf{u}}\mathbf{B} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{A} + \mathbf{u}\mathbf{B}+\bar{\boldsymbol{\alpha}}) + \mathbf{u}\times(\boldsymbol{\omega}\mathbf{B}^T+\mathbf{u}\mathbf{C}+\bar{\boldsymbol{\beta}}) = 0,$$
$$\dot{\boldsymbol{\omega}}\mathbf{B}^T+\dot{\mathbf{u}}\mathbf{C} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{B}^T+\mathbf{u}\mathbf{C}+\bar{\boldsymbol{\beta}}) = 0 \quad (10.16)$$

or, if we introduce the notation

$$\mathbf{M} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \boldsymbol{\omega}\mathbf{A} + \mathbf{u}\mathbf{B}+\bar{\boldsymbol{\alpha}}, \quad (10.17)$$

and

$$\mathbf{p} = \frac{\partial T}{\partial \mathbf{u}} = \boldsymbol{\omega}\mathbf{B}^T+\mathbf{u}\mathbf{C}+\bar{\boldsymbol{\beta}}, \quad (10.18)$$

in the alternative form

$$\dot{\mathbf{M}} + \boldsymbol{\omega} \times \mathbf{M} + \mathbf{u} \times \mathbf{p} = 0,$$
$$\dot{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p} = 0. \quad (10.19)$$

Equation (10.16) are quite complicated. An obvious disadvantage is that they are not solved with respect to the derivatives. Every scalar equation of motion may contain the six components of the derivatives $\dot{\boldsymbol{\omega}}$ and $\dot{\mathbf{u}}$. Following Lamb, we also note that those equations admit three integrals of motion:

1. Jacobi's integral, the homogeneous quadratic part of the Lagrangian

$$I_1 = \frac{1}{2}(\boldsymbol{\omega}\mathbf{A}\cdot\boldsymbol{\omega}+2\mathbf{u}\mathbf{B}\cdot\boldsymbol{\omega} + \mathbf{u}\mathbf{C}\cdot\mathbf{u}). \quad (10.20)$$

2. From the second equation in (10.19), it follows that the magnitude of the vector $\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}}$ is conserved.

$$I_2 = |\mathbf{p}|^2 = |\boldsymbol{\omega}\mathbf{B}^T+\mathbf{u}\mathbf{C}+\bar{\boldsymbol{\beta}}|^2.$$

3. Also, using both Eq. (10.19), we get

$$I_3 = \mathbf{M}\cdot\mathbf{p} = (\boldsymbol{\omega}\mathbf{A} + \mathbf{u}\mathbf{B} + \bar{\boldsymbol{\alpha}})\cdot(\boldsymbol{\omega}\mathbf{B}^T+\mathbf{u}\mathbf{C}+\bar{\boldsymbol{\beta}}).$$

The system of Eq. (10.16) was used in the treatment of certain simple cases and is usually transformed to the Hamiltonian variables. Using a Hamiltonian (see, e.g. [41]):

$$\begin{aligned} H &= \mathbf{M} \cdot \omega - T \\ &= \frac{1}{2}(\mathbf{M}\tilde{\mathbf{a}} \cdot \mathbf{M} + 2\mathbf{M}\tilde{\mathbf{b}} \cdot \mathbf{p} + \mathbf{p}\tilde{\mathbf{c}} \cdot \mathbf{p}) + \tilde{\alpha} \cdot \mathbf{M} + \tilde{\beta} \cdot \mathbf{p} \end{aligned} \tag{10.21}$$

the equations of motion acquire the form

$$\dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}}, \dot{\mathbf{p}} = \mathbf{p} \times \frac{\partial H}{\partial \mathbf{M}}, \tag{10.22}$$

or in explicit form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times (\mathbf{M}\tilde{\mathbf{a}} + p\tilde{\mathbf{b}}^T + \tilde{\alpha}) + \mathbf{p} \times (\mathbf{M}\tilde{\mathbf{b}} + \mathbf{p}\tilde{\mathbf{c}} + \tilde{\beta}), \\ \dot{\mathbf{p}} &= \mathbf{p} \times (\mathbf{M}\tilde{\mathbf{a}} + p\tilde{\mathbf{b}}^T + \tilde{\alpha}). \end{aligned} \tag{10.23}$$

Integrals of motion take the simple form:

$$\begin{aligned} I_1 &= H, \\ I_2 &= \mathbf{p}^2, \\ I_3 &= \mathbf{M} \cdot \mathbf{p}. \end{aligned}$$

The last form of equations is used in most recent works, e.g. [41].

10.3 On Different Forms of the Equations of Motion

The traditional equations of Kirchhoff and Lamb suffer some disadvantages that in most cases lead to their treatment for most of their history in isolation from other problems of rigid body dynamics. They also involve the non-symmetric matrix **b**. Although this matrix can be reduced to symmetric form by shifting the origin to the central point of the body, the presence of non-symmetry complicates the equations either in Lagrangian or Hamiltonian forms. In most recent works some simplifying restrictions on the parameters are assumed, such as $\mathbf{b} = \mathbf{0}$ (e.g. [151, 263]). Equation (10.11) has also the disadvantage that their solution gives the vector quantity **M**, which has no direct interpretation in terms of the motion unless transformed to an expression involving the angular velocity $\boldsymbol{\omega}$ and the vector **p** constant in space.

If Eqs. (10.2) (or (10.11)) are written in the frame of reference attached to the principal axes of a matrix **A** (or $\tilde{\mathbf{a}}$), they involve 15 parameters characterizing the shape of the body.

If Eqs. (10.15) (or (10.22)) are written in the frame of reference attached to the principal axes of a matrix **A** (or $\tilde{\mathbf{a}}$), they involve 24 parameters characterizing the shape of the body and, for a perforated body, circulations of the fluid along irreducible contours on its surface.

10.4 A New Form of the Equations of Motion

Here, we present with minor modification a new form of the equations of motion of a general body in a liquid, which was derived in our work [383]. We note first that in the Lagrangian (10.1), the Cartesian coordinates (X, Y, Z) of the origin of the system of axes fixed in the body relative to the inertial system are cyclic variables, since the resultant of forces acting on the body-and-fluid system vanishes. We now ignore those coordinates using the vector cyclic integral

$$\frac{\partial T}{\partial \mathbf{u}} = \boldsymbol{\omega}\mathbf{B}^T + \mathbf{u}\mathbf{C} + \tilde{\boldsymbol{\beta}} = \mathbf{p}, \tag{10.24}$$

where p is a vector whose components are constant in space and hence satisfies the Poisson equation

$$\dot{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p} = \mathbf{0}. \tag{10.25}$$

Now, solving the relation (10.24) in $\mathbf{u}$ we obtain

$$\mathbf{u} = (\mathbf{p} - \bar{\boldsymbol{\beta}} - \boldsymbol{\omega}\mathbf{B}^T)\mathbf{C}^{-1} \tag{10.26}$$

and we proceed to form Routh's function

$$\begin{aligned} R &= T - \mathbf{u} \cdot \frac{\partial L}{\partial \mathbf{u}} \\ &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} + (\boldsymbol{\kappa} + \mathbf{p}\tilde{\mathbf{K}}) \cdot \boldsymbol{\omega} - \mathbf{a} \cdot \mathbf{p} - \frac{1}{2}\mathbf{p}\mathbf{J} \cdot \mathbf{p} \end{aligned} \tag{10.27}$$

where $\mathbf{I}, \tilde{\mathbf{K}}, \mathbf{J}$ are the constant 3×3 matrices and $\boldsymbol{\kappa}$, $\mathbf{a}$ are the constant vectors given by

$$\begin{aligned} \mathbf{I} &= \mathbf{A} - \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B}, \\ \mathbf{J} &= \mathbf{C}^{-1}, \\ \tilde{\mathbf{K}} &= \mathbf{C}^{-1}\mathbf{B}, \end{aligned} \tag{10.28}$$

$$\mathbf{a} = -\bar{\boldsymbol{\beta}}\mathbf{C}^{-1}, \tag{10.29}$$

$$\boldsymbol{\kappa} = \bar{\boldsymbol{\alpha}} - \bar{\boldsymbol{\beta}}\mathbf{C}^{-1}\mathbf{B}. \tag{10.30}$$

As seen from (10.30), the matrices $\mathbf{I}$, $\mathbf{J}$ are symmetric but $\tilde{\mathbf{K}}$, in general, is not. Let $\tilde{\mathbf{K}}_s$ and $\tilde{\mathbf{K}}_a$ be the symmetric and antisymmetric parts of $\bar{\mathbf{K}}$, so that

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_s + \tilde{\mathbf{K}}_a, \tag{10.31}$$

$$\tilde{\mathbf{K}}_s = \frac{1}{2}[\mathbf{C}^{-1}\mathbf{B} + (\mathbf{C}^{-1}\mathbf{B})^T] \equiv -\frac{1}{2}\mathbf{K}, \tag{10.32}$$

$$\tilde{\mathbf{K}}_a = \frac{1}{2}[\mathbf{C}^{-1}\mathbf{B} - (\mathbf{C}^{-1}\mathbf{B})^T]. \tag{10.33}$$

Here we introduced a constant matrix $\mathbf{K} = -[\mathbf{C^{-1}B} + (\mathbf{C^{-1}B})^{\mathbf{T}}]$. Inserting (10.31) into (10.27), we can write

$$R = R_0 + \mathbf{p}\tilde{\mathbf{K}}_a \cdot \boldsymbol{\omega}, \tag{10.34}$$

where

$$R_0 = \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} + (\boldsymbol{\kappa} - \frac{1}{2}\mathbf{pK}) \cdot \boldsymbol{\omega} - \mathbf{a} \cdot \mathbf{p} - \frac{1}{2}\mathbf{pJ} \cdot \mathbf{p}. \tag{10.35}$$

We now show that the antisymmetric part $\mathbf{K}_a$ does not contribute to the equations of motion. In fact, the last term of (10.34) is

$$\begin{aligned}\mathbf{pK}_a \cdot \boldsymbol{\omega} &= (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)\begin{pmatrix} 0 & -K_{a3} & K_{a2} \\ K_{a3} & 0 & -K_{a1} \\ -K_{a2} & K_{a1} & 0 \end{pmatrix} \cdot \boldsymbol{\omega} \\ &= (\mathbf{p} \times \mathbf{k}_a) \cdot \boldsymbol{\omega}\end{aligned}$$

where we introduced the vector $\mathbf{k}_a = (K_{a1}, K_{a2}, K_{a3})$ constant in the body axes. Thus, we have

$$\begin{aligned}\mathbf{pK}_a \cdot \boldsymbol{\omega} &= \mathbf{k}_a \cdot (\boldsymbol{\omega} \times \mathbf{p}) \\ &= -\mathbf{k}_a \cdot \dot{\mathbf{p}} \\ &= \frac{d}{dt}(-\mathbf{k}_a \cdot \mathbf{p}).\end{aligned}$$

Thus, the extra term in (10.34) is a nugatory term and has no contribution to the equations of motion (e.g. [305]). The Routhian R_0 gives full description of the rotational motion of the body. Euler's equation for this motion can be deduced in a simple way. With an eye on future applications, we present that in detail. In fact, the equation of motion about the third axis of the body system is

$$\frac{d}{dt}(\frac{\partial R_0}{\partial \dot{\varphi}}) - \frac{\partial R_0}{\partial \varphi} = 0. \tag{10.36}$$

This gives

$$\frac{d}{dt}(\frac{\partial R_0}{\partial \boldsymbol{\omega}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{\varphi}}) - \frac{\partial R_0}{\partial \boldsymbol{\omega}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \varphi} - \frac{\partial R_0}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial \varphi} = 0.$$

But from formulas of Chap. 2, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial \dot{\varphi}} = \mathbf{k}, \frac{\partial \boldsymbol{\omega}}{\partial \varphi} = -\mathbf{k} \times \boldsymbol{\omega}, \frac{\partial \mathbf{p}}{\partial \varphi} = -\mathbf{k} \times \mathbf{p}, \tag{10.37}$$

and thus we get

$$\mathbf{k} \cdot [(\frac{\partial R_0}{\partial \boldsymbol{\omega}}\dot{)} + \boldsymbol{\omega} \times \frac{\partial R_0}{\partial \boldsymbol{\omega}} + \mathbf{p} \times \frac{\partial R_0}{\partial \mathbf{p}}] = 0,$$

so that the vector equation of motion can be written as

$$(\frac{\partial R_0}{\partial \boldsymbol{\omega}}\dot{)} + \boldsymbol{\omega} \times \frac{\partial R_0}{\partial \boldsymbol{\omega}} + \mathbf{p} \times \frac{\partial R_0}{\partial \mathbf{p}} = \mathbf{0}.. \tag{10.38}$$

Now, inserting the expression (10.35) for the Routhian, we obtain

$$(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}\dot{)} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}) + \mathbf{p} \times [-\frac{1}{2}\boldsymbol{\omega}\mathbf{K} - (\mathbf{a} + \mathbf{p}\mathbf{J})] = \mathbf{0}, \tag{10.39}$$

As $\mathbf{I}, \boldsymbol{\kappa}$ and $\mathbf{K}$ are constants in the body, the last equation becomes

$$\dot{\boldsymbol{\omega}}\mathbf{I} - \frac{1}{2}\dot{\mathbf{p}}\mathbf{K} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}) - \frac{1}{2}\mathbf{p} \times \boldsymbol{\omega}\mathbf{K} = \mathbf{p} \times (\mathbf{a} + \mathbf{p}\mathbf{J}),$$

and using Poisson's equation in the second term

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K}) + \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{p})\mathbf{K} + \frac{1}{2}\boldsymbol{\omega}\mathbf{K} \times \mathbf{p} = \mathbf{p} \times (\mathbf{a} + \mathbf{p}\mathbf{J}). \tag{10.40}$$

Here, using the identity

$$(\boldsymbol{\omega} \times \mathbf{p})\mathbf{K} + \boldsymbol{\omega}\mathbf{K} \times \mathbf{p} = \boldsymbol{\omega} \times (\mathbf{p}[tr(\mathbf{K})\boldsymbol{\delta} - \mathbf{K}]),$$

valid for any two vectors $\boldsymbol{\omega}$, $\mathbf{p}$ and symmetric matrix $\mathbf{K}$, we write the final form of the equations of motion

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} + \mathbf{p}\bar{\mathbf{K}}) &= \mathbf{p} \times (\mathbf{a} + \mathbf{p}\mathbf{J}), \\ \dot{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p} &= \mathbf{0}. \end{aligned} \tag{10.41}$$

where $\bar{\mathbf{K}} = \frac{1}{2}tr(\mathbf{K})\boldsymbol{\delta} - \mathbf{K}$, which is the same as the relation between $\mathbf{I}$ and $\bar{\mathbf{I}}$ in Chap. 1.

Equation (10.41) admit three first integrals:

$$
\begin{aligned}
I_1 &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{a}\cdot\mathbf{p} + \frac{1}{2}\mathbf{p}\mathbf{J}\cdot\mathbf{p},\\
I_2 &= \mathbf{p}^2,\\
I_3 &= (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\mathbf{p}\mathbf{K})\cdot\mathbf{p}.
\end{aligned}
\tag{10.42}
$$

The vectors $\boldsymbol{\kappa}$ and $\mathbf{a}$, resulting from the circulation of the fluid in the body perforations vanish for a simply connected body, in which case Eq. (10.41) reduce to a form equivalent to Kirchhoff's equations.

When referred to principal axes of the matrix $\mathbf{I}$, Eq. (10.41) in the general case involve only 21 parameters, compared to 24 in (10.22). The parameters of the original problem can be expressed by inverting the relations (10.30) as:

$$
\begin{aligned}
\mathbf{A} &= \mathbf{I} - (\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a)\mathbf{J}^{-1}(-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{K} + \tilde{\mathbf{K}}_{\mathbf{a}}),\\
\mathbf{B} &= \mathbf{J}^{-1}(-\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\mathbf{C} &= \mathbf{J}^{-1},\\
\tilde{\mathbf{K}} &= -\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a,\\
\bar{\boldsymbol{\alpha}} &= \boldsymbol{\kappa} + \mathbf{a}\mathbf{J}^{-1}(-\frac{1}{2}\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\bar{\boldsymbol{\beta}} &= -\mathbf{a}\mathbf{J}^{-1},
\end{aligned}
\tag{10.43}
$$

so that we retain, if we like, the three elements of the antisymmetric matrix, and thus also the full set of 24 (18) parameters of the original Lamb (Kirchhoff) formulation.

Remark *The observation that the antisymmetric part K_a of the matrix $C^{-1}B$ has no contribution to the equations of motion, except entering into the symmetric matrix A and the vector $\bar{\alpha}$, eliminates the necessity in several works to translate the origin O fixed in the body to the so-called central point of the body, or to assume the symmetry of the matrices and thus, unnecessarily, restricting the possible forms of the body. Calculation of the coefficient matrices can be done at a suitable point from the point of view of calculation and the characteristics of the rotational motion are then constructed free of the choice of the origin.*

Remark *The same observation resolves once for all a situation that the Hamiltonian equations based on the original Kirchhoff and Lamb equations that one can need to perform a canonical transformation to the Hamiltonian, to the equations of motion and to the integrals of motion, so that after the transformation the integrals of motion take a relatively simpler form. An example of such situation is the case found originally by Sokolov in [335]. A canonical transformation introduced by Borisov and Mamaev in [39] was used to simplify the Hamiltonian and to give the integral a simpler form.*

10.5 Steklov and Kharlamov Analogies and Their Generalization

The problem of motion of a rigid body in a liquid has been considered for a part of its history in complete isolation of other problems of motion of a rigid body about a fixed point.

As will be seen in more detail in the coming chapters, the equations of motion in their full form (10.41) derived from the Routhian (10.27) can be interpreted as equations of motion about a fixed point of a heavy, magnetized and electrically charged body bearing a rotor and influenced by an axially symmetric combination of three classical fields. More precisely, the second equation of (10.41) resembles Poisson's equation met in several previous chapters. This equation describes the space time derivative of a vector $\mathbf{p}$ constant in space, referred to the body system. Let us take a unit vector γ in the direction of $\mathbf{p}$, so that $\mathbf{p} = p_0\gamma$ and thus equations (10.41) take the form

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} + p_0\gamma\bar{\mathbf{K}}) = \gamma\times(p_0\mathbf{a}+p_0^2\gamma\mathbf{J}),\\ p_0(\dot{\gamma} + \boldsymbol{\omega} \times \gamma) = \mathbf{0}. \end{aligned} \tag{10.44}$$

Let us first assume that $p_0 \neq 0$. In that case one can absorb this constant in the definitions for $\mathbf{a}$, $\mathbf{J}$ and $\bar{\mathbf{K}}$.

10.5.1 The Equivalent Problem of Motion About a Fixed Point

Alternatively, one can choose the units of measurement so that p_0 becomes unity. Finally, Eq. (10.41) can be written in the form of equations of motion of a rigid body about a fixed point with the vector γ fixed in space, i.e.

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} + \gamma\bar{\mathbf{K}}) = \gamma\times(\mathbf{a}+\gamma\mathbf{J}),\\ \dot{\gamma} + \boldsymbol{\omega} \times \gamma = \mathbf{0}. \end{aligned} \tag{10.45}$$

This system of equations may be obtained from (10.41) by the replacement

$$(\mathbf{p},\bar{\mathbf{K}},\mathbf{a},\mathbf{J}) \to (p_0\gamma,\bar{\mathbf{K}}/p_0,\mathbf{a}/p_0,\mathbf{J}/p_0^2), \tag{10.46}$$

so that if for some parameters $\mathbf{I},\boldsymbol{\kappa},\bar{\mathbf{K}},\mathbf{a},\mathbf{J}$ one has a solution $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{t})$ and $\gamma = \boldsymbol{\Gamma}(t)$ of the equivalent system (10.45), then one can obtain a solution $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{t})$, $p = p_0\gamma$ of (10.41) through the replacement of parameters

$$(\bar{\mathbf{K}}, \mathbf{a}, \mathbf{J}) \rightarrow (p_0\bar{\mathbf{K}}, p_0\mathbf{a}, p_0^2\mathbf{J}). \tag{10.47}$$

In this way, in the solution of the problem (10.41) an additional parameter p_0 is added. Returning to the problem of motion of a body in a liquid in the original formulation, we obtain a solution containing five parameters more than the solution of the equivalent problem.

Let us now turn to the excluded case $p_0 = 0$. In that case (10.44) reduces to

$$\dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}) = \mathbf{0}, \tag{10.48}$$

which are the equations of motion of a free gyrostat fixed from one point. Those are the integrable equations already discussed in Chap. 5 under the name of Joukovsky and Volterra. This justifies the use of Eq. (10.45) in the generic case.

Equation (10.45) can be derived from the Lagrangian

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + (\boldsymbol{\kappa}-\frac{1}{2}\gamma\mathbf{K})\cdot\boldsymbol{\omega}-\mathbf{a}\cdot\gamma-\frac{1}{2}\gamma\mathbf{J}\cdot\gamma, \tag{10.49}$$

which is the last form of the Routhian R in (10.27). They admit the set of three integrals corresponding to (10.42), which are now written as

$$\begin{aligned} I_1 &= \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{a}\cdot\gamma+\frac{1}{2}\gamma\mathbf{J}\cdot\gamma = h, \\ I_2 &= \gamma^2 = 1, \\ I_3 &= (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} - \frac{1}{2}\gamma\mathbf{K})\cdot\gamma = f \end{aligned} \tag{10.50}$$

where h, f are arbitrary integration constants.

The six-dimensional problem of motion of the rigid body in the liquid is thus reduced to another problem of motion of a body about a fixed point, having only three degrees of freedom. This problem is described by Eq. (10.49) and has the integrals (10.50). The Lagrangian of the new problem (the Routhian R of the original problem) involves the angular velocity $\boldsymbol{\omega}$ and the vector $\boldsymbol{\gamma}$ constant in space. The forces acting on this virtual body can be interpreted as having a scalar potential

$$V = \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma, \tag{10.51}$$

and a vector potential

$$\mathbf{l} = \boldsymbol{\kappa}-\frac{1}{2}\gamma\mathbf{K}. \tag{10.52}$$

From now on, to conform with the previous simpler problems and with future study of more complex problems, we shall write the Lagrangian (10.49) as

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{l}\cdot\boldsymbol{\omega} - V, \tag{10.53}$$

and Eq. (10.45) as

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\mu}) &= \boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.54}$$

where

$$V = \mathbf{a}\cdot\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma}, \tag{10.55}$$

$$\boldsymbol{\mu} = \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}. \tag{10.56}$$

In this form, each of the terms appearing in the equations of motion (10.54) of a rigid body in a liquid can be given concrete alternative interpretation:

(a) The vector $\mathbf{a}$ constant in the body, compared with formulas in Chap. 3, can be interpreted as the term $Mg\mathbf{r}_0$, the product of the weight of the equivalent body in a uniform gravity field g in the direction of $(-\boldsymbol{\gamma})$ and the position vector $\mathbf{r}_0$ of the centre of mass of that body.

(b) The vector $\boldsymbol{\kappa}$, also constant in the body, can be interpreted as a gyrostatic momentum of a symmetric rotor fixed from its axis of symmetry and rotating about it with a constant angular rate (Compare with Chap. 5).

(c) The potential term $\frac{1}{2}\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma}$ has a form, similar to that of the potential of a far Newtonian centre of attraction (Compare with Chap. 6), but can be interpreted in that way only when the matrices $\mathbf{J}$ and $\mathbf{I}$ are proportional $\mathbf{J} = \lambda\mathbf{I}$. For an arbitrary matrix $\mathbf{J}$, this term can be given interpretation as partially due to an attraction centre and partially as due to the electric interaction of a far Coulomb centre on the line parallel to $\boldsymbol{\gamma}$ and passing through the origin O, fixed in the present analogy, on a set of electric charges fixed in the equivalent body. In this interpretation, the matrix $\mathbf{J}$ is proportional to the inertia matrix of the electric charges on the equivalent body.

(d) The term $\boldsymbol{\gamma}\bar{\mathbf{K}}$ of the vector $\boldsymbol{\mu}$ can be interpreted as a result of the Lorentz effect of a uniform magnetic field parallel to $\boldsymbol{\gamma}$ on the electric charge distribution on the body (see e.g. [139]). This effect will be considered in more detail later in this book.

Conclusion: *The above considerations show that the overall effect of the hydrodynamic forces exerted by the fluid on the body can be replaced, as to their effect on the rotational motion of the body, by a set of relatively simple gravitational and electromagnetic interactions.*

By analogy or equivalence between the two problems here we mean full isomorphism of their equations of motion.[1] This analogy, pointed out in 1986 [383], generalizes the limited earlier analogies due to Steklov and Kharlamov:

---

[1] A weaker type of equivalence will be treated below involves isomorphism on the level of Routh-reduced equations of motion. The full Lagrangian systems are not isomorphic to each other, but any integrable case of one of them leads to an integrable case of the other.

10.5.2 Steklov's Analogy

In [345] (1895) and [348] (1902) noted that if in Kirchhoff's equations in Clebsch's form (10.12) one sets $\tilde{\mathbf{b}} = \mathbf{0}$, $\tilde{\mathbf{c}} = \epsilon \tilde{\mathbf{a}}$ those equations become identical with the equations of motion of a rigid body about a fixed point while acted upon by approximate Newtonian field in the integrable case when the body is fixed from its centre of mass (Case 2 of Chap. 6). In the terminology of Eq. (10.45) Steklov's analogy concerns the case $\mathbf{J} = \epsilon \mathbf{I}$, $\bar{\mathbf{K}} = \mathbf{0}$, $\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$.

10.5.3 Kharlamov's Analogy

In 1963, Kharlamov [192] generalized Steklov's analogy to the case of a perforated body, allowing non-zero vectors $\tilde{\alpha}$, $\tilde{\beta}$ in (10.23) and requiring only that $\tilde{\mathbf{b}} = \mathbf{0}$, $\tilde{c} = \epsilon \tilde{\mathbf{a}}$. For Eq. (10.54) Kharlamov's analogy requires that:

$$\mathbf{J} = \epsilon \mathbf{I}, \bar{\mathbf{K}} = \mathbf{0},$$

under which the problem of motion of a body in a liquid is analogous to the motion of a gyrostat about a fixed point, under the action of approximate Newtonian field of a centre (See Chap. 6).

10.6 Completing the Solution

10.6.1 Solution of the Equivalent Problem

Solving the system of equations of motion (10.45) we determine, as functions of the time t, the vectors $\boldsymbol{\omega}(t)$ and $\boldsymbol{\gamma}(t)$. In the alternative problem we regard the vector $\boldsymbol{\gamma}(=\frac{\mathbf{p}}{p_0})$, constant in space, as the unit vector pointing vertically upwards, take the Z-axis in that direction and measure the angle of precession ψ in the plane orthogonal to it. As in the classical problem (see 3.9), this determines the Eulerian angles of nutation and proper rotation θ and φ as

$$\theta = \cos^{-1} \gamma_3, \varphi = \tan^{-1} \frac{\gamma_1}{\gamma_2} \tag{10.57}$$

while the precession angle ψ is expressed by the quadrature

$$\psi = \psi_0 + \int_0^t \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2} dt, \tag{10.58}$$

ψ_0 is an integration constant. This completes the solution of the equivalent problem of motion about a fixed point, which is also the solution of the rotational part of the body in a liquid.

10.6.2 Solution of the Original Problem

Suppose that for the parameters $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}, \mathbf{a}, \mathbf{J}$ the equivalent problem (10.45) has a solution $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{t})$ and $\gamma = \boldsymbol{\Gamma}(t)$. The rotational motion of the body in the liquid is the same as in the previous subsection.

Conditions on the parameters of the original parameters are obtained by applying (10.47) to (10.43). This gives

$$\begin{aligned}
\mathbf{A} &= \mathbf{I} - (\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a)\mathbf{J}^{-1}(-\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\mathbf{B} &= \frac{1}{p_0^2}\mathbf{J}^{-1}(-\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\mathbf{C} &= \frac{1}{p_0^2}\mathbf{J}^{-1},\\
\tilde{\mathbf{K}} &= -\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a,\\
\bar{\boldsymbol{\alpha}} &= \boldsymbol{\kappa} + \frac{1}{p_0}\mathbf{a}\mathbf{J}^{-1}(-\frac{1}{2}p_0\mathbf{K} + \tilde{\mathbf{K}}_a),\\
\bar{\boldsymbol{\beta}} &= -\frac{1}{p_0}\mathbf{a}\mathbf{J}^{-1}.
\end{aligned} \tag{10.59}$$

The velocity of the point O taken as origin is determined from (10.26) through the formula

$$\begin{aligned}
\mathbf{u} &= \mathbf{a} + \boldsymbol{\Gamma}\mathbf{J} - \boldsymbol{\Omega}\tilde{\mathbf{K}}^T\\
&= \mathbf{a} + \boldsymbol{\Gamma}\mathbf{J} - \boldsymbol{\Omega}(\frac{1}{2}\mathbf{K} - \mathbf{K}_a)\\
&= \mathbf{a} + \boldsymbol{\Gamma}\mathbf{J} - \frac{1}{2}\omega\mathbf{K} + \boldsymbol{\Omega} \times \mathbf{k}_a.
\end{aligned} \tag{10.60}$$

In the last formula, one can easily recognize the term $\boldsymbol{\omega} \times \mathbf{k}_a$ as the only origin-dependent term. It represents the velocity of a unique point of the body whose position vector relative to O is $\mathbf{k}_a$. In the sequel, this point will be called the *proper* central point of the body. In contrast to the settled notation of the central point as the point at which the matrix B is symmetric, the proper central point has direct dynamical significance. If we take this point of the body as the origin, the matrix $\tilde{\mathbf{K}}$ would be symmetric. Taking (10.47) into account, the velocity $\mathbf{u}_{cp}$ of the central point is

$$\mathbf{u}_{cp} = p_0\mathbf{a}+p_0^2\boldsymbol{\Gamma}\mathbf{J}-p_0\boldsymbol{\Omega}\mathbf{K}. \tag{10.61}$$

That is origin-independent and depends only on the angular velocity and the orientation of the body.

For a given solution of the equations of motion, the position vector of the central point of the body can be found by quadratures

$$\begin{aligned}\mathbf{r}' = (X', Y', Z') &= \mathbf{r}'_0 + \int_0^t \mathbf{u}_{cp} dt \\ &= p_0\mathbf{a}t+p_0^2(\int_{\mathbf{0}}^{\mathbf{t}} \boldsymbol{\Gamma}(t)dt)\mathbf{J}-p_0(\int_{\mathbf{0}}^{\mathbf{t}} \boldsymbol{\Omega}(t)dt)\mathbf{K}.\end{aligned} \tag{10.62}$$

This yields the projections of the position vector of the origin of the body system relative to the origin of the inertial coordinate system on the axes of the body system. To express the position vector of the central point of the body referred to the inertial system, we write

$$\mathbf{r} = (\mathbf{r}' \cdot \boldsymbol{\alpha}, \mathbf{r}' \cdot \boldsymbol{\beta}, \mathbf{r}' \cdot \boldsymbol{\gamma}).$$

That is

$$\mathbf{r} = \mathbf{r}'\mathbf{R} \tag{10.63}$$

in terms of the rotation matrix $\mathbf{R}$, which can be constructed using the expressions (10.57) and (10.58) as shown in Chap. 2.

In the rest of this chapter we shall deal with the equivalent problem, returning to the original problem only occasionally, when some important assertions are to be made concerning the original problem. This is made here as a way of accommodating the problem of motion of a body in a liquid in the hierarchy on the top of problems of the previous chapters. A higher level in this hierarchy will be added in the next chapter.

10.7 Uniform Translational-Rotational Motion of a Body in a Liquid (Permanent Rotations of a Body with a Fixed Point About a Vertical Axis)

We now put forward a more general motion than that of Sect. 10.1.2, to find all possible permanent stationary (time-independent) motions. That is all solutions of (10.45) with the pair $(\boldsymbol{\omega}, \boldsymbol{\gamma})$ constant in the body and also in space. Substituting $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\gamma}} = \mathbf{0}$ in (10.45), we get

$$\begin{aligned}\boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}) &= \boldsymbol{\gamma}\times(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J}), \\ \boldsymbol{\omega} \times \boldsymbol{\gamma} &= \mathbf{0}.\end{aligned} \tag{10.64}$$

From the second equation, we can express the angular velocity in the form

$$\boldsymbol{\omega} = \omega_0 \boldsymbol{\gamma}, \tag{10.65}$$

where ω_0 is some proportionality constant and inserting this in the first equation, we obtain

$$\boldsymbol{\gamma}\times[\omega_0^2\boldsymbol{\gamma}\mathbf{I}+\omega_0(\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})-(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})]=\mathbf{0}. \tag{10.66}$$

This condition determines the vector $\boldsymbol{\gamma}$, which characterizes the possible stationary motion in the sense that at any moment the body rotates about an axis parallel to $\boldsymbol{\gamma}$ and passing through the proper central point.

Equation (10.66) has an obvious and direct geometric meaning:

For each, arbitrarily given, real ω_0, the vector $\boldsymbol{\gamma}$ characterizing the possible stationary motion lies along one of the lines drawn from the proper central point to intersect at right angle the surface

$$\boldsymbol{\Phi} = \frac{1}{2}\gamma(\omega_0^2\mathbf{I}+\omega_0\bar{\mathbf{K}}-\mathbf{J})\cdot\gamma + (\omega_0\boldsymbol{\kappa}-\mathbf{a})\cdot\gamma{=}{=}const. \tag{10.67}$$

Here *fl* is an inhomogeneous quadratic function. The surface is a quadric referred to an origin (the proper central point of the body) different from its centre. In the case of a simply connected body $\boldsymbol{\kappa}=\mathbf{a}=\mathbf{0}$ and then γ becomes one of the eigenvector of the matrix $\omega_0^2\mathbf{I}-\mathbf{2}\omega_0\bar{\mathbf{K}}-\mathbf{J}$, which are known to be three in number and orthogonal to each other (See Sect. 10.1.2). The same conclusion can be reached also when $\boldsymbol{\kappa}$, $\mathbf{a}$ are non-zero parallel vectors and ω_0 is chosen such that $\omega_0\boldsymbol{\kappa}-\mathbf{a}=\mathbf{0}$. In the general case of a Multiply connected (perforated) body no such general rule can be stated. When this surface is an ellipsoid, for arbitrary ω_0, $\boldsymbol{\kappa}$ and $\mathbf{a}$ only two lines are guaranteed to be drawn from the origin to intersect the surface orthogonally. Those are points on the surface, nearest and farthest from the origin. Only one such line is guaranteed when the surface is one-sheeted and extending to infinity.

The vector Eq. (10.66) can be written in the form of three scalar equations, but only two of those equations are independent. In fact, multiplying (10.66) scalarly by each of the vectors $\boldsymbol{\gamma}\mathbf{I}$ and $(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})$, one obtains two different expressions for the angular speed

$$\omega_0 = \frac{\boldsymbol{\gamma}\cdot[\boldsymbol{\gamma}\mathbf{I}\times(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})]}{\boldsymbol{\gamma}\cdot[\boldsymbol{\gamma}\mathbf{I}\times(\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]} = \frac{\boldsymbol{\gamma}\cdot[(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})\times(\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]}{\boldsymbol{\gamma}\cdot[\boldsymbol{\gamma}\mathbf{I}\times(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})]}. \tag{10.68}$$

Equality of the two expressions for ω_0 determines the locus of the vector $\boldsymbol{\gamma}$ in the form

$$\begin{aligned} &\{\boldsymbol{\gamma}\cdot[\boldsymbol{\gamma}\mathbf{I}\times(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})]\}^2- \\ &-\{\boldsymbol{\gamma}\cdot[\boldsymbol{\gamma}\mathbf{I}\times(\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]\}\{\boldsymbol{\gamma}\cdot[(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J})\times(\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}})]\} \\ =&\ 0. \end{aligned} \tag{10.69}$$

This equation is non-homogeneous of degree six, and it represents a surface fixed in the body. This surface intersects the Poisson sphere in some spherical curve. The line joining the fixed point to each point of that spherical curve generates the cone of possible axes of permanent rotations. One readily recognizes the following special cases:

(1) From (10.66) we find that pure translations ($\omega_0 = 0$) are possible if and only if γ is a generator of the cone $\mathbf{a} \cdot (\gamma \times \gamma\mathbf{J}) = 0$. This equation resembles that of Staude's cone, except for replacing the inertia matrix $\mathbf{I}$ by the matrix $\mathbf{J}$, that appears in the potential.
(2) For a simply connected body ($\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$) Eq. (10.69) of degree six in γ becomes homogeneous, and hence represents a cone. This result was obtained by Minkowski [284] in 1888.
(3) For a gyrostat moving about a fixed point in a uniform gravity field, $\mathbf{J} = \bar{\mathbf{K}} = \mathbf{0}$, (10.69) becomes, as already seen in Chap. 5,

$$[\mathbf{a} \cdot (\gamma \times \gamma\mathbf{I})]^2 - [\boldsymbol{\kappa} \cdot (\gamma \times \gamma\mathbf{I})][\mathbf{a} \cdot (\boldsymbol{\kappa} \times \gamma)] = 0.$$

(4) In the special case collinear gyrostatic momentum and centre of mass and proportional matrices $\bar{\mathbf{K}}, \mathbf{J}$ such that $\bar{\mathbf{K}} = \epsilon\mathbf{J}$, $\boldsymbol{\kappa} = \epsilon\mathbf{a}$, the cone of permanent rotation axes reduces to Staude's cone for the classical problem. Shortly below, we shall see that this is a result of certain symmetry of the equations of motion, which allows for a rotation transformation.
(5) For the classical problem of motion of a body ($\mathbf{J} = \bar{\mathbf{K}} = \boldsymbol{\kappa} = \mathbf{0}$) it gives Staude's cone described by the equation $\mathbf{a} \cdot (\gamma \times \gamma\mathbf{I}) = 0$ [343].

Remark The above analysis applies mostly to the equivalent problem of motion of a rigid body about a fixed point under the action of potential and gyroscopic forces, described by the equations of motion (10.45) or (10.54). In the problem of motion of a body in a liquid, as explained above, the body rotates with the constant angular speed ω_0 about an axis parallel to γ and passing through the proper central point, while the latter moves with the uniform velocity

$$\mathbf{u}_{cp} = p_0\mathbf{a} + p_0^2\gamma\mathbf{J} - p_0\omega_0\gamma\mathbf{K}). \tag{10.70}$$

The position vector of the central point of the body can be expressed in the form

$$\begin{aligned}
\mathbf{r}_{cp} &= \mathbf{r}_{cp0} + \int_0^t \mathbf{u}_{cp}\, dt\mathbf{R} \\
&= \mathbf{r}_{cp0} + \mathbf{u}_{cp} \int_0^t dt\mathbf{R} \\
&= \mathbf{r}_{cp0} + \frac{1}{\omega_0}[\mathbf{a} + \gamma(\mathbf{J} - \omega_0\mathbf{K})] \begin{pmatrix} \sin(\omega_0 t) & 1 - \cos(\omega_0 t) & 0 \\ \cos(\omega_0 t) - 1 & \sin(\omega_0 t) & 0 \\ 0 & 0 & \omega_0 t \end{pmatrix}.
\end{aligned} \tag{10.71}$$

10.8 Stationary Motions About an Axis Inclined to the Vertical

Unlike the classical problem and its generalization to the heavy gyrostat, the problem of motion of a body in a liquid admits another type of motions in which $\boldsymbol{\omega}$ is constant (in space and in the body), but in a direction different from that of $\boldsymbol{\gamma}$. Let us take the z-axis of the body coordinate system along that direction. One can write

$$\boldsymbol{\omega} = \Omega \mathbf{k}, \Omega = \text{const}\,. \tag{10.72}$$

Equations of motion give

$$\begin{aligned} \Omega \mathbf{k} \times (\Omega \mathbf{k}\mathbf{I}+\boldsymbol{\kappa} + \gamma \bar{\mathbf{K}}) = \gamma \times (\mathbf{a} + \gamma \mathbf{J}), \\ \dot{\gamma} + \Omega \mathbf{k} \times \gamma = \mathbf{0}. \end{aligned} \tag{10.73}$$

The motion can be described by Euler's angles in the usual notation: $\psi = \psi_0, \theta = \theta_0, \phi = \Omega t$. In virtue of the symmetry of the problem about the Z-axis and without loss of generality, one can take $\psi_0 = 0$. The unit vector $\boldsymbol{\gamma}$ can be expressed as

$$\gamma = (\sin\theta_0 \sin\Omega t, \sin\theta_0 \cos\Omega t, \cos\theta_0). \tag{10.74}$$

This can be easily shown to satisfy the second equation in (10.73). Substituting in the first equation and equating coefficients of similar terms in powers of $\sin\Omega t$ and $\cos\Omega t$, we arrive at the following set of conditions:

$$\begin{aligned} J_{12} = J_{13} = J_{23} = 0, J_{22} = J_{11}, \\ a_1 = a_2 = 0, a_3 + \Omega \tilde{K}_{11} - (J_{11} - J_{33}) \cos\theta_0 = 0, \\ \bar{K}_{12} = 0, \bar{K}_{22} = \bar{K}_{11}, \\ \kappa_1 + \Omega I_{13} + \cos\theta_0 \bar{K}_{13} = 0, \kappa_2 + \Omega I_{23} + \cos\theta_0 \bar{K}_{23} = 0. \end{aligned} \tag{10.75}$$

In the generic case, we can write

$$\begin{aligned} \mathbf{J} = \begin{pmatrix} J_{11} & 0 & 0 \\ 0 & J_{11} & 0 \\ 0 & 0 & J_{33} \end{pmatrix}, \bar{\mathbf{K}} = \begin{pmatrix} \bar{K}_{11} & 0 & \bar{K}_{13} \\ 0 & \bar{K}_{11} & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33} \end{pmatrix}, \\ \mathbf{a} = (0, 0, -\Omega \bar{K}_{11} + \cos\theta_0 (J_{11} - J_{33})), \\ \boldsymbol{\kappa} = (-\Omega I_{13} - \cos\theta_0 \bar{K}_{13}, -\Omega I_{23} - \cos\theta_0 \bar{K}_{23}, \kappa_3). \end{aligned} \tag{10.76}$$

where $J_{11}, J_{33}, \bar{K}_{11}, \bar{K}_{33}, \bar{K}_{13}, \bar{K}_{23}, I_{13}, I_{23}, \kappa_3$ are arbitrary parameters. Note that the axis of a permanent rotation of the present type must be a principal axis of the matrix $\mathbf{J}$, while the eigenvalues corresponding to the other two principal axes are equal. The virtual centre of mass (the vector $\mathbf{a}$) should also lie on the axis of rotation.

The last two Eq. (10.76) determine the pair of vectors $\mathbf{a}, \boldsymbol{\kappa}$ depending on the angular velocity Ω and the angle θ_0, which can be given arbitrary values. On the other hand, if one regards $\mathbf{a}, \boldsymbol{\kappa}$ as given parameters, the last equations reduce to three equations in two unknowns Ω and θ_0, for the solution of which a condition on the parameters of the body should be imposed. The parameters of the body must also satisfy the obvious restriction $|\cos\theta_0| \leq 1$. The solution of (10.75) exhibits several special and degenerate cases, some of which will be summed up in the exercises.

10.9 A Several-Parameter Particular Solution

A result, which will be presented in the next chapter, was obtained in [389] in the context of the problem of motion of a body about a fixed point under the action of an axially symmetric combination of forces. A special case of this result reduces to a case of motion of a body in a liquid (in fact, the alternative problem) and gives a quite general particular solution of that problem, in the sense of the number of parameters retained in it. That is a solution satisfying three invariant relations. It can be formulated as the following

Theorem 10.1 *Let in (10.45)*

$$\begin{aligned} \mathbf{J} &= -\mathbf{MIM} + \alpha\mathbf{M} + \varepsilon\boldsymbol{\delta}, \\ \bar{\mathbf{K}} &= \beta\mathbf{M} - \alpha\boldsymbol{\delta} + \mathrm{tr}(\mathbf{MI})\boldsymbol{\delta} - \mathbf{IM} - \mathbf{MI}, \\ \boldsymbol{\kappa} &= \mathbf{m}(\beta\boldsymbol{\delta} - \mathbf{I}), \\ \mathbf{a} &= \mathbf{m}(\alpha\boldsymbol{\delta} - \mathbf{IM}). \end{aligned} \tag{10.77}$$

where $\mathbf{M}, \mathbf{m}$ *are a constant real* 3×3 *symmetric matrix and a vector, respectively,* $\alpha, \beta, \varepsilon$ *are constants. Then,*

(1) (10.45) admits a solution, which satisfies the relations

$$\boldsymbol{\omega} = \boldsymbol{\gamma}\mathbf{M} + \mathbf{m}, \tag{10.78}$$

and γ *is a solution of Poisson's equation, which now takes the form*

$$\dot{\boldsymbol{\gamma}} + (\boldsymbol{\gamma}\mathbf{M} + \mathbf{m}) \times \boldsymbol{\gamma} = \mathbf{0}. \tag{10.79}$$

(2) In the generic case, the solution ($\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$) is expressed in terms of elliptic functions of time.

Proof (1) On substituting (10.77)–(10.79) into (10.45) and using the identity in Appendix 11.1, the first equation turns into identity.

(2) Assuming that $\det(\mathbf{M}) \neq \mathbf{0}$ and using the relation inverse to (10.78), one can write Eq. (10.79) as

$$\dot{\boldsymbol{\omega}}\mathbf{M}^{-1} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{M}^{-1} - m\mathbf{M}^{-1}) = \mathbf{0}.$$

This is the equation of motion of a free gyrostat with inertia matrix $\mathbf{M}^{-1}$ and gyrostatic momentum $-\mathbf{m}\mathbf{M}^{-1}$. This characterizes the case discussed in Chap. 5 under the name of Joukovsky–Volterra's case. Referring equations to the principal axes of $\mathbf{M}^{-1}$ by a suitable rotation, the solution is determined in terms of elliptic functions of time. ■

This completes building for the alternative problem (10.45), which involves 21 parameters, a solution depending on 15 of those parameters. One can also go back through (10.43) to build the relevant solution of the problem of motion of a body in a liquid (Eq. (10.41)) and including the parameter p_0 and the three elements of the anti-symmetric part of **B**.

The solution established by theorem 1 generalizes by the presence of several extra-parameters a former solution obtained by Kharlamov in [197]. It also generalizes another solution obtained by Kharlamova [205], while studying the motion of a rigid body about a fixed point in an approximate Newtonian field without gyroscopic forces, except the constant gyrostatic momentum. The choices in both works [197, 205] correspond to a matrix M which is diagonal and hence commuting with **I**. The relation of the variables to time was established only in some special cases, where very restrictive conditions were imposed on the parameters. Much older partial results were obtained by Steklov, who considered the case $\mathbf{m} = \mathbf{0}$ and **M** diagonal in the principal axes of inertia [345].

Solutions on invariant relations of the general form (10.79) (with non-diagonal **M**) were considered in the much later papers [128, 129] (See also [125]). In those papers, no reference is made to our relevant result in [389], published 14 years earlier. Conditions that the dynamical equations of motion are satisfied along with the given invariant relations are obtained by the (brute force) method of solving algebraic equations. Expressions obtained in [129] are not transparent, and there is no comparison with previous results.

10.10 Alternative Hamiltonian Formulation

Equations of motion (10.45) can be put in Hamiltonian form in two ways. On one hand, using canonical variables, such as Euler's angles and momenta conjugate to them is hopelessly complicated for analytical considerations. On the other hand, one can introduce the angular momentum of the system described by the Routhian (10.27)

$$\mathbf{M} = \frac{\partial R}{\partial \boldsymbol{\omega}} = \boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} + \gamma\bar{\mathbf{K}}, \tag{10.80}$$

as phase variable instead of $\boldsymbol{\omega}$, so that

$$\boldsymbol{\omega} = (\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{\mathbf{K}})\mathbf{I}^{-1}. \tag{10.81}$$

The Hamiltonian corresponding to the Lagrangian (10.49) as a function in $\mathbf{M}$ and γ is

$$
\begin{aligned}
H &= \frac{1}{2}\omega\mathbf{I}\cdot\omega + \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma \\
&= \frac{1}{2}(\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{\mathbf{K}})\mathbf{I}^{-1}\cdot(\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{\mathbf{K}}) + \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma \\
&= \frac{1}{2}\mathbf{M}\mathbf{I}^{-1}\cdot\mathbf{M} - (\boldsymbol{\kappa} + \gamma\bar{\mathbf{K}})\mathbf{I}^{-1}\cdot\mathbf{M} \\
&\quad + (\mathbf{a} + \boldsymbol{\kappa}\mathbf{I}^{-1}\bar{\mathbf{K}}^T)\cdot\gamma + \frac{1}{2}\gamma(\mathbf{J} + \bar{\mathbf{K}}\mathbf{I}^{-1}\bar{\mathbf{K}}^T)\cdot\gamma,
\end{aligned}
\tag{10.82}
$$

so that the equations of motion can be written as

$$
\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M}\times\frac{\partial H}{\partial \mathbf{M}} + \gamma\times\frac{\partial H}{\partial \gamma}, \\
\dot{\gamma} &= \gamma\times\frac{\partial H}{\partial \mathbf{M}}.
\end{aligned}
\tag{10.83}
$$

Note that

$$
\begin{aligned}
\frac{\partial H}{\partial \mathbf{M}} &= (\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{K})\mathbf{I}^{-1} = \boldsymbol{\omega}, \\
\frac{\partial H}{\partial \gamma} &= \mathbf{a} + \gamma\mathbf{J} - (\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{\mathbf{K}})\mathbf{I}^{-1}\bar{\mathbf{K}}^T,
\end{aligned}
$$

or, in the expanded form

$$
\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M}\times(\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{\mathbf{K}})\mathbf{I}^{-1} + \gamma\times\frac{\partial H}{\partial \gamma}, \\
\dot{\gamma} &= \gamma\times(\mathbf{M} - \boldsymbol{\kappa} - \gamma\bar{\mathbf{K}})\mathbf{I}^{-1}.
\end{aligned}
\tag{10.84}
$$

For the Hamiltonian equations, the integrals of motion take the simplest form

$$
\begin{aligned}
I_1 &= H = h, \\
I_2 &= \mathbf{M}\cdot\gamma = f, \\
I_3 &= \gamma\cdot\gamma = 1.
\end{aligned}
\tag{10.85}
$$

This situation makes use of the Hamiltonian form of equations favourable in certain situations. However, in other situations and for most of our purposes, the Lagrangian formalism of the equations of motion remains the favourable choice.

Throughout this book, we adhere to the use of Lagrangian formalism. We owe the reader some explanation for that. In early times of Hamiltonian mechanics, the formulation of mechanical problems stemmed directly from the physical setting. In the Hamiltonian describing the motion of a particle under the action of certain forces, each term of the Hamiltonian usually had its definite and unambiguous meaning. The situation in modern days is different. Some integrable Hamiltonians of the structure (10.82) were recently obtained by searching the relevant coefficients in a general ansatz. Two factors come into play even in the simpler cases when the description of motion is given a priori in Hamiltonian form:

(1) The Hamiltonian and the equations of motion derived from it are not unique for one and the same physical problem. This makes classification of integrable cases in Hamiltonian formulation rather problematic. As a matter of fact, to decide whether two Hamiltonians describe the same mechanical system practically reduces to computing the functions V and $\boldsymbol{\mu}$ as the quantities that remain invariant under all gauge transformations (canonical transformations linear in momenta in Hamiltonian terms).
(2) In the Euler–Poisson variables, it is possible to tell about physical interpretation of various terms of the potential. For example, terms linear in $\boldsymbol{\gamma}$ represent the potential of the heavy body in a constant gravity field. The centre of mass of the body is uniquely determined by terms in the Lagrangian linear in $\boldsymbol{\gamma}$. Other terms can be identified as a result of gravitational, electric or magnetic potential, but in the transformed Hamiltonian form terms of various degrees are totally disguised.

To illustrate the above points, we use as an example the case introduced by Sokolov [336] with the Kowalevski configuration $A = B = 2$, $C = 1$. The original Hamiltonian describing this case is

$$
\begin{aligned}
H_1 = &\frac{1}{4}(M_1^2 + M_2^2 + 2M_3^2) + \frac{1}{2}M_3(c_1\gamma_1 + c_2\gamma_2) + \frac{1}{2}\gamma_3(c_1M_1 + c_2M_2) \\
&+ (c_1\gamma_2 - c_2\gamma_1)^2 - (c_1^2 + c_2^2)\gamma_3^2.
\end{aligned} \tag{10.86}
$$

Calculating the Lagrangian corresponding to this Hamiltonian (10.86) and using the Legendre transformation

$$
\begin{aligned}
\boldsymbol{\omega} \equiv (p, q, r) &= \frac{\partial H_1}{\partial \mathbf{M}} \\
&= (\frac{1}{2}M_1 + \frac{1}{2}c_1\gamma_3, \frac{1}{2}M_2 + \frac{1}{2}c_2\gamma_3, M_3 + \frac{1}{2}(c_1\gamma_1 + c_2\gamma_2)),
\end{aligned} \tag{10.87}
$$

we find

$$
\begin{aligned}
L_1 &= \mathbf{M} \cdot \frac{\partial H_1}{\partial \mathbf{M}} - H_1 \\
&= p^2 + q^2 + \frac{r^2}{2} \\
&\quad - c_1(p\gamma_3 + \frac{1}{2}r\gamma_1) - c_2(q\gamma_3 + \frac{1}{2}r\gamma_2) \\
&\quad - \frac{9}{8}[(c_1\gamma_2 - c_2\gamma_1)^2 - (c_1^2 + c_2^2)\gamma_3^2] + \frac{1}{8}(c_1^2 + c_2^2)(\gamma_1^2 + \gamma_2^2 + \gamma_3^2).
\end{aligned} \tag{10.88}
$$

In the last expression one can eliminate its last term, since the spherically symmetric term does not contribute to the equations of motion. Thus, the Lagrangian L_1 has a potential part

$$V = \frac{9}{8}[(c_1\gamma_2 - c_2\gamma_1)^2 - (c_1^2 + c_2^2)\gamma_3^2]. \tag{10.89}$$

On the other hand, the gyroscopic terms of L_1 correspond to the choice of the vector $\boldsymbol{l}$ as

$$\boldsymbol{l} = -(c_1\gamma_3, c_2\gamma_3, \frac{1}{2}c_1\gamma_1 + \frac{1}{2}c_2\gamma_2).$$

This uniquely determines the vector

$$\begin{aligned}\boldsymbol{\mu} &= -\nabla[\frac{3}{2}(c_1\gamma_1 + c_2\gamma_2)\gamma_3] \\ &= -\frac{3}{2}(c_1\gamma_3, c_2\gamma_3, c_1\gamma_1 + c_2\gamma_2).\end{aligned} \tag{10.90}$$

The mechanical system under consideration is completely characterized by the pair of functions V and $\boldsymbol{\mu}$. The complementary integral of motion in the Euler–Poisson variables can be written as

$$I_4 = Z_1 Z_2, \tag{10.91}$$

$$Z_1 = (r - 1/2\, a_1\, \gamma_1 - 1/2\, a_2\, \gamma_2)\,,$$

$$\begin{aligned}Z_2 &= 1/4\,(2r - a_1\,\gamma_1 - a_2\,\gamma_2)\,[4\,p^2 + 4\,q^2 + (2r - a_1\,\gamma_1 - a_2\,\gamma_2)^2] \\ &+ 2\,\,(2\,pa_1 + 2\,qa_2)\,(2\,p\gamma_1 + 2\,q\gamma_2) + \,(a_1\,\gamma_1 + a_2\,\gamma_2)\,(2r - a_1\,\gamma_1 - a_2\,\gamma_2)^2 \\ &+ 1/2\,(a_1\,\gamma_1 + a_2\,\gamma_2)^2\,(2r - a_1\,\gamma_1 - a_2\,\gamma_2) \\ &- 1/2\,\left(a_1^2 + a_2^2\right)\gamma_3\,[8\,p\gamma_1 + 8\,q\gamma_2 + \gamma_3(2r - a_1\,\gamma_1 - a_2\,\gamma_2)].\end{aligned} \tag{10.92}$$

where we have set $a_1 = 3c_1, a_2 = 3c_2$.

An alternative Hamiltonian describing the same system was introduced by Borisov and Mamaev [39], using a linear transformation of the phase variables, which preserves the Poisson brackets and simplifies the Hamiltonian to

$$H_2 = \frac{1}{4}(M_1^2 + M_2^2 + 2M_3^2) + \frac{1}{2}M_3(a_1\gamma_1 + a_2\gamma_2) + \frac{1}{4}(a_1^2 + a_2^2)(\gamma_1^2 + \gamma_2^2). \tag{10.93}$$

The relation between $\boldsymbol{\omega}$ and $\mathbf{M}$ for this Hamiltonian is

$$\begin{aligned}\boldsymbol{\omega} &= \frac{\partial H_2}{\partial \mathbf{M}} \\ &= (\frac{1}{2}M_1, \frac{1}{2}M_2, M_3 + \frac{1}{2}(a_1\gamma_1 + a_2\gamma_2)),\end{aligned} \tag{10.94}$$

and by direct calculation of the corresponding Lagrangian

$$
\begin{aligned}
L_2 == p^2 + q^2 + \frac{r^2}{2} \\
- \frac{1}{2} r(a_1\gamma_1 + a_2\gamma_2) \\
+ \frac{1}{8}[(a_1\gamma_2 + a_2\gamma_1)^2 + 2(a_1^2 + a_2^2)\gamma_3^2].
\end{aligned}
\tag{10.95}
$$

Note that its second line gives

$$
\boldsymbol{l} = (0, 0, \frac{1}{2}(a_1\gamma_1 + a_2\gamma_2)),
$$

which leads to the same $\boldsymbol{\mu}$ as in (10.90). The two Lagrangians $L_{1,2}$ are in fact equivalent. Their difference is

$$
\begin{aligned}
L_1 - L_2 &= c_1(r\gamma_1 - p\gamma_3) + c_2(r\gamma_2 - q\gamma_3) - (c_1^2 + c_2^2)(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \\
&= \frac{d}{dt}(c_2\gamma_1 - c_1\gamma_2) - (c_1^2 + c_2^2)(\gamma_1^2 + \gamma_2^2 + \gamma_3^2),
\end{aligned}
$$

i.e a gauge term and a central potential term. Both terms do not contribute to the equations of motion.

In contrast to the clarity and the physical relevance of the Lagrangian approach, none of the Hamiltonians H_1 and H_2 reflects the real nature of the potential (The terms quadratic in γ_i). Physical characteristics of the mechanical system are disguised in Hamiltonian form.

On the other hand, different Hamiltonian equations of motion are obtained using the Hamiltonians H_1, H_2. Also, each form of the Hamiltonians corresponds to a different form of the complementary integral, which can be constructed by substituting (10.87) and (10.94), respectively, in (10.91).

The change of the phase variables $\{\mathbf{M},\boldsymbol{\gamma}\} \to \{\bar{\mathbf{M}}, \boldsymbol{\gamma}\}$ which transforms H_1 into H_2 can be obtained by comparing (10.94) with (10.87), in the form

$$
\bar{M}_1 = M_1 + c_1\gamma_3, \bar{M}_2 = M_2 + c_2\gamma_3, \bar{M}_3 = M_3 - c_1\gamma_1 - c_2\gamma_2. \tag{10.96}
$$

This is identical to the (canonical) transformation given by Borisov and Mamaev in [39] (See also [336]).

10.11 The Uniform Precession Transformation [383]

In its full final form (10.45), the problem of motion of a body in a liquid is at the top of a hierarchy, consisting of the problems considered in the previous chapters, involving the gyrostatic effect, the Newtonian potential term and the uniform gravity field as special cases. Consequently, every integrable case or solution of the above problems may have a generalization in the frame of the present one. This situation will be made clear in the tables of integrable cases provided below in this chapter.

A remarkable feature of Eq. (10.45) for the body in liquid, which is not enjoyed by any of the three simpler problems of Sects. 10.3–10.6, is their invariance under the uniform precession transformation, which we are going to describe now. This transformation was firstly introduced for the problem of motion of a body in a liquid in [383].

10.11.1 Direct Derivation

In the equations of motion (10.45), we perform the transformation of the variables $\boldsymbol{\omega}$ to a new set of variables $\boldsymbol{\omega}'$ by the relation

$$\boldsymbol{\omega} = \boldsymbol{\omega}' - n\boldsymbol{\gamma}, \tag{10.97}$$

containing the free real parameter n. Substituting in (10.45) we obtain

$$\begin{aligned}(\dot{\boldsymbol{\omega}}' - n\dot{\boldsymbol{\gamma}})\mathbf{I} + (\boldsymbol{\omega}' - n\boldsymbol{\gamma}) \times [(\boldsymbol{\omega}' - n\boldsymbol{\gamma})\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}) = \boldsymbol{\gamma} \times (\mathbf{a} + \boldsymbol{\gamma}\mathbf{J}),\\ \dot{\boldsymbol{\gamma}} + (\boldsymbol{\omega}' - n\boldsymbol{\gamma}) \times \boldsymbol{\gamma} = \mathbf{0}.\end{aligned} \tag{10.98}$$

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}} - n\boldsymbol{\gamma}\mathbf{I}) = n(\dot{\boldsymbol{\gamma}}\mathbf{I} + \boldsymbol{\gamma} \times \boldsymbol{\omega}'\mathbf{I}) + \boldsymbol{\gamma} \times (\mathbf{a} + \boldsymbol{\gamma}\mathbf{J})\\ + n\boldsymbol{\gamma} \times (- n\boldsymbol{\gamma}\mathbf{I} + \boldsymbol{\kappa} - 2\boldsymbol{\gamma}\bar{\mathbf{K}}),\\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0}.\end{aligned} \tag{10.99}$$

Using Poisson's equation to express $\dot{\boldsymbol{\gamma}}$ and noting that

$$(\boldsymbol{\omega}' \times \boldsymbol{\gamma})\mathbf{I} + \boldsymbol{\omega}'\mathbf{I} \times \boldsymbol{\gamma} = \boldsymbol{\omega}' \times \boldsymbol{\gamma}[tr(\mathbf{I})\delta - \mathbf{I}]$$

we give (10.99) the form

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}} + 2n\boldsymbol{\gamma}\bar{\mathbf{I}}) = \boldsymbol{\gamma} \times [\mathbf{a} + n\boldsymbol{\kappa} + \boldsymbol{\gamma}(J - 2n\bar{\mathbf{K}} - n^2\mathbf{I})],\\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0},\end{aligned} \tag{10.100}$$

which can be readily put in the final form

$$\begin{aligned}\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} + \boldsymbol{\gamma}\bar{\mathbf{K}}') = \boldsymbol{\gamma} \times (\mathbf{a}' + \boldsymbol{\gamma}\mathbf{J})\\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0},\end{aligned} \tag{10.101}$$

after introducing the notation

$$\mathbf{K}' = \mathbf{K} + 2n\mathbf{I}, \bar{\mathbf{K}}' = \bar{\mathbf{K}} + 2n\bar{\mathbf{I}},$$
$$\mathbf{a}' = \mathbf{a} + n\boldsymbol{\kappa},$$
$$\mathbf{J}' = \mathbf{J} - n\mathbf{K} - n^2\mathbf{I}. \tag{10.102}$$

A look at the two sets of Eqs. (10.45) and (10.101) reveals that they have the same structure in terms of the two sets of variables $\{\boldsymbol{\omega}, \gamma\}$ and $\{\boldsymbol{\omega}', \gamma\}$ and that they differ only in the values of parameters $\mathbf{a}, \mathbf{J}, \mathbf{K}$, which are transformed to $\mathbf{a}', \mathbf{J}', \mathbf{K}'$, respectively, containing the extra-parameter n. When one sets $n = 0$, $\boldsymbol{\omega}' = \boldsymbol{\omega}$ and the three primed matrix-parameters reduce to their original (unprimed) values. It is an easy exercise to show that the consecutive application of two transformations with parameters n_1, n_2 is equivalent to the application of one transformation with the parameter $n_1 + n_2$.

10.11.2 Lagrangian Derivation

Consider the problem described by the equations of motion (10.45) derived from the Lagrangian (10.49). Let us affect the transformation. It can be readily checked that this transformation changes the Lagrangian (10.49) to the similar form

$$L' = \frac{1}{2}\omega'\mathbf{I}\cdot\omega' + \boldsymbol{l}'.\omega' - V', \tag{10.103}$$

where

$$V' = V + n\boldsymbol{l}\cdot\gamma - \frac{1}{2}n^2\gamma\mathbf{I}\cdot\gamma, V = \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma,$$
$$\boldsymbol{l}' = \mathbf{l} - n\gamma\mathbf{I}, \boldsymbol{l} = \boldsymbol{\kappa} - \frac{1}{2}\gamma\mathbf{K}. \tag{10.104}$$

and renders the equations of motion (10.45) to

$$\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}'\times(\boldsymbol{\omega}'\mathbf{I} + \bar{\ }') = \gamma\times\frac{\partial V'}{\partial\gamma},$$
$$\dot{\gamma} + \boldsymbol{\omega}'\times\gamma = \mathbf{0} \tag{10.105}$$

where

$$\boldsymbol{\mu}' = \boldsymbol{\mu} + 2n\gamma\bar{\mathbf{I}}$$
$$= \boldsymbol{\kappa} + \gamma\bar{\mathbf{K}} + 2n\gamma\bar{\mathbf{I}}.$$

Equation (10.105) admit the integrals

$$I_1 \equiv \frac{1}{2}\boldsymbol{\omega}'\mathbf{I}\cdot\boldsymbol{\omega}' + \mathbf{a}'\cdot\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\gamma}\mathbf{J}'\cdot\boldsymbol{\gamma} = h',$$
$$I_2 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\kappa} - \frac{1}{2}\boldsymbol{\gamma}\bar{\mathbf{K}}')\cdot\boldsymbol{\gamma} = f',$$
$$I_3 = \boldsymbol{\gamma}^2 = 1, \tag{10.106}$$

with the constants

$$h' = h + nf,\ f' = f. \tag{10.107}$$

10.11.3 Physical and Mechanical Significance of the Transformation

The transformation (10.97) was used by Tisserand in [353] (See also [354]) to illustrate the effect of Coriolis and centrifugal forces on the motion of a rigid body with one point fixed on the rotating earth. It was implicitly used by other authors (e.g. [21, 51]) while studying the stability of relative equilibria in the problem of motion of a satellite in a circular orbit. It was applied only to the integrals of motion, but the transformed equations were not obtained and the full significance of the transformation was not revealed.

The transformation was applied for the first time to the full equations of motion of a charged and magnetized body in [378], where all its properties were revealed. It was also applied to the problem of motion of a satellite in a circular orbit to obtain its equations of motion relative to the orbital frame [382], in a form that resembles equations of motion of a rigid body about a fixed point under the action of given potential and gyroscopic forces. The invariance of the equations of motion of the body in a liquid under this transformation was first recognized in our work [383].

The presence of the parameter n in the transformed Lagrangian and the transformed equations of motion, in the framework of the equivalent physical problem, turns on a simultaneous combination of three physical effects:

(1) The effect of displacing the centre of mass of the body by an amount $n\boldsymbol{\kappa}$, proportional to the gyrostatic moment.

When $\mathbf{a}$ is proportional to $\boldsymbol{\kappa}$, say, $\mathbf{a} = m\boldsymbol{\kappa}$, then one can choose $n = -m$, so that $\mathbf{a}' = \mathbf{0}$ and thus getting rid of the uniform gravity field in the transformed problem.

(2) The matrix $\mathbf{K}$ is changed by the amount $n\mathbf{I}$. This can be interpreted as the matrix of coefficients in the vector potential of a static, on the body, charge distribution

whose inertia matrix is proportional to the inertia matrix of the distribution of mass in the body and subject to the Lorentz forces due to a uniform magnetic field of intensity $\mathcal{B} = -2n$ in the direction of $\boldsymbol{\gamma}$.

If the matrix $\mathbf{K}$ is proportional to $\mathbf{I}$, say, $\mathbf{K} = m\mathbf{I}$, then the regular precession transformation can be used to make $\mathbf{K}' = \mathbf{0}$ by taking $n = m$.

(3) The matrix $\mathbf{J}$ of coefficients of the quadratic part of the potential is modified by adding two terms: $-n\mathbf{K} - n^2\mathbf{I}$. The potential resulting from those terms can be interpreted as due to magnetized (electrically charged) parts of the body influenced by a magnetic (electric) field with second-harmonic potential.

In the next sections, we shall use the uniform precession transformation in the two ways: to construct more general solutions containing the parameter n from known simpler ones and to simplify some other cases of motion by using that parameter to reduce the number of physical constants in them, in order to facilitate obtaining their solutions.

As a quick illustrative example, we work out an explicit solution of Euler's case generalized by the uniform precession transformation. For the transformed motion

$$V = -\frac{1}{2}n^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}, \qquad \boldsymbol{\mu} = 2n\boldsymbol{\gamma}\bar{\mathbf{I}}. \tag{10.108}$$

The equations of motion for this case take the form

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + 2n\boldsymbol{\gamma}\bar{\mathbf{I}}) &= -n^2\boldsymbol{\gamma}\times\boldsymbol{\gamma}\mathbf{I}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.109}$$

They admit the complementary integral

$$A^2(p + n\gamma_1)^2 + B^2(q + n\gamma_2)^2 + C^2(r + n\gamma_3)^2 = G^2. \tag{10.110}$$

On one hand, Eq. (10.109) characterizes an integrable case of motion of the body in a liquid, which lies on the intersection of the cases of Clebsch and Steklov (See Table 10.2 below). On the other hand, in the framework of the equivalent problem, they describe the motion of a body under the influence of potential and Lorentz' forces. A family of explicit solutions of this case[2] can be written down immediately by transforming the solution constructed for Euler's case in Chap. 4 Sect. 4.1.

[2] In Euler's case we have solved only the dynamical equations of motion and adopted a very special solution of Poisson's equations in which the vectors $\boldsymbol{\gamma}$ and $\mathbf{G}$ are parallel. Hoever, the transformation applies equally well to the general solution of thewhole Euler-Poisson system.

$$p = \pm(\mu - \frac{n}{D}A)\sqrt{\frac{D(D-C)}{A(A-C)}}\mathrm{cn}\lambda(t-t_0),$$

$$q = (\mu - \frac{n}{D}B)\sqrt{\frac{D(D-C)}{B(B-C)}}\mathrm{sn}\lambda(t-t_0),$$

$$r = \pm(\mu - \frac{n}{D}C)\sqrt{\frac{D(A-D)}{C(A-C)}}\mathrm{dn}\lambda(t-t_0), \tag{10.111}$$

$$\gamma_1 = \pm\sqrt{\frac{A(D-C)}{D(A-C)}}\mathrm{cn}\lambda(t-t_0),$$

$$\gamma_2 = \sqrt{\frac{B(D-C)}{D(B-C)}}\mathrm{sn}\lambda(t-t_0),$$

$$\gamma_3 = \pm\sqrt{\frac{C(A-D)}{D(A-C)}}\mathrm{dn}\lambda(t-t_0), \tag{10.112}$$

where λ, D, μ and k, the modulus of elliptic functions, are the same as in Chap. 4 Sect. 4.1. The motion of the body is quite different from that in Euler's case. For example, choosing $n = \frac{\mu D}{A}$ we make $p \equiv 0$. The angular velocity lies permanently in the yz-plane, and it is still expressed in elliptic functions of time.

10.11.4 Uniform Precession Transformation in Hamiltonian Formalism

The expression (10.82) gives the Hamiltonian corresponding to the Lagrangian (10.49). Let H' be the Hamiltonian corresponding to the Lagrangian L' in (11.6), i.e. the Lagrangian obtained from (10.49) by the replacement $\boldsymbol{\omega} \to \boldsymbol{\omega}' = \boldsymbol{\omega} + n\boldsymbol{\gamma}$. It can be shown by direct calculation that

$$H' = H + n\mathbf{M}\cdot\boldsymbol{\gamma}. \tag{10.113}$$

The precession transformation is equivalent to adding the term $n\mathbf{M}\cdot\boldsymbol{\gamma}$ to the Hamiltonian, which is the precession parameter n multiplied by the areas integral (the second integral in (10.85)). The transformed Hamiltonian is a constant of motion

$$H' = h' = h + nf, \tag{10.114}$$

in agreement with (10.107). We also have

$$\begin{aligned}\boldsymbol{\omega}' &= \frac{\partial H'}{\partial \mathbf{M}} \\ &= \frac{\partial H}{\partial \mathbf{M}} + n\boldsymbol{\gamma} \\ &= \boldsymbol{\omega} + n\boldsymbol{\gamma}.\end{aligned} \tag{10.115}$$

Moreover, it is easy to show that the transformed Hamiltonian H' in (10.113) produces the same equations of motion as the original Hamiltonian H. In fact, using H' in (10.83), we obtain the equations

$$\begin{aligned}\dot{\mathbf{M}} &= \mathbf{M} \times \frac{\partial H'}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H'}{\partial \boldsymbol{\gamma}} \\ &= \mathbf{M} \times (\frac{\partial H}{\partial \mathbf{M}} + n\boldsymbol{\gamma}) + \boldsymbol{\gamma} \times (\frac{\partial H}{\partial \boldsymbol{\gamma}} + n\mathbf{M}) \\ &= \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \frac{\partial H'}{\partial \mathbf{M}} \\ &= \boldsymbol{\gamma} \times (\frac{\partial H}{\partial \mathbf{M}} + n\boldsymbol{\gamma}) \\ &= \boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}},\end{aligned} \tag{10.116}$$

which are identical to the original Eq. (10.83).

In contrast to the transformed Lagrangian (10.103), the transformed Hamiltonian (10.113) does not reveal any of the physical effects of the uniform precession transformation, which we listed in the last subsection. Thus, *the part of the physical effects induced by the uniform precession transformation in the problem is completely hidden by the Hamiltonian form of the equations of motion.* The Hamiltonian formalism identifies the whole family of mechanical systems depending on the arbitrary parameter n into a single Hamiltonian system. The Hamiltonian flow on the integral manifold of that system is the same for all physical problems, which differ only in the value of n. In the problem of motion of a body in a liquid that is a family of bodies with differing shape characteristics, but in the alternative problem it means a body subject to a family of potential and gyroscopic forces depending on n. As usual in the search for integrable cases, one assumes only the Hamiltonian form of the equations of motion and tries to determine the coefficients in a general form (ansatz) of the Hamiltonian and the complementary integral. In Hamiltonians constructed in this way, a term of the form $n\mathbf{M} \cdot \boldsymbol{\gamma}$ is missing and the dynamical behaviour of the original physical system will be determined up to a precessional motion with a constant rate.

10.12 Generalization of General Integrable Cases

The most important consequence of the form-invariance of equations of motion under the transformation (10.97) is the possibility it opens to generalize general (unconditional) and conditional integrable cases and also particular solutions through adding the precession parameter n into their structure, and thus enriching the physical problem by adding new physical effects. In the present section, we formulate this result for general integrable cases and give concrete examples of its applications. Conditional and particular cases are discussed in the next sections.

Theorem 10.2 *Let for some set of parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}', \mathbf{a}', \mathbf{J}'$, *Eq.* (10.105) *admit a complementary integral* $I_4 = I_4(\omega', \gamma)$, *so that they become integrable for arbitrary initial conditions, and let their solution be* $\{\boldsymbol{\omega} = \Omega(t), \gamma = \Gamma(t)\}$. *Then Eq.* (10.54) *are also integrable for arbitrary initial conditions, for the set of values of the parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}, \mathbf{a}, \mathbf{J}$:

$$\begin{aligned} \tilde{\mathbf{K}} &= \tilde{\mathbf{K}}' - 2n\mathbf{I}, \\ \mathbf{a} &= \mathbf{a}' - n\boldsymbol{\kappa}, \\ \mathbf{J} &= \mathbf{J}' + n\mathbf{K} + n^2\mathbf{I}, \end{aligned} \tag{10.117}$$

their complementary integral is

$$I_4 = I_4(\boldsymbol{\omega} + n\boldsymbol{\gamma}, \boldsymbol{\gamma}), \tag{10.118}$$

and their general solution is $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t) + n\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$. *It contains the additional arbitrary real parameter* n. *When* $n = 0$, *the generalized solution renders to the original solution.*

Any one of the hierarchy of integrable cases provided in the previous chapters admits a generalization as a case of the motion of a rigid body in a liquid. Transformed cases are of the same type (general or conditional) as the original ones. Examples are given in the next subsections: In Sokolov's case, the introduction of the parameter n results in a new integrable case. Even when the parameter n already enters in the structure of a known system, like in the case due to Rubanovsky [317], the regular precession transformation can be used to simplify the process of construction of an explicit solution of the equations of motion in terms of time. In such cases, the solution can be found first for the simpler case $n = 0$ and then generalized by that transformation for arbitrary n by the formulas in the last theorem.

10.12.1 Generalization of the Integrable Case Found by Sokolov

In 2002, Sokolov [336] obtained an integrable case of the rigid body in a liquid, which adds a parameter c to a former case by Yehia [380] (1986). The body has the Kowalevski configuration $A = B = 2C$. The centre of mass lies in the equatorial plane. The functions $V, \mathbf{l}, \boldsymbol{\mu}$ and the integrals I_3 and I_4 are given, according to the order followed in this book, as

$$\begin{aligned}
V &= C[kc\gamma_1 + a_2\gamma_2 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2)],\\
\mathbf{l} &= C(2c\gamma_3, 0, k - c\gamma_1),\ \boldsymbol{\mu} = C(c\gamma_3, 0, k + c\gamma_1),
\end{aligned} \tag{10.119}$$

$$\begin{aligned}
I_3 &= 2(p\gamma_1 + q\gamma_2) + (r + k + c\gamma_1)\gamma_3,\\
I_4 &= \left[p^2 - q^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r - k)\right]^2\\
&\quad + \left[2pq - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r - k)\right]^2\\
&\quad + 2k(r - k + c\gamma_1)\left[p^2 + q^2 + 2cp\gamma_3\right]\\
&\quad - 2kc^2\{2\gamma_3[2p\gamma_1 + c\gamma_1\gamma_3 + 2q\gamma_2 + r\gamma_3]\\
&\quad + k\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + c\gamma_1)\} - 4a_2kq\gamma_3.
\end{aligned} \tag{10.120}$$

It was pointed out in [411] (2003) that the parameter n can be added to this case to produce a non-trivial generalization, represented by the formulas

$$\begin{aligned}
V &= C[kc\gamma_1 + a_2\gamma_2 - nk\gamma_3 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2)\\
&\quad - nc\gamma_1\gamma_3 - \frac{n^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)],\\
\mathbf{l} &= C(2c\gamma_3 + 2n\gamma_1, 2n\gamma_2, k - c\gamma_1 + n\gamma_3),\\
\boldsymbol{\mu} &= C(c\gamma_3 - n\gamma_1, -n\gamma_2, k + c\gamma_1 - 3n\gamma_3),
\end{aligned} \tag{10.121}$$

$$\begin{aligned}
I_3 &= 2(p\gamma_1 + q\gamma_2) + (r + k + c\gamma_1)\gamma_3 + n(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2),\\
I_4 &= \left[(p + n\gamma_1)^2 - (q + n\gamma_2)^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r + n\gamma_3 - k)\right]^2\\
&\quad + \left[2(p + n\gamma_1)(q + n\gamma_2) - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r + n\gamma_3 - k)\right]^2\\
&\quad + 2k(r + n\gamma_3 - k + c\gamma_1)\left[(p + n\gamma_1)^2 + (q + n\gamma_2)^2 + 2c(p + n\gamma_1)\gamma_3\right]\\
&\quad - 2kc^2\{2\gamma_3[2(p + n\gamma_1)\gamma_1 + c\gamma_1\gamma_3 + 2(q + n\gamma_2)\gamma_2 + (r + n\gamma_3)\gamma_3]\\
&\quad + k\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + n\gamma_3 + c\gamma_1)\} - 4a_2k(q + n\gamma_2)\gamma_3.
\end{aligned} \tag{10.122}$$

Comparing (10.121), (10.122), (10.119), (10.120) we note that, unlike in Sokolov's case, the centre of mass in the generalized case does not lie in the equatorial plane since it has three non-zero coordinates. Also, the vector potential $\mathbf{l}$ and the gyroscopic vector $\boldsymbol{\mu}$ do not lie in a meridional plane as in Sokolov's case.

Remark. On the other hand, the regular precession transformation can be used in the reverse direction. For example, to seek the explicit solution of the equations of motion or to study the stability of a given motion, it suffices to study equations of motion for the Sokolov case. The solution may be extended to the generalized system with the extra-parameter n and the conclusion about stability will be the same before and after adding this parameter to the system.

10.12.2 Steklov's Case and Its Generalizations

One of the first known integrable cases of Kirchhoff equations (The version of (10.45) with $\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$)

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\gamma}\bar{\mathbf{K}}) &= \boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{J}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.123}$$

describing the motion of a rigid body with a singly connected surface in a liquid was discovered in 1895 by Steklov [345]. It corresponds to the choice

$$\mathbf{J} = \mathbf{0}. \tag{10.124}$$

Using (10.123), (10.124) we calculate the derivative

$$\frac{d}{dt}\{\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2 + \boldsymbol{\gamma}\bar{\mathbf{K}} \cdot (\boldsymbol{\omega}\mathbf{I})\} = -\boldsymbol{\omega} \cdot [(\boldsymbol{\omega}\times\boldsymbol{\gamma})\bar{\mathbf{K}}\mathbf{I}].$$

Under the condition

$$\bar{\mathbf{K}}\mathbf{I} = \varepsilon\boldsymbol{\delta},$$

where $\boldsymbol{\delta}$ is the unit matrix, or, equivalently,

$$\bar{\mathbf{K}} = \varepsilon\mathbf{I}^{-1} \tag{10.125}$$

the right-hand side of the last equation vanishes, which leads to Steklov's complementary integral of motion

$$\frac{1}{2}|\boldsymbol{\omega}\mathbf{I}|^2 + \varepsilon\boldsymbol{\omega} \cdot \boldsymbol{\gamma} = \text{const}\,. \tag{10.126}$$

This is the classical case of Steklov. When $\varepsilon = 0$, it turns into Euler's case.

Kharlamov [192, 197] investigated the full equations of motion which describe the problem of motion of a body with a multi-connected surface equivalent to (10.45) but in certain modified Clebsch variables. We write them in our form (10.45)

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\kappa}+\boldsymbol{\gamma}\bar{\mathbf{K}}) &= \boldsymbol{\gamma}\times(\mathbf{a}+\boldsymbol{\gamma}\mathbf{J}), \\ \dot{\boldsymbol{\gamma}}+\boldsymbol{\omega}\times\boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.127}$$

In those equations, the analog of the gyrostatic momentum and the centre of mass are present. Using what can be called a "brute force" method, Kharlamov found a generalization of Steklov's result under the conditions

$$a_1 + n\kappa_1 = a_3 + n\kappa_3 = 0, a_2 = \kappa_2 = 0 \tag{10.128}$$

and expressed the complementary integral as a quadratic polynomial in the variables [197]. Somewhat later, Rubanovsky [320] by a similar method replaced Kharlamov's non-symmetrical conditions $a_2 = \kappa_2 = 0$ by the less restrictive and more symmetric condition

$$a_2 + n\kappa_2 = 0, \tag{10.129}$$

so that the two vectors $\mathbf{a}$ and $\boldsymbol{\kappa}$ are now proportional, i.e.

$$\mathbf{a} = -n\boldsymbol{\kappa}. \tag{10.130}$$

Let us now consider a system of equations of motion containing only the arbitrary gyrostatic vector $\boldsymbol{\kappa}$, in addition to Eq. (10.123) and take Steklov's condition (10.128) into account. We write them as

$$\begin{aligned} \dot{\boldsymbol{\omega}}'\mathbf{I}+\boldsymbol{\omega}'\times(\boldsymbol{\omega}'\mathbf{I}+\boldsymbol{\kappa}+\varepsilon\mathbf{I}^{-1}) &= \mathbf{0}, \\ \dot{\boldsymbol{\gamma}}+\boldsymbol{\omega}'\times\boldsymbol{\gamma} &= \mathbf{0}. \end{aligned} \tag{10.131}$$

It can be easily verified that this system admits the following complementary integral, which generalizes (10.126):

$$\frac{1}{2}|\boldsymbol{\omega}'\mathbf{I}+\boldsymbol{\kappa}|^2 + \varepsilon\boldsymbol{\omega}'\cdot\boldsymbol{\gamma} = \text{const}\,. \tag{10.132}$$

Now we apply the regular precession transformation to the system (10.131) and its integral (10.132). Equation (10.131) transforms to (10.127), in which

$$\bar{\mathbf{K}} = -\frac{1}{2}\varepsilon\mathbf{I}^{-1} + n\bar{\mathbf{I}}, \mathbf{J} = -n^2\mathbf{I}, a = -n\boldsymbol{\kappa}. \tag{10.133}$$

This gives at once Rubanovsky's generalization of Kharlamov's result. It also enables to write the complementary integral in the very simple and transparent way:

$$I_4 = \frac{1}{2}[\boldsymbol{\omega}\mathbf{I} + n\boldsymbol{\gamma}\mathbf{I}+\boldsymbol{\kappa}]^2 + \varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\gamma} = \text{const}\,. \tag{10.134}$$

Although this case was noted by Rubanovsky, the method used here helped write the integral I_4 in this simple form. A notable advantage of the regular precession transformation is that one can use it here in the reverse way. To construct the explicit solution in terms of functions in time, it is sufficient to do that for the case $n = 0$. This means to construct the solution of the system (10.131) with the fourth integral (10.132). Having completed this task, i.e. having found $\boldsymbol{\omega}' = \Omega(t),\ = \Gamma(t)$, one can just write down the solution for the generalized case

$$\boldsymbol{\omega} = \Omega(t) - n\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t). \tag{10.135}$$

Explicit time solution of the full case (10.131) is not constructed to the present moment. This was achieved only in two particular cases:

(1) In Steklov's case $\boldsymbol{\kappa} = \mathbf{0}$, by Kötter in terms of theta functions of two variables [235].
(2) In Joukovsky's case $\varepsilon = 0$, the solution was obtained by Volterra [366] in terms of Weierstrass functions, which are complex functions in t. An alternative solution in terms of *real* Jacobi's elliptic functions was constructed by Wittenburg [369].

Despite the interest in applying methods of modern algebraic geometry (e.g. [71]), the general solution for the full basic case $\varepsilon|\boldsymbol{\kappa}|\neq 0$ was not considered.

10.13 Generalization of Conditional Integrable Cases

Theorem 10.3 *Let for some set of parameters* $\mathbf{I}, \boldsymbol{\kappa}, \bar{\mathbf{K}}', \mathbf{a}', \mathbf{J}'$, *Eq.* (10.105) *be integrable on the integral level* $I_2 = f_0$ *with the complementary integral* $I_4 = I_4(\omega', \gamma)$, *and let their solution be* $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$. *Then Eq.* (10.54) *is also integrable on the same integral level* $I_2 = f_0$, *for the set of values of the parameters* $\mathbf{I}, \boldsymbol{\kappa}, \tilde{\mathbf{K}}, \mathbf{a}, \mathbf{J}$,

$$\begin{aligned} \mathbf{K} &= \mathbf{K}' - 2n\mathbf{I}, \\ \mathbf{a} &= \mathbf{a}' - n\boldsymbol{\kappa}, \\ \mathbf{J} &= \mathbf{J}' + n\mathbf{K} + n^2\mathbf{I}, \end{aligned}$$

their complementary integral is

$$I_4 = I_4(\boldsymbol{\omega} + n\boldsymbol{\gamma}, \boldsymbol{\gamma}),$$

and their general solution on that level is $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t) + n\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$. *It contains the additional arbitrary real parameter* n. *When* $n = 0$, *the generalized solution renders to the original solution.*

There are two conditionally integrable cases known presently in the problem of motion of a body in a liquid. We now demonstrate how the uniform precession transformation works on one of them, namely, the Goryachev–Chaplygin hierarchy of cases. The second hierarchy is based on a conditional subcase of Kowalevski's case and the integrable problem of a body in a liquid found by Chaplygin. The last problem will be treated in more detail later in this chapter.

10.13.1 Generalization of Goryachev–Chaplygin's, Sretensky's and Sokolov–Tsiganov Cases

The first and most famous conditional integrable case of Goryachev and Chaplygin of the classical problem (See Chap. 4 Sect. 4.4) was built in 1900–1901 for a body satisfying the conditions $A = B = 4C$ and $z_0 = 0$. This case was generalized through the addition of a gyroscope along the axis of dynamical symmetry by Sretensky in 1963 [341]. For more details, see Chap. 5 Sect. 5.3. Sokolov and Tsiganov [337] (2002) added two more parameters. In our way of writing, the last case corresponds to the choice

$$V = C[a_1\gamma_1 + a_2\gamma_2 + \frac{1}{2}(c_2\gamma_1 - c_1\gamma_2)^2], \tag{10.136}$$

and

$$\boldsymbol{\mu} = C(c_1\gamma_3, c_2\gamma_3, \kappa + c_1\gamma_1 + c_2\gamma_2). \tag{10.137}$$

This case can be readily generalized by the regular precession transformation to include the parameter n as follows:

$$\begin{aligned} V &= C[a_1\gamma_1 + a_2\gamma_2 - n\kappa\gamma_3 + \frac{1}{2}(c_2\gamma_1 - c_1\gamma_2)^2 \\ &\quad - n\gamma_3(c_1\gamma_1 + c_2\gamma_2) - \frac{n^2}{2}(4\gamma_1^2 + 4\gamma_2^2 + \gamma_3^2)], \\ \boldsymbol{\mu} &= C(c_1\gamma_3 - n\gamma_1, c_2\gamma_3 - n\gamma_2, \kappa + c_1\gamma_1 + c_2\gamma_2 - 7n\gamma_3). \end{aligned} \tag{10.138}$$

The transformation adds several terms to the potential, including linear and quadratic terms, and some linear terms to $\boldsymbol{\mu}$. The complementary integral for the generalized case is

$$\begin{aligned} I_4 &= (r - \kappa + c_1\gamma_1 + c_2\gamma_2 + n\gamma_3)[(p + n\gamma_1 + \frac{1}{2}c_1\gamma_3)^2 + (q + n\gamma_2 + \frac{1}{2}c_2\gamma_3)^2] \\ &\quad + \gamma_3[(\kappa c_1 - a_1)(p + n\gamma_1) + (\kappa c_2 - a_2)(q + n\gamma_2)] \\ &\quad + \frac{1}{2}\gamma_3^2[\kappa(c_1^2 + c_2^2) - c_1 a_1 - c_2 a_2]. \end{aligned} \tag{10.139}$$

Explicit time solution for this case is not found yet. But to find this solution, it suffices to express solution for the special case $n = 0$, i.e. for the case found by Sokolov and Tsiganov [337].

10.14 Generalizations of Particular Solvable Cases

All the twelve particular solvable cases of the classical problem presented in Chap. 8 can be immediately generalized by the regular precession transformation. All those cases produce new cases in the problem of motion of a rigid body in liquid. The same remark fully applies for all the known solvable cases of the motion of a gyrostat (See Chaps. 13 and 14) and also any particular solution known in the problem of motion of a body in a liquid. The parameter n can be added to all those cases, with all possible implications on the nature of forces acting on the body.

We shall not make a complete list of those generalized cases, but we shall provide some of the most illustrative examples. We first formulate the following theorems:

Theorem 10.4 *Let for some set of parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}', \mathbf{a}', \mathbf{J}'$ *and initial conditions* $\boldsymbol{\omega} = \boldsymbol{\Omega}_0, \; = \Gamma_0$, *the Eq.* (10.105) *admit a particular solution* $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(\mathbf{t}), \gamma = \boldsymbol{\Gamma}(\mathbf{t})\}$, *then for the set of values of the parameters* $\mathbf{I}, \boldsymbol{\kappa}, \mathbf{K}, a, J$ *and for the initial conditions* $\boldsymbol{\omega} = \boldsymbol{\Omega}_0 + n\boldsymbol{\Gamma}_0, \gamma = \boldsymbol{\Gamma}_0$ *Eq.* (10.54) *admit a particular solution* $\{\boldsymbol{\omega} = \boldsymbol{\omega}(t) + n\boldsymbol{\Gamma}(t), \gamma = \boldsymbol{\Gamma}(t)\}$ *containing the additional arbitrary real parameter* $n. When n = 0$, *the generalized particular solution renders to the original particular solution.*

Corollary. *Any motion of a body in a liquid whose angular velocity has the form* $\boldsymbol{\omega} = \omega_1\gamma + \boldsymbol{\omega}_2$, *i.e. involves a component* ω_1 *in the direction of* γ, *can be reduced by using the transformation (10.97) with* $n = -\omega_1$ *to a motion with angular velocity* $\omega = \omega_2$ *and vise versa. In particular,*

1- A uniform (permanent) rotation about a vertical axis can be reduced to a position of equilibrium.

2- A regular precession with a vertical precession axis can be reduced to a permanent rotation about the configuration axis (fixed in the body), which becomes fixed in space.

3- The so-called semi-regular precession (composed of pendulum-like motion of the rigid body about a horizontal axis and a uniform rotation about the vertical) can be reduced to pendulum-like motion, parallel to a fixed plane.

Theorem 10.5 *All properties of the first solution in the last theorem, like stability in the sense of Lyapunov, stability in (or by) the first approximation, instability and periodicity*[3] *are passed to the second solution.*

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[3] Here, periodicity relates only to the Euler-Poisson variables $\boldsymbol{\omega}, \boldsymbol{\gamma}$. The motion is periodic relative to the body system of axes. The motion can be periodic in space only under commensurability condition between the periods of the relative and the precessional motions.

The proof follows immediately from the fact that the stability, instability or periodicity of one of the pairs of solutions $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$ and $\{\boldsymbol{\omega} = \boldsymbol{\Omega}(t) + n\boldsymbol{\Gamma}(t), \boldsymbol{\gamma} = \boldsymbol{\Gamma}(t)\}$ implies the same to the other pair.

This theorem allows a great simplification to the study of properties of motion, as in the last corollary, a permanent rotation reduces to an equilibrium. Also, a regular precession (a periodic motion) reduces to a uniform rotation. Note that the equations of variation for the precession are periodic in time, while those for uniform rotation are of constant coefficients and hence their analysis is much simpler.

We give here only a few illustrative examples to show how the transformation can be used in the direct or in the reverse directions, to generalize a given case or to simplify it. We present results partly in the framework of the problem of motion of a rigid body about a fixed point and partly in the equivalent problem, according to our analogy described earlier in this chapter, of motion of a body in a liquid.

10.14.1 Example 1. Equilibria and Permanent Rotations About a Vertical Axis

It is evident that a position of equilibrium of the body governed by Eq. (10.45) can be transformed by the regular precession transformation. The image for a given finite real n is a permanent rotation about an axis fixed in the body and taking a vertical position. Conversely, a permanent rotation can always be reduced to a relative equilibrium in a coordinate system moving with the same precession speed as the body.

Equilibria. Consider an equilibrium position of the system (10.45). Those are the solutions $\{\boldsymbol{\omega} = \mathbf{0}, \boldsymbol{\gamma} = \boldsymbol{\gamma}_0\}$ where $\boldsymbol{\gamma}_0$ satisfies

$$\boldsymbol{\gamma}_0 \times (\mathbf{a} + \boldsymbol{\gamma}_0 \mathbf{J}) = \mathbf{0}. \tag{10.140}$$

1. For the classical problem, when $\mathbf{J} = \mathbf{0}, \mathbf{a} \neq \mathbf{0}$, there are two equilibria: the upper and lower equilibria of the centre of mass above or below the fixed point.

2. When $\mathbf{a} = \mathbf{0}, \mathbf{J} \neq \mathbf{0}$, there are six equilibria, in which one of the principal axes of the matrix $\mathbf{J}$ is directed along or against the vector $\boldsymbol{\gamma}$.

3. In the generic case $\boldsymbol{\gamma}_0$ satisfies a relation $\mathbf{a} + \boldsymbol{\gamma}_0 \mathbf{J} = \breve{\ }\boldsymbol{\gamma}_0$, so that

$$\boldsymbol{\gamma}_0 = -\mathbf{a}(\mathbf{J} - \lambda\boldsymbol{\delta})^{-1}, \tag{10.141}$$

where λ is a root of the sixth-degree equation

$$\left|\mathbf{a}(\mathbf{J} - \lambda\boldsymbol{\delta})^{-1}\right|^2 = 1. \tag{10.142}$$

In this case also we have a maximal number of six positions of equilibrium and minimum number of two.

Permanent rotations. Permanent rotations were discussed in detail in Sect. 10.7. In a coordinate system moving with the same precession speed as the body, the permanent rotation looks like an equilibrium position, which is determined from the Eq. (10.140) but with the transformed parameters $\mathbf{a}'$ and $\mathbf{J}'$, *i.e.*

$$\gamma_0 \times (\mathbf{a}'+\gamma_0\mathbf{J}') = \mathbf{0}. \tag{10.143}$$

Substituting those parameters from (10.102) into the last relation, we get the condition for a permanent rotation as

$$\gamma_0 \times [\mathbf{a} + n\boldsymbol{\kappa} + \gamma_0(\mathbf{J} - n\mathbf{K} - n^2\mathbf{I})] = \mathbf{0}. \tag{10.144}$$

This equation can now be compared with the condition for the permanent rotation (10.66). They become identical, provided we take $n = -\omega_0$.

10.14.2 Example 2. Permanent Rotations About a Tilted Axis and Precessional Motions About the Vertical

Consider a precessional motion, in which the angular velocity of the body is given by

$$\boldsymbol{\omega} = \Omega_0\mathbf{e} + \Omega_1\gamma, \tag{10.145}$$

where Ω_0, Ω_1 are constants and $\mathbf{e}$ is a unit vector fixed in the body at the fixed point O. The body rotates about $\mathbf{e}$ with angular velocity Ω_0, while this axis precesses about the vertical with angular velocity Ω_1. Using the transformation (10.97) with the choice $n = -\Omega_1$, we have

$$\boldsymbol{\omega}' = \Omega_0\mathbf{e}. \tag{10.146}$$

The precessional motion is reduced by this transformation to a uniform rotation about an axis fixed in the body and in space and inclined to the vertical at a fixed angle. That is the permanent rotational motion described in Sect. 10.8. Similarly, a permanent rotational motion with angular velocity (10.146) can be transformed by the inverse transformation $n = \Omega_1$ to the precessional motion.

Solutions of the equations of motion corresponding to regular precessions were investigated in [123] (See also [125]). The conditions for existence of such precession are obtained in a quite complicated form (conditions (18) in [123]). Those conditions can be easily shown to be equivalent to conditions (10.75) followed by the rotation transformation (10.97). According to the said above, one could consider only uniform rotational motions about an axis fixed in space and inclined to the vertical. The whole class of precessional motions generated by transforming uniform rotations about an inclined axis using (10.97) with the parameter n taking all real values are equivalent to that rotation and, moreover, have the same properties, for example, as concerns stability of the motions.

10.14.3 Example 3. generalization of grioli's precession [402, 405]

On a dynamical basis, Grioli established the possibility of a regular precession of the heavy rigid body about a non-vertical axis under certain conditions on the parameters of the body [138]. Guliaev derived the full explicit solution of this case [141] (see also [256]). We present the necessary details in brief. The solution differs from that of Guliaev only in that we have assigned a certain value for the initial time moment, so that the solution becomes more transparent.

Let the axes be arranged such that $A \geq B \geq C$. For

$$\mathbf{a}' = (a, 0, c), \boldsymbol{\kappa} = \mathbf{0}, \mathbf{K}' = \mathbf{J}' = \mathbf{0}, \tag{10.147}$$

where $a\sqrt{B-C} = c\sqrt{A-B}$, the system of Eq. (10.105) admits a particular solution (See 8.10)

$$\begin{aligned}
p' &= \frac{\Omega}{s}(a - c\cos(\Omega t)), q' = \Omega\sin(\Omega t), r' = \frac{\Omega}{s}(c + a\cos(\Omega t)),\\
\gamma_1 &= -\frac{\Omega^2}{s^2}[Cc\cos(\Omega t) + (B-C)a\sin^2(\Omega t)],\\
\gamma_2 &= \frac{\Omega^2}{s^3}\sin(\Omega t)[(Aa^2 + Cc^2) - (A-C)ac\cos(\Omega t)],\\
\gamma_3 &= \frac{\Omega^2}{s^2}[Aa\cos(\Omega t) + (A-B)c\sin^2(\Omega t)],
\end{aligned} \tag{10.148}$$

where $s = \sqrt{a^2+c^2}$, $\Omega^2 = \frac{s}{\sqrt{(A-B+C)^2+(A-B)(B-C)}}$. This solution corresponds to a regular precession of the body. The angular velocity $\boldsymbol{\omega}'$ can be written as the sum of two terms

$$\boldsymbol{\omega}' = \Omega\boldsymbol{\zeta} + \Omega\boldsymbol{\alpha}, \tag{10.149}$$

where $\boldsymbol{\zeta}$, $\boldsymbol{\alpha}$ are two unit vectors: the first fixed in the body (orthogonal to a circular section of the inertia ellipsoid) and the second fixed in space [141], so that in the body system

$$\boldsymbol{\zeta} = (\frac{a}{s}, 0, \frac{c}{s}), \boldsymbol{\alpha} = (-\frac{c}{s}\cos(\Omega t), \sin(\Omega t), \frac{a}{s}\cos(\Omega t)). \tag{10.150}$$

Note that $\boldsymbol{\zeta}$ is orthogonal to $\boldsymbol{\alpha}$ and that $\boldsymbol{\alpha}$ is inclined to the upward vertical vector $\boldsymbol{\gamma}$ at a fixed angle δ,

$$\cos\delta = \frac{A - B + C}{\sqrt{(A - B + C)^2 + (A - B)(B - C)}}. \tag{10.151}$$

The body rotates with the uniform velocity Ω around the vector $\boldsymbol{\zeta}$ fixed in it, while that vector rotates with the same velocity Ω about the direction $\boldsymbol{\alpha}$ fixed in space.

We now consider another case of motion of the same body as above, but we will replace V', $\boldsymbol{\mu}'$ by

$$\begin{aligned}
&\mathbf{a}' = (a, 0, c),\ \boldsymbol{\kappa} = \mathbf{0},\ \mathbf{K}' = \mathbf{J}' = \mathbf{0},\\
&V = a\gamma_1 + c\gamma_3 - \frac{1}{2}n^2(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\\
&\boldsymbol{\mu} = n((B + C - A)\gamma_1, (C + A - B)\gamma_2, (A + B - C)\gamma_3)
\end{aligned} \tag{10.152}$$

where, for simplicity, n is taken as a constant. It is easy to verify that applying the substitution $\boldsymbol{\omega} = \boldsymbol{\omega}' + n\boldsymbol{\gamma}$ transforms (10.152) into (10.147). Thus, the system with (10.152) admits a particular solution representing Grioli's precession uniformly rotated with speed n about the vertical. In this solution $\gamma_1, \gamma_2, \gamma_3$ are the same as in (10.148), while

$$\begin{aligned}
p &= \frac{\Omega}{s}(a - c\cos(\Omega t)) - \frac{n\Omega^2}{s^2}[Cc\cos(\Omega t) + (B - C)a\sin^2(\Omega t)],\\
q &= \Omega\sin(\Omega t) + \frac{n\Omega^2}{s^3}\sin(\Omega t)[(Aa^2 + Cc^2) - (A - C)ac\cos(\Omega t)],\\
r &= \frac{\Omega}{s}(c + a\cos(\Omega t)) + \frac{n\Omega^2}{s^2}[Aa\cos(\Omega t) + (A - B)c\sin^2(\Omega t)].
\end{aligned} \tag{10.153}$$

This case is a non-trivial generalization of Grioli's result [138]. The whole picture of Grioli's precession about the inclined axis precesses about the vertical at an arbitrary angular speed n. The resulting motion admits two interpretations as a motion of a body in liquid [383] or a motion of a charged body under potential and Lorentz forces as described in Sect. 2.2 above. It is noteworthy that this gives a new result in both interpretations.

The angular velocity $\boldsymbol{\omega} = \Omega(\boldsymbol{\zeta} + \boldsymbol{\alpha}) + n\boldsymbol{\gamma}$ no longer has constant magnitude as was the case in Grioli's precession. The resulting motion is not a regular precession. Although $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are periodic functions of time, the motion is not in general periodic in space for arbitrary values of n. However, if $\frac{n}{\Omega}$ is rational the body returns periodically to its initial position. As far as we know, such motions have not been considered previously. This solution can be generalized by adding a rotor to the body along the normal to a circular cross-section of the ellipsoid of inertia of the body. The solution for the last case was found by Keis [167] and rediscovered by Kharlamova. The resulting case of motion of a heavy gyrostat can be transformed using the regular precession transformation to a case of motion of another body by inertia in a liquid or a case of motion of an electrically charged body in gravity and magnetic fields. Formulas received will generalize (10.152), (10.153).

10.14.4 *Example 4. Regularly Precessing Pendulum*

By this motion, we mean a generalization of the motion of a physical pendulum, such that the axis of the pendulum rotation performs regular precession about the vertical. A near, but different, term "semi-regular precession" was coined by Grioli [139] in certain problems lower in the hierarchy than that of the present chapter. The same name was used later by many authors, e.g. [357]. (See also Gorr [119] and for more detail [126]). This name refers to the motion in which the body rotates with a time-dependent (i.e. non-uniform) angular velocity about an axis fixed in it, while this axis makes regular precession about an axis fixed in space. Thus, the regularly precessing pendulum motion is a type of semi-regular precession, but the last may comprise motions that do not fit in the pendulum type in addition to a regular precession.

Conditions for existence of semi-regular precessions of a rigid body in a liquid involving a pendulum-like motion about an axis fixed in the body and regular precession about the (virtual) vertical were first found in [247] where solutions of the equations of motion were sought such that the angular velocity has the form

$$\dot{\varphi}\mathbf{e} + n\boldsymbol{\gamma}, \tag{10.154}$$

where n is a constant, $\boldsymbol{\gamma}$ is the unit vector along the (virtual) vertical and $\mathbf{e}$ is a unit vector constant in the body. This formula is substituted into the equations of motion, compatibility conditions are found and then a differential equation is obtained for the determination of φ.

Independently, and slightly later, of [247] the same motions were considered more comprehensively in the Ph.D. Thesis [148]. In this work not only conditions for the existence of pendulum motions and their transformed version (the semi-regular precession) are obtained, but also a detailed study was made on the orbital stability of certain special cases of those motions.

In our presentation of the precessing pendulum motion, we shall use the method used in [148]. The study of the motion is made in two steps:

A) The motion is studied in a rotating reference frame in which the body performs a pendulum motion about a horizontal axis fixed in this system. Conditions necessary for performing this motion are found on the transformed parameters of the body $\mathbf{I}, \mathbf{a}', \mathbf{J}', \boldsymbol{\kappa}, \bar{\mathbf{K}}'$.

B) The motion is transformed back to the inertial frame, the regular precession will be added. The relevant conditions on the original parameters of the body are obtained from

$$\boldsymbol{\omega}' = \boldsymbol{\omega} + n\boldsymbol{\gamma}. \tag{10.155}$$

10.14.4.1 Pendulum Motion

Consider the motion of the body as a physical pendulum, taking place around a principal axis of the inertia matrix of the body-liquid system, while this axis keeps a permanent horizontal position (Fig. 10.2).

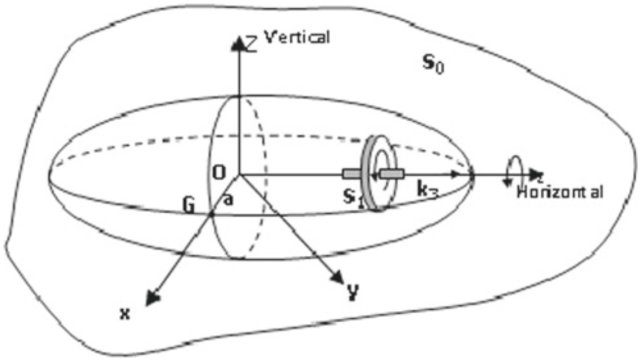


Fig. 10.2 The body configuration of pendulum motion. G the centre of mass, S the rotor with gyrostatic moment k3

Let us choose the third principal axis to be the axis of proper rotation with variable angular velocity $\dot{\varphi}$. The solution corresponding to this motion with $\theta = \frac{\pi}{2}$ is

$$\begin{aligned} \boldsymbol{\omega} &= (p, q, r) = (0, 0, \dot{\varphi}), \\ \boldsymbol{\gamma} &= (\sin\varphi, \cos\varphi, 0). \end{aligned} \tag{10.156}$$

We shall find conditions on the matrices $\bar{\mathbf{K}}$, $\mathbf{J}$ and the vectors $\boldsymbol{\kappa} = (k_1, k_2, k_3)$ and $\mathbf{a} = (a_{1,} a_2, a_3)$ that allow the body to perform pendulum motion. Substituting into Eq. (10.45), the Poisson equations are identically satisfied and the first two dynamical equations give:

$$\begin{aligned} &-\dot{\varphi}[\bar{K}_{12}\sin\varphi - 2\bar{K}_{22}\cos\varphi + k_2] - \cos\varphi[J_{13}\sin\varphi + J_{23}\cos\varphi + a_3] = 0, \\ &\dot{\varphi}[\bar{K}_{11}\sin\varphi - 2\bar{K}_{12}\cos\varphi + k_1] + \sin(\varphi)[J_{13}\sin\varphi + J_{23}\cos\varphi + a_3] = 0, \end{aligned} \tag{10.157}$$

while the third dynamical equation is replaced by the energy integral

$$\frac{1}{2}C\dot{\varphi}^2 + \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma = E, \tag{10.158}$$

where E is the energy constant of the motion.

A combination of (10.157) gives

$$\dot{\varphi}[k_1\sin\varphi - k_2\cos\varphi - (\bar{K}_{11} - \bar{K}_{22})\sin 2\varphi - 2\bar{K}_{12}\cos 2\varphi] = 0,$$

which leads to the conditions

$$\begin{aligned} k_1 &= k_2 = 0, \\ \bar{K}_{12} &= 0, \bar{K}_{22} = \bar{K}_{11}. \end{aligned} \tag{10.159}$$

Now, getting back to (10.157) we obtain one equation

$$\bar{K}_{11}\dot{\varphi} + J_{13}\sin\varphi + J_{23}\cos\varphi + a_3 = 0. \tag{10.160}$$

This equation in φ gives a law of motion contradicting the pendulum law, and hence should be satisfied as an identity. We obtain, in addition to (10.159), the conditions

$$\bar{K}_{11} = 0, a_3 = 0, J_{13} = J_{23} = 0. \tag{10.161}$$

Summing up, for this version one can write the parameters of the problem in the form:

$$\mathbf{a} = (a_1, a_2, 0), \boldsymbol{\kappa} = (0, 0, k_3),$$
$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{12} & J_{22} & 0 \\ 0 & 0 & J_{33} \end{pmatrix}, \tag{10.162}$$
$$\bar{\mathbf{K}} = \begin{pmatrix} 0 & 0 & \bar{K}_{13} \\ 0 & 0 & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33} \end{pmatrix}, \mathbf{K} = \begin{pmatrix} \bar{K}_{33} & 0 & -\bar{K}_{13} \\ 0 & \bar{K}_{33} & -\bar{K}_{23} \\ -\bar{K}_{13} & -\bar{K}_{23} & 0 \end{pmatrix}. \tag{10.163}$$

Those conditions mean that the centre of mass of the body lies in the xy-plane, perpendicular to the axis of pendulum rotation (the z-axis), which is a principal axis also of the matrix $\mathbf{J}$. The angle of proper rotation φ can be determined as an elliptic function of time by inverting the integral, obtained by separating variables in (10.158),

$$t = \int \frac{d\varphi}{\sqrt{2(E - a_1\sin\varphi - a_2\cos\varphi - J_{12}\sin\varphi\cos\varphi) - J_{11}\sin^2\varphi - J_{22}\cos^2\varphi}}. \tag{10.164}$$

This formula contains the energy constant E, which takes all real values on the interval $[V_-, \infty)$, V_- being the minimum of the potential V on the Poisson sphere. Pendulum motions constitute a family of periodic motions of two types: vibrational motions reversing their direction every half-period time and complete rotational motions going on in one direction all the time.

10.14.4.2 The Precessing Pendulum

Conditions (10.163) for existence of pendulum-like motion of the body in a liquid (or in the equivalent generalized problem) can now be generalized to generate conditions for the semi-regular precession. One can now transform the pendulum-like motion about its axis fixed in space to add the parameter n to the solution. Every pendulum-like motion generates a family of semi-regular precessions, with n taking all real values. The parameter n enters in the transformed conditions (10.154) according to the transformation formulas (10.102). Finally, we can write the parameters of the body, in order that the body can perform a semi-regular motion composed of a pendulum motion and a precession with angular velocity n :

$$\mathbf{a}' = (a_1, a_2, nk_3), \boldsymbol{\kappa} = (0, 0, k_3),$$
$$\bar{\mathbf{K}}' = \begin{pmatrix} 2n(B+C-A) & 0 & \bar{K}_{13} \\ 0 & 2n(C+A-B) & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33}+2n(A+B-C) \end{pmatrix}$$
$$\mathbf{J}' = \begin{pmatrix} J_{11}-n^2A-n\bar{K}_{33} & J_{12} & n\bar{K}_{13} \\ J_{12} & J_{22}-n^2B-n\bar{K}_{33} & n\bar{K}_{23} \\ n\bar{K}_{13} & n\bar{K}_{23} & J_{33}-n^2C \end{pmatrix}. \qquad (10.165)$$

In the last formula, we used the relation $\mathbf{K} = \mathrm{tr}(\bar{\mathbf{K}})-\bar{\mathbf{K}}$ to obtain the transformed parameters from (10.102).

Conditions (10.165) are well-ordered and much transparent than conditions in [247], where conditions for the semi-regular precession are given just as relations between the parameters.

10.14.4.3 The Space Picture of the Motion

The integral (10.164) is elliptic and can be evaluated using formulas from [130]. When $J_{12} = 0$ and $a_2 = 0$, i.e. when the vector $\mathbf{a}$ (the centre of mass) lies on a principal axis of inertia and $\mathbf{J}$, $\mathbf{I}$ have common principal axes, the integral becomes simpler and $\gamma_1 = \sin\varphi$ can be determined in terms of Jacobian elliptic function in time. This was done in [148], where also the translational motion was studied. It turned out that the central point can draw several types of trajectories in space.

In the special case when the parameters of the body satisfy the conditions

$$A = B,$$
$$J_{11} = J_{22} = \varepsilon A, J_{33} = \varepsilon C,$$
$$K_{13} = K_{23} = 0, K_{33} = -nC,$$

one finds

$$\gamma_1 = -1 + 2\,\mathrm{sn}^2\,v, \gamma_2 = 2\,\mathrm{sn}\,v\,\mathrm{cn}\,v,$$
$$p = -n\gamma_1, q = -n\gamma_2, r = \frac{2}{k}\sqrt{\frac{a_1}{C}}\,\mathrm{dn}\,v, \qquad (10.166)$$

where

$$v = \sqrt{\frac{a_1}{C}}\frac{t}{k} \qquad (10.167)$$

and the modulus of the elliptic functions

$$k = \sqrt{\frac{4a_1}{2h + 2a_1 - A(\varepsilon + n^2)}}. \tag{10.168}$$

Note that we have chosen the case of fast pendulum (in which the pendulum rotates with variable angular velocity in one direction).

From (10.166), we also get

$$\dot{\psi} = \frac{p\gamma_1 + q\gamma_2}{\gamma_1^2 + \gamma_2^2} = -n,$$

so that we can write

$$\psi = -nt,$$

and thus we arrive at the following expressions for the base vectors in space

$$\begin{aligned} \boldsymbol{\alpha} &= (\gamma_2 \cos nt, -\gamma_1 \cos nt, -\sin nt), \\ \boldsymbol{\beta} &= (-\gamma_2 \sin nt, \gamma_1 \sin nt, -\cos nt). \end{aligned} \tag{10.169}$$

The velocity of the central point of the body can now be written from (10.60) as

$$\mathbf{u} = \mathbf{a} + \gamma \mathbf{J} - \frac{1}{2}\omega\, \mathbf{K}.$$

The space components of the velocity with respect to some inertial system of axes $\xi\eta\zeta$ are

$$\dot{\xi} = \mathbf{u} \cdot \boldsymbol{\alpha}, \dot{\eta} = \mathbf{u} \cdot \boldsymbol{\beta}, \dot{\zeta} = \mathbf{u} \cdot \boldsymbol{\gamma},$$

and can now be evaluated:

$$\begin{aligned} \frac{d\xi}{dt} &= F(t) \sin nt + 2a_1 \operatorname{sn} v \operatorname{cn} v \cos nt \\ \frac{d\eta}{dt} &= F(t) \cos nt - 2a_1 \operatorname{sn} v \operatorname{cn} v \sin nt \\ \frac{d\zeta}{dt} &= (\varepsilon + n^2)A - a_1 + nC + 2a_1 \operatorname{sn}^2 v, \end{aligned} \tag{10.170}$$

where

$$F(t) = n(k_3 + \frac{2}{k}\sqrt{Ca_1} \operatorname{dn} v).$$

By integrating (10.170) with respect to time, we obtain

$$\xi = -C(\frac{k_3}{C} + \frac{2}{k}\sqrt{\frac{a_1}{C}}\,\mathrm{dn}(\sqrt{\frac{a_1}{C}}\frac{t}{k}))\cos nt,$$

$$\eta = C(\frac{k_3}{C} + \frac{2}{k}\sqrt{\frac{a_1}{C}}\,\mathrm{dn}(\sqrt{\frac{a_1}{C}}\frac{t}{k}))\sin nt,$$

$$\zeta = [(\varepsilon + n^2)A - a_1 + nC + \frac{2a_1}{k^2}(1 - \frac{E(k)}{K(k)})]t - \frac{2}{k}\sqrt{\frac{a_1}{C}}Z(\sqrt{\frac{a_1}{C}}\frac{t}{k}) \quad (10.171)$$

where $K(k)$, $E(k)$ are complete elliptic integrals and Z is Jacobi's Zeta function of the same modulus k.

The functions dn, Z have period

$$T_1 = 2kK(k)\sqrt{\frac{C}{a_1}}, \quad (10.172)$$

while the trigonometric terms have period

$$T_2 = \frac{2\pi}{n}. \quad (10.173)$$

The position vector of the central point of the body P (say) is not periodic in time in general, but its projection on the $\xi\eta$-plane can be a closed curve if the ratio $\frac{T_1}{T_2}$ is a rational number.

Now we are ready to describe the space picture of the motion of the body. The body performs the periodic pendulum motion about its horizontal z-axis while this axis precesses with a uniform angular speed n in the (virtual) horizontal plane. The motion of the central point P of the body traces a space curve of helicoidal type about a (virtual) vertical axis. The radial distance ρ of P from the ζ-axis of the curve

$$\rho = k_3 + \frac{2}{k}\sqrt{Ca_1}\,\mathrm{dn}\,v \quad (10.174)$$

changes periodically, while rotating about the vertical ζ-axis with the same angular speed n of the body about the vertical Z-axis. As to its horizontal motion, the body moves around the ζ-axis and rotates in such a way that one face of the body is always directed to that axis. In celestial mechanics, this regime of motion is called $1-1$ rotation.

From the expression (10.174), we note that the effect of the gyrostatic moment k_3 appears in the motion of the central point as an increase (decrease) of the radial distance between that point and the ζ-axis. This means widening or narrowing the lateral dimensions of the helicoidal trajectory, according to the sign of k_3.

The motion of the central point in the ζ-direction is not periodic in general, due to the presence of a secular term (linear in t). After each (orbital) revolution about the ζ-axis the central point elevates (or descends) a certain distance above (or below) the horizontal plane that passed through P at the initial moment $t = 0$. The space path of P is helicoidal-like.

However, if the coefficient of t in the secular term vanishes, i.e. if

$$(\varepsilon + n^2)A - a_1 + nC + \frac{2a_1}{k^2}(1 - \frac{E(k)}{K(k)}) = 0, \tag{10.175}$$

then the vertical motion of P is periodic with period T_1. If, moreover, T_1 and T_2 are commensurable, say, $\frac{T_1}{N_1} = \frac{T_2}{N_2}$, then the space trajectory of P closes after a number N of revolutions $N = \text{LCM}(N_1, N_2)$ (The least common multiple of the two numbers).

The following figures illustrate the shapes of some space trajectories of the central point. In all of them we suppose that (10.175) is satisfied and set $C = 1$ and substitute$\sqrt{a_1} = 2kK(k)/T1$, so that the equation of the space curve becomes

$$\begin{aligned}
\xi &= -[k_3 + \frac{2}{k}\sqrt{a_1}\,\text{dn}(2K(k)\frac{t}{T_1})]\cos(2\pi\frac{t}{T_2}),\\
\eta &= [k_3 + \frac{2}{k}\sqrt{a_1}\,\text{dn}(2K(k)\frac{t}{T_1})]\sin(2\pi\frac{t}{T_2}),\\
\zeta &= -\frac{2}{k}\sqrt{a_1}Z(2K(k)\frac{t}{T_1}).
\end{aligned} \tag{10.176}$$

The following Fig. 10.3 shows the space orbit of the central point of the body for values of the parameters:

$$k_3 = 1, k = 0.9, n = 1, a_1 = k^2 = 0.81.$$

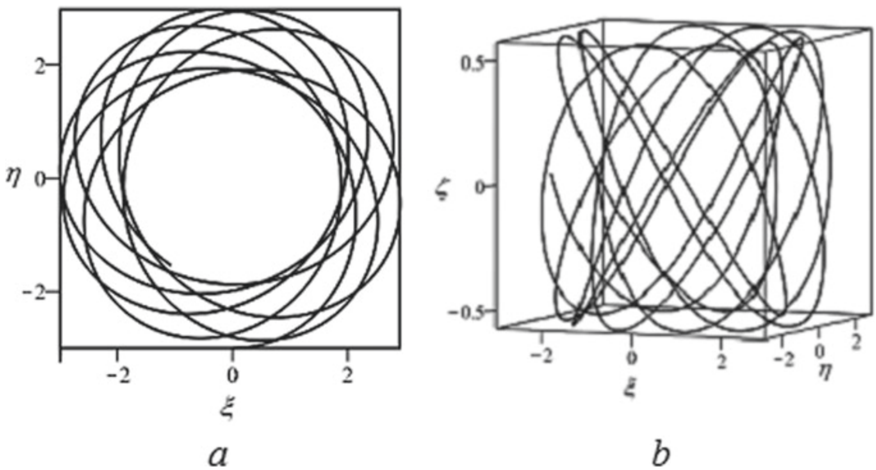


Fig. 10.3 Space trajectory of pendulum. **a** An upper view **b** A side view

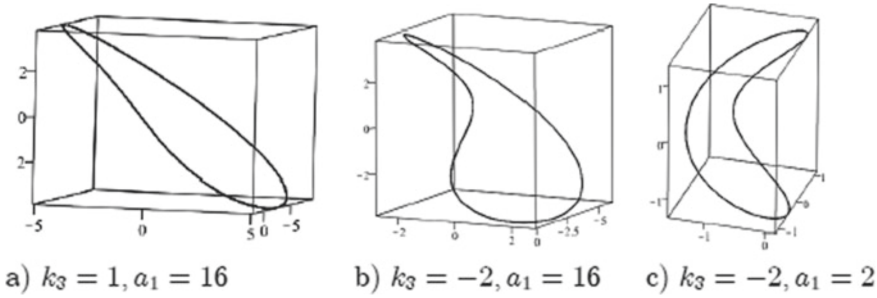


Fig. 10.4 $T_1 = T_2$

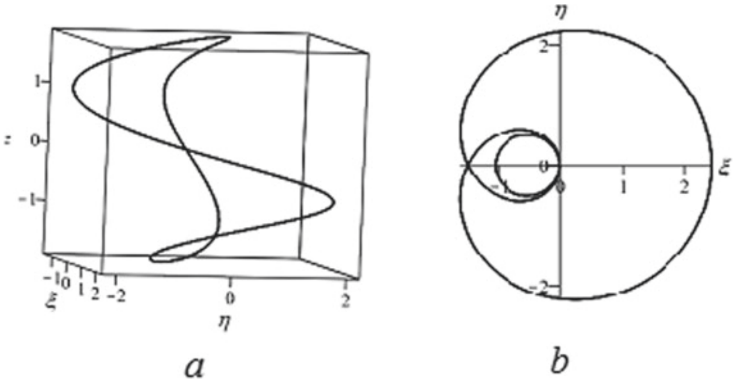


Fig. 10.5 $k_3 = -3, a_1 = 4, T_1 = 2T_2 (N_1 = 2, N_2 = 1)$. **a** Side view. **b** Projection on $\xi\eta$-plane

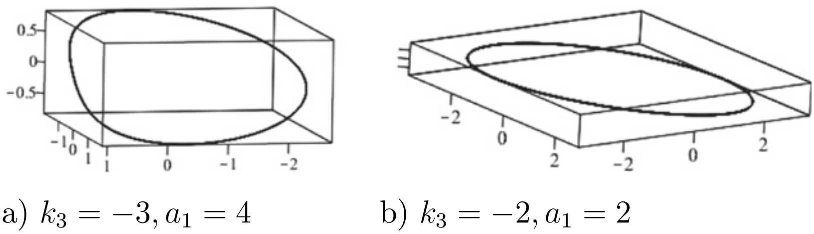


Fig. 10.6 Side view for different values of k_3 and a_1

The shapes of some closed space curves are shown for different values of the parameters k_3, k and the integers N_1 and N_2. They show the diversity of the forms of trajectories, even for a very limited set of initial conditions (Figs. 10.4, 10.5, 10.6, 10.7, 10.8).

Fast pendulum rotations $k = 0.99$,

Slower rotation $k = 0.5$ $N_1 = 1, N_2 = 1$

$N_1 = 2, N_2 = 1$

The case of a simple body $k_3 = 0$:

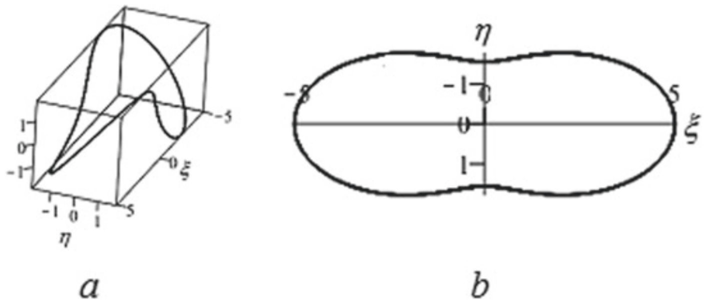


Fig. 10.7 $k_3 = -3, a_1 = 4, T_2 = 2T_1(N_1 = 1, N_2 = 2)$. **a** side view **b** Projection on $\xi\eta$-plane

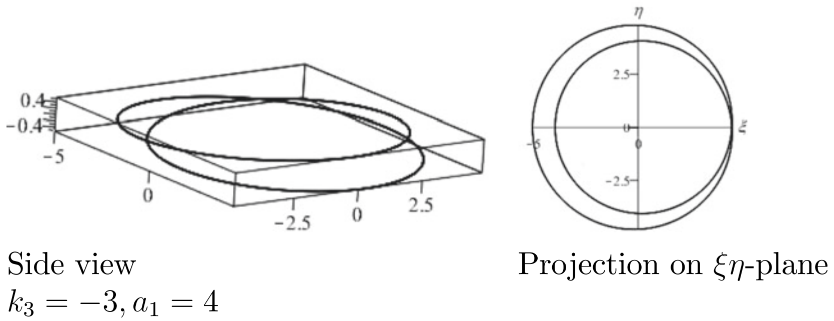
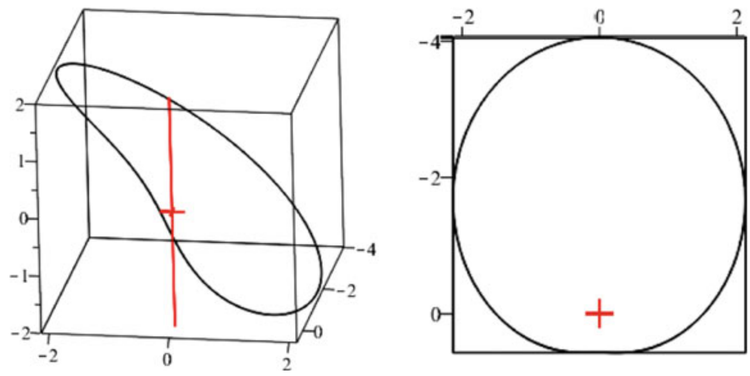


Fig. 10.8 $k_3 = -3, a_1 = 4$ **a** Side view **b** Projection on $\xi\eta$-plane

It has been noticed before that the presence of the gyrostatic momentum k_3 affects only the lateral dimensions of the trajectory of the body. We now examine some periodic motions of the body in the absence of k_3. To keep the variation of the radial distance ρ somewhat large, we give the modulus of elliptic functions k the value 0.99. The following figures are obtained by taking $a_1 = 4$. The values of N_1 and N_2 are given for each figure. The motion of the body consists of

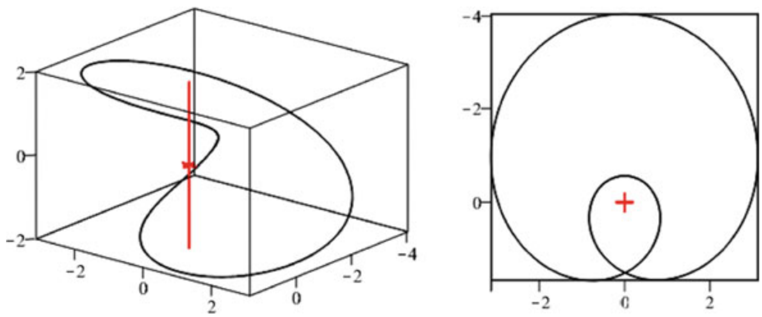
- Complete rotations of the body as a physical pendulum, with period of rotation T_1, about its z-axis, which is always horizontal (orthogonal to the virtual vertical γ) and directed to the ζ-axis.
- A 1-1 regime of rotation about the ζ-axis (One side of the body always faces that axis) of periodic time $T_2 = \frac{2\pi}{n}$.
- Radial displacement of the central point from the ζ-axis of periodic time T_1 (The same as the periodic time of the pendulum).
- Oscillations of the central point in the direction of the ζ-axis (the virtual vertical) of periodic time T_1.

Figure 10.9 depicts the case of equal T_1 and T_2. The space trajectory of the central point of the body closes after a single rotation about the ζ-axis. The central point ascends from the lowest point to the highest point on the part of the trajectory near to the ζ-axis and then descends on the farther part to the first point. At the same time,



Upper side view Projection on $\xi\eta$-plane

Fig. 10.9 $T_1 = T_2$



Upper side view Projection on $\xi\eta$-plane

Fig. 10.10 $T_1 = 2T_2$

the body completes a pendulum rotation cycle in its vertical plane which rotates, in turn, so that the axis of the pendulum motion remains all the time directed to the ζ-axis. In Fig. 10.10, at the time of a complete pendulum rotation cycle the body completes two rotations about the ζ-axis, giving always the same face to that axis. The central point of the body ascends along the narrower loop and descends along the wider one.

In Fig. 10.11, each pendulum rotation cycle corresponds to one vertical oscillation but corresponds to ten precession cycles associated with ten loops around the ζ-axis forming a ten-loop solenoid. The central point of the body ascends along the narrow part of the solenoid and descends along the wider part to the lowest point. Figure 10.12 shows the reverse case $T_2 = 10T_1$. The time of one complete precession cycle of the body and rotation of its central point is enough for ten cycles of the vertical and lateral vibrations of the central point and ten complete cycles of the pendulum vibration.

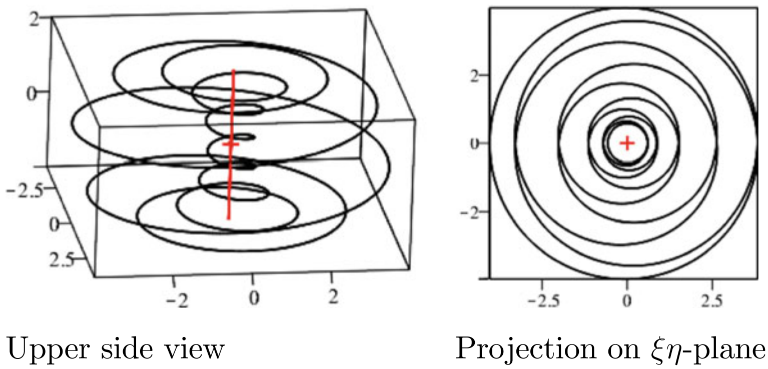


Fig. 10.11 $T_1 = 10T_2$

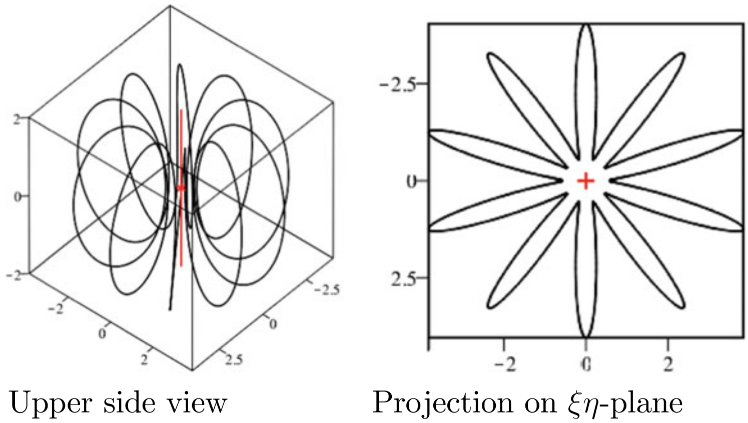


Fig. 10.12 $T_2 = 10T_1$

Remark: It is now time to repeat our previous assertion, that all regular pendulum precessions generated by a certain pendulum motion share all qualitative properties with that motion. For example, the condition for the (orbital) stability of the horizontal axis of semi-regular precession is the same as the condition of stability of the pendulum-like motion generating it, for $n = 0$. Conditions for stability of precessional motion are obtained from the former conditions by replacing the original parameters by the primed ones, which involve the precession parameter n. Some information about orbital stability of pendulum-like motions in the two problems of this chapter will be included in the exercises.

10.15 Tables of Integrable Cases of Motion of a Rigid Body in a Liquid

In this section, we provide tables of general and conditional integrable cases of motion of a rigid body by inertia in an ideal incompressible liquid, infinitely extending in all directions and resting at infinity. results are displayed for the case of a body bounded by a multi-connected surface, i.e. for a perforated body. This case is characterized by the presence of the two vectors **a** and $\boldsymbol{\kappa}$ in our equations in the framework of the equivalent problem of motion about a fixed point of a rigid body acted upon by potential and gyroscopic forces. To follow the same pattern as in previous and coming chapters, we have chosen to put the problem of the present chapter in the context of the second problem. In the tables, all integrable cases of the problem of motion of a body in a liquid are presented in their most general form, as cases of integrability of our new Eq. 10.45 and in terms of the parameter matrices and vectors **I**, **J**, **K**,$\boldsymbol{\kappa}$ and **a**. To express the integrability conditions and the integrals of motion in terms of the original (Kirchhoff-Lamb) parameters, one should use (10.43) or (10.30) as needed.

As to the classification of cases in each table, we organized cases according to the degree of the complementary integral as a function in the components of the angular velocity.

For each case, we provide

(1) The full hierarchy of earlier cases, to which the given case reduces under relevant conditions on the parameters.
(2) The conditions on the parameters on the body and fields, under which the case is valid.
(3) The potential function V.
(4) The vector functions $\boldsymbol{l}$ and $\boldsymbol{\mu}$, which describe the gyroscopic forces acting on the body: The first enters in the Lagrangian and the second in the equations of motion.
(5) The explicit forms of the first integrals I_3 and I_4 in the Euler-Poisson variables.
(6) A Hamiltonian function H is given for each case, together with the corresponding form of the complementary integral in terms of the variables ($\mathbf{M}$,γ) (See Sect. 10.10).

Remark 16 Regarding the fact that most integrable cases are obtained by using inverse method, different hamiltonians may be constructed for one and the same case.

10.15.1 General Integrable Cases

The number of known general integrable cases in the two equivalent problems of the present chapter is seven. Most of them are solutions of Thomson-Lamb equations. They developed historically from solutions of the simpler cases of integrability of the Kirchhoff equations and, in cases, from cases of integrability of problems lower in the hierarchy, which are presented in the previous chapters.

Remark: The regular precession transformation parameter n figures in five of the seven integrable cases, namely, cases 2,3,5,6 and 7. If n is introduced in cases 1 and 4, it can be absorbed in other parameters of the problem and can give no new effects. In cases 2,3 and 5, the parameter n appeared at some stage in their development and not necessarily from the first discovery of the case. In the remaining cases (number 6,7), the introduction of that parameter in [411] was a significant generalization of the case found by Sokolov [336].

Table 10.1 General integrable cases

1	The case of axi-symmetric body Generalization of Lagrange's case Kirchhoff [219] (1870) (see also [220]) $a_3 = 0$, $\kappa_3 = 0$
	$A = B$, $V = a_3\gamma_3 + \frac{1}{2}[b_1(\gamma_1^2 + \gamma_2^2) + b_3\gamma_3^2]$, $\mathbf{l} = (K_1\gamma_1, K_1\gamma_2, K_3\gamma_3 + \kappa)$, $\boldsymbol{\mu} = (-K_3\gamma_1, -K_3\gamma_2, (K_3 - 2K_1)\gamma_3 + \kappa)$, $I_3 = A(p\gamma_1 + q\gamma_2) + (Cr + \kappa)\gamma_3 + K_1(\gamma_1^2 + \gamma_2^2) + K_3\gamma_3^2)$, $I_4 = Cr + \kappa + K_3\gamma_3$
	$H = \frac{M_1^2+M_2^2}{2A} + \frac{M_3^2}{2C} - \frac{K_1}{A}(M_1\gamma_1 + M_2\gamma_2) - \frac{M_3}{C}(K_3\gamma_3 + \kappa)$ $+a_3^*\gamma_3 + b_1^*(\gamma_1^2 + \gamma_2^2) + b_3^*\gamma_3^2$,
	$I_4 = M_3$

where a_3^*, b_1^* and b_3^* are constants

2	Clebsch [55] (1870). Euler [79] (1758). $n = b = 0$
	$V = (b - \frac{1}{2}n^2)(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$ $\mathbf{l} = n(A\gamma_1, B\gamma_2, C\gamma_3)$ $\boldsymbol{\mu} = n((A - B - C)\gamma_1, (B - C - A)\gamma_2, (C - A - B)\gamma_3)$ $I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 + n(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$, $I_4 = \frac{1}{2}[A^2(p + n\gamma_1)^2 + B^2(q + n\gamma_2)^2 + C^2(r + n\gamma_3)^2]$ $-b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$.
	$H = \frac{1}{2}(M_1^2/A + M_2^2/B + M_3^2/C) + b(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$ $-n(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$,
	$I_4 = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) - b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$.

Somewhat later, after [55], the special version ($n = 0$) of this case case was found, apparently independently, by Tisserand [354] (1891) and Brun [47] (1893) in the context of the motion of a body acted upon by approximate Newtonian gravitational forces (See Chap. 6). At that time the Steklov analogy, described above in this chapter, between the two problems was still unknown.

	3	Clebsch [55] (1870) -
		$A = B = C$ $V = \frac{1}{2}A(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2)$ $\mathbf{l} = nA\boldsymbol{\gamma}, \boldsymbol{\mu} = -nA\boldsymbol{\gamma}$ $I_3 = A(p\gamma_1 + q\gamma_2 + r\gamma_3),$ $I_4 = A[c_1(p + n\gamma_1)^2 + c_2(q +$ $n\gamma_2)^2 + c_3(r + n\gamma_3)^2$ $-(c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2)]$
		$H = \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2)$ $-n\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,,$
		$I_4 = c_1M_1^2 + c_2M_2^2 + c_3M_3^2$ $-A[c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2]$

In cases 2 and 3, the parameter n invokes gyroscopic terms due to circulation of the liquid through perforations. Setting $n = 0$ makes the body simply connected.

	4	Rubanovsky [317] (1968), Kharlamov ($\kappa_2 = 0$) [192] (1963), Steklov ($\boldsymbol{\kappa} = \mathbf{0}$) [344] (1893), Joukowsky ($a = 0$) [163] (1885), Euler ($\kappa = \mathbf{0}$, $a = 0$) [79] (175	
		$\mathbf{I} = \mathrm{diag}(A, B, C), \bar{\mathbf{I}} = \frac{1}{2}tr(\mathbf{I}) - \mathbf{I},$ $\mathbf{J} = [tr(\mathbf{I}^{-1})\boldsymbol{\delta} - \mathbf{I}^{-1}]$ $= \mathrm{diag}(\frac{1}{B} + \frac{1}{C}, \frac{1}{A} + \frac{1}{C}, \frac{1}{A} + \frac{1}{B})$ $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3),$ $V = -n(\boldsymbol{\kappa} \cdot \boldsymbol{\gamma} - a\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot \gamma) - \frac{1}{2}n^2\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma},$ $\boldsymbol{l} = \boldsymbol{\kappa} + a\boldsymbol{\gamma}\mathbf{J} + n\boldsymbol{\gamma}\mathbf{I},$ $\boldsymbol{\mu} = \boldsymbol{\kappa} + 2a\boldsymbol{\gamma}\mathbf{I}^{-1} - 2n\boldsymbol{\gamma}\bar{\mathbf{I}},$ $I_3 = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l}) \cdot ,$ $I_4 = \frac{1}{2} \mid (\boldsymbol{\omega} + n\boldsymbol{\gamma})\mathbf{I} + \boldsymbol{\kappa} \mid^2 - 2a\boldsymbol{\omega} \cdot \boldsymbol{\gamma}$	
		$H = \frac{1}{2}(\mathbf{M} - \check{} - a\boldsymbol{\gamma}\mathbf{J})\mathbf{I}^{-1} \cdot (\mathbf{M} - \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J})$ $-n\mathbf{M} \cdot ,$	
		$I_4 = \frac{1}{2} \mid \mathbf{M} - a\boldsymbol{\gamma}\mathbf{J} \mid^2 - a\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot (\mathbf{M} - \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J}).$	

5	Rubanovsky [317] (1968) Lyapunov [267] ($a_1 = a_2 = a_3 = 0$) (1893)
	$A = B = C$, $V = C\{a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3$ $\quad -\frac{1}{2}[(bc + b_0)\gamma_1^2 + (ca + b_0)\gamma_2^2 + (ab + b_0)\gamma_3^2]\}$, $\mathbf{l} = -\frac{1}{2}C((b + c)\gamma_1, (c + a)\gamma_2, (a + b)\gamma_3)$, $\boldsymbol{\mu} = C(a\gamma_1, b\gamma_2, c\gamma_3)$, $I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 + \frac{1}{2}(a\gamma_1^2 + b\gamma_2^2 + c\gamma_3^2)$, $I_4 = \frac{1}{2}[(b + c)\, p^2 + (c + a)\, q^2 + (a + b)\, r^2] - abc(p\frac{\gamma_1}{a} + q\frac{\gamma_2}{b} + r\frac{\gamma_3}{c})$ $\quad +a_1(p + a\gamma_1) + a_2(q + b\gamma_2) + a_3(r + c\gamma_3)$.
	$H = \frac{1}{2C}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}[(b + c)M_1\gamma_1 + (c + a)M_2\gamma_2 + (a + b)M_3\gamma_3]$ $\quad +C(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3)$ $\quad -\frac{C}{8}[(a^2 + 2bc)\gamma_1^2 + (b^2 + 2ac)\gamma_2^2 + (c^2 + 2ab)\gamma_3^2]$,
	$I_4 = (b + c)M_1^2 + (c + a)M_2^2 + (a + b)M_3^2$ $\quad +C\{[(b^2 + c^2)\gamma_1 + 2a_1]M_1 + [(a^2 + c^2)\gamma_2 + 2a_2]M_2 + [(a^2 + b^2)\gamma_3 + 2a_3]M_3\}$ $\quad +\frac{C^2}{4}[(b + c)(b - c)^2\gamma_1^2 + (c + a)(c - a)^2\gamma_2^2 + (a + b)(a - b)^2\gamma_3^2]$ $\quad +C^2[(a + b + c)(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3) + 2(a_1a\gamma_1 + a_2b\gamma_2 + a_3c\gamma_3)]$.

The parameter b_0 has meaning in the problem of motion of a body in a liquid. In the alternative problem it is immaterial.

6	Yehia [411] (2003) Sokolov [336] $n = 0$ (2002) Yehia [380] $n = c = 0$ (1986) Kowalevski [238] $n = c = \kappa = 0$ (1889)
	$A = B = 2C$, $\boldsymbol{\kappa} = C(0, 0, \kappa)$, $V = C[\kappa c\gamma_1 + a_2\gamma_2 - n\kappa\gamma_3 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2) - nc\gamma_1\gamma_3$ $\quad -\frac{n^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)]$, $\mathbf{l} = C(2c\gamma_3 + 2n\gamma_1, 2n\gamma_2, \kappa - c\gamma_1 + n\gamma_3)$, $\boldsymbol{\mu} = C(c\gamma_3 - n\gamma_1, -n\gamma_2, \kappa + c\gamma_1 - 3n\gamma_3)$,
	$I_3 = 2(p\gamma_1 + q\gamma_2) + (r + \kappa + c\gamma_1)\gamma_3 + n(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)$,
	$I_4 = \left[(p + n\gamma_1)^2 - (q + n\gamma_2)^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r + n\gamma_3 - \kappa)\right]^2$ $\quad + \left[2(p + n\gamma_1)(q + n\gamma_2) - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r + n\gamma_3 - \kappa)\right]^2$ $\quad +2\kappa(r + n\gamma_3 - \kappa + c\gamma_1)\left[(p + n\gamma_1)^2 + (q + n\gamma_2)^2 + 2c(p + n\gamma_1)\gamma_3\right]$ $\quad -2\kappa c^2\{2\gamma_3[2(p + n\gamma_1)\gamma_1 + c\gamma_1\gamma_3 + 2(q + n\gamma_2)\gamma_2 + (r + n\gamma_3)\gamma_3]$ $\quad +\kappa\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + n\gamma_3 + c\gamma_1)\} - 4a_2\kappa(q + n\gamma_2)\gamma_3$.
	$H = \frac{1}{2C}(\frac{M_1^2}{2} + \frac{M_2^2}{2} + M_3^2) - c\gamma_3M_1 + (c\gamma_1 - \kappa)M_3 + Ca_2\gamma_2$ $\quad -n(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$
	$I_4 = [\frac{M_1^2 - M_2^2}{4} + Cc\,(-M_1\, z + M_3\, x) + \left(a_2\, y + c^2\right) C^2]^2$ $\quad +[\frac{M_1M_2}{2} + Cc\,(-M_2\, z + M_3 y) - C^2a_2\, x]^2 + \frac{1}{2}Ck\,(M_3 - 2Ck)\left(M_1^2 + M_2^2\right)$ $\quad +C^2k[-2Ca_2M_2\; z - 2M_2\;(M_1\, y - M_2\, x)\, c - 2Cc^2M_3]$

	7	The parameter n is added here to B-M-S result. Borisov, Mamaev and Sokolov $n = s = 0$ [39] (2001) Sokolov [336] $n = 0$ (2001) Kowalevski $n = m = 0$ (1888)	
		$A = B = 2C,$ $V = C\{s(c_1\gamma_1 + c_2\gamma_2) + \frac{1}{2}m^2[(c_1\gamma_1 + c_2\gamma_2)^2 - (c_1^2 + c_2^2)\gamma_3^2]$ $+nm\gamma_3(c_2\gamma_1 - c_1\gamma_2)\gamma_3 - \frac{n^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)\},$ $\mathbf{l} = C(2n\gamma_1, 2n\gamma_2, m(c_2\gamma_1 - c_1\gamma_2) + n\gamma_3),$ $\boldsymbol{\mu} = C(-mc_2\gamma_3 - n\gamma_1, -mc_1\gamma_3 - n\gamma_2, m(c_2\gamma_1 - c_1\gamma_2) - 3n\gamma_3),$	
		$I_3 = 2(p\gamma_1 + q\gamma_2) + [r + m(c_2\gamma_1 - c_1\gamma_2)]\gamma_3 + n(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2),$	
		$I_4 = A_4^2 + B_4^2,$ where $A_4 = (p + n\gamma_1)^2 - (q + n\gamma_2)^2 + s(c_2\gamma_2 - c_1\gamma_1)$ $+m(r + n\gamma_3)(c_2\gamma_1 + c_1\gamma_2) + m^2(c_2^2\gamma_2^2 - c_1^2\gamma_1^2),$ $B_4 = 2(p + n\gamma_1)(q + n\gamma_2) - s(c_1\gamma_2 + c_2\gamma_1)$ $-m(r + n\gamma_3)(c_1\gamma_1 - c_2\gamma_2) - m^2(c_1\gamma_2 + c_2\gamma_1)(c_1\gamma_1 + c_2\gamma_2)]^2.$	
		$H = \frac{1}{2C}[\frac{M_1^2+M_2^2}{2} + M_3^2] + m(c_1\gamma_2 - c_2\gamma_1)M_3 + Cs(c_1\gamma_1 + c_2\gamma_2)$ $-Cm^2(c_1^2 + c_2^2)\gamma_3^2 - n\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,,$	
		$I_4 = [\frac{M_1^2-M_2^2}{4C^2} + \frac{m(c_1\gamma_2+c_2\gamma_1)}{C}M_3 + s(-c_1\gamma_1 + c_2\gamma_2) - m^2(c_1^2 + c_2^2)(\gamma_1^2 - \gamma_2^2)]^2$ $+[\frac{M_1M_2}{2C^2} - \frac{m(c_1\gamma_1-c_2\gamma_2)}{C}M_3 - s(c_1\gamma_2 + c_2\gamma_1) - 2m^2(c_1^2 + c_2^2)\gamma_1\gamma_2]^2.$	

Strictly speaking, case 7 is related to case 6 and can be considered as its special case. We prefer, for future uses (See Chap. 12), to consider case 7 as independent case in its most general form containing maximum number of parameters.

In case 7, the integral I_4 is the sum of two squares. It is the squared modulus of the complex quantity

$$A_4 + iB_4 = [p + iq + n(\gamma_1 + i\gamma_2)]^2 - (c_1 + ic_2)(\gamma_1 + i\gamma_2)[s + im(r + n\gamma_3) + m^2(c_1\gamma_1 + c_2\gamma_2)]. \tag{10.177}$$

The quantities A_4, B_4 satisfy the relations

$$\dot{A}_4 = [r + n\gamma_3 + m(c_1\gamma_2 - c_2\gamma_1)]B_4,\ \dot{B}_4 = -[r + n\gamma_3 + m(c_1\gamma_2 - c_2\gamma_1)]A_4, \tag{10.178}$$

so that the set of conditions

$$\{A_4 = 0,\ B_4 = 0\}$$

define an invariant manifold.

When $s = 0$, case 7 renders to the case discussed in Sect. 10.10 and the integral becomes expressible, as in (10.91), in the form of the product of two functions one linear and the other cubic in velocities.

10.15.2 Conditional Integrable Cases on the Level $f = 0$

Two conditional integrable cases are known. Those cases can be generalized, as will be shown later, using an arbitrary function $\nu(\boldsymbol{\gamma})$ instead of n. Nevertheless, we write them down here with that parameter, as it adds physically significant terms to both problems considered in this chapter: the problem of motion of a body in a liquid and the alternative problem of motion under the action of potential and gyroscopic forces.

Table 10.2 Cases integrable on the level

1	Parameter n is added here to the result of Sokolov and Tsiganov $n = 0$ Sokolov-Tsiganov [338], 2002, $n = c_1 = c_2 = 0$ Sretensky [341], 1963, $n = c_1 = c_2 = \kappa = 0$ Goryachev-Chaplygin
	$A = B = 4C$, $V = C[a_1\gamma_1 + a_2\gamma_2 - n\kappa\gamma_3 + \frac{1}{2}(c_2\gamma_1 - c_1\gamma_2)^2$ $\quad -n\gamma_3(c_1\gamma_1 + c_2\gamma_2) - \frac{n^2}{2}(4\gamma_1^2 + 4\gamma_2^2 + \gamma_3^2)]$, $\mathbf{l} = C(\frac{c_1}{2}\gamma_3 + 4n\gamma_1, \frac{c_2}{2}\gamma_3 + 4n\gamma_2, \frac{1}{2}(c_1\gamma_1 + c_2\gamma_2) + n\gamma_3)$, $\boldsymbol{\mu} = C(c_1\gamma_3 - n\gamma_1, c_2\gamma_3 - n\gamma_2, \kappa + c_1\gamma_1 + c_2\gamma_2 - 7n\gamma_3)$, $I_3 = 4p\gamma_1 + 4q\gamma_2 + [r + \kappa + c_1\gamma_1 + c_2\gamma_2]\gamma_3 + n(4\gamma_1^2 + 4\gamma_2^2 + \gamma_3^2)$, $I_4 = (r - \kappa + c_1\gamma_1 + c_2\gamma_2 + n\gamma_3)\times$ $\quad \times[(p + n\gamma_1 + \frac{1}{2}c_1\gamma_3)^2 + (q + n\gamma_2 + \frac{1}{2}c_2\gamma_3)^2]$ $\quad +\gamma_3[(\kappa c_1 - a_1)(p + n\gamma_1) + (\kappa c_2 - a_2)(q + n\gamma_2)]$ $\quad +\frac{1}{2}\gamma_3^2[\kappa(c_1^2 + c_2^2) - c_1a_1 - c_2a_2]$
	$H = \frac{1}{2C}(\frac{M_1^2}{4} + \frac{M_2^2}{4} + M_3^2) + (-\kappa + 2c_1\gamma_1 + 2c_2\gamma_2)M_3$ $\quad -\gamma_3(c_1M_1 + c_2M_2) + C(a_1\gamma_1 + a_2\gamma_2)$ $\quad -n(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$,
	$I_4 = [(M_3 - 2C\kappa + 4C(c_1\gamma_1 + c_2\gamma_2)](M_1^2 + M_2^2)$ $\quad -4C^2\gamma_3(a_1M_1 + a_2M_2)$

$I_3 = f = 0$

In [337, 338], Sokolov and Tsiganov did not give the complementary integral for this full case. The above formulas are adjusted from [41] (English edition 2017).

2	Yehia [386] 1987, $n=\kappa=0$ Chaplygin [53] 1903, $n=\kappa=b_1=b_2=0$ Kowalevski [238] 1888 (Special case $f=0$),		
	$A=B=2C$, $V=C[a_1\gamma_1+a_2\gamma_2-n\kappa\gamma_3+b_1(\gamma_1^2-\gamma_2^2)+2b_2\gamma_1\gamma_2$ $-\frac{n^2}{2}(2\gamma_1^2+2\gamma_2^2+\gamma_3^2)]$, $\boldsymbol{l}=C(2n\gamma_1,2n\gamma_2,\kappa+n\gamma_3)$, $\boldsymbol{\mu}=C(-n\gamma_1,-n\gamma_2,\kappa-3n\gamma_3)$,		
	$I_3=2p\gamma_1+2q\gamma_2+(r+\kappa)\gamma_3+n(2\gamma_1^2+2\gamma_2^2+\gamma_3^2)$, $I_4=[(p+n\gamma_1)^2-(q+n\gamma_2)^2-a_1\gamma_1+a_2\gamma_2+b_1\gamma_3^2]^2$ $+[2(p+n\gamma_1)(q+n\gamma_2)-a_1\gamma_2-a_2\gamma_1+b_2\gamma_3^2]^2$ $+2\kappa(r+n\gamma_3-\kappa)[(p+n\gamma_1)^2+(q+n\gamma_2)^2]$ $-4\kappa\gamma_3[(p+n\gamma_1)(a_1+b_1\gamma_1+b_2\gamma_2)$ $+(q+n\gamma_2)(a_2+b_2\gamma_1-b_1\gamma_2)]$.		
	$H=\frac{1}{2C}(\frac{M_1^2}{2}+\frac{M_2^2}{2}+M_3^2)-\kappa M_3$ $+C[a_1\gamma_1+a_2\gamma_2+b_1(\gamma_1^2-\gamma_2^2)+2b_2\gamma_1\gamma_2]$ $-n(M_1\gamma_1+M_2\gamma_2+M_3\gamma_3)$, $I_4=[\frac{M_1^2-M_2^2}{4C^2}-a_1\gamma_1+a_2\gamma_2+b_1\gamma_3^2]^2$ $+[\frac{M_1M_2}{2C^2}-a_1\gamma_2-a_2\gamma_1+b_2\gamma_3^2]^2$ $-\frac{\kappa}{2C^3}(2C\kappa-M_3)(M_1^2+M_2^2)$ $-\frac{2\kappa}{C}\gamma_3[M_1(a_1+b_1\gamma_1+b_2\gamma_2)+M_2(a_2+b_2\gamma_1-b_1\gamma_2)]$.		

10.16 Further Studies on Integrable Cases

In the set of integrable cases in the dynamics of a body in a liquid, the presence of the complementary (fourth) integral makes it possible to perform several analytical and qualitative investigations on each one of the integrable cases, something we are not able to do in the generic case missing the fourth integral. Those investigations include the final explicit solution of the equations of motion in terms of time, bifurcation and topological classification of orbits of the integrable system on its integral manifold and the stability analysis of certain motions like stationary and periodic motions.

In this section, we try to give a quick review of some of those investigations performed for the problem of motion of a body in a liquid.

10.16.1 Separation of Variables, Explicit Solutions and

Separation of the variables was performed most easily in the general integrable cases of Euler and Lagrange of the classical problem. Both cases were reduced to elliptic quadratures and hence the explicit solution was expressed in terms of elliptic functions of time. The general integrable case of Kowalevski and the conditional case of Goryachev and Chaplygin were reduced to hyper-elliptic quadratures (For more information see Chap. 4). Explicit time solution of various integrable cases of motion of a body in a liquid was investigated by several authors. The present status of this aspect is summarized in the following:

General integrable cases:

(1) Kirchhoff reduced the case of simply connected body ($a_3 = \kappa = 0$) to an elliptic quadrature and expressed some particular motions in terms of elliptic functions [219]. Detailed analysis of the general solution in elliptic functions was performed by Halphen [146] and Greenhill [135, 136]. The full general case 1 of Table 10.1 can be easily solved also in terms of elliptic functions of time. This is shown in Chap. 12 Sect. 12.1 as a special version of a more general separable case of Lagrange's type (Case 7 of Table 10.1).

(2) The two cases 2 and 3 discovered by Clebsch were shown by Kötter [233] in 1892 to have their general solution in terms of Theta functions in two variables. The special version $f = 0$ of the asymmetric case of Clebsch was solved, in the same set of functions by Weber, somewhat earlier using separation of variables in Hamilton–Jacobi equation [367]. The version $f = 0$ of the spherical Clebsch case is equivalent to Neumann's problem solved in Theta functions of two variables (See Chap. 9 Sect. 9.7.3). Equations of motion in the Lax pair form and generalization to n-dimensional space are briefly discussed in [306]. Some later trials led to separation of variables in a much more complicated form, e.g. [260, 273]. Recently, the full version and Weber's one have been reconsidered in [95, 271].

(3) Steklov and Lyapunov subcases of cases 4 and 5 are conjugate in the same sense as the two cases of Clebsch. A solution of those subcases proposed by Kötter in Theta functions of two variables [235] was presented in a very compact and complicated way, which led to some controversies between his contemporaries. Tsiganov [360] reconsidered the separation problem for Steklov and Lyapunov subcases and recently, in [363], the full versions of cases 4, 5 due to Rubanovsky. However, no explicit formulas are given. Thus, the separation of variables for the Rubanovsky cases cannot be considered complete, except for the Steklov and Lyapunov subcases separated by Kötter.

(4) As concerns case 6, only the lower level of this hierarchy (Kowalevski's case) is separated by Kötter [232, 234]. The status of the second level (Yehia's gyrostat) is described in Chap. 5. A successful procedure like that followed by Kötter has not been found. Separation of variables is not yet achieved for that level and for the next one (Sokolov's generalization of Yehia's gyrostat). Note that if the solution of the Sokolov case ($n = 0$) is constructed, the solution of the last level with $n \neq 0$ is the same, as concerns the vector $\boldsymbol{\gamma}(t)$. The vector $\boldsymbol{\omega}(t)$ is then readily obtained by applying the regular precession transformation (See Sect. 10.11).

Topological classification of the case of Yehia's gyrostat in the uniform gravity field (See Chap. 5) is discussed in detail in the book of M. Kharlamov et al. [184] (See also [185]). The generalized version when $n \neq 0$ and for non-zero Sokolov parameter c, was not investigated until now.

(5) Variable separation for the Sokolov case in the hierarchy 7 (without a gyrostatic momentum) was proposed in [227]. It generalizes the one used for Kowalevski's case by Kowalevski and Kötter. Explicit separation and expressions for dynamical variables were given in [70] , in terms of two intermediate variables, which are expressed in genus-2 Theta functions. In the last level of the hierarchy, after the introduction of the parameter n, the solution is obtained by applying the uniform precession transformation. The same separation variables of [227] were used in [186] for detailed investigation of the integral manifolds and their bifurcation and also complete description of the phase topology of this case.

Conditional integrable cases

(1) Separation of variables is known for the first two levels of the hierarchy. For Goryachev–Chaplygin's see Chap. 4 Sect. 4.4 and for Sretensky's level see Chap. 5. Explicit time solution for the full case (1) of Table 10.2 is not found yet.

(2) The second case of Table 10.2 involves 6 significant parameters $a_1, a_2, b_1, b_2, \varkappa, n$, of which the last one can be set equal to zero for variable separation and it can be restored in the system by the uniform precession transformation. Separation of variables and explicit expressions for the dynamical variables are known in the following subcases:

a- The special version $f = 0$ of Kowalevski's case ($n = \varkappa = a_2 = b_1 = b_2 = 0$). By a rotation of the x, y axes fixed in the body by a constant arbitrary angle in their plane, one can construct a solution in which both coefficients a_1, a_2 are present.

b- Chaplygin [53] first established the integrability, on the level $f = 0$, of the case $n = \varkappa = a_1 = a_2 = b_2 = 0$, that describes the motion of a simply connected body in a liquid (with only one parameter b_1 present in the potential). Then he achieved a separation of variables for this case and expressed the dynamical variables in terms of two parameters s_1, s_2, each of which can be expressed as an elliptic function of t. This solution is presented in detail in the next section. By a rotation of the x, y axes fixed in the body at a constant arbitrary angle in their plane, one can construct a solution in which both coefficients b_1, b_2 are present.

c- From the results of [416], it follows that the problem of motion of a rigid body with $A = 2C$ and arbitrary B, subject to forces with potential

$$V = a_1\gamma_1 + b_1(\gamma_1^2 - \gamma_2^2),$$

under the additional restrictions

$$q = 0, f = 0,$$

is solvable in elliptic functions of time.

10.16.2 Topological Classification of Integrable Cases

The classical integrable cases of the problem of motion of a rigid body in a liquid served as a fertile land for the application of methods of algebraic geometry and topology, created specially for the study of the phase space of integrable systems. Topological classification offers an alternative, which determines the picture of the foliation of the Liouville tori and hence sheds some light on the general (qualitative) features of motion that can hardly be obtained from the explicit solution of complicated problems. General methods of the study of bifurcation of integral manifolds and phase topology of the integrable cases of the classical problem and gyrostat motion were developed by M. Kharlamov (See e.g. [183, 185]). Steklov's case was investigated by Bogoyavlensky and Ivakh [33].

Fomenko constructed what may be called "Morse theory of integrable Hamiltonian systems" [87–90], building on previous results of many authors including, in particular, works of Smale. This theory was further developed by Fomenko, his colleagues and coworkers (e.g. [34, 35, 92, 303]). Topological classification is made for several two- and multi-dimensional integrable Hamiltonian systems. Most interesting, in particular, are Hamiltonian systems with two degrees of freedom. Those include reductions of higher dimensional systems with cyclic coordinates. A theory of topological invariants of such systems was developed, which gives their classification up to Liouville equivalence, i.e. up to deformation of Liouville tori. For basic information about the theory and some applications to rigid body dynamics, the reader is referred to papers in [90], the works cited above and references therein. In this subsection, some results about topological classification are pointed out parallel to information about explicit solution for each integrable case of the dynamics of a rigid body in a liquid. It has to be said here that topological classification is not a characteristic property of an integrable system. Chaplygin's case of rigid body in a liquid, discussed in the next section, is an example.

10.17 Chaplygin's Case of Integrability

In [53], Chaplygin discovered the conditional case, integrable on the zero level ($f = 0$) of the areas integral and like Kowalevski's case valid under the condition $A = B = 2C$ and characterized by the choice

$$V = Cc(\gamma_1^2 - \gamma_2^2),\ \boldsymbol{\mu} = \mathbf{0}, \tag{10.179}$$

in the equations of motion (10.54), which, in this case take the form

$$\begin{aligned} 2\dot{p} - qr &= 2c\gamma_2\gamma_3, \\ 2\dot{q} + pr &= 2c\gamma_1\gamma_3, \\ \dot{r} &= -4c\gamma_1\gamma_2, \end{aligned}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0. \tag{10.180}$$

The four integrals of motion are

$$\begin{aligned}
p^2 + q^2 + \frac{1}{2}r^2 + c(\gamma_1^2 - \gamma_2^2) &= h, \\
2p\gamma_1 + 2q\gamma_2 + r\gamma_3 &= 0, \\
\gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1, \\
(p^2 - q^2 + c\gamma_3^2)^2 + 4p^2q^2 &= K^2.
\end{aligned} \tag{10.181}$$

They involve two arbitrary constants h and K. The sign of K is immaterial and, without loss of generality, we can assume that $K \geq 0$.

Chaplygin's case is highly interesting for many reasons. In particular,

(1) It was the second conditional case in rigid body dynamics, after the Goryachev–Chaplygin case of the classical problem (See Chap. 3 Sect. 3.4).
(2) It turned out that separation of variables is much simpler than in the former case and leads to explicit expressions of the Euler-Poisson in terms of elliptic functions. In fact, it is a rare example of dynamical problem, with a relatively simple solution that can be explicitly written in terms of two sets of elliptic functions, which have two independent moduli.
(3) Because, as will be seen later in Chap. 2, it can be brought to equivalence with a completely different problem. Namely, it is that of motion of a body with the Kowalevski configuration about a fixed point, while acted upon by two irreducible uniform fields.
(4) It was the subject of many later generalizations, as will be seen in Chap. 13.

Chaplygin's case was a favourite subject for topological analysis by many authors. Topology of the iso-energy surfaces, bifurcation diagrams in the plane $\{K^2, h\}$ and topological classification of the Liouville tori are studied in [300, 322]. More detailed topological analysis can be found in [295]. In [91], topological equivalence of Chaplygin's case is established with two other problems, the Euler case of rigid body dynamics and Jacobi's problem of geodesics on an ellipsoid. Nevertheless, it seems that not much is done in the literature dealing with the explicit analytical forms of the solution or the qualitative properties of motion. For all those reasons, we now give a somewhat detailed description of the solution and possible types of motion of the body.

10.17.1 Separation of Variables

We give here the expressions for the Euler–Poisson variables in terms of Chaplygin's separation variables. Some more details on the separation process can be found in [53].

$$p = 1/2\,\frac{\sqrt{2}\sqrt{K}\sqrt{s_1-1}\sqrt{1-s_2}}{\sqrt{s_1-s_2}},$$
$$q = 1/2\,\frac{\sqrt{2}\sqrt{K}\sqrt{s_1+1}\sqrt{1+s_2}}{\sqrt{s_1-s_2}},$$
$$r = \frac{\sqrt{2}[\sqrt{\left(s_1^2-1\right)(\beta-cs_2)}+\sqrt{\left(1-s_2^2\right)(cs_1-\alpha)}]}{s_1-s_2}), \qquad (10.182)$$

$$\gamma_1 = \frac{[\sqrt{(s_1+1)(1-s_2)(\beta-cs_2)}-\sqrt{(1+s_2)(s_1-1)(cs_1-\alpha)}]}{\sqrt{2c}\,(s_1-s_2)},$$
$$\gamma_2 = \frac{[\sqrt{(s_1-1)(1+s_2)(\beta-cs_2)}+\sqrt{(1-s_2)(s_1+1)(cs_1-\alpha)}]}{\sqrt{2c}\,(s_1-s_2)},$$
$$\gamma_3 = -\frac{\sqrt{2K}}{\sqrt{c(s_1-s_2)}}, \qquad (10.183)$$

where $\alpha = h + K$, $\beta = h - K$. Note that $\alpha \geq \beta$ and equality occurs when $K = 0$. The two variables s_1, s_2 are solutions of the equations

$$\dot{s}_1 = -\sqrt{2\left(s_1^2-1\right)(cs_1-\alpha)},$$
$$\dot{s}_2 = -\sqrt{2(1-s_2^2)(\beta-cs_2)}. \qquad (10.184)$$

It is not hard to see that for a real solution of those equations $s_1 \in [\max(\alpha, 1), \infty]$, while $s_2 \in [-1, \min(\beta, 1)]$.

Now, for more visibility of the results, one can choose the units of measuring time, so that the constant $c = 1$. It is essential to find the conditions of repeated roots in the under-root polynomials in (10.184). Those are, respectively,

$$h = -K \pm 1, h = K \pm 1. \qquad (10.185)$$

For simplicity, we introduce the parameters

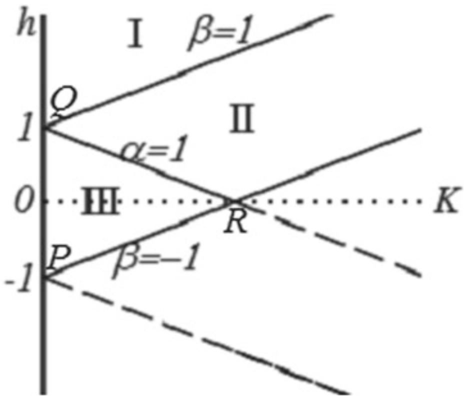
$$\alpha = h + K, \beta = h - K. \qquad (10.186)$$

The bifurcation diagram in the Kh-plane is shown in Fig. 10.13.

There are three admissible regions I, II and III, inscribed by solid lines. In those regions we have

$$-1 < 1 < \beta < \alpha \text{ in region I,}$$
$$-1 < \beta < 1 < \alpha \text{ in region II,}$$
$$-1 < \beta < \alpha < 1 \text{ in region III.}$$

Fig. 10.13 Bifurcation diagram for Chaplygin's case



Inside each region the analytical form of the solution of (10.184) and its qualitative properties do not change, and so does the topological type of the invariant two-dimensional manifold which consists of tori on which the trajectories are wind in the phase space. Only crossing the boundaries between those regions, those properties can change.

By integrating (10.184), it is not hard to obtain the following formulas for s_1 and s_2 in terms of time:

$$s_1 = \alpha + (\alpha - 1)\frac{\mathrm{sn}^2(\sqrt{\frac{\alpha+1}{2}}t, k_1)}{\mathrm{cn}^2(\sqrt{\frac{\alpha+1}{2}}t, k_1)}, \qquad k_1 = \sqrt{\frac{2}{\alpha+1}}, \quad \alpha > 1,$$
$$= 1 + (1-\alpha)\frac{\mathrm{sn}^2(t,\ \nu_1)}{\mathrm{cn}^2(t,\ \nu_1)}, \qquad \nu_1 = \sqrt{\frac{\alpha+1}{2}}, \qquad \alpha < 1, \tag{10.187}$$

and

$$s_2 = -1 + 2\,\mathrm{sn}^2(\sqrt{\frac{\beta+1}{2}}\tau, k_2), \qquad k_2 = \sqrt{\frac{2}{\beta+1}}, \quad \beta > 1,$$
$$= -1 + (\beta+1)\,\mathrm{sn}^2(\tau,\ \nu_2), \qquad \nu_2 = \sqrt{\frac{\beta+1}{2}}, \quad \beta < 1. \tag{10.188}$$

Here $\tau = t - t_0$, t_0 is an arbitrary constant. Note that $0 \le t_0 < T$, where T is the period of the Jacobi elliptic functions of moduli k_2. A similar constant appears in the first two formulas is set equal to zero without loss of generality by choosing the initial time moment. The generic motion is quasi-periodic, but becomes periodic when the periods T_1 and T_2 of the two sets of Jacobi's functions are commensurable.

According to the general Liouville-Arnold theorem for completely integrable hamiltonian systems, the integral manifold of the Chaplygin system, corresponding to a fixed pair of the parameters $\{K, h\}$, is a 2-torus or a union of such tori, each

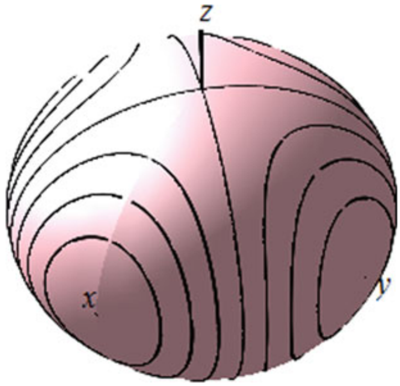


Fig. 10.14 Iso-potentials on the Poisson sphere, the same as zero-velocity curves

of which is filled (winded) by quasi-periodic phase trajectories. On each torus, a trajectory is singled out by the value of the parameter t_0. The projection of a trajectory of the system on the Poisson sphere is the trajectory of the apex of the vector $\boldsymbol{\gamma}$, during the motion of the body, on that sphere. That is what we try to clarify in the following subsections.

10.17.2 Forms of Motion on the Poisson Sphere

A look at Eq. (10.180) reveals that they have six equilibrium positions, in which an end of one of the principal axes of inertia is directed vertically upwards. Two positions correspond to potential minima $V = -1$ at the points $\boldsymbol{\gamma} = (0, \pm 1, 0)$, two correspond to potential saddle point $V = 0$ at $\boldsymbol{\gamma} = (0, 0, \pm 1)$ and the last two correspond to potential maxima $V = 1$ at $\boldsymbol{\gamma} = (\pm 1, 0, 0)$.

From the energy integral in (10.181), one can see that any real possible motion or equilibrium must satisfy the condition

$$V = \gamma_1^2 - \gamma_2^2 \le h. \tag{10.189}$$

The region determined by this condition on the Poisson sphere is called *the region of possible motions.* On its boundary $\gamma_1^2 - \gamma_2^2 = h$, the angular velocity of the body vanishes. If exists, this boundary is named *the zero-velocity curve ZVC*.

Figure 10.14 depicts iso-potential lines on the Poisson sphere. At the minimum value of the energy parameter $h = -1$, the ZVC is composed of two opposite points, corresponding to two stable equilibrium positions[4] of the body with either ends of

[4] Here we mean the alternative problem of motion about a fixed point. In the Chaplygin problem, it corresponds to a steady translational motion of the body in the liquid.

the y-axis directed along the upward vertical (The vector $\boldsymbol{\gamma}$). As h increases, namely, for $h \in (-1, 0)$ ZVC consists of two components, each of which is closed around one of the ends of the y-axis. The region of possible motion is composed of the two areas inside the two components of the ZVC. The value $h = 0$ is a critical one. At this value, the ZVC renders to a pair of great circles intersecting on the z-axis and the two regions meet at the two ends of the z-axis. For greater values of $h \in (0, 1)$, the two components of the ZVC become closed around the tips of the x-axis and the region of possible motion is the whole sphere with the exception of the two regions inscribed by the ZVC. For $h = 1$, the region of possible motion is the whole sphere and the equilibrium is possible with the x-axis in vertical position. Finally, for $h > 1$, the region of possible motion is the whole sphere and no ZVC exists.

In the bifurcation diagram Fig. 10.13, one can readily see that the least value of the energy parameter for a possible motion is $h = -1$ at P and at this point $K = 0$. Those values correspond to two equilibrium positions $\boldsymbol{\gamma} = (0, \pm 1, 0)$ at two potential minima. The y-axis is then directed up or down the positive Z-axis fixed in space. The trajectory of the apex of y consists of two points. If we move in the bifurcation diagram on the boundary PQ, the trajectory becomes an arc of the great circle $\gamma_3 = 0$, corresponding to a periodic pendulum-like motion about the z-axis. As we approach the point Q, the motion becomes asymptotic to one of the two equilibrium positions at the two potential maxima at $\boldsymbol{\gamma} = (\pm 1, 0, 0)$. At all points beyond Q, the trajectory is the whole circle, corresponding to complete uni-directional plane (pendulum-like) rotations about the z-axis.

A similar pattern is noted also on the line PR. The motion begins as pendulum-like vibration about the x-axis with increasing amplitude that reaches $\pi/2$ at R, where the motion becomes asymptotic to the equilibrium positions at $\boldsymbol{\gamma} = (0, 0, \pm 1)$, the saddle points of the potential. Beyond R, the motion is a pendulum complete rotation about the x-axis.

An exceptional family of motions corresponds to parameters on the segment RQ. The motion begins as a pendulum-like vibration about the y-axis with increasing amplitude that reaches $\pi/2$ at Q, where it becomes asymptotic to the equilibrium positions at two potential maxima at $\boldsymbol{\gamma} = (\pm 1, 0, 0)$.

Finally, on the critical line $h = K + 1$, the motion is asymptotic to pendulum-like complete rotations about the y-axis.

10.17.3 Explicit Solution

Now, substituing relevant expressions for s_1 and s_2 from (10.187), (10.188) into (10.182), (10.183), one can write down all the Euler-Poisson variables as functions of time. Doing that, one has to choose the signs of the radicals $\sqrt{s_1 - 1}$, $\sqrt{1 - s_2}$, $\cdots$ and$\sqrt{s_1 - s_2}$. However, one has to take only the combinations of signs which are compatible with the areas integral, the second one in (10.181). To make it more definite, we first note that the equations of motion and the integrals of motion enjoy the property of being invariant under each of the simultaneous changes of signs of the tuples

$$
\begin{aligned}
&1)\{\gamma_1, \gamma_2, \gamma_3\},\\
&2)\{p, q, \gamma_3\},\\
&3)\{r, \gamma_1, \gamma_2, t\}.
\end{aligned}
\tag{10.190}
$$

Some more changes can be obtained as products of those three. Example is the change $\{\boldsymbol{\omega}(t), \boldsymbol{\gamma}(t)\} \to \{-\boldsymbol{\omega}(-t), \boldsymbol{\gamma}(t)\}$, obtainable as the product of the three changes. Alternatively, the last change follows from a general principle, the time-reversibility of the motion of natural mechanical systems, acted upon by purely potential forces.

We now write down the final forms in the three zones of the primary solution, obtained by giving all radicals in (10.182), (10.183) a positive sign. In all illustrating examples in the accompanying figures below, we have set $t_0 = 0$, i.e. we have projected one trajectory of the infinite number on an integral torus of the problem. Note that those figures were plotted on different time intervals, sufficient for suitable visualization (Figs. 10.15, 10.16, 10.17).

10.17.3.1 In Zone I ($1 \le \beta \le \alpha < \infty$,)

$$
\begin{aligned}
p &= \frac{\sqrt{K(\alpha-1)}\,\mathrm{cn}\,(\nu_2\tau, k_2)}{\sqrt{\Delta_1}},\\
q &= \frac{\sqrt{K(\alpha+1)}\,\mathrm{dn}\,(\nu_1 t, k_1)\,\mathrm{sn}\,(\nu_2\tau, k_2)}{\sqrt{\Delta_1}},\\
r &= \frac{\sqrt{2(\alpha-1)}}{\Delta_1}[2\mathrm{sn}(\nu_1 t, k_1)\,\mathrm{cn}(\nu_1 t, k_1)\,\mathrm{sn}\,(\nu_2\tau, k_2)\,\mathrm{cn}\,(\nu_2\tau, k_2)\\
&\quad+\sqrt{(\alpha+1)(\beta+1)}\,\mathrm{dn}\,(\nu_2\tau, k_2)\,\mathrm{dn}(\nu_1 t, k_1)],
\end{aligned}
\tag{10.191}
$$

$$
\begin{aligned}
\gamma_1 &= \frac{1}{\Delta_1}[\sqrt{(\alpha+1)(\beta+1)}\,\mathrm{cn}(\nu_1 t, k_1)\,\mathrm{dn}(\nu_1 t, k_1)\,\mathrm{cn}\,(\nu_2\tau, k_2)\ \mathrm{dn}\,(\nu_2\tau, k_2)\\
&\quad-(\alpha-1)\,\mathrm{sn}\,(\nu_2\tau, k_2)\ \mathrm{sn}(\nu_1 t, k_1)],\\
\gamma_2 &= \frac{\sqrt{\alpha-1}}{\Delta_1}[\sqrt{(\alpha+1)}\,\mathrm{sn}(\nu_1 t, k_1)\,\mathrm{dn}(\nu_1 t, k_1)\ \mathrm{cn}\,(\nu_2\tau, k_2)\\
&\quad+\sqrt{(\beta+1)}\,\mathrm{sn}\,(\nu_2\tau, k_2)\ \mathrm{dn}\,(\nu_2\tau, k_2)\ \mathrm{cn}(\nu_1 t, k_1)\,],\\
\gamma_3 &= -\frac{\sqrt{2}\sqrt{K}\,\mathrm{cn}(\nu_1 t, k_1)}{\sqrt{\Delta_1}},
\end{aligned}
\tag{10.192}
$$

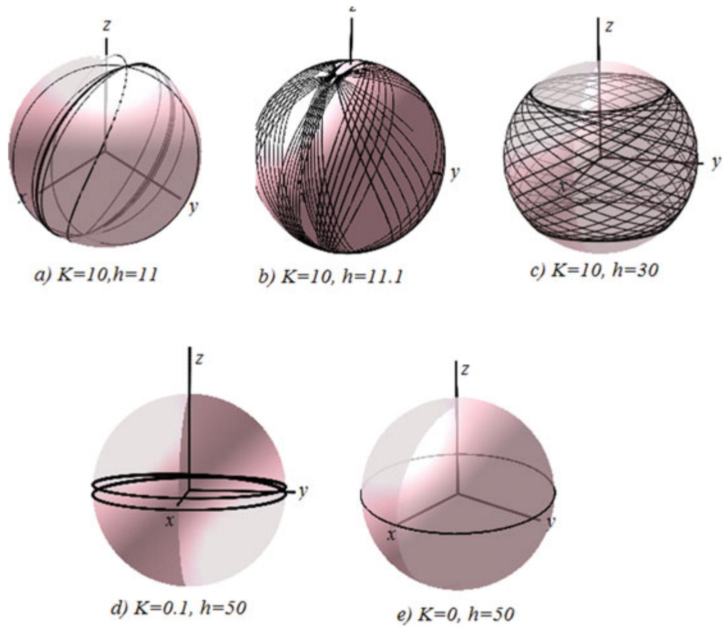


Fig. 10.15 Examples of trajectories from zone I

where

$$\Delta_1 = (\alpha + 1)\,\mathrm{dn}^2\,(\nu_1 t, k_1) - 2\,\mathrm{cn}^2\,(\nu_1 t, k_1)\,\mathrm{sn}^2\,(\nu_2 \tau, k_2)\,. \tag{10.193}$$

10.17.3.2 In Zone II ($-1 \leq \beta \leq 1 \leq \alpha < \infty$)

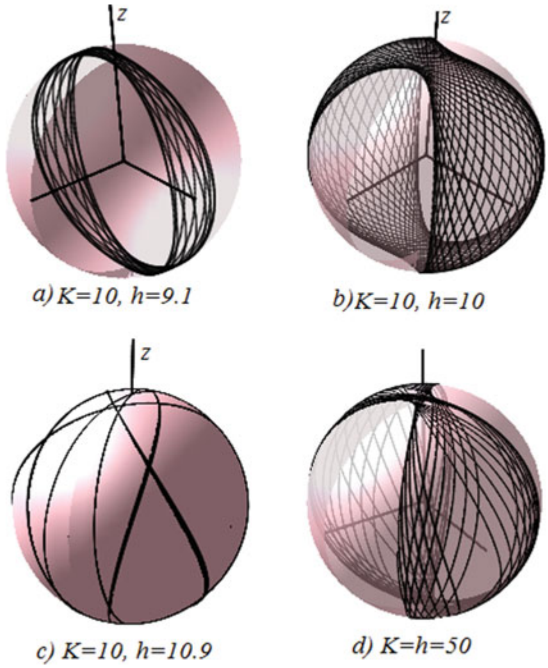
$$p = \frac{\sqrt{K(\alpha - 1)}\,\mathrm{dn}\,(\tau, \nu_2)}{\sqrt{\Delta_2}},$$

$$q = \frac{\sqrt{K(\alpha + 1)(\beta + 1)}\,\mathrm{dn}\,(\nu_1 t, k_1)\,\mathrm{sn}\,(\tau, \nu_2)}{\sqrt{2\Delta_2}},$$

$$r = \frac{\sqrt{2(\alpha - 1)(\beta + 1)}}{\Delta_2}$$

$$[\sqrt{2}\,\mathrm{sn}(\nu_1 t, k_1)\,\mathrm{cn}(\nu_1 t, k_1)\,\mathrm{sn}\,(\tau, \nu_2)\,\mathrm{dn}\,(\tau, \nu_2) + \sqrt{\alpha + 1}\,\mathrm{cn}\,(\tau, \nu_2)\,\mathrm{dn}(\nu_1 t, k_1)], \tag{10.194}$$

Fig. 10.16 Examples of zone II



$$\gamma_1 = -\frac{\sqrt{\beta+1}}{2\Delta_2}[-2\sqrt{\alpha+1}\,\mathrm{dn}(\nu_1 t, k_1)\ \mathrm{dn}\,(\tau,\nu_2)\ \mathrm{cn}\,(\tau,\nu_2)\ \mathrm{cn}(\nu_1 t, k_1)$$

$$+\sqrt{2}\ (\alpha-1)\,\mathrm{sn}\,(\tau,\nu_2)\ \mathrm{sn}(\nu_1 t, k_1)],$$

$$\gamma_2 = \frac{\sqrt{2(\alpha-1)}}{2\Delta_2}[\sqrt{2(\alpha+1)}\,\mathrm{sn}(\nu_1 t, k_1)\,\mathrm{dn}(\nu_1 t, k_1)\ \mathrm{dn}\,(\tau,\nu_2)$$

$$+(\beta+1)\,\mathrm{sn}\,(\tau,\nu_2)\ \mathrm{cn}\,(\tau,\nu_2)\ \mathrm{cn}(\nu_1 t, k_1)\,],$$

$$\gamma_3 = -\frac{\sqrt{2}\sqrt{K}\,\mathrm{cn}(\nu_1 t, k_1)}{\sqrt{\Delta_2}}, \tag{10.195}$$

where

$$\Delta_2 = (\alpha+1)\,\mathrm{dn}^2\,(\nu_1 t, k_1) - (\beta+1)\,\mathrm{cn}^2\,(\nu_1 t, k_1)\,\mathrm{sn}^2\,(\tau,\nu_2)\,. \tag{10.196}$$

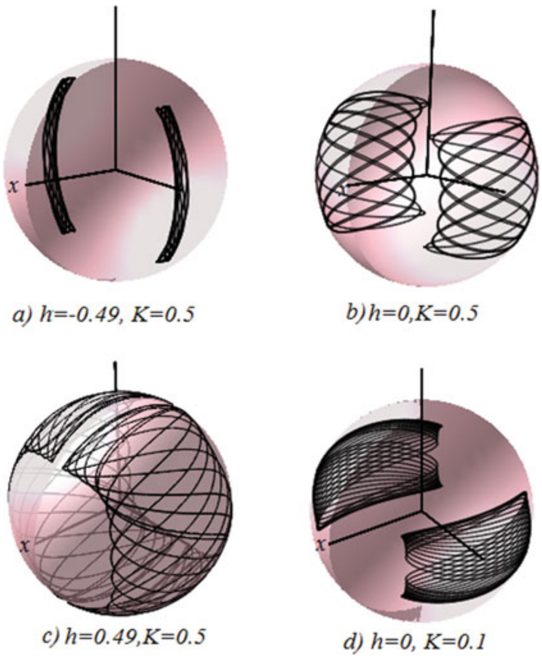


Fig. 10.17 Examples of trajectories in zone III

10.17.3.3 In Zone III ($-1 \le \beta \le \alpha \le 1 < \infty$)

$$
\begin{aligned}
p &= \frac{\sqrt{K(1-a)}\,\mathrm{sn}\,(t,\nu_1)\,\mathrm{dn}\,(\tau,\nu_2)}{\sqrt{\Delta_3}},\\
q &= \frac{\sqrt{K(\beta+1)}\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2)}{\sqrt{2\Delta_3}},\\
r &= \frac{\sqrt{2(1-\alpha)(\beta+1)}}{\Delta_3}[\mathrm{sn}\,(t,\nu_1)\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{cn}\,(\tau,\nu_2)\\
&\quad + \mathrm{cn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2)\,\mathrm{dn}\,(\tau,\nu_2)],
\end{aligned}
\tag{10.197}
$$

$$
\begin{aligned}
\gamma_1 &= \frac{\nu_2}{\Delta_3}[(a-1)\,\mathrm{sn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2) + 2\,\mathrm{cn}\,(t,\nu_1)\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{cn}\,(\tau,\nu_2)\,\mathrm{dn}\,(\tau,\nu_2)]\\
\gamma_2 &= \frac{\sqrt{2(1-a)}}{2\Delta_3}[2\,\mathrm{dn}\,(t,\nu_1)\,\mathrm{dn}\,(\tau,\nu_2)\\
&\quad + (\beta+1)\,\mathrm{sn}\,(t,\nu_1)\,\mathrm{sn}\,(\tau,\nu_2)\,\mathrm{cn}\,(t,\nu_1)\,\mathrm{cn}\,(\tau,\nu_2)]\\
\gamma_3 &= -\frac{\sqrt{2K}\,\mathrm{cn}(t,\nu_1)}{\sqrt{\Delta_3}},
\end{aligned}
\tag{10.198}
$$

where

$$\Delta_3 = 2\,\mathrm{dn}^2(t,\nu_1) - (\beta+1)\,\mathrm{sn}^2(\tau,\nu_2)\,\mathrm{cn}^2(t,\nu_1)\,. \tag{10.199}$$

10.18 Integrability Issues

To begin with, let us note that just as in the problems considered in the previous chapters, the equations of motion (10.45) satisfy Jacobi's divergence condition, which may be written as

$$\frac{\partial}{\partial \boldsymbol{\omega}} \cdot \dot{\boldsymbol{\omega}} + \frac{\partial}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} = 0$$

and thus we need only one general integral (involving a new arbitrary constant) to complete the integration of the problem of motion.

The problem of motion of a body by inertia in an ideal fluid is described by Lagrangian and Hamiltonian equations, isomorphic by the analogy introduced in this chapter to the equations describing the motion of a rigid body about a fixed point under the action of an axi-symmetric combination of three classical fields. The last problem has three degrees of freedom and thus requires for complete integrability the existence of a fourth integral independent of the three known ones. In any set of generalized coordinates, say, Euler's angles, the geometric integral degenerates into an identity and we are left with three integrals, the number of integrals required for complete integrability in the sense of Liouville. Thus, Jacobi's and Liouville's approaches lead to the same requirement.

Neither Eqs. (10.45) and (10.41) nor the equivalent Thomson-Lamb equations were investigated in their full form for the existence of a fourth (complementary) integral. The situation is somewhat better for Kirchhoff's equations, which describe the motion of a body bounded by a simply connected surface. We give here only brief account of the various research on this matter.

10.18.1 Results Concerning Kirchhoff's Equations

10.18.1.1 The Case of Tri-Axial Ellipsoid of the Matrix $\bar{\mathbf{a}}$

Existence of a real-analytic fourth integral: One of the notable results is due to Kozlov and Onishchenko [246] (See also [41]), who used Eq. (10.12) to establish that when the matrices $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}$ are simultaneously diagonal, i.e.

$$\begin{aligned}
\bar{\mathbf{a}} &= \mathrm{diag}(a_1, a_2, a_3),\\
\bar{\mathbf{b}} &= \mathrm{diag}(b_1, b_2, b_3),\\
\bar{\mathbf{c}} &= \mathrm{diag}(c_1, c_2, c_3),
\end{aligned}$$

and under the condition that $a_1 \neq a_2 \neq a_3 \neq a_1$, there exists no real-analytic complementary integral of (10.12) independent of the three known general integrals (10.13), except in the two cases when the following necessary relations hold between the matrices:

$$\text{A)}\ \frac{c_1 - c_3}{a_2} + \frac{c_2 - c_1}{a_3} + \frac{c_3 - c_2}{a_1} = 0,\ \bar{\mathbf{b}} = \mathbf{0}, \tag{10.200}$$

Conditions (A) are also sufficient for integrability. They correspond to Clebsch's integrable case (Case 2 of Table 10.1 above) under the restriction $n = 0$, n the regular precession transformation parameter.[5]

$$\text{B)}\ \frac{b_1 - b_3}{a_2} + \frac{b_2 - b_1}{a_3} + \frac{b_3 - b_2}{a_1} = 0. \tag{10.201}$$

This condition is necessary but not sufficient. The classical case of Steklov (Case 5 of Table 10.1 above, with $n = 0$, $\boldsymbol{\kappa} = \mathbf{0}$) satisfies this condition and existence of the fourth integral is secured by the additional restriction $\bar{\mathbf{c}} = \mathbf{0}$.

Branching of solution: The large number of parameters involved in the Thomson-Lamb equations of motion of a body in a liquid has become an obstacle for further analytical studies of those equations. In spite of its huge success in the classical problem, the approach used by Kowalevski [238] to isolate possible cases in which the solution of equations of motion has only poles as critical points in the complex t-plane does not seem efficient in the problem of motion of a body in a liquid. However, analogous result was established for Kirchhoff's equations:

Under the condition that $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ are simultaneously diagonal and $a_1 \neq a_2 \neq a_3 \neq a_1$, *the general solution of (10.12) is meromorphic only for the cases of Clebsch and Steklov* [316].

In both cases, the complementary integral is known and the explicit time solution is expressed in terms of Theta functions.

Existence of a single-valued or algebraic fourth integral: The investigation of existence of a single-valued complementary integral was performed in [36]. It turned out that when $a_1 \neq a_2 \neq a_3 \neq a_1$, branching of solutions is an obstacle for existence of single-valued integrals. It is shown that

Under the condition that $\bar{a}$, $\bar{b}$, $\bar{c}$ are simultaneously diagonal and $a_1 \neq a_2 \neq a_3 \neq a_1$, the cases of Clebsch and Steklov are the only cases, when Eq. (10.12) admit a single-valued fourth integral.

5 In fact, the condition $\bar{\mathbf{b}} = \mathbf{0}$, is over-restrictive. The result holds when $\bar{\mathbf{b}}$ is propertional to $\bar{\mathbf{I}} = \frac{1}{2}\,\mathrm{tr}(\mathbf{I})\delta - \mathbf{I} = \frac{1}{2}\,\mathrm{tr}(\bar{\mathbf{a}}^{-1})\delta - \bar{\mathbf{a}}^{-1}$. Compare with Case 2 of Table 10.1. The full form, consistent with that in Table 10.1, was given in [36].

The existence of a polynomial integral of Kirchhoff was also considered in some recent works [263, 428], following a line due to Darboux.

10.18.1.2 Case when ā Has an Ellipsoid of Revolution

Preserving the assumption of diagonal three matrices and adding the restriction $a_1 = a_2$, Sadetov [327, 328] has shown that a complementary algebraic integral of the equations of motion does not exist, except in the Kirchhoff case (Case 1 of Table 10.1) and the special versions of the cases of Clebsch under the extra-condition $a_1 = a_2$.

Remark: As we have seen, in all methods used to investigate integrability of the problem, diagonality of all matrices was a common assumption. This situation greatly reduced the efficiency of those methods. None of them has pointed out a new integrable case. Ironically, in the case which generalizes the classical case of Kowalevski, found later by Sokolov, the matrix K has off-diagonal elements. It was not predicted by any method, but came as a result of the application of a brute force method. An ansatz of an integral of degree 4 was used and a symbolic program was used to solve the resulting conditions on the coefficients and on the system parameters.

10.19 Remark Concerning Particular Solutions of the Problem

The above tables of general and conditional integrable cases of Thomson-Lamb and Kirchhoff equations give a complete up-to-date list and full identification of those cases. Although we also know a large number of particular exact solutions, we have not tried to make a complete list of them. At present, some of those solutions are scattered in journal papers. We have described the most important of those solutions in the present chapter, as examples on solutions of various forms of the equations of motion and also in examples of application of the regular precession transformation.

The largest collection of particular exact solutions of problems of motion of a body in a liquid may be found in books of Gorr and co-authors [121, 125, 126]. Cases are classified by the nature of motion: permanent rotations, regular precessions, semi-regular precessions and so on. However, those books concentrate more on the research of the Donetsk group and in general on results published in Russian. Some results may have been disguised by the use of various sets of variables and may need careful revision. In general, further effort is needed to compare, complete, classify and tabulate all existing results.

10.20 The Donetsk School of Mechanics and Its Attitude to Competing Works

Be not proud because thou art learned; but discourse with the ignorant man as with the sage. For no limit can be set to skill, neither is there any craftsman that possesseth full advantages
Ptah-Hotep (2880 BC) [72]

Although founded by Euler and developed by the basic works of D'Alembert, Poisson and Lagrange, the field of dynamics of a rigid body acted upon by various forces suffered from stagnation for almost a century. Over that period, the search for integrable cases or particular solutions didn't lead to any notable results, even in the simplest problem, the classical problem of motion of a body about a fixed point under its own weight, more than Euler's and Lagrange's cases.

The first breakthrough in the classical problem was made, in 1888, by Kowalevski, who discovered the third integrable case. To that she was not led by a physical or mechanical conservation law, as in the previous two cases, but was led only by a purely mathematical property of the solution of the equations of motion. Kowalevski's success encouraged a number of several of the classics in mathematics and mechanics to invest huge efforts in the same problem. Over the next two decades, the search of such eminent scientists as Joukovsky, Lyapunov, Steklov, Chaplygin and Goryachev produced several integrable cases and particular solutions not only in the classical problem, but also in the gyrostat problem and the problem of motion of a body in a liquid. We have listed those results in relevant chapters of this book. It can be noted that of the eight solutions known up to the first decades of the twentieth century, four cases were found by Russian authors. The next four decades have brought no significant changes in the status of the field, but in 1948, Grioli announced the discovery of a regular precession about an axis inclined to the vertical.

Donetsk school headed by P.V. Kharlamov has made a significant advance in the subject of rigid body dynamics in the period extending from the mid-fifties to the late eighties. For most of this period the Donetsk school comprised a large number of coworkers who worked on all aspects of the classical problem and its generalizations, and especially, the problem of motion of the gyrostat. The group made several notable achievements: three new particular solutions of the classical problem raised the ratio of exact particular solutions constructed by Russian-writing authors to seven cases out of a total of twelve known at the present time. Donetsk school's success was exclusive in the problem of motion of a heavy gyrostat about a fixed point. As pointed out in Sect. 15 of Chap. 5, a considerable part of our present knowledge of exact particular solutions of the equations of motion of a gyrostat belongs to that school. Those are mostly cases generalizing known ones of the classical problem, but a few ones have no analogs in the classical problem. However, the Donetsk school did not find any general integrable cases of the gyrostat problem. This is a key remark to which we shall return later.

The group used a "brute force" policy in the search for exact solutions. Problems are scanned for the possibility of admitting a solution of a prescribed form. Each of the resulting cases certainly required a high cost of manual calculations. The view of an expression or a solution with coefficients written in one or two pages was normal and mostly expected. In the classical field of rigid body dynamics no easy results are left. The group earned credibility and authority in the area of constructing exact solutions and renewed the spirit that prevailed at the turn of the 19th to the 20th century, when the field of rigid body dynamics was, almost completely, a Russian-language science.

Researchers from the Donetsk school have shown that the few results announced in the thirties by Field, Corliss and Fabbri mainly repeat or are special cases of the former results of Russian authors. Also, a result of Mertsalov (1946) was shown to be in error. The overall performance of the research group was more than successful. This gave the group a sense of responsibility to the Russian heritage: they kept its competence among other schools of mechanics and turned into custodian not only of the Russian contributions but also of the whole subject of rigid body dynamics.

In the mid-eighties, the author introduced the simple transformation discussed in detail in Sect. 10.11, which led to an automatic generalization of all general and conditional integrable cases as well as particular solutions of the classical problem and its generalizations by inserting an additional parameter n that invokes a simultaneous combination of potential and gyroscopic forces. Results have interpretations as new integrable and solvable cases in the problem of motion of a body in a liquid. Nearly at the same time, the author devised a method for constructing two-dimensional integrable systems that admit a complementary integral, polynomial in velocity. This method had two main advantages. Firstly, it produced systems living on Riemannian manifolds and not only on flat spaces. Secondly, those systems are time-irreversible, and thus accommodate reductions of 3D systems with a cyclic integral. Those two advantages made the method able to obtain a new integrable system that needed some restrictions to produce a case of motion of the gyrostat, which turned out to be the long-waited generalization of the historical Kowalevski's case, by adding a rotor to the body along its axis of dynamical symmetry (For details, see Chap. 5 Sect. 5.6). There were some other new results, like the new form of the equations of motion of a body in a liquid, which we presented in detail in Sect. 10.4.

The new results have shocked the Donetsk school in more than one way. On one hand, a significant contribution came from outside the Donetsk school. On the other hand, no brute force was used, nor needed, in obtaining those results. The Donetsk school behaved in reaction to the appearance of the new results in a strange way. We give here few brief quotations from the publications of members of the Donetsk school of mechanics, to show to what extent some scientific criticism can go when a strongly overconfident group of researchers have full control over a well-known scientific journal. A rebuttal of some of those criticizing publications was published in 2001 in [405], too late, after some comments were included in Mathematical Reviews [281] and Zentralblatt (Zbl 1025.70007). After the publication of our article [405], it seems that the Donetsk group, at last, realized that they were in error, nevertheless, no one of the authors of the aggressive publications came out to declare that. The direct aggressive series of criticism was stopped. Only in few occasions they were resumed, and mainly indirectly (e.g. [203]).

After the detailed presentation of the problem of motion of a body in a liquid in this chapter, one can hardly need any comments on the claims in the papers of the Donetsk group. However, we find it necessary to pick up some of the most offensive claims. In fact, there are some lessons here to be learned from them.

10.20.1 The Attitude to the Uniform Precession Transformation

In a series of publications, which were brought to our attention only in the fall of 1996, the authors claimed that this method of generalization is void, meaningless and leads to nothing new [200, 211, 216, 217, 292, 350]. We give few quotations, referring the reader to original sources.

(A) In a rare example of unjustified criticism, one of those authors (Kharlamov P.V. [200]) stated, after appraising the criticism in [216, 217, 350], that this criticism:

> "*restores the truth in the most difficult problems of the dynamics of a rigid body, and cleans the field of study from rubbish introduced by faulty and illiterate papers of H. M. Yehia ...* " [200].

The same quotation was included in the review (MR 93g: 01040), written for Mathematical Reviews by Konosevich, the colleague of the authors in the same institute.

(B) Another author writes [350]:

> *"H. Yehia has announced so significant results, that, in case they were true, the state of the classical problems of rigid body dynamics could have radically changed. ...*
> *Astonishing is the lightness with which the achievements of the greatest scientists including prominent nationals were "generalized" in a single stroke by means of a trivial change of variables and introducing a nonsignificant parameter to the system. But neither V. A. Steklov, N. E. Joukovsky, S. A. Chaplygin nor G. V. Kolossov can defend themselves against Yehia's generalizations"* .

(C) As was explained by Kharlamov in [200] (The same reasoning also in [216, 217, 350]), the main point of their criticism is the following:

"Let the system of differential equations

$$\dot{x}_i = X_i(x_1, x_2, ..., x_n), i = 1, ..., n \tag{10.202}$$

be transformed by means of the invertible substitution

$$y_i = y_i(x_1, x_2, ..., x_n; \nu), i = 1, ..., n \tag{10.203}$$

to the form

$$\dot{y}_i = Y_i(y_1, y_2, ..., y_n; \nu), i = 1, ..., n. \tag{10.204}$$

If (10.202) admits an integral

$$I(x_1, x_2, ..., x_n) = const. \tag{10.205}$$

then (10.204) has the integral

$$J(y_1, y_2, ..., y_n; \nu) = const. \tag{10.206}$$

obtained from (10.205) by the substitution (10.203). Even a beginner in Mathematics can realize that the factiously introduced parameter ν in (10.204) and (10.206) can have no significant meaning, since it can be eliminated again by the use of the inverse transformation of (10.203). Thus Yehia's idea to generalize in this way all the known results in the dynamics of rigid bodies (belonging to Joukovsky, Kowalevski and others) is empty and meaningless".

(1) The first lesson here to learn is that a scientific journal not owned and edited by that research group, could not allow the use of words like "*rubbish, faulty and illiterate*" to describe the publications of a competing author.
(2) The second is that over-confidence caused the whole group to deny or disbelieve scientific achievements of others.
(3) The third lesson is that the review data bases Mathematical Reviews and Zentralblatt sometimes adopt the easy solution: to assign each of the members of a certain scientific institution to review the publications of other members, and thus allowing less probability of fair reviews and objective evaluations. In our case, each of the members of the IAMM (Institute of Applied Mathematics and Mechanics) reviewed other members' works. The circle is thus closed: The Authors are the Editors of the Journal (Mekh. Tverd. Tela) and reviewers of their articles and, at last, the reviewers of their publications for the MR and Zbl bases.[6]

10.20.2 The Attitude to the Equations of Motion in the Form (10.45)

In 2001, Kharlamov P.V., Mozalevskaya G.V. and Lesina M.E. published the paper [203], in which the equations of motion of a body in a liquid are observed to be written in four different forms, as per the choice of the principal variables in the equations. The first is the classical Tomson-Lamb Eq. (10.16) using the variables $\boldsymbol{\omega}, \mathbf{u}$. The second is (10.23) using $\mathbf{M}, \mathbf{p}$. In fact, they use a slightly modified form due to Kharlamov [192], praising this form as being chosen by Chaplygin and Kharlamov in their research and giving it the term "principal (main) representation" of the equations of motion. The third form uses $\boldsymbol{\omega}$ and $\mathbf{p}(\gamma)$, which are in fact our equations presented in Sect. 10.4 and deduced originally in 1986 [383], but they are not presented in [203] in full form. The fourth form uses $\mathbf{M}, \mathbf{u}$ and is termed as the worst choice. As the authors tried to give references and names for the first two forms, they pass by the third form of the equations without giving any references nor referring to any

[6] In fact, articles were rejected from publication in the Russian journal PMM J. Appl. Math. Mech. (See [200]). Namely, this rejection evoked the publication of the whole series of papers in "Mekh. Tverd. Tela".

authority in the field. In particular, our 1986 paper [383], which is most relevant to this context was not mentioned nor cited in [203].

It may be interesting in this context to recall the next quotation from the Zentralblatt review (Zbl 1025.70007) concerning the above paper [203] and just repeating all the claims advanced in that paper:

"*It is noticed, that the objective factors, that characterize the given mechanical object, should be separated from the subjective factors, brought by the investigator into the mathematical model of this object. In this connection it is shown, that the equations used by H.M. Yehia in some of his papers are not new, but they are partial cases of Kirchhoff's equations. It is also noticed that some of the generalizations of known integrable cases given by H.M. Yehia are not new too, but they can be obtained from the initial integrable cases by coordinate transformation. To avoid such mistakes the authors suggest that all results in this area should be compared with the corresponding results for the main form*[7] *of Kirchhoff's equations.*"

Reviewer: Boris Ivanovich Konosevich (Donetsk)

It is notable here that the reviewer and the authors of the article [203] are members of the same institute.

10.21 Exercises

(1) A solid of revolution moves through a liquid and its kinetic energy T is given by

$$T = \frac{1}{2}[A(p^2 + q^2) + Cr^2 + A'(u_1^2 + u_2^2) + C'u_3^2].$$

Prove that the steady motion given by

$$p = q = 0, r = \Omega, u_1 = u_2 = 0, u_3 = v$$

is stable in the linear approximation, provided

$$\Omega^2 = 4v^2 \frac{AC'(A' - C')}{A'C^2}.$$

[Lamb]

(2) Show that in the classical problem of motion of a heavy rigid body fixed from on point the permanent rotations around a tilted axis (Sect. 10.8) is possible, only when the body is fixed from its centre of mass, and the axis of rotation is a principal axis of inertia of the body at the fixed point.

[7] The "main form" means the second representation, i.e. the one used by Kharlamov (See the last paragraph).

(3) A body fixed from its centre of mass moves under the action of forces with potential $V = \frac{1}{2}\sum J_{ij}\gamma_i\gamma_j$, γ_i are the direction cosines of a certain line fixed in space. Show that a uniform rotation of the body about an axis inclined to that line is possible only when the axis of rotation is a common principal axis of the matrices **J**, **I** and this axis takes a horizontal position.

(4) A body bounded by a simply connected surface is moving in an ideal incompressible fluid, infinitely extending in all directions and at rest at infinity. The equations of motion have the form (10.45), with $\boldsymbol{\kappa} = \mathbf{a} = \mathbf{0}$. Show that uniform rotation about the z-axis of the body is possible only if the three matrices have the form:

$$\mathbf{J} = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{11} & 0 \\ 0 & 0 & J_{33} \end{bmatrix},$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{11} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix},$$

$$\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & \frac{K_{11}}{2(J_{33}-J_{11})}K_{13} \\ I_{12} & I_{22} & \frac{K_{11}}{2(J_{33}-J_{11})}K_{23} \\ \frac{K_{11}}{2(J_{33}-J_{11})}K_{13} & \frac{K_{11}}{2(J_{33}-J_{11})}K_{23} & I_{33} \end{bmatrix},$$

provided $J_{33} \neq J_{11}$, $K_{11} \neq 0$, the angle θ_0 is chosen arbitrarily and the angular speed of rotation

$$\Omega = -\frac{2(J_{33} - J_{11})\cos\theta_0}{K_{11}}.$$

(5) Consider the critical cases of exercise 4: $K_{11} = 0$, $J_{33} \neq J_{11}$ and $K_{11} = 0$, $J_{33} = J_{11}$.

(6) Show that the consecutive application of two transformations with parameters n_1, n_2 is equivalent to the application of one transformation with the parameter $n_1 + n_2$.

(7) In Sect. 10.14.4, when $\bar{K}_{11} \neq 0$ use Eq. (10.160) to show that the relation between the rotation angle φ and time is determined from the equation

$$t = -\bar{K}_{11}\int \frac{d\varphi}{J_{13}\sin\varphi + J_{23}\cos\varphi + a_3} \tag{10.207}$$

under the condition that the parameters of the body are given by

$$\mathbf{a} = a_3(\frac{-CJ_{13}}{K_{11}^2}, \frac{-CJ_{23}}{K_{11}^2}, 1), \boldsymbol{\kappa} = (0, 0, \kappa_3),$$

$$\mathbf{J} = \begin{pmatrix} J_{11} & -\frac{CJ_{13}J_{23}}{K_{11}^2} & J_{13} \\ -\frac{CJ_{13}J_{23}}{K_{11}^2} & J_{11} + \frac{C}{K_{11}^2}(J_{13}^2 - J_{23}^2) & J_{23} \\ J_{13} & J_{23} & J_{33} \end{pmatrix},$$

$$\bar{\mathbf{K}} = \begin{pmatrix} \bar{K}_{11} & 0 & \bar{K}_{13} \\ 0 & \bar{K}_{11} & \bar{K}_{23} \\ \bar{K}_{13} & \bar{K}_{23} & \bar{K}_{33} \end{pmatrix}. \tag{10.208}$$

Show that in contrast to the case of pendulum motion, this motion has a definite energy value, which depends on the parameters of the body

$$E = \frac{1}{2}[J_{11} + \frac{C}{K_{11}^2}(J_{13}^2 + a_3^2)]. \tag{10.209}$$

(8) Starting from the Lagrangian (10.49) of the generalized problem (10.45) (the Routhian of the problem of motion of a body in liquid after ignoring the cyclic translational coordinates):

(a) Ignore the angle of precession retaining the Poisson variables (the components of $\boldsymbol{\gamma}$) as redundant configurational variables.
(b) Apply Hamilton's principle in the form of Jacobi to the reduced time-irreversible Routhian system. Equations of motion are deduced from a variational problem of the type $\delta \int R dt = 0$. Applying Maupertuis' principle to eliminate the time differential from the variational problem.
(c) Use γ_1 as the independent variable and obtain the following second-order differential in γ_3, to which the equations of motion of the body in a liquid are reduced on the integral level $\{I_1 = h, I_2 = f\}$ [384]:

$$\begin{aligned} & D(1 - \gamma_1^2 - \gamma_3^2)\gamma_3'' + C\gamma_3(1 - \gamma_3^2) \\ & - \gamma_1[A - (A + 2C)\gamma_3^2]\gamma_3' + \gamma_3[C - (C + 2A)\gamma_1^2]\gamma_3'^2 \\ & - A\gamma_1(1 - \gamma_1^2)\gamma_3'^3 \\ & - \frac{\rho}{ABCD}\{C\gamma_3[(A - B)(A + B - C)\gamma_1^2 + B(B - C)(1 - \gamma_3^2)] \\ & \qquad + A\gamma_1[(B - C)(B + C - A)\gamma_3^2 + B(A - B)(1 - \gamma_1^2)]\gamma_3'\} \\ & + \frac{\rho}{2ABC(h - V_1)}[\frac{\partial V_1}{\partial \gamma_3}(\lambda + \mu\gamma_3') - \frac{\partial V_1}{\partial \gamma_1}(\mu + \nu\gamma_3')] \\ & + \frac{\rho^{3/2}}{ABC\sqrt{aD^3(h - V_1)}} \\ & \times \{f[(A - B)(A + B - C)\gamma_1^2 - B(A - B + C) + (C - B)(B + C - A)\gamma_3^2] \\ & \quad + \Lambda\} \\ & = 0, \end{aligned} \tag{10.210}$$

where

$$\rho = \lambda + 2\mu\gamma_3' + \nu\gamma_3'^2,$$
$$\lambda = C[B(1-\gamma_3^2) + (A-B)\gamma_1^2],$$
$$\mu = AC\gamma_1\gamma_3$$
$$\nu = A[B(1-\gamma_1^2) + (C-B)\gamma_3^2].$$

and

$$V_1 = \mathbf{a}\cdot\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma + \frac{1}{2D}[f - \boldsymbol{\kappa}\cdot\gamma + \frac{1}{2}\mathbf{K}\cdot\gamma]^2,$$
$$\Lambda = D\boldsymbol{\kappa}\cdot\gamma\mathbf{I} + \boldsymbol{\kappa}\cdot\gamma[D\,\mathrm{tr}(\mathbf{I}) - 2|\gamma\boldsymbol{I}|^2]$$
$$+ |\gamma\mathbf{I}|^2\gamma\mathbf{K}\cdot\gamma + D[\mathrm{tr}(\mathbf{K})D - \gamma\mathbf{IK}\cdot\gamma - \mathrm{tr}(\mathbf{I})\gamma\mathbf{K}\cdot\gamma].$$

(9) Under conditions (10.163) a solution of the orbital equation in the previous exercise is possible in the form $\gamma_3 = 0$.

[This characterizes the precessing pendulum motion, including the pendulum motion about a fixed axis.]

(10) Let the particle of unit mass and unit electric charge moving on the fixed smooth ellipsoid

$$Ax^2 + By^2 + Cz^2 = 1$$

be acted upon by forces with potential

$$V = \frac{1}{2}[\frac{k}{A^2x^2 + B^2y^2 + C^2z^2} + \frac{J^2}{(A^2x^2 + B^2y^2 + C^2z^2)^2}]$$

where k, J are constants, and effective magnetic field H whose component H_n orthogonal to the surface is given by

$$H_n = J\frac{[A^2(B+C-A)x^2 + B^2(C+A-B)y^2 + C^2(A+B-C)z^2]}{[A^2x^2 + B^2y^2 + C^2z^2]^{5/2}}.$$

Show that this system admits in addition to Jacobi's integral, the quadratic integral

$$I = (A^2x^2 + B^2y^2 + C^2z^2)(A\dot{x}^2 + B\dot{y}^2 + C\dot{z}^2) - k\frac{A^3x^2 + B^3y^2 + C^3z^2}{A^2x^2 + B^2y^2 + C^2z^2}$$
$$+ 2J\frac{[BC(B-C)yz\dot{x} + CA(C-A)zx\dot{y} + AB(A-B)xy\dot{z}]}{A^2x^2 + B^2y^2 + C^2z^2}$$
$$+ J^2\frac{[A^2(B+C-A)x^2 + B^2(C+A-B)y^2 + C^2(A+B-C)z^2]}{(A^2x^2 + B^2y^2 + C^2z^2)^2},$$

and is consequently integrable.

[Use the Lagrangian (10.53) with the choice

$$V = \frac{1}{2}b(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),$$
$$\mathbf{l} = n(A\gamma_1, B\gamma_2, C\gamma_3),$$

which characterize Clebsch's case of tri-axial body (Case 2 of Table 10.1). After Routhian reduction by the cyclic variable ψ perform Minkowsky change of variables. For detailed solution see [412].

11. A pendulum of unit length whose bulb has unit mass and carries a unit electric charge is moving under the influence of forces whose potential is $V(\mathbf{r})$ and a magnetic field $\mathbf{H}(\mathbf{r})$. Show that the equations of motion on the unit sphere can be written in the form [404]:

$$\mathbf{r}\times\ddot{\mathbf{r}} = H_r\dot{\mathbf{r}} - \mathbf{r}\times\frac{\partial V}{\partial \mathbf{r}}, \tag{10.211}$$

$\mathbf{r} = (x, y, z)$ is the position vector of the bulb, $H_r = \mathbf{H}\cdot\mathbf{r}$ is the radial component of the magnetic field. The motion is completely determined by the two scalar functions V and H_r. The two cases of motion of a dynamically spherical body in a liquid generate the following two cases of motion of a particle on the sphere:

(1) The case corresponding to Clebsch's case.

It is characterized by the pair of functions

$$\begin{aligned} V &= ax^2 + by^2 + cz^2, \\ H_r &= f. \end{aligned} \tag{10.212}$$

The second integral of motion for this case can be obtained from Clebsch's integral substiuting $\boldsymbol{\omega}\rightarrow f\mathbf{r} - \mathbf{r}\times\dot{\mathbf{r}}$ (compare to (2.33)).

$$\begin{aligned} I = {} & a(y\dot{z} - z\dot{y} - fx)^2 + b(z\dot{x} - x\dot{z} - fy)^2 + c(x\dot{y} - y\dot{x} - fz)^2 \\ & - (bcx^2 + cay^2 + abz^2). \end{aligned} \tag{10.213}$$

This case is a non-separable generalization of the well-known separable Neumann integrable problem [294] by the presence of the gyroscopic forces and reduces to it when $f = 0$.

(2) The case corresponding to the Rubanovsky–Lyapunov case

$$V = s_1 x + s_2 y + s_3 z - \frac{abc}{2}(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c})$$
$$+ \frac{1}{8}[2f + (b+c)x^2 + (c+a)y^2 + (a+b)z^2]^2,$$
$$H_r = f + \frac{1}{2}[a + b + c - 3(ax^2 + by^2 + cz^2)]. \qquad (10.214)$$

The second integral of motion is

$$I = (b+c)(y\dot{z} - z\dot{y} - Nx)^2 + (c+a)(z\dot{x} - x\dot{z} - Ny)^2$$
$$+ (a+b)(x\dot{y} - y\dot{x} - Nz)^2 + s_1[(N+a)x + z\dot{y} - y\dot{z}]$$
$$+ s_2[(N+b)y + x\dot{z} - z\dot{x}] + s_3[(N+c)z + y\dot{x} - x\dot{y}]$$
$$- (bcx^2 + cay^2 + abz^2) \qquad (10.215)$$

where $N = f + \frac{1}{2}[(b+c)x^2 + (c+a)y^2 + (a+b)z^2]$.

Chapter 11
The General Problem of Motion of a Rigid Body Acted upon by a Coaxial Combination of Potential and Gyroscopic Forces

11.1 Introduction

In the last chapter, we have seen that the problem of motion of a body in a liquid or, more precisely, the alternative problem of motion of a body about a fixed point, while acted by magnetic, electric and Lorentz forces, lies on the top of a hierarchy of problems, each of which generalizes the one below it. In this chapter, we extend this hierarchy upwards, by allowing general axi-symmetric potential and gyroscopic forces to act on the body. The fact that problems on that level of complication were not treated in the literature in no way means that such problems have little physical significance. A natural reason is that the grave theoretical difficulties met in as simple as the classical problem gave the impression that difficulties will grow with the degree of complication of forces applied to the body. Fortunately, it turned out that certain symmetries grow with the complication, opening wide chances to achieve far-reaching results. In fact, one can go along the line of thinking that led to the precession transformation in the last chapter, but this time replacing the constant precession speed n with a function $\nu(\gamma)$. Under different circumstances, this type of transformation keeps the equations of motion of the new problem form-invariant, leading to construct new integrable/solvable cases from all known cases of the previous chapters. To this end in this chapter, we shall use two different types of transformations which can be applied to all the known integrable cases to generate from them new ones of the most complicated structure ever seen, while preserving integrability either general or restricted to a certain level of the areas integral. Some of the new cases can be given definite and non-trivial physical interpretation. In this respect a word of warning is due. As stressed in previous chapters, we are dealing with physical models, which have their obvious limitations. Both relativistic effects and the radiation from accelerated electric charges are permanently neglected.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_11

11.2 Equations of Motion

Now we assume a rigid body moving about a fixed point, while subject to conservative (time-independent) potential and gyroscopic forces of the most general form with a common axis of symmetry OZ fixed in space and passing through the fixed point O of the body. The Lagrangian has the form

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\omega + \boldsymbol{l}.\omega - V, \tag{11.1}$$

in which $V = V(\gamma_1, \gamma_2, \gamma_3), \boldsymbol{l} = \boldsymbol{l}(\gamma_1, \gamma_2, \gamma_3)$. The precession angle ψ is a cyclic coordinate. The corresponding cyclic integral is

$$\frac{\partial L}{\partial \dot{\psi}} = \frac{\partial L}{\partial \boldsymbol{\omega}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{\psi}} = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l}) \cdot \gamma = f. \tag{11.2}$$

To write down the dynamical (Euler-like) equations of motion in the body System, we first deduce the equation corresponding to the angle φ (the proper rotation angle around the z-axis fixed in the body):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0.$$

That is

$$\frac{d}{dt}(Cr + l_3) - [(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\frac{\partial \boldsymbol{\omega}}{\partial \varphi} + \boldsymbol{\omega}\cdot\frac{\partial \boldsymbol{l}}{\partial \varphi} - \frac{\partial V}{\partial \varphi}] = 0,$$

and after expressing derivatives w.r.t. φ in terms of derivatives w.r.t. $\boldsymbol{\gamma}$, it can be written as

$$C\dot{r} + (B - A)pq + p[\frac{\partial(l \cdot \gamma)}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \boldsymbol{\gamma}} \cdot \boldsymbol{l}] - q[\frac{\partial(\boldsymbol{l} \cdot \boldsymbol{\gamma})}{\partial \gamma_1} - \gamma_1 \frac{\partial}{\partial \boldsymbol{\gamma}} \cdot \boldsymbol{l}]$$

$$-(\gamma_1 \frac{\partial V}{\partial \gamma_2} - \gamma_2 \frac{\partial V}{\partial \gamma_1}) = 0.$$

The last equation can be given the form

$$\mathbf{k} \cdot \{\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times[\boldsymbol{\omega}\mathbf{I} + \frac{\partial(\mathbf{l} \cdot \gamma)}{\partial \gamma} - (\frac{\partial}{\partial \gamma} \cdot \boldsymbol{l})\gamma] - \gamma \times \frac{\partial V}{\partial \boldsymbol{\gamma}}\} = 0.$$

Now, we note that nothing in the curly bracket depends on the unit vector **k** figuring before that bracket. This vector can be replaced in the last equation by any of the other two unit vectors **i** and **j**. Thus, we can write the dynamical equation in the final vector form

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I}+\boldsymbol{\mu}) = \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}}, \tag{11.3}$$

where

$$\begin{aligned} \boldsymbol{\mu} &= \boldsymbol{l}+(\boldsymbol{\gamma}\times\frac{\partial}{\partial \boldsymbol{\gamma}}) \times \boldsymbol{l} \\ &\equiv \frac{\boldsymbol{\partial}}{\boldsymbol{\partial\gamma}}(\boldsymbol{l}\cdot\boldsymbol{\gamma}) - (\frac{\boldsymbol{\partial}}{\boldsymbol{\partial\gamma}}\cdot\boldsymbol{l})\boldsymbol{\gamma}. \end{aligned} \tag{11.4}$$

Equation (11.3) and Poisson's equation constitute the system

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\mu}) &= \boldsymbol{\gamma}\times\frac{\partial V}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} &= \mathbf{0}, \end{aligned} \tag{11.5}$$

of six first-order equations in 6 unknowns, which generalizes the equations of motion in all problems considered in the previous chapters. We shall refer to V and $\boldsymbol{l}$ as the scalar and vector potentials, respectively, and to $\boldsymbol{\mu}$ as the gyroscopic vector.

It is easy to check that the system (11.5) satisfies Jacobi's condition for the last integrating multiplier

$$\frac{\partial \dot{\boldsymbol{\omega}}}{\partial \boldsymbol{\omega}}+\frac{\partial \dot{\boldsymbol{\gamma}}}{\partial \boldsymbol{\gamma}} \equiv 0.$$

Hence, for its integration one needs a single additional integral of motion I_4 besides the three general integrals, which we write in the form

$$\begin{aligned} I_1 &\equiv \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + V = h, \\ I_2 &= \boldsymbol{\gamma}^2 = 1, \\ I_3 &= (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\boldsymbol{\gamma} = f. \end{aligned} \tag{11.6}$$

Those are the energy integral or, more precisely, Jacobi's integral, the geometric integral and the cyclic integral corresponding to the coordinate ψ. The last can be found as

$$I_3 = \frac{\partial L}{\partial \dot{\psi}} = \frac{\partial L}{\partial \boldsymbol{\omega}}\cdot\frac{\partial \boldsymbol{\omega}}{\partial \dot{\psi}} = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\boldsymbol{\gamma}.$$

The solution of the system (11.5) determines only $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ as functions of t. This completely determines only the angles $\theta = \cos^{-1}\gamma_3$ and $\varphi = \tan^{-1}\frac{\gamma_1}{\gamma_2}$. To obtain ψ, one has to use the cyclic integral (11.2) together with formulas of Chap. 2 to express $\dot{\psi}$ in the form

$$\dot{\psi} = \frac{1}{\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}}[f - \boldsymbol{l}\cdot\boldsymbol{\gamma} - \frac{(A-B)\gamma_1\gamma_2\dot{\gamma}_3 - C(\gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2)}{1-\gamma_3^2}]. \tag{11.7}$$

The angle of precession is found by integrating this relation with respect to time, and this completes the solution.

Remark: It must be noted here that the gyroscopic vector $\boldsymbol{\mu}$, which enters the equations of motion (11.5), is unique for any physical problem, but the vector potential $\boldsymbol{l}$ is not. In fact, as was noted before in Chap. 10, a term of the type

$$-\frac{d\chi(\gamma)}{dt} = -\frac{d\chi}{d\gamma} \cdot \frac{d\gamma}{dt} = \boldsymbol{\omega}\cdot(\gamma\times\frac{d\chi}{d\gamma})$$

can be added to the Lagrangian without changing the equations of motion. Thus, the vector $\boldsymbol{l}$ can be determined only up to a term of the form

$$\boldsymbol{l}_0 = \gamma\times\frac{d\chi}{d\gamma}, \tag{11.8}$$

in which χ is an arbitrary function of γ.

11.3 Relation to Grioli's and Kharlamov's Equations

11.3.1 Grioli's Equations

Grioli [139] considered the system of equations of motion

$$\begin{aligned} \dot{\omega}\mathbf{I} + \omega\times[\omega I + m(\omega, \gamma)] &= \gamma\times\frac{\partial V(\gamma)}{\partial\gamma}, \\ \dot{\gamma} + \omega\times\gamma &= \mathbf{0}, \end{aligned} \tag{11.9}$$

as a generalization of the classical problems of motion of a rigid body about a fixed point including a general potential function $V(\gamma)$ and a general gyroscopic term $\mathbf{m}(\omega, \gamma)$. He answered the question: for which $\mathbf{m}$ does this system admit an areas integral? In fact, one can use (11.9) to deduce the relation

$$\frac{d}{dt}(\omega\mathbf{I}\cdot\gamma) + \mathbf{m}\cdot\dot{\gamma} = \mathbf{0}. \tag{11.10}$$

If $\mathbf{m}$ is expressible in the form

$$\mathbf{m} = \frac{\partial F(\gamma)}{\partial\gamma} + \Phi(\boldsymbol{\omega}, \gamma)\gamma, \tag{11.11}$$

where F and Φ are scalar functions of their arguments, then (11.9) admits the areas integral

$$\omega\mathbf{I}\cdot\gamma + F(\gamma) = f \text{ (arbitrary constant).} \tag{11.12}$$

Grioli did not study further the system (11.9) with $\mathbf{m}$ as in (11.11). Although such system preserves the sum of kinetic and potential energies, it cannot be presented in Lagrangian or Hamiltonian form. The powerful techniques of analytical dynamics are inapplicable to that system.

11.3.2 M. Kharlamov's Equations

M. Kharlamov considered the same question of existence of areas integral, but demanded that the system (11.9) had Lagrangian structure [173]. He was led to the same form of the gyroscopic function as (11.11), but with velocity-independent Φ, so that

$$\boldsymbol{\mu}=\frac{\partial F(\gamma)}{\partial \gamma}+\Phi(\gamma)\gamma, \tag{11.13}$$

where F, Φ are two arbitrary functions of γ.

We now prove that gyroscopic terms in the equations of motion (11.5) can be determined in two equivalent ways, either by giving the vector $\boldsymbol{l}(\gamma)$ or the pair of scalar functions F and Φ :

(1) Let $\boldsymbol{l}(\gamma)$ be given, then

$$F=\boldsymbol{l}\cdot\gamma, \tag{11.14a}$$

$$\Phi=\frac{\partial}{\partial\gamma}\cdot\boldsymbol{l}. \tag{11.14b}$$

Note that a gauge-term vector $\boldsymbol{l}_0$ in the form (11.8) gives no contribution to any of those functions.

(2) Let $\boldsymbol{l}(\gamma)$, $\boldsymbol{l}'(\gamma)$ be two solutions of (11.14a) and (11.14b) for given F and Φ. The difference

$$\boldsymbol{\lambda}=\boldsymbol{l}'-\boldsymbol{l} \tag{11.15}$$

satisfies the equations

$$\boldsymbol{\lambda}\cdot\gamma=\mathbf{0},\frac{\partial}{\partial\gamma}\cdot\boldsymbol{\lambda}=0.$$

The general solution of the first equation is

$$\boldsymbol{\lambda}=\gamma\times\mathbf{s}(\gamma), \tag{11.16}$$

and inserting this into the second equation we get

$$\frac{\partial}{\partial\gamma}\cdot(\gamma\times\mathbf{s}(\gamma))=-\gamma\cdot(\frac{\partial}{\partial\gamma}\times\mathbf{s})=0.$$

This is a single under-determined linear partial differential equation in the three components of $\mathbf{s}$. Its solution involving two arbitrary functions χ and N is

$$\mathbf{s} = \frac{\partial \chi}{\partial \gamma} + N(\gamma)\boldsymbol{\gamma}. \tag{11.17}$$

Inserting this expression into (11.16) and using (11.15), we can write

$$\begin{aligned} \boldsymbol{l}' &= \boldsymbol{l} + \gamma \times [\frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{\gamma}} + N(\gamma)\boldsymbol{\gamma}] \\ &= \boldsymbol{l} + \gamma \times \frac{\partial \boldsymbol{\chi}}{\partial \gamma}. \end{aligned} \tag{11.18}$$

Thus, replacing $\boldsymbol{l}$ by $\boldsymbol{l}'$ in the Lagrangian (11.1) adds to L a term of the form

$$\boldsymbol{\omega} \cdot (\gamma \times \frac{\partial \boldsymbol{\chi}}{\partial \gamma}) = \frac{\partial \chi}{\partial \gamma} \cdot (\boldsymbol{\omega} \times \gamma) = -\frac{d\chi}{dt},$$

which is a nugatory term, having no contribution to the equations of motion. Kharlamov's form (11.13) for the vector $\boldsymbol{\mu}$ is equivalent to our form (11.4).

11.4 Potential of, and Torques on, a Heavy, Magnetized and Electrically Charged Body

The model of an absolutely rigid body as such is a purely mathematical model. All ordinary materials suffer deformation under stresses applied to them. Nevertheless, this model has proved practical, useful and comfortable in the study of a wide spectrum of physical and mechanical problems. In this section, we formulate the equations of motion of a rigid body about a fixed point in a much wider physical setting, taking into account classical interactions, all at a time. In addition to its mass distribution acted upon by gravitational forces, assume that the body has some magnetized parts and carries some electric charges. The body is also subject to electric and magnetic fields.

The picture to be drawn here for the rigid body and physical effects on it should not be taken as literally describing a real body with usual properties as electric insulation or conductivity, magnetic permeability or other properties that change its physical characteristics when its orientation changes under the action of external fields. Our aim here is to construct a mathematical model that would lead to tractable equations of motion of the rigid body in the presence of all the classical physical interactions. To this end, we make some necessary simplifying assumptions:

1- The main part of the body (the carrier body), which is fixed from the origin O, has neither electrical nor magnetic properties, so that it does not interfere with the interaction between the external fields and the magnets and electric charges carried by the body.

2- The physical characteristics of the rigid body are constant in it. They do not change with time, with the change of the body's orientation in space, nor with the change of internal forces in the body. Thus, the body may carry a distribution of immovable electric charges and some permanently magnetized parts, also fixed in it. Magnetization of the body can also arise due to the presence of steady electric circuits in the body. An electric motor whose axis is fixed in the body generates in its working mode a magnetic moment due to electric current in its coil, equivalent to a permanent magnet, and a constant gyrostatic moment due to the steady rotation of the coil.

3- It is well known that, according to the laws of classical physics, an accelerated electric charge emits electromagnetic radiation. This was established by Larmor [254] in 1897 (see also [161]). The total energy of motion of the body decreases with time. The maximum acceleration attained by a point of the body will be assumed small enough to justify neglecting this effect.

Under those conditions, the following effects on the body will be taken into account:

(1) A torque arises due to the gravitational field $\mathbf{g}$ of a certain distribution of gravitating sources, fixed in the inertial system of axes $OXYZ$, O being the fixed point of the body. Gravitational forces are derivable from a scalar potential $V_g(X, Y, Z)$ by the relation $\mathbf{g} = -\nabla V_g$. The gravitational potential is harmonic, i.e. satisfies Laplace's equation in the inertial coordinate system outside gravitating sources. The potential of the body, due to the gravitational field, has the form

$$V_G = \int V_g(X, Y, Z)dm,$$

where dm is the mass element at the point $\mathbf{r}(X, Y, Z)$ of the body and integration is extended on the space domain occupied by the body. Referring to the system of axes $Oxyz$ fixed in the body, we have $\mathbf{r} = (x, y, z)$ and hence the potential can be written as

$$V_G = \int V_g(\mathbf{r} \cdot \boldsymbol{\alpha}, \mathbf{r} \cdot \boldsymbol{\beta}, \mathbf{r} \cdot \boldsymbol{\gamma})dm, \tag{11.19}$$

$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ being the basic unit vectors in the inertial space.

(2) The external electric field $\mathbf{E}$, derived from the potential V_e by the relation $\mathbf{E} = -\nabla V_e$, acts on the electric charges on the body in a similar way. The electric potential of the body is

$$V_E = \int V_e(\mathbf{r} \cdot \boldsymbol{\alpha}, \mathbf{r} \cdot \boldsymbol{\beta}, \mathbf{r} \cdot \boldsymbol{\gamma})de. \tag{11.20}$$

(3) The external magnetic field $\mathbf{H} = -\nabla V_m$ acts on the magnetized parts of the body. Note that we use the magnetic field $\mathbf{H}$ rather than the magnetic induction $\mathbf{B}$,

since the body is considered as having unit permeability. Also, for simplicity, we consider the magnetized part of the body as composed of a set of short magnets (dipoles). If $\boldsymbol{m}_i$ is the magnetic dipole moment at the point $\mathbf{r}_i(X, Y, Z)$, the potential of the body due to the scalar magnetic interaction is

$$\begin{aligned} V_M &= \Sigma \boldsymbol{m}_\mathbf{i} \cdot \nabla V_m(\mathbf{r}_i \cdot \boldsymbol{\alpha}, \mathbf{r}_i \cdot \boldsymbol{\beta}, \mathbf{r}_i \cdot \boldsymbol{\gamma}) \\ &= -\Sigma \boldsymbol{m}_\mathbf{i} \cdot \mathbf{H}(\mathbf{r}_i \cdot \boldsymbol{\alpha}, \mathbf{r}_i \cdot \boldsymbol{\beta}, \mathbf{r}_i \cdot \boldsymbol{\gamma}). \end{aligned} \tag{11.21}$$

(4) The external magnetic field also exerts the velocity-dependent Lorentz forces on the electric charge distribution in the body. The moment of those forces about the origin is[1]

$$\mathbf{M}_H = \int \mathbf{r} \times [(de\frac{d\mathbf{r}}{dt}) \times \mathbf{H}],$$

where the velocity of the point of the body in space $\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$. We have

$$\begin{aligned} \mathbf{M}_H &= \int \mathbf{r} \times [(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{H}]de \\ &= \int (\mathbf{r} \cdot \mathbf{H})\boldsymbol{\omega} \times \mathbf{r}de \\ &= \boldsymbol{\omega} \times \int (\mathbf{r} \cdot \mathbf{H})\mathbf{r}de. \end{aligned} \tag{11.22}$$

This means that the vector $\boldsymbol{\mu}$ in the equations of motion (11.3) may be written in the form

$$\boldsymbol{\mu} = \boldsymbol{\kappa} - \int (\mathbf{r} \cdot \mathbf{H})\mathbf{r}de. \tag{11.23}$$

For certain purposes, e.g. to construct a Lagrangian for the problem of motion, the magnetic field can also be derived from a vector potential $\mathcal{A}$, which is also assumed time-independent, according to the formula $\mathbf{H} = \nabla \times \mathcal{A}$. The vector potential $\boldsymbol{l}$ of the body may be written as

$$\boldsymbol{l} = \boldsymbol{\kappa} + \int \mathbf{r} \times \mathcal{A}de, \tag{11.24}$$

while $\boldsymbol{\mu}$ can be derived from $\boldsymbol{l}$ according to (11.4).

For the purpose of giving a concrete example, let us consider the following physical situation.

Let the principal body of a gyrostat be carrying a permanent distribution of electric charges and the system be subject to

(1) A uniform magnetic field $\mathbf{H}$ in the Z-direction, i.e. $\mathbf{H} = H\boldsymbol{\gamma}$.

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[1] Here MKS units are used. In Gaussian units de should be divided by the velocity of light c (e.g. [44]).

(2) An electric field whose potential is $a_1 Z + \frac{1}{2}a_2 Z^2$.
(3) A gravitational field with another quadratic potential $b_1 Z + \frac{1}{2}b_2 Z^2$.

It should be noted that those forms of the electric and gravitational potentials appear as a second approximation of the potentials of a general rigid body (or gyrostat) in arbitrary coaxially symmetric electric and gravitational fields, by including the second harmonics. The same applies for the case of fields due to a distant axi-symmetric and symmetrically situated invariable body.

According to (11.23), we write

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{\kappa} - H\int(\mathbf{r}\cdot\gamma)\mathbf{r}de \\ &= \boldsymbol{\kappa} - 2H\gamma\bar{\mathbf{I}}_e,\end{aligned} \tag{11.25}$$

where $\bar{\boldsymbol{I}}_e = \frac{1}{2}(tr\mathbf{I}_e)\delta - \mathbf{I}_e$, $\mathbf{I}_e$ is the inertia matrix of the distributions, and δ is the unit matrix. The corresponding vector potential is

$$\boldsymbol{l} = \boldsymbol{\kappa} + \frac{1}{2}H\gamma\mathbf{I}_e. \tag{11.26}$$

On the other hand, the total potential of the system is (ignoring an insignificant constant)

$$\begin{aligned}V &= \int[a_1\mathbf{r}.\gamma + \frac{1}{2}a_2(\mathbf{r}.\gamma)^2]de \\ &\quad + \int[b_1\mathbf{r}.\gamma + \frac{1}{2}b_2(\mathbf{r}.\gamma)^2]dM \\ &= \mathbf{a}.\gamma + \frac{1}{2}\gamma\mathbf{J}\cdot\gamma\end{aligned} \tag{11.27}$$

where $\mathbf{J} = -a_2\mathbf{I}_e - b_2\mathbf{I}$, $\mathbf{a} = a_1\int\mathbf{r}de + b_1 M\mathbf{r}_c$, M the mass of the system and $\mathbf{r}_c$ its centre of mass. As seen in Chap. 10, formulas (11.25)–(11.27) characterize the problem of motion of a body in a liquid.

The effect of Lorentz forces on the motion of a rigid body was considered only in very few works (e.g. [22, 139, 140, 378, 382]). In a number of more recent works, similar problems were considered, repeating to a great extent previous results as introducing the inertia matrix of electric charges, e.g. [430–433].

Expressions analogous to the above ones can be derived for more complicated forms of the magnetic field. In the case when the scalar potential of the external magnetic field can be expressed as a second-degree harmonic polynomial,

$$V_M = a_1 Z + a_2(3Z^2 - r^2). \tag{11.28}$$

The vector $\boldsymbol{\mu}$ can be expressed as

$$\boldsymbol{\mu} = \int [a_1 \mathbf{r} \cdot \gamma + 2a_2(3(\mathbf{r} \cdot \gamma)^2 - r^2)]\mathbf{r}de. \tag{11.29}$$

In expanded form, one may write

$$\begin{aligned}
\mu_1 = & -2a_2(I_{xxx} + I_{xyy} + I_{xzz}) + a_1(I_{xx}\gamma_1 + I_{xy}\gamma_2 + I_{xz}\gamma_3) \\
& +6a_2(I_{xxx}\gamma_1^2 + I_{xyy}\gamma_2^2 + I_{xzz}\gamma_3^2 + 2I_{xxy}\gamma_1\gamma_2 + 2I_{xxz}\gamma_1\gamma_3 + 2I_{xyz}\gamma_2\gamma_3) \\
\mu_2 = & -2a_2(I_{xxy} + I_{yyy} + I_{yzz}) + a_1(I_{xy}\gamma_1 + I_{yy}\gamma_2 + I_{yz}\gamma_3) \\
& +6a_2(I_{xxy}\gamma_1^2 + I_{yyy}\gamma_2^2 + I_{yzz}\gamma_3^2 + 2I_{xyy}\gamma_1\gamma_2 + 2I_{xyz}\gamma_1\gamma_3 + 2I_{yyz}\gamma_2\gamma_3) \\
\mu_3 = & -2a_2(I_{xxz} + I_{yyz} + I_{zzz}) + a_1(I_{xz}\gamma_1 + I_{yz}\gamma_2 + I_{zz}\gamma_3) \\
& +6a_2(I_{xxz}\gamma_1^2 + I_{yyz}\gamma_2^2 + I_{zzz}\gamma_3^2 + 2I_{xyz}\gamma_1\gamma_2 + 2I_{xzz}\gamma_1\gamma_3 + 2I_{yzz}\gamma_2\gamma_3)
\end{aligned} \tag{11.30}$$

where, for example, $I_{xx} = \int x^2 de$, $I_{xyz} = \int xyzde$ and so forth are the second- and third-degree moments of the charge distribution.

11.5 On General and Conditional Integrable Cases in Rigid Body Dynamics

As explained in previous chapters of this book, we call a problem *general integrable* if I_4 exists for arbitrary initial conditions and *conditional integrable* if it admits a fourth integral I_4 only on a single level f of the cyclic integral I_3 (in many cases $f = 0$) but for all initial conditions compatible with that level. In both types of integrable problems, the solution can be reduced to quadratures through the application of Liouville's theorem or Jacobi's theorem to the reduced two-dimensional Hamiltonian system. It is thus sufficient to point out the fourth integral to ensure integrability in those cases. In some cases, it becomes possible to construct a quantity constant only under other restrictions on the initial state of motion, which do not fit as conditions on the integral level of I_3. Then one cannot apply Liouville's theorem to construct the solution and a procedure for accomplishing this task should be indicated separately. In such cases, we deal with *particular solutions* of the problem.

Equations of motion of the form (11.5) cover a wide range of applications in rigid body dynamics. Special cases are the classical problem of motion of a heavy body, its generalizations to the case of a gyrostat moving under potential and Lorentz forces. We recall that they cover also the Routhian reduction of the problem of motion of a body in a liquid, in which the body has no fixed point. In many cases, Eq. (11.5) with reasonably behaving functions V can be interpreted as characterizing gravitational, electric and magnetic interactions and $\boldsymbol{\mu}$ as the Lorentz force exerted by the magnetic field on some electric charges resting on the body. However, this is not always the case. In some problems that happen to be integrable, such interpretation is not possible, due to the presence of singularities that cannot be exhibited by the potentials of real bodies. Detailed examples of integrable problems of both types will be considered in the next two chapters, Chap. 12 and Chap. 13.

11.6 Transformation of the Equations of Motion

In the preceding chapter, we have applied the transformation $\boldsymbol{\omega} = \boldsymbol{\omega}' + \nu\boldsymbol{\gamma}$, where ν is a constant, to a system of the type (11.5) and its form-invariance is used to generate integrable cases containing ν as a parameter. Here we shall develop this idea, by replacing the constant ν by a function $\nu(\gamma)$. In fact, the substitution

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \nu\boldsymbol{\gamma}, \quad \nu = \nu(\gamma_1, \gamma_2, \gamma_3) \tag{11.31}$$

leaves the invariant form of the Poisson equation in (11.5), transforming it to

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0}, \tag{11.32}$$

while the areas integral in (11.6) takes the form

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l} + \nu\boldsymbol{\gamma}\mathbf{I}) \cdot \boldsymbol{\gamma} = f. \tag{11.33}$$

Substituting in the Eulerian part of the equations of motion, using (11.32) and rearranging terms, we get

$$\begin{aligned} &\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\mu} + 2\nu\boldsymbol{\gamma}\mathbf{I} - \nu(tr\mathbf{I})\boldsymbol{\gamma} + \boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma}\frac{\partial \nu}{\partial \boldsymbol{\gamma}} - (\boldsymbol{\gamma}\mathbf{I} \cdot \frac{\partial \nu}{\partial \boldsymbol{\gamma}})\boldsymbol{\gamma}) \\ &= \boldsymbol{\gamma} \times [\frac{\partial V}{\partial \boldsymbol{\gamma}} - \nu\,\boldsymbol{\mu} - \nu^2\boldsymbol{\gamma}\mathbf{I} + (\boldsymbol{\omega}'\mathbf{I} \cdot \boldsymbol{\gamma})\frac{\partial \nu}{\partial \boldsymbol{\gamma}}]. \end{aligned} \tag{11.34}$$

On the level $I_3 = f$ (say), we substitute $\boldsymbol{\omega}'\mathbf{I} \cdot \boldsymbol{\gamma}$ from (11.33) and after some manipulations write the equations of motion in the final form:

$$\begin{aligned} \dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\mu}') &= \boldsymbol{\gamma} \times \frac{\partial V'}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} &= \mathbf{0}, \end{aligned} \tag{11.35}$$

where

$$\begin{aligned} \boldsymbol{\mu}' &= \boldsymbol{\mu} + \frac{\partial}{\partial \boldsymbol{\gamma}}(\nu\mathbf{I} \cdot \boldsymbol{\gamma}) - [\frac{\partial}{\partial \boldsymbol{\gamma}} \cdot (\nu\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma}, \\ &\equiv \boldsymbol{\mu} - 2\nu\boldsymbol{\gamma}\bar{\boldsymbol{I}} + \boldsymbol{\gamma}\mathbf{I} \times (\frac{\partial \nu}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}) \end{aligned}$$

$$V' = V + \nu(f - \boldsymbol{l}\cdot\gamma) - \frac{1}{2}\nu^2\gamma\mathbf{I}\cdot\gamma, \tag{11.36}$$

and $\bar{\boldsymbol{I}}=\frac{1}{2}tr(\mathbf{I})\boldsymbol{\delta}-\mathbf{I}$. From the first of Eq. (11.36) and comparing with (11.4), we can also write the transformation law for the vector $\boldsymbol{l}$ as

$$\boldsymbol{l}' = \boldsymbol{l}+\nu\gamma\mathbf{I}. \tag{11.37}$$

Thus, the transformation (11.31) preserves the form of the equations of motion on a fixed level of I_3, changing only V, $\boldsymbol{\mu}$ (or $\boldsymbol{l}$) to V', $\boldsymbol{\mu}'$ (or $\boldsymbol{l}'$). The value f of I_3 enters in the potential V' as a parameter. The solution of the transformed equations of motion (11.35) can be obtained from that of (11.5) through the substitution (11.31).

The system of Eq. (11.35) admits the linear integral

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l}')\cdot \gamma = f,$$

equivalent to (11.33), and also the energy (Jacobi's) integral

$$\frac{1}{2}\boldsymbol{\omega}'\mathbf{I} \cdot \boldsymbol{\omega}' + V' = h.$$

On the one hand, the transformed system (11.36) can be viewed as the equations of motion of the original system as in (11.5), as seen by an observer resting in the reference frame moving with the position-dependent angular velocity $\nu(\gamma_1, \gamma_2, \gamma_3)$. The new terms that appeared in the transformed system are the inertial forces due to the rotation of the frame.

On the other hand, there is a different and more constructive way of looking at (11.36). We shall make use of the situation that the transformation preserves the form of the equations of motion to understand the transformed equations on their own as describing the motion of a second body in the inertial frame under the forces determined by V', $\boldsymbol{\mu}'$. In other words, we consider the system (11.36) as formally generalizing (11.5) to which it reduces when $\nu = 0$. However, this will not prevent us from relating the solutions of the two systems by the (formal) transformation (11.31). This duality in interpretation is the key to understanding the present method. From now on, we will mostly regard the system (11.36) as a generalization of (11.5) rather than a transformed form of it.

Remark 17 A curious note may be in place here. In certain cases, it is possible from Eq. (11.36) to choose the function ν so that V' vanishes. This means that in those cases, when the resulting $\nu(\gamma)$ is a real function, the original forces with potential V can be replaced by purely gyroscopic forces in a properly chosen rotating coordinate frame. However, we shall not follow this line, since it seemingly has no practical consequences.

11.7 Maximal Reduction of the Order of the Equations of Motion

The method used in Chap. 9 Sect. 9.2 and an exercise of Chap. 10 can be used here in the most general case of potential and gyroscopic forces to obtain a second-order orbital equation connecting two of the geometric variables γ_i. The Lagrangian of the problem of motion will be taken in the form (11.1), namely

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \boldsymbol{l}\cdot\boldsymbol{\omega} - V, \tag{11.38}$$

in the redundant configurational variables $\psi, \gamma_1, \gamma_2, \gamma_3$, subject to the holonomic condition

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{11.39}$$

The angular velocity may be written as

$$\boldsymbol{\omega} = \dot{\psi}\boldsymbol{\gamma} + \mathbf{N}, \tag{11.40}$$

where

$$\begin{aligned}\mathbf{N} &= \dot{\theta}\mathbf{n} + \dot{\varphi}\mathbf{k}\\ &= -\frac{\dot{\gamma}_3}{\sqrt{1-\gamma_3^2}}(\cos\varphi, -\sin\varphi, 0) + \frac{\gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2}{\gamma_1^2+\gamma_2^2}\mathbf{k}\\ &= \frac{(-\gamma_2\dot{\gamma}_3, \gamma_1\dot{\gamma}_3, \gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2)}{1-\gamma_3^2}.\end{aligned} \tag{11.41}$$

As a result of cyclicity of the Lagrangian in the variable ψ, we have the integral

$$\frac{\partial L}{\partial \dot{\psi}} = \frac{\partial L}{\partial \boldsymbol{\omega}}\cdot\frac{\partial \boldsymbol{\omega}}{\partial \dot{\psi}} = (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{l})\cdot\boldsymbol{\gamma} = f. \tag{11.42}$$

Multiplying (11.40) scalarly by $\boldsymbol{\gamma}\mathbf{I}$ and using (11.42), we obtain

$$\dot{\psi} = \frac{1}{D}(f - \boldsymbol{l}\cdot\boldsymbol{\gamma} - \boldsymbol{\gamma}\mathbf{I}\cdot\mathbf{N}),\ D = \boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}. \tag{11.43}$$

Then, ignoring ψ we construct the Routhian

$$R = \frac{ABC}{2D}\dot{\boldsymbol{\gamma}}\mathbf{I}^{-1}\cdot\dot{\boldsymbol{\gamma}} + \boldsymbol{l}^{*}\cdot\dot{\boldsymbol{\gamma}} - V^{*}, \tag{11.44}$$

where

$$V^* = V(\gamma) + \frac{1}{2D}(f - \boldsymbol{l}\cdot\boldsymbol{\gamma})^2,$$
$$\boldsymbol{l}^* = \frac{1}{D}[\boldsymbol{\gamma}\mathbf{I}\times\boldsymbol{l} + f\frac{\partial}{\partial\dot{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}\mathbf{I}\cdot\mathbf{N})]. \quad (11.45)$$

Just as in the preceding chapters, applying Maupertuis' principle to (11.44) and eliminating γ_2, we arrive at the following second-order differential equation in $\gamma_3(\gamma_1)$, to which the equations of motion of the problem are reduced on the integral level $\{I_1 = h, I_2 = 1, I_3 = f\}$[2] [384]:

$$\begin{aligned}
&D(1-\gamma_1^2-\gamma_3^2)\gamma_3'' + C\gamma_3(1-\gamma_3^2)\\
&-\gamma_1[A-(A+2C)\gamma_3^2]\gamma_3' + \gamma_3[C-(C+2A)\gamma_1^2]\gamma_3'^2\\
&-A\gamma_1(1-\gamma_1^2)\gamma_3'^3\\
&-\frac{\rho}{ABCD}\{C\gamma_3[(A-B)(A+B-C)\gamma_1^2 + B(B-C)(1-\gamma_3^2)]\\
&\qquad +A\gamma_1[(B-C)(B+C-A)\gamma_3^2 + B(A-B)(1-\gamma_1^2)]\gamma_3'\}\\
&+\frac{\rho}{2ABC(h-V^*)}[\frac{\partial V^*}{\partial\gamma_3}(\lambda+\mu\gamma_3') - \frac{\partial V^*}{\partial\gamma_1}(\mu+\nu\gamma_3')]\\
&+\frac{\rho^{3/2}}{ABC\sqrt{aD^3(h-V^*)}}\\
&\times\{f[(A-B)(A+B-C)\gamma_1^2 - B(A-B+C) + (C-B)(B+C-A)\gamma_3^2]\\
&\quad +\Lambda\}\\
=\ &0,
\end{aligned} \quad (11.46)$$

where

$$\begin{aligned}
\rho &= \lambda + 2\mu\gamma_3' + \nu\gamma_3'^2,\\
\lambda_1 &= C[B(1-\gamma_3^2) + (A-B)\gamma_1^2],\\
\lambda_2 &= AC\gamma_1\gamma_3,\\
\lambda_3 &= A[B(1-\gamma_1^2) + (C-B)\gamma_3^2],
\end{aligned} \quad (11.47)$$

and

$$\begin{aligned}
V^* &= V(\boldsymbol{\gamma}) + \frac{1}{2D}(f - \boldsymbol{l}\cdot\boldsymbol{\gamma})^2,\\
\Lambda &= D^2\boldsymbol{\gamma}\cdot[\frac{\partial}{\partial\boldsymbol{\gamma}}\times(\frac{\boldsymbol{l}\times\boldsymbol{\gamma}\mathbf{I}}{D})]\\
&\equiv D^2\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot[\frac{1}{D}\gamma\times(\boldsymbol{\gamma}\mathbf{I}s\times\boldsymbol{l})].
\end{aligned} \quad (11.48)$$

---

[2] The positive sign of the square root in (11.46) corresponds the choice of positive sign of the root in (11.49). If this choice is reversed, Eq. (11.46) is not changed, provided the signs of f and $\boldsymbol{l}$ are reversed. This is a consequence of the invariance of the system (11.5) with respect to the replacement $t, \boldsymbol{\omega}, \boldsymbol{\mu} \rightarrow -t, -\boldsymbol{\omega}, -\boldsymbol{\mu}$.

As should be expected, one can verify that a gauge term $\boldsymbol{l}_0$ (11.8) does not contribute to the two functions V^* and Λ.

Now, having a solution $\gamma_3 = \gamma_3(\gamma_1)$ of the orbital Eq. (11.46), one can obtain the dependence of γ_1 on time by inverting the integral

$$t = \int \sqrt{\frac{\lambda_1 + 2\lambda_2\gamma_3' + \lambda_3\gamma_3'^2}{2D(h - V^*)(1 - \gamma_1^2 - \gamma_3^2)}} d\gamma_1, \tag{11.49}$$

and substituting in γ_3, the last is determined in terms of time and then γ_2 is found from the geometric integral. This completes determination of $\boldsymbol{\gamma}$ and hence the two Eulerian angles θ and φ as functions of t. Thus, we have shown the equivalence of the reduced Eq. (11.46) to the equations of motion (11.5) on the integral level $\{h, f\}$, provided γ_3' and γ_3'' are well defined, i.e. excluded are only trajectories along which γ_1 takes a constant value.

It should be noticed here that the three functions V^*, $\boldsymbol{l}^*$ and Λ, which occur in (11.44) and (11.46), are all invariant with respect to the transformation (11.31). This can be easily verified by replacing the pair $(V, \boldsymbol{l})$ in them by the pair $(V', \boldsymbol{l}')$. This means that, on the integral level $\{h, f\}$, the Routhian (11.44), the orbital Eq. (11.46) and the expressions of $\boldsymbol{\gamma}$, θ and φ do not change by the variable rotation transformation (11.31). We shall use this property later in several situations.

To completely determine the position of the body in space, one has to find an expression for the precession angle ψ by integrating (11.43), which involves the vector potential $\boldsymbol{l}$. Using (11.40), one can express the angular velocity $\boldsymbol{\omega}$ in the form

$$\boldsymbol{\omega} = \frac{1}{D}[\dot{\boldsymbol{\gamma}} \times \boldsymbol{\gamma}\mathbf{I} + (f - \boldsymbol{l} \cdot \boldsymbol{\gamma})\boldsymbol{\gamma}]. \tag{11.50}$$

Not only all the Euler–Poisson variables are thus determined as functions of time, but also the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$. For the transformed system (11.35), regarding (11.37), this process gives

$$\begin{aligned} \dot{\psi}' &= \frac{1}{D}(f - \boldsymbol{l}' \cdot \boldsymbol{\gamma} - \boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{N}) \\ &= \dot{\psi} - \nu, \\ \boldsymbol{\omega}' &= \boldsymbol{\omega} - \nu\boldsymbol{\gamma}, \end{aligned} \tag{11.51}$$

which coincides with (11.31).

11.7.1 The Case of Complete Dynamical Symmetry

For the purpose of future use, we now write down in expanded form the Routhian (11.44) in the special case when the inertia ellipsoid of the body at the fixed point becomes a sphere. Then, from (11.44) we have

$$
\begin{aligned}
R = & \frac{1}{2}A(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2) \\
& -[l_1(\gamma_2\dot{\gamma}_3 - \gamma_3\dot{\gamma}_2) + l_2(\gamma_3\dot{\gamma}_1 - \gamma_1\dot{\gamma}_3) + (l_3 + \frac{f\gamma_3}{\gamma_1^2 + \gamma_2^2})(\gamma_1\dot{\gamma}_2 - \gamma_2\dot{\gamma}_1)] \\
& -[V(\boldsymbol{\gamma}) + \frac{1}{2A}[f - l_1\gamma_1 - l_2\gamma_2 - l_3\gamma_3]^2]. \qquad (11.52)
\end{aligned}
$$

11.8 Extensions of Integrable Problems

As a direct application of the transformed equations, we can readily deduce the following theorems which construct integrable extensions of the known integrable problems and connect the solutions of the generalized systems to those of the original problems.

Theorem 11.1 *Let the system (11.5) with certain $V(\gamma)$ and $\mu(\gamma)$ corresponding to vector potential $l(\gamma)$, be general integrable, for arbitrary initial conditions, with the complementary integral $I_4 = F(\omega, \gamma)$. Then, upon replacing V, μ by*

$$
\begin{aligned}
V' &= V + \nu(b - \boldsymbol{l} \cdot \boldsymbol{\gamma}) - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma}, \\
\boldsymbol{\mu}' &= \boldsymbol{\mu} + \frac{\partial}{\partial \boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\mathbf{I} \cdot \gamma) - [\frac{\partial}{\partial \boldsymbol{\gamma}} \cdot (\nu\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma} \qquad (11.53)
\end{aligned}
$$

where $\nu = \nu(\gamma)$ is an arbitrary function and b a new parameter, the new system is integrable on the level

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l} + \nu\boldsymbol{\gamma}\mathbf{I}) \cdot \boldsymbol{\gamma} = b. \qquad (11.54)$$

This theorem allows one to generate from an unconditional case (integrable for arbitrary initial conditions) a conditional case integrable on a single level of the areas integral I_3, but with additional potential and gyroscopic forces involving an arbitrary function $\nu(\boldsymbol{\gamma})$ and an arbitrary parameter b more than the original integrable problem. To illustrate the feasibility of the generalized problem, one can calculate for it the reduced potential. One gets

$$V^* = V + (b - f)\nu + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2. \qquad (11.55)$$

The extra-parameter b enters in Eq. (11.35), in the equations of motion derived from the Routhian (11.44) as well as in the orbital Eq. (11.46). The extended problem may not be integrable for arbitrary initial conditions. However, on the single level $f = b$, the reduced potential reduces to that of the original problem. The extended problem involves one more physical parameter b and under the dynamical condition $f = b$, it becomes integrable and its solution has the same number of parameters as in the solution of the original problem.

A quick example can be readily given by the simplest extension of Euler's case of the motion of a body under no torques. Let us take $V = 0, \boldsymbol{l} = 0$ and choose $\nu = n + n_1\gamma_1$, so that

$$V' = b\nu - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I}\cdot\gamma, \boldsymbol{l}' = \nu\boldsymbol{\gamma}\mathbf{I}. \tag{11.56}$$

A family of solutions of the transformed problem can be written down generalizing formulas (10.111), (10.112) of Chap. 10 by replacing n by ν in that solution. The resulting solution is valid on the level $f = b$.

Theorem 11.2 *Let the system (11.5) with certain $V(\gamma)$ and $\boldsymbol{\mu}(\gamma)$ corresponding to vector potential $\boldsymbol{l}(\gamma)$, be general integrable (for arbitrary initial conditions). Let also V have the structure*

$$V = V_0 + b_1V_1 + ... + b_kV_k, \tag{11.57}$$

where $V_i, i = 0...k$ and $\boldsymbol{l}$ are functions of $\boldsymbol{\gamma}$ not involving any of the parameters $b_1, ..., b_k$ and the complementary integral be

$$I_4 = F(\boldsymbol{\omega}, \gamma; b_1, ..., b_k). \tag{11.58}$$

Then, upon replacing $V, \boldsymbol{\mu}(\boldsymbol{l})$ by

$$\begin{aligned} V' &= V_0 + b_1V_1 + ... + b_kV_k - \nu\mathbf{l}\cdot\gamma - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I}\cdot\gamma, \\ \boldsymbol{\mu}' &= \boldsymbol{\mu} + \frac{\partial}{\partial\boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\mathbf{I}\cdot\gamma) - [\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot(\boldsymbol{\nu}\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma}, \\ (\boldsymbol{l}' &= \boldsymbol{l} + \nu\boldsymbol{\gamma}\mathbf{I}), \end{aligned} \tag{11.59}$$

where $\nu = n_1V_1 + ... + n_kV_k$ and n_i are new constants, the new system is unconditionally integrable with the areas integral

$$I_3 = (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{l}')\cdot\gamma = f, \tag{11.60}$$

and for it the complementary integral is

$$I_4 = F(\boldsymbol{\omega}' + \nu\gamma, \gamma; b_1 - n_1I_3, ..., b_k - n_kI_3). \tag{11.61}$$

In fact, comparing the reduced potentials for the original problem characterized by the pair $\{V, \boldsymbol{l}\}$ with that of the extended problem characterized by the pair $\{V', \boldsymbol{l}'\}$ in (11.59), we find

$$\boldsymbol{l}'^* = \boldsymbol{l}^*,$$

$$V^* = V_0 + b_1 V_1 + ... + b_k V_k + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2,$$

$$V'^* = V_0 + b_1 V_1 + ... + b_k V_k - f(n_1 V_1 + ... + n_k V_k) + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2$$
$$= V_0 + (b_1 - f n_1) V_1 + ... + (b_k - f n_k) V_k + \frac{1}{2D}(f - \boldsymbol{l} \cdot \gamma)^2. \quad (11.62)$$

The potentials V^*, V'^* in (11.62) are identical in form. The only difference is that each b_i is replaced by $b_i' = b_i - f n_i$, $i = 1, ..., k$, and hence follows integrability and the form of the integral (11.61). The set of solutions of the extended problem is the same as that of the original problem. Notable here is the coupling between the constants which characterize the physical problem, and hence appear in the equations of motion, and a dynamical constant of motion I_3,which appears in the process of integrating those equations. In fact, the phase portrait and phase trajectories of the new integrable problems are different from their original counterparts provided $f \neq 0$.

Theorem 11.2 generates from an unconditional case integrable for arbitrary initial conditions another unconditional case also integrable for arbitrary initial conditions. The new system involves $k + 1$ parameters $n_0, n_1, ..., n_k$ more than the old one and renders to it when one puts $n_0 = n_1 = ... = n_k = 0$. According to the problem setting, the new parameters invoke additional forces in the equations of motion, which can be given concrete physical interpretation.

The presence of I_3 in the expression for I_4 in the transformed problem may lead in certain cases to notable changes in the structure of the integral. For example, we shall see below a case in which the degree of the quadratic I_4 is raised to 3 because of the appearance of I_3 in the coefficients of the quadratic terms.

Theorem 11.3 If $\{\omega = \Omega(t, \omega^\circ, \gamma^\circ), \gamma = \Gamma(t, \omega^\circ, \gamma^\circ)\}$, is the general solution of the first system satisfying the arbitrary initial conditions $\{\omega = \omega^\circ, \quad \gamma = \gamma^\circ\}$, then for arbitrary $\nu(\gamma)$ the solution of the second system, satisfying the initial conditions $\{\omega' = \omega'^\circ, \quad \gamma = \Gamma^\circ\}$, is

$$\{\boldsymbol{\omega}' = \boldsymbol{\Omega}(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\gamma^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ)$$
$$-\nu(\boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\gamma^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ))\boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\gamma^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ),$$
$$\gamma = \boldsymbol{\Gamma}(t, \boldsymbol{\omega}'^\circ + \boldsymbol{\nu}(\gamma^\circ)\boldsymbol{\gamma}^\circ, \boldsymbol{\gamma}^\circ)\}. \quad (11.63)$$

Theorem 11.4 If the first system admits any particular solution $\{\omega = \Omega(t), \quad \gamma = \Gamma(t)\}$, then for arbitrary $\nu(\gamma)$ the second system admits the solution $\{\omega' = \Omega(t) - \nu(\Gamma(t))\Gamma(t), \quad \gamma = \Gamma(t)\}$.

The last theorem follows from the fact that the solution of the second system for the Poisson variables $\boldsymbol{\gamma}$ is not affected by the function $\nu(\boldsymbol{\gamma})$.

In the following chapter, we discuss the consequences of the above theorems in application to known solvable problems of rigid body dynamics. Theorem 11.1 ensures the integrability of the problem (11.35) on the level f of the cyclic integral and for arbitrary $\nu(\gamma)$ whenever the corresponding problem (11.5) is integrable, either for arbitrary initial conditions or only on a fixed level of the cyclic integral. Theorem 11.2 relates the explicit solutions of the two problems. Theorem 11.3 enables the generalization, by means of including the function ν, of particular solutions of (11.5), i.e. solutions not involving any arbitrary constants or involving a number of constants of motion less than needed to guarantee integrability.

11.9 Transformations of Cyclic Variables

In Sects. 11.6 and 11.8, we have introduced the variable precession transformations that leave the invariant form of the Euler–Poisson equations of motion. We were also able to use this transformation, specially designed for rigid body dynamics under the influence of axi-symmetric forces, to construct integrable extensions of known cases. It turns out that the same transformation can be attained in a completely different way, applicable to any system whose structure involves cyclic coordinates. The basic idea is that for such system to be integrable, all that matters is the structure of its Routhian equations of motion after ignoring the cyclic coordinates. We use a simple observation that several Lagrangian mechanical systems that have different Lagrangian and Routhian functions can be reduced to one and the same set of Routhian equations in the palpable part of the generalized coordinates. Clearly, this will be the case if the Routhians of those systems differ only by constant terms that may depend only on the cyclic constants, but not on any of the palpable coordinates or velocities.

Consider the mechanical system of $n + k$ degrees of freedom, of which k degrees are cyclic, characterized by the time-independent Lagrangian

$$L = L(q_1 ..., q_n, \dot{q}_1 ..., \dot{q}_n, \dot{Q}_1, ..., \dot{Q}_k). \tag{11.64}$$

The system admits the cyclic integrals

$$\frac{\partial L}{\partial \dot{Q}_i} = f_i, \qquad i = 1, ..., k. \tag{11.65}$$

Let us consider another system with the Lagrangian

$$L' = L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, \dot{Q}'_1 + \nu_1, ..., \dot{Q}'_k + \nu_k) - \sum_{i=1}^{k} \beta_i \nu_i (q_1 ..., q_n), \tag{11.66}$$

where β_i are certain constants and ν_i are certain functions of the palpable coordinates $q_1 ..., q_n$. We notice that the system (11.66) is time-independent with the cyclic variables $Q'_1, ..., Q'_k$. This system can be considered as a transformation of (11.64) through the linear time-independent transformation of the cyclic variable rates

$$\dot{Q}_i = \dot{Q}'_i + \nu_i(q_1, ..., q_n). \tag{11.67}$$

Consider the motion of the system (11.66) on the same level of the cyclic integrals as in (11.65), i.e.

$$\frac{\partial L'}{\partial \dot{Q}'_i} = f_i, \qquad i = 1, ..., k. \tag{11.68}$$

This is the transformed form of (11.65) according to (11.67).

Now, let R and R' be the Routhians of the two systems, then their difference

$$\begin{aligned} R' - R &= L - \sum_{i=1}^{k} \beta_i \nu_i - \sum_{i=1}^{k} \dot{Q}'_i f_i - (L - \sum_{i=1}^{k} \dot{Q}_i f_i) \\ &= \sum_{i=1}^{k} (\dot{Q}_i - \dot{Q}'_i) f_i - \beta_i \nu_i \\ &= \sum_{i=1}^{k} (f_i - \beta_i)\nu_i. \end{aligned} \tag{11.69}$$

The Routhian equations of motion (see, for example, [305, 368]) of the system characterized by (11.64), (11.65) will be identical to those obtained for the transformed system (11.66), (11.68) if we set $\{f_i = \beta_i, i = 1, ..., k\}$. In other words, under the last conditions, the arbitrary functions ν_i do not affect the solution for the non-cyclic coordinates.

From the above considerations we draw the following theorems:

1. For constant $\{\nu_i = n_i, i = 1, ..., k\}$. In this case the right-hand side of (11.69) is constant, and one can take $\{\beta_i = 0\}$. Equations for $q_1..., q_n$ are identical from R' and R.

Theorem 11.5 *If the Lagrangian (11.64) is general integrable (for arbitrary initial conditions), then the Lagrangian*

$$L' = L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, \dot{Q}'_1 + n_1, ..., \dot{Q}'_k + n_k) \tag{11.70}$$

is also integrable for arbitrary initial conditions.

It is not hard to see that this theorem applied to the problem of motion of a body about a fixed point under the action of axi-symmetric fields, i.e. with one cyclic coordinate ψ (the angle of precession), leads to the uniform precession transformation introduced in Chap. 10. Note that in this method, we have not used the property of invariance of the form of Euler–Poisson equations.

Exercise [405]: Apply the last theorem to exercise 5 of Chap. 9, using the transformation $\dot{\psi} \to \dot{\psi} + n$, $\dot{\varphi} \to \dot{\varphi} + N$, n, N constants. Show that the transformed integrable Lagrangian is

$$L' = \frac{1}{2}[(A + mz^2)(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + C(\dot{\psi}\cos\theta + \dot{\varphi})^2 + \dot{z}^2]$$
$$+n[\left(A + mz^2\right)\sin^2\theta\dot{\psi} + C\cos\theta(\dot{\psi}\cos\theta + \dot{\varphi})] + CN(\dot{\psi}\cos\theta + \dot{\varphi})$$
$$-\{V(z) - \frac{n^2}{2}[\left(A + mz^2\right)\sin^2\theta + C\cos^2\theta] - nNC\cos\theta\}. \quad (11.71)$$

Note that the transformation engenders, among other effects, the presence of a gyrostatic momentum CN along the axis of symmetry and uniform field potential $-nNC\gamma_3$.

2. For variable $\{\nu_i = \nu_i(q_1 ..., q_n), i = 1, ..., k\}$

Theorem 11.6 *If the system with the Lagrangian*

$$L = L(q_1 ..., q_n, \dot{q}_1 ..., \dot{q}_n, \dot{Q}_1, ..., \dot{Q}_k) \quad (11.72)$$

is integrable for arbitrary initial conditions, then the system whose Lagrangian is

$$L' = L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, \dot{Q}'_1 + \nu_1, ..., \dot{Q}'_k + \nu_k) - \sum_{i=1}^{k} \beta_i \nu_i(q_1 ..., q_n) \quad (11.73)$$

is integrable for arbitrary functions ν_i and arbitrary constants $\{\beta_i\}$ on the level

$$\{\frac{\partial L'}{\partial \dot{Q}'_i} = \beta_i, \qquad i = 1, ..., k\} \quad (11.74)$$

of the cyclic integrals.

It should be stressed again that the integrability of the system with Lagrangian (11.73) in the last theorem is conditional, i.e. valid only for initial conditions consistent with the restriction (11.74), even though the original system (11.72) is integrable for arbitrary initial conditions. In application to dynamics of a rigid body about a fixed point in an axi-symmetric field, this theorem reproduces Theorem 1 of the previous section, which generates a conditional integrable extension from a general one.

There are, however, very important situations when the new system can be made integrable for all initial conditions. This depends on the structure of the potential part of the Lagrangian.

For the sake of clarity and for future applications, we consider in detail the case of a generalized natural system with three degrees of freedom, of which one is cyclic. Let

$$L = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) + (c_1\dot{q}_1 + c_2\dot{q}_2)\dot{Q} + \frac{1}{2}c_3\dot{Q}^2$$
$$+b_1\dot{q}_1 + b_2\dot{q}_2 + b_3\dot{Q} - V, \quad (11.75)$$

where a_{ij}, b_i, c_i, V depend only on q_1, q_2, so that Q is a cyclic variable. On an arbitrary level of the cyclic integral

$$c_1\dot{q}_1 + c_2\dot{q}_2 + c_3\dot{Q} + b_3 = f \tag{11.76}$$

the Routhian has the form

$$R = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2$$
$$+b_1\dot{q}_1 + b_2\dot{q}_2 - V. \tag{11.77}$$

Now we perform in (11.75) the transformation

$$\dot{Q} = \nu + \dot{Q}', \nu = \nu(q_1, q_2). \tag{11.78}$$

According to the last theorem, we get the new Lagrangian

$$L' = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) + (c_1\dot{q}_1 + c_2\dot{q}_2)(\dot{Q}' + \nu) + \frac{1}{2}c_3(\dot{Q}' + \nu)^2$$
$$+b_1\dot{q}_1 + b_2\dot{q}_2 + b_3(\dot{Q} + \nu) - V, \tag{11.79}$$

integrable on the level of the cyclic integral

$$c_1\dot{q}_1 + c_2\dot{q}_2 + c_3(\dot{Q}' + \nu) + b_3 = f. \tag{11.80}$$

Now, ignoring the cyclic coordinate Q' in (11.79) with the aid of this integral, one obtains the Routhian

$$R' = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2$$
$$+b_1\dot{q}_1 + b_2\dot{q}_2 - V + f\nu. \tag{11.81}$$

Note that

$$R' = R + f\nu. \tag{11.82}$$

Let the system with the Lagrangian (11.75) be integrable. This implies the integrability of the system described by the Routhian (11.77), which should admit a complementary integral, independent of the Jacobi integral (the Hamiltonian). The transformed system with Lagrangian L' is not necessarily integrable. This is clearly seen from the relation (11.82) between the Routhians R and R'. When ν is not a constant, the two systems have different Routhian equations for the palpable coordinates. The following curious situation arises, which enables us to construct a wide class of extended integrable problems.

Let the potential V in (11.75) have the structure

$$V = V_0 + \sum a_i v_i \tag{11.83}$$

where $\{a_i\}$ are arbitrary constants and V_0, v_i are certain functions in the palpable generalized coordinates q_1, q_2. Let, further, the system (11.75) be integrable for arbitrary initial conditions. This means that, besides the three general integrals, the Routhian equations of this system admit a complementary general integral, which will depend on the set of constants $\{a_i\}$, say

$$I_4 = F(q_1, q_2, \dot{q}_1, \dot{q}_2, f; a_1, a_2, ...). \tag{11.84}$$

If, moreover, we choose ν in the transformation (11.78) in the form

$$\nu = \sum n_i v_i \tag{11.85}$$

and substitute this and (11.83) in (11.77) and (11.81), we put the two Routhians in the form

$$\begin{aligned} R = {} & \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2 \\ & + b_1\dot{q}_1 + b_2\dot{q}_2 - V_0 - \sum a_i v_i, \end{aligned} \tag{11.86}$$

and

$$\begin{aligned} R' = {} & \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2) - \frac{1}{2c_3}[c_1\dot{q}_1 + c_2\dot{q}_2 + b_3 - f]^2 \\ & + b_1\dot{q}_1 + b_2\dot{q}_2 - V_0 - \sum A_i v_i, \end{aligned} \tag{11.87}$$

where $A_i = a_i - f n_i$. The only difference between the two is that $\{a_i\}$ are replaced by $\{A_i\}$.

The Routhian equations of motion for the transformed problem are also generally integrable. They admit the integral

$$\begin{aligned} I_4' &= F(q_1, q_2, \dot{q}_1, \dot{q}_2, f; A_1, A_2, ...) \\ &= F(q_1, q_2, \dot{q}_1, \dot{q}_2, f; a_i - f n_i, a_2 - f n_2, ...). \end{aligned} \tag{11.88}$$

Moreover, the corresponding Lagrangians are also integrable. The integrals I_3 and I_3' can be obtained by substituting the parameter f from (11.76) and (11.80), respectively. It is remarkable that this substitution replaces some constant coefficients of the complementary integral of the transformed problem by ones depending on f, which can be replaced by its expression involving the velocity variables. The presence of the added parameters $\{n_i\}$ changes the structure of the integral. Naturally, the transformed system is a physical generalization of the original one and when all $\{n_i\}$ vanish one goes back to the original system.

The above three ways of generalizing integrable systems with cyclic coordinates can be applied to rigid body dynamics. In the particular case of axi-symmetric fields, they give the same results as described in the theorems of the last section. The method using cyclic variables furnishes a great advantage. It does not require invariance of the Euler–Poisson equations and it will be used in various situations in the sequel, while dealing with the motion of the body in an asymmetric combination of fields.

Chapter 12
The Most General Integrable Cases in Rigid Body Dynamics

In this chapter, we present a small, but most exotic, set of general and conditional integrable cases, which constitute currently the uppermost level of the hierarchy of integrable cases in rigid body dynamics. That level is inaccessible for all direct methods used in mechanics in the past. Methods which investigate the existence of analytical or polynomial integrals and the existence of single-valued solutions of the equations of motion are equally hopeless in facing such a wide class of problems. Here, we speak about several-parameter generalizations of six of the seven integrable cases in the dynamics of a body in a liquid in two different ways, general and conditional, by applying Theorems 2 and 3 of the last chapter.

12.1 General Integrable Cases

Listed in Table 12.1 are the most general and most exotic integrable cases known up-to-date of the problem of motion of a rigid body about a fixed point under the action of an axi-symmetric combination of conservative potential and gyroscopic forces. Their generality results from the extra number of parameters (an arbitrary function in case 7) included in their structure. The first five of the seven general integrable cases, namely cases occupying positions 1–5 in Table 12.1, are obtained by applying Theorem 2 of the preceding chapter to construct unconditional integrable generalizations of all but one of the integrable cases of Chap. 10 concerning the motion of a body in a fluid. Depending on the structure of the potential, a number of additional parameters, ranging up to 4, is added to the structure of each case. The case number 6 of Table 12.1 is obtained by applying the same Theorem 2 to a general integrable case found by Yehia and Bedwehy. The latter generalizes the classical Kowalevski case by adding a singular term $\frac{\varepsilon}{\sqrt{1-\gamma_3^2}}$ to the heavy body potential.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_12

Table 12.1 General integrable extensions of general integrable cases

1	Yehia [398] (1997), Oreshkina [301], Clebsch [55]: $n_1 = 0$, Brun [47]: $n = n_1 = 0$, Tisserand [354]: $n = n_1 = 0$
	$V = (A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)\{b - \frac{1}{2}[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]^2\}$, $\nu = n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$, $\boldsymbol{l} = [n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\boldsymbol{\gamma}\mathbf{I}$, $\mu_1 = \gamma_1\{(A - B - C)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$ $\quad +2n_1[A(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\}$, $\mu_2 = \gamma_2\{(B - C - A)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$ $\quad +2n_1[B(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\}$, $\mu_3 = \gamma_3\{[(C - A - B)[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$ $\quad +2n_1[B(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - (A^2\gamma_1^2 + B^2\gamma_2^2 + C^2\gamma_3^2)]\}$
	$I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3$ $\quad +[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)](A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$
	$I_4 = A^2\{p + \gamma_1[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2$ $\quad +B^2\{q + \gamma_2[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2$ $\quad +C^2\{r + \gamma_3[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]\}^2$ $\quad -2(b - n_1 I_3)(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$
	$I_3 = M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3$, $H = 1/2\,(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}) + b\left(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2\right)$ $\quad - (M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,[n + n_1(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)]$
	$I_4 = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2)$ $\quad -ABC(b - n_1 I_3)(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$

The last of those cases, number 7, is the ultimate generalization of Lagrange's case to the most general case in which the proper rotation angle φ is a cyclic coordinate and the fourth integral is the cyclic integral.

We also give relevant supplementary information and some characteristic properties of the cases provided in the table of the last section. Some clarifications are made about the present status of the explicit solution of each of the generalized cases as per the progress made in solving their primitive counterparts at the lower levels of the hierarchy.

12.1.1 Table of General Integrable Extensions of General Integrable Cases

This case includes one parameter n_1 more than Clebsch's case and two parameters n, n_1 more than Brun's problem. As established in the last chapter, the explicit solution for this case in terms of time can be obtained by the variable precession transformation (11.31) from the basic solution $n = n_1 = 0$, given by Kötter in terms of theta functions of two arguments for the first integrable case of Clebsch (see Chap. 10).

In [301], an integrable case of M. Kharlamov's equations was constructed. It admits a fourth integral quadratic in the angular velocities, with coefficients depending on $\boldsymbol{\gamma}$. To this end, the author used an ansatz for the quadratic integral, and used consistency conditions with the equations of motion. The resulting expressions are quite complicated and lack transparency. The author has neither noted the possibility of transforming this case to Clebsch's case of motion of a body in a liquid, nor even any relation to Clebsch's case as a special case. Consequently, to the end of her paper, the author states that the existence of the fourth algebraic integral "means the possibility, in principle, to reduce the problem to quadrature". Our method gives an effortless constructive way to build the explicit solution by applying the variable precession transformation to Kötter's solution.

2	Yehia [398] (1997), Kharlamova L. [214] (1990): $\frac{n_1}{c_1} = \frac{n_2}{c_2} = \frac{n_3}{c_3}$ Clebsch's case of spherical symmetry [55]: $n_1 = n_2 = n_3 = 0$.
	$B = C = A$, $V = \frac{1}{2}A[c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2 - (n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)^2]$, $\nu = n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2$, $\boldsymbol{l} = A\nu\boldsymbol{\gamma}$, $\mu_1 = -A\gamma_1[n + n_1\gamma_1^2 + \gamma_2^2(3n_2 - 2n_1) + \gamma_3^2(3n_3 - 2n_1)]$, $\mu_2 = -A\gamma_2[n + \gamma_1^2(3n_1 - 2n_2) + n_2\gamma_2^2 + (3n_3 - 2n_2)\gamma_3^2]$, $\mu_3 = -A\gamma_3[n + \gamma_1^2(3n_1 - 2n_3) + \gamma_2^2(3n_2 - 2n_3) + n_3\gamma_3^2]$
	$I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 + n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2$,
	$I_4 = (c_1 - 2n_1 I_3)(p + \nu\gamma_1)^2 + (c_2 - 2n_2 I_3)(q + \nu\gamma_2)^2$ $+ (c_3 - 2n_3 I_3)(r + \nu\gamma_3)^2$ $-[(2n_2 I_3 - c_2)(2n_3 I_3 - c_3)\gamma_1^2 + (2n_3 I_3 - c_3)(2n_1 I_3 - c_1)\gamma_2^2$ $+ (2n_1 I_3 - c_1)(2n_2 I_3 - c_2)\gamma_3^2]$
	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3$, $H = \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2)$ $-(n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$
	$I_4 = \left(Ac_1 - 2n_1 I_3^*\right)M_1^2 + \left(Ac_2 - 2n_2 I_3^*\right)M_2^2 + \left(Ac_3 - 2n_3 I_3^*\right)M_3^2$ $-A[\left(2n_2 I_3^* - Ac_2\right)\left(2n_3 I_3^* - Ac_3\right)\gamma_1^2 + \left(2n_3 I_3^* - Ac_3\right)\left(2n_1 I_3^* - Ac_1\right)\gamma_2^2$ $+ \left(2n_1 I_3^* - Ac_1\right)\left(2n_2 I_3^* - Ac_2\right)\gamma_3^2]$

In this case, the body has spherical dynamical symmetry. The basic case $n_1 = n_2 = n_3 = 0$ is a case of the motion of a body in a liquid (Case 3 of Table 10.1 of Sect. 10.15). It is closely related to the other Clebsch's integrable case with a tri-axial body in the same problem. The solution of this case can be expressed in terms of theta functions of two variables [233] (see also [71]) and so, in principle, will be the present generalization. However, this point needs a closer examination, as the present case presents some unusual and rarely met characteristic properties.

(1) The presence of the three extra parameters n_1, n_2, n_3, which we assume different, raises the degree of the polynomial potential $V(\gamma)$ from 2 to 4 and the degree of the components of $\boldsymbol{\mu}$ from 1 to 3. The combination of forces acting on the body has turned into a much complicated one, compared to the original.

(2) The presence of the same parameters raises the degree of the complementary polynomial integral I_4 as a function of the angular velocity components in the six-dimensional phase space $\{\boldsymbol{\omega}, \boldsymbol{\gamma}\}$ of the three-dimensional problem from 2 to 3. However, on every level of the integral I_3, say, $I_3 = f$, the complementary integral I_4 becomes of the second degree. This may be reformulated in the language of analytical dynamics in the following way: The reduced equations of motion of the problem under consideration after ignoring the cyclic coordinate ψ admit a quadratic complementary integral I_4 in the other two Eulerian angles $\{\dot{\theta}, \dot{\varphi}\}$.

(3) An exceptional case arises, when the two sets of constants are proportional

$$\frac{n_1}{c_1} = \frac{n_2}{c_2} = \frac{n_3}{c_3} = \lambda \text{ (say)}.$$

Then I_4 takes the form

$$\begin{aligned} I_4 = (1 - 2\lambda I_3)\,\{&c_1(p + \nu\gamma_1)^2 + c_2(q + \nu\gamma_2)^2 + c_3(r + \nu\gamma_3)^2 \\ &- (1 - 2\lambda I_3)\,[c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2]\}. \end{aligned}$$

For arbitrary value of I_3, one can cancel the first factor $(1 - 2\lambda I_3)$ to obtain for I_4 the expression

$$\begin{aligned} I_4 = {}&c_1(p + \nu\gamma_1)^2 + c_2(q + \nu\gamma_2)^2 + c_3(r + \nu\gamma_3)^2 \\ &- (1 - 2\lambda I_3)\,[c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2], \end{aligned}$$

which is quadratic in the velocities, but has some linear terms. This is the case of a quadratic integral found in [214] in a more complicated and less transparent way.

3	Yehia [398] (1997), Rubanovsky [317]: $n_1 = n_2 = n_3 = 0$ Lyapunov [267]: $n_1 = n_2 = n_3 = a_1 = a_2 = a_3 = 0$
	$B = C = A$, $\nu = n + n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, $V = A\{a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 - \frac{1}{2}(bc\gamma_1^2 + ca\gamma_2^2 + ab\gamma_3^2)$ $\quad + \frac{1}{2}\nu[(b + c)\,\gamma_1^2 + (c + a)\,\gamma_2^2 + (a + b)\,\gamma_3^2] - \frac{1}{2}\nu^2\}$, $\boldsymbol{l} = A[-\frac{1}{2}((b + c)\gamma_1, (c + a)\gamma_2, (a + b)\gamma_3) + \nu\boldsymbol{\gamma}]$ $\mu_1 = A[n_1 + \gamma_1(a + n - 2\nu)]$, $\mu_2 = A[n_2 + \gamma_2(b + n - 2\nu)]$, $\mu_3 = A[n_3 + \gamma_3(c + n - 2\nu)]$
	$I_3 = (p\gamma_1 + q\gamma_2 + \gamma_3 r) - \frac{1}{2}[\,(b + c)\,\gamma_1^2 + (c + a)\,\gamma_2^2 + (a + b)\,\gamma_3^2] + \nu$ $I_4 = \frac{1}{2}\{\,(b + c)\,(p + \nu\gamma_1)^2 + (c + a)\,(q + \nu\gamma_2)^2 + (a + b)\,(r + \nu\gamma_3)^2\}$ $\quad + (-n_1 I_3 + a_1)\,(p + \nu\gamma_1 + a\gamma_1)$ $\quad + (-n_2 I_3 + a_2)\,(q + \nu\gamma_2 + b\gamma_2)$ $\quad + (-n_3 I_3 + a_3)\,(r + \nu\gamma_3 + c\gamma_3)$ $\quad -(bcp\gamma_1 + caq\gamma_2 + abr\gamma_3) - \nu(bc\gamma_1^2 + ca\gamma_2^2 + ab\gamma_3^2)$

	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3,$ $H = \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}[(b+c)M_1\gamma_1 + (c+a)M_2\gamma_2 + (a+b)M_3\gamma_3]$ $+[(Aa_1 - n_1 I_3^*)\gamma_1 + (Aa_2 - n_2 I_3^*)\gamma_2 + \gamma_3(Aa_3 - n_3 I_3^*)]$ $-\frac{A}{8}[(a^2 + 2bc)\gamma_1^2 + (b^2 + 2ca)\gamma_2^2 + (c^2 + 2ab)\gamma_3^2]$
	$I_4 = (b+c)[M_1 + \frac{A}{2}(b+c)\gamma_1]^2 + (c+a)[M_2 + \frac{A}{2}(c+a)\gamma_1]^2$ $+(a+b)[M_3 + \frac{A}{2}(a+b)\gamma_1]^2$ $+(Aa_1 - n_1 I_3^*)[2M_1 + A(2a+b+c)\gamma_1]$ $+(Aa_2 - n_2 I_3^*)[2M_2 + A(a+2b+c)\gamma_2]$ $+(Aa_3 - n_3 I_3^*)[2M_3 + A(a+b+2c)\gamma_3]$ $-A\{bc\gamma_1[2M_1 + A(b+c)\gamma_1] + ca\gamma_2[2M_2 + A(c+a)\gamma_2]$ $+ab\gamma_3[2M_3 + A(a+b)\gamma_3]\}$

Lyapunov's case $s_1 = s_2 = s_3 = n_1 = n_2 = n_3 = 0$ [267] was solved by Kötter, as well as the related Steklov case, in terms of theta functions of two arguments [235]. This solution will cover the case $s_1 = s_2 = s_3 = 0$ for arbitrary n_1, n_2, n_3. It is obvious that to express the solution in the most general case by applying the variable precession transformation, it suffices to obtain the solution for the basic case $n = n_1 = n_2 = n_3 = 0$, $s_1 s_2 s_3 \neq 0$. This was not done up to the present time.

4	Yehia [398] (1997), Yehia [383] $n_1 = n_2 = 0$, Yehia [380]: $n = n_1 = n_2 = 0$, Kowalevski [238]: $k = n = n_1 = n_2 = 0$
	$A = B = 2C,$ $V = C[a_1\gamma_1 + a_2\gamma_2 - \kappa\gamma_3\nu - \frac{1}{2}\nu^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)],$ $\nu = n + n_1\gamma_1 + n_2\gamma_2.$ $\mu_1 = C\left(-n\gamma_1 - n_1\gamma_1^2 + 2n_1\gamma_2^2 + n_1\gamma_3^2 - 3n_2\gamma_1\gamma_2\right),$ $\mu_2 = C\left(-\gamma_2 n + 2n_2\gamma_1^2 - n_2\gamma_2^2 + n_2\gamma_3^2 - 3n_1\gamma_1\gamma_2\right),$ $\mu_3 = C(\kappa - 3n\gamma_3 - 5n_1\gamma_1\gamma_3 - 5n_2\gamma_2\gamma_3)$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + (r+\kappa)\gamma_3 + \nu\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)$
	$I_4 = [(p+\nu\gamma_1)^2 - (q+\nu\gamma_2)^2 - (a_1 - n_1 I_3)\gamma_1 + (a_2 - n_2 I_3)\gamma_2]^2$ $+[2(p+\nu\gamma_1)(q+\nu\gamma_2) - (a_1 - n_1 I_3)\gamma_2 - (a_2 - n_2 I_3)\gamma_1]^2$ $+2\kappa(r + \nu\gamma_3 - \kappa)[(p+\nu\gamma_1)^2 + (q+\nu\gamma_2)^2]$ $-4\kappa\gamma_3[(a_1 - n_1 I_3)(p+\nu\gamma_1) + (a_2 - n_2 I_3)(q+\nu\gamma_2)]$
	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3,$ $H = \frac{1}{2C}(\frac{M_1^2+M_2^2}{2} + M_3^2) - kM_3 + C(a_1\gamma_1 + a_2\gamma_2)$ $-(n + n_1\gamma_1 + n_2\gamma_2)(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$
	$I_4 = [\frac{M_1^2 - M_2^2}{4C^2} - (a_1 - \frac{n_1}{C}I_3^*)\gamma_1 + (a_2 - \frac{n_2}{C}I_3^*)\gamma_2]^2$ $+[\frac{M_1 M_2}{2C^2} - (a_1 - \frac{n_1}{C}I_3^*)\gamma_2 - (a_2 - \frac{n_2}{C}I_3^*)\gamma_1]^2$ $+\frac{k}{C^2}(\frac{M_3}{2C} - k)(M_1^2 + M_2^2)$ $-\frac{2k}{C^2}\gamma_3[(a_1 - \frac{n_1}{C}I_3^*)M_1 + (a_2 - \frac{n_2}{C}I_3^*)M_2]$

In view of the presence of the coefficients a_1, a_2 in the linear terms of V and n_1, n_2 in the function ν, it is evident that by rotating the xy axes one can eliminate one of those four coefficients. Suppose we have eliminated a_2, then n_2 will remain there, although it will not appear if we apply our method to the basic potential without a_2. Thus, we shall keep the four terms and the resulting case will neither repeat nor be included in the other generalization of Kowalevski's case in the line leading to case number 5.

The Euler–Poisson variables in Kowalevski's case were expressed by Kowalevski herself in terms of hyper-elliptic functions of time [238]. The solution was somewhat simplified and systematized by Kötter [232, 234] (see also [256]). Explicit expressions for the six variables in terms of the separation variables can also be found in [113, 256]. Many qualitative and global properties of motion are discussed in [240] The same problem was treated in a large number of recent works using methods of modern algebraic geometry and the inverse scattering method (e.g. [71, 145] and references cited therein). Of special interest is the work [152], which relates the Kowalevski case to a special version ($f = 0$) of Clebsch's case by means of a rational transformation and thus explicit solutions for the first can be obtained from that of the other. The same idea was realized for our generalization of Kowalevski's case to the gyrostat by Gavrilov, who related it to the full case of Clebsch ($f \neq 0$) solvable in Theta functions of two arguments [110]. Thus, it becomes evident that the generalized case under discussion here is, in principle, solvable in terms of the same class of functions. However, a direct separation of Yehia's gyrostat is not achieved yet (see Chap. 5, Sect. 5.6).

5	Yehia [411] Sokolov [336] $n = n_2 = 0$ Yehia [380] $n = n_2 = c = 0$ Kowalevski [238]$n = n_2 = c = \kappa = 0$
	$A = B = 2C,$ $\nu = n + n_2\gamma_2,$ $V = C[\kappa c\gamma_1 + a_2\gamma_2 - \nu(\kappa + c\gamma_1)\gamma_3 - \frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2)$ $\quad -\frac{\nu^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)],$ $\boldsymbol{l} = C(2\nu\gamma_1, 2\nu\gamma_2, \kappa + \nu\gamma_3 + c\gamma_1),$ $\mu_1 = C(c\gamma_3 - n\gamma_1 - 3n_2\gamma_1\gamma_2),$ $\mu_2 = C[-n\gamma_2 + n_2\left(2\gamma_1^2 - \gamma_2^2 + \gamma_3^2\right)],$ $\mu_3 = C(\kappa + c\gamma_1 - 3n\gamma_3 - 5n_2\gamma_2\gamma_3)$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + (r + \kappa + c\gamma_1)\gamma_3$ $\quad +(n + n_2\gamma_2)(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2),$ $I_4 = \{(p + \nu\gamma_1)^2 - (q + \nu\gamma_2)^2 + (a_2 - n_2 I_3)\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r + \nu\gamma_3 - \kappa)\}^2$ $\quad +\{2(p + \nu\gamma_1)(q + \nu\gamma_2) - (a_2 - n_2 I_3)\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r + \nu\gamma_3 - \kappa)\}^2$ $\quad +2\kappa(r + \nu\gamma_3 - \kappa + c\gamma_1)[(p + \nu\gamma_1)^2 + (q + \nu\gamma_2)^2 + 2c(p + \nu\gamma_1)\gamma_3]$ $\quad -4\kappa\gamma_3(a_2 - n_2 I_3)(q + \nu\gamma_2)$ $\quad -2\kappa c^2[2\gamma_3 I_3 - \kappa\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r + \nu\gamma_3 + c\gamma_1)]$
	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3,$ $H = \frac{1}{2C}(\frac{M_1^2 + M_2^2}{2} + M_3^2) - (\kappa + c\gamma_1)M_3 + C(a_2\gamma_2 + 2c\kappa\gamma_1 - c^2\gamma_3^2)$ $\quad -\nu(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$
	$I_4 = [\frac{M_1^2 - M_2^2}{4C^2} + (a_2 - \frac{n_2}{C}I_3^*)\gamma_2 + c(\frac{M_3}{C} - 2\kappa)\gamma_1 - c^2(\gamma_1^2 - \gamma_2^2)]^2$ $\quad +[\frac{M_1 M_2}{2C^2} - (a_2 - \frac{n_2}{C}I_3^*)\gamma_1 + c(\frac{M_3}{C} - 2\kappa)\gamma_2 - 2c^2\gamma_1\gamma_2]^2$ $\quad +\kappa(\frac{M_3}{C} - 2\kappa)[\frac{M_1^2 + M_2^2}{2C^2} + 2c\gamma_3\frac{M_3}{C}] - 2\kappa\gamma_3(a_2 - \frac{n_2}{C}I_3^* + 2c^2\gamma_2)\frac{M_2}{C}$ $\quad -\frac{2\kappa c^2}{C}[2\gamma_1\gamma_3 M_1 - (\gamma_1^2 + \gamma_2^2)M_3]$

Variable separation for the basic Sokolov's case, $n = n_2 = 0$, was obtained in [227] and explicit expressions for dynamical variables are constructed in [70], in terms of two intermediate variables, which are expressed in genus-2 Theta functions. At the present level of the hierarchy, after the introduction of the parameters n, n_2, the solution is obtained by applying the relevant precession transformation.

6	Yehia [398] (1997), Yehia–Bedwehy [419]: $n = n_1 = n_2 = N = 0$, Kowalevski [238]: $\varepsilon = n = n_1 = n_2 = N = 0$
	$A = B = 2C$, $\nu = n + n_1\gamma_1 + n_2\gamma_2 + \frac{N}{\sqrt{1-\gamma_3^2}}$
	$V = C[a_1\gamma_1 + a_2\gamma_2 + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}} - \frac{1}{2}\nu^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)]$
	$\boldsymbol{l} = C\nu(2\gamma_1, 2\gamma_2, \gamma_3)$, $\mu_1 = C[-n\gamma_1 + n_1\left(2\gamma_2^2 - \gamma_1^2 + \gamma_3^2\right) - 3n_2\gamma_1\gamma_2 - \frac{N\gamma_1}{(1-\gamma_3^2)^{\frac{3}{2}}}]$, $\mu_2 = C[-n\gamma_2 + n_2\left(2\gamma_1^2 - \gamma_2^2 + \gamma_3^2\right) - 3n_1\gamma_1\gamma_2 - \frac{N\gamma_2}{(1-\gamma_3^2)^{\frac{3}{2}}}]$, $\mu_3 = -C[(3n + 5n_1\gamma_1 + 5n_2\gamma_2)\gamma_3 + \frac{N\gamma_3}{\sqrt{1-\gamma_3^2}}]$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + r\gamma_3 + \nu(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)$
	$I_4 = [(p + \nu\gamma_1)^2 - (q + \nu\gamma_2)^2 - (a_1 - n_1 I_3)\gamma_1 + (a_2 - n_2 I_3)\gamma_2]^2$ $+[2(p + \nu\gamma_1)(q + \nu\gamma_2) - (a_1 - n_1 I_3)\gamma_2 - (a_2 - n_2 I_3)\gamma_1]^2$ $+2\frac{(\varepsilon - N I_3)}{\sqrt{1-\gamma_3^2}}[(p + \nu\gamma_1)^2 + (q + \nu\gamma_2)^2] + \frac{(\varepsilon - N I_3)^2}{1-\gamma_3^2}$
	$I_3^* = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3$, $H = \frac{1}{2C}(\frac{M_1^2 + M_2^2}{2} + M_3^2) - \nu(M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)$ $+C[a_1\gamma_1 + a_2\gamma_2 + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}}]$
	$I_4 = [\frac{M_1^2 - M_2^2}{4C^2} - (a_1 - \frac{n_1}{C} I_3^*)\gamma_1 + (a_2 - \frac{n_2}{C} I_3^*)\gamma_2]^2$ $+[\frac{M_1 M_2}{2C^2} - (a_1 - \frac{n_1}{C} I_3^*)\gamma_2 - (a_2 - \frac{n_2}{C} I_3^*)\gamma_1]^2$ $+\frac{M_1^2 + M_2^2}{2C^2}[\frac{\varepsilon - \frac{N}{C} I_3^*}{\sqrt{1-\gamma_3^2}}] + \frac{[\varepsilon - \frac{N}{C} I_3^*]^2}{1-\gamma_3^2}$

Separation variables and expressions of the dynamical variables in terms of them are constructed in [218] for the conditional case 11 of Table 13.1 of Chap. 13, which covers the Yehia–Bedwehy only on the level $f = 0$. To cover the present full general case, explicit solution of the full Yehia–Bedwehy case for $f \neq 0$ is needed. This was not achieved until now. Note that in cases 1–6, the constant I_3 (or I_3^*) figuring in the expression for I_4 can be substituted by its expression in each case as a function in the components of $\boldsymbol{\omega}$(or $\mathbf{M}$) and $\boldsymbol{\gamma}$.

Remark: *It would be highly interesting to study how this phenomenon of coupling constants changes many results and conclusions obtained for all integrable cases of motion of a body in a liquid and, in particular, the portrait of the integrals of motion, bifurcation diagrams and the topological classifications of integral manifolds. Those questions are presently open for all the above six cases in Table 12.1.*

7	Generalization of Lagrange's case. Two cyclic coordinates ψ and φ. $B = A$, $V = V_0(\gamma_3)$, $\boldsymbol{l}=(\ell\gamma_1, \ell\gamma_2, l_3)$, $\boldsymbol{\mu} = (-l_3'\gamma_1, -l_3'\gamma_2, l_3 - 2\gamma_3\ell + (\gamma_1^2+\gamma_2^2)\ell')$, $V_0(\gamma_3), \ell(\gamma_3), l_3(\gamma_3)$ arbitrary functions, and l_3', ℓ' denote derivative w.r.t. to γ_3
	$I_3 = A(p\gamma_1 + q\gamma_2) + Cr\gamma_3 + (\gamma_1^2+\gamma_2^2)\ell + l_3\gamma_3$
	$I_4 = Cr + l_3$
	$I_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3$, $H = \frac{1}{2}(\frac{M_1^2+M_2^2}{A} + \frac{M_3^2}{C}) - [\frac{\ell}{A}(M_1\gamma_1 + M_2\gamma_2) + \frac{l_3}{C}M_3]$ $+V_0(\gamma_3) + \frac{1}{2A}(\gamma_1^2+\gamma_2^2)\ell^2 + \frac{l_3^2}{2C}$
	$I_4 = M_3$

This is the most general case of Lagrange's type. The body admits not only dynamical but also physical symmetry about its z-axis. The equations of motion can be easily reduced to quadratures. In fact, the integrals I_3 and I_4 can be written as

$$\begin{aligned} I_3 &= A(p\gamma_1 + q\gamma_2) + \ell(\gamma_1^2+\gamma_2^2) + (Cr + l_3)\gamma_3 = f, \\ I_4 &= Cr + l_3 = j, \end{aligned} \tag{12.1}$$

where f, j are the integral constants. The first can be reduced to the relation

$$\dot\psi = \frac{1}{A}[\frac{f - j\gamma_3}{1-\gamma_3^2} - \ell], \tag{12.2}$$

while I_4 gives

$$C(\dot\psi\gamma_3 + \dot\varphi) + l_3 = j. \tag{12.3}$$

From here we find, using (12.2),

$$\dot\varphi = \frac{1}{C}(j - l_3) - \frac{\gamma_3}{A}[\frac{f - j\gamma_3}{1-\gamma_3^2} - \ell]. \tag{12.4}$$

Thus, we have expressed $\dot\psi$ and $\dot\varphi$ in terms of γ_3. To obtain the relation with time, we use the energy (in fact, Jacobi's) integral

$$\frac{1}{2}[A(p^2+q^2) + Cr^2] + V = h.$$

That is

$$A(\dot\theta^2 + \sin^2\theta\dot\psi^2) + \frac{(j-l_3)^2}{C} = 2(h - V),$$

which can be given the final form

$$\dot{\gamma}_3^2 = \frac{1}{A}(1-\gamma_3^2)[2(h-V) - \frac{(j-l_3)^2}{C}] - \frac{1}{A^2}[f - j\gamma_3 - \ell(1-\gamma_3^2)]^2. \quad (12.5)$$

Denoting the right-hand side of this relation by $F(\gamma_3)$, one can make a separation of variables and find the quadrature

$$t = \int \frac{d\gamma_3}{\sqrt{F(\gamma_3)}}, \quad (12.6)$$

which may be used to express $\gamma_3 = \cos\theta$ as a function in time, and hence (12.2) and (12.4) can be integrated to obtain the angles ψ and φ, respectively.

The above formulas are direct generalization of their lower counterparts in the hierarchy, beginning from Lagrange's top to Kirchhoff's case of the motion of a body in a liquid (Case 1 of Table 10.1 of Sect. 10.15). For Lagrange's top, as we have seen in Sect. 4.2, $F(\gamma_3)$ is a cubic function and γ_3 can be expressed in elliptic functions of time. In the case of a multi-connected symmetric body in a liquid, we have

$$\begin{aligned} V &= a_3\gamma_3 + \frac{1}{2}[b_1(1-\gamma_3^2) + b_3\gamma_3^2], \\ \ell &= K_1, l_3 = K_3\gamma_3 + \kappa, \end{aligned} \quad (12.7)$$

so that

$$\begin{aligned} F(\gamma_3) = &\frac{1}{A}(1-\gamma_3^2)[2h + b_1 - 2a_3\gamma_3 + (b_1 - b_3)\gamma_3^2 \\ &- \frac{1}{C}(j - \kappa - K_3\gamma_3)^2] - \frac{1}{A^2}[f - j\gamma_3 - K_1(1-\gamma_3^2)]^2 \end{aligned}$$

is a polynomial of the fourth degree and hence γ_3 is also an elliptic function of time.

Kirchhoff reduced the case of simply connected body ($a_3 = \kappa = 0$) to an elliptic quadrature and expressed some particular motions in terms of elliptic functions [219]. Detailed analysis of the general solution of the last special case in elliptic functions was performed by Greenhill [135]. This solution was not extended to the case of multi-connected body, but it can be noted that the presence of the constant gyrostatic term κ and the parameter a_3 changes the distribution of the roots of F and hence affects the picture of motion. As far as we know, this case was not studied in detail.

Thus, of all known results remains without the present type of generalization only one case, namely the case of a body in a liquid found by Rubanovsky [317] that includes as special versions an earlier case due to Steklov [345] and the case of a torque-free gyrostat considered by Joukovsky [163] and Volterra [366]. Due to the situation that in this case the basic potential is zero, no more terms can be added by the present method. However, as we have seen in Chap. 11, a generalization involving an arbitrary function and a parameter has been applied to this case in the next section, but to produce a conditional integrable case from it.

12.1.2 About the Hamiltonian Formulation

We have shown in Chap. 10 that, when $\nu(\gamma) = n = \text{const}$, the transformed problems are generalizations of the original ones, without bringing any mathematical complication to the solution process. The original and transformed problems are described by one and the same set of Hamiltonian equations. This is a consequence of the fact that the Hamiltonian after transformation is $H' = H + n\mathbf{M}\cdot\gamma$, i.e. a combination of the original Hamiltonian and the areas integral.

We shall show now that when $\nu(\gamma)$ is chosen according to one of the two procedures described in Theorems 11.1–2, the Hamiltonian flow of the generalized problem is really different from that of the original problem and hence the first problem is a genuine generalization of the second.

In the tables of extended integrable cases, we have adopted the choice to identify every case by the expressions of functions V and $\boldsymbol{\mu}$, which are unique and completely characterize the physical setting for that case. An expression for the vector potential $\boldsymbol{l}$ is also given in the tables, so that the Lagrangian for each case can be readily constructed. The Hamiltonian of a problem can be obtained as described in previous chapters. To this end one can write

$$H = \boldsymbol{\omega} \cdot \frac{\partial L}{\partial \boldsymbol{\omega}} - L, \tag{12.8}$$

and eliminate $\boldsymbol{\omega}$ using the momentum variables

$$\mathbf{M} = \frac{\partial L}{\partial \boldsymbol{\omega}}. \tag{12.9}$$

The equations of motion take the form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M}\times \frac{\partial H}{\partial \mathbf{M}} + \gamma \times \frac{\partial H}{\partial \gamma}, \\ \dot{\gamma} &= \gamma \times \frac{\partial H}{\partial \mathbf{M}}, \end{aligned} \tag{12.10}$$

and their integrals are

$$\begin{aligned} I_1 &= H, \\ I_2 &= \gamma^2 = 1, \\ I_3 &= \mathbf{M}\cdot\gamma = f. \end{aligned} \tag{12.11}$$

In order to illustrate our point of view described above, we now give examples of the extended cases in the Hamiltonian formalism.

(1) The first example is case 1 of Table 12.1, which involves one parameter n_1 of the new type in addition to the uniform precession parameter n. The Hamiltonian for that case is

$$\begin{aligned}H &= 1/2\,(\frac{M_1^2}{A}+\frac{M_2^2}{B}+\frac{M_3^2}{C})+b\left(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2\right)\\ &\quad -(M_1\,\gamma_1+M_2\,\gamma_2+M_3\,\gamma_3)\,[n+n_1(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)]\\ &= H_c-(M_1\,\gamma_1+M_2\,\gamma_2+M_3\,\gamma_3)\,[n+n_1(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)],\end{aligned} \tag{12.12}$$

where H_c is the original Hamiltonian of the Clebsch case. The equations of motion are

$$\begin{aligned}\dot{\mathbf{M}} &= \mathbf{M}\times\{\frac{\partial H_c}{\partial \mathbf{M}}-[n+n_1(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)]\boldsymbol{\gamma}\}\\ &\quad +\boldsymbol{\gamma}\times\{\frac{\partial H_c}{\partial \boldsymbol{\gamma}}-[n+n_1(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)]\mathbf{M}-2n_1(\mathbf{M}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}\mathbf{I}\},\\ &= \mathbf{M}\times\frac{\partial H_c}{\partial \mathbf{M}}+\boldsymbol{\gamma}\times[\frac{\partial H_c}{\partial \boldsymbol{\gamma}}-2n_1(\mathbf{M}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}\mathbf{I}],\end{aligned} \tag{12.13}$$

and

$$\begin{aligned}\dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma}\times\{\frac{\partial H_c}{\partial \mathbf{M}}-[n+n_1(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)]\boldsymbol{\gamma}\}\\ &= \boldsymbol{\gamma}\times\frac{\partial H_c}{\partial \mathbf{M}}.\end{aligned} \tag{12.14}$$

The fourth integral is

$$\begin{aligned}I_4 &= \frac{1}{2}(M_1^2+M_2^2+M_3^2)\\ &\quad -ABC[b-n_1\,(M_1\,\gamma_1+M_2\,\gamma_2+M_3\,\gamma_3)](BC\gamma_1^2+CA\gamma_2^2+AB\gamma_3^2)\\ &= I_{4c}+n_1ABC\,(M_1\,\gamma_1+M_2\,\gamma_2+M_3\,\gamma_3)\,(BC\gamma_1^2+CA\gamma_2^2+AB\gamma_3^2).\end{aligned} \tag{12.15}$$

From (12.13), we see that when $f \neq 0$ and $n_1 \neq 0$, then H_c and I_{4c} are no more integrals of motion. The Hamiltonian flow is deformed and the overall picture of the trajectories in the phase space of the new problem is different from that of Clebsch's case.

On the other hand, when $n_1 = 0$ ($\nu = n$)

$$H = H_c - nI_3,$$

so that the Hamiltonian is a linear combination of the two integrals H_c and I_3 with constant coefficients. In that case, from (12.13), (12.14), we see that the flow defined by the Hamiltonian (12.12) is identical with the flow corresponding to Clebsch's Hamiltonian H_c and the integral takes the form of Clebsch. The same holds also if consideration is restricted to the level $I_3 = f = 0$.

(2) The second example is the extension of Clebsch's spherically symmetric case. That is case 2 in Table 12.1. Let H_s and I_{4s} be the original Clebsch spherical Hamiltonian and the corresponding fourth integral. We have

$$\begin{aligned} H_s &= H_s(\mathbf{M}, \boldsymbol{\gamma}, c_1, c_2, c_3) \\ &= \frac{1}{2A}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2), \\ I_{4s} &= c_1 M_1^2 + c_2 M_2^2 + c_3 M_3^2 - (c_2 c_3 \gamma_1^2 + c_3 c_1 \gamma_2^2 + c_1 c_2 \gamma_3^2). \end{aligned} \tag{12.16}$$

For the extended integrable system with three different parameters n_1, n_2, n_3,

$$\begin{aligned} H &= H_s - (n + n_1\gamma_1^2 + n_2\gamma_2^2 + n_3\gamma_3^2)\,(M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)\,, \\ &= H_s(\mathbf{M}, \boldsymbol{\gamma}, c_1 - 2n_1 I_3, c_2 - 2n_2 I_3, c_3 - 2n_3 I_3). \end{aligned} \tag{12.17}$$

The equations of motion have the form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \frac{\partial H_s}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times [\frac{\partial H_s}{\partial \boldsymbol{\gamma}} - 2(\mathbf{M} \cdot \boldsymbol{\gamma})(n_1\gamma_1, n_2\gamma_2, n_3\gamma_3)], \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \frac{\partial H_s}{\partial \mathbf{M}}, \end{aligned} \tag{12.18}$$

and the complementary integral is

$$\begin{aligned} I_4 = {} & (c_1 - 2n_1 I_3)\, M_1^2 + (c_2 - 2n_2 I_3)\, M_2^2 + (c_3 - 2n_3 I_3)\, M_3^2 \\ & - A[(2n_2 I_3 - c_2)\,(2n_3 I_3 - c_3)\,\gamma_1^2 + (2n_3 I_3 - c_3)\,(2n_1 I_3 - c_1)\,\gamma_2^2 \\ & + (2n_1 I_3 - c_1)\,(2n_2 I_3 - c_2)\,\gamma_3^2] \\ = {} & I_{4s}(\mathbf{M}, \boldsymbol{\gamma}, c_1 - 2n_1 I_3, c_2 - 2n_2 I_3, c_3 - 2n_3 I_3). \end{aligned} \tag{12.19}$$

The deformation caused by the presence of the three parameters n_1, n_2, n_3 and I_3 in the Hamiltonian flow on any level $I_3 = f \neq 0$ is obvious. The solution of the new problem of motion is obtained from that of the original case by replacing the original physical parameters c_1, c_2, c_3 by their new values involving three new physical parameters and the dynamical parameter f.

It is remarkable that the fourth integral (12.19) is cubic in the momenta in the whole phase space, due to the presence of I_3 in the coefficients but becomes quadratic in momenta on any fixed level of I_3. It also reduces to Clebsch's quadratic integral when $n_1 = n_2 = n_3 = 0$.

12.2 Conditional Integrable Deformations of General Integrable Cases

By Theorem 1 in Chap. 11, Sect. 11.6, all the general and conditional integrable cases of integrability of the previous hierarchy of problems admit a generalization using the transformation (11.31) to conditional cases involving the arbitrary function $\nu(\gamma_1, \gamma_2, \gamma_3)$. The explicit solution of the equations of motion in each case can be obtained from the solution of the original case, if the last is known, by Theorem 3. We apply this procedure here to all the general integrable cases, which were the subject of generalization in the preceding chapter, but this time including the case of Steklov–Rubanovsky, which was not amenable to the generalization as an unconditional case. As the structure of potential in the basic cases is not significant for the present type of generalization, we consider only one branch of the Kowalevski–Sokolov hierarchy, so that the total number of cases is still 7.

12.2.1 Table of Cases

In Table 12.2, we list the deformations of the known general integrable cases as conditional cases on a fixed level of the areas integral. For each case we provide

– conditions, if any, on the inertia matrix of the body,

– the pair of scalar potential V and vector $\boldsymbol{\mu}$, figuring in the equations of motion,

– the vector $\boldsymbol{l}$ which enters in the Lagrangian or Hamiltonian of the problem and in the structure of the cyclic integral,

– the cyclic integral I_3 itself,

– and, finally, the complementary integral.

– Each of the systems in the following table is integrable on the level $I_3 = \beta$. For each case, we also add the forms of the Hamiltonian function and the complementary integral in terms of momenta. In the Hamiltonian formulation, the cyclic integral has the same form for all integrable cases

$$I_3 = M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3 = \beta.$$

Table 12.2 Conditional integrable extensions of general integrable cases, valid on the level $I_3 = \beta$. $\nu = \nu(\gamma_1, \gamma_2, \gamma_3)$ is an arbitrary function

1	Case of Clebsch's type
	$V = (b - \frac{1}{2}\nu^2)(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) + \beta\nu$, $\mathbf{l}=\nu\boldsymbol{\gamma}\mathbf{I}$, $\boldsymbol{\mu} = \frac{\partial}{\partial\gamma}(\nu\boldsymbol{\gamma}\mathbf{I}\cdot\gamma) - [\frac{\partial}{\partial\gamma}\cdot(\nu\boldsymbol{\gamma}\mathbf{I})]\gamma$
	$I_3 = \boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\gamma} + \nu(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) = \beta$
	$I_4 = \frac{1}{2}[A^2(p+\nu\gamma_1)^2 + B^2(q+\nu\gamma_2)^2 + C^2(r+\nu\gamma_3)^2$ $-b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$
	$I_3 = M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3$, $H = 1/2\,(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}) + b\left(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2\right)$ $+[\beta - (M_1\,\gamma_1 + M_2\,\gamma_2 + M_3\,\gamma_3)]\nu$
	$I_4 = M_1^2 + M_2^2 + M_3^2$ $-b(BC\gamma_1^2 + CA\gamma_2^2 + AB\gamma_3^2)$

In all the cases under consideration, one can show that

$$\frac{dI_4}{dt} = (I_3 - \beta)\Phi(\boldsymbol{\omega}, \boldsymbol{\gamma}),$$

where Φ is a different function for each case. Thus, I_4 becomes an integral under the condition $I_3 = \beta$. As examples we give explicit results for two cases from the table.

Remark: *For the conditional integrable cases presented in Table 12.2, the Hamiltonian of the extended system is*

$$H' = H + \nu(\beta - M \cdot \gamma),$$

where H is the original Hamiltonian (before the transformation). The equations of motion are

$$\dot{\mathbf{M}} = \mathbf{M}\times\frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}} + (\beta - \mathbf{M}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma} \times \frac{\partial \nu}{\partial \boldsymbol{\gamma}},$$
$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}}. \tag{12.20}$$

Although the equations of motion in all cases of Table 12.2 depend on the parameter β and the (non-constant) function $\nu(\gamma)$, on the single level $f = \beta$ the Hamiltonian flow of the new conditional integrable problem becomes identical with the flow in the original unconditional integrable problem. In the last problem, the parameter f is arbitrary, while in the former the additional parameter β is present but regarding the dynamical condition $f = \beta$ both problems have the same number of parameters.

2	Case of Clebsch's type of spherical symmetry
	$A = B = C$, $V = \beta\nu + \frac{1}{2}(b_1\gamma_1^2 + b_2\gamma_2^2 + b_3\gamma_3^2) - \frac{1}{2}C\nu^2$, $\mathbf{l} = C\nu\boldsymbol{\gamma}$, $\boldsymbol{\mu} = C[\frac{\partial\nu}{\partial\boldsymbol{\gamma}} - (\nu + \boldsymbol{\gamma}\cdot\frac{\partial\nu}{\partial\boldsymbol{\gamma}})\boldsymbol{\gamma}]$
	$I_3 = C(p\gamma_1 + q\gamma_2 + r\gamma_3 + \nu) = \beta$
	$I_4 = C[b_1(p + \nu\gamma_1)^2 + b_2(q + \nu\gamma_2)^2 + b_3(r + \nu\gamma_3)^2]$ $-(b_2b_3\gamma_1^2 + b_1b_3\gamma_2^2 + b_1b_2\gamma_3^2)$
	$H = \frac{1}{2C}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(b_1\gamma_1^2 + b_2\gamma_2^2 + b_3\gamma_3^2)$ $+[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]\nu$
	$I_4 = b_1M_1^2 + b_2M_2^2 + b_3M_3^2 - C(b_2b_3\gamma_1^2 + b_1b_3\gamma_2^2 + b_1b_2\gamma_3^2)$

3	Rubanovsky–Lyapunov type [317]
	$A = B = C$ $V = C\{a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 - \frac{1}{2}(bc\gamma_1^2 + ca\gamma_2^2 + ab\gamma_3^2)$ $+\nu\{\beta + \frac{1}{2}[(b+c)\gamma_1^2 + (c+a)\gamma_2^2 + (a+b)\gamma_3^2]\} - \frac{1}{2}\nu^2\}$ $\boldsymbol{l} = C[\nu\boldsymbol{\gamma} - \frac{1}{2}((b+c)\gamma_1, (c+a)\gamma_2, (a+b)\gamma_3)]$, $\boldsymbol{\mu} = C[(a\gamma_1, b\gamma_2, c\gamma_3) + \frac{\partial\nu}{\partial\boldsymbol{\gamma}} - (\nu + \boldsymbol{\gamma}\cdot\frac{\partial\nu}{\partial\boldsymbol{\gamma}})\boldsymbol{\gamma}]$
	$I_3 = (p\gamma_1 + q\gamma_2 + r\gamma_3 + \nu) - \frac{1}{2}[(b+c)\gamma_1^2 + (c+a)\gamma_2^2 + (a+b)\gamma_3^2] = \beta$

	$I_4 = \frac{1}{2}[(b+c)(p+\nu\gamma_1)^2 + (c+a)(q+\nu\gamma_2)^2 + (a+b)(r+\nu\gamma_3)^2]$ $+a_1[p+(\nu+a)\gamma_1] + a_2[q+(\nu+b)\gamma_2] + a_3[r+(\nu+c)\gamma_3]$ $-[bc(p+\nu\gamma_1)\gamma_1 + ca(q+\nu\gamma_2)\gamma_2 + ab(r+\nu\gamma_3)\gamma_3]$
	$H = \frac{1}{2C}(M_1^2+M_2^2+M_3^2) + \frac{1}{2}[(b+c)M_1\gamma_1 + (c+a)M_2\gamma_2 + (a+b)M_3\gamma_3]$ $+C(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3)$ $-\frac{C}{8}[(a^2+2bc)\gamma_1^2 + (b^2+2ac)\gamma_2^2 + (c^2+2ab)\gamma_3^2],$ $+[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]\nu$
	$I_4 = (b+c)M_1^2 + (c+a)M_2^2 + (a+b)M_3^2$ $+C\{[(b^2+c^2)\gamma_1 + 2a_1]M_1 + [(a^2+c^2)\gamma_2 + 2a_2]M_2 + [(a^2+b^2)\gamma_3$ $+2a_3]M_3\} + \frac{C^2}{4}[(b+c)(b-c)^2\gamma_1^2 + (c+a)(c-a)^2\gamma_2^2 + (a+b)(a-b)^2\gamma_3^2]$ $+C^2[(a+b+c)(a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3) + 2(a_1a\gamma_1 + a_2b\gamma_2 + a_3c\gamma_3)]$

4	Case of Steklov–Rubanovsky's type
	$V = \nu(\beta - \boldsymbol{\kappa}\cdot\boldsymbol{\gamma} + a[tr(\mathbf{I}^{-1})\lvert\boldsymbol{\gamma}\rvert^2 - \boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}]) - \frac{1}{2}\nu^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma},$ $\boldsymbol{l} = \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J} + \nu\boldsymbol{\gamma}\mathbf{I}, \mathbf{J} = [tr(\mathbf{I}^{-1})\boldsymbol{\delta} - \mathbf{I}^{-1}]$ $\boldsymbol{\mu} = \boldsymbol{\kappa} + 2a\boldsymbol{\gamma}\mathbf{I}^{-1} + \frac{\partial}{\partial\boldsymbol{\gamma}}(\nu\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}) - [\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot(\nu\boldsymbol{\gamma}\mathbf{I})]\boldsymbol{\gamma}$
	$I_3 = [\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J} + \nu\boldsymbol{\gamma}\mathbf{I}]\cdot\boldsymbol{\gamma} = \beta$
	$I_4 = \frac{1}{2}\lvert\boldsymbol{\omega}\mathbf{I} + \nu\boldsymbol{\gamma}\mathbf{I} + \boldsymbol{\kappa}\rvert^2 + 2a(\boldsymbol{\omega}\cdot\boldsymbol{\gamma} + \nu)$
	$H = \frac{1}{2}(\mathbf{M} - \boldsymbol{\kappa} - \boldsymbol{\gamma}\mathbf{J})\mathbf{I}^{-1}\cdot(\mathbf{M} - \boldsymbol{\kappa} - \boldsymbol{\gamma}\mathbf{J})$ $+[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]\nu$
	$I_4 = \frac{1}{2}\lvert\mathbf{M} - a\boldsymbol{\gamma}\mathbf{J}\rvert^2 - a\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot(\mathbf{M} - \boldsymbol{\kappa} - a\boldsymbol{\gamma}\mathbf{J})$

For this case, one can show that

$$\frac{dI_4}{dt} = (I_3 - \beta)[(\boldsymbol{\omega}\mathbf{I} + \nu\boldsymbol{\gamma}\mathbf{I} + \boldsymbol{\kappa})\cdot(\frac{\partial\nu}{\partial\boldsymbol{\gamma}}\times\boldsymbol{\gamma}) + a\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot(\frac{\partial\nu}{\partial\boldsymbol{\gamma}}\times\boldsymbol{\gamma})].$$

(1) For any function ν, I_4 is an integral on the level $I_3 = \beta$ and the dynamics is conditionally integrable. On the other hand, when $\nu(\boldsymbol{\gamma}) = n$ (a constant) the terms in the square bracket vanish and this case becomes integrable for arbitrary initial conditions and coincides with the Rubanovsky–Steklov case of motion of a body in liquid.

5	Case of Kowalevski–Yehia–Sokolov type
	$A = B = 2C,$ $V = \beta\nu + C[\kappa c\gamma_1 + a_2\gamma_2 - \nu\kappa\gamma_3$ $-\frac{c^2}{2}(\gamma_1^2 + 2\gamma_3^2) - \nu c\gamma_1\gamma_3$ $-\frac{\nu^2}{2}(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)]$
	$\boldsymbol{l} = C(2c\gamma_3 + 2\nu\gamma_1, 2\nu\gamma_2, \kappa - c\gamma_1 + \nu\gamma_3),$ $\boldsymbol{\mu} = C\{(c\gamma_1, 0, \kappa + c\gamma_3) + \frac{\partial}{\partial\boldsymbol{\gamma}}[(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)\nu]$ $-[5\nu + 2\gamma_1\frac{\partial\nu}{\partial\gamma_1} + 2\gamma_2\frac{\partial\nu}{\partial\gamma_2} + \gamma_3\frac{\partial\nu}{\partial\gamma_3}]\boldsymbol{\gamma}\},$ $I_3 = C[2p\gamma_1 + 2q\gamma_2 + (r+\kappa)\gamma_3 + c\gamma_1\gamma_3 + (2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)\nu] = \beta$
	$I_4 = \left[(p+\nu\gamma_1)^2 - (q+\nu\gamma_2)^2 + a_2\gamma_2 + c^2\gamma_2^2 + c\gamma_1(r+\nu\gamma_3-\kappa)\right]^2$ $+\left[2(p+\nu\gamma_1)(q+\nu\gamma_2) - a_2\gamma_1 - c^2\gamma_1\gamma_2 + c\gamma_2(r+\nu\gamma_3-\kappa)\right]^2$ $+2\kappa(r+\nu\gamma_3-\kappa+c\gamma_1)\left[(p+\nu\gamma_1)^2 + (q+\nu\gamma_2)^2 + 2c(p+\nu\gamma_1)\gamma_3\right]$ $-4a_2\kappa(q+\nu\gamma_2)\gamma_3$ $-2\kappa c^2\{2\gamma_3[2(p+\nu\gamma_1)\gamma_1 + c\gamma_1\gamma_3 + 2(q+\nu\gamma_2)\gamma_2 + (r+\nu\gamma_3)\gamma_3]$ $+\kappa\gamma_3^2 - (\gamma_1^2 + \gamma_2^2 + 2\gamma_3^2)(r+\nu\gamma_3+c\gamma_1)\}$

	$I_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 = \beta$, $H = \frac{1}{2C}(\frac{M_1^2+M_2^2}{2} + M_3^2) - (\kappa + c\gamma_1)M_3 + C(a_2\gamma_2 + 2c\kappa\gamma_1 - c^2\gamma_3^2)$ $+\nu[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]$
	$I_4 = [\frac{M_1^2-M_2^2}{4C^2} + a_2\gamma_2 + c(\frac{M_3}{C} - 2\kappa)\gamma_1 - c^2(\gamma_1^2 - \gamma_2^2)]^2$ $+[\frac{M_1M_2}{2C^2} - a_2\gamma_1 + c(\frac{M_3}{C} - 2\kappa)\gamma_2 - 2c^2\gamma_1\gamma_2]^2$ $+\kappa(\frac{M_3}{C} - 2\kappa)[\frac{M_1^2+M_2^2}{2C^2} + 2c\gamma_3\frac{M_3}{C}] - 2\kappa\gamma_3(a_2 + 2c^2\gamma_2)\frac{M_2}{C}$ $-\frac{2\kappa c^2}{C}[2\gamma_1\gamma_3M_1 - (\gamma_1^2 + \gamma_2^2)M_3]$

6	Yehia [398], Yehia–Bedwehy [419]: $\nu = 0$, Kowalevski [238]: $\nu = a = 0$
	$A = B = 2C$
	$V = \beta\nu + C[a_1\gamma_1 + a_2\gamma_2 + \frac{a}{\sqrt{1-\gamma_3^2}} - \frac{1}{2}\nu^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)]$
	$\boldsymbol{l} = C\nu(2\gamma_1, 2\gamma_2, \gamma_3)$, $\boldsymbol{\mu} = C\{\frac{\partial}{\partial\boldsymbol{\gamma}}[\nu(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2] - [5\nu + \frac{\partial\nu}{\partial\boldsymbol{\gamma}} \cdot (2\gamma_1, 2\gamma_2, \gamma_3)]\boldsymbol{\gamma}\}$
	$I_3 = C[2p\gamma_1 + 2q\gamma_2 + r\gamma_3 + \nu(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)] = \beta$
	$I_4 = [(p + \nu\gamma_1)^2 - (q + \nu\gamma_2)^2 - a_1\gamma_1 + a_2\gamma_2]^2$ $+[2(p + \nu\gamma_1)(q + \nu\gamma_2) - a_1\gamma_2 - a_2\gamma_1]^2$ $+2\frac{a}{\sqrt{1-\gamma_3^2}}[(p + \nu\gamma_1)^2 + (q + \nu\gamma_2)^2] + \frac{a^2}{1-\gamma_3^2}$

7	Case of Lagrange's type
	$B = A$, $V = V_0(\gamma_3) + \beta\nu - \nu[(\gamma_1^2 + \gamma_2^2)\ell + l_3\gamma_3] - \frac{1}{2}\nu^2[A + (C - A)\gamma_3^2]$, $\boldsymbol{l} = ((\ell + A\nu)\gamma_1, (\ell + A\nu)\gamma_2, l_3 + C\nu\gamma_3)$, $\boldsymbol{\mu} = \frac{\partial}{\partial\boldsymbol{\gamma}}(\boldsymbol{l} \cdot \boldsymbol{\gamma}) - (\frac{\partial}{\partial\boldsymbol{\gamma}} \cdot \boldsymbol{l})\boldsymbol{\gamma}$, $V_0(\gamma_3), \ell(\gamma_3), l_3(\gamma_3), \nu(\gamma_1, \gamma_2, \gamma_3)$ arbitrary functions
	$I_3 = A(p\gamma_1 + q\gamma_2) + (Cr + l_3)\gamma_3 + (\gamma_1^2 + \gamma_2^2)\ell + \nu[A + (C - A)\gamma_3^2] = \beta$
	$I_4 = C(r + \nu\gamma_3) + l_3$
	$I_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3$, $H = \frac{1}{2}(\frac{M_1^2+M_2^2}{A} + \frac{M_3^2}{C}) - [\frac{\ell}{A}(M_1\gamma_1 + M_2\gamma_2) + \frac{l_3}{C}M_3]$ $+V_0(\gamma_3) + \frac{1}{2A}(\gamma_1^2 + \gamma_2^2)\ell^2 + \frac{l_3^2}{2C} + \nu[\beta - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]$
	$I_4 = M_3$

Note that the precession angle ψ is cyclic and the corresponding generalized momentum is the integral of motion I_3. The proper rotation angle φ is no longer cyclic in general, but becomes cyclic on the level $I_3 = \beta$. In fact, one may calculate

$$\frac{\partial L}{\partial\varphi} = (I_3 - \beta)\frac{\partial\nu}{\partial\varphi}.$$

The cyclic integral I_4 is conditional on the level $I_3 = \beta$. One can easily find

$$\frac{dI_4}{dt} = (I_3 - \beta)(\gamma_2\frac{\partial\nu}{\partial\gamma_1} - \gamma_1\frac{\partial\nu}{\partial\gamma_2}). \tag{12.21}$$

This expression vanishes when either $I_3 = \beta$ or $\nu = \nu(\gamma_1^2 + \gamma_2^2)$, which is equivalent to $\nu = \nu(\gamma_3)$. Case 7 is integrable for arbitrary function $\nu(\gamma_1, \gamma_2, \gamma_3)$ on the level $I_3 = \beta$, but becomes unconditional when $\nu = \nu(\gamma_3)$.

12.2.2 Example of Physical Application

Consider the simple original case of Kowalevski, obtained from case 5 of Table 12.2 by setting $\kappa = c = 0$. We shall transform this case using

$$\nu = \lambda\gamma_3, \beta = C\lambda a.$$

The generalized case will be characterized by

$A = B = 2C$
$V = C[a_1\gamma_1 + \lambda a\gamma_3 - \frac{\lambda^2}{2}\gamma_3^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right)]$
$\boldsymbol{l} = C\lambda\gamma_3(2\gamma_1, 2\gamma_2, \gamma_3)$, $\boldsymbol{\mu} = C\lambda(-2\gamma_1\gamma_3, -2\gamma_2\gamma_3, 2 - 3\gamma_3^2)$
$I_3 = C[2p\gamma_1 + 2q\gamma_2 + r\gamma_3 + \gamma_3(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)] = Ca$
$I_4 = [(p + \lambda\gamma_1\gamma_3)^2 - (q + \lambda\gamma_2\gamma_3)^2 - a_1\gamma_1]^2$ $+[2(p + \lambda\gamma_1\gamma_3)(q + \lambda\gamma_2\gamma_3) - a_1\gamma_2]^2$
$H = \frac{1}{4C}(M_1^2 + M_1^2 + 2M_1^2) + C(a_1\gamma_1 + \lambda a\gamma_3)$ $+\lambda\gamma_3[Ca - (M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3)]$, $I_4 = [M_1^2 - M_2^2 - 4C^2a_1\gamma_1]^2$ $+4[M_1M_2 - 2C^2a_1\gamma_2]^2$

The potential is modified by the addition of two terms. The first $C\lambda a\gamma_3$ means a displacement of the centre of mass of the body in the z-direction (normal to the equatorial plane) to the point $(x_0 = Ca_1/Mg, 0, z_0 = \frac{C\lambda a\gamma_3}{Mg})$ and the second term is quartic in $\boldsymbol{\gamma}$. The last may be written as

$$-\frac{C\lambda^2}{2}\gamma_3^2\left(2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right) = -\frac{C\lambda^2}{2}\gamma_3^2\left(2 - \gamma_3^2\right).$$

The new problem involves also the gyroscopic moments described by the vector $\boldsymbol{l}$ and $\boldsymbol{\mu}$ and it is integrable on the level $I_3 = Ca$. On setting $\lambda = 0$, one recovers the general integrable case of Kowalevski.

Note that the parameter λ does not appear in the Hamiltonian form of I_4. Note also that from the Hamiltonian function of the new problem one cannot recognize the acting potential and gyroscopic forces, which are clearly defined by the potential V and the vector $\boldsymbol{\mu}$.

Chapter 13
Miscellaneous Cases Integrable on a Single Level of the Areas Integral

In this chapter, we collect certain sets of conditional integrable cases of different origins and of various characters, which do not belong to one of the problems discussed in previous chapters that have definite physical interpretation, but they are unified only by being valid on a single level of the areas integral, and mostly on $f = 0$. The first set consists of the separable cases investigated in Chap. 9 above. Another set began to appear in a work of Goryachev [117], who used an inverse method to find potentials that admit existence of a complementary integral of the third or fourth degrees in the components of the angular velocity, as modifications of the known cases at that time. Those are Kowalevski's case of a heavy body with a fixed point and Chaplygin's case of a rigid body in a liquid and Goryachev–Chaplygin's case of the classical problem. The search led in [117] to the new cases:

(1) A conditional case, on the level $f = 0$, under the condition $A = B = 2C$, with the potential

$$V = a_1\gamma_1 + a_2\gamma_2 + b_1(\gamma_1^2 - \gamma_2^2) + 2b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}.$$

This adds the singular term $\frac{\lambda}{2\gamma_3^2}$ to a former result of Chaplygin [53], which combines the potentials of Kowalevski's classical case and Chaplygin's case of a body in a liquid. The complementary integral for this case is of the fourth degree.

(2) Another case under the condition $A = B = \frac{4}{3}C$, with the potential

$$V = \frac{a + b\gamma_1}{\gamma_3^{\frac{2}{3}}},$$

which admits a cubic integral.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_13

> A variety of modifications and generalizations of known conditional integrable cases have accumulated in the last three decades, mainly in the works of the author and coworkers, as a result of the introduction of a new method of construction of two-dimensional integrable systems living on Riemannian manifolds and whose integrals are polynomials in velocities [381]. The resulting systems usually involved a large number of parameters, which could be adjusted to identify the metric of the problem with that of the reduced system of rigid body dynamics after ignoring the precession angle as in Chap. 9. Here we shall not try to make any physical interpretation of the potential and gyroscopic terms in each case, even though they are mostly generalizations of some of the cases presented in the previous chapters in natural physical settings.

The most natural and comfortable classification of the relatively large number (22) of known conditional integrable cases is the classification by the degree of the polynomial complementary integral in every case. We shall follow this classification here. We also give full-time sequence of each hierarchy of overlapping cases.

Although most of those cases do not have physical interpretation, due to strange singularities in the potentials, we give full up-to-date list of them. As known cases are scattered in the literature, we believe this step, made here for the first time, can play a definite role in future development of the subject. As will be seen below, some of those cases have already stimulated further studies on the separation of variables and also on topological classification.

Remark 18 We have used the uniform and variable precession transformations in some previous chapters. In the present one, those transformations will not be applied. The reason is that in conditional cases on the level $f = 0$ this transformation may be easily applied using an arbitrary function $\nu(\gamma)$ as discussed in Sect. 12.2 of the preceding chapter.

13.1 Cases with a Quadratic Integral

It is well-known that a natural (time-reversible) mechanical system of two degrees of freedom which admits an integral quadratic in velocities and independent of the energy integral must be Liouville separable system in some generalized coordinates. The dynamics of a rigid body acted upon by pure potential forces is time-reversible on the zero-level of the cyclic integral. When gyroscopic forces are present, equations of motion become irreversible and Liouville separability is lost. In the present section, we list the three known types of potentials that admit a complementary quadratic integral and hence admit Liouville separation and also three non-separable cases with a quadratic integral.

13.1.1 Separable Integrable Potentials

From the Minkowski analogy between the motion of a rigid body about a fixed point and the motion of a material point on the inertia ellipsoid of the body at the fixed point, it follows that certain potentials exist, which allow separation of variables on the zero level of the areas integral. Those are of three types:

(1) Potentials separable in elliptic coordinates on the tri-axial ellipsoid ($A \neq B \neq C$) Chap. 9 Sect. 9.7.2.
(2) Potentials separable in spherical coordinates for a dynamically symmetric body ($A = B$), including the case of dynamical spherical symmetry at the fixed point Chap. 9 Sect. 9.7.1.
(3) Potentials separable in sphero-conic (elliptic) coordinates on the sphere of inertia in the case of complete dynamical symmetry ($A = B = C$) Chap. 9 Sect. 9.7.3.

For all three types of separable cases $\boldsymbol{\mu} = \mathbf{0}$, $f = 0$ and u, v, F, G, g are arbitrary functions of their arguments (Table 13.1).

Table 13.1 Conditional integrable cases. The first 20 cases are valid on the level $f = 0$

1	Separable in elliptic coordinates on the ellipsoid of inertia [31, 224, 381]
	$V = \frac{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}{\sqrt{\beta}}[u(\alpha+\sqrt{\beta})+v(\alpha-\sqrt{\beta})]$, $\alpha = AB+BC+CA-ABC(\frac{\gamma_1^2}{A}+\frac{\gamma_2^2}{B}+\frac{\gamma_3^2}{C}) = ABC[tr(\mathbf{I}^{-1})-\boldsymbol{\gamma}\mathbf{I}^{-1}\cdot\boldsymbol{\gamma}]$, $\beta = \alpha^2 - 4ABCD, \quad D = A\gamma_1^2+B\gamma_2^2+C\gamma_3^2 \equiv \boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}$.
	$I_4 = A^2p^2+B^2q^2+C^2r^2+\frac{1}{\sqrt{\beta}}[(\alpha-\sqrt{\beta})v(\alpha-\sqrt{\beta})+(\alpha+\sqrt{\beta})u(\alpha+\sqrt{\beta})]$
2	Separable in spherical coordinates on the Poisson sphere $B = A$
	$V = F(\gamma_3)+\frac{A-(A-C)\gamma_3^2}{A(1-\gamma_3^2)}G(\frac{\gamma_1}{\gamma_2}) \equiv F(\gamma_3)+\frac{A-(A-C)\gamma_3^2}{A\gamma_2^2}g(\frac{\gamma_1}{\gamma_2}), \qquad G(\frac{\gamma_1}{\gamma_2}) = \frac{g(\frac{\gamma_1}{\gamma_2})}{1+(\frac{\gamma_1}{\gamma_2})^2}$
	$I_4 = Cr^2+2G(\frac{\gamma_1}{\gamma_2})$
3	Separable in sphero-conic coordinates on the Poisson sphere [390, 391] $A = B = C$,
	$V = \frac{[u(\alpha'-\sqrt{\beta'})+v(\alpha'+\sqrt{\beta'})]}{\sqrt{\beta'}}$, $\alpha' = a+b+c-(a\gamma_1^2+b\gamma_2^2+c\gamma_3^2),\ \beta' = \alpha'^2-4abc(\frac{\gamma_1^2}{a}+\frac{\gamma_2^2}{b}+\frac{\gamma_3^2}{c})$.
	$I_4 = (ap^2+bq^2+cr^2) + \frac{1}{\sqrt{\beta'}}[(\alpha'+\sqrt{\beta'})u(\alpha'-\sqrt{\beta'})+(\alpha'-\sqrt{\beta'})v(\alpha'+\sqrt{\beta'})]$

A special case of this separable potential, equivalent to $F(\gamma_3) = 0$, was pointed out also by Kolossov in [224], but the general potential seems to be unnoticed in the literature. This type of potential appears as a part of the potential in some integrable generalizations of Kowalevski's case, which admit an integral quartic in velocities. The quartic integral in those cases can be written as the square of the quadratic integral in the separable

13.1.2 Non-separable Cases with a Quadratic Integral

4	Yehia [414] Separable ($K = 0$), Subcase of Steklov's ($s = 0$)
	$V = \frac{sS}{2ABC\sqrt{S^2-4ABCD}}$, $S = A(B+C)\gamma_1^2 + B(C+A)\gamma_2^2 + C(A+B)\gamma_3^2$ $\boldsymbol{\mu} = -K(\frac{\gamma_1}{A}, \frac{\gamma_2}{B}, \frac{\gamma_3}{C})$, $\boldsymbol{l} = \frac{K}{2ABC}(A(B+C)\gamma_1, B(C+A)\gamma_2, C(A+B)\gamma_3)$
	$I_4 = \frac{1}{2}(A^2p^2 + B^2q^2 + C^2r^2) - K(p\gamma_1 + q\gamma_2 + r\gamma_3) + \frac{s}{\sqrt{S^2-4ABCD}}$
5	Yehia [394]
	$V = n\frac{A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2}{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}$ $\boldsymbol{\mu} = \frac{\partial F}{\partial \boldsymbol{\gamma}} - \Phi\boldsymbol{\gamma}$ $F = J\frac{A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2}{\sqrt{ABC(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)}}$
	$\Phi = J\frac{2(A+B+C)v_0-3v_0^2+\frac{2ABC}{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}}{\sqrt{ABC(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2)}}$
	$I_3 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 + F$ I_4 constructed in elliptic coordinates and not in Euler-Poisson variables. See [394]
6	Yehia [401] Separable ($N = 0$) Subcase of Lyapunov's ($K = 0$)
	$A = B = C$ $V = C[-\frac{N^2abc}{2}(\frac{\gamma_1^2}{a} + \frac{\gamma_2^2}{b} + \frac{\gamma_3^2}{c}) + \frac{K}{\sqrt{\beta'}}]$ $\mathbf{l} = -\frac{1}{2}C((b+c)\gamma_1, (c+a)\gamma_2, (a+b)\gamma_3)$ $\mu = C(a\gamma_1, b\gamma_2, c\gamma_3)$
	$I_4 = \frac{1}{2}[(b+c)\,p^2 + (c+a)\,q^2 + (a+b)\,r^2]$ $-Nabc(p\frac{\gamma_1}{a} + q\frac{\gamma_2}{b} + r\frac{\gamma_3}{c}) + \frac{1}{2}K\frac{a+b+c+a\gamma_1^2+b\gamma_2^2+c\gamma_3^2}{\sqrt{\beta'}}$

13.2 Cases with a Cubic Integral

The list of such cases comprises only three items:

7	Yehia and Elmandouh 2016 [424] (c_1, c_2 added)	
	Yehia 2002 [409]	$c_1 = c_2 = 0$
	Sokolov and Tsiganov 2002 [337]	$e_0 = e_1 = \lambda = 0$
	Yehia [395] 1996 (Independently of [226])	$e_0 = e_1 = c_1 = c_2 = 0$
	Komarov and Kuznetsov [226] 1987	$e_0 = e_1 = c_1 = c_2 = a_2 = 0$
	Sretensky [341] 1963	$e_0 = e_1 = c_1 = c_2 = \lambda = 0$
	Goryachev [118] 1916 Goryachev-Chaplygin [115] 1900, [52] 1901	$e_0 = e_1 = c_1 = c_2 = k = 0$ $e_0 = e_1 = c_1 = c_2 = k = \lambda == 0$

	$A = B = 4C,$ $\boldsymbol{l} = C\left(0, 0, k + c_1\gamma_1 + c_2\gamma_2 + e_0(\frac{2}{\gamma_3^4} - \frac{1}{\gamma_3^2}) + \frac{e_1}{\gamma_1^2+\gamma_2^2}\right),$ $\mu_1 = C[c_1\gamma_3 + 2e_0\frac{\gamma_1(4-\gamma_3^2)}{\gamma_3^5} - \frac{2e_1\gamma_1\gamma_2}{(\gamma_1^2+\gamma_2^2)^2}],$ $\mu_2 = C[c_2\gamma_3 + 2e_0\frac{\gamma_2(4-\gamma_3^2)}{\gamma_3^5} - \frac{2e_1\gamma_1\gamma_2}{(\gamma_1^2+\gamma_2^2)^2}],$ $\mu_3 = C[k + c_1\gamma_1 + c_2\gamma_2 + e_0\frac{\gamma_2(2-\gamma_3^2)}{\gamma_3^4} + \frac{e_1}{\gamma_1^2+\gamma_2^2}],$ $V = C\,\{a_1\gamma_1 + a_2\gamma_2 - c_1c_2\gamma_1\gamma_2$ $+\frac{c_2^2}{2}\gamma_1^2 + \frac{c_1^2}{2}\gamma_2^2 + \frac{\lambda}{\gamma_3^2}$ $+e_0\left(\gamma_3^2 - 2\right)\frac{c_1\gamma_1+c_2\gamma_2}{\gamma_3^4} - e_0^2\frac{4-8\gamma_3^2+5\gamma_3^4}{2\gamma_3^8}$ $+e_1\frac{k+e_0-\gamma_1c_1-\gamma_2c_2}{\gamma_1^2+\gamma_2^2} - \frac{1}{2}e_1^2\frac{4\gamma_1^2+4\gamma_2^2+1}{\left(\gamma_1^2+\gamma_2^2\right)^2}\}$
	$I_3 = 4p\gamma_1 + 4q\gamma_2$ $+[r + k + c_1\gamma_1 + c_2\gamma_2 + \frac{e_1}{\gamma_1^2+\gamma_2^2} + e_0(\frac{2}{\gamma_3^4} - \frac{1}{\gamma_3^2})]\gamma_3 = 0,$
	$I_4 = \left[r - k + c_1\gamma_1 + c_2\gamma_2 + \frac{e_0(2-\gamma_3^2)}{\gamma_3^4} - \frac{e_1(8\gamma_1^2-1)}{(\gamma_1^2+\gamma_2^2)}\right]\Big\{[p + \frac{c_1}{2}\gamma_3]^2$ $+[q + \frac{c_2\gamma_3}{2}]^2 + \frac{\lambda}{2\gamma_3^2} + k(\frac{e_0}{\gamma_3^4} - \frac{e_1}{2}) - (\frac{e_1}{2} + \frac{e_0(2-\gamma_3^2)}{2\gamma_3^4})r$ $+(c_1\gamma_1 + c_2\gamma_2)\,[\frac{e_1}{2} + \frac{e_0(2-\gamma_3^2)}{2\gamma_3^4}] - \frac{e_0^2(3\gamma_3^4-6\gamma_3^2+4)}{2\gamma_3^8}$ $+\frac{e_1}{\gamma_1^2+\gamma_2^2}[\frac{e_1(1-8\gamma_1^2)}{2} + \frac{8\gamma_1^2(\gamma_3^2-2)-2\gamma_3^4+9\gamma_3^2-8}{2\gamma_3^4}]\Big\}$ $-\gamma_3[(2e_1c_1 - c_1k + a_1)(p + \frac{c_1\gamma_3}{2})$ $+(2e_1c_2 - c_2k + a_1)(q + \frac{c_2}{2}\gamma_3)]$ $+k\Big\{(c_1\gamma_1 + c_2\gamma_2)[\frac{e_0}{\gamma_3^2} + \frac{e_1(1-2\gamma_3^2)}{\gamma_1^2+\gamma_2^2}] + \frac{4e_1\gamma_1^2(e_0-2e_1\gamma_3^2)}{\gamma_3^2(\gamma_1^2+\gamma_2^2)}$ $-\frac{4e_1\gamma_3}{\gamma_1^2+\gamma_2^2}(p\gamma_1 + q\gamma_2)\Big\}$ $-\frac{8e_0e_1c_2\gamma_2\gamma_1^2}{\gamma_3^2(\gamma_1^2+\gamma_2^2)} + (a_1\gamma_1 + a_2\gamma_2)[\frac{e_0}{\gamma_3^2} - \frac{e_1}{\gamma_1^2+\gamma_2^2}] + 4e_1\gamma_1^2[\frac{2e_0^2}{\gamma_3^6} + \frac{\lambda}{\gamma_3^2(\gamma_1^2+\gamma_2^2)}]$ $-\frac{8e_1c_1\gamma_1^3}{\gamma_1^2+\gamma_2^2}[e_1 + \frac{e_0}{\gamma_3^2}] + \frac{4e_0e_1^2\gamma_1^2}{\gamma_3^4(\gamma_1^2+\gamma_2^2)^2}[8\gamma_1^2(\gamma_3^2 - 2) - (\gamma_1^2 + \gamma_2^2)(\gamma_3^2 - 4)]$ $+\frac{e_1(c_1^2+c_2^2)}{9(\gamma_1^2+\gamma_2^2)}[18\gamma_1^2\gamma_3^2 - 9\gamma_3^4 + 13\gamma_3^2 - 4] + \frac{8e_1^3\gamma_1^3(1-4\gamma_1^2)}{(\gamma_1^2+\gamma_2^2)^2}$ $+\frac{2e_1\gamma_3}{\gamma_1^2+\gamma_2^2}\{q[c_2(5\gamma_1^2 - \gamma_2^2) - 2c_1\gamma_1\gamma_2]$ $+p[c_1(3\gamma_1^2 + \gamma_2^2) - 2c_2\gamma_1\gamma_2]\}$ $-\frac{16e_1\gamma_1\gamma_2}{\gamma_1^2+\gamma_2^2}\left[pq - \frac{e_1c_2}{2}\gamma_1\right] + \frac{8e_1(\gamma_1^2-\gamma_2^2)}{\gamma_1^2+\gamma_2^2}q^2$

The second case with a cubic complementary integral is due to Goryachev [117]. It is characterized by the following

8	Goryachev 1915 [117].
	$A = B = \frac{4}{3}C,$
	$\boldsymbol{l} = \boldsymbol{\mu} = \boldsymbol{0},$
	$V = \frac{a\gamma_1+b\gamma_2+c}{\gamma_3^{\frac{2}{3}}},$
	$I_3 = \frac{4}{3}(p\gamma_1 + q\gamma_2) + r\gamma_3 = 0,$
	$I_4 = 2r(p^2 + q^2) + r^3 - 2a\gamma_3^{\frac{1}{3}}p + r\frac{a+b\gamma_1}{\gamma_3^{\frac{2}{3}}}.$

Although having no obvious physical meaning, this case has received a growing interest in the last years [45, 361]. It turns out to be the first example of a mechanical system whose complex invariant varieties are strata of Jacobians of a non-hyper-elliptic curve, here a trigonal curve of genus 3 [45].

Goryachev's case 8 has been generalized to the following one involving two (vector and scalar) potentials (Yehia 2002 [409]):

8*	$\boldsymbol{l} = (0, 0, \kappa + \frac{1}{\gamma_1^2+\gamma_2^2}[3\kappa + e_0\gamma_3^{\frac{2}{3}} + \frac{e_1(2+\gamma_3^2)}{\gamma_3^{\frac{2}{3}}}])$, $V = \frac{a\gamma_1+b\gamma_2+c}{\gamma_3^{\frac{2}{3}}} + \frac{1}{(\gamma_1^2+\gamma_2^2)^2}[\frac{e_0^2(4-7\gamma_3^2)}{6\gamma_3^{\frac{2}{3}}} - \frac{e_1^2(13\gamma_3^4-8\gamma_3^2+4)}{2\gamma_3^{\frac{4}{3}}} - e_0e_1(5\gamma_3^2-2) + \frac{3\kappa^2\gamma_3^2}{2}(\gamma_3^2-4) - 3e_0\kappa\gamma_3^{\frac{8}{3}} - 3\kappa e_1\gamma_3^{\frac{4}{3}}(\gamma_3^2+2)]$.

where κ, e_0 and e_1 are arbitrary constants. Note that the constant gyrostatic momentum κ is a coupling constant for some potential and gyroscopic terms.

For this generalization, one can easily write

$$I_3 = \frac{4}{3}(p\gamma_1 + q\gamma_2) + \{r + \kappa + \frac{1}{\gamma_1^2+\gamma_2^2}[3\kappa + e_0\gamma_3^{\frac{2}{3}} + \frac{e_1(2+\gamma_3^2)}{\gamma_3^{\frac{2}{3}}}]\}\gamma_3 = 0,$$

but the complementary cubic integral is not yet expressed in the Euler–Poisson variables (See [409]).

Cases 7 and 8 were constructed as special cases of an integrable many-parameter system with a cubic integral under some restrictions on those parameters.

9	Yehia 2000 [404]. Gaffet 1998 [96, 97] (The equivalent problem for a particle on a sphere).
	$A = B = C$, $V = \frac{K}{(\gamma_1\gamma_2\gamma_3)^{\frac{2}{3}}}$, $\mathbf{l} = \mathbf{0}$.
	$I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 = 0$, $I_4 = Apqr - 2K(\gamma_1\gamma_2\gamma_3)^{\frac{1}{3}}(\frac{p}{\gamma_1} + \frac{q}{\gamma_2} + \frac{r}{\gamma_3})$.

13.3 Cases with a Quartic Integral

The thirteen presently known cases with a quartic integral are characterized by the Kowalevski configuration $A = B = 2C$. They are mostly (but not all) generalizations on the level $f = 0$ of the two classical cases: Kowalevski's case of a heavy body and Chaplygin's case of a body in a liquid. It is curious to note that the main potential in most of those cases is composed of the basic potential present in Kowalevski's or Chaplygin's cases or both of them and some additional terms that belong to separable potentials discussed in Chap. 9 Sect. 9.7.1. When alone, the last terms give rise to a quadratic integral, instead of the quartic one.

Those cases are divided into five types, presented in the following subsections.

13.3.1 Cases Stemming from Kowalevski's Case

Four cases of this type are listed here:

10	Yehia 1996 [396], $\lambda = 0$ Yehia-Bedwehy 1987, (unconditional case) [419] (unconditional case), $\lambda = \varepsilon = 0$ Kowalevski 1888 [238] (unconditional case).
	$A = B = 2C$, $V = C[a_1\gamma_1 + a_2\gamma_2 + \frac{\varepsilon}{\sqrt{\gamma_1^2+\gamma_3^2}} + \frac{\lambda}{\gamma_3^2}]$, $\boldsymbol{l} = \boldsymbol{\mu} = \mathbf{0}$.
	$I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 = 0$, $I_4 = [p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{2\gamma_3^2}]^2$ $+[2pq - a_1\gamma_2 - a_2\gamma_1 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$ $+\varepsilon[\frac{(p+n\gamma_1)^2+(q+n\gamma_2)^2}{\sqrt{\gamma_1^2+\gamma_3^2}} + \frac{\varepsilon}{\gamma_1^2+\gamma_3^2} + \frac{2\lambda\sqrt{\gamma_1^2+\gamma_3^2}}{\gamma_3^2}]$
11	Yehia 2006 [413] $\nu_1 = \delta_2 = 0$: Yehia-Bedwehy 1987 [419] $A = B = 2C$. $V = C[a_1\gamma_1 + \frac{\lambda}{\gamma_3^2} + \frac{\varepsilon}{\sqrt{1-\gamma_3^2}} + \frac{2-\gamma_3^2}{\gamma_2^2}(\delta_2 + \frac{\nu_1\gamma_1}{\sqrt{1-\gamma_3^2}})]$. $\boldsymbol{l} = \boldsymbol{\mu} = \mathbf{0}$.
	$I_3 = p\gamma_1 + q\gamma_2 + r\gamma_3 = 0$, $I_4 = [p^2 - q^2 - a_1\gamma_1 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}]^2 + [2pq - a_1\gamma_2 - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$ $+[\delta_2\frac{\gamma_3^2}{\gamma_2^2} + \frac{\varepsilon+\nu_1\gamma_1\frac{\gamma_3^2}{\gamma_2^2}}{\sqrt{1-\gamma_3^2}}][2p^2 + 2q^2 + \delta_2\frac{\gamma_3^2}{\gamma_2^2} + \frac{\varepsilon+\nu_1\gamma_1\frac{\gamma_3^2}{\gamma_2^2}}{\sqrt{1-\gamma_3^2}}]$ $+\frac{2\lambda\varepsilon\sqrt{1-\gamma_3^2}}{\gamma_3^2} - \frac{2}{\gamma_2^2}(a_1\gamma_3^2 + \lambda\gamma_1)(\delta_2\gamma_1 + \nu_1\sqrt{1-\gamma_3^2})$

The main result of [413] was the construction of an integrable system of two degrees of freedom living on a Riemannian (or pseudo-Riemannian[1]) manifold and admitting an integral of degree four in velocities. Cases 11, 14 were obtained as special cases under suitable restrictions of this twenty-one-parameter system that render the metric to that of the Routhian of the rigid body dynamics. Case 18 below was obtained by further development of the method of [413].

Separation variables and expressions of the dynamical variables in terms of them are constructed for case 11 in [218] (See also [137]), without treating the inversion of the resulting quadratures.

[1] In differential geometry, that is a manifold whose metric is not necessarily positive-definite

12	Yehia, Elmandouh 2011 [422] $c = 0$. Yehia-Bedwehy (unconditional case) [419] $\lambda = 0$. Sokolov (unconditional case)
	$A = B = 2C$, $V = C[a\gamma_1 + \frac{c^2}{2}(\gamma_1^2 - \gamma_3^2) + \frac{\varepsilon}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{\lambda}{\gamma_3^2}]$, $\boldsymbol{\mu} = Cc(0, \gamma_3, \gamma_2)$, $I_4 = (p^2 - q^2 - a\gamma_1 - cr\gamma_2 - c^2\gamma_1^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2})^2$ $+(2pq - a\gamma_2 + cr\gamma_1 - c^2\gamma_1\gamma_2 - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2})^2$ $+\varepsilon[\frac{2(p^2-q^2)}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{\varepsilon}{\gamma_1^2+\gamma_2^2} + 2\sqrt{\gamma_1^2 + \gamma_2^2}(c^2 + \frac{\lambda}{\gamma_3^2})]$

13	Yehia, Elmandouh 2011 [422], $\lambda = 0$: the integral I_4 becomes unconditional and gives Sokolov's [336].
	$A = B = 2C$, $V = C[\kappa c\gamma_1 + a_2\gamma_2 + \frac{c^2}{2}(\gamma_2^2 - \gamma_3^2) + \frac{\lambda}{\gamma_3^2}]$, $\boldsymbol{l} = C\,(0, 0, k + c\gamma_1)$, $\boldsymbol{\mu} = C(c\gamma_3, 0, \kappa + c\gamma_1)$, $I_4 = \left[p^2 - q^2 + a_2\gamma_2 + c^2\gamma_2^2 - c\gamma_1(\kappa - r) - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}\right]^2$ $+\left[2pq - a_2\gamma_1 - c^2\gamma_1\gamma_2 - c\gamma_2(\kappa - r) - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}\right]^2$ $+2\kappa(r - \kappa + c\gamma_1)[p^2 + q^2 + 2cp\gamma_3 + \lambda(1 + \frac{1}{\gamma_3^2}) - \gamma_3\left(2qa_2 + c^2\kappa\gamma_3\right)]$ $-2\kappa c^2[4\gamma_3(p\gamma_1 + q\gamma_2) - (1 - \gamma_3^2)(r + c\gamma_1)]$.

13.3.2 Cases Stemming from Chaplygin's Case

Four cases of this type are listed in the following table:

14	Yehia 2006 [413] $\delta_1 = \delta_2 = 0$, Yehia 2003 [411] $\delta_1 = \delta_2 = \rho = 0$, Goryachev [118] $\delta_1 = \delta_2 = \rho = \lambda = 0$, Chaplygin [53]
	$A = B = 2C$, $V = C[b_1(\gamma_1^2 - \gamma_2^2) + \frac{\lambda}{\gamma_3^2} + \rho(\frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6})$ $+(2 - \gamma_3^2)(\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2})]$, $\boldsymbol{l} = \boldsymbol{\mu} = \mathbf{0}$, $I_3 = 2p\gamma_1 + 2q\gamma_2 + r\gamma_3 = 0$, $I_4 = [p^2 - q^2 + b_1\gamma_3^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}]^2 + 4[pq - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$ $+2(p^2 + q^2)\{\rho[\frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6}] + \gamma_3^2[\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2}]\} + \rho\frac{(\gamma_1^2+\gamma_2^2)^2}{\gamma_3^{12}}(\rho - 2\lambda\gamma_3^4)$ $+\frac{2\rho b_1(\gamma_1^2-\gamma_2^2)}{\gamma_3^4} - 2b_1\gamma_3^4[\frac{\delta_1}{\gamma_1^2} - \frac{\delta_2}{\gamma_2^2}] + \gamma_3^4[\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2}]^2$ $-2(\rho + \lambda\gamma_3^4)\frac{1-\gamma_3^2}{\gamma_3^4}[\frac{\delta_1}{\gamma_1^2} + \frac{\delta_2}{\gamma_2^2}]$.

Together with case 11, this case was constructed in [413] (See comment next to case 11). Separation variables and quadratures were constructed for this case (and for case 11) in [137].

15	Yehia-Elmandouh 2013 [423] $K=0$ Yehia [411] (Sect. 4.2.3) 2003 $K=\rho=0$ Goryachev [118] 1916 $K=\rho=\lambda=0$ Chaplygin [53] 1903
	$A=B=2C$ $V = C\{k\left[2d\gamma_1\gamma_2 + c(\gamma_1^2-\gamma_2^2)\right]$ $+K^2[2cd\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2) + \frac{d^2}{2}(\gamma_3^4+4\gamma_1^2\gamma_2^2)$ $-c^2(\gamma_3^2(\gamma_1^2+\gamma_2^2)+2\gamma_1^2\gamma_2^2)] + \frac{\lambda}{\gamma_3^2} + \rho(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})\},$ $\mathbf{l} = C(0, 0, K[d(\gamma_2^2-\gamma_1^2)+2c\gamma_1\gamma_2]),$ $\boldsymbol{\mu} = C(2K\gamma_3(c\gamma_2-d\gamma_1), 2K\gamma_3(d\gamma_2+c\gamma_1),$ $K[d(\gamma_2^2-\gamma_1^2)+2c\gamma_1\gamma_2]).$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + \{r + K\left[d(\gamma_2^2-\gamma_1^2)+2c\gamma_1\gamma_2\right]\}\gamma_3,$ $I_4 = \{p^2 - q^2 + ck\gamma_3^2 + \gamma_3^2[Kdr + cK^2(c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2)]$ $-\frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}\}^2$ $+\{2pq + dk\gamma_3^2 + [dK^2(c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2) - Kcr]\gamma_3^2$ $-\frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}\}^2$ $+\rho\{\frac{2(\gamma_3^2-1)}{\gamma_3^6}[p^2+q^2]$ $-\frac{2Kr}{\gamma_3^4}[2c\gamma_1\gamma_2 + d(\gamma_2^2-\gamma_1^2)]$ $+\frac{(1-\gamma_3^2)^2}{\gamma_3^{12}}(\rho - 2\lambda\gamma_3^4)$ $+K^2[2c^2(\frac{1}{\gamma_3^4}-\frac{2}{\gamma_3^2}) + 8\frac{(d^2-c^2)\gamma_1^2\gamma_2^2 + cd\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2)}{\gamma_3^4}]$ $+\frac{2k}{\gamma_3^4}[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2]\}.$

16	Yehia and Elmandouh 2016 [425] $\kappa = 0$: Special case of Yehia and Elmandouh [423]
	$A=B=2C$ $V = C\{\kappa[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2] + \kappa K[2c\gamma_1\gamma_2 - d(\gamma_1^2-\gamma_2^2)]$ $+K^2\{2cd\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2) - c^2[\gamma_3^2(\gamma_1^2+\gamma_2^2)+2\gamma_1^2\gamma_2^2]$ $+\frac{d^2}{2}(\gamma_3^4+4\gamma_1^2\gamma_2^2)\} + \frac{\lambda}{\gamma_3^2}\},$
	$\mathbf{l} = C(0, 0, \kappa + K[2c\gamma_1\gamma_2 - d(\gamma_1^2-\gamma_2^2)]),$
	$\boldsymbol{\mu} = C(2K\gamma_3(c\gamma_2-d\gamma_1), 2K\gamma_3(c\gamma_1+d\gamma_2), \kappa + K[2c\gamma_1\gamma_2 + d(\gamma_2^2-\gamma_1^2)]),$
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + (r + \kappa + K[2c\gamma_1\gamma_2 + d(\gamma_2^2-\gamma_1^2)])\gamma_3,$
	$I_4 = \left\{p^2 - p^2 + ck\gamma_3^2 + cK^2\gamma_3^2[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2]\right.$ $\left.+dK[2\kappa - \gamma_3^2(3\kappa - r)] - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}\right\}^2$ $+\left\{2pq + kd\gamma_3^2 + dK^2\gamma_3^2[2d\gamma_1\gamma_2 + c(\gamma_1^2-\gamma_2^2)]\right.$ $\left.+cK[\gamma_3^2(3\kappa - r) - 2\kappa] - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}\right\}^2$ $+2\kappa[r - \kappa - K(2c\gamma_1\gamma_2 + d(\gamma_2^2-\gamma_1^2))]\left\{p^2 + q^2 + \lambda(1+\frac{1}{\gamma_3^2})\right.$ $\left.+\gamma_3^2[K^2(c^2+d^2)(\gamma_3^2-1) - 2d\kappa K + \kappa c]\right\}$

	$-4\kappa\gamma_3\{[2K\kappa(c\gamma_1+2d\gamma_2)-k(2c\gamma_2-d\gamma_1)](q+n\gamma_2)$ $+\gamma_2(p+n\gamma_1)(2c\kappa K+dk)\}$ $-8\kappa\left\{c^2K\gamma_3^2[\kappa K(\gamma_3^2-1)-k\gamma_1\gamma_2]\right.$ $+c\{2\kappa dK^2\gamma_1\gamma_2\gamma_3^2+K[\frac{1}{2}kd\gamma_3^4-2\lambda\gamma_1\gamma_2$ $+dk\gamma_3^2(\gamma_1^2-\frac{1}{2})]-\frac{1}{2}k\kappa\gamma_3^2\}$ $\left.-2\kappa d^2K^2\gamma_1^2\gamma_3^2+K[2\lambda d\gamma_1^2+d(\kappa^2+\lambda)\gamma_3^2]\right\}$

17	Yehia 2003 [411] Goryachev 1916 [118], $\rho=0$. Chaplygin 1903 [53] $\lambda=\rho=0$.
	$A=B=2C$,
	$V=C[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2+\frac{\lambda}{\gamma_3^2}+\rho(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})]$,
	$\boldsymbol{l}=\boldsymbol{\mu}=\mathbf{0}$,
	$I_3=2p\gamma_1+2q\gamma_2+r\gamma_3=0$,
	$I_4=[p^2-q^2+c\gamma_3^2-\frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}]^2+[2\,pq+d\gamma_3^2-2\,\frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$ $+2\,\rho\,[(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})\left(p^2+q^2\right)+c\,\frac{(\gamma_1^2-\gamma_2^2)}{\gamma_3^4}+2\,d\,\frac{\gamma_2\,\gamma_1}{\gamma_3^4}-\lambda\frac{(1-\gamma_3^2)^2}{\gamma_3^8}]$ $+\rho^2\,\frac{(1-\gamma_3^2)^2}{\gamma_3^{12}}$.

Elmandouh (2015) [73] introduced a two-parameter generalization of this case by adding singular terms into the vector and scalar potentials:

17*	$V=C[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2+\frac{\lambda}{\gamma_3^2}+\rho(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})+\frac{(\gamma_3^2-2)\gamma_3^2}{2\gamma_1^2}(\frac{\nu_1}{\gamma_1}+\frac{\nu_2}{\gamma_2})^2]$,
	$\boldsymbol{l}=C(0,0,\frac{2-\gamma_3^2}{\gamma_1}(\frac{\nu_1}{\gamma_1}+\frac{\nu_2}{\gamma_2}))$.

The fourth integral is also given in the Euler–Poisson variables in [73].

13.3.3 Cases Combining Kowalevski's and Chaplygin's Cases

Two cases are listed in the following table:

18	Yehia 2012 [415], $\delta=0$ Goryachev 1917 [118], $\delta=\lambda=a_1=0$ Chaplygin 1903 [53], $\delta=\lambda=a_2=0$ Kowalevski 1888 [238],
	$A=B=2C$, $V=2C[a_1\gamma_1+a_2(\gamma_1^2-\gamma_2^2)+\frac{\lambda}{\gamma_3^2}+\delta\frac{2-\gamma_3^2}{\gamma_2^2}]$, $\boldsymbol{l}=\boldsymbol{\mu}=\mathbf{0}$,
	$I_3=2p\gamma_1+2q\gamma_2+r\gamma_3=0$, $I_4=[p^2-q^2-a_1\gamma_1+a_2\gamma_3^2-\frac{\lambda(\gamma_1^2-\gamma_2^2)}{\gamma_3^2})^2$ $+(2pq-a_1\gamma_2-\frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2})^2$ $+\frac{\delta}{\gamma_2^2}\{2[(p^2+q^2]\gamma_3^2-2a_1\gamma_1\gamma_3^2-2\lambda\gamma_1^2+2a_2+\frac{\delta\gamma_3^4}{\gamma_2^2}\}$.

The Goryachev subcase ($\delta = 0$) of case 18 has led to complex separation variables in [337, 359].

19	Yehia 1996 [396], $\lambda = 0$. Yehia 1987 [386], $k = 0$. Goryachev [118], $k = a_2 = \lambda = 0$ Chaplygin 1903 [53], $k = b_1 = b_2 = \lambda = 0$ Kowalevski 1888 [238],
	$V = C[a_1\gamma_1 + a_2\gamma_2 + b_1(\gamma_1^2 - \gamma_2^2) + b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}]$, $\boldsymbol{l} = \boldsymbol{\mu} = C(0, 0, k)$.
	$I_3 = 2p\gamma_1 + 2q\gamma_2 + (r + k)\gamma_3 = 0$, $I_4 = [p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2 + b_1\gamma_3^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{2\gamma_3^2}]^2$ $+[2pq - a_1\gamma_2 - a_2\gamma_1 + b_2\gamma_3^2 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}]^2$ $+k\{(r - k)[2(p^2 + q^2) + \lambda(1 + \frac{1}{\gamma_3^2})]$ $-4\gamma_3[(a_1 + b_1\gamma_1 + b_2\gamma_2)p + q(a_2 + b_2\gamma_1 - b_1\gamma_2)]\}$.

Elmandouh (2015) [74], added a parameter e, which engenders singular terms in the vector and scalar potentials, as follows:

19*	$V = C\{a_1\gamma_1 + a_2\gamma_2 + b_1(\gamma_1^2 - \gamma_2^2) + b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}$ $-\frac{e\gamma_3^2}{\gamma_1^2}[k - \frac{e(2\gamma_2^2+\gamma_3^2)}{2\gamma_1^2}]\}$, $\boldsymbol{l} = C\left(0, 0, k + \frac{e(1+\gamma_2^2)}{\gamma_1^2}\right)$, $\boldsymbol{\mu} = C\left(\frac{-2e\gamma_3}{\gamma_1^3}(1 + \gamma_2^2), \frac{2e\gamma_2\gamma_3}{\gamma_1^2}, k + \frac{e(1+\gamma_2^2)}{\gamma_1^2}\right)$,

The complementary integral was also provided in [74].

Separation of variables for the Chaplygin level of the above hierarchies was attained by Chaplygin himself. For detailed solution see Chap. 10 Sect. 10.16. The version $\kappa \neq 0$ of this case was considered in Tsiganov's work [358], where an assertion is made that separated variables are constructed for what the author calls "the Kowalevski–Goryachev–Chaplygin gyrostat". However, the proposed separated variables are complex functions of the physical variables. It remains an open problem how to construct real solutions using complex hyper-elliptic quadratures [358]. **Note that** the title and references in that work have brought certain confusion, which was commented in our note [410].

In [416], it was shown that the problem of motion of a rigid body, with $A = 2C$ and arbitrary B, subject to forces with potential containing one Kowalevski term, one Chaplygin term together with the singular Goryachev term

$$V = a_1\gamma_1 + b_1(\gamma_1^2 - \gamma_2^2) + \frac{c_1}{\gamma_3^2}, \tag{13.1}$$

under the additional restrictions $q = 0$, $f = 0$, is solvable in elliptic functions of time. The solution is the same in case 17, when $A = B = 2C$, $n = \kappa = a_2 = b_2 = 0$

under the additional restriction $q = 0$. Without this restriction, the solution corresponding to the last potential (13.1) is not known at this moment.

The subcase with the potential

$$V = b_1(\gamma_1^2 - \gamma_2^2) + \frac{c_1}{\gamma_3^2}, \tag{13.2}$$

common between hierarchies #14-19, has attracted more attention. It is called by some authors the "Goryachev system". Ryabov found real separation variables for this case in [323], reduced its integration to hyper-elliptic quadratures and studied the phase topology for positive values of the parameters, i.e. when the integral surfaces are compact. The case of negative values of the parameters, when the integral surfaces become non-compact, is treated by Nikolaenko [297], who has also shown that Goryachev's system is Liouville equivalent to other integrable cases in rigid body dynamics, according to the value of the energy parameter on the admissible energy interval $[h_{min}, \infty)$ [296].

13.3.4 A Case with a Quartic Integral Outside the Above Classification

20	Yehia 2003 [411], $f = 0$,
	$A = B = 2C$, $V = \frac{a\gamma_3}{(\gamma_1^2+\gamma_2^2)^{\frac{3}{4}}} + \frac{b}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{\gamma_3\sqrt{c\gamma_1+d\gamma_2+\sqrt{(c^2+d^2)(\gamma_1^2+\gamma_2^2)}}}{\sqrt{\gamma_1^2+\gamma_2^2}}$, $\boldsymbol{\mu} = \mathbf{0}$.

This is a case of algebraic potential, which involves three singular terms of different fractional powers. It reminds the case of fractional power potential and third-degree integral due to Goryachev. The fourth integral for this case can be expressed in terms of Euler's angles, using formulas provided in [411], but it is not constructed yet in the Euler-Poisson variables.

13.3.5 Two Conditional Cases Valid on a Single, Not Necessary Zero, Level of the Linear Integral [421]

This case adds to Yehia's gyrostat the parameter m, figuring in potential and gyroscopic terms, and turns into it when $m = 0$. The quartic integral is expressed in terms of Euler's angles, but not in Euler-Poisson variables [421] (Table 13.2).

Table 13.2 Conditional cases on a single level of the linear integral $f = \alpha(\alpha arbitrary)$

1	$A = B = 2C$,
	$V = a\gamma_1 + b\gamma_2 - \frac{m}{2(\gamma_1^2+\gamma_2^2)}\left[2(k-m) - 2\alpha\gamma_3 + \frac{m}{\gamma_1^2+\gamma_2^2}\right]$,
	$\boldsymbol{l} = (0, 0, k + \frac{m}{\gamma_1^2+\gamma_2^2})$
2	$V = a\gamma_1 + b\gamma_2 - \frac{k}{2(\gamma_1^2+\gamma_2^2)}\left[-2\alpha\gamma_3 + \frac{k}{\gamma_1^2+\gamma_2^2}\right] + \frac{\lambda+\gamma_3\sqrt{\frac{c^2+d^2}{2}\sqrt{\gamma_1^2+\gamma_2^2}+\frac{c^2-d^2}{2}\gamma_2+cd\gamma_1}}{\sqrt{\gamma_1^2+\gamma_2^2}}$,
	$\boldsymbol{l} = (0, 0, k + \frac{k}{\gamma_1^2+\gamma_2^2})$.

Case 2 generalizes the case of Yehia and Bedwehi. In both cases, the added new terms are all singular at the two positions $\boldsymbol{\gamma} = (0, 0, \pm 1)$.

13.4 Integrable Extensions of Conditional Integrable Cases

As remarked in the beginning of this chapter, the method of transformation with an arbitrary function $\nu(\gamma_1, \gamma_2, \gamma_3)$ used in Sect. 12.2 of the last chapter is also applicable to all conditional integrable cases, valid on the zero level of the cyclic integral, i.e. to the 20 cases of this type listed in the last three sections.

We shall not give here a list of generalizations of the conditional cases. Most of those cases involve singular terms that are not likely to get acceptable physical interpretation. Physical effects of the transformation are here immaterial and will remain at present just as parts of mathematical models. Moreover, unlike the generalized cases introduced in Chap. 12, the flexibility offered by the presence of the areas constant as an arbitrary parameter is here lost. The transformed integrable problems and their original counterparts share the same Hamiltonian. To this kind of extension of conditional integrable cases applies the argument of Borisov and Mamaev [41], mentioned in the last section of the preceding chapter and they need not to be considered unless for some reason it becomes necessary to use a concrete form of the function ν in the transformation.

Chapter 14
The Rigid Body Acted upon by a Skew Combination of Fields

The difficulties encountered in the study of the classical problem of motion of a heavy rigid body have not been a stimulant for systematically considering problems of rigid body dynamics in more general settings. In spite of its actuality and importance for scientific and technological purposes and although it is a direct logical generalization of the classical problem, the problem of motion of a rigid body under the action of asymmetric forces was left aside and was considered only occasionally.

In what may be called a historical exception, the quest for integrable cases began with constructing the set of integrals in Brun's problem of motion of a body in an asymmetric gravitational field [48]. This is one of the most complicated motions of a rigid body. More than seven decades elapsed before Bogoyavlensky made a resurrection of this problem, completing the proof of its integrability and indicating a way for its integration [29]. He also gave a general integrable case, in which a spherical body moves under the action of three skew fields. The first general integrable case of a gyrostat acted upon by two skew uniform fields was found in [380] for the body with Kowalevski's configuration and under a restriction coupling the intensities of the fields. Shortly later, this problem was shown to be integrable without that restriction.

In this chapter, we consider the most general problem of motion of a rigid body about a fixed point in an essentially new and general setting, comprising all types of asymmetric, but conservative force fields. We give the first ever systematic presentation of this topic together with exhausting lists of presently known integrable cases, which far exceed in number the three cases described above. Those new integrable cases were obtained mainly by two methods, both depending on the existence of a cyclic coordinate. The first is based on the equivalence between two problems on the level of Routhian reduction after ignoring a cyclic coordinate in each. The second uses the transformation of cyclic coordinates, as explained in Sect. 11.9, to construct a generalization of known cases. We shall get acquainted with those methods along the way.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1_14

14.1 Equations of Motion

Assume that the moving body is acted upon by the most general combination of conservative potential and gyroscopic forces, described by the Lagrangian:

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \boldsymbol{l}.\boldsymbol{\omega} - V, \tag{14.1}$$

where $\mathbf{I} = \text{diag}(A, B, C)$ is the inertia matrix of the body. The scalar and vector potentials $V, \boldsymbol{l}$ depend only on the Eulerian angles through the nine direction cosines $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$.

The Lagrangian (14.1) describes a conservative system with three degrees of freedom, which admits the Jacobi integral (the Hamiltonian of the system)

$$I_1 \equiv H = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + V = \text{const.} \tag{14.2}$$

The equations of motion of a rigid body are usually written in the Euler– Poisson variables $\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. For the present problem this form, corresponding to (14.1), is written as

$$\begin{aligned}\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\mu}) = \boldsymbol{\alpha}\times\frac{\partial V}{\partial\boldsymbol{\alpha}} + \boldsymbol{\beta}\times\frac{\partial V}{\partial\boldsymbol{\beta}} + \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}},\\ \dot{\boldsymbol{\alpha}} + \boldsymbol{\omega}\times\boldsymbol{\alpha} = \mathbf{0}, \dot{\boldsymbol{\beta}} + \boldsymbol{\omega}\times\boldsymbol{\beta} = \mathbf{0}, \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0}\end{aligned} \tag{14.3}$$

where $\mathbf{I}$ is the inertia tensor of the body at the fixed point and

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{l} + (\boldsymbol{\alpha}\times\frac{\partial}{\partial\boldsymbol{\alpha}} + \boldsymbol{\beta}\times\frac{\partial}{\partial\boldsymbol{\beta}} + \boldsymbol{\gamma}\times\frac{\partial}{\partial\boldsymbol{\gamma}})\times\boldsymbol{l}\\ &\equiv \frac{\partial}{\partial\boldsymbol{\alpha}}(\boldsymbol{l}\cdot\boldsymbol{\alpha}) + \frac{\partial}{\partial\boldsymbol{\beta}}(\boldsymbol{l}\cdot\boldsymbol{\beta}) + \frac{\partial}{\partial\boldsymbol{\gamma}}(\boldsymbol{l}\cdot\boldsymbol{\gamma}) - (\frac{\partial}{\partial\boldsymbol{\alpha}}\cdot\boldsymbol{l})\boldsymbol{\alpha} - (\frac{\partial}{\partial\boldsymbol{\beta}\cdot}\boldsymbol{l})\boldsymbol{\beta} - (\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\boldsymbol{l})\boldsymbol{\gamma}\\ &\quad - 2\boldsymbol{l}.\end{aligned} \tag{14.4}$$

Equations (14.3), (14.4) were derived in [390]. They generalize to the case of arbitrary forces Eq. (11.3) derived in Chap. 11 for the case of axi-symmetric forces and can be seen to reduce to them when $\boldsymbol{l}$ and V depend only on $\boldsymbol{\gamma}$. Moreover, it is not hard to show that a part $\boldsymbol{l}_0$ of the vector $\boldsymbol{l}$ has no contribution to the vector $\boldsymbol{\mu}$, if it is derived from a scalar function $\chi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ by the formula

$$\boldsymbol{l}_0 = \boldsymbol{\alpha}\times\frac{\partial\chi}{\partial\boldsymbol{\alpha}} + \boldsymbol{\beta}\times\frac{\partial\chi}{\partial\boldsymbol{\beta}} + \boldsymbol{\gamma}\times\frac{\partial\chi}{\partial\boldsymbol{\gamma}}. \tag{14.5}$$

This part adds a gauge term $\frac{d\chi}{dt}$ in the Lagrangian. The last formula generalizes (11.8) for axi-symmetric fields.

Exercise: Show by direct calculation that the special form of the vector potential $\boldsymbol{l}_0$ (14.5) gives zero gyroscopic vector $\boldsymbol{\mu}$ in (14.4).

The number of integrals of motion necessary for solving the equations of motion.

The system (14.25) is composed of 12 first-order differential equations. It can be easily checked that they satisfy the Jacobi (zero divergence) condition. According to Jacobi's theorem, one must find $12 - 2 = 10$ integrals, of which we know only seven, Jacobi's integral and six geometric integrals

$$|\boldsymbol{\alpha}| = \mathbf{1},\, |\boldsymbol{\beta}| = \mathbf{1},\, |\boldsymbol{\gamma}| = \mathbf{1},\, \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \mathbf{0}, \boldsymbol{\beta}\cdot\boldsymbol{\gamma} = \mathbf{0}, \boldsymbol{\alpha}\cdot\boldsymbol{\gamma} = \mathbf{0}. \tag{14.6}$$

The areas integral, which played an important role in our study of the classical problem, does not exist in the present problem. Thus, it remains to find three more integrals of motion.

Another approach widely used in analytical mechanics applies only to Hamiltonian systems, but it has proved more effective for our purpose. That is applying a theorem due to Liouville (see, for example, [368]), which states that "*For a Hamiltonian system with n degrees of freedom, the knowledge of n integrals of motion, which are in involution is sufficient for completely solving the equations of motion*". Such systems are usually named "completely integrable systems".

To apply Liouville's theorem, we first note that the mechanical system under consideration, i.e. the rigid body in the most general fields, is a Hamiltonian system of 3 degrees of freedom. In principle, one can use the three Eulerian angles as generalized coordinates and the generalized momenta conjugate to them to write equations of motion in the Hamiltonian form. In those coordinates, the geometric integrals turn into identities, and we are left only with one general integral of motion I_1. In our case $n = 3$, so that three integrals are sufficient for integration of the problem. As one integral is known, we need only two other integrals for integrability, independent of the energy integral I_1 and in involution, instead of three demanded by Jacobi's theorem.

It turns out to be more advantageous to keep writing equations of motion in the symmetric and in most cases algebraic form (14.3). Compared to Hamiltonian equations in canonical variables, this form is found to be of greater help in finding expressions for the integrals of motion and in constructing explicit solutions to the problems under consideration.

14.1.1 Interpretation of Forces

Different terms of Eq. (14.3) in their general form may be interpreted in most cases in one or more of the following (or other) ways.

The potential V can be understood as due to the scalar interactions of a gravitational field with the mass distribution in the body, an electric field with a permanent distribution of electric charges and a magnetic field with some magnetized parts or steady currents in electric circuits on the body. A constant term $\boldsymbol{\kappa}$ of the vectors $\boldsymbol{\mu}$ and $\boldsymbol{l}$ is the gyrostatic momentum, while the variable terms of $\boldsymbol{\mu}$ may appear as a result of the Lorentz effect of the magnetic field on the electric charges. Let $\mathcal{B}$ and $\mathcal{A}$ be the intensity of the magnetic field and the vector potential of this field at the point $\mathbf{r}$ of the body where the current charge element de is placed. In that case, one can write the vector $\mathbf{l}$ as (for details, see [382])

$$\mathbf{l} = \boldsymbol{\kappa} + \int \mathbf{r} \times \mathcal{A} de,$$

while $\boldsymbol{\mu}$ can be derived from $\mathbf{l}$ according to (14.4) or constructed directly in the form [382]:

$$\begin{aligned} \boldsymbol{\mu} &= \boldsymbol{\kappa} - \int (\mathbf{r} \cdot \boldsymbol{\mathcal{B}})\mathbf{r} de \\ &= \boldsymbol{\kappa} + \int (\mathbf{r} \cdot \frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{r}})\mathbf{r} de \end{aligned} \tag{14.7}$$

where Ω is the scalar magnetic potential. In several cases of interest for future application, Ω can be expressed as a sum

$$\Omega(X, Y, Z) = \Omega_1(X, Y, Z) + \cdots + \Omega_N(X, Y, Z), \tag{14.8}$$

of homogeneous harmonic polynomials up to the Nth degree; the formula (14.4) can be replaced by

$$\boldsymbol{\mu} = \boldsymbol{\kappa} + \sum_{s=1}^{N} s \int \boldsymbol{\Omega}_s(X, Y, Z)\mathbf{r} de.$$

Now, expressing $\mathbf{r}$ in the moving body axes xyz, we get

$$\boldsymbol{\mu} = \boldsymbol{\kappa} + \sum_{s=1}^{N} s \int \Omega_s(\mathbf{r} \cdot \boldsymbol{\alpha}, \mathbf{r} \cdot \boldsymbol{\alpha}, \mathbf{r} \cdot \boldsymbol{\alpha})\mathbf{r} de, \tag{14.9}$$

i.e. components are polynomials in the direction cosines. In most cases of physical interest the functions $V, \boldsymbol{\mu}$ are polynomials in the direction cosines, but this will not be assumed in general, since we shall deal also with some cases involving non-polynomial terms.

In the following sections, we survey the few known integrable cases in asymmetric fields. In case of existence of a cyclic variable, we show some tricks that enable constructing integrable extensions of those cases. Several cases of definite physical interest are obtained in this way.

14.1.2 The Motion of a Magnetizable Rigid Body in an Ideal Fluid and In a Uniform Magnetic Field

Under the assumption that the body and the fluid are linearly magnetizable, it was shown in [50] that the equations of motion may be derived from a Hamiltonian, which generalizes that of Clebsch's Hamiltonian (10.10) for the Kirchhoff problem (see Chap. 10). The equivalent Lagrangian may be written as

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \boldsymbol{\gamma}\mathbf{B}\cdot\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\gamma}\mathbf{C}\cdot\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\delta}\mathbf{D}\cdot\boldsymbol{\delta} + \mathbf{J}\cdot\boldsymbol{\delta}, \tag{14.10}$$

in which the additional matrix $\mathbf{D}$ and vector $\mathbf{J}$ are determined by the shape of the body and $\boldsymbol{\delta}$ denotes the magnetic field intensity, constant in space.

A direct generalization of this Lagrangian is

$$\begin{aligned} L = {} & \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + (\boldsymbol{\kappa} + \boldsymbol{\alpha}\mathbf{B}' + \boldsymbol{\beta}\mathbf{B}'' + \boldsymbol{\gamma}\mathbf{B})\cdot\boldsymbol{\omega} \\ & + \frac{1}{2}\boldsymbol{\alpha}\mathbf{C}'\cdot\boldsymbol{\alpha} + \frac{1}{2}\boldsymbol{\beta}\mathbf{C}''\cdot\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\gamma}\mathbf{C}\cdot\boldsymbol{\gamma} + \mathbf{a}\cdot\boldsymbol{\alpha} + \mathbf{b}\cdot\boldsymbol{\beta} + \mathbf{c}\cdot\boldsymbol{\gamma}. \end{aligned} \tag{14.11}$$

This generalizes all physical problems discussed above in this book, including the motion of a multi-connected body in a liquid and the motion of an electrified body in a superposition of three classical fields, whose potentials are at most quadratic in the direction cosines. Some integrable problems introduced below fit as integrable versions of the last Lagrangian.

14.1.3 Example: The Motion of a Satellite in a Circular Orbit

The centre of mass of a small satellite is moving in a circular orbit around a spherical planet with angular velocity Ω. We now write down the equations for its rotational motion in the orbital frame. Let R be the radius of the satellite orbit and μ be Gauss' constant of the planet. Designate by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, respectively, the unit vectors in the directions of the tangent to the circular orbit, the orthogonal to its plane and the radial in that plane from the centre of the planet. The condition of balance of the satellite in the radial direction is

$$\frac{(\Omega R)^2}{R} = \frac{\mu}{R^2},$$

so that we have

$$\Omega^2 = \frac{\mu}{R^3}. \tag{14.12}$$

The kinetic energy of the satellite is

$$T = \frac{1}{2}M(\Omega R)^2 + \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega}, \tag{14.13}$$

where $\boldsymbol{\omega}$ is the angular velocity in the inertial frame. On the other hand, the potential of the satellite may be written, regarding Eq. (6.5), in the form

$$V = \frac{3}{2}\Omega^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}. \tag{14.14}$$

Neglecting a constant term, the Lagrangian may be written as

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} - \frac{3}{2}\Omega^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}. \tag{14.15}$$

This Lagrangian describes the motion in the coordinate frame with origin at the centre of mass of the satellite and axes with fixed directions in space.

Denote by $\boldsymbol{\omega}'$ the angular velocity in the orbital frame. We have

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \Omega\boldsymbol{\beta}. \tag{14.16}$$

For detailed history of the "uniform precession" transformation (14.16) in rigid body dynamics, the reader is referred to Chap. 10, Sect. 10.11.3. Substituting in (14.15), we obtain the new Lagrangian in the orbital frame

$$\begin{aligned} L &= \frac{1}{2}(\boldsymbol{\omega}' + \Omega\boldsymbol{\beta})\mathbf{I}\cdot(\boldsymbol{\omega}' + \boldsymbol{\Omega}\boldsymbol{\beta}) - \frac{3}{2}\Omega^2\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma} \\ &= \frac{1}{2}\boldsymbol{\omega}'\mathbf{I}\cdot\boldsymbol{\omega}' + \Omega\boldsymbol{\beta}\mathbf{I}\cdot\boldsymbol{\omega}' - \frac{1}{2}\Omega^2(3\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma} - \boldsymbol{\beta}\mathbf{I}\cdot\boldsymbol{\beta}). \end{aligned} \tag{14.17}$$

Here, one can introduce Eulerian angles in the orbital frame and obtain three equations of motion in terms of them. Alternatively, one may derive the equations of motion from the Lagrangian (14.17) using (14.3) and (14.4) to obtain, as in [382],

$$\dot{\boldsymbol{\omega}}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} - 2\Omega\boldsymbol{\beta}\bar{\mathbf{I}}) = 3\Omega^2\boldsymbol{\gamma} \times \boldsymbol{\gamma}\mathbf{I} - \Omega^2\boldsymbol{\beta} \times \boldsymbol{\beta}\mathbf{I}. \tag{14.18}$$

Augmenting this equation with two others, expressing constancy of the two unit vectors $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ in the orbital frame:

$$\dot{\boldsymbol{\beta}} + \boldsymbol{\omega}'\times\boldsymbol{\beta} = \mathbf{0},\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}'\times\boldsymbol{\gamma} = \mathbf{0}. \tag{14.19}$$

In this way, the problem of motion of a satellite in a circular orbit is reduced to the problem of motion of a body about a fixed point, governed by Eqs. (14.18) and (14.19), i.e. to a special version of (14.10).

14.2 The Rigid Body (Gyrostat) Acted upon by More than One Uniform Field

We shall first make some clarification. In classical physics, there are three distinct force fields: gravity acts on mass, magnetic field acts on magnetized parts of the body and electric field acts on electric charges. However, the presence of electric charges leads to the appearance of two new effects:

(a) Accelerated electric charges produce electromagnetic radiation. This causes attenuation of the total energy of the body carrying them, from the classical point of view.

(b) Moving electric charges interact with magnetic field and result in velocity-dependent Lorentz forces.

If the velocity and acceleration of points of the body are small enough to be neglected over some period of time, then over that period we can consider the motion of the heavy, magnetized body carrying electric charges subject to the three uniform fields, gravity, magnetic and electric.

We first write down the potential of the system

$$V = M\mathbf{r}_0 \cdot \mathbf{g} + \mathbf{m} \cdot \mathbf{H} + \mathbf{Q} \cdot \mathbf{E}$$

where $\mathbf{r}_0$, $\mathbf{m}$ denote the position vector of the centre of mass and the magnetic moment of the magnetized parts of the body and $\mathbf{Q} = \Sigma q\mathbf{r}$ is the dipole moment of electric charges carried by the body and $\mathbf{g}$, $\mathbf{H}$, $\mathbf{E}$ are the three fields: gravity, magnetic and electric, respectively. The first three vectors $\mathbf{r}_0$, $\mathbf{m}$, $\mathbf{Q}$ are fixed in the body, while the field vectors $\mathbf{g}$, $\mathbf{H}$, $\mathbf{E}$ are fixed in space. If we fix a system of axes in space with the orthonormal basis $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, the potential can be rewritten in the form

$$V = \mathbf{a} \cdot \boldsymbol{\alpha} + \mathbf{b} \cdot \boldsymbol{\beta} + \mathbf{c} \cdot \boldsymbol{\gamma}. \tag{14.20}$$

This is the most general form of the potential of a rigid body in the three classical fields. It is the potential of three irreducible fields, i.e. that cannot be reduced to two, as long as the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are not coplanar. This means that the matrix

$$\mathbb{M} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{14.21}$$

has rank 3. Equivalently, the determinant

$$\det(\mathbb{M}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0. \tag{14.22}$$

Expressing (14.20) in Euler–Rodrigues' parameters using expression (2.25) for the rotation matrix, one converts the potential V to a quadratic form in $\lambda_0, \lambda_1, \lambda_2, \lambda_3$. In the generic case, a rotation of the four-dimensional axes, which is equivalent to rotating both space and body axes, determines a single coordinate frame in the body, in which only diagonal elements of the rotation matrix are present. This means that the potential in those axes has the form

$$V = a\alpha_1 + b\beta_2 + c\gamma_3. \tag{14.23}$$

This process is completely independent of that of determining the principal axes of inertia and thus simultaneous use of the last formula of the potential with principal axes of inertia requires some restrictions on the parameters of the body. However, for a body with complete dynamical (spherical) symmetry, the general potential (14.20) can always be reduced to that form.

When $\operatorname{rank}(\mathbb{M}) = 2$, the potential is due to two fields and in the general case can be written as

$$V = \mathbf{a} \cdot \boldsymbol{\alpha} + \mathbf{b} \cdot \boldsymbol{\beta}, \tag{14.24}$$

and, without loss of generality, one can take $\mathbf{a} = (a_1, a_2, 0)$ and $\mathbf{b} = (b_1, b_2, 0)$ provided

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0.$$

Lastly, when $\operatorname{rank}(\mathbb{M}) = 1$, the potential is due to only one field and in the general case can be chosen to be the gravity field. The problem reduces to the classical problem discussed in previous chapters.

The equations of motion of the body (gyrostat) moving about a fixed point in the presence of three physical effects can be written in the form:

$$\begin{aligned} \dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa}) &= \boldsymbol{\alpha} \times \mathbf{a} + \boldsymbol{\beta} \times \mathbf{b} + \boldsymbol{\gamma} \times \mathbf{c}, \\ \dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}, \dot{\boldsymbol{\beta}} + \boldsymbol{\omega} \times \boldsymbol{\beta} &= \mathbf{0}, \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}. \end{aligned} \tag{14.25}$$

The system of Eq. (14.25) admits the energy integral

$$I_1 = \frac{1}{2}\boldsymbol{\omega}\mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{a} \cdot \boldsymbol{\alpha} + \mathbf{b} \cdot \boldsymbol{\beta} + \mathbf{c} \cdot \boldsymbol{\gamma}. \tag{14.26}$$

Thus, to obtain a general integrable case of motion of the body acted upon by two or three uniform fields, one must know two more integrals independent of I_1.

The classical problem has benefited from the interest of several eminent mathematicians, who investigated analytical properties of the solution of Euler–Poisson equations in the complex t-plane and conditions of existence of first integrals with various analytical structures: polynomial, analytic or meromorphic or single-valued in the Euler–Poisson variables. In contrast to all that, no effort was made to extend any of those lines of investigation to problems involving more than one field.

In the following two sections, we give a brief description of the integrable cases and particular solutions in both versions of the problem.

14.2.1 The Motion of a Body Acted upon by Two Uniform Fields

This is the case of motion of a heavy magnetized rigid body (gyrostat) about a fixed point under the action of two uniform gravity and magnetic fields. Equations of motion can be written in the form:

$$\begin{aligned}\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa}) = \boldsymbol{\alpha} \times \mathbf{a} + \boldsymbol{\beta} \times \mathbf{b},\\ \dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}, \dot{\boldsymbol{\beta}} + \boldsymbol{\omega} \times \boldsymbol{\beta} = \mathbf{0}.\end{aligned} \tag{14.27}$$

The investigation of such system and construction of integrable motions seem to be initiated by Bogoyavlensky. In [30, 31], he has shown that Eq. (14.27) under the Kowalevski condition $\mathbf{I} = C\,\mathrm{diag}(2, 2, 1)$ for $\mathbf{a} = (a, 0, 0), b = (0, b, 0), \boldsymbol{\kappa}=\mathbf{0}$ is completely integrable on the invariant relation (written here in our usual notation)

$$\begin{aligned} z_1 &= p^2 - q^2 - a\alpha_1 + b\beta_2 = 0,\\ z_2 &= 2pq - (a\alpha_2 + b\beta_1) = 0. \end{aligned} \tag{14.28}$$

It turned out that this is a particular case of the unrestricted integral (14.31) below and reminds a generalization of the Appelrot families of solutions to the case of a simple body in two fields (without a gyrostatic moment).

In a much later work [69], it was claimed that the conditional classical integrable case of Goryachev and Chaplygin admits a generalization to the problem of two fields with two conditions of the areas integral type. It was shown in [406] that conditions of this type lead only to pendulum-like motion and that the presence of the second field has no significance.

14.2.1.1 The Hierarchy of Yehia and Reyman and Semenov–Tian–Shansky

We now turn to general integrable cases of the present problem. In our note [380], we have considered the problem of motion of the gyrostat acted upon by forces with the unsymmetrical potential

$$V = C(a_1\alpha_1 + a_2\alpha_2 + b_1\beta_1 + b_2\beta_2 + c_1\gamma_1 + c_2\gamma_2). \tag{14.29}$$

The gyrostatic moment was taken in the form

$$\boldsymbol{\mu} = C\kappa\mathbf{k}, \tag{14.30}$$

directed along the axis of dynamical symmetry of the body. It was found that this problem admits the general fourth-degree integral

$$\begin{aligned} I_3 = {} & [p^2 - q^2 - a_1\alpha_1 + a_2\alpha_2 - b_1\beta_1 + b_2\beta_2 - c_1\gamma_1 + c_2\gamma_2]^2 \\ & +[2pq - (a_1\alpha_2 + a_2\alpha_1 + b_1\beta_2 + b_2\beta_1 + c_1\gamma_2 + c_2\gamma_1)]^2 \\ & +2\kappa(r-\kappa)(p^2+q^2) \\ & -4\kappa[p(a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3) + q(a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3)]. \end{aligned} \tag{14.31}$$

The integral (14.31) generalizes Kovalevskaya's integral by including two arbitrary constant fields and a gyrostatic momentum. However, the knowledge of this integral is not sufficient to establish complete integrability of the case (14.29–14.30) since the potential (14.29) does not allow a linear integral in general. Another integral is needed. In the same work [380], two integrable cases have been pointed out, characterized by the existence of a linear integral in each case:

(a) The case of axi-symmetric potential $V = c_1\gamma_1 + c_2\gamma_2$ admits the familiar areas integral

$$I_2 = 2(p\gamma_1 + q\gamma_2) + (r+\kappa)\gamma_3, \tag{14.32}$$

corresponding to the angle of precession ψ as a cyclic variable. This case is discussed in detail in Sect. 5.6.

(b) The case $V = a_1(\alpha_1 - \chi\beta_2) + a_2(\alpha_2 + \chi\beta_1)$ $(\chi = \pm 1)$ is characterized by the integral

$$I_{2,\chi} = 2(p\gamma_1 + q\gamma_2) + (r+\kappa)(\gamma_3 + \chi), \tag{14.33}$$

which is also linear in velocities and corresponds to the cyclic variable $\varphi \pm \psi$, according to the chosen value for χ. Those cases are equivalent and can be obtained from each other by reversing the sign of the vector **fi**. Thus one can consider only the choice $\chi = 1$. Note that, unlike case a, this case does not include Kowalevski's case as a special version. The two fields are coupled together and can vanish only simultaneously.

An explicit solution for case (a) is commented in Chap. 5. Case (b) in its full generality is not solved yet. In [189], the solution of a special version of this case, namely when $I_2 = f = 4\kappa$, was reduced to real elliptic quadratures. We shall return to case (b) later in this chapter. It is in fact closely connected to another problem of a completely different nature, namely Chaplygin's case of motion of a body in liquid, modified by the Goryachev singular potential term $\frac{\varepsilon}{\gamma_3^2}$. This connection will be extensively studied in Sect. 14.5 below.

The full system with potential (14.29) and gyrostatic momentum (14.30) turned out to be integrable. This result was established by Reyman and Semenov–Tian–Shansky in [311] (see also [26]), where a quadratic integral was constructed for arbitrary parameters of the two fields. In the present notation, this integral can be written as

$$\begin{aligned} I_2 = {} & [(\mathbf{G}\cdot\boldsymbol{\alpha})\mathbf{a} + (\mathbf{G}\cdot\boldsymbol{\beta})\mathbf{b} + (\mathbf{G}\cdot\boldsymbol{\gamma})\mathbf{c}]^2 \\ & + 2(r-\kappa)[\Delta_1\mathbf{G}\cdot\boldsymbol{\alpha} + \Delta_2\mathbf{G}\cdot\boldsymbol{\beta} + \Delta_3\mathbf{G}\cdot\boldsymbol{\gamma}] \\ & + 4\mathbf{k}\cdot[\Delta_1(\mathbf{b}\times\boldsymbol{\gamma} + \boldsymbol{\beta}\times\mathbf{c}) + \Delta_2(\mathbf{c}\times\boldsymbol{\alpha} + \boldsymbol{\gamma}\times\mathbf{a}) \\ & \qquad + \Delta_3(\mathbf{a}\times\boldsymbol{\beta} + \boldsymbol{\alpha}\times\mathbf{b})], \end{aligned} \tag{14.34}$$

where $\mathbf{G} = \boldsymbol{\omega}\mathbf{I} + \boldsymbol{\kappa}$ is the total angular momentum and $\Delta_1 = b_1c_2 - b_2c_1$, $\Delta_2 = c_1a_2 - c_2a_1$, $\Delta_3 = a_1b_2 - a_2b_1$.

The three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in (14.29) are not redundant, since we deal with only two irreducible fields, this form is symmetric and also comfortable in obtaining the above two limiting cases. In case a) we have $I_2 = \mathbf{c}^2(\mathbf{G}\cdot)^2$, which reduces to the linear integral (14.32). In case (b) one can easily show that

$$I_2 = -a^2[I_{2,\chi}^2 + 4\chi\kappa I_{2,\chi} - 4I_1 + 2\kappa^2],$$

so that $I_{2,\chi}$ is really an integral.

To put the integral (14.34) in its simplest form, one can normalize the potential to the form

$$V = a\alpha_1 + b\beta_2.$$

The integral takes the form

$$I_2 = a^2(\mathbf{G}\cdot\boldsymbol{\alpha})^2 + b^2(\mathbf{G}\cdot\boldsymbol{\beta})^2 + 2ab[(r-\kappa)(\mathbf{G}\cdot\boldsymbol{\gamma}) + 2(b\alpha_1 + a\beta_2)]. \tag{14.35}$$

To construct this general integrable case, the authors of [311] introduced a Lax representation of the equations of motion, using a Lax pair with a spectral parameter. The solution of the complexified equations is pointed out in terms of Theta functions. However, without a real separation of variables, those results are not useful in the study of bifurcation and qualitative properties of motion. Some works were devoted to the topological analysis of this case, e.g. [177, 187]. It is worthy to mention also M. Kharlamov's work, where a generalization of the second and third Appelrot classes was pointed out as a two-parameter family of doubly periodic motions [175] and generalization of the fourth class in [188]. Separation of variables for generalized fourth Appelrot class is discussed in [179, 180]. Extensions of the Appelrot classes for the generalized gyrostat in two fields were investigated in [182]. The stability of the stationary motions in some special versions of this problem was studied in [149] and recently in [159].

Worth mentioning is the work [84], in which the authors put forward a systematic way of the derivation of the algebraic curves of separation of variables for the classical Kovalevskaya on top and its generalizations, starting from the spectral curve of the corresponding Lax representation found by Reyman and Semonov–Tian–Shansky. Recently, in a conference[1] the author learned from Yu. N. Fedorov that further results were obtained including expressions of the physical variables in terms of separation variables for the generic Reyman and Semenov–Tian–Shansky case and its special cases of symmetry (the case a of a single field and the case b of two fields of equal intensities). However, most of these results are not published yet.

14.2.1.2 A Generalization of the Last Hierarchy

Strictly speaking, the case presented here does not belong to the above hierarchy of cases in two uniform fields with the Reyman Semenov–Tian–Shansky on its top. The deformation parameter ε_1 simultaneously evokes forces with quadratic potential in addition to a change in the application centres $\mathbf{r}_1$, $\mathbf{r}_2$ of the two uniform fields to become

$$\mathbf{r}_1 = \varepsilon_0 a(1,0,0) + \varepsilon_1 a\kappa(0,1,0),$$
$$\mathbf{r}_2 = \varepsilon_0 b(0,1,0) - \varepsilon_1 b\kappa(1,0,0).$$

Note that this change keeps the application centres of the uniform fields in the body orthogonal to each other. The changes are coupled by the gyrostatic moment parameter κ and disappear when $\kappa = 0$. We list this case here in the most suitable place for it in this section, since it is, like its predecessor R-STS case, of three degrees of freedom (with no cyclic coordinates). The remaining two sections will be devoted to integrable problems with cyclic coordinates.

Sokolov and Tsiganov [337] ε_1 introduced (2002). Reyman and Semenov–Tian–Shansky [311] $\varepsilon_1 = 0, ab \neq 0$ (1987), See also [26] (1989). Yehia [380] $\varepsilon_1 = 0, b = \pm a$ and $\varepsilon_1 = 0, ab = 0$ (1986).
$A = B = 2C$, $V = -\{\varepsilon_0(a\alpha_1 + b\beta_2) + \varepsilon_1\kappa(a\alpha_2 - b\beta_1) + \varepsilon_1^2[a^2\alpha_3^2 + b^2\beta_3^2 + \frac{1}{2}(a\alpha_2 + b\beta_1)^2]\}$, $\boldsymbol{l} = (2\varepsilon_1 b\beta_3, -2\varepsilon_1 a\alpha_3, \kappa + \varepsilon_1(a\alpha_2 - b\beta_1))$, $\boldsymbol{\mu} = (\varepsilon_1 b\beta_3, -\varepsilon_1 a\alpha_3, \kappa - \varepsilon_1(a\alpha_2 - b\beta_1))$.
$I_2 = a^2[2p\alpha_1 + 2q\alpha_2 + (r+\kappa)\alpha_3 - \varepsilon_1(b\alpha_3\beta_1 + a\alpha_3\alpha_2 - 2b\alpha_1\beta_3)]^2$
$+b^2[2p\beta_1 + 2q\beta_2 + (r+\kappa)\beta_3 + \varepsilon_1(a\alpha_2\beta_3 + b\beta_1\beta_3 - 2a\alpha_3\beta_2)]^2$
$+2[r - \kappa + \varepsilon_1(a\alpha_2 - b\beta_1)][2p\gamma_1 + 2q\gamma_2 + \gamma_3(r+\kappa) + \varepsilon_1(a\alpha_2 - b\beta_1 - a\alpha_3\gamma_2 + b\gamma_1\beta_3)]$
$-4\varepsilon_0(b\beta_2 + a\alpha_1)$
$+4\varepsilon_1[2aq\alpha_3 - 2bp\beta_3 - (a\alpha_2 - b\beta_1)(\kappa + r) - \varepsilon_1((a\alpha_2 - b\beta_1)^2 + 2a^2\alpha_3^2 + 2b^2\beta_3^2)]$

(continued)

[1] "Classical mechanics, dynamical systems and mathematical physics" on the occasion of Academician Valery V. Kozlov's 70th birthday. Steklov Mathematical Institute of RAS in Moscow, 20-24.01.2020.

$$
\begin{aligned}
I_3 = {} & [p^2 - q^2 + \varepsilon_1{}^2 \left(-a^2\alpha_1^2 + b^2\beta_2^2\right) - \varepsilon_1 (a\alpha_2 + b\beta_1)(\kappa - r) + \varepsilon_0 (a\alpha_1 - b\beta_2)]^2 \\
& + [2pq - \varepsilon_1^2(a\alpha_2 + b\beta_1)(a\alpha_1 + b\beta_2) + \varepsilon_1(a\alpha_1 - b\beta_2)(\kappa - r) + \varepsilon_0(a\alpha_2 + b\beta_1)]^2 \\
& - 2\kappa(\kappa - r + \varepsilon_1 a\alpha_2 - \varepsilon_1 b\beta_1)\left(p^2 + q^2\right) \\
& - 2\kappa\{-\varepsilon_1 r[\varepsilon_1 \left((a\alpha_1 - b\beta_2)^2 + (a\alpha_2 + b\beta_1)^2\right) + 2bp\beta_3 - 2a\,q\alpha_3] \\
& - 2q[\varepsilon_1{}^2 \left(-(a\alpha_2 + b\beta_1)a\alpha_3 - 2b^2\beta_2\beta_3\right) + \varepsilon_1\kappa a\alpha_3 + \varepsilon_0 b\beta_3] \\
& + 2p[\varepsilon_1{}^2 \left((a\alpha_2 + b\beta_1)b\beta_3 + 2a^2\alpha_1\alpha_3\right) + \kappa\varepsilon_1 b\beta_3 - \varepsilon_0 a\alpha_3] \\
& + \varepsilon_1{}^2[\{\varepsilon_1 (a\alpha_2 - b\beta_1) + \kappa\}\{(a\alpha_2 + b\beta_1)^2 + (a\alpha_1 - b\beta_2)^2\} \\
& + 2\kappa\left(a^2\alpha_3^2 + b^2\beta_3^2\right)]\}.
\end{aligned}
$$

The integrability of this case was established in 2002 by Sokolov and Tsiganov who constructed a Lax pair for it. Although explicit forms of the integrals of motion follow from the Lax pair, no effort was made towards constructing those integrals, a step taken later by Kharlamov and Ryabov. The two integrals are written here following Ryabov [324], but as usual in this book, we normalize the two fields and do some further simplifications. The deformation parameters $\varepsilon_0, \varepsilon_1$ are arbitrary, so that the above formulas unify the two cases listed in [324] as different, in a form from which they are obtained as special cases $\varepsilon_0 = 1$ and $\varepsilon_1 = 1$, respectively.

This integrable case, which occupies the top of its hierarchy, has been studied only in very few works. Neither separation of variables nor the phase topology was investigated in the full case. In certain circumstances, the full system of three degrees of freedom can be reduced to a lower dimensional system on invariant manifolds. In [324] two invariant four-dimensional submanifolds were pointed out, on which the original system reduces almost everywhere to a Hamiltonian one with two degrees of freedom. The system of equations specifying one of the invariant submanifolds is a generalization of the invariant relations (14.28) for the integrable Bogoyavlensky case, which, in turn, generalizes the first Appelrot class. The method of critical subsystems, devised by M. Kharlamov in [175, 176], is used to this end. The phase topology is described and bifurcation diagrams are given for those subsystems.

Further study of the bifurcation diagrams led in [304] to the construction of certain periodic solutions of the problem of motion in two fields. For those solutions, phase variables are expressed as algebraic functions of a single auxiliary variable and a set of constants. This auxiliary variable satisfies a differential equation which can be integrated in elliptic functions of time.

It can be readily seen that when $b = 0$, the integral I_2 reduces to the simple form of the areas integral

$$2(p\alpha_1 + q\alpha_2) + (r + \kappa - a\varepsilon_1\alpha_2)\alpha_3.$$

It may be also shown, using the energy and geometric integrals, that in the special cases $b = \mp a$, I_2 renders to the linear integral

$$2[(p \pm a\varepsilon_1\beta_3)\gamma_1 + (q - a\varepsilon_1\alpha_3)\gamma_2] + [r + \kappa + a\varepsilon_1(\alpha_2 \pm \beta_1)](\gamma_3 \pm 1).$$

The last expression generalizes (14.33) and reduces to it when $\varepsilon_1 = 0$.

14.2.2 The Motion of a Body Acted upon by Three Irreducible Uniform Fields

It seems that the general problem of motion under the action of three fields has escaped attention and is still in that state. This may be caused by the very limited advancement in its apparently simpler version of just two fields. The problem was not investigated for integrability, so that no integrable cases are known and it is not known if they exist. The only exception is the case of a body whose ellipsoid of inertia at the fixed point is a sphere.

As far as we know, the list of works dealing with this problem for an asymmetric body consists exclusively of three papers [149, 417, 418]. The first is concerned with the determination of equilibrium positions in the potential of the special form (14.23) and studying their stability in the case of a dynamically symmetric body. The other two deal with regular precessional motion of Grioli's type, studied in Chap. 8 in the presence of a single gravity field. It turns out that this type of motion persists to exist, under certain conditions, in the general problem of motion in three irreducible fields and even when one of the three uniform fields is replaced by a linear one [153].

In this subsection, we give a brief description of the only known integrable case of a spherical body acted upon by three irreducible uniform fields and two of its special versions of definite interest. Interested readers may find the particular solution in the above-mentioned papers.

14.2.2.1 Bogoyavlensky's Case of Dynamically Spherical Body in Three Uniform Fields

Bogoyavlensky has noted that the dynamics of the body of spherical dynamical symmetry $A = B = C$ is integrable for the general potential (14.20). The problem was reduced in quaternion variables to Neumann's problem on the sphere S^3 [31]. In the quaternion space, the integrals of motion are all quadratic in the velocities and the explicit solution can be found in terms of the Riemannian Theta functions in three variables, as follows from that of the Neumann system [290–294].

It is possible, as explained earlier in this section, to choose the axes of the space frame to reduce the potential to the much simpler form

$$V = C(a\alpha_1 + b\beta_2 + c\gamma_3), \tag{14.36}$$

containing only three parameters. This simple form of the potential enables us to express the integrals of the problem in the Euler–Poisson variables, which was not undertaken in [31]. We get, as in [407],

$$\begin{aligned}
I_2 &= \frac{1}{2}[apP + bqQ + crR] + bc\alpha_1 + ca\beta_2 + ab\gamma_3, \\
I_3 &= 1/2\{a^2[p^2 + P^2] + b^2[q^2 + Q^2] + c^2[r^2 + R^2]\} \\
&\quad + bcpP + caqQ + abrR \\
&\quad + 2a\left(b^2 + c^2\right)\alpha_1 + 2b\left(c^2 + a^2\right)\beta_2 + 2c\left(a^2 + b^2\right)\gamma_3,
\end{aligned} \tag{14.37}$$

where $P = \boldsymbol{\omega}\cdot\boldsymbol{\alpha}$, $Q = \boldsymbol{\omega}\cdot\boldsymbol{\beta}$, $R = \boldsymbol{\omega}\cdot\boldsymbol{\gamma}$. Two special cases of particular interest arise.

14.2.2.2 Case of a Linear Integral [407]

When $|a| = |b|$, without loss of generality, we can assume the potential in the form

$$V = C[a(\alpha_1 - \beta_2) + c\gamma_3]. \tag{14.38}$$

This choice makes the variable $\psi + \varphi$ cyclic and leads to the linear integral

$$I_3 = \boldsymbol{\omega}\cdot\boldsymbol{\gamma} + r, \tag{14.39}$$

and the complementary is

$$I_2 = \frac{1}{2}[a(pP - qQ) + crR] - ca(\alpha_1 - \beta_2) - a^2\gamma_3.$$

14.2.2.3 Case of Three Linear Integrals [380]

When the three coefficients are equal in modulus $|a| = |b| = |c| = a$, one can take

$$V = Aa(\alpha_1 + m\beta_2 + m'\gamma_3), \tag{14.40}$$

where $m^2 = m'^2 = 1$. In that case, along with the energy integral

$$I_1 = \frac{1}{2}A(p^2 + q^2 + r^2) + Aa(\alpha_1 + m\beta_2 + m'\gamma_3),$$

one can write three linear integrals

$$I_2 = \boldsymbol{\omega}\cdot\boldsymbol{\alpha} - mp = c_1,\ I_3 = \boldsymbol{\omega}\cdot\boldsymbol{\beta} - m'q = c_2,\ I_4 = \boldsymbol{\omega}\cdot\boldsymbol{\gamma} - mm'r = c_3. \tag{14.41}$$

As shown in [385] (see also case 1 in Table 14.5 of Sect. 14.5.4 below), explicit solution of this case can be written in terms of elliptic functions of time. It can also be noted that this case is super-integrable, i.e. it has one superfluous integral (say) I_4 more than the three needed for integrability. The trajectories of the integrable rigid body motions are closed curves in the space of phase variables $\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma},$ resulting from the intersection of Liouville tori with the integral surface $I_4 = c_3$, and hence motion on them is generically periodic.

14.3 Integrable Cases of a Body with a Homogeneous Quadratic Potential

14.3.1 Brun's Case of the Asymmetric Body in an Asymmetric Gravitational Field

Let an arbitrary body fixed from its centre of mass be acted upon by a gravitational field whose potential is $V_0 = N_1 X^2 + N_2 Y^2 + N_3 Z^2$, i.e. a three-dimensional generalization of (6.7) in the elementary Brun problem of Chap. 6. The potential of the body in that field can be written as

$$V = \frac{1}{2}(a\boldsymbol{\alpha}\mathbf{I} \cdot \boldsymbol{\alpha} + b\boldsymbol{\beta}\mathbf{I} \cdot \boldsymbol{\beta} + c\boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma}). \tag{14.42}$$

This potential can be interpreted alternatively as an approximate form of the potential due to three sufficiently distant centres of Newtonian or Coulomb interactions. However, unlike the problem of motion in three skew fields, the three terms in (14.42) are redundant. In fact, we have

$$\begin{aligned}\boldsymbol{\alpha}\mathbf{I} \cdot \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{I} \cdot \boldsymbol{\beta} + \boldsymbol{\gamma}\mathbf{I} \cdot \boldsymbol{\gamma} &= \Sigma I_{ij}(\alpha_i\alpha_j + \beta_i\beta_j + \gamma_i\gamma_j) \\ &= \Sigma I_{ij}\delta_{ij} = tr(\mathbf{I}),\end{aligned}$$

so that the potential can be equivalently written as

$$V = \frac{1}{2}[(a-c)\boldsymbol{\alpha}\mathbf{I} \cdot \boldsymbol{\alpha} + (b-c)\boldsymbol{\beta}\mathbf{I} \cdot \boldsymbol{\beta}]. \tag{14.43}$$

Nevertheless, keeping all terms preserves an obvious degree of symmetry and easily gives all special cases of axial symmetry.

Brun found three integrals of motion [48]. The first is the total energy integral

$$I_1 = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + V, \tag{14.44}$$

and the second is

$$I_2 = (A^2p^2 + B^2q^2 + C^2r^2) - ABC(a\boldsymbol{\alpha}\mathbf{I}^{-1} \cdot \boldsymbol{\alpha} + b\boldsymbol{\beta}\mathbf{I}^{-1} \cdot \boldsymbol{\beta} + c\boldsymbol{\gamma}\mathbf{I}^{-1} \cdot \boldsymbol{\gamma}). \tag{14.45}$$

As the field has no axis of symmetry, the areas integral does not exist in this problem. Brun found the third integral which turned out to be also quadratic and can be written in the form

$$I_3 = a(\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\alpha})^2 + b(\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\beta})^2 + c(\boldsymbol{\omega}\mathbf{I}\cdot\gamma)^2 \\ + ABC[bc\boldsymbol{\alpha}\mathbf{I}^{-1}\cdot\boldsymbol{\alpha} + ca\boldsymbol{\beta}\mathbf{I}^{-1}\cdot\boldsymbol{\beta} + ab\gamma\mathbf{I}^{-1}\cdot\gamma). \tag{14.46}$$

This integral is written in a more complicated form in [41]. It is easy to check that the integrals I_2 and I_3 reduce, respectively, to one integral of Tisserand's type (6.13) and an areas integral in cases of axial symmetry of the gravitational field, i.e. whenever two of the parameters a, b, c are equal (or vanish).

In 1910, Goryachev [116] pointed out several cases in which all three integrals of motion are quadratic. However, all of them turn out to be special cases of Brun's general problem described above.

Although Brun has indicated the three integrals of motion, sufficient for integrability in the sense of Liouville, he did not state that the problem of motion is integrable. This step was completed long later by Bogoyavlensky in [29] where he used Lax pair formulation of the equations of motion to prove integrability and to obtain explicit formulas expressing angular velocities of the body in terms of θ-functions of four variables restricted to a three-dimensional manifold. As may be expected, the complex formulas obtained are too complicated to have any impact on the study of qualitative properties of the motion. In contrast to the case of a symmetric field, no further studies were performed on this case using the integrals of motion obtained more than a century ago.

Remark: Brun's problem of Chap. 6 has a curious generalization as follows [29]:

Assume that a rigid body moves freely in a gravitational field with homogeneous quadratic potential

$$V_g = \frac{1}{2}(a\xi^2 + b\eta^2 + c\zeta^2)$$

in the inertial frame. Let (X, Y, Z) be the coordinates of the centre of mass and $\mathbf{r}$ be the position vector of the element of mass dm referred to the body system. The potential of the body is

$$\begin{aligned} V &= \frac{1}{2}\int [a(X+\mathbf{r}\cdot\boldsymbol{\alpha})^2 + b(Y+\mathbf{r}\cdot\boldsymbol{\beta})^2 + c(Z+\mathbf{r}\cdot\gamma)^2]dm \\ &= \frac{1}{2}M(aX^2+bY^2+cZ^2) + \frac{1}{2}\sum(a\alpha_i\alpha_j + b\beta_i\beta_j + c\gamma_i\gamma_j)\bar{\mathbf{I}}_{ij} \\ &= \frac{1}{2}M(aX^2+bY^2+cZ^2) - \frac{1}{2}(a\boldsymbol{\alpha}\mathbf{I}\cdot\boldsymbol{\alpha} + b\boldsymbol{\beta}\mathbf{I}\cdot\boldsymbol{\beta} + c\gamma\mathbf{I}\cdot\gamma) \\ &\quad + \frac{3}{2}\,\mathrm{tr}(\mathbf{I})(a+b+c). \end{aligned} \tag{14.47}$$

The last term is constant and can be omitted. The Lagrangian of the motion takes the form

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - \frac{1}{2}M(aX^2 + bY^2 + cZ^2)$$
$$+ \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \frac{1}{2}(a\boldsymbol{\alpha}\mathbf{I}\cdot\boldsymbol{\alpha} + b\boldsymbol{\beta}\mathbf{I}\cdot\boldsymbol{\beta} + c\boldsymbol{\gamma}\mathbf{I}\cdot\boldsymbol{\gamma}).$$

Thus, the equations of translational motion of the centre of mass and the rotational motion about the centre of mass are completely separate. The first is solved as simple harmonic motion and the second is Brun's case just discussed above in this subsection.

Remark: Burov and Subkhankulov [50] pointed out a new integrable case of (14.10). Since in this case $\mathbf{B} = \mathbf{J} = \mathbf{0}$ and $\mathbf{C}$, $\mathbf{D}$ are proportional to $\mathbf{I}$, the effect of magnetization brings the same term as a gravitational source and the case is equivalent to Brun's case in asymmetric gravitational field.

14.3.2 Case of Dynamically Spherical Body

Bogoyavlensky [30] pointed out the integrability of the motion of a spherical body in the fields whose potential may be written in the form

$$V = \frac{1}{2}(k_1\boldsymbol{\alpha}\mathbf{J}\cdot\boldsymbol{\alpha} + k_2\boldsymbol{\beta}\mathbf{J}\cdot\boldsymbol{\beta} + k_3\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma}), \tag{14.48}$$

in which $\mathbf{J}$ is an arbitrary (nonsingular) symmetric matrix and $k_{1,2,3}$ are arbitrary constants. Explicit expressions for integrals were not given in [30], but were provided in [380] in a form that keeps non-diagonal coefficient matrix. Without losing generality, we diagonalize that matrix here to be $\operatorname{diag}(k_1, k_2, k_3)$.

$$\begin{aligned} I_2 &= C\boldsymbol{\omega}\mathbf{J}\cdot\boldsymbol{\omega} - \det(\mathbf{J})[k_1\boldsymbol{\alpha}\mathbf{J}^{-1}\cdot\boldsymbol{\alpha} + k_2\boldsymbol{\beta}\mathbf{J}^{-1}\cdot\boldsymbol{\beta} + k_3\boldsymbol{\gamma}\mathbf{J}^{-1}\cdot\boldsymbol{\gamma}], \\ I_3 &= C[k_1(\boldsymbol{\omega}\cdot\boldsymbol{\alpha})^2 + k_2(\boldsymbol{\omega}\cdot\boldsymbol{\beta})^2 + k_3(\boldsymbol{\omega}\cdot\boldsymbol{\gamma})^2] \\ &\quad - [k_2k_3\boldsymbol{\alpha}\mathbf{J}\cdot\boldsymbol{\alpha} + k_3k_1\boldsymbol{\beta}\mathbf{J}\cdot\boldsymbol{\beta} + k_1k_2\boldsymbol{\gamma}\mathbf{J}\cdot\boldsymbol{\gamma}]. \end{aligned} \tag{14.49}$$

It can be easily shown that, as in Brun's case, the three constants k_i can always be reduced to two, so that one term of the potential is redundant. Also, the integrals I_2, I_3 reduce to Clebsch's type of spherical body in a liquid and an areas integral as two of the coefficients k_1, k_2 become equal.

The Lagrangian of the last problem can be written as $L = \frac{1}{2}C!^2 - V$. In quaternion space, using (2.46), this reads

$$L = 2C\dot{\mathbf{\Lambda}}^2 + V(\mathbf{\Lambda}),\ |\mathbf{\Lambda}|^2 = 1, \tag{14.50}$$

where V is a homogeneous quartic polynomial in $\mathbf{\Lambda}$, obtained by substituting (2.25) in (14.48). This Lagrangian describes the motion of a particle on the three-dimensional sphere S^3, pointed out simultaneously by Bogoyavlensky [32] and in a more detailed direct way, by Kozlov [243].

14.4 The Motion of an Axi-Symmetric Body Under the Action of Asymmetric Forces

In the last two sections, we collected the well-known integrable cases of more or less classical problems of motion of a rigid body or a gyrostat under potential forces having linear or quadratic asymmetric potential. In this section, we present integrable cases of motion under more general potential and gyroscopic forces. The body is assumed to have physical, not only dynamical, axial symmetry about its z-axis, so that $B = A$ and the proper rotation angle φ is a cyclic coordinate. In order to keep aside generalization of Lagrange's case, we assume that the precession angle ψ is a palpable coordinate. We first search for versions satisfying this condition among the known integrable cases. The next step is to apply the transformation of cyclic velocities established in Chap. 11 Sect. 11.9 to build more general integrable cases.

14.4.1 Description of the Problem

For such problem, the Lagrangian can depend only on the angles θ, ψ or, equivalently, on the three direction cosines $\alpha_3, \beta_3, \gamma_3$. It can be written as

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{l}\cdot\boldsymbol{\omega} - V, \tag{14.51}$$

where the potential V and the quantities $\boldsymbol{l}\cdot\boldsymbol{\gamma}$, $\boldsymbol{l}\cdot\mathbf{n}$, $\boldsymbol{l}\cdot\mathbf{k}$ depend only on $\alpha_3, \beta_3, \gamma_3$. We denote any vector $\mathbf{v}$ by $\bar{\mathbf{v}}$ when it is referred to the inertial reference frame, so that

$$\bar{\mathbf{v}} = \mathbf{v}\mathbf{R} = (\mathbf{v}\cdot\boldsymbol{\alpha})\boldsymbol{\alpha} + (\mathbf{v}\cdot\boldsymbol{\beta})\boldsymbol{\beta} + (\mathbf{v}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}.$$

Then L becomes

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \bar{\boldsymbol{l}}\cdot\bar{\boldsymbol{\omega}} - V(\alpha_3, \beta_3, \gamma_3) \tag{14.52}$$

and the cyclic integral has the form

$$Cr + l_3 = Cr + \bar{\boldsymbol{l}}\cdot\bar{\mathbf{k}} = f. \tag{14.53}$$

In order to describe the motion of axially symmetric body, we shall use, in addition to the cyclic angle of proper rotation φ, the direction cosines $\alpha_3, \beta_3, \gamma_3$ of the z-axis of symmetry of the body, as redundant coordinates. Those are the components of the unit vector $\mathbf{k}$ referred to the inertial coordinate frame. We know that the space time derivative of $\bar{\mathbf{k}}$ is

$$\frac{d\bar{\mathbf{k}}}{dt} = \bar{\boldsymbol{\omega}} \times \bar{\mathbf{k}},$$

so that

$$\bar{\boldsymbol{\omega}} = \bar{\mathbf{k}} \times \frac{\mathbf{d}}{\mathbf{d}\bar{\mathbf{t}}} + r\bar{\mathbf{k}}. \tag{14.54}$$

We also have $\tan\psi = -\frac{\alpha_3}{\beta_3}$ and hence

$$\dot{\psi} = \frac{\alpha_3\dot{\beta}_3 - \beta_3\dot{\alpha}_3}{\alpha_3^2 + \beta_3^2}. \tag{14.55}$$

After some manipulations, using both equations in (14.52), we obtain as in [390] the Routhian

$$\begin{aligned} R = &\frac{1}{2}A(\dot{\alpha}_3^2 + \dot{\beta}_3^2 + \dot{\gamma}_3^2) \\ &+ \bar{\boldsymbol{l}}_1(\beta_3\dot{\gamma}_3 - \gamma_3\dot{\beta}_3) + \bar{\boldsymbol{l}}_2(\gamma_3\dot{\alpha}_3 - \alpha_3\dot{\gamma}_3) + (\bar{\boldsymbol{l}}_3 + \frac{f\gamma_3}{\alpha_3^2 + \beta_3^2})(\alpha_3\dot{\beta}_3 - \beta_3\dot{\alpha}_3) \\ &- [V + \frac{(f - \bar{\boldsymbol{l}}_1\alpha_3 - \bar{\boldsymbol{l}}_2\beta_3 - \bar{\boldsymbol{l}}_3\gamma_3)^2}{2C}]. \end{aligned} \tag{14.56}$$

The problem thus reduces to that of motion of a particle of mass A on the unit sphere

$$|k|^2 = \alpha_3^2 + \beta_3^2 + \gamma_3^2 = 1, \tag{14.57}$$

under the action of certain potential and gyroscopic forces.

Comparing the last Routhian, describing what we shall call problem 1, with another Routhian (11.52), one easily realizes that both Routhians have the same structure. The second Routhian describes problem 2, the reduction of the problem of motion of a spherical body in an axi-symmetric combination after ignoring the cyclic angle of precession. The two problems can be identified by applying the transformation

$$\begin{aligned} f \to f, t \to -t, \\ \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \to \mathbf{k} = (\alpha_3, \beta, \gamma_3), \\ \boldsymbol{l} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k} \to \bar{\boldsymbol{l}} = \bar{l}_1\boldsymbol{\alpha} + \bar{l}_2\boldsymbol{\beta} + \bar{l}_3\boldsymbol{\gamma}, \\ V(\boldsymbol{\gamma}) \to V(\mathbf{k}) - \frac{A-C}{2AC}(f - \bar{\boldsymbol{l}}_1\alpha_3 - \bar{\boldsymbol{l}}_2\beta_3 - \bar{\boldsymbol{l}}_3\gamma_3)^2. \end{aligned} \tag{14.58}$$

Thus, the problem of motion of an axially symmetric body in a skew combination of fields is brought into equivalence with that of motion of a body of spherical dynamical symmetry in a coaxial combination of fields. However, equations of motion are isomorphic for the two problems on the level of Routhian reduction of each. The full

Lagrangian systems of equations are not necessarily isomorphic, and hence also the explicit solution in terms of the time of those systems.

14.4.1.1 Applications

Maximal reduction of order of the equations of motion. As done in Sect. 11.7, the equations of motion can be reduced, for arbitrary V and $\boldsymbol{l}$ and on an arbitrary fixed integral level $\{h, f\}$, to a single second-order differential equation for γ_3 in α_3, say. This may be accomplished by using Maupertuis' principle and eliminating β_3 in virtue of the geometric integral (14.57). An explicit form of the reduced equation is given in [390], where an example of the reduced equation is given for the problem of motion of a symmetric satellite on a circular orbit. Interested reader is referred to that work.

Equivalent integrable problems. The analogy just established can be used to construct integrable cases in problem 1, which is much less studied, from well-known integrable cases of problem 2. From the last relation in (14.58), it is clear that in general the resulting cases will be conditional one, since the cyclic constant f figures in the potential in a significant way. However, a general integrable case produces a general equivalent one in two circumstances:

(a) When $\left|\bar{l}\right| = 0$. Then the constant potential term $\frac{A-C}{2AC} f^2$ can be ignored, or

(b) when the body has spherical dynamical symmetry, and then the extra terms in the equivalent potential disappear.

We shall meet both circumstances in the application below.

In the next subsection, we shall construct two integrable cases by applying the above method. For space considerations, we write down these two cases after the application of a transformation of the cyclic velocity $\dot{\varphi} = \dot{\varphi}' + \nu$, where $\nu = \nu(\alpha_3, \beta_3, \gamma_3)$. This is equivalent to the change

$$r = r' + \nu. \tag{14.59}$$

This transformation, as established in Chap. 11, Sect. 11.9, generically leads to conditional integrable cases, but for suitably chosen ν leads to the construction of general integrable cases. Note that the choice $\nu = n = \text{const}$ adds a gyrostatic moment Cn along the axis of symmetry of the body.

14.4.2 General Integrable Cases

14.4.2.1 A General Integrable Case Relevant to Brun's Case

Among the integrable cases presented hitherto in this chapter, only one admits an axially symmetric version amenable to the application of the transformation (14.59). In the case due to Brun (Sect. 14.3.1), the body is in general asymmetric and moves under forces whose potential has a certain quadratic form in the nine direction cosines given by Eq. (14.42). The restriction of this problem to the case of an axi-symmetric body $B = A$ gives

$$\begin{aligned} V &= \frac{1}{2}(a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2) \\ \boldsymbol{l} &= (0, 0, 0), \end{aligned} \tag{14.60}$$

where a, b, c are certain constants. If we transform this case using

$$\nu = n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2 \tag{14.61}$$

we get a new integrable one that is listed in Table 14.1 case 1.

Note that I_3 is, in general, of the third degree in the angular velocities, since coefficients of quadratic terms involve I_2 linearly. However, when $n_1 : n_2 : n_3 :: a : b : c$ a constant factor can be cancelled out and I_3 becomes of the second degree. If we set $n_1 = n_2 = n_3 = 0$ then the new case generalizes the original one merely by the addition of a gyrostatic momentum $\kappa = Cn$ along the axis of dynamical symmetry. The present case was first obtained in [407]. A restricted version of it, in which the body exhibits spherical dynamical symmetry, was obtained in [400] by a different method.

14.4.2.2 A General Integrable Case Relevant to Lyapunov's Hierarchy

The second integrable case is valid for a body of spherical dynamical symmetry $A = B = C$. It corresponds to choice nu as in (14.61). The obtained new integrable case is inserted in Table 14.1 case 2.

Note that the parameter n engenders an arbitrary constant gyrostatic momentum $\kappa = Cn$ along the axis of symmetry z of the body. This case was obtained in [400] using a transformation of the case 3 of Table 12.1 in Chap. 12. The solution of this version is not known, except in the very special case $s_1 = s_2 = s_3 = n_1 = n_2 = n_3 = 0$ analogous to Lyapunov's case of motion of a body in liquid solved by Kötter in terms of Theta functions of two arguments [235].

Physical interpretation of the special version $n_1 = n_2 = n_3 = 0$ is given in [390] as motion of an axi-symmetric body carrying electric charges fixed in it, in the presence of a magnetic field whose potential has the form $V_m = \frac{1}{2}(J_1X^2 + J_2Y^2 + J_3Z^2)$.

14.4.2.3 Examples of Physical Interpretation

From the considerations of Sect. 14.1, we see that a physical interpretation of the obtained cases is possible within the framework of motion of charged, magnetized bodies in the presence of non-uniform combination of the three classical fields. Due to the abundance of physical parameters representing the three distributions and the coef-

Table 14.1 General integrable cases

1	Yehia [407] (2001) $C = A$ (The spherical body) Yehia [400] (1998) $n = n_1 = n_2 = n_3 = 0$. Yehia [390] (1988).
	$V = \frac{1}{2}C[(a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2) - \frac{1}{2}(n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2)^2]$, $\boldsymbol{l} = (0, 0, C(n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2))$, $\mu_1 = -2C(n_1\alpha_1\alpha_3 + n_2\beta_1\beta_3 + n_3\gamma_1\gamma_3)$, $\mu_2 = -2C(n_1\alpha_2\alpha_3 + n_2\beta_2\beta_3 + n_3\gamma_2\gamma_3)$, $\mu_3 = C(n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2)$,
	$I_2 = C(r + n + n_1\alpha_3^2 + n_2\beta_3^2 + n_3\gamma_3^2)$, $I_3 = (a - 2n_1I_2)[A(p\alpha_1 + q\alpha_2) + C(r + \nu)\alpha_3]^2$ $\quad +(b - 2n_2I_2)[A(p\beta_1 + q\beta_2) + C(r + \nu)\beta_3]^2$ $\quad +(c - 2n_3I_2)[A(p\gamma_1 + q\gamma_2) + C(r + \nu)\gamma_3]^2$ $\quad -A[(b - 2n_2I_2)(c - 2n_3I_2)\alpha_3^2 + (c - 2n_3I_2)(a - 2n_1I_2)\beta_3^2$ $\quad +(a - 2n_1I_2)(b - 2n_2I_2)\gamma_3^2]$.
2	Yehia [400] (1998) $n = n_1 = n_2 = n_3 = 0$. Arbitrary body axes. Yehia [392] (1989) $n = n_1 = n_2 = n_3 = 0$. Yehia [390] (1988).
	$V = C\{s_1\alpha_3 + s_2\beta_3 + s_3\gamma_3 - \frac{1}{2}(bc\alpha_3^2 + ca\beta_3^2 + ab\gamma_3^2)$ $\quad -\frac{1}{2}(n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)^2$ $\quad +\frac{1}{2}(n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)[(b + c)\,\alpha_3^2 + (c + a)\,\beta_3^2 + (a + b)\,\gamma_3^2]\}$, $\boldsymbol{l} = C\{-\frac{1}{2}[(b + c)\alpha_3\boldsymbol{\alpha} + (c + a)\beta_3\boldsymbol{\beta} + (a + b)\gamma_3\boldsymbol{\gamma}]$ $\quad +(n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)\mathbf{k}\}$ $\mu_1 = -C[(n_1\alpha_1 + n_2\beta_1 + n_3\gamma_1) + a\alpha_3\alpha_1 + b\beta_3\beta_1 + c\gamma_3\gamma_1]$, $\mu_2 = -C[(n_1\alpha_2 + n_2\beta_2 + n_3\gamma_2) + a\alpha_3\alpha_2 + b\beta_3\beta_2 + c\gamma_3\gamma_2]$, $\mu_3 = C[n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3 - (a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2)]$.
	$I_2 = r + n - \frac{1}{2}[(b + c)\,\alpha_3^2 + (c + a)\,\beta_3^2 + (a + b)\,\gamma_3^2]$ $\quad +n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3$, $I_3 = \frac{1}{2}\{(b + c)\,[\boldsymbol{\omega}.\boldsymbol{\alpha} + (n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)\alpha_3]^2$ $\quad + (c + a)\,[\boldsymbol{\omega}.\boldsymbol{\beta} + (n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)\beta_3]^2$ $\quad + (a + b)\,[\boldsymbol{\omega}.\boldsymbol{\gamma} + (n + n_1\alpha_3 + n_2\beta_3 + n_3\gamma_3)\gamma_3]^2\}$ $\quad + (s_1 - n_1I_2)\,[\boldsymbol{\omega}.\boldsymbol{\alpha} + (a + n + n_1\alpha_3 + n_2\beta_3 + \gamma_3n_3)\alpha_3]$ $\quad + (s_2 - n_2I_2)\,[\boldsymbol{\omega}.\boldsymbol{\beta} + (b + n + n_1\alpha_3 + n_2\beta_3 + \gamma_3n_3)\beta_3]$ $\quad + (s_3 - n_3I_2)\,[\boldsymbol{\omega}.\boldsymbol{\gamma} + (c + n + n_1\alpha_3 + n_2\beta_3 + \gamma_3n_3)\gamma_3]$ $\quad -abc\{[\boldsymbol{\omega}.\boldsymbol{\alpha} + (n + n_1\alpha_3 + n_2\beta_3 + \gamma_3n_3)\alpha_3]\frac{\alpha_3}{a}$ $\quad +[\boldsymbol{\omega}.\boldsymbol{\beta} + (n + n_1\alpha_3 + n_2\beta_3 + \gamma_3n_3)\beta_3]\frac{\beta_3}{b}$ $\quad +[\boldsymbol{\omega}.\boldsymbol{\gamma} + (n + n_1\alpha_3 + n_2\beta_3 + \gamma_3n_3)\gamma_3]\frac{\gamma_3}{c}\}$.

ficients of the three potentials, it should be easy to adjust those parameters to match the potential in each case and, moreover, in a variety of choices.

We shall carry out detailed examples of the less obvious adjustment of the scalar magnetic potential Ω and the charge distribution to meet the Lorentz effect giving rise to the vector $\boldsymbol{\mu}$ in each case. This will be done for the two cases of Table 14.1 in this section. It is easy to verify that the most general harmonic second-degree polynomial potential can be reduced by a rotation transformation to the form

$$\Omega = a_1 X + a_2 Y + a_3 Z + \frac{1}{2}(a_{11} X^2 + a_{22} Y^2 + a_{33} Z^2), \tag{14.62}$$

with coefficients subject to the single condition

$$a_{11} + a_{22} + a_{33} = 0, \tag{14.63}$$

insuring that Ω is harmonic. According to (14.9), one can write

$$\begin{aligned} \mu_1 &= \int x F(\mathbf{r}\cdot\boldsymbol{\alpha}, \mathbf{r}\cdot\boldsymbol{\beta}, \mathbf{r}\cdot\boldsymbol{\gamma}) de, \\ \mu_2 &= \int y F(\mathbf{r}\cdot\boldsymbol{\alpha}, \mathbf{r}\cdot\boldsymbol{\beta}, \mathbf{r}\cdot\boldsymbol{\gamma}) de, \\ \mu_3 &= \boldsymbol{\sigma} + \int z F(\mathbf{r}\cdot\boldsymbol{\alpha}, \mathbf{r}\cdot\beta, \mathbf{r}\cdot\boldsymbol{\gamma}) de, \end{aligned} \tag{14.64}$$

where $\boldsymbol{\sigma}$ is a gyrostatic moment directed along the z-axis and

$$F(X, Y, Z) = a_1 X + a_2 Y + a_3 Z + a_{11} X^2 + a_{22} Y^2 + a_{33} Z^2. \tag{14.65}$$

To guarantee that φ is cyclic, we assume the distribution of electric charges on the body to be axi-symmetric around the z-axis. In virtue of symmetry $\int x^{\varepsilon_1} y^{\varepsilon_2} z^{\varepsilon_3} de$ is symmetric in ε_1 and ε_2, and it vanishes whenever ε_1 or ε_2 is odd. We denote the remaining integrals as

$$\int x^2 de = \int y^2 de = J, \int z^2 de = J', \int x^2 z de = \int y^2 z de = K, \int z^3 de = K'.$$

One finally gets

$$\begin{aligned} \mu_1 &= J(a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1) + K(a_{11}\alpha_3\alpha_1 + a_{22}\beta_3\beta_1 + a_{33}\gamma_3\gamma_1), \\ \mu_2 &= J(a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2) + K(a_{11}\alpha_3\alpha_2 + a_{22}\beta_3\beta_2 + a_{33}\gamma_3\gamma_2), \\ \mu_3 &= \sigma + J'(a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3) + K'(a_{11}\alpha_3^2 + a_{22}\beta_3^2 + a_{33}\gamma_3^2). \end{aligned} \tag{14.66}$$

For case 1. Comparing the components μ_i in case 1 to (14.66), we find that the uniform part (a_1, a_2, a_3) of the external magnetic field must vanish, so that the potential (14.62) should be homogeneous quadratic. In addition to the symmetry around the z-axis, the charge distribution should satisfy the single condition $K' = -\frac{1}{2}K$, i.e.

$$\int (x^2 + 2z^2) z de = 0. \tag{14.67}$$

The coefficients in case 1, in terms of the parameters of the body and field, have the form

$$n = \frac{\sigma}{C}, n_1 = -\frac{Ka_{11}}{2C}, n_2 = -\frac{Ka_{22}}{2C}, n_3 = -\frac{Ka_{33}}{2C}. \tag{14.68}$$

This completes interpretation of the first case. Note that the transformation parameters $n, n_1, \ldots$ have their distinct contributions to the physical problem.

For case 2. Equating the coefficients in case 2 and (14.66), we find that the charge distribution must satisfy the following two conditions $J' = -J$, $K' = K$, i.e.

$$\int z^2 de = -\int x^2 de, \tag{14.69}$$

$$\int x^2 z de = \int z^3 de, \tag{14.70}$$

and, without any loss of generality, one can express the parameters of case 2 in terms of the parameters of the body and external magnetic field in the form:

$$\begin{aligned} n_1 &= -\frac{Ja_1}{A}, n_2 = -\frac{Ja_2}{A}, n_3 = -\frac{Ja_3}{A}, \\ n &= \frac{\sigma}{A}, \\ a &= -Ka_{11}, b = -Ka_{22}, c = -Ka_{33}. \end{aligned} \tag{14.71}$$

If we define the moments of inertia of the distribution by A_e, B_e, C_e, then from symmetry we have the equality $A_e = B_e$. The condition (14.69) imposed on the second moments of the charge distribution can be put in the form $A_e = B_e = 0$. This is not a serious restriction, since electric charge distribution, unlike mass, can take positive and negative densities.

14.4.3 *Conditional Integrable Cases*

14.4.3.1 Two Cases Valid on a Single Level of the Cyclic Integral

The integrals I_2, I_3 can be verified directly for each case (Table 14.2).

14.4.3.2 Conditional Case on the Level $f = 0$

For example, the special cases $F_2(x) = F_1(x) = \frac{x^2}{8}$ and $F_2(x) = F_1(x) = \frac{1}{x}$ lead, respectively, to $V = \frac{1}{2}\Delta_1$, $\frac{1}{2\Delta_2}$. The complementary integral I_3 in the two cases, respectively, is

Table 14.2 Cases valid for arbitrary $\nu(\alpha_3, \beta_3, \gamma_3)$ on the level $I_2 = \varepsilon$

1	$B = A,\ \nu = \nu(\alpha_3, \beta_3, \gamma_3)$, $V = \frac{1}{2}(a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2) + \varepsilon\nu - \frac{1}{2}C\nu^2$, $\boldsymbol{l} = (0, 0, C\nu)$, $\boldsymbol{\mu}$ is found from $\boldsymbol{l}$ according to (14.4).
	$I_2 = C(r + \nu) = \varepsilon$ $I_3 = a[A(p\alpha_1 + q\alpha_2) + C(r + \nu)\alpha_3]^2 + b[A(p\beta_1 + q\beta_2) + C(r + \nu)\beta_3]^2 + c[A(p\gamma_1 + q\gamma_2) + C(r + \nu)\gamma_3]^2 - A[bc\alpha_3^2 + ca\beta_3^2 + ab\gamma_3^2]$
2	$A = B = C,\ \nu = \nu(\alpha_3, \beta_3, \gamma_3)$, $V = C\{s_1\alpha_3 + s_2\beta_3 + s_3\gamma_3 - \frac{1}{2}(bc\alpha_3^2 + ca\beta_3^2 + ab\gamma_3^2) + \varepsilon\nu - \frac{1}{2}\nu^2 + \frac{1}{2}\nu[(b + c)\,\alpha_3^2 + (c + a)\,\beta_3^2 + (a + b)\,\gamma_3^2]\}$, $\boldsymbol{l} = C\{-\frac{1}{2}[(b + c)\alpha_3\boldsymbol{\alpha} + (c + a)\beta_3\boldsymbol{\beta} + (a + b)\gamma_3\boldsymbol{\gamma}] + \nu\mathbf{k}]$ $\boldsymbol{\mu}$ is found from $\boldsymbol{l}$ according to (14.4).
	$I_2 = r - \frac{1}{2}[(b + c)\,\alpha_3^2 + (c + a)\,\beta_3^2 + (a + b)\,\gamma_3^2] + \nu = \varepsilon$, $I_3 = \frac{1}{2}\{(b + c)\,(\boldsymbol{\omega}.\boldsymbol{\alpha} + \nu\alpha_3)^2 + (c + a)\,(\boldsymbol{\omega}.\boldsymbol{\beta} + \nu\beta_3)^2 + (a + b)\,(\boldsymbol{\omega}.\boldsymbol{\gamma} + \nu\gamma_3)^2\} + s_1(\boldsymbol{\omega}.\boldsymbol{\alpha} + \nu\alpha_3) + s_2(\boldsymbol{\omega}.\boldsymbol{\beta} + \nu\beta_3) + s_3(\boldsymbol{\omega}.\boldsymbol{\gamma} + \nu\gamma_3) - abc\{(\boldsymbol{\omega}.\boldsymbol{\alpha} + \nu\alpha_3)\frac{\alpha_3}{a} + (\boldsymbol{\omega}.\boldsymbol{\beta} + \nu\beta_3)\frac{\beta_3}{b} + (\boldsymbol{\omega}.\boldsymbol{\gamma} + \nu\gamma_3)\frac{\gamma_3}{c}\}$.

$$
\begin{aligned}
I_3 &= A[a(\boldsymbol{\omega} \cdot \boldsymbol{\alpha})^2 + b(\boldsymbol{\omega} \cdot \boldsymbol{\beta})^2 + c(\boldsymbol{\omega} \cdot \boldsymbol{\gamma})^2] - \Delta_2 \\
&= A[a(p\alpha_1 + q\alpha_2)^2 + b(p\beta_1 + q\beta_2)^2 + c(p\gamma_1 + q\gamma_2)^2 - \Delta_2,
\end{aligned} \tag{14.72}
$$

and

$$
I_3 = A[a(\boldsymbol{\omega} \cdot \boldsymbol{\alpha})^2 + b(\boldsymbol{\omega} \cdot \boldsymbol{\beta})^2 + c(\boldsymbol{\omega} \cdot \boldsymbol{\gamma})^2] + \frac{a + b + c - \Delta_1}{\Delta_2}. \tag{14.73}
$$

The above case is separable in sphero-conic coordinates on the sphere. Another type of potential separable in spherical coordinates can be written as

$$
V = u(\gamma_3) + \frac{v(\frac{\alpha_3}{\beta_3})}{1 - \gamma_3^2} = u(\gamma_3) + \frac{v_1(\frac{\alpha_3}{\beta_3})}{\beta_3^2}, \tag{14.74}
$$

where u, v and v_1 are arbitrary functions. This potential, analogous to that of Chap. 9, Sect. 9.7.1, will not be considered further here (Table 14.3).

Table 14.3 A separable case valid on the level $I_2 = 0$

1	Bogoyavlensky [31] (1986): $F_2 = F_1$ is polynomial $B = A$, $V = \frac{F_1(\alpha^* - \sqrt{\beta^*}) - F_2(\alpha^* + \sqrt{\beta^*})}{\sqrt{\beta^*}}$, $\alpha^* = a + b + c - \Delta_1$, $\beta^* = \alpha^{*2} - 4abc\Delta_2$, $\Delta_1 = a\alpha_3^2 + b\beta_3^2 + c\gamma_3^2$, $\Delta_2 = bc\alpha_3^2 + ca\beta_3^2 + ab\gamma_3^2$.
	$I_2 = r = 0$, $I_3 = A[a(\boldsymbol{\omega} \cdot \boldsymbol{\alpha})^2 + b(\boldsymbol{\omega} \cdot \boldsymbol{\beta})^2 + c(\boldsymbol{\omega} \cdot \boldsymbol{\gamma})^2] +$ $+ \frac{1}{\sqrt{\beta^*}}[(\alpha^* + \sqrt{\beta^*})F_1(\alpha^* - \sqrt{\beta^*}) - (\alpha^* - \sqrt{\beta^*})F_2(\alpha^* + \sqrt{\beta^*})]$.

14.5 Motion of a Body with Combined (Quaternion) Symmetry

14.5.1 Introduction

Hitherto, we have studied integrable problems of two types of symmetry:

1- Symmetry of the fields around an axis fixed in space corresponding to a cyclic variable ψ and leading to the areas integral.

2- Complete or physical symmetry of the body about an axis fixed in it. The last symmetry corresponds to a cyclic variable φ (the angle of proper rotation), and it leads to conservation of the generalized momentum conjugate to this angle.

In the present section, we present some integrable cases of motion of a rigid body acted upon by a combination of non-coaxial fields under a different condition. We assume that the Lagrangian of the problem admits the type of symmetry met earlier in case b of Sect. 14.2.1.1, corresponding to a cyclic variable $\psi \pm \varphi$. For determinacy we consider only the case when $\psi + \varphi$ is a cyclic variable. The other case is completely analogous. A useful interpretation of this symmetry follows easily from formulas (2.41). It is symmetry with respect to rotation of the plane $\lambda_3\lambda_0$ about its origin in the space of Hamilton–Rodrigues' parameters (the quaternion space restricted to the unit sphere). Similar rotation in the plane $\lambda_1\lambda_2$ corresponds to $\varphi - \psi$ as a cyclic coordinate. We have used for those types of symmetry the name "combined symmetry", since it combines the condition of dynamical symmetry of the body and a certain condition of symmetry of the applied fields. In several works, M. Kharlamov referred to it as "singular symmetry" (see, for example, [189]) or S^1-symmetry [213].

Systematic treatment of this type of symmetry was initiated in [391]. That was based on the observation that one of the new integrable cases introduced in [380], namely case b of Sect. 14.2.1.1, concerned a problem of motion of a heavy magnetized body-gyrostat with the Kovalevskaya configuration $A = B = 2C$ in a combination of uniform gravity and magnetic fields admitting this symmetry. Following up the general case when the problem of motion admits this type of symmetry, it turned out that the reduced Routhian equations of motion of such problem in the space of Hamilton–Rodrigues' parameters can be brought into isomorphism with the reduction of the, much better studied, problem of motion of a dynamically axi-symmetric body in a coaxial combination of fields. This situation revealed exotic relation between known integrable cases in both problems and also furnished a way to construct new integrable cases of the first problem based on the analogy with known ones of the second. This will be detailed in the coming subsection. However, the lists here are not as complete as for the former two problems. Since the isomorphism is only on the level of reduced problems, the determination of complementary integrals of the original problem is not an easy matter. Some of the integrable cases listed below are left without this integral explicitly written down. Though it certainly exists, its determination needs further work.

Finally, as systems with cyclic variables are amenable to generalizations of the type introduced in Sect. 11.9, in the following subsection we apply what we called cyclic velocity transformations to obtain new cases involving a larger number of physical parameters—or a function—in their structure. For such application, we have chosen those cases for which the complementary integral is known in the Euler–Poisson variables. To have complete lists one needs some additional work.

14.5.2 Routhian Reduction

Let a rigid body-gyrostat with $A = B$ be in motion under the action of forces with potential V admitting combined symmetry and gyroscopic moment l_3 compatible with the present type of symmetry (This may also include a gyrostatic moment directed along the axis of dynamical symmetry.). In terms of Hamilton–Rodrigues parameters, we have

$$V = V(\lambda_1, \lambda_2, \lambda), l_3 = l_3(\lambda_1, \lambda_2, \lambda),$$

where $\lambda = \sqrt{\lambda_0^2 + \lambda_3^2}$. We shall use the redundant coordinates λ_1, λ_2 and the polar coordinates λ, Ψ in the $\lambda_0\lambda_3$ plane, satisfying the condition

$$\lambda_1^2 + \lambda_2^2 + \lambda^2 = 1. \tag{14.75}$$

The Lagrangian of this system is

$$L = \frac{1}{2}[A(p^2 + q^2) + Cr^2] + l_3 r - V. \tag{14.76}$$

In terms of the quaternions, it can be put in the form

$$L = 2[A(\dot{\lambda}_1^2 + \dot{\lambda}_2^2 + \dot{\lambda}_3^2 + \dot{\lambda}_0^2) - (A - C)(\lambda_0\dot{\lambda}_3 - \lambda_3\dot{\lambda}_0 + \lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2)^2]$$
$$+2l_3(\lambda_0\dot{\lambda}_3 - \lambda_3\dot{\lambda}_0 + \lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2) - V$$
$$= 2[A(\dot{\lambda}_1^2 + \dot{\lambda}_2^2 + \dot{\lambda}^2 + \lambda^2\dot{\Psi}^2) - (A - C)(\lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2 + \lambda^2\dot{\Psi})^2]$$
$$+2l_3(\lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2 + \lambda^2\dot{\Psi}) - V. \tag{14.77}$$

The variable Ψ is cyclic and corresponds to the cyclic integral

$$I_2 = \lambda^2\{4[A + (C - A)\lambda^2]\dot{\Psi} + 4(C - A)(\lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2) + 2l_3\} = f. \tag{14.78}$$

Ignoring the cyclic variable, we obtain the Routhian [391]

$$R = \frac{1}{4}(L - f\dot{\Psi})$$
$$= \frac{A^2C}{2D}(\frac{\dot{\lambda}_1^2 + \dot{\lambda}_2^2}{A} + \frac{\dot{\lambda}^2}{C}) + \frac{1}{D}[\frac{f}{4}(C - A) + \frac{A}{2}l_3](\lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2)$$
$$-[\frac{V}{4} + \frac{1}{2\lambda^2 D}(\frac{f}{4} - \frac{1}{2}\lambda^2 l_3)^2], \tag{14.79}$$

where $D = A(\lambda_1^2 + \lambda_2^2) + C\lambda_3^2$.

Reduced equations of motion in quaternions $(\lambda_1, \lambda_2, \lambda)$ as redundant coordinates may be derived simply from the last Routhian subject to the geometric condition (14.75). However, this is of secondary importance. This Routhian may be used in various ways, keeping in mind the obvious resemblance to the Routhian (11.44) in redundant coordinates $(\gamma_1, \gamma_2, \gamma_3)$ if one adds to the latter the dynamical symmetry condition $B = A$. Simple examples are

(1) **The analog of Staude rotations:** Stationary solutions of the system described by (14.79) are those for which $\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda} = 0$. This may be rewritten as $\dot{\theta} = 0$, $\dot{\psi} = \dot{\varphi}$, which characterizes a precessional motion of the body, whose figure axis is the z-axis and precession axis is the Z-axis fixed in space.

(2) **The maximal reduction of order:** Exactly as in Sect. 11.7, the equations of motion can be reduced, for arbitrary V and l_3 and on arbitrary fixed integral level $\{h, f\}$, to a single second-order differential equation in λ, λ_1. This may be accomplished by using Maupertuis' principle and eliminating λ_2. An explicit form of such reduced equation for the case $l_3 = \text{const} = k_3$ (a gyrostatic momentum along the symmetry axis) is given in [391], where some applications of it are pointed out. Interested readers may consult that work.

14.5.3 Equivalence of Two Problems

Now, we turn to the most useful application of the Routhian (14.79). Denote by R' the Routhian of the problem of motion under coaxial fields considered in Sect. 11.7, but with potentials $V'(\gamma)$ and $\boldsymbol{l}' = (0, 0, l_3'(\gamma))$. If we add Setting $A = B$ in the Routhian

(11.44), resulting from ignoring the angle of precession, takes the form

$$R' = \frac{A^2C}{2D'}\left(\frac{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}{A} + \frac{\dot{\gamma}_3^2}{C}\right) + \frac{1}{D'}\left(Al_3' + \frac{f'C\gamma_3}{\gamma_1^2 + \gamma_2^2}\right)(\gamma_2\dot{\gamma}_1 - \gamma_1\dot{\gamma}_2)$$
$$-[V' + \frac{1}{2D'}(f' - l_3'\gamma_3)^2], \tag{14.80}$$

where $D' = A(\gamma_1^2 + \gamma_2^2) + C\gamma_3^2$.

Now, if we compare the Routhians (14.79) and (14.80), we note a curious similarity in their structures. They can be made identical, through a change of variables $\to (\lambda_1, \lambda_2, \lambda)$, if we add the following conditions on the potentials of the two problems:

$$\frac{f}{4}(C - A) + \frac{A}{2}l_3 = Al_3'(\lambda_1, \lambda_2, \lambda) + \frac{f'C\lambda}{\lambda_1^2 + \lambda_2^2},$$
$$\frac{V}{4} + \frac{1}{2\lambda^2 D}\left(\frac{f}{4} - \frac{1}{2}\lambda^2 l_3\right)^2 = V'(\lambda_1, \lambda_2, \lambda) + \frac{1}{2D}[f' - \lambda l_3'(\lambda_1, \lambda_2, \lambda)]^2. \tag{14.81}$$

For the time being, one would regard f, f' as the constant values of the linear integrals of motion in the two problems (not the integrals themselves, which depend on the phase variables). Note that if $l_3'()$ is a smooth function on the Poisson sphere, then the potentials l_3 and V have a singularity on the great circle $\lambda = 1(\lambda_0^2 + \lambda_3^2 = 1)$ on the unit quaternion sphere. Since such behaviour is not favourable for a real problem, then, for the sake of simplicity, we shall impose the additional condition $f' = 0$.

Finally, let a case B (say) of the axi-symmetric problem be given, on the level $f' = 0$ of the linear integral, with the pair (V', l_3'). Then the corresponding problem A (say) with combined symmetry will be characterized by the pair (V, l_3) given by

$$l_3 = 2l_3' + \frac{f(A - C)}{2A},$$
$$V = 4V' + \frac{f}{A}l_3' - \frac{f^2}{8A\lambda^2}. \tag{14.82}$$

Here the λ's are substituted, using formulas of Chap. 2, from the table

$$\lambda_1^2 = \frac{1 + \alpha_1 - \beta_2 - \gamma_3}{4},$$
$$\lambda_2^2 = \frac{1 - \alpha_1 + \beta_2 - \gamma_3}{4},$$
$$\lambda_1\lambda_2 = \frac{\alpha_2 + \beta_1}{4},$$
$$\lambda^2 = \frac{1 + \gamma_3}{2}.$$

Thus, in problems amenable to the above construction the potentials V, l_3 are functions of the three quantities: $\alpha_1 - \beta_2, \alpha_2 + \beta_1, \gamma_3$.

The inverse of the transformation (14.82) may be written as

$$
\begin{aligned}
l_3' &= \frac{1}{2}l_3 - \frac{f(A-C)}{4A},\\
V' &= \frac{1}{4}V - \frac{f}{8A}l_3 + \frac{f^2}{32A\gamma_3^2}.
\end{aligned}
\tag{14.83}
$$

The dynamical parameter f figures in the two functions, so that the integrable case is generically conditional. However, in certain cases it is possible to construct also general integrable cases. This occurs mostly when the potential V' contains in its structure some arbitrary parameter, so that f can be absorbed in it. Examples will be provided below.

The isomorphism of problems A and B was first established in [391] (1988) for the simpler case when both l_3, l_3' are constant gyroscopic momenta directed along the axis of dynamical symmetry. In the same work, the appearance of the Goryachev singular term $\frac{a}{\gamma_3^2}$ in problem B was associated with the cyclic constant of problem A. In [403], the isomorphism was extended to the case when l_3, l_3' are variable quantities. In both works, the isomorphism concerned the two problems A and B on the level of their Routhian reductions. The original full problems, before ignoring the cyclic variable in each, are of course equivalent, but, in general, they are not isomorphic. In [37] of 1997 (see also their book [38] of 1999), Borisov and Mamaev found explicit transformation between the phase variables in the two reduced problems, and thus established isomorphism between them. However, their result was applied only in two cases of purely potential forces ($l_3 = 0$), because they usually do not write equations of motion for the asymmetric fields problem A in the presence of skew gyroscopic forces (the Poisson bracket equivalent of Eq. (14.3)). It may be noted here that the comment in [41] (Sect. 4.1) "... a little earlier H. Yehia [623] (1988) had used a restricted version (for $M_3 \pm N_3 = 0$)" is misleading, since the real condition was $f' = 0$. The latter is the areas integral in problem B.

Alternative formulation:

When written in terms of Euler's angles, the Lagrangian (14.76) becomes

$$
L = \frac{1}{2}[A(\dot\theta^2 + \sin^2\theta\dot\psi^2) + C(\dot\psi\cos\theta + \dot\varphi)^2] + (\dot\psi\cos\theta + \dot\varphi)l_3 - V. \tag{14.84}
$$

We shall now introduce a change of variables

$$
\theta = 2\Theta, \psi = \Psi - \Phi, \varphi = \Psi + \Phi. \tag{14.85}
$$

This will change (14.84) to

$$
\begin{aligned}
L = {}& 2\{A[\dot\Theta^2 + \sin^2\Theta\cos^2\Theta(\dot\Psi - \dot\Phi)^2] + C(\sin^2\Theta\dot\Phi + \cos^2\Theta\dot\Psi)^2\}\\
&+2l_3(\sin^2\Theta\dot\Phi + \cos^2\Theta\dot\Psi) - V.
\end{aligned}
\tag{14.86}
$$

Note that V and l_3 are now functions of Θ and Φ only. The cyclic integral corresponding to the cyclic variable Ψ is

$$4\cos^2\Theta D\dot{\Psi} - 4(A-C)\sin^2\Theta\cos^2\Theta\dot{\Phi} + 2l_3\cos^2\Theta = \text{const} = f \tag{14.87}$$

where $D = A\sin^2\Theta + C\cos^2\Theta$. Thus one can write

$$\dot{\Psi} = \frac{f - 2l_3\cos^2\Theta + 4(A-C)\sin^2\Theta\cos^2\Theta\dot{\Phi}}{4\cos^2\Theta D}. \tag{14.88}$$

This can be used to ignore the cyclic variable Ψ and construct the Routhian

$$\begin{aligned} R &= \frac{1}{4}(L - f\dot{\Psi}) \\ &= \frac{1}{2}A(\dot{\Theta}^2 + \frac{C}{D}\sin^2\Theta\dot{\Phi}^2) + \sin^2\Theta[f(C-A) + 2Al_3]\frac{\dot{\Phi}}{4D} \\ &\quad -[\frac{V}{4} + \frac{(f - 2\cos^2\Theta l_3)^2}{32\cos^2\Theta D}]. \end{aligned} \tag{14.89}$$

It is curious to remark that the linear term in this Routhian (14.89) (and also in (14.79)) vanishes whenever $f(C-A) + 2Al_3 = 0$. That is, for a body with constant gyrostatic momentum $\boldsymbol{l} = (0, 0, k_3)$, on the level $f = \frac{2Ak_3}{A-C}$, the reduced equations of motion become time-reversible. This property is characteristic of the motion of the body in pure potential forces as was shown in Chap. 9. On the other hand, it may be said that the parameter f partially engenders a gyrostatic momentum component along the axis of dynamical symmetry, enough to annul the effect of the existing gyrostatic moment on the motion of the reduced system. This situation is used in [189] to get a real separation of variables and reduce the solution of the reduced equations of motion of the body in two coupled uniform fields to elliptic quadratures. We shall return to this point shortly later.

The Routhian of problem B in Euler's angles is

$$\begin{aligned} R' &= \frac{1}{2}A(\dot{\theta}^2 + \frac{C}{D'}\sin^2\theta\dot{\varphi}^2) + \frac{\dot{\varphi}}{D'}(fC\cos\theta + Al_3'\sin^2\theta) \\ &\quad -[V' + \frac{(f' - l_3'\cos\theta)^2}{2D'}]. \end{aligned} \tag{14.90}$$

Formulas corresponding to (14.81) are

$$\begin{aligned} l_3 &= 2l_3' + \frac{f(A-C)}{2A} + \frac{2Cf'\cos\Theta}{A\sin^2\Theta}, \\ V &= 4V' + \frac{fl_3'}{A} - \frac{f^2}{8A\cos^2\Theta} \\ &\quad + \frac{f'}{A^2\sin^2\Theta}[2f'(A+C) + (Cf - 4Al_3')\cos\Theta + \frac{2Cf'}{\sin^2\Theta}]. \end{aligned} \tag{14.91}$$

In terms of the Euler angles of the original problem A, the last formulas read

$$\begin{aligned} l_3 &= 2l_3'(\frac{\theta}{2}, \frac{\varphi-\psi}{2}) + \frac{f(A-C)}{2A} + \frac{4Cf'\cos\frac{\theta}{2}}{A(1-\cos\theta)}, \\ V &= 4V'(\frac{\theta}{2}, \frac{\varphi-\psi}{2}) + \frac{f}{A}l_3'(\frac{\theta}{2}, \frac{\varphi-\psi}{2}) - \frac{f^2}{4A(1+\cos\theta)} \\ &\quad + \frac{2f'}{A^2(1-\cos\theta)}[2f'(A+C) + (Cf - 4Al_3'(\frac{\theta}{2}, \frac{\varphi-\psi}{2}))\cos\frac{\theta}{2} + \frac{2Cf'}{1-\cos\theta}]. \end{aligned} \tag{14.92}$$

Those formulas turned out to be useful and fast in converting integrable cases of problem B to their counterparts of problem A, especially when $f' = 0$. They were effectively used in the construction of most of the integrable cases listed below.

14.5.4 Basic Equivalent Integrable Problems

The most useful application of the isomorphism established above is that for any integrable case of problem B there corresponds an integrable case of problem A, and vice versa. As problem B is much more studied, the isomorphism will mostly work in the direction from B to A. Among the conditional integrable cases of Chap. 13, all cases valid for $B = A$ and $l_1 = l_2 = 0$ satisfy the requirement for problem B and all have equivalent cases in problem A. However, not all the resulting cases are equally sound. As noticed in Chap. 2, potentials in problem A that involve terms odd in the quaternion variables λ_1, λ_2 are double-valued on the group of rotations SO_3. This fact renders of no physical significance the equivalent of the two classical integrable cases of Kowalevski and Goryachev–Chaplygin. In fact, those cases have the same potential $V' = a\gamma_1 + b\gamma_2$ and hence the equivalent cases have the potential $V = a\lambda_1 + b\lambda_2$. It is curious that the full general integrable case of Kowalevski has as its equivalent a conditional integrable case, valid only on the level $f = 0$, while its conditional version with $f' = 0$ containing the singular Goryachev term $\frac{\varepsilon}{\gamma_3^2}$ is a general integrable case. The cyclic constant f and the arbitrary constant ε are connected by the relation $\varepsilon = \frac{f^2}{32A}$. Integrable cases involving quaternion (double-valued) potentials are first met in [391] (see also [403]). They are considered in more detail in [41],where they are called quaternion integrable cases.

One of the integrable cases equivalent to the well-known case of problem B exhibits at once several characteristics of the correspondence between the two problems. Consider case 17 of Table 13.1 in Chap. 13, i.e. the case characterized by the potentials:

$$\begin{aligned} \boldsymbol{l} &= C(0, 0, \kappa'), \\ V &= C[a_1\gamma_1 + a_2\gamma_2 - n\kappa\gamma_3 + b_1(\gamma_1^2 - \gamma_2^2) + 2b_2\gamma_1\gamma_2 + \frac{\lambda}{2\gamma_3^2}], \end{aligned}$$

to which we applied the condition $n = 0$, in order to make it amenable to the above transformation. For this case, formulas (14.82) give

$$\begin{aligned} \boldsymbol{l} &= C(0, 0, \kappa' + \frac{f}{4C}), \\ V &= 4C[a_1\lambda_1 + a_2\lambda_2 + b_1(\lambda_1^2 - \lambda_2^2) + 2b_2\lambda_1\lambda_2 + \frac{\varepsilon}{2\lambda^2}] \\ &\quad - \frac{f}{16}(\kappa' + \frac{f}{4C}) + \frac{f^2}{64C\lambda^2}. \end{aligned}$$

Renaming arbitrary constants and ignoring the constant term in V, one gets

$$\begin{aligned} \boldsymbol{l} &= C(0, 0, \kappa), \\ V &= C[a_0\lambda_1 + b_0\lambda_2 + 2a(\lambda_1^2 - \lambda_2^2) + 4b\lambda_1\lambda_2 + \frac{c}{2\lambda^2}], \end{aligned} \tag{14.93}$$

in which all parameters are now arbitrary and free of any conditions. This signifies that

(1) No restriction is there on f, i.e. the integrable case (14.93) is unconditional. It is valid for all feasible initial conditions,[2] though the equivalent integrable case of problem B is only conditionally integrable, on the zero areas integral level.
(2) The parameter f is absorbed with κ' in the arbitrary gyrostatic momentum $\kappa = \kappa' + \frac{f}{4C}$.
(3) The two terms $2a(\lambda_1^2 - \lambda_2^2) + 4b\lambda_1\lambda_2$ engendered by Chaplygin's potential for the body moving in a liquid yield in problem A, after expressing them in direction cosines, the terms

$$a(\alpha_1 - \beta_2) + b(\alpha_2 + \beta_1). \tag{14.94}$$

This is the potential in case B of Sect. 14.2.1.1 above, the first case of its type, discovered in 1986 [380].
(4) The last term in the potential gives $\frac{c}{2\lambda^2} = \frac{c}{2\cos^2(\theta/2)} = \frac{c}{1+\gamma_3}$. This term presents a singularity at $\gamma_3 = -1$. An interpretation will be given below for it.
(5) Finally, there remain the first two terms, namely

$$a_0\lambda_1 + b_0\lambda_2.$$

[2] When $c \neq 0$, one has to exclude the positions that satisfy $\lambda = 0 (\theta = \pi)$ from the configuration space.

Table 14.4 Conditional integrable cases for which $\varphi + \psi$ is cyclic. For all cases, the cyclic integral is $A(p\gamma_1 + q\gamma_2) + Cr(\gamma_3 + 1) = 0$

1	Separable in spherical coordinates Θ, Φ. Analog of case 2 Chap. 13, using the Routhian (14.89). $B = A, \boldsymbol{l} = \boldsymbol{\mu} = 0$, $V = F_1(\Theta) + \frac{D}{C\sin^2\Theta} F_2(\Phi)$ $\quad = V_1(\gamma_3) + \frac{A+C+(A-C)\gamma_3}{C(1-\gamma_3)} F(\frac{\alpha_1-\beta_2}{1-\gamma_3})$. V_1, F arbitrary functions. $I_2 = A(p\gamma_1 + q\gamma_2) + Cr(\gamma_3 + 1) = 0$ I_3 Not constructed.
2	Separable in sphero-conical coordinates on the reduced quaternion sphere (14.75). Analog of case 3 Chap. 13. $A = B = C, \boldsymbol{l} = \boldsymbol{\mu} = 0$, $V = \frac{1}{v}[V_1(u - v) + V_2(u + v)]$, $I_2 = A(p\gamma_1 + q\gamma_2) + Cr(\gamma_3 + 1) = 0$ I_3 Not constructed.

If alone, those terms designate a general integrable case that imitates Kowalevski's potential in the quaternion space. Since those terms are not single-valued on the group of rotations, no physical significance can be adhered to them. In the tables of integrable cases below, we shall ignore such terms (odd terms in the rotation quaternion) from consideration.

In the present subsection, we list some equivalent integrable cases corresponding to different cases tabulated in Chap. 13. Cases are presented in their basic form as implied by corresponding cases of problem B. Some generalizations are left to the next subsection. We ignore cases that degenerate under the dynamical symmetry condition or the condition $l_1 = l_2 = 0$ and, in other cases, we also ignore terms that are double-valued in the direction cosines. Regarding those considerations, the total number of different cases is much less than cases of type B (Table 14.4).

V_1, V_2 are arbitrary functions of their arguments,

$$\begin{aligned} u &= a + b + c - (a\lambda_1^2 + b\lambda_2^2 + c\lambda_3^2) \\ &= \frac{3}{4}(a+b) + \frac{c}{2} + \frac{1}{4}(b-a)(\alpha_1 - \beta_2) + \frac{1}{4}(a+b-2c)\gamma_3, \\ v^2 &= u^2 - 4(bc\lambda_1^2 + ca\lambda_2^2 + ab\lambda^2) \\ &= u^2 - 2ab(1+\gamma_3) + c(a-b)(\alpha_1 - \beta_2) - c(a+b)(1-\gamma_3), \end{aligned} \quad (14.95)$$

a, b, c are arbitrary constants. For example, the choice $V_2(x) = -V_1(x) = -x^2$ leads to a potential of the form

$$V = a'(\alpha_1 - \beta_2) + b'\gamma_3.$$

Another example is $V_2(x) = -V_1(x) = 1/x$. This yields the potential

$$V = \frac{d}{2ab(1+\gamma_3) - c(a-b)(\alpha_1 - \beta_2) + c(a+b)(1-\gamma_3)},$$

where d is a constant.

The solution is reduced in [385] to quadratures, elliptic for linear V and hyper-elliptic for higher degree polynomial potentials.

2	Yehia [407] (2001) $A = B = C$, $\boldsymbol{l} = \boldsymbol{\mu} = 0$, $V = A[a_1(\alpha_1 - \beta_2) + a_2\gamma_3]$. $I_2 = \boldsymbol{\omega} \cdot \boldsymbol{\gamma} + r$, $I_3 = a_1(p\boldsymbol{\omega} \cdot \boldsymbol{\alpha} - q\boldsymbol{!} \cdot \mathbf{fi}) + a_2 r\boldsymbol{\omega} \cdot \boldsymbol{\gamma} - 2a_1[a_2(\alpha_1 - \beta_2) + a_1\gamma_3]$

Although this is a special case of that of Bogoyavlensky with three quadratic integrals (see Sect. 14.2.2.2), this case was not noticed in [32]. It was singled out in [407], where the integrals were also given.

3	Yehia [403] (2000), Borisov and Mamaev [40] Yehia [380] (1986), [391] (1988) $c = 0$. $A = B = 2C$, $\boldsymbol{l} = \boldsymbol{\mu} = C(0, 0, \kappa)$, $V = C[a(\alpha_1 - \beta_2) + b(\alpha_2 + \beta_1) + \frac{c}{1+\gamma_3}]$,
	$I_2 = 2(p\gamma_1 + q\gamma_2) + (r + \kappa)\gamma_3 + r$, $I_3 = [p^2 - q^2 - a(\alpha_1 + \beta_2) + b(\alpha_2 - \beta_1)]^2 + [2pq - b(\alpha_1 + \beta_2) - a(\alpha_2 - \beta_1)]^2$ $+2\kappa(r - \kappa)(p^2 + q^2) - 4k[p(a\alpha_3 + b\beta_3) + q(b\alpha_3 - a\beta_3)]$ $+\frac{2c}{1+\gamma_3}[p^2 + q^2 + \kappa r - \kappa^2 + \frac{c}{2(1+\gamma_3)}]$.

History: The full case 3 ($c \neq 0$) was presented in 2000 with integrals in the Euler–Poisson variables. In 2001, it was independently found, but in a different set of Hamiltonian variables based on quaternions as configuration variables [40].

The special case $c = 0$ was found first in [380], together with its integrals I_2 and I_3. Neither the full nor the special integrable case has known general explicit solution yet in terms of time. An exception is the case when, in addition to the condition $c = 0$, one can rotate the xy-axes in order to make $b = 0$. Then the potentials become

$$V = Ca(\alpha_1 - \beta_2), \boldsymbol{l} = C(0, 0, \kappa). \tag{14.96}$$

The integration of the Routhian reduction of the problem characterized by the last potentials was investigated in [189]. It was reduced on the level $I_2 = f = 4\kappa$ to two hyper-elliptic quadratures. When $f = 4\kappa = 0$, the quadratures become elliptic, and resemble those for Chaplygin's case of a body in a liquid (see Chap. 10, Sect. 10.17). It can be seen easily that the equivalent case in problem B corresponds according to the choice

$$l_3' = 0,$$
$$V' = a'(\gamma_1^2 - \gamma_2^2) + \frac{\kappa^2}{8\gamma_3^2} + \text{const}\,. \tag{14.97}$$

This is the subcase of case 17 of Table 13.1 in Chap. 13 that was considered by Ryabov in [323] and reduced to hyper-elliptic quadratures.

It should be noted that the two types of separation of problem B in [323] and problem A in [189] are quite different. In the first, all phase variables in the (generalized) Euler–Poisson equations of motion are determined in terms of the separation variables and then the cyclic precession angle is found by integrating equation of the type (11.7). In the second, separation is performed only for the reduced (Routhian) equations and the cyclic coordinate is determined by another quadrature. The angular velocity is to be derived from the Eulerian angles for the original problem A.

Physical interpretation of the full case 3 may be given as follows [403]:

(1) Let an axi-symmetric gyroscope with axial moment of inertia I_G placed along the axis of dynamical symmetry of the body be kept rotating with uniform angular velocity Ω. The vector $\boldsymbol{\mu}$ is the gyrostatic moment due to this rotor, provided we take $C\kappa = I_G\Omega$.

(2) Let the body have total mass M, centre of mass $\mathbf{r}_0 = (x_0, y_0, 0)$ lying in its equatorial plane. Suppose the body also contains some magnetized parts with total magnetic moment $\mathbf{m} = (m_1, m_2, 0)$. Let the system be moving in the presence of a uniform gravity field g **a**nd a horizontal magnetic field H**fi**. Its potential will be

$$g(x_0\alpha_1 + y_0\alpha_2) + H(m_1\beta_1 + m_2\beta_2).$$

This expression can be identified with the first two terms of V if we take $\mathbf{m} \cdot \mathbf{r}_0 = 0$ and $|g\mathbf{r}_0| = |H\mathbf{m}|$, so that $Ca = gx_0$, $Cb = gy_0$.

(3) Consider two points $P(0, 0, 1)$ fixed in the body on the z-axis (its axis of dynamical symmetry) and Q on the fixed Z-axis at $Z = -1$. The distance $PQ = 2\cos\frac{\theta}{2}$. The singular potential term in case 3 is thus inversely proportional to the square of the distance PQ.

It is curious to note that the singular term $\frac{1}{\sqrt{1-\gamma_3^2}} = \frac{1}{\sin\Theta}$ *(Yehia–Bedwehy term) in problem B corresponds in problem A to* $\frac{1}{\sin\frac{\theta}{2}}$, *which is proportional to* $\frac{1}{PR}$, *where* R *is the other pole of the fixed sphere opposite to* Q. *Such interaction may be interpreted as due, for example, to Coulomb interaction.*

Among the cases of Chap. 13, some have physical equivalent cases and others correspond to purely “quaternion” cases. Example of last type is the equivalent of case Chap. 13#8. This is a complicated conditional integrable case ($f = 0$), since the potential does not include the singular Goryachev term $\frac{1}{\gamma_3^2}$. This case includes as a special case, $\kappa = e_0 = e_1 = 0$, the original Goryachev singular potential, which gives rise to the purely quaternionic equivalent potential

$$V = \frac{a\lambda_1 + b\lambda_2 + c}{\lambda^{\frac{2}{3}}}. \tag{14.98}$$

A similar case characterized by

$$\begin{aligned} V &= \frac{a}{(\lambda_1^2\lambda_2^2\lambda^2)^{1/3}} \\ &= \frac{a^*}{\{(1+\gamma_3)[(1-\gamma_3)^2 - (\alpha_1-\beta_2)^2]\}^{1/3}} \end{aligned} \tag{14.99}$$

results from case Chap. 13#9 (Gaffet's case). Note that this singular potential is single-valued, in the real sense, on SO_3 and hence physical interpretation is possible in certain regions, excluding singularities at $\theta = 0, \pi$ and positions at which $\sin(\varphi - \psi) = 0$.

The equivalent of case Chap. 13#14 is characterized by the potential

$$V = b(\alpha_1 - \beta_2) + \frac{c + d\gamma_3}{1+\gamma_3} + (3-\gamma_3)[\frac{d_1}{\alpha_1 - \beta_2 - \gamma_3 + 1} + \frac{d_2}{\alpha_1 - \beta_2 + \gamma_3 - 1}]. \tag{14.100}$$

Case Chap. 13#16 generates the general integrable case:

$$\begin{aligned} l_3 &= \kappa + \frac{K}{2}[d(\alpha_1 - \beta_2) + c(\alpha_2 + \beta_1)], \\ V &= \frac{1}{2}[(k^*Kd - Jc)(\alpha_1 - \beta_2) + (k^*Kc + Jd)(\alpha_2 + \beta_1) + \frac{2\varepsilon}{1+\gamma_3} \\ &\quad + \frac{K^2}{8}[(c^2 - d^2)(\alpha_1 - \beta_2)^2 + 2cd(\alpha_1 - \beta_2)(\alpha_2 + \beta_1) + (c^2 + 2d^2)\gamma_3^2 + 2c^2\gamma_3], \end{aligned} \tag{14.101}$$

where k^* is a renaming of the arbitrary constant k. This case generalizes case 3 of Table 14.5 above and reduces to it when $K = 0$. For this case the complementary integral is not found yet. If one sets $\varepsilon = 0$, then V becomes a polynomial in the direction cosines.

Table 14.5 General integrable cases for which $\varphi + \psi$ is cyclic. For all cases, the cyclic integral is $I_2 = A(p\gamma_1 + q\gamma_2) + (Cr + l_3)(\gamma_3 + 1) = f$

1	Yehia [385] (1987) $A = B = C, \boldsymbol{l} = \boldsymbol{\mu} = 0$, $V = F(\lambda_1^2) = V(\alpha_1 - \beta_2 + \gamma_3)$, This case is super-integrable with three linear integrals: $I_2 = \boldsymbol{\omega} \cdot \boldsymbol{\gamma} + r$, $I_3 = \boldsymbol{\omega} \cdot \boldsymbol{\alpha} + p$, $I_4 = \boldsymbol{\omega} \cdot \boldsymbol{\beta} - q$.

14.5.5 Generalization Through Transformation

As in previous problems with cyclic coordinates, the transformation $(\dot{\psi}, \dot{\varphi}) \to (\dot{\psi} + \nu, \dot{\varphi} + \nu)$ changes (p, q, r) to $(p + \nu\gamma_1, q + \nu\gamma_2, r + \nu(\gamma_3 + 1))$ and renders the pair $(\boldsymbol{l}, V)$ to the new pair

$$\boldsymbol{l}' = \boldsymbol{l} + \nu(A\gamma_1, A\gamma_2, C(\gamma_3 + 1)),$$
$$V' = V(\theta, \psi - \varphi) + \nu(f - \boldsymbol{l} \cdot \boldsymbol{\gamma} - l_3) - \frac{\nu^2}{2}[A(\gamma_2^2 + \gamma_1^2) + C(\gamma_3 + 1)^2]. \tag{14.102}$$

The two systems described by the $(\boldsymbol{l}, V)$ and $(\boldsymbol{l}', V')$ are mathematically equivalent. From the physical point of view, the latter system involves several changes compared to the first. An interesting consequence of this equivalence is that any integrable case of (14.84) always generates a more general integrable case of (14.102) containing the additional function ν, which can be chosen according to the free parameters in the structure of the potential. Of special interest is the few parameter generalization introduced in Chap. 11 and applied above to cases of other types of symmetry. Listed below are the two generalized cases, for which the integrals of motion are explicitly known in terms of the Euler–Poisson variables. Other basic cases can be generalized in the same way.

The linear integral I_2 corresponds to the cyclic variable $\psi + \varphi$. The integral I_3, though became more complicated, is still of fourth degree.

The explicit solution of this case can be deduced as described in the above sections from that of the basic version $n = n_1 = n_2 = n_3 = 0$. Let the last solution be $\psi = \Psi(t), \theta = \Theta(t), \varphi = \Phi(t)$. The solution of the full generalized case can be readily written as $\psi(t) = \Psi(t) - \int \nu(t)dt, \theta(t) = \Theta(t), \varphi(t) = \Phi(t) - \int \nu(t)dt$, where $\nu(t) = n + (1 - \cos\Theta(t))(n_1 \cos(\Psi(t) - \Phi(t)) + n_2 \sin(\Psi(t) - \Phi(t)))$. Note that the palpable coordinates θ, $\psi - \varphi$ are not affected by the extra parameters n, n_1, n_2 and n_3.

The next case is a generalization of case 2 of Table 14.5.

2	Yehia [407] (2001) $n = n_1 = n_2 = 0$. Special case of [31] (1986).
	$A = B = C$, $\nu = n + n_1(\alpha_1 - \beta_2) + n_2\gamma_3$, $V = A\{a(\alpha_1 - \beta_2) + c\gamma_3 - (1 + \gamma_3)[n + n_1(\alpha_1 - \beta_2) + n_2\gamma_3]^2\}$ $\mu_1 = A[-n\gamma_1 + 2n_1(\beta_1\gamma_2 - \alpha_1\gamma_1) - n_2\gamma_1(1 + 2\gamma_3)]$, $\mu_2 = A[-n\gamma_2 + 2n_1(\beta_2\gamma_2 - \alpha_2\gamma_1) - n_2\gamma_1(1 + 2\gamma_3)]$, $\mu_3 = A[n(1 - \gamma_3) - 2n_1\gamma_3(\alpha_1 - \beta_2) + n_2(1 - \gamma_3)(1 + 2\gamma_3)]$,
	$I_2 = \boldsymbol{\omega} \cdot \boldsymbol{\gamma} + r + 2(1 + \gamma_3)[n + n_1(\alpha_1 - \beta_2) + n_2\gamma_3]$, $I_3 = (a - n_1 I_2)[(p + \nu\gamma_1)(\boldsymbol{\omega} \cdot \boldsymbol{\alpha} + \nu\alpha_3) - (q + \nu\gamma_2)(\boldsymbol{\omega} \cdot \boldsymbol{\beta} + \nu\beta_3)]$ $+(c - n_2 I_2)[r + \nu(\gamma_3 + 1)][\boldsymbol{\omega} \cdot \boldsymbol{\gamma} + \nu(1 + \gamma_3)]$ $-2(a - n_1 I_2)(c - n_2 I_2)(\alpha_1 - \beta_2) - 2(a - n_1 I_2)^2\gamma_3$.

Table 14.6 Generalized integrable cases

1	Yehia [411] (2003), Yehia [407] (2001). $n_3 = 0$.
	$A = B = 2C$, $\nu = n + n_1(\alpha_1 - \beta_2) + n_2(\alpha_2 + \beta_1) + \frac{n3}{1+\gamma_3}$, $\boldsymbol{l} = C(2\nu\gamma_1, 2\nu\gamma_2, \kappa + \nu(1+\gamma_3))$, $\mu_1/C = -n\gamma_1 + n_1(\alpha_2\gamma_2 - 2\alpha_1\gamma_1 + 3\beta_1\gamma_2) + n_2(\beta_2\gamma_2 - 2\beta_1\gamma_1 - 3\alpha_1\gamma_2)$, $\mu_2/C = -n\gamma_2 + n_1(-\beta_1\gamma_1 + 2\beta_2\gamma_2 - 3\alpha_2\gamma_1) + n_2(\alpha_1\gamma_1 - 2\alpha_2\gamma_2 - 3\beta_2\gamma_1)$, $\mu_3/C = \kappa - n_3 + n(1 - 3\gamma_3) + n_1[\beta_3\gamma_2 - \alpha_3\gamma_1 + 4\gamma_3(\beta_2 - \alpha_1)]$ $\quad -n_2[\alpha_3\gamma_2 + \beta_3\gamma_1 + 4\gamma_3(\alpha_2 + \beta_1)]$, $V = C\{a_1(\alpha_1 - \beta_2) + a_2(\alpha_2 + \beta_1) + \frac{a_3}{1+\gamma_3} - \frac{\nu^2}{2}[2(\gamma_2^2 + \gamma_1^2) + (\gamma_3 + 1)^2]$ $\quad -\kappa(1+\gamma_3)[n + n_1(\alpha_1 - \beta_2) + n_2(\alpha_2 + \beta_1)]\}$. $I_2 = 2[(p + \nu\gamma_1)\gamma_1 + (q + \nu\gamma_2)\gamma_2]$ $\quad +(1+\gamma_3)\{r + (\gamma_3 + 1)[n + n_1(\alpha_1 - \beta_2) + n_2(\alpha_2 + \beta_1)] + \kappa + n_3\}$ $I_3 = [(p + \nu\gamma_1)^2 - (q + \nu\gamma_2)^2 - (a_1 - n_1 I_2)(\alpha_1 + \beta_2) + (a_2 - n_2 I_2)(\alpha_2 - \beta_1)]^2$ $\quad +[2(p + \nu\gamma_1)(q + \nu\gamma_2) - (a_2 - n_2 I_2)(\alpha_1 + \beta_2) - (a_1 - n_1 I_2)(\alpha_2 - \beta_1)]^2$ $\quad +2\kappa[r - \kappa + \nu(1+\gamma_3)][(p + \nu\gamma_1)^2 + (q + \nu\gamma_2)^2]$ $\quad -4\kappa\{(p + \nu\gamma_1)[(a_1 - n_1 I_2)\alpha_3 + (a_2 - n_2 I_2)\beta_3]$ $\quad +(q + \nu\gamma_2)[(a_2 - n_2 I_2)\alpha_3 - (a_1 - n_1 I_2)\beta_3]\}$ $\quad +\frac{2(a_3 - n_3 I_2)}{1+\gamma_3}\{(p + \nu\gamma_1)^2 + (q + \nu\gamma_2)^2 + \kappa[r + \nu(1+\gamma_3) - \kappa] + \frac{(a_3 - n_3 I_2)}{2(1+\gamma_3)}\}$

This case contains the extra parameters n, n_1, n_2. Note that the integral I_3 is a polynomial of the third degree in the angular velocities. It reduces to a quadratic form when either $n_1 : n_2 :: a : c$ or $n_1 = n_2 = 0$. As the case $n = n_1 = n_2 = 0$ reduces to a special version of Bogoyavlensky's case, the general solution of the last full case can also be expressed in terms of the hyper-elliptic Theta functions (Table 14.6).

Appendix A
Some Useful Identities

(1) The following identity has proved very useful in reshaping expressions in several places of this book, especially, in Chap. 10, dealing with the regular precession transformation, and Chap. 11, dealing with the general precession transformation.

Let $\mathbf{a}$, $\mathbf{b}$ be two three-dimensional vectors, $\mathbf{K}$ arbitrary 3×3 matrix and $\boldsymbol{\delta}$ the unit matrix, then the following identity holds:

$$(\mathbf{a} \times \mathbf{b})\mathbf{K} + \mathbf{a}\mathbf{K}^T \times \mathbf{b} = \mathbf{a} \times (\mathbf{b}[tr(\mathbf{K})\delta - \mathbf{K^T}]). \tag{A.1}$$

The proof is straightforward.

The special case when $\mathbf{K}$ is a symmetric matrix, the above identity becomes

$$(\mathbf{a} \times \mathbf{b})\mathbf{K} + \mathbf{a}\mathbf{K} \times \mathbf{b} = \mathbf{a} \times (\mathbf{b}[tr(\mathbf{K})\delta - \mathbf{K}]). \tag{A.2}$$

(2) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three three-dimensional vectors, $\mathbf{A}$ nonsingular symmetric 3×3 matrix. Then

$$\mathbf{a} \cdot (\mathbf{bA} \times \mathbf{cA}) = \mathbf{a}(adj(\mathbf{A})) \cdot (\mathbf{b} \times \mathbf{c}) \tag{A.3}$$

where, for nonsingular $\mathbf{A}$, $adj(\mathbf{A}) \equiv det(\mathbf{A})\mathbf{A}^{-1}$.

The proof is straightforward, if one takes $\mathbf{A}$ in its diagonal form.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1

Appendix B
Kowalevski's Case: Appelrot's Four Classes of Simple Motions

A full account of the Appelrot classes and their degeneracies would require too much space, not available in the present context. In this appendix, we give an outline of those classes, based on the original work of Appelrot and on the review in [108]. Following Appelrot, to have less parameters in the solution, as $a = Mgx_0/C \neq 0$, one can choose units so that $a = 1$.

Conditions for each of those classes are given in the following table (F is defined in (4.59)).

The condition for class IV is somewhat complicated, and it may be useful for some purposes to express it in the parametric form

$$f = -2h\rho + 2\rho^3, K = 1 - 2h\rho^2 + 3\rho^4. \tag{B.1}$$

In the three-dimensional space $\{f, k, h\}$ or, more precisely, in the half $k \geq 0$ of that space, those conditions represent certain surfaces on which coalescence of roots of the polynomial Φ occurs. Those surfaces, called bifurcation surfaces, separate zones of space, which correspond to ultra-elliptic solutions of one and the same type and also the same qualitative character of motion.

M. P. Kharlamov studied the question for which values of the first integrals of motion, i.e. for which parameters $\{f, k, h\}$, those integrals become dependent in the sense that the rank of the Jacobian matrix

$$\frac{\partial(I_1, I_2, I_3, I_4)}{\partial(\boldsymbol{\omega}, \boldsymbol{\gamma})}$$

falls under 4, and then, according to the theory of integrable systems, determine singular (degenerate) Liouville tori. It turned out that those values define exactly the same surfaces listed in Table B.1, which correspond to the four Appelrot classes of simplest motions.

A quite detailed description of the integral manifolds of Kowalevski's case and their bifurcation are presented in [183]. As usual in such complicated dynamical problems, bifurcation diagrams for the present case are illustrated in the plane of

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1

Table B.1 Appelrot classes of simplest motions

Class	Conditions
I	$k = 0\ (e_4 = e_5)$
II	$f^2 - 2h - 2k = 0\ (F(e_5) = 0)$
III	$f^2 - 2h + 2k = 0\ (F(e_4) = 0), kf^2 \leq a^2$
IV	$[\frac{27}{2} f^2 - h(h^2 + 9 - 9k^2)]^2 - (h^2 - 3 + 3k^2)^3 = 0$, ($F(s)$ has a double root)

two parameters k, h for fixed values for areas parameter f. Those are sections of the bifurcation surfaces by selected planes $f = \text{const}$. Full presentation of those results is not possible in view of the scope of the present book.

The nature of the solution in those critical cases depends on the way in which roots of the polynomial Φ coalesce together. When one of the admissible intervals shrinks to a point at the double root r_1 (say), s_1 is a constant $s_1 = r_1$ at all times. The other variable s_2 varies on another admissible interval as an elliptic function of time. The term *particularly remarkable* motion was coined by Appelrot to characterize motions on which s_1 or s_2 keeps a constant value all the time, while the other one changes on an interval as an elliptic function of time or one of its degenerate forms. Appelrot classes also contain motions for which s_1 and s_2 are expressible in terms of elliptic functions or their degenerations. Those correspond to the coalescence of two roots bounding two different admissible intervals. Asymptotic motions usually appear in this class.

B.1 The First Class of Simplest Motions (Known as Delone's Case)

This case, characterized by the condition $k = 0$, was first investigated by Delone [60]. As it contains only "particularly remarkable" motions, the solution is expressed in terms of one variable, which is expressible as an elliptic function of time. The fourth integral of motion (4.57) becomes

$$(p^2 - q^2 - a\gamma_1)^2 + (2pq - a\gamma_2)^2 = 0. \tag{B.2}$$

As all the variables are real, this Eq. (B.2) holds only if

$$p^2 - q^2 - a\gamma_1 = 0, \tag{B.3}$$

$$2pq - a\gamma_2 = 0. \tag{B.4}$$

We should note that we have two integrals of the motion that do not contain arbitrary constants instead of the Kowalevski's case which is the general integrable case. The three classical integrals of motion are

$$2(p^2 + q^2) + r^2 + 2a\gamma_1 = 2h, \tag{B.5}$$

$$2(p\gamma_1 + q\gamma_2) + r\gamma_3 = f, \tag{B.6}$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{B.7}$$

Thus, we have five Eqs. (B.3)–(B.7) in six variables, and they can be solved to express five of those variables in terms of the remaining one. The two Eqs. (B.3)–(B.4) give

$$\gamma_1 = \frac{p^2 - q^2}{a}, \qquad \gamma_2 = \frac{2pq}{a}. \tag{B.8}$$

Inserting the two expressions (B.8) in the geometric integral (B.7), we obtain

$$\gamma_3 = \varepsilon\sqrt{1 - \frac{(p^2+q^2)^2}{a^2}}, \varepsilon = \pm 1. \tag{B.9}$$

Substituting in the energy integral, we get

$$r = \delta\sqrt{2(h - 2p^2)}, \delta = \pm 1. \tag{B.10}$$

Finally, Eq. (B.6) can be written as

$$r\gamma_3 = f - 2(p\gamma_1 + q\gamma_2).$$

Squaring both sides and using the above expressions, we get

$$2h(p^2+q^2)^2 - 4fap(p^2+q^2) + a^2(f^2 - 2h + 4p^2) = 0.$$

The last equation yields

$$q = \pm\sqrt{-p^2 + \frac{a}{h}[2fp \pm \sqrt{2(f^2-2h)(2p^2-h)}]}. \tag{B.11}$$

Thus, all the variables are expressed in terms of one variable p. Inserting the obtained results in Eq. (4.52), we arrive at the differential equation

$$2\dot{p} = \pm\sqrt{2h - 4p^2}\sqrt{-p^2 + \frac{a}{h}[2fp \pm \sqrt{2(f^2-2h)(2p^2-h)}]},$$

which, on separating the variables, gives the relation

$$t = t_0 \pm 2\int \frac{dp}{\sqrt{2h-4p^2}\sqrt{-p^2 + \frac{a}{h}[2fp \pm \sqrt{2(f^2-2h)(2p^2-h)}]}}. \tag{B.12}$$

Performing the transformation

$$p = -\sqrt{2h}\frac{x}{1+x^2} \tag{B.13}$$

reduces the relation (B.12) to

$$\frac{t-t_0}{2} = \pm\int \frac{dx}{\sqrt{a\sqrt{1-\frac{f^2}{2h}}(1-x^4) - 2hx^2 - \frac{2}{h}fax(1+x^2)}}. \tag{B.14}$$

The integral in the last expression is an elliptic integral and can be inverted giving x as an elliptic function of the time t.

B.1.1 A Special Case

Appelrot noted also the following type of motion which deserves special consideration.

Let the third body axis z at some moment of time t_0 occupy a vertical position, i.e. $\gamma_3 = \pm 1$. From the geometric integral, it follows that at the same moment $\gamma_1 = \gamma_2 = 0$ and, consequently, from (B.8), $p = q = 0$. This means that at $t = t_0$, the motion of the body is momentarily similar to that of a sleeping gyroscope with apex directed upwards or downwards. Let the angular velocity component r at that moment have the value r_0. From (4.54), we get

$$h = \frac{1}{2}r_0^2,\ f = \pm r_0, \tag{B.15}$$

and hence

$$t = 2\int \frac{dp}{\sqrt{r_0^2 - 4p^2}\sqrt{-p^2 \pm \frac{4a}{r_0^2}p]}}. \tag{B.16}$$

It can be shown using the above formulas that the subsequent motion is periodic, i.e. the body returns periodically to its initial state of motion with vertical z-axis position. When $r_0^2 < 8a$, the body periodically passes through the two opposite vertical positions, while for $r_0^2 > 8a$ the z-axis passes only through one of the two vertical positions and gets from it a maximal inclination at an angle $\sin^{-1}\frac{8a}{r_0^2}$.

The special case under conditions (B.15) enjoys another property of definite interest. Using (3.32) and (B.8)–(B.11), one can express the precession velocity in the form

$$\begin{aligned}\dot{\psi} &= \\ &= \frac{ap}{p^2+q^2} \\ &= \frac{r_0}{2}.\end{aligned} \tag{B.17}$$

The projection of the z-axis on the horizontal plane passing through the fixed point rotates about that point with uniform speed $\frac{r_0}{2}$.

B.1.2 A Case of Rational Solution

In [285] (1896), Mlodz'evsky constructed the following rare and interesting solution, in which all the six Euler–Poisson variables are rational functions. We present it in a slightly modified form to conform with our choice of the parameters in this book.

$$p = \frac{\varepsilon \left(u^4 + 6\,u^2 - 3\right)}{u^4 + 2\,u^2 + 9}, q = 8\frac{\varepsilon\sqrt{2}u}{u^4 + 2\,u^2 + 9}, r = -\frac{\varepsilon\sqrt{2}\left(u^4 - 6\,u^2 - 15\right)}{u^4 + 2\,u^2 + 9}, \tag{B.18}$$

$$\begin{aligned}\gamma_1 &= 1/3\,\frac{\sqrt{3}\left(u^8 + 12\,u^6 + 30\,u^4 - 164\,u^2 + 9\right)}{\left(u^4 + 2\,u^2 + 9\right)^2},\\ \gamma_2 &= 16/3\,\frac{\sqrt{3}\left(u^4 + 6\,u^2 - 3\right)\sqrt{2}u}{\left(u^4 + 2\,u^2 + 9\right)^2},\\ \gamma_3 &= -1/3\,\frac{\sqrt{3}\sqrt{2}\left(u^4 - 2\,u^2 - 11\right)}{u^4 + 2\,u^2 + 9},\end{aligned} \tag{B.19}$$

where $\varepsilon = \frac{1}{\sqrt[4]{3}}$, $u = \frac{t}{\sqrt[4]{3}}$. This solution is valid on the integral level $k = 0$, $h = \sqrt{3}$, $f = \frac{4}{3^{3/4}}$. It asymptotically approaches, as $t \to \pm\infty$, the uniform rotation

$$\begin{aligned}\boldsymbol{\omega} &= \varepsilon(1, 0, -\sqrt{2})\\ \boldsymbol{\gamma} &= (\frac{1}{\sqrt{3}}, 0, -\sqrt{\frac{2}{3}}),\end{aligned} \tag{B.20}$$

i.e. a permanent rotation with angular velocity $\varepsilon\sqrt{3}$ about the vertical unit vector $\boldsymbol{\gamma}$.

Figure B.1 shows the trajectory of the apex traversed by the upward vertical during the motion of the body on the Poisson sphere fixed in it in the direction PQR, P being

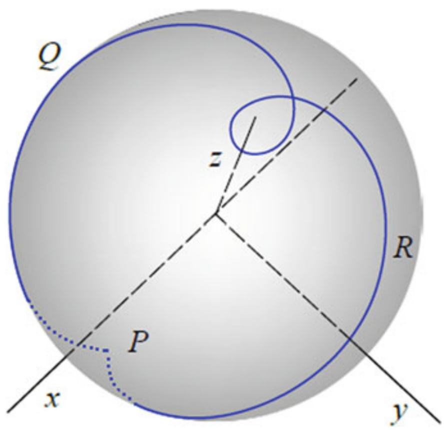

Fig. B.1 The path of the apex (Hidden line dotted) its position corresponding to uniform rotation (B.20). From the time-reversibility of motion in a potential field it follows that another motion exists, in which the apex traverses its path in the opposite sense PRQ. This happens on the integral level $k = 0, h = \sqrt{3}, f = -\frac{4}{3^{3/4}}$.

B.2 The Second Class of Simplest Motions

This class, under the condition $h = f^2/2 - k$, like the first one, consists of particularly remarkable solutions. The special case $f = 0$ is much simpler. It can be shown that $p = r = \gamma_2 = 0$. This characterizes the pendulum-like motion of the body around its y-axis which occupies a fixed horizontal position.

In the generic case $f \neq 0$, the solution can be written as

$$p = \frac{1}{f}(\mu \sin u - 1), q = \frac{\mu}{f} \cos u, r = 2\dot{u},$$
$$\gamma_1 = \frac{2}{f^2} \sin u(\mu - \sin u), \gamma_2 = \frac{2}{f^2} \cos u(\mu - \sin u), \gamma_3 = \frac{2}{f}\dot{u}, \quad \text{(B.21)}$$

where $\mu = \sqrt{1 + kf^2}$ and the auxiliary variable u is related to time by the relation

$$t = \pm \int \frac{du}{\sqrt{\frac{f^2}{4} - \frac{1}{f^2}(\mu - \sin u)^2}}. \quad \text{(B.22)}$$

Using the substitution $\tan \frac{u}{2} = x$, one can invert the last relation in terms of elliptic functions of time.

Exercise: Show that in all motions of Appelrot's second class the body precesses about the fixed point with a uniform rate $\dot{\psi} = \frac{f}{2}$.

B.3 The Third Class of Simplest Motions

For this class, under the condition

$$h = f^2/2 + k, \quad \text{(B.23)}$$

two types of solutions exist.

The Generic Solution

The solution of the equations of motion can be written in terms of two auxiliary variables, in the form:

$$
\begin{aligned}
p &= \sqrt{k}\sin u\cos v - \frac{1}{2} f\cos u,\\
q &= \sqrt{k}\sin u\sin v - \dot{u},\\
r &= -2\sqrt{k}\cos u\cos v - f\sin u,\\
\gamma_1 &= -2\sqrt{k}\dot{u}\sin u\sin v + \cos u(1 + f\sqrt{k}\sin u\cos v) - \cos^2 u(\frac{f^2}{2} + 2k\sin^2 v),\\
\gamma_2 &= (2\sqrt{k}\sin u\cos v - f\cos u)\dot{u} + \cos u\sin v(f\sqrt{k}\sin u + 2k\cos u\cos v),\\
\gamma_3 &= 2k\sin u\cos u - 2\sqrt{k}\cos u\sin v\dot{u} - \sin u - f\sqrt{k}\cos^2 u\cos v,
\end{aligned} \tag{B.24}
$$

where u satisfies the equation

$$\dot{u}^2 = \cos u[1 - (k + \frac{f^2}{4})\cos u], \tag{B.25}$$

and hence can be expressed as an elliptic function of time, and v is determined from

$$\dot{v} = \sqrt{k}\cos u\cos v + \frac{f}{2}\sin u. \tag{B.26}$$

The Particularly Remarkable Solution

It can be checked that under the condition (B.23), the equations of motion admit a solution identical with that given for the second class by formulas (B.21)–(B.22), but with the change of the sign of k, so that this time $\mu = \sqrt{1 - kf^2}$. This particularly remarkable solution is valid only under the condition $kf^2 < 1$.

B.4 The Fourth Class of Simplest Motions

Like in the third class, here we also have particularly remarkable motions and generic motions. One of the first turned out to be a special case of the periodic motion presented in Sect. 8.5 corresponding to Kowalevski's configuration $A = B = 2C$. The generic motion is partially studied in [9]. More detailed analysis of those motions was performed in [66] (see also [67]), by means of transforming the equations of motion to a new set of variables

$$
\begin{aligned}
p &= \rho - x/M,\\
q &= -y/M,\\
r &= 2z + 4x\gamma/M,\\
\gamma_1 &= 2\alpha - 4(x^2 - y^2)\gamma^2/M^2 + 2\rho x(2\rho x - 1)/M,\\
\gamma_2 &= 2\beta - 8xy\gamma^2/M^2 - 2\rho y(2\rho x - 1)/M,\\
\gamma_3 &= 2(2\rho x - 1)\gamma/M + 2\rho z.
\end{aligned} \tag{B.27}
$$

Using this transformation and the integrals of motion (4.54), (4.57), the equations of motion (4.52), (4.53) can be reduced to a somewhat simpler system whose solution can be written in the following form.

The variable z is determined from the relation

$$t = \int \frac{dz}{F(z)},\ \ F(z) = \frac{1}{4} - \rho^2 z^2 - [z^2 + \frac{1}{2}(\rho^2 - h)]^2, \tag{B.28}$$

and α, β by the expressions

$$\alpha = z^2 + \frac{1}{2}(\rho^2 - h), \beta = \sqrt{F(z)}, \tag{B.29}$$

provided ρ and h are determined from two conditions $P(\rho) = P'(\rho) = 0,\ P = -\rho^4 + 2h\rho^2 - 2f\rho + 1 - k^2$.

In the non-degenerate cases, the variable z oscillates between two real roots of $F(z)$, so that z, α, β are elliptic functions of time. The remaining variables are given by the expressions

$$\begin{aligned} y &= \eta\sqrt{\Phi(z)}, \\ x &= -[\frac{\rho}{2L_2} + \frac{\beta\eta}{\sqrt{\Phi(z)}} + \frac{z}{L_2}\sqrt{\Phi(z)}\dot{\eta}], \\ \gamma &= \frac{\rho z}{2L_2} + \frac{1}{L_2}(\rho^2 + \alpha)\sqrt{\Phi(z)}\dot{\eta}, \end{aligned} \tag{B.30}$$

where $\Phi(z) = z^2 - L_2$, η is an auxiliary variable satisfying the relation

$$\int \frac{d\eta}{\sqrt{L_1 - L_2 L_3 \eta^2}} + \int \frac{dt}{\Phi(z(t))} = \text{const}, \tag{B.31}$$

and

$$\begin{aligned} L_1 &= \frac{1}{16}(2h - 2\rho^2), \\ L_2 &= \frac{1}{4}(2h - 6\rho^2), \\ L_3 &= 4\rho^2 L_1 - \frac{1}{4}. \end{aligned} \tag{B.32}$$

Classification of motions corresponding to the constructed solution according to the values of parameters $L_{1,2,3}$ was performed in [66] (see also [67]), into eight cases

$$\begin{aligned} (I) L_1 > 0, L_2 > 0, L_3 > 0, \\ (II) L_1 > 0, L_2 < 0, L_3 < 0, \\ (III) L_1 > 0, L_2 < 0, L_3 > 0, \end{aligned}$$

$$
\begin{aligned}
(IV) L_1 > 0, L_2 > 0, L_3 < 0, \\
(V) L_1 < 0, \\
(VI) L_1 = 0, \\
(VII) L_2 = 0, \\
(VIII) L_3 = 0.
\end{aligned} \tag{B.33}
$$

A simplified particular version of this solution is also presented in [41].

B.5 Intersection of the Four Classes

It can be shown that under the conditions

$$k = 0, h = 2, f = 2, \tag{B.34}$$

common to the four classes, the polynomial in (4.59) three equal roots = 2 and two roots = 1.

$$\Phi(s) = (1 - s)^2(2 - s)^3. \tag{B.35}$$

But it is simpler, since this is a particular case of Delone's solution, to use formulas of Sect. B.1 to obtain for p the equation

$$\dot{p}^2 = p(1 - p)^2(1 + p), \tag{B.36}$$

so that we can write the generic solution of the equations of motion in the form

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{((e^{\sqrt{2}t} - 1)^2, \pm 2\sqrt{2}(e^{\sqrt{2}t} - 1)e^{t/\sqrt{2}}, \pm 8\,(e^{\sqrt{2}t} + 1)e^{t/\sqrt{2}})}{e^{2\sqrt{2}t} + 6\,e^{\sqrt{2}t} + 1}, \\
\boldsymbol{\gamma} &= (\frac{(e^{\sqrt{2}t} - 1)^2(e^{2\sqrt{2}t} - 10\,e^{\sqrt{2}t} + 1)}{(e^{2\sqrt{2}t} + 6\,e^{\sqrt{2}t} + 1)^2}, \pm 4\,\frac{\sqrt{2}(e^{\sqrt{2}t} - 1)^3 e^{t/\sqrt{2}}}{(e^{2\sqrt{2}t} + 6\,e^{\sqrt{2}t} + 1)^2}, \\
&\quad \pm 4\,\frac{(e^{\sqrt{2}t} + 1)e^{t/\sqrt{2}}}{e^{2\sqrt{2}t} + 6\,e^{\sqrt{2}t} + 1}).
\end{aligned} \tag{B.37}
$$

Here, we note that q, γ_2 are odd functions in t and the other four quantities are even. This solution as $t \to \pm\infty$ asymptotically approaches the solution $\boldsymbol{\omega} = (1, 0, 0)$, $\boldsymbol{\gamma}=(1, 0, 0)$, which corresponds to the solution $p = 1$ of Eq. (B.36). That is a uniform rotation about the first axis (bearing the centre of mass of the body), while this axis preserves vertical upward position.

Figure B.2 depicts the path $PQRSP$ of the apex of $\boldsymbol{\gamma}$, corresponding to the positive sign in (B.37), on the Poisson sphere. It begins at $t = -\infty$ at P and ends also at P at $t = \infty$. At $t = 0$ the z-axis passes through the vertical position. The trace

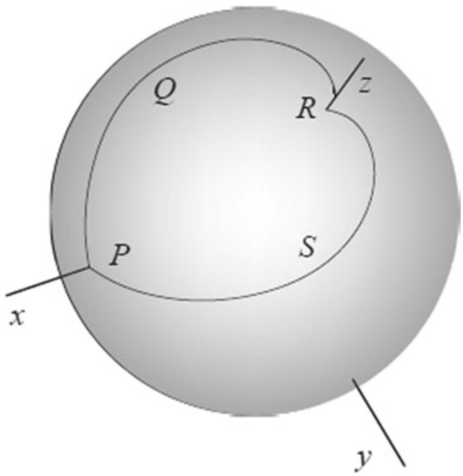

Fig. B.2 The path traced by the apex of γ on the Poisson sphere

of the apex for the second solution, corresponding to the negative sign, is a mirror reflection of the first in the xy-plane.

Note that, due to the invariance of the equations of motion under the change $(\boldsymbol{\omega}, \boldsymbol{\gamma}, t) \rightarrow (-\boldsymbol{\omega}, \boldsymbol{\gamma}, -t)$, two other solutions exist for the parameter values

$$k = 0, h = 2, f = -2.$$

In one of them, the apex traces the same curve in the retrograde sense $PSRQP$ with reversed angular velocity. The other is interpreted similarly.

Appendix C
Particularly Simple Classes of Motions in Goryachev–Chaplygin's Case

Just as in the case of Kowalevski, the analytical solution of the equations of motion in the generic Goryachev–Chaplygin case does not offer any help in imagining or simulating the motion of the body. This situation gives high importance to the study of the extremal cases of motion, when the six-degree polynomials in (4.77) have repeated roots. For simplicity and without loss of generality, we set $C = 1, a = 1$. The discriminant equation is the same for both polynomials

$$G^2[2(E-1)^3 - 27G^2][2(E+1)^3 - 27G^2] = 0. \tag{C.1}$$

Its solution (the bifurcation diagram) can be represented in the plane of parameters (G, E) by the following three parametric equations:

$$\begin{array}{ll} \text{I} & G = 0, \quad E > -1, \\ \text{II} & G = \frac{1}{2}\sigma^3,\ E = 1 + \frac{3}{2}\sigma^2, \quad \sigma \in (-\infty, \infty), \\ \text{III} & G = \frac{1}{2}\sigma^3,\ E = -1 + \frac{3}{2}\sigma^2,\ \sigma \in (-\infty, \infty). \end{array}$$

Motions corresponding to those three branches constitute for the case of Goryachev–Chaplygin the analog of Appelrot classes for Kowalevski's case. The bifurcation diagram is displayed graphically in Fig. C.1.

The (G, E) plane is divided by the three branches into 5 open regions. The motion is impossible in the region Ω_0 under curve III. Each of the symmetric regions Ω_1, Ω_2 is composed of two components separated by line I. In each of the two regions, the number of real roots of the under-root polynomials do not change and, consequently, the qualitative character of the motion does not change. The bifurcation diagram was discussed in [183] and in a slightly modified way in [166].

We shall write the solution on the two boundaries II and III of the regions of hyper-elliptic solutions. On those boundaries the solution is expressible in elliptic functions of time. As usual on those boundaries, it is simpler to ignore the Eqs. (4.77) and try to find a solution directly from the equations of motion (4.64) under relevant conditions on the parameters of the system.

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1

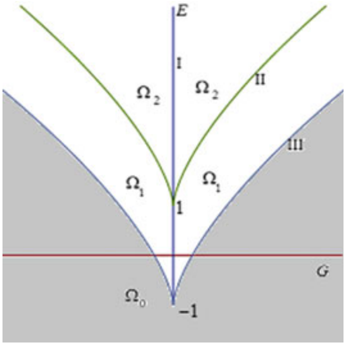

Fig. C.1 Regions of possible motion Ω_1 and Ω_2, each composed of two components separated by the line I

C.1 Solution on the Boundary I (The Case of Goryachev)

C.1.0.1 The Subcase of Goryachev

This is the special version of this case that was first found by Goryachev [115] (1900) and immediately generalized to its full just presented form by Chaplygin [52] (1901). This version is characterized by the pair of invariant relations

$$p^2 + q^2 - bp^{2/3} = 0, \sqrt[3]{p}\gamma_3 - br = 0, \tag{C.2}$$

where b is an arbitrary parameter. In fact, this parameter is an integration constant that can be obtained by considering the following in virtue of (4.64)

$$\frac{d}{dt}(\frac{p^2 + q^2}{p^{2/3}}) = -\frac{q}{2p^{5/3}}[r(p^2 + q^2) - p\gamma_3]. \tag{C.3}$$

Comparing the right-hand side with (4.65), we conclude that the quantity between brackets becomes an integral of motion

$$\frac{p^2 + q^2}{p^{2/3}} = b$$

provided

$$r(p^2 + q^2) - p\gamma_3 = G = 0, \tag{C.4}$$

and from the last two relations the second invariant relation in (C.2) follows. Thus, the case $G = 0$ is exactly Goryachev's version of the Goryachev–Chaplygin case.

C.1.1 The Solution

Full analytical solution for the subcase of Goryachev was first obtained in Chaplygin's work [52] using the above separation variables (4.77). The solution expressed in terms of Weierstrass' elliptic functions is not so simple to reveal properties of the motion. Somewhat simpler reduction to quadratures was derived by Dokshevich [67]. A more transparent and finalized explicit solution was found by Gashenenko [107]. This will be written here, divided into three cases, according to the values of the energy parameter E.

C.1.1.1 When $E \in [-1, 1)$

We have

$$
\begin{aligned}
p &= 2b^{3/4}\sqrt{2s_1^3}\frac{\mathrm{cn}^3\tau}{(1+s_1^2\mathrm{cn}^4\tau)^{3/2}},\\
q &= b^{3/4}\sqrt{2s_1}\frac{\mathrm{cn}\tau(1-s_1^2\mathrm{cn}^4\tau)}{(1+s_1^2\mathrm{cn}^4\tau)^{3/2}},\\
r &= -\frac{4s_1\mathrm{sn}\tau\mathrm{cn}\tau\mathrm{dn}\tau}{1+s_1^2\mathrm{cn}^4\tau},\\
\gamma_1 &= \frac{1}{(1+s_1^2\mathrm{cn}^4\tau)^2}[E(1+s_1^4\mathrm{cn}^8\tau-6s_1^2\mathrm{cn}^4\tau)-4s_1\mathrm{cn}^2\tau(\sqrt{1+4b^3-E^2}-b^{3/2})\\
&\qquad +4s_1^3\mathrm{cn}^6\tau(\sqrt{1+4b^3-E^2}+b^{3/2})],\\
\gamma_2 &= \frac{-1}{(1+s_1^2\mathrm{cn}^4\tau)^2}[\sqrt{1+4b^3-E^2}(1+s_1^4\mathrm{cn}^8\tau-6s_1^2\mathrm{cn}^4\tau)\\
&\qquad +4Es_1\mathrm{cn}^2\tau(1-s_1^2\mathrm{cn}^4\tau)-2b^{3/2}(1-s_1^4\mathrm{cn}^8\tau)],\\
\gamma_3 &= -2b^{3/4}\sqrt{2s_1}\frac{\mathrm{sn}\tau\mathrm{dn}\tau}{(1+s_1^2\mathrm{cn}^4\tau)^{1/2}},
\end{aligned}
\tag{C.5}
$$

where

$$
s_1 = \frac{E+1}{\sqrt{1+4b^3-E^2}+2b^{3/2}}, \tau = \frac{1}{2}(t-t_0) \tag{C.6}
$$

and the modulus of the elliptic functions

$$
\nu = \sqrt{\frac{E+1}{2}}. \tag{C.7}
$$

The last does not depend on the parameter b and hence the period of the solution is independent of b. Note that when the energy parameter takes its lowest value $E = -1$, then the solution renders to the lower equilibrium position:

$$p = q = r = 0, \gamma = (-1, 0, 0). \tag{C.8}$$

C.1.1.2 When $E > 1$

The above formulas become

$$\begin{aligned}
p &= 2b^{3/4}\sqrt{2s_1^3}\frac{\mathrm{dn}^3\tau}{(1+s_1^2\mathrm{dn}^4\tau)^{3/2}},\\
q &= b^{3/4}\sqrt{2s_1}\frac{\mathrm{dn}\tau(1-s_1^2\mathrm{dn}^4\tau)}{(1+s_1^2\mathrm{dn}^4\tau)^{3/2}},\\
r &= -\frac{4s_1\mathrm{sn}\tau\mathrm{cn}\tau\mathrm{dn}\tau}{1+s_1^2\mathrm{dn}^4\tau},\\
\gamma_1 &= \frac{1}{(1+s_1^2\mathrm{dn}^4\tau)^2}[E(1+s_1^4\mathrm{dn}^8\tau-6s_1^2\mathrm{dn}^4\tau)-4s_1\mathrm{dn}^2\tau(\sqrt{1+4b^3-E^2}-b^{3/2})\\
&\qquad +4s_1^3\mathrm{dn}^6\tau(\sqrt{1+4b^3-E^2}+b^{3/2})],\\
\gamma_2 &= \frac{-1}{(1+s_1^2\mathrm{dn}^4\tau)^2}[\sqrt{1+4b^3-E^2}(1+s_1^4\mathrm{dn}^8\tau-6s_1^2\mathrm{dn}^4\tau)\\
&\qquad +4Es_1\mathrm{dn}^2\tau(1-s_1^2\mathrm{dn}^4\tau)-2b^{3/2}(1-s_1^4\mathrm{dn}^8\tau)],\\
\gamma_3 &= -2b^{3/4}\sqrt{2s_1}\frac{\mathrm{sn}\tau\mathrm{cn}\tau}{(1+s_1^2\mathrm{dn}^4\tau)^{1/2}},
\end{aligned} \tag{C.9}$$

where s_1 is the same as above, the modulus of the elliptic functions $k = \sqrt{\frac{2}{E+1}} = 1/\nu$ and $\tau = \frac{1}{2k}(t - t_0)$.

We note that the two classes of motion are periodic and are spanning between two types of pendulum-like motions of the body, which correspond to the values $b = 0, b = \infty$. Those are motions in which the centre of mass describes a pendulum motion in the xy(or xz)-plane that takes a vertical position. For pendulum-like motions in general, see Chap. 8, Sect. 8.2.

When $E = 1$: For this class of motions $s_1 = \frac{1}{2b^{3/2}}$, and the solution can be obtained as a limiting case from either of (C.5) or (C.9) as $\nu = k = 1$. Elliptic functions are replaced by hyperbolic functions. It can be also verified that the solution is asymptotic as $t \to \pm\infty$ to the upper equilibrium position characterized by

$$p = q = r = 0, \gamma = (1, 0, 0). \tag{C.10}$$

C.2 Solution on Boundary II

For this solution, expressions (C.13) and Eq. (C.14) are replaced by [99]:

$$
\begin{aligned}
p &= \frac{1}{4}(\sigma + r)\sqrt{\sigma(r-\sigma)}, \\
q &= \frac{1}{4}\sqrt{-\sigma R_1(r)}, \\
\gamma_1 &= 1 - \frac{1}{2}(r-\sigma)(2\sigma + r), \\
\gamma_2 &= -\frac{1}{2}\sqrt{(\sigma - r)R_1(r)}, \\
\gamma_3 &= \sqrt{\sigma(r-\sigma)},
\end{aligned} \tag{C.11}
$$

$$
\dot{r} = \frac{1}{2}\sqrt{(\sigma - r)R_1(r)},\ R_1(r) = -4(\sigma + r) + (2\sigma + r)^2(r - \sigma). \tag{C.12}
$$

Note that the equations and the integrals of motion in the Goryachev–Chaplygin case are invariant under the change of signs of the part of variables p, q, γ_3 and hence another solution is obtained from (C.11) by changing those signs.

The cubic R_1 has the discriminant

$$
\Delta = -4[\sigma^4 + \frac{4}{3}\sigma^2 + \frac{16}{27}] < 0
$$

and thus it always has three real roots. It is sufficient to consider the case $\sigma > 0$. One can also notice that $R_1(\infty) > 0$, $R_1(\sigma) < 0$, $R_1(0) < 0$, $R_1(-2\sigma) > 0$, so that R_1 has one positive root r^* (say) $(r^* > \sigma)$ and two negative ones. To have all physical quantities real, the variable r changes on the interval $[\sigma, r^*]$, where it is expressible as an elliptic function of time. In the limit as $s \to 0$, this interval shrinks to zero, $E = 1$ and the solution (C.11) becomes

$$
\boldsymbol{\omega} = \mathbf{0}, \boldsymbol{\gamma} = (1, 0, 0),
$$

which represents the upper equilibrium position. That is the intersection of boundaries I and II.

C.3 Solution on Boundary III

Five of the Euler–Poisson variables are expressed in terms of r [124]:

$$
\begin{aligned}
p &= \frac{1}{4}(2\sigma + r)\sqrt{\sigma(\sigma - r)}, \\
q &= -\frac{1}{4}\sqrt{\sigma R(r)}, \\
\gamma_1 &= -1 + \frac{1}{2}(\sigma - r)(2\sigma + r),
\end{aligned}
$$

$$\gamma_2 = -\frac{1}{2}\sqrt{(\sigma - r)R(r)},$$
$$\gamma_3 = -\sqrt{\sigma(\sigma - r)}, \tag{C.13}$$

while r is determined, as an elliptic function of time, from the differential equation

$$\dot{r} = \frac{1}{2}\sqrt{(\sigma - r)R(r)},\ R(r) = 4(\sigma + r) - (2\sigma + r)^2(\sigma - r). \tag{C.14}$$

Another solution can be obtained by changing the signs of p, q and γ_3. The cubic polynomial R has the discriminant

$$\Delta = 4[(\sigma^2 - \frac{2}{3})^2 + \frac{4}{27}] > 0,$$

so that it has only one real root. Let us denote this root by r_1. We have 3 cases:

(1) When $\sigma > 0$ then $r_1 < \sigma$ and r varies on the interval $[r_1, \sigma]$. r is elliptic function with period $T = 4\int_{r_1}^{\sigma} \frac{dr}{\sqrt{(\sigma-r)R(r)}}$.

(2) When $\sigma < 0$ then $r_1 > \sigma$ and r varies on the interval $[\sigma, r_1]$. r is elliptic function with period $T = 4\int_{\sigma}^{r_1} \frac{dr}{\sqrt{(\sigma-r)R(r)}}$.

(3) When $\sigma = 0$ then $r_1 = 0$ and (C.14) takes the form

$$\dot{r} = \frac{1}{2}\sqrt{(-r^2)(4 + r^2)}. \tag{C.15}$$

This has only the solution $r = 0$, which leads only to the solution $\boldsymbol{\omega} = (0, 0, 0)$, $\boldsymbol{\gamma} = (-1, 0, 0)$, which characterizes the lower equilibrium position of the body.

Appendix D
Gyrostatic Generalization of the Appelrot Classes

The question arises, whether it is possible to generalize the four Appelrot classes of motion, which we have described in Sect. 3.3, to construct analogs of them involving non-zero gyrostatic momentum, which admits simpler solutions than the generic case. This was investigated by M. Kharlamov. In view of the absence of variable separation for Yehia's case, Kharlamov searched for the critical set of parameters in the space of parameters of the problem $\{h, f, k, \kappa\}$, on which the four integrals of motion become functionally dependent. It turned out that all solutions corresponding to the critical set were already isolated and shown to be expressible in terms of elliptic functions long before the complementary integral I_4 (in case 3 of Table 5.1 of Chap. 5) was discovered. This came as a result of the use of the method of invariant relations as we shall see below.

(1) **The first class** of Appelrot motions in Kowalevski's case (case of Delone) completely disappears in the gyrostatic analog. The integral $I_4^{(K)} = (p^2 - q^2 - a_1\gamma_1 + a_2\gamma_2)^2 + (2pq - a_1\gamma_2 - a_2\gamma_1)^2 = k^2$ is a sum of two squared terms and when $k = 0$ each of those terms is zero. The integral of motion is replaced by two invariant relations that determine Delone's case. In the presence of the gyrostatic momentum κ, the integral

$$I_4 = I_4^{(K)} + 2\kappa\,(r - \kappa)\,(p^2 + q^2) - 4\kappa\gamma_3(a_1 p + a_2 q)$$

contains terms of different structure and hence loses that property.

(2) **The second and third classes:**

The condition

$$\begin{aligned}
&K^2[\left(f^2 - 2\,h\right)^2 - 4\,K] = \\
&= \kappa^2\{4\kappa^6\,(1 - h^2) - 4\kappa^4\,\left(-2\,f^2h^2 - 4\,h^3 + Kh + 3\,f^2 + 3\,h\right) \\
&\quad +\kappa^2[-16\,h^4 - 16\,f^2h^3 + \left(4\,f^4 - 32\,K + 24\right)h^2 - 8\,f^2\,(K - 3/2)\,h \\
&\quad\quad -12\,f^4 + K^2 + 18\,K - 27]
\end{aligned}$$

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1

$$+ (-16\,K + 32)\,h^3 + 16\,f^2\,(K-3)\,h^2 + \left(-4\,f^4 K + 24\,f^4 + 20\,K^2 - 36\,K\right) h$$
$$+2\,f^2 \left(-2\,f^4 + K^2 + 9\,K\right)\} \tag{D.1}$$

generalizes cases II and III in Table 4.3 of Chap. 4 for $\kappa \neq 0$ and unifies them in one case. A suitable parametrization of this condition is

$$\begin{aligned} f^2 &= 2\,h + \kappa^2 - 4\,\kappa^2 s^2 - s^{-1}, \\ K &= -2\;\kappa^2 \left(h - 1/2\,\kappa^2 - 2\,s\right) - \kappa^4 + 1/4\,s^{-2}. \end{aligned} \tag{D.2}$$

The corresponding solution of the equations of motion was described in [198]. It will be written here in the form given in [181] and also in [184].

Let us fix a value f for the areas integral and let $s \neq 0$ be a constant. The equations of motion (5.19) admit a solution

$$\begin{aligned} p &= -\frac{f}{2s} - \varkappa\rho Y, \\ q &= -\rho\sqrt{s}R, \\ r &= \kappa + 2\varkappa X, \\ \gamma_1 &= \frac{2\kappa s X + f\rho Y}{2\varkappa} - 2\varkappa^2 Y^2, \\ \gamma_2 &= -2\varkappa Y\sqrt{s}R, \\ \gamma_3 &= \frac{fX - 2\kappa s\rho Y}{2\varkappa}, \end{aligned} \tag{D.3}$$

in which

$$\begin{aligned} \varkappa &= \frac{f^2}{4} + s^2\kappa^2,\, \rho^2 = 1 - \frac{2\varkappa^2}{s}, \\ (X, Y) &= \{ \begin{matrix} (\cos\sigma, \sin\sigma), & \rho^2 \geqslant 0 \\ (\cosh\sigma, i\sinh\sigma), & \rho^2 < 0 \end{matrix} \\ R^2 &= \frac{1}{2}[(X + \frac{\kappa}{\varkappa})^2 + (\rho Y + \frac{f}{2s\varkappa})^2 - 1], \end{aligned} \tag{D.4}$$

and the relation with time is determined from the equation

$$\dot{\sigma}^2 = \mathrm{sgn}(\rho^2) s R^2, \tag{D.5}$$

which can be solved in elliptic functions of time.

(3) **The fourth class:**

The condition on the parameters h, f, K for this class can be written in the following form, which reduces to (B.1) when $\kappa = 0$,

$$\begin{aligned}&\frac{2}{27}\{[\frac{27f^2}{2}-h\left(h^2-9K+9\right)]^2-\left(h^2+3K-3\right)^3\}\\&+\kappa^2[+\frac{1}{4}f^2\left(12h^2-6h\kappa^2+\kappa^4\right)(K-1)^2\left(\kappa^2-4h\right)\\&+(K-1)\left(-1/8\,\kappa^6+h\kappa^4-3h^2\kappa^2+4h^3-9f^2\right)]\\=&\ 0.\end{aligned} \tag{D.6}$$

It is more convenient to use parametrization

$$\begin{aligned}f^2&=s^2\left(2h-\kappa^2-2s\right),\\K&=1+\left(h-1/2\,\kappa^2\right)^2-4s\left(h-1/2\,\kappa^2\right)+3s^2.\end{aligned} \tag{D.7}$$

The particularly remarkable motion. The particular solution with two linear relations (see Chap. 8, Sect. 8.15.1.4) in the problem of motion of a heavy gyrostat generalizes the solution due to Bobylev and Steklov (Sect. 8.5 this chapter). If one sets $A=B=2C$ and $\kappa_1=\kappa_2=y_0=0$, then one is left with a subcase of the Kowalevski–Yehia case. The solution of the equations of motion in this case can be written in the form (see e.g. [184]):

$$\begin{aligned}p&=p_0,q=0,\\\gamma_1&=h-p_0^2-\frac{1}{2}r^2,\\\gamma_2&=\sqrt{R(r)},\\\gamma_3&=p_0(\kappa-r),\end{aligned} \tag{D.8}$$

where r is obtained by inverting the elliptic quadrature

$$\begin{aligned}t&=\int\frac{dr}{\sqrt{R(r)}},\\R(r)&=-\frac{1}{4}r^4+(h-2p_0^2)r^2+2\kappa p_0^2r+1-(h-p_0^2)^2-\kappa^2p_0^2,\end{aligned} \tag{D.9}$$

and we have, for simplicity, set units of measurement so that $C=1$ and $a_1=1$. The solution can thus be expressed in terms of elliptic functions of time. This gives the gyrostatic generalization of the fourth class of particularly remarkable motions of Appelrot.

The Non-degenerate Solutions

Modifying the transformation of variables (B.27) introduced for Kowalevski's case by Dokshevich in [66], Gashenenko [101] used the transformation

$$\begin{aligned}p&=\rho-x/M,\\q&=-y/M,\end{aligned}$$

$$\begin{aligned}
r &= 2z + \kappa + 4x\gamma/M,\\
\gamma_1 &= 2\alpha - 4(x^2 - y^2)\gamma^2/M^2 + 2\rho x(2\rho x - 1)/M - 4\kappa\gamma x/M,\\
\gamma_2 &= 2\beta - 8xy\gamma^2/M^2 - 2\rho y(2\rho x - 1)/M - 4\kappa\gamma y/M,\\
\gamma_3 &= 2(2\rho x - 1)\gamma/M + 2\rho z,
\end{aligned} \tag{D.10}$$

where ρ and other constants of motion are subject to two conditions $P(\rho) = P'(\rho) = 0$, $P = -\rho^4 + (2h - \kappa^2)\rho^2 - 2f\rho + 1 - K$.

Using this transformation, (D.7) and the integrals (5.20), the equations of motion (5.19) can be reduced to a somewhat simpler system whose solution can be written in the following form.

The variable z is determined from the relation

$$t = \int \frac{dz}{F(z)},\ F(z) = \frac{1}{4} - \rho^2 z^2 - [(z - \frac{\kappa}{2})^2 + \frac{1}{2}(\rho^2 - h)]^2, \tag{D.11}$$

and α, β by the expressions

$$\begin{aligned}
\alpha &= (z - \frac{\kappa}{2})^2 + \frac{1}{2}(\rho^2 - h),\\
\beta &= \sqrt{F(z)}.
\end{aligned} \tag{D.12}$$

In the non-degenerate cases, the variable z oscillates between two real roots of $F(z)$, so that z, α, β are elliptic functions of time. The remaining variables are given by the expressions

$$\begin{aligned}
y &= \eta\sqrt{\Phi(z)},\\
x &= -[\frac{\rho}{2L_2} + \frac{\beta\eta}{\sqrt{\Phi(z)}} - \frac{1}{L_2}(z - \kappa)\sqrt{\Phi(z)}\dot{\eta}],\\
\gamma &= \frac{\rho z}{2L_2} + \frac{\kappa\beta\eta}{\sqrt{\Phi(z)}} + \frac{1}{L_2}(\rho^2 + \alpha)\sqrt{\Phi(z)}\dot{\eta},
\end{aligned} \tag{D.13}$$

where $\Phi(z) = (z - \kappa)^2 - L_2$, η is an auxiliary variable satisfying the relation

$$\int \frac{d\eta}{\sqrt{L_1 - L_2 L_3 \eta^2}} + \int \frac{dt}{\Phi(z(t))} = \text{const}, \tag{D.14}$$

and

$$\begin{aligned}
L_1 &= \frac{1}{16}(2h - 2\rho^2 - \kappa^2),\\
L_2 &= \frac{1}{4}(2h - 6\rho^2 - \kappa^2),\\
L_3 &= 4(\rho^2 + \kappa^2)L_1 - \frac{1}{4}.
\end{aligned} \tag{D.15}$$

Classification of motions corresponding to the constructed solution according to the values of parameters $L_{1,2,3}$ was performed in [100, 101] into eight cases

$$
\begin{aligned}
(I) L_1 &> 0, L_2 > 0, L_3 > 0, \\
(II) L_1 &> 0, L_2 < 0, L_3 < 0, \\
(III) L_1 &> 0, L_2 < 0, L_3 > 0, \\
(IV) L_1 &> 0, L_2 > 0, L_3 < 0, \\
(V) L_1 &< 0, \\
(VI) L_1 &= 0, \\
(VII) L_2 &= 0, \\
(VIII) L_3 &= 0.
\end{aligned} \tag{D.16}
$$

It is noted there that in cases I, II doubly periodic are possible, in cases III, IV, VII, $VIII$ motions asymptotically tend to those described in (D.8)–(D.9), while in cases V and VI motions different from (D.8)–(D.9) are impossible. The case of a double root of $F(z)$ is also considered. The effect of the presence of the gyrostatic momentum on those motions was not considered.

Appendix E
The Conditional Case of Sretensky

Separation of variables in Sretensky's case is obtained by inserting the expressions (5.21) into the quadratures (4.77). The discriminant equation of the resulting polynomial leads to three classes of critical sets [172]

$$\begin{aligned} \text{I)} \quad & G = 0, \\ \text{II)} \quad & \Delta = 27[G + \frac{2}{3}\kappa(E-1) - (\frac{\kappa}{3})^3]^2 - 8(E - 1 + \frac{\kappa^2}{6})^3 = 0, \\ \text{III)} \quad & \Delta_* = 27[G + \frac{2}{3}\kappa(E+1) - (\frac{\kappa}{3})^3]^2 - 8(E + 1 + \frac{\kappa^2}{6})^3 = 0. \end{aligned} \tag{E.1}$$

Obviously, those generalize the three classes I, II, III given by (C.1) in Goryachev–Chaplygin's case and reduce to them when $\kappa = 0$.

Class I ($G = 0$) generalizes Goryachev's case whose full solution was given in Chap. 4, Sect. 4.1. M. Kharlamov has shown that when $\kappa \neq 0$, the analog of Goryachev's integral doesn't exist and hence the method of solution of Goryachev's case cannot be generalized. As far as we know, the explicit solution of class I was not considered.

Classes II and III ($\Delta = 0$, $\Delta_* = 0$) were considered in [172], but full solution is given only for class III. For this class, the parametrization

$$E = -1 + \frac{3}{2}\sigma^2 + 2\sigma\kappa + \frac{1}{2}\kappa^2, G = \frac{1}{2}\sigma^2(1+\kappa)$$

was used and formulas (C.13) and (C.14) are then replaced by

$$\begin{aligned} p &= \frac{1}{4}(2\sigma + \kappa + r)\sqrt{\sigma(\sigma + \kappa - r)}, \\ q &= -\frac{1}{4}\sqrt{\sigma R(r)}, \end{aligned}$$

H. M. Yehia, *Rigid Body Dynamics*, Advances in Mechanics and Mathematics 45,
https://doi.org/10.1007/978-3-030-96336-1

$$\begin{aligned}\gamma_1 &= -1 + \frac{1}{2}(\sigma + \kappa - r)(2\sigma + \kappa + r),\\ \gamma_2 &= -\frac{1}{2}\sqrt{(\sigma + \kappa - r)R(r)},\\ \gamma_3 &= -\sqrt{\sigma(\sigma + \kappa - r)},\end{aligned} \tag{E.2}$$

and

$$\dot{r} = \frac{1}{2}\sqrt{(\sigma + \kappa - r)R(r)}, \tag{E.3}$$
$$R(r) = r^3 + (\kappa + 3\sigma)r^2 + (4 - 2\sigma\kappa - \kappa^2)r + (\sigma + \kappa)[4 - (2\sigma + \kappa)^2].$$

Note that another solution is valid that can be obtained from (E.2) by changing the signs of p, q, γ_3.

Some details of this solution are discussed in [172]. We point out here only two rare and quite interesting versions of the solution when all the variables are algebraic functions of time. They occur when $R(r)$ has a triple root and require the value $\kappa = \frac{2}{\sqrt{3}}$. They can be written in the form

$$\begin{aligned}(p, q, r) &= (\pm 1/9\, \frac{\sqrt{6}t\left(t^2 + 9\right)}{\left(t^2 + 3\right)^{3/2}}, \pm \frac{2\sqrt{2}}{\left(t^2 + 3\right)^{3/2}}, -4\, \frac{\sqrt{3}\left(t^2 - 6\right)}{9\left(t^2 + 3\right)}),\\ (\gamma_1, \gamma_2, \gamma_3) &= (1/3\, \frac{t^4 + 18\,t^2 - 27}{\left(t^2 + 3\right)^2}, 8\, \frac{\sqrt{3}t}{\left(t^2 + 3\right)^2}, \mp 2/3\, \frac{\sqrt{2}t}{\sqrt{t^2 + 3}}).\end{aligned} \tag{E.4}$$

Those solutions represent two motions, which, as $t \to \pm\infty$, asymptotically approach the uniform rotations characterized by the solutions

$$\begin{aligned}(p, q, r)_{-\infty} &= (\mp\frac{\sqrt{6}}{9}, 0, -\frac{4\sqrt{3}}{9}),\\ P = (\gamma_1, \gamma_2, \gamma_3)_{-\infty} &= (\frac{1}{3}, 0, \pm\frac{2}{3}\sqrt{2}),\end{aligned} \tag{E.5}$$

and approach as $t \to \infty$ the values

$$\begin{aligned}(p, q, r)_{\infty} &= (\pm\frac{\sqrt{6}}{9}, 0, -\frac{4\sqrt{3}}{9}),\\ Q = (\gamma_1, \gamma_2, \gamma_3)_{\infty} &= (\frac{1}{3}, 0, \mp\frac{2}{3}\sqrt{2}).\end{aligned} \tag{E.6}$$

In Fig. E.1, we visualize the motion corresponding to the last pair of solutions by depicting the trajectory of the apex of the vertical upward unit vector γ during the motion on the Poisson sphere. The apexes draw two halves of a Fig. 8 curve. The first half (in red colour) begins from $P(t = -\infty)$, goes through R and ends at $Q(t = \infty)$.

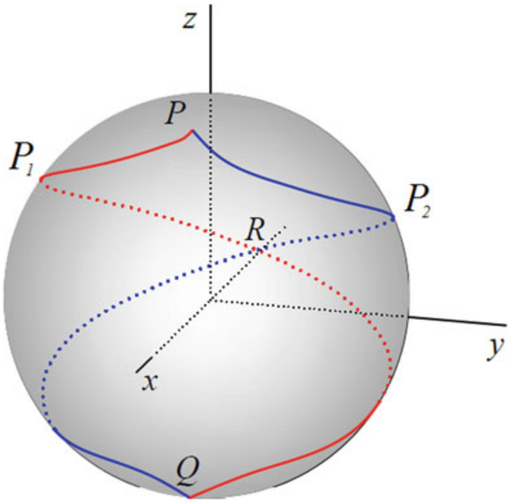

Fig. E.1 The trajectory of the apex on Poisson's sphere (Hidden lines dotted)

The second (PP_1RQ, in blue colour) begins at $Q(t=-\infty)$ and through R ends at $P(t=\infty)$. Note that the motion is not time-reversible, due to the presence of the gyrostatic momentum κ. On the other hand, since the equations of motion (5.5)–(5.6) are invariant under the simultaneous change of signs of the quantities $\boldsymbol{\omega}$, $\boldsymbol{\kappa}$, t, when $\kappa=-\frac{2}{\sqrt{3}}$ two other motions exist, in which the apex traverses each of the two paths described above in the retrograde sense.

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