Multiple Criteria Decision Making

Salvatore Greco
Vincent Mousseau
Jerzy Stefanowski
Constantin Zopounidis *Editors*

# Intelligent Decision Support Systems

## Combining Operations Research and Artificial Intelligence - Essays in Honor of Roman Słowiński

Springer

# Multiple Criteria Decision Making

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Jerzy Stefanowski • Constantin Zopounidis
Editors

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# Preface

This special collective book is dedicated to Professor Roman Słowiński on the occasion of his 70th birthday this year—2022.

Professor Roman Słowiński (Fig. 1) is a widely recognized scientist due to his work in the fields of operations research, multiple-criteria decision-making, optimization, computer science and artificial intelligence. In particular, he strongly participated in the creation and development of the domain of intelligent decision support systems, which combines elements of operations research and artificial intelligence. This research thread became the leading idea of this book and the inspiration for its title.

In his long-term scientific career, which began at the Poznań University of Technology in 1974, Roman Słowiński undertook various thematically basic theoretical and applied research. This is reflected in more than several hundred published scientific texts that are well appreciated in the international research community, which is visible by the very high number of citations, and on this basis, Roman Słowiński is among the world's highest-cited researchers in computer science and operations research, occupying the first position in Poland.

Professor Roman Słowiński is an authority, mentor, and inspiration for many researchers all over the world. He opened the door to real scholarship to many people, including us.

He actively participated in international collaborations. The number of his co-authors, many of them from outside Poland, exceeds 150. Throughout his career, he has been visiting professor or researcher at many well-known universities. He has participated in many conferences and has been invited several times to give keynote speeches. We illustrate this with the attached photos (Figs. 2, 3, 4 and 5).

In addition to his position as a university professor and researcher, he served in a number of organizational capacities. In particular, he is the vice president of the Polish Academy of Science. For many years, he has also served as one of the editors-

**Fig. 1** Professor Roman Słowiński

**Fig. 2** Roman Słowiński together with Salvatore Greco, Constantin Zopounidis, and Pekka Korhonen during 22nd Conference of the International Society on Multiple Criteria Decision Making, June 17–21, 2013, Malaga, Spain

in-chief of the well-known scientific journal—the European Journal of Operational Research. It is also worth emphasizing that in the 1990s, he founded the Intelligent Decision Support Systems lab at Poznań University of Technology, which he has led until now.

Let us also mention that he has been awarded several times for his extraordinary achievements with various prestigious recognitions, both in Poland and in the international research societies or universities.

**Fig. 3** Bernard Roy and Roman Słowiński in Poznan in 1992

**Fig. 4** Roman Słowiński, on the right, together with (from the left to the right) Jerzy Stefanowski, Shusaku Tsumoto, Lofti Zadeh, Wojciech Ziarko and Zdzislaw Pawlak during 4th Int. Workshop on Rough Sets, Fuzzy Sets and Machine Discovery, Tokyo, 1996

**Fig. 5** Jerzy Stefanowski, Benedetto Matarazzo, Jan Weglarz, Roman Słowiński and Jerzy Stefanowski during EURO XVI Conference in Brussels, 1998

In the first chapter of this volume, following this preface, we present in more detail main research contributions of Roman Słowiński, so we will not repeat it now. Furthermore, the second chapter also contains more information on his scientific biography.

This book contains 21 chapters written by several of Roman Słowiński's collaborators, his previous PhD students, and friends, which cover various issues related to his research interests and contributions to such fields as multiple criteria decision aiding, multi-objective optimization methods, intelligent decision support systems, and uncertainty managing in artificial intelligence. These chapters also well demonstrate Roman Słowiński's influence on theory and practice of intelligent decision support systems, and we hope that the readers could find them stimulating and enjoyable.

Finally, we know that Roman Słowiński will continue to be a very active scientist in the next years. We wish him many more successes and happy years of health, with his family and in our community.

| | |
|---|---|
| Catania, Italy | Salvatore Greco |
| Gif sur Yvette, France | Vincent Mousseau |
| Poznan, Poland | Jerzy Stefanowski |
| Chania, Greece | Constantin Zopounidis |

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We take this opportunity to thank all contributors for submitting their papers to this edited book. Their joint efforts and good co-operation with us have enabled to successfully finalize the project of this volume.

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# Chapter 1
# Roman Słowiński and His Research Program: Intelligent Decision Support Systems Between Operations Research and Artificial Intelligence

**Salvatore Greco, Vincent Mousseau, Jerzy Stefanowski, and Constantin Zopounidis**

**Abstract** This chapter is aimed to present the genesis and the development of the scientific research activity of Roman Słowiński considering his contributions in Operations Research, Multiple Criteria Decision Aiding, and Artificial Intelligence in the perspective of Intelligent Decision Support Systems. We try to reproduce his vision of intelligent decision support systems, which he largely initiated in the scientific community, and his ideas and achievements in this field. The impact of his contacts and collaborations with other researchers was also important in creating the field. On the one hand, it was formed on the interaction with three great scientific personalities such as Jan Węglarz, Bernard Roy, and Zdzisław Pawlak, and on the other hand, it was developed during the years with a very large and diversified network of cooperators from his own University as well as from other research teams distributed in many different countries. In addition, we provide an in-depth bibliometric analysis of Roman Słowiński's published papers. We introduce also

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the contributions of this book discussing their relation with the work of Roman Słowiński.

## 1.1 Introduction

This volume is a Festschrift in honor of Roman Słowiński for his 70th birthday. Roman Słowiński is one of the most influential scholars in the fields of Computer Science, Operations Research, Artificial Intelligence, and Decision Science over the last forty years. Through the contributions of some of the many colleagues that along the years have been cooperating and interacting with him, this volume aims to shed light, on the whole life scientific project of Roman Słowiński: combining Operations Research and Artificial Intelligence into Intelligent Decision Support Systems.

The basic idea of Intelligent Decision Support System is often acknowledged to the early contribution of Herbert Simon [88, 90], one of the fathers of Artificial Intelligence and 1978 Nobel Prize for Economics, that proposed to integrate intelligent capabilities in decision support systems (for a general reconstruction of Herbert Simon's contribution to decision support systems see, e.g., [80]). In one illuminating paper [89], considering the widespread interest in expert system at the end of the 80s due to the availability of minicomputers manifested in those years, Herbert Simon discussed about combination of Artificial Intelligence (AI) and Operations Research (OR), proposing "to think of expert systems and the data bases that support them as matrices within which we can apply an amalgamation of AI and OR analytic and problem-solving methods." More precisely, after observing that "in the decade after 1955, the tools of AI were applied side by side with OR tools to problems of management" and that "after about 1960, AI and OR went their separate ways," Herbert Simon proposed what can be seen as his vision for intelligent decision support systems: "We should aspire to increase the impact of MS/OR [Management Science/Operations Research] by incorporating the AI kit of tools that can be applied to ill-structured, knowledge-rich, nonquantitative decision domains that characterize the work of top management and that characterize the great policy decisions that face our society." In the program drafted by Herbert Simon we could notice the inspirations for the agenda developed in his almost fifty years of scientific activity by Roman Słowiński, even if we were not able to find a single citation of that paper in the 14 books and more than 400 articles published by him. Consequently, it is interesting to reconstruct how Herbert Simon's vision became the scientific program of Roman Słowiński through his meeting, acquaintance, collaboration, and friendship with three scholars who shaped operations research, decision aiding, and computer science: Jan Węglarz, Bernard Roy, and Zdzisław Pawlak.

Jan Węglarz, who was a teacher and the supervisor for the master thesis and the PhD of Roman Słowiński at the Poznań University of Technology, introduced him to OR and, in particular, to project scheduling. The results obtained by Roman Słowiński in this domain, some of them together with Jan Węglarz, were so strikingly relevant and innovative that Roman Słowiński and Jan Węglarz, together with Jacek Błażewicz, another brilliant student of Jan Węglarz, were awarded with 1991 EURO Gold medal, the highest distinction within OR in Europe. Observe that EURO Gold medal is in general considered as the award of a lifelong research activity and Roman Słowiński got this prize when he was still not forty years old. In fact, project scheduling was for him an excellent starting point rather than a point of arrival. From the point of view of intelligent decision support systems, starting from project scheduling had two positive aspects for Roman Słowiński:

1. Scheduling with its use of heuristics, as recognized by Herbert Simon in the above mentioned article, was one of the few domains in which, application of AI continued to be coupled with OR even when the trajectories of AI and OR diverged; in this perspective, the first years of research activity shaped the mindset of Roman Słowiński to work on intelligent decision support systems (and, in fact, one of his last papers on scheduling was proposing a decision support system for applications in that domain [100]);
2. Research on scheduling permitted to meet Bernard Roy, one of the founders of OR in Europe, that introduced Roman Słowiński to multiple criteria decision aiding, probably the domain in which he devoted most of his efforts and capacities.

Roman Słowiński met Bernard Roy in 1977, participating, together with Jan Węglarz, at the congress of Association Française pour la Cybernétique, Économique et Technique (AFCET). Bernard Roy was interested in the presentation of Roman Słowiński and wanted to speak with him. This was the beginning of a scientific cooperation and a sincere friendship of the two researchers developed for forty years until the death of Bernard Roy. Let us remember that Bernard Roy, one of the major promoters of OR techniques in Europe, gave fundamental contributions in graph theory and scheduling, but, overall, he is the founder of the European School of Multiple Criteria Decision Aiding (MCDA), proposing, beyond the famous ELECTRE methods, many important contributions to the overall approach and the scientific foundation of MCDA. The basic reference in MCDA is the Bernard Roy's book "Méthodologie multicritère d'aide à la décision" [82] that Roman Słowiński translated in Polish [84]. To synthesize the basic idea of MCDA promoted by Bernard Roy we can consider the following frame, taken by a paper written together with Roman Słowiński [86]: an MCDA method "should be seen as a tool for going deeper into the decision problem, for exploring various possibilities, interpreting them, debating and arguing, rather than a tool able to make the decision. We suppose further that the model of preferences used by the method is, at least partially, co-constructed through interaction between the analyst and the decision maker (or his representative). This co-construction should account for the consequences on which the actions will be judged and for value systems related to the decision context."

A specific preoccupation of Bernard Roy was related to the imprecision, uncertainty, ill determination of the performances assigned by considered criteria to the actions under evaluation (see, e.g., [83]). Again we can find the attention to the "ill-structured, knowledge-rich, nonquantitative decision domains" of the above mentioned article of Herbert Simon.

With respect to the interest to handle ambiguity and uncertainty in the data, Roman Słowiński had the possibility to interact and to cooperate with another great personality, this time, directly related to artificial intelligence: Zdzisław Pawlak, one of the pioneers of Artificial Intelligence and Computer Science with worldwide influence. Zdzisław Pawlak is specifically well known for his rough set theory [74] but he also significantly contributed to many branches of theoretical computer science, mathematics, and logic [79]. In the preface of his seminal book [75], presenting rough set theory he writes: "This book is devoted to some areas of Artificial Intelligence: knowledge, imprecision, vagueness, learning, induction and others" and also "The main issue we are interested in is reasoning from imprecise data, or more specifically, discovering relationship between them." Again, you can see that Zdzisław Pawlak was sharing with Herbert Simon and Bernard Roy the same interest in the imperfection of data. Roman Słowiński met Zdzisław Pawlak in the middle of eighties of the last century and he was immediately interested as explained by himself [94]: "Personally, I am grateful to Professor Pawlak for revealing me the concept of rough set at the beginning of its conception. Together with my brother Krzysztof, a surgeon, we had the privilege of working with him on the first real-world application of rough set theory—verification of indications for the treatment of duodenal ulcer by HSV." These joint scientific meetings also initiated next Roman Słowiński's work on generalizations of the rough set theory to take into account various types of imprecision and on a more general exploitation of rules, resulting from the simplification of decision tables, see, e.g., [64, 95–98]. Recall that Roman Słowiński with his cooperators organized in Kiekrz near Poznań the first international workshop on rough sets theory and was an editor of one first multi-authors monograph on rough sets theory and intelligent decision support systems [93]. Then, he has continued to work on the rough set theory of Zdzisław Pawlak combining it with the MCDA ideas of Bernard Roy, proposing many original contributions and advances relevant both for rough set theory and MCDA, orienting both of them in an AI perspective.

The above considerations show that the genesis of Roman Slowinski's scientific project of combining Operations Research and Artificial Intelligence in Intelligent Decision Support Systems is the result of the interaction of the great scientific personality of Roman Słowiński with the three great scientific personalities of Jan Węglarz, Bernard Roy and Zdzisław Pawlak. This permitted to realize the vision drafted by Herbert Simon, the founder of intelligent support systems, in his short paper that very probably was unknown to all of them (we were not able to find any citation of that paper also in the scientific productions of Jan Węglarz, Bernard Roy, and Zdzisław Pawlak).

A confirmation that what we have reconstructed corresponds to the scientific project Roman Słowiński has been pursuing can be meaningfully found in the following frame of an interview he recently released [29]:

> [...] computer systems were and are created with the intention of decision support. [...] Its development accelerated after World War II with the transition of military operational techniques to civilian operational research, which became a scientific discipline practiced at the border of economics and computer science. In 1950, Alan Turing posed the question

> of whether machines could think in the pages of the journal Mind, and from 1956 the term "artificial intelligence" began to be used to refer to computer systems that relieve humans of certain intellectual tasks, that is, support their decisions. This is how operations research and artificial intelligence came together to form intelligent decision support systems, which also owe their effectiveness to the high computing power of computers and the Internet, which is a huge database. A feature of AI is learning from observations accumulated in large data sets. As Herbert Simon put it in the 1980s, learning allows adaptive changes to a system that make it perform the same or similar task more efficiently next time. An intelligent decision support system uses AI precisely in the aspect of learning from data about the decision situation, thus making the user more familiar with the situation, as AI discovers regularities, anomalies, cause-effect relationships, that is, it discovers knowledge useful in decision making. [...] The data provided to the system may be incomplete, inaccurate, subject to random fluctuations, and partially contradictory. Different mathematical theories and models, such as probability calculus and statistics, rough set theory, or fuzzy set theory, deal with extracting knowledge from such "imperfect" data.

We want to conclude this section with some other words from another interview of Roman Słowiński [71] clarifying his specific "humanistic" approach to the intelligent decision support system domain:

> These systems [intelligent decision support systems] help humans to better understand the decision-making situation and recommend solutions that are consistent with the preferences of a given person or group of people, that is, with their value system. These systems are interactive, because there is a dialogue between the machine, which works according to an algorithm, and the user: we need to let the machine know our preferences, then the machine algorithm will produce a recommendation that is consistent with the model of those preferences, and the human can either accept that recommendation or provide new preference information in response, and such a process loops until a satisfactory recommendation is obtained. Such a decision support process is said to be "human in the loop of the system."

## 1.2 Short Biographical Notes

Roman Słowiński was born in Poznań, Poland, on 16 March 1952 into the family of Lech Słowiński, a professor of Polish philology, and Melania née Michalska. He received his MSc degree from the Electric Faculty of the Poznan University of Technology in 1974, followed by his doctorate (PhD) in 1977 under the supervision of Jan Węglarz. In 1981 he obtained the degree of higher doctorate (DSc) (Habilitated doctor in Polish system) from the same University. He attained the national rank of professor in 1989, and since 1991 he has held the position of Full Professor at the Poznań University of Technology. Since 2003 he is also a Professor at the Systems Research Institute of the Polish Academy of Sciences in Warsaw. Between 1995 and 1997 he hold the European Chair at the University of Paris Dauphine. He was also an invited professor at the Swiss Federal Institute of Technology in Lausanne, the University of Catania, Polytech' Mons, University of Michigan-Ann Arbor, Yokohama National University, Université Laval-Québec, University of Missouri-Columbia, University of Osaka, Polytech' Tours, Ecole Centrale Paris, and many others.

In 1989 Roman Słowiński founded the Laboratory of Intelligent Decision Support Systems (IDSS; http://idss.cs.put.poznan.pl/site/idss-en.html) at the Institute of Computing Science, Poznań University of Technology. He has been chairing IDSS continuously until now. Furthermore he created and led a new Master of Science specialization at Poznan University of Technology devoted to Intelligent Decision Support Systems (the first Master program in this domain in Poland).

He became a Corresponding Member of the Polish Academy of Sciences in 2004, and an Ordinary Member in 2013. Since 2013 he has been a member of Academia Europaea. In 2011–2018 he held the post of Chairman of the Poznań Branch of the Polish Academy of Sciences. From 2015 till 2020 he was an elected Chairman of the Committee on Computer Science of the Polish Academy of Sciences. In 2019 he was elected by the General Assembly of the Polish Academy of Sciences to the post of Vice President of the Academy for the 2019–2022 term.

Other positions he has held include:

- Deputy Director of the Institute of Control Engineering, Poznań University of Technology (1984–1987),
- Vice Dean of the Electric Faculty, Poznań University of Technology (1987–1993),
- Professeur en chaire européenne at Université Paris-Dauphine (2003–2009),
- President (2010–2012) and fellow (since 2015) of the International Rough Set Society;
- Vice president of Polish Operational and Systems Research Society—POSRS;
- Member of the scientific council of Polish Artificial Intelligence Society—PSSI;
- Expert panel member of the European Research Council, PE6-Computer Science (2009–2013);
- The Vice-chairman of the Social Advisory Council at the Archbishop of Poznan.

Roman Słowiński has received many acknowledgments and prestige awards. We have already mentioned the prestigious EURO Gold Medal that he was given in 1991. In 1997 he received the Edgeworth-Pareto Award by International Society on Multiple Criteria Decision Making. Poland awarded Roman Słowiński with the Annual Prize of the Foundation for Polish Science, the highest scientific honor in the Country. He was also given the Scientific Award of the President of the Polish Academy of Sciences (2016) and the Scientific Award of the Prime Minister of Poland for creating a scientific school of Intelligent Decision Support Systems (2020). In 2016, he also received the Scientific Award of the President of the Polish Academy of Sciences. Furthermore, he was honored in his hometown, Poznan. The society of the 19th century Polish positivist Hipolit Cegielski gave him the title of “Outstanding Personage of Organic Work” (2017), and the City Council of Poznan awarded him the title of “Distinguished Citizen of the City of Poznan” (2018). In 2021 Roman Słowiński has been promoted “Officier dans l’Ordre des Palmes académiques” by the French Prime Minister to reward his research work in the field of computer science and his unfailing commitment to the development of Franco-Polish relations in the academic and scientific field.

Roman Słowiński is Doctor Honoris Causa of Polytechnic Faculty of Mons (2000), University Paris Dauphine (2001) and Technical University of Crete (2008). He is also a Honorary Professor of the Nanjing University of Aeronautics and Astronautics (2018).

Roman Słowiński is fellow of IEEE (the Institute of Electrical and Electronics Engineers—2017), IRSS (International Rough Set Society—2015), INFORMS (the Institute for Operations Research and the Management Sciences—2019), and IFIP (International Federation for Information Processing—2019). He was also an active member of Polish Information Processing Society (PTI) and Polish Artificial Intelligence Society (PSSI).

Since 1999 Roman Słowiński has been contributing to the development and promotion of Operational Research as Coordinating Editor of the European Journal of Operational Research (Elsevier, CiteScore=8.5), which, also due to his incessant work, is now a premier journal in Operations Research. In years 1998–2003, he was editor of Decision Analysis Section of the International Journal on Fuzzy Sets and Systems. Currently, he is on editorial board of twenty scientific journals.

Since 2007 Roman Słowiński has been a coordinator of the EURO Working Group on Multiple Criteria Decision Aiding, succeeding Bernard Roy, who founded it in 1975. He has also been President of the INFORMS Section on Multiple Criteria Decision Making for the two years period 2020–2021.

## 1.3 Advice and Supervision of New Researchers

In his academic career, Roman Słowiński has been a mentor and advisor to many young researchers and students. So far, he has promoted 26 doctoral students, many of whom have continued their scientific careers, achieving postdoctoral habilitation degrees (13 in Poland) or professorships. They were successively: Eduardo R. Fernandez (the first of Roman Słowiński's supervised PhD student that defended his thesis in 1987), Marek Kurzawa, Wiktor Treichelt, Mariusz Boryczka, Piotr Zielczyński, Piotr Czyżak, Jerzy Stefanowski, Jacek Żak, Andrzej Jaszkiewicz, Maciej Hapke, Piotr Zielniewicz, Krzysztof Krawiec, Robert Susmaga, Paweł Kominek, Maciej Komosiński, Jacek Jelonek, Szymon Wilk, Roman Pindur, Irmina Masłowska, Bartłomiej Prędki, Izabela Szczęch, Wojciech Kotłowski, Krzysztof Dembczyński, Jerzy Błaszczyński, Miłosz Kadziński, and Marcin Szeląg. Many of them have already raised their own doctoral students, so it can be stated that Roman Słowiński is more than the scientific father of many, many researchers.

Let us emphasize that Roman Słowiński in his cooperation with PhD students and younger colleagues not only plays the role of a mentor but also is even more of an advisor and partner, leaving to his collaborators a large space of research freedom encouraging them to be independent in a perspective of a very open discussion. He also allows them to explore interesting research problems, which are sometime distant from his main interests.

Furthermore Roman Słowiński reviewed many dissertations, both in Poland and abroad, and collaborated with many researchers contributing to the development of their careers. In the Sect. 1.5 we will provide a bibliometric review of his joint publications and discuss his research impact.

Roman Słowiński has also always encouraged and motivated colleagues, especially if young and at the beginning of their scientific carriers. Very often these colleagues started some scientific project with Roman Słowiński and became his friends beyond the scientific activity. For instance Constantin Zopounidis met Roman in Lamsade, the laboratory of Université Paris Dauphine and the CNRS founded by Bernard Roy as a researcher centre in Operational Research and Decision Aiding, in the early 1980s. In that occasion, after a discussion about multicriteria decision aiding and financial modeling, Roman Słowiński asked Constantin Zopounidis to write a paper for the Journal "Foundations of Control Engineering," a quarterly peer-reviewed international journal published by Poznań University of Technology since 1975—that since 1990 changed name becoming "Foundations of Computing and Decision Sciences." Constantin Zopounidis wrote an article entitled "A Multicriteria decision making methodology for the evaluation of the risk of failure and an application" [107], which was published in 1987. It was one of the first papers of Constantin Zopounidis that after started a cooperation with Roman Słowiński on the subject of that paper resulting in the publications of several articles in prestigious journals [22, 99, 101].

## 1.4 The Main Research Contributions of Roman Słowiński

As explained in Sect. 1.1, the scientific carrier of Roman Słowiński started in the area of project scheduling. In that domain he gave relevant contributions related to multiple category resources, multiple job modes, multiple criteria, and uncertainty. More in detail, his contributions were related to Resource Constrained Project Scheduling [8, 73, 106] and scheduling problems with preemptive activities (jobs) to be scheduled on unrelated parallel machines (processors) with additional limited discrete resources [91]. A specific mention deserves the pioneering contributions on fuzzy scheduling [52] and, overall, multiobjective scheduling considering conflicting criteria such as project duration or maximum lateness [92, 100].

In fact, also due to the cooperation with Bernard Roy, MCDA became, in a second moment, the most relevant research area of Roman Słowiński that has given many diversified contribution in different directions:

- interactive multiobjective optimization: in this context we remember a classical interactive multiobjective optimization method [56] in which the decision maker's preference is modeled through the piecewise linear utility model of UTA method [55], an interactive method in which an outranking relation is used to select solutions to be proposed to the decision maker [58], a visual interactive method, called FLIP, handling multiobjective linear programming problems with fuzzy coefficients in the objective functions and the constraints [17], interactive procedures based on an achievement scalarizing function based on weights of objectives compatible with preference information supplied by the user [57, 60]; in this domain, a specific mention deserves the contribution in the heuristics for multiobjective optimization: in this perspective let us remember the Pareto simulated annealing for fuzzy multiobjective combinatorial optimization [53], and overall, several methodologies to drive evolutionary multiobjective optimization algorithm towards the most desirable region of the Pareto front using preference information supplied by the decision maker such as NEMO methods [9, 10] based on ordinal and robust ordinal regression and some methods based on contraction of preference cones [61];

- ELECTRE methods: Roman Słowiński has proposed some relevant extensions of the ELECTRE methods introducing two new effects called reinforced preference and counter-veto effect [85] and specific procedures permitting to handle hierarchy of criteria [14, 18]. Roman Słowiński has also introduced the induction of parameters of ELECTRE methods from decision preferences through ordinal regression approach [68]. Roman Słowiński has participated also to the development of very reliable decision support systems for the applications of ELECTRE methods [69]. Let us mention one of the first and most well-known real-world applications of ELECTRE methods [87], related to the use of ELECTRE III [81] for programming a water supply system (WSS) for a rural area. Roman Słowiński has been also working on the axiomatic basis of ELECTRE methods [102]. Let us also remember a very comprehensive and updated state of the art on the ELECTRE methods [28].
- robust ordinal regression: in fact this can be considered a specific MCDA approach originally proposed by Roman Słowiński [44, 47]. Differently from the classical ordinal regression [55] that aimed to represent preferences expressed by the DM with a single specification of a given decision model, very often the additive value function, robust ordinal regression takes into account the whole set of specifications of the considered model that are compatible with the preference information supplied by the DM. Robust ordinal regression, originally proposed taking into account additive value functions used to handle ranking problems [44], was extended and generalized in a vast diversity of directions considering representation of intensity of preferences [27], sorting problems based on additive value functions [46], ELECTRE methods [48], hierarchy of criteria in value functions [13] and ELECTRE methods [14, 15], interaction of criteria in value functions [51] and ELECTRE methods [16], and so on.

Data analysis based on rough set theory is the third pillar of the Roman Słowiński's research activity. We have already mentioned that he did the first real-world medical applications of the rough set theory [78] based on the development, together with his cooperators, of the first software to apply rough set approach [96]. Roman Słowiński proposed also some remarkable extensions such as the rough approximation based on a similarity relation being only reflexive that generalizes the classical indiscernibility relation of original rough set theory which, being an equivalence relation, is not only simply reflexive but also symmetric and transitive [98]. He studied also rough approximation in case of missing or imprecise data [95], a methodology to apply the decision rules obtained from rough set approach on the basis of their closeness to objects to be classified [97]. Roman Słowiński applied rough set theory also in business and finance [22, 101].

From the beginning of the 1990s Roman Słowiński started to be interested in applying rough set approach to multiple criteria decision aiding [76, 77]. The proposed approach was based on the idea of supporting decisions by means of the "if ..., then ..." rules obtained from preference information supplied by the decision maker in terms of example of decisions. In fact, the information requested, the examples of decisions, as well as the information supplied, the decision rules, were simple and easily understandable for human. However, its restriction was noticed in the application of rough set theory to multiple criteria decision aiding. Rough set theory is based on the indiscernibility relation that holds between two objects if they have the same evaluation with respect to all considered attributes, so that if two objects are indiscernible they should be assigned to the same class. Instead, multiple criteria decision making is based on preference relations that respect a dominance principle for which if one alternative $a$ has an evaluation not worse than another alternative $b$ on all the considered attributes, then $a$ has to be considered at least as good as $b$. These considerations matured gradually in a certain number of years with a sequence of steps. First, the idea that the object of a rough approximation has to be a preference relation rather than a classification was taken into consideration [30]. The approach gave some interesting results, but in a first moment still the usual indiscernibility was continued to be applied. The necessity to pass from indiscernibility to dominance was in a second moment matured, at the beginning with a specific "single level" dominance used to approximate a preference relation [33]. The clear idea of substituting indiscernibility with dominance and on this basis reformulating the whole rough set theory appeared for the first time in a paper in which rough set approximation was used to the problem of bankruptcy evaluation [31], further developed and extended in [34], and finally published in its definitive form called Dominance-based Rough Set Approach (DRSA) in [37] which is one of the most cited papers both in the domains of rough set theory and multiple criteria decision aiding. In the following years, DRSA has continued to be extended and applied in many domains thanks also to the algorithms [35, 105] and software [6] developed by Roman Słowiński and his cooperators. Among the most relevant extensions of DRSA proposed by the same Roman Słowiński let us remember

- the variable consistency DRSA [3, 36] permitting to discover strong patterns in the data through a relaxation of the dominance principle,
- an extension of DRSA permitting to handle missing values [32],
- an extension of DRSA to handle decisions under uncertainty and time preferences [45],

- an interactive multiobjective optimization method in which the preference information supplied by the decision maker is represented in terms of DRSA "if…, then …" decision rule [43],
- the stochastic DRSA [63] resulting in a probabilistic model for ordinal classification problems based on a statistical approach relating DRSA with machine learning approach [62].

In addition to the main research areas discussed above, Roman Słowiński with his cooperators contributed to many problems in the field of artificial intelligence, in particular, knowledge discovery from data and machine learning. We will mention just some of them:

- new methods of evaluating rules discovered from data, including studies on values rule evaluation measures [49], in particular, starting from an initial idea presented in the paper co-authored also by Zdzisław Pawlak [39], promoting Bayesian confirmation measures and investigating their formal properties [50],
- new methodologies for interacting with human analysts to evaluate and select the most meaningful decision rules induced with the rough set approach [11], taking also into account paradigms of multiple criteria perspective [103],
- an approach to evaluate the most important sets of elementary conditions in rules by an adaptation of set functions used in cooperative game theory [42],
- online inductions and evaluation of rule classifiers [11, 40],
- selection of the most relevant subsets of features in the data processing steps of knowledge discovery [104, 105] as well as in combination with artificial neural network learning [59],
- new algorithms for discovering rules with handling semantically correlated attributes [19, 38, 40],
- specialized rule-based ensemble classifiers [2, 4, 5, 20, 21],
- nonparametric ordinal classification with monotonic constrains [62],
- procedures to interpret classifiers with an additional dialogue with users [7].

Finally, it is necessary to mention the involvement of Roman Słowiński in many practical applications of the above-discussed methods, in particular, in the field of medicine [26, 66, 67], environmental protection [12, 24, 61], medical [25, 101] and technical diagnostics [72], banking and finance [22, 31, 101], as well as in preferences learning, e.g., in the domain of customer satisfaction [1, 41].

## 1.5 A Bibliometric Analysis of the Research Activity and Impact of Roman Słowiński

Roman Słowiński is an exceptionally active and creative author of many scientific texts. While writing them, he collaborated with many co-authors, inspired other researchers, and influenced research conducted in many fields, as we discussed in the earlier sections of this chapter. In order to discuss his creative writing activity, we present below a bibliometric analysis of his publications.

Let us recall that bibliometrics studies the publishing material quantitatively [23]. Here we will focus on an individual author by analyzing such indicators as the number of publications, their total citations, cites of the most visible papers, the $h$—index [54]. Furthermore we will identify the journals and other publication sources where he published most often together with the fields and the most important topics/keywords of his contributions. This part is somehow inspired by earlier types of such bibliometric analysis undertaken in such studies as, for instance, [65]. We will also determine his most frequent co-authors and comment on his research network and other publishing activities.

This analysis is done using the following bibliography databases and web search engines: Scopus (Elsevier's abstract and citation database), Web of Sciences (currently maintained by Clarivate Analytics), and Google Scholar. It is also supported by considering some summary information provided by DBLP—a computer science bibliography website. The search process was carried out in October 5, 2021.

### *1.5.1 Numbers of Publications and Citations*

Roman Słowiński has published over several hundred texts depending on the bibliography source. These numbers and citations are presented in Table 1.1. Note that Web of Sciences Core Collection is the most restrictive to selected journals and main conferences while Scopus contains the larger collections of the good quality materials. On the other hand Google Scholar indexes other conferences and electronic materials, therefore its database and the number of found publication links are usually clearly bigger.

The numbers of publications in each of these bibliography sources are very high. If one analyzes the annual distributions of their numbers—see figures available in Scopus or Web of Sciences interface—they increased since beginning of the 90' (4–6 per year), to approximately 10–14 between 2000 and 2010, and an even higher numbers more recently. The highest number of 22 publications occurred in 2012 year. These are, in any case, very high indicators proving his extremely rich scientific activity.

**Table 1.1** Numbers of Roman Słowiński's publications, their citations and $h$ index

| Database system | No. publications | Total no. of citations | No. of citing documents | $h$-index |
|---|---|---|---|---|
| Scopus | 325 | 14,727 | 8036 | 61 |
| Web of Sciences | 275 | 10,894 | 6238 | 51 |
| Google Scholar | > 500 | 30,378 | – | 87 |
| Dblp | 264 | – | – | – |

The interesting information on the different types of publications may be found in dblp website. According to it Roman Słowiński co-authored:

- 117 journal papers,
- 114 conferences papers,
- 17 chapters in books and collections,
- 8 being editor or co-editor of multi-author's monographs.

This is somehow consistent with a similar analysis of the author output panel available in Scopus (called document by source view) where also the numbers of different journals and conferences papers are approximately the same (although with advantage for journals). Following this Scopus report the Springer Lecture Notes in Computer Science including Subseries in Artificial Intelligence is the most numerous outlet for the publication of conference papers and chapters (87 publications—26.9%—for all 325 papers). Nevertheless one should notice a relatively very high number of Roman Słowiński's publications in many journals. The most frequent journals are analyzed in the next subsection.

The total numbers of citations presented in Table 1.1 are very high and should be regarded as definitely above the average values of even very good researchers. Moreover its dynamics is constantly increasing since the 90s—which is illustrated in Fig. 1.1 as the number of citations received in each year. As one can notice, the number of citations has increased significantly though time since the end of 70s in the previous century. Following Scopus summaries Slowinski's publications were receiving around several hundreds of citations per year till 2003, then the numbers

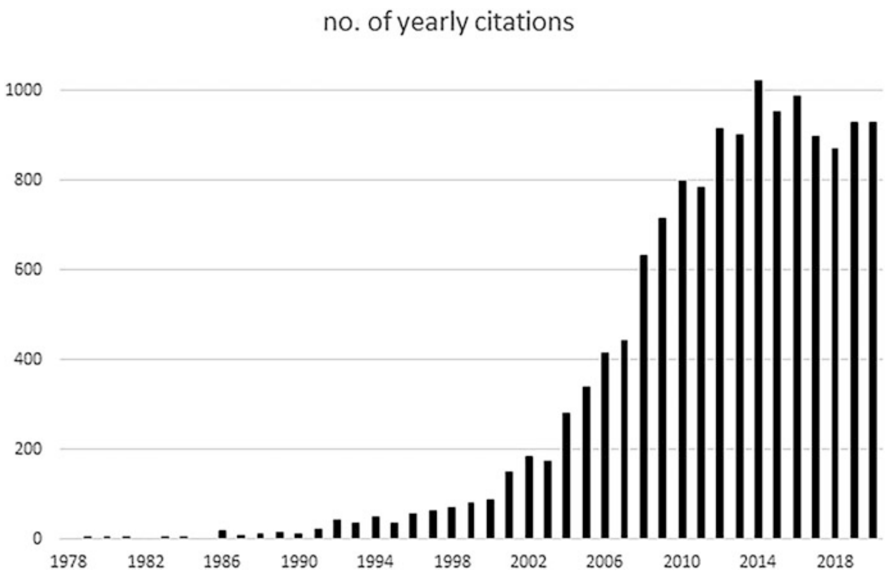


**Fig. 1.1** A number of annual citations of papers

increased to nearly one thousand—for instance, it was 1021 in year 2014 and this number remains relatively stable at a level slightly above 900 from this moment. Following Google Scholar the corresponding values are higher and currently reach values in the range of 1400–1500 citations per year.

The visibility and impact of Roman Słowiński's publications is clearly reflected by values of his $h$ index in all examined bibliography databases. They are very high and, in particular, 51 for Web of Science, 62 for Scopus, and 87 for Google Scholar. Let us notice that values of bibliometric indices allow to classify Roman Słowiński as one of the world's most recognizable scientists. For instance, Guide2research webpage (currently changed to research.com) contains 2021 7th edition of top scientists ranking for computer science and Roman Słowiński is placed on 504 position in the world ranking and as the first researcher in Polish subranking.[1] Following the discussion in [70] is also recognized as one of the most cited Polish researchers in Artificial Intelligence.

### 1.5.2 The Most Cited Papers

The papers of Roman Slowinski received a high number of citations. The 20 most cited positions in Scopus and Web of Science are the following (we list them according to Scopus ranking and additionally show the number of citations from Web of Science in brackets + an average per year):

1. Greco, S., Matarazzo, B., Slowinski, R.: Rough sets theory for multicriteria decision analysis. European Journal of Operational Research, 2001—1281 citations (1034 in WoS; 49.24 per year),
2. Pawlak, Z., Grzymala-Busse, L., Slowinski, R., Ziarko, W.: Rough sets. Communications of the ACM, 1995—834 citations (582 in WoS, 21.56),
3. Slowinski, R., Vanderpooten, D.: A generalized definition of rough approximations based on similarity. IEEE Transactions on Knowledge and Data Discovery, 2000—800 citations (656 in WoS; 29.82 per year),
4. Greco, S., Matarazzo, B., Slowinski, R.: Rough approximation of a preference relation by dominance relations. European Journal of Operational Research, 1999—389 citations (311 in WoS; 13.52 per year),
5. Pawlak, Z., Slowinski, R.: Rough set approach to multi-attribute decision analysis. European Journal of Operational Research, 1994—375 citations (271 in WoS; 9.68 per year),
6. Greco, S., Matarazzo, B., Slowinski, R.: Rough approximation by dominance relations. International Journal of Intelligent Systems, 2002—367 citations (288 in WoS; 14.4 per year),

[1] https://research.com/scientists-rankings/computer-science/2021/pl.

7. Dimitras, A., Slowinski, R., Susmaga, R., Zopounidis, C.: Business failure prediction using rough sets. European Journal of Operational Research, 1999—360 citations (276 in WoS; 12 per year),
8. Greco, S., Matarazzo, B., Slowinski, R.: Rough sets methodology for sorting problems in presence of multiple attributes and criteria. European Journal of Operational Research, 2002—346 citations (290 in WoS; 14.5 per year),
9. Greco, S., Mousseau, V., Slowinski, R.: Ordinal regression revisited: Multiple criteria ranking using a set of additive value functions. European Journal of Operational Research, 2002—305 citations (277 in WoS; 19.79 per year),
10. Mousseau, V., Slowinski, R.: Inferring an ELECTRE TRI model from assignment examples. Journal of Global Optimization, 1998—272 citations (228 in WoS; 9.5 per year),
11. Jelonek, J., Krawiec, K., Slowinski, R.: Rough set reduction of attributes and their domains for neural networks, 1995—207 citations (117 in WoS; 4.33 per year),
12. Figueira, J.R. Greco, S., Roy, B., Slowinski, R.: An Overview of ELECTRE Methods and their Recent Extensions, Journal of Multi-criteria Decision Analysis, 2013—194 citations (162 in WoS; 18 per year),
13. Mousseau, V., Slowinski, R., Zielniewicz, P.: A user-oriented implementation of the ELECTRE-TRI method integrating preference elicitation support, Computers and Operation Research, 2000—194 citations (158 in WoS; 7.18 per year),
14. Blaszczynski, J., Greco S., Slowinski, R.: Multi-criteria classification—a new scheme for applications of dominance-based decision rules. European Journal of Operational Research, 2007—175 citations (135 in WoS; 9 per year),
15. Laengle, S. et al.: Forty years of the European Journal of Operational Research: A bibliometric overview. European Journal of Operational Research, 2017—173 citations (148 in WoS; 29.6 per year),
16. Blaszczynski, J., Slowinski, R., Szelag, M.: Sequential covering rule induction algorithm for variable consistency rough set approaches. Information Science, 2011—173 citations (149 in WoS; 13.6 per year),
17. Slowinski, R.: A multicriteria fuzzy linear programming method for water supply system development planning. Fuzzy Sets and Systems, 1986—172 citations (155 in WoS; 4.31 per year),
18. Figueira, J.R. Greco, S., Slowinski, R.: Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. European Journal of Operational Research, 2009—141 citations (128 in WoS; 9.85 per year),
19. Greco, S., Matarazzo, B., Slowinski, R., Stefanowski, J.: Variable consistency model of dominance-based rough sets approach. Proc. RSCTC, Springer LNCS 2001—139 citations (no in WoS),
20. Jaszkiewicz, A., Slowinski, R.: The 'Light Beam Search' approach - an overview of methodology and applications. European Journal of Operational Research, 1999—123 citations (105 in WoS, 4.57 per year).

The citation rankings in Scopus and WoS are slightly different, although nearly the same papers are present in both Top 20 positions. In case of Web of Science the differences are: Roy. B, Slowinski R.: Questions guiding the choice of a multicriteria decision aiding method. EURO Journal of Decision Process 2013 (112 citations, ordered as 19 in the ranking) ; then Slowinski, R.: Two approaches to problems of resource allocation among project activities. A comparative study. Journal of the Operational Research Society 1980 and Corrente S., Greco S., Slowinski, R.: Robust ordinal regression in preference learning and ranking. Machine Learning 2013 (both 98 citations). Google Scholar in its Top 20 shows also: Slowinski, R., Zopounidis, C.: Application of the rough set approach to evaluation of bankruptcy risk. Intelligent Systems in Accounting, Finance and Management, 1995; and Greco, S., Matarazzo, B., Slowinski, R.: The use of rough sets and fuzzy sets in MCDM, a chapter in Multicriteria decision making, Springer 1999, and a Polish translation by Roman Słowiński of Bernard Roy's book Multicriteria Decision Aid [84]. All these papers are highly cited, in particular, first positions in this ranking around 1000 times—which is an extraordinary result.

Nearly all of the listed top 20 cited papers were published in journals. Only few publications are conference papers (Scopus) or book chapters. European Journal of Operational Research is the dominated place for publications (over 10 times per 20 positions), while others are selected as single positions. Most of them also contribute to fields of operational research, decision analysis, fuzzy sets, or intelligent systems. The most frequent key words and topics include: rough set theory and rough approximations, multicriteria decision analysis, dominance relations, ordinal regression, ELECTRE method, preferences, and decision rules.

### *1.5.3 Journals and Other Sources of Publications*

Unlike the previous section (only top cited publications), now we present global summaries of all publications. Firstly using Scopus Author Output panel we list main journals and other sources of publications. Table 1.2 presents names of journals of book series where Roman Slowinski published papers at least 3 times in the period 1977–2021.

According to this summary table Roman Słowiński is author of a high number of publications in journals from Operation Research, Artificial Intelligence, and Soft Computing. Many of the listed journals are very prestigious ones with demanding acceptance rates and high impact factors. For instance, the impact factor is 5.334 for European Journal Of Operational Research, 7.084 for Omega, 6.795 for Information Sciences, around 3.8 for Fuzzy Sets And Systems and International Journal of Approximate Reasoning. Publication on such journals permitted that Roman Słowinski's articles could be read and cited by many researchers worldwide, which explains also the large number of citations in the earlier list of papers presented in the previous subsection. Springer series Lecture Notes In Computer Science

**Table 1.2** Names of journals or book series where Roman Slowinski published papers at least 3 times—source Scopus Author output

| Name of journal/book | How many |
|---|---|
| Lecture Notes In Computer Science | 87 |
| European Journal Of Operational Research | 37 |
| Information Sciences Journal | 9 |
| Lecture Notes In Artificial Intelligence | 8 |
| Omega (United Kingdom) | 8 |
| Fuzzy Sets And Systems | 7 |
| Control And Cybernetics | 6 |
| Decision Support Systems | 6 |
| Fundamenta Informaticae | 6 |
| International Journal of Approximate Reasoning | 5 |
| Annals Of Operations Research | 4 |
| Communications In Computer and Information Science | 4 |
| International Series In Operations Research And Management Science | 4 |
| Studies In Computational Intelligence | 4 |
| Computers And Operations Research | 3 |
| Engineering Applications Of Artificial Intelligence | 3 |
| Infor | 3 |
| Journal Of Global Optimization | 3 |
| Knowledge Based Systems | 3 |
| Springer Handbook Of Computational Intelligence | 3 |

refers to mainly conference proceedings in similar fields of Artificial Intelligence, Computational Intelligence, or Rough Sets Theory.

### *1.5.4 Main Co-Authors*

Roman Słowiński cooperated with many researchers from different countries. Depending on the bibliography systems their number is listed up to 150 ones. Below in Table 1.3 we list the names of the most frequent collaborators.

One can notice that some of the names already occurred in the list of the most cited papers of Roman Słowiński. Besides these co-authors Roman Słowiński collaborated with many other researchers coming also from France, Germany, Belgium, Spain, Portugal, Greece, Finland, Mexico, Brazil, and many other countries. All these reports clearly demonstrate that he was able to create a strong research networks of many researchers coming from different countries, not only from his

**Table 1.3** Names of the most frequent co-authors of Roman Słowiński papers indexed in Scopus—the total number of authors 150 and papers 325

| Name | Country | Co-authored documents |
|---|---|---|
| Salvatore Greco | Italy | 167 |
| Matarazzo Benedetto | Italy | 66 |
| Blaszczynski Jerzy | Poland | 43 |
| Kadzinski Milosz | Poland | 31 |
| Corrente Salvatore | Italy | 20 |
| Dembczynski Krzysztof | USA, Poland | 20 |
| Kotlowski Wojciech | Poland | 18 |
| Stefanowski Jerzy | Poland | 18 |
| Wilk Szymon | Poland | 16 |
| Michalowski Wojtek | Canada | 15 |
| Szelag Marcin | Poland | 15 |
| Inuiguchi Masahiro | Japan | 13 |
| Mousseau Vincent | France | 10 |

homeland Poland and his PhD students there. Many of his co-authors have also independently published a high number of papers, which are also read and cited, what is reflected by their own high $h$ indices and numbers of citations. This creates an extensive network of subsequent authors and shows how Roman Słowiński has influenced so many researchers around the world.

### *1.5.5 Most Contributed Topics*

Using Scopus authors profile the most contributed topics (keywords) in recent Roman Słowiński's publications are: multiple criteria, ELECTRE methods, attribute reduction, rough sets, fuzzy rough sets, multiobjective evolutionary algorithm, multiobjective optimization, and Pareto front. On the other hand using additional tools offered by Web of Science one can find the report on research categories to which Roman Słowiński's published articles shown in Table 1.4.

The similar report from Scopus shows that his publications are mostly concerned with general categories Operations Research, Management Science, and Computer Sciences with Artificial Intelligence.

**Table 1.4** Names of general fields associated with types of publications—source Web of Science

| Web of Science categories | Record count no. | % of 275 |
|---|---|---|
| Computer Science Artificial Intelligence | 119 | 43.273 |
| Operations Research | 84 | 30.545 |
| Management Science | 54 | 19.636 |
| Computer Science & Information Systems | 42 | 15.273 |
| Computer Science Theory Methods | 40 | 14.545 |
| Mathematics Applied | 21 | 7.636 |
| Computer Science Interdisciplinary Applications | 14 | 5.091 |
| Automation Control Systems | 12 | 4.364 |
| Engineering Electrical Electronic | 12 | 4.364 |
| Computer Science Cybernetics | 10 | 3.636 |
| Engineering Multidisciplinary | 7 | 2.545 |
| Statistics Probability | 7 | 2.545 |
| Medical Informatics | 6 | 2.182 |
| Environmental Sciences | 5 | 1.818 |
| Engineering Industrial | 4 | 1.455 |
| Social Sciences Mathematical Methods | 4 | 1.455 |
| Economics | 3 | 1.091 |
| Mathematics Interdisciplinary Applications | 3 | 1.091 |
| Biochemistry | 2 | 0.727 |

## 1.6 Volume's Contributions

The chapters of this book constitute a set of contributions which are fairly representative of the past and current research themes on which Roman Słowiński had a substantial influence. The set of authors who are involved in this book as authors correspond to renowned colleagues that had collaborations with Roman Słowiński or were influenced by his works. We are grateful to all contributors coming from laboratories from all over the world.

The book content, besides this Preface and the following contribution presenting the scientific activity of Roman Słowiński, is organized into three main parts related to the following domains in turn: Multicriteria Decision Aid, Multiobjective optimization, Intelligent Decision Support Systems handling uncertainty in knowledge management.

Jan Węglarz, drawing extensively from his personal memories first as a mentor and then as a colleague and friend, reconstructs the scientific career of Roman Słowiński, from his initial fundamental contributions in project scheduling, through his research on multiple criteria decision aiding and rough set theory due to his encounters with Bernard Roy and Zdzisław Pawlak, and continuing with all his subsequent activity up to the present day full of many diverse international collaborations.

Part I of the book, devoted to Multiple Criteria Decision Aiding, is composed of twelve contributions.

Salvatore Corrente, José Figueira, and Salvatore Greco present a reasoned survey of the contributions given by Roman Słowiński to ELECTRE methods, a well-known family of multiple criteria decision aiding methods originally proposed by Bernard Roy and his cooperators.

Hannele Wallenius and Jyrki Wallenius discuss potential opportunities for the Decision Science/MCDM community in view of the technology mega trends and other trends characterizing the current world rapid changes. In this perspective they envisage space for promising contributions in domains such as recommender systems and search engines, matching algorithms big data, indices for sustainable investing, and mass decision support tools to help online purchases.

Michael Doumpos, Evangelos Grigoroudis, Nikolaos Matsatsinis, and Constantin Zopounidis provide an overview of the preference disaggregation paradigm, covering the existing methodologies in this area, the main types of decision models used in a disaggregation context (i.e., value functions, outranking, and rule-based models), and existing formulations for ranking and sorting problems. The chapter also reviews the recent applications, as well as methodological advances focusing on robustness issues.

Eyke Hullermeier and Christophe Labreuche elaborate two important developments in the realm of multicriteria decision aid, which have attracted increasing attention: first, the idea of leveraging methods from preference learning for the data-driven (instead of human-centric) construction of decision models, and second, the use of hierarchical instead of “flat” decision models. Finally, an approach based on these two ideas is illustrated by means of a concrete example, namely the learning of tree-structured combinations of the Choquet integral as a versatile aggregation function.

Adiel Teixeira de Almeida Filho, Julio Cezar Soares Silva, Diogo Ferreira de Lima Silva, and Luciano Ferreira give an overview of preference learning techniques for the credit rating of financial assets focusing on the country and corporate credit risk. The authors discuss multiple criteria decision methods helping the investors to understand in a transparent way the assignment of financial assets to different risk classes.

Eduardo Fernandez, Jorge Navarro, and Efrain Solares present two novel methods to address multicriteria ordinal classification problems on the basis of interval value functions, which are used to represent preferences of decision makers that hesitate about the precise value of criteria weights and criterion scores. The authors prove that the proposed methods satisfy the following basic consistency properties for sorting problems: Unicity, Independence, Homogeneity, Monotonicity, Conformity, and Stability.

Aida Valls and Antonio Moreno review the extensions of the ELECTRE multiple criteria decision aiding methods developed by the ITAKA research group that they coordinate. In particular, they present three versions of the concordance and discordance indices of ELECTRE methods that permit to handle multi-valued linguistic scales and the hierarchy of criteria.

Miłosz Kadziński reviews Robust Ordinal Regression, a relevant multiple criteria methodology introduced by Roman Słowiński, which incorporates indirect preference information in the form of decision examples and verify the consequences of applying all compatible instances of an assumed preference model. The contribution discusses different aspects of the methodology, recalls significant extensions, and lists selected real-world applications.

Yves Meinard and Alexis Tsoukiàs in their chapter entitled "What Is Legitimate Decision Support?" discuss the aspects of the organizational context, the overall problem situation, the environment, culture, history, which could play an important role in supporting decisions. Based on the literature review, they propose a general theory of legitimacy, adapted to decision support contexts, encompassing the relevant contributions they identified in the literature.

Philippe Fortemps and Marc Pirlot present a multicriteria outranking method to handle sorting problems, that is, classification in preference ordered classes, in which the information is not complete. The discussed method is an extension of the MR-Sort, a multicriteria outranking method constituting a simplified variant of ELECTRE TRI method.

Luis Dias and Miłosz Kadziński examine possibility of creating the meta-rankings of journals publishing Multiple Criteria Decision Aiding research based on multiple ratings coming from expert panels, which use different qualitative scales. Their approach exploits Benefit-of-Doubt composite indicators for heterogeneous qualitative scales, derived from Data Envelopment Analysis. They applied it to rankings about 50 journals, including also the European Journal of Operational Research for which Roman Slowinski has been editor in chief since 1999.

Eleftherios Siskos and Yannis Siskos propose a multicriteria evaluation methodology, which is based on a synergy of the outranking method PROMETHEE II and the Robust Simos method for the elicitation of criteria importance weights. The evaluation system operates via a robustness control algorithm, called "Bipolar Robustness Control," which measures and progressively improves the robustness of both the evaluation model and the ranking results. The net outranking flow, given by the PROMETHEE II method, indicates the degree of superiority or inferiority of a country, compared to the average e-government performance in Europe.

Part II of the book includes three contributions on Multiobjective Optimization.

Juergen Branke, Andrzej Jaszkiewicz, and Piotr Zielniewicz review and summarize the research of Roman Słowiński, members of his Laboratory, and his main collaborators related to the use of the decision maker's preferences in metaheuristic and evolutionary algorithms for multiobjective optimization.

Margaret M. Wiecek and Philip J. de Castro discuss multiobjective optimization problems focusing on their decomposition into subproblems with a smaller number of criteria and their coordination to solve the original problem. Decomposing in subproblems permits to improve interaction with decision makers enabling the trade-off elicitation in lower dimensional spaces. In this perspective a comprehensive process and guidance for developing a decomposition-coordination technique is proposed.

Masahiro Inuiguchi proposes a robust treatment of possibilistic linear programming problems with linear membership functions introducing necessity fractional optimization models as optimization models with robust constraints. Those problems are reduced to a semi-infinite linear programming problem that can be solved approximately by a linear programming technique. Moreover, under some conditions on functions associated with the necessity measures, the problem can be reduced to a usual linear programming problem or solved by a relaxation procedure with a bisection method.

Part III of the book, devoted to Intelligent Decision Support Systems handling uncertainty in knowledge management, contains four contributions.

Jerzy Błaszczyński, Salvatore Greco, Benedetto Matarazzo, and Marcin Szelag provide a comprehensive view on basic ideas and main trends of the Dominance-based Rough Set Approach, which is one of the main methodological contribution of Roman Słowiński. Besides a systematic presentation of main variants of this proposal, this chapter contains an interesting section on the rationale and history of its gradual development.

Patrick G. Clark, Jerzy W. Grzymala-Busse, Zdzislaw S. Hippe, and Teresa Mroczek discuss probabilistic generalizations of rough set theory, which deals with missing attribute values in incomplete data tables. The authors consider two global and local probabilistic approximations of the sets of examples, which are constructed from maximal consistent blocks. Then, they study their influence on the complexity of induced sets of rules. The presented research fits well with Roman Słowiński's interests in various generalizations of the rough set theory.

Izabela Szczech, Robert Susmaga, Dariusz Brzezinski, and Jerzy Stefanowski present a survey on confirmation measures that are indices representing the impact of evidence an E on a hypothesis H with respect to decision rules. The paper takes into consideration the idea, due to Roman Słowiński and his cooperators, that confirmation measures can be used as interestingness measure of rules induced from data and then can be applied in different tasks.

Dympna O'Sullivan, Szymon Wilk, Martin Michalowski, Hugh O'Sullivan, Marc Carrier, and Wojtek Michalowski discussed their approach to combining adherence to therapy and patient preference models for evaluation of therapies in patient-centered care. More precisely, each patient has a preference model that defines preferences for specific therapies. The adherence and patient preference models are constructed from preferential information elicited using multiple criteria decision analysis methods and they are represented as value functions. Then, they discuss the clinical scenarios illustrating the use of this approach.

This concludes our brief overview of the chapters of this volume.

## 1.7 Final Remarks

This book aims to honor Roman Słowiński's outstanding achievements by collecting various contributions from his colleagues, collaborators, previous PhD students, and friends. As we shown in the previous section, in terms of content, the chapters of this book are related to the topics of Intelligent Decision Support, Multicriteria

Decision Analysis, Operation Research, and Artificial Intelligence but also include more personal comments from the authors about working with Roman Słowiński. Note that Roman Słowiński is not only an outstanding scientist but also an open-minded man with a great and wonderful personality with a positive attitude towards other people and the world and he acts according to deep values.

While the characteristics of this multi-author monograph correspond to the German term Festschrift, i.e., a special book published on the occasion of the significant birthday, we do not regard it retrospectively as a summary of Roman Słowiński's long and fruitful career. Knowing him personally, we think prospectively and expect his further work proposing many original ideas to contribute to the development of many areas of research continuing to inspire the work of a rich network of researchers.

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# Chapter 2
# Roman's Scientific Trajectory: A Retrospective with an Emphasis on the Beginning

**Jan Węglarz**

**Abstract** The scientific trajectory of Roman Słowiński began in 1972, when he started an individual study program under my supervision at the Poznań University of Technology. This chapter is a retrospective of Roman's early career from my point of view. It begins with his master thesis, through doctorate and habilitation, until 1989, when he obtained the title of professor at the age of 37 and created his own Laboratory of Intelligent Decision Support Systems known worldwide.

I met Roman Słowiński for the first time in October 1971 at the Electrical Engineering Faculty of the Poznań University of Technology. He was a student of the master's degree in "Automatic Control", and I conducted exercises of "Control Theory" as an assistant at the Institute of Automatic Control. I quickly realized that, together with Jacek Błażewicz and Wojtek Cellary, he belonged to a group of outstanding students who want to learn something more than it was provided for in the study program. Thus, at the beginning of 1972, I offered them individual lectures, in which I introduced them to selected areas of mathematics that, to my opinion, needed to be completed in order to attack research problems. I did it on the basis of my own experience which I gained while studying mathematics at the Adam Mickiewicz University of Poznań, completed in 1969, and while working on my doctoral thesis. We agreed that this would be more effective than completing a full math studies, as in my case.

It is worth noting that my master thesis at the Poznań University of Technology and my doctoral dissertation (both under the supervision of Prof. Zdzisław Bubnicki from the Wrocław University of Technology) concerned the problems of optimal control of the so-called complexes of operations. The basic problem was to perform in a minimum time a set of $n$ independent tasks (operations), the start and end states of which were known, as well as their processing speeds being piecewise-continuous, increasing functions of the continuously divisible resource available in

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a limited amount. Although it was a problem in the area of optimal control theory, its specificity made the known methods of this theory unsuitable, whereas properties of optimal schedules could be proved using the theory of convex sets and Minkowski functionals. The specificity of this problem, apart from the task models, was that the tasks were performed simultaneously using a limited resource. This drew my attention to other problems of this type, in which sets of precedence-related tasks (activities), i.e., activity networks, require different types and categories of resources and task performance characteristics depend on resource allocations. In this way, I entered the area of operations research, more precisely project scheduling, which I found very attractive, in particular, in the field of resource management in computer systems. While focusing personally on the problem described at the beginning with some important generalizations (e.g., doubly constrained resources, like energy-power), I encouraged my younger colleagues to address other problems of this type. In particular, Roman, for his master thesis, faced problems in which the activity network is composed of divisible, i.e., preemptive activities using limited, discrete resources of different categories. It is worth noting that considering preemptive activities allows for the formulation of many variants of these problems as linear programming (LP) problems. The most natural variant to tackle with was the problem of Resource Constrained Project Scheduling (RCPS), frequently considered in operations research for its practical utility. In the classical RCPS problem, resource requirements of non-preemptive activities concern known types and amounts of limited renewable resources. We considered the RCPS problem in which activities are preemptive. For this problem we developed an original algorithm, called ARSME, which, based on a concise description of the activity precedence constraints, automatically generated the LP problem and then found the schedule using the Revised Simplex Method. This schedule was optimal for a given ordering of nodes of an activity-on-node network. Finding the globally optimal schedule required solving LP problems for all possible node orders and permitted to assess the quality of node-ordering heuristics. The globally optimal schedule gives also a lower bound on the project processing time in case of non-preemptive activities, which is useful to assess the quality of heuristics proposed for the computationally harder classical RCPS problem. Description of this algorithm, together with its program in Fortran and results of a computational experiment was the content of our first publication in a world-class journal [29].

It is worth mentioning that one of the reviewers of the said article, who appreciated its importance, disclosed himself and helped us prepare the final version. It was James H. Patterson from the University of Missouri-Columbia, with whom we established scientific cooperation (e.g., [16, 17]) and a friendship that continues up to now.

For his doctoral thesis, Roman considered another variant of scheduling problems with preemptive activities (jobs) to be scheduled on unrelated parallel machines (processors) with additional limited discrete resources. In this problem, the activity processing time depends on the type of machine and the amount of allocated resources. This type of resource requirements, initiated in the works of our team in late 70s, is called multiple job-modes and is presently a standard in RCPS problems. Roman proposed a two-phase method consisting of an LP problem for finding the minimum project processing time, and a polynomial-in-time algorithm for re-arranging the pieces of activities assigned to particular job-modes, so as to get a feasible schedule [21]. His procedure dealt, moreover, with minimization of the number of activity preemptions, as a secondary performance measure.

At the stage of preparation for the habilitation degree, which grants in Poland the right of advising doctoral students, Roman worked on scheduling problems involving arbitrary discrete resources. These were: renewable resources, like machines, processors, manpower; non-renewable resources, like energy, raw materials, money; and doubly constrained resources which involved constraints on availability in every moment and over a period of time, like electric power and energy. Consideration of all these resource categories in one model led him to multi-criteria scheduling problems, with competing time and cost criteria, and Roman became one of the pioneers in this research area [22]. His interest in multi-criteria decisions has also another origin-in November 1977, we participated together (at our own expense) in the Congress of Association Française pour la Cybernétique, Économique et Technique (AFCET) in Versailles, chaired by Bernard Roy-the founder of the European School of Multiple Criteria Decision Aiding (MCDA). Bernard listened to Roman's presentation in French and invited him to a discussion afterwards. Seeing Bernard's sympathetic interest in our research, we invited him and his wife Françoise to Poland, which was realized a year later. I was happy to observe that this meeting started a very fruitful collaboration between Roman and Bernard, which lasted 40 years and turned into a deep friendship including their families.

A turning moment in Roman's career was his six-month visit to the LAMSADE laboratory at Univeristé Paris Dauphine in 1980/81, led by Bernard Roy, where he learned about the MCDA methodology and applied it to scheduling problems [23]. In total, Roman spent five years at French universities, mainly at Paris Dauphine, where in 2001 he obtained a honorary doctorate.

A summary of the first fourteen years of our joint multidirectional research on scheduling under resource constraints has been included in the book [1] which, for many reasons, was a breakthrough for us. First of all, it was the first monograph in world literature defining such a wide spectrum of these problems and their mutual relations. Secondly, it showed our team as an important research center in which the synergy of research activities was achieved. Last but not least, this book appeared at the end of the first decade of the pontificate of Pope John Paul II, who, also because of the faith uniting us, has been for us a moral authority and an unattainable model of conduct. We expressed this by offering the Holy Father during one of the audiences a copy of our book with a dedication. It also seems to us that the book was the main argument for awarding three of its four authors with the EURO Gold Medal (Aachen, 1991), the first and so far the only one awarded to Poles.

Our joint research on RCPS problems evolved later to expert systems for scheduling under resource constraints and resulted in a system being an example of a rising interest in interactions between operations research and artificial intelligence [27]. This was our last joint publication. In 1989, when Roman received a professor title, we decided to establish separate laboratories. Roman's laboratory took the name "Intelligent Decision Support Systems" and opened a master degree with that name. Since then, Roman has followed his own scientific way, developing his wonderful doctoral school.

Still in eighties, Roman led a research project on water supply system programming in Poland, which raised his interest in fuzzy set modeling of subjective uncertainty [24]. He also applied the MCDA methodology in this project [20] that was implemented in practice.

Another milestone in the Roman's scientific trajectory was his meeting in mid-eighties with Prof. Zdzisław Pawlak-the founder of rough set theory. In 1992, Roman organized the First International Workshop on Rough Set Theory and Applications in Poznań, and in the same year, he edited the first handbook on rough sets [25]. He also brought this research topic to his collaborators in France [26]. Rough set theory has proven to be a very useful tool for preference learning from partially inconsistent examples of people's decisions. For some time he worked on this topic with Pawlak, publishing, e.g., the first paper on application of the rough set methodology to reasoning about medical data [19] and then a review on the rough set approach to multi-attribute decision analysis [18], however, a real burst of creativity in the latter topic came with Roman's new PhD student Salvatore Greco from Catania. Together with Benedetto Matarazzo, they have done pioneering works on handling jointly various aspects of imprecise, vague, ambiguous, and granular information in reasoning about ordinal data. This led them to development of, the so-called, Dominance-based Rough Set Approach (DRSA)-a new way of preference modeling using logical "if…, then…" decision rules [6]. This approach was applied to support multiple criteria decision, group decision, and decision under risk and uncertainty. For this breakthrough achievement Roman was awarded the 2005 Prize of the Foundation for Polish Science, considered as the most prestigious Polish scientific award.

DRSA is enriching the traditional way of modeling preferences in decision analysis. The traditional preference models have the form of either a utility function or a system of binary relations-their underlying theories are Multi-Attribute Utility Theory (MAUT) and Outranking Theory, respectively. These models often show a limited capacity of preference representation, they require an important cognitive effort on the part of the decision maker providing preference information necessary to build these models, and their decision recommendations are not easy to explain to the users. DRSA replaces the utility and outranking models by a set of monotonic decision rules induced from data structured using the dominance-based rough set concept. They constitute an intelligible preference model that is non-compensatory and able to represent interactions between the attributes. Rules identify values that drive decision maker's decisions-each rule is a scenario of a causal relationship between evaluations on a subset of attributes and a comprehensive judgment. It has been proved on the axiomatic level that the proposed decision rule preference model is the most general among all known models, so it has a greater capacity of preference representation than utility functions (including Choquet integral and Sugeno integral) and outranking relations [28]. Decision rules represent knowledge whose value can be assessed by Bayesian confirmation measures, as shown in

[7, 10]. It is worth underlining that DRSA has solid algebraic and topological foundations [9, 11] and it has also been interpreted in terms of empirical risk minimization typical for machine learning [14].

A good example of the practical use of rough set rules is the Mobile Emergency Triage (MET) system that aids physicians in making triage decisions in the emergency department of a hospital. The system's mobile component is designed to work on handheld computers. It has been tested at the Children's Hospital of Eastern Ontario in Ottawa [15].

Another original methodological proposal of Roman and Salvatore, working at this stage with their own PhD students, is the robust ordinal regression approach to constructive preference learning [4]. It assumes that a preference model of a given type (utility function, binary relation, or set of decision rules) is learned from preference information provided by decision makers in terms of decision examples. Since, in general, the number of instances of the supposed preference model compatible with the available preference information is infinite, the application of all compatible instances on the set of considered alternatives leads to necessary and possible preference relations. When the preference information is enriched, the necessary and possible preference relations become closer each other. This way of preference learning appeared to be particularly useful in interactive evolutionary multiobjective optimization for both continuous and combinatorial problems [2, 13]. It is also worth mentioning a very innovative application of the robust ordinal regression in case of a hierarchical structure of criteria [3], group decision [8], and decision under risk and uncertainty [5, 12].

The impact of Roman's publications is witnessed by a high number of citations. He also enjoys the best bibliometric score among all computer scientists working in Poland. What is, however, the most important, he is a man of flawless reputation in moral matters. In a common opinion, his research activity is characterized not only by technical proficiency, but also by creativity, vision and a sense of what is and will be important. This all has been combined with an extraordinary dynamism, scientific productivity, organizational proficiency and also a very skillful mentoring of the young generation of researchers. Roman has been the supervisor of 26 PhD students in the area of intelligent decision support, in which various tools and techniques of operations research and artificial intelligence have been developed and combined in an innovative way. 16 from among his former PhD students are now professors in academia and are renown scientists with their own research teams and doctoral students. How could I not be proud of my former student and one of my closest friends to see how wonderfully he uses the talents given to him?

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# Chapter 3
# ELECTRE Methods: A Survey on Roman Słowiński Contributions

**Salvatore Corrente, Josè Rui Figueira, and Salvatore Greco**

**Abstract** ELECTRE family is composed of a set of very well-known and appreciated multiple criteria decision aiding methods. The ELECTRE methods, proposed by Bernard Roy and his cooperators since the middle of 1960s, have been constantly improved and enriched during the years. In this context, Roman Słowiński has given many relevant contributions both from the methodological and the operational point of view. We present a reasoned survey of these contributions underlining their interest for real life applications.

## 3.1 Introduction

Roman Słowiński met for the first time Bernard Roy at the AFCET conference organized by the same Bernard Roy in Versailles on November 21–24, 1977. In that conference Bernard Roy wanted to speak with Roman Słowiński. Since then, until the death of Bernard Roy in 2017, the two researchers have continued to be in contact with a regular scientific cooperation and many occasions of personal meetings. After that, Roman Słowiński, invited by Bernard Roy, visited regularly LAMSADE, the laboratory that he founded at the Paris Dauphine University. At the beginning, Roman Słowiński planned to work with Bernard on scheduling problems. Indeed, scheduling, being the original specialization of Roman Słowiński,

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was a domain in which Bernard Roy gave some pioneer contributions. However, in the 70s Bernard Roy was very interested in Multiple Criteria Decision Aiding (MCDA) and in LAMSADE there were some of the most brilliant researchers in that domain such as Yannis Siskos and Eric Jacquet-Lagrèze. Roman started then to be interested in decision aiding and to collaborate with Bernard and his cooperators on this domain. The interest of Roman Słowiński for MCDA, and especially for the constructivist approach and the ELECTRE methods proposed by Bernard Roy, has been maintained during the years. Inspired by the thought and the reflections of Bernard Roy, Roman Słowiński has produced a relevant corpus of results in MCDA, contributing substantially to shape the current MCDA state of the art. The approach of Roman Słowiński to MCDA problems has been always motivated by the possible real world applications. In this perspective, he has always searched for that clarity and meaningfulness of the technical proposals that should permit the best comprehension of the proposed methods not only for the analyst but also and especially for the decision makers. Indeed, if according with the lesson of Bernard Roy MCDA should provide concepts, techniques, and instruments to reflect on the decision problem constructing a decision model with the cooperation of the decision maker, clarity and meaningfulness are unavoidable. It appears also that clarity and meaningfulness are the main ingredients that give their specific elegance to all the contributions of Roman Słowiński to MCDA.

Because of the above considerations, in this chapter we propose a survey of the scientific production of Roman Słowiński related to the ELECTRE methods, trying to highlight the above-mentioned interest for real world applications, clarity, meaningfulness, and elegance.

The chapter is structured as follows. Next section recalls the basic concepts of ELECTRE methods. The following section presents the most relevant contributions of Roman Słowiński to ELECTRE methods. The last section contains the conclusions.

## 3.2 A Brief Overview on ELECTRE Main Concepts

ELECTRE methods are based on the principles of the so-called European School of Multiple Criteria Decision Aiding (MCDA; [14, 32, 45]). In MCDA a set of actions $A = \{a, b, \ldots\}$ is evaluated on a coherent family of criteria $G = \{g_1, \ldots, g_m\}$ to deal with a decision making problem that could be of four different types: choice, ranking, sorting, or description [42]. Several ELECTRE methods have been developed during the years to deal with such type of problems (see [21, 22] for two surveys on ELECTRE methods and their application areas). However, independently on the type of problems they are applied to, the methods belonging to the ELECTRE family are based on outranking relations $S$ where an action $a$

outranks an action $b$ ($aSb$) iff $a$ is at least as good as $b$. The construction of such outranking relation $aSb$ is based on the fact the two different tests are satisfied:

- *Concordance test*: a majority of criteria in $G$ should be in favor of the outranking of $a$ over $b$,
- *Non-discordance test:* none of the criteria in $G$ should oppose too strongly to the outranking of $a$ over $b$.

Depending on the considered method, the two tests are performed in a different way. In the following, we will recall how the concordance index $C(a,b)$ and the partial discordance index $d_j(a,b)$ are built since most of the ELECTRE methods are based on their computation.

The construction of $C(a,b)$ is based on the following steps:

- For each criterion $g_j \in G$ and for each ordered pair of actions $(a,b) \in A \times A$, the *partial concordance index* $\varphi_j(a,b)$ is computed

$$\varphi_j(a,b) = \begin{cases} 1 & \text{if } g_j(b) - g_j(a) \leqslant q_j, \\ 0 & \text{if } g_j(b) - g_j(a) \geqslant p_j, \\ \frac{p_j - [g_j(b) - g_j(a)]}{p_j - q_j} & \text{if } q_j < g_j(b) - g_j(a) < p_j, \end{cases} \tag{3.1}$$

  where $q_j$ and $p_j$ denote, respectively, the indifference and the preference threshold on $g_j$. $q_j$ represents the maximum difference $g_j(b) - g_j(a)$ compatible with the indifference between $a$ and $b$ on $g_j$, while $p_j$ represents the minimum difference $g_j(b) - g_j(a)$ compatible with a strict preference of $b$ over $a$ on $g_j$ (more details on the meaning and definition of the thresholds can be found in [47]). In general, the Decision Maker (DM) has to define these thresholds only if he wishes to. Moreover, the thresholds should be dependent on the action presenting the lowest or the greatest evaluation on $g_j$ between $a$ and $b$. However, to simplify the description and without loss of generality, in the following, we shall assume that they are fixed for all pairs of actions. $\varphi_j(a,b)$ is a non-increasing function of $g_j(b) - g_j(a)$ and it represents how much $a$ is at least as good as $b$ on $g_j$. $\varphi_j(a,b) \in [0,1]$ and, the greater $\varphi_j(a,b)$, the more $g_j$ is in favor of the outranking of $a$ over $b$;
- For each ordered pair of actions $(a,b) \in A \times A$, the *concordance index* $C(a,b)$ is computed as follows

$$C(a,b) = \sum_{j=1}^{m} w_j \varphi_j(a,b). \tag{3.2}$$

  In the previous equation, $w_j$ is the weight of $g_j$ and it represents the importance of $g_j$ in the family of criteria $G$. Without loss of generality it is assumed that $w_j$'s are such that $w_j > 0$ for all $g_j \in G$ and $\sum_{j=1}^{m} w_j = 1$. $C(a,b) \in [0,1]$ and the greater $C(a, b)$, the greater the strength of the coalition of criteria in favor of the assertion that $a$ outranks $b$.

Regarding, instead, the *partial discordance index* $d_j(a, b)$, it is computed for each criterion $g_j$ as follows:

$$d_j(a,b) = \begin{cases} 1 & \text{if } g_j(b) - g_j(a) \geqslant v_j, \\ 0 & \text{if } g_j(b) - g_j(a) \leqslant p_j, \\ \frac{[g_j(b)-g_j(a)]-p_j}{v_j-p_j} & \text{if } p_j < g_j(b) - g_j(a) < v_j. \end{cases} \tag{3.3}$$

In the previous equation, $v_j$ is the veto threshold on $g_j$ and it represents the minimum difference $g_j(b) - g_j(a)$ incompatible with the outranking of $a$ over $b$. $d_j(a, b) \in [0, 1]$ and it is a non-decreasing function of $g_j(b)-g_j(a)$. The greater the $d_j(a, b)$, the more the $g_j$ is against the outranking of $a$ over $b$. As mentioned above, if $g_j(b) - g_j(a) \geqslant v_j$, then $a$ cannot outrank $b$ independently on their comparison with respect to the other criteria in $G$. Also in this case, the DM is not obliged to provide $v_j$ for all criteria but only to those he would assign this power.

The outranking relation $aSb$ is therefore built on the basis of the indices previously defined and the different methods differ on the way they define and exploit such relation. Two main outranking relations are used in the ELECTRE methods:

1. $aSb$ iff $C(a, b) \geqslant \lambda$ and $g_j(b) - g_j(a) \leqslant v_j, \forall g_j \in G$. This is the outranking relation used in the ELECTRE Iv method [40],
2. $aSb$ iff $\sigma(a, b) \geqslant \lambda$ where

$$\sigma(a,b) = C(a,b) \prod_{g_j : d_j(a,b) > C(a,b)} \frac{1 - d_j(a,b)}{1 - C(a,b)} \tag{3.4}$$

is the credibility index defining how much credible is the outranking of $a$ over $b$. This is the outranking relation used in ELECTRE III [41], ELECTRE Tri-B [55], ELECTRE Tri-C [2], and ELECTRE Tri-nC [3]. In particular, in ELECTRE III, the credibility indices are aggregated to build two complete rankings of the actions by a descending and an ascending distillation and, finally, a partial ranking of the considered actions is obtained as the intersection of the previous two rankings.

Let us observe that in both the outranking relations, $\lambda \in ]0.5, 1]$ is called *cutting level* and it represents, in 1., the minimum value of $C(a, b)$ necessary (but not sufficient) to have the outranking of $a$ over $b$, while in 2., it represents the minimum value of $\sigma(a, b)$ necessary and sufficient to have the same outranking.

## 3.3 Roman Słowinski Contributions

In this section we shall briefly present the papers containing contributions given by Roman Słowiński to ELECTRE methods.

### 3.3.1 *"Inferring an ELECTRE Tri Model from Assignment Examples" and "A User-Oriented Implementation of the ELECTRE-TRI Method Integrating Preference Elicitation Support"*

In sorting problems, a set of actions $A$ evaluated on a set of criteria $G$ have to be assigned to a set of categories $C_1, \ldots, C_p$ preferentially ordered with respect to the preferences of the DM (it is assumed that $C_1$ and $C_p$ are the categories containing the worst and the best actions, respectively). ELECTRE Tri-B [55] is the first ELECTRE method built to deal with sorting problems. In ELECTRE Tri-B categories are delimited by reference profiles $b_h$, $h = 0, \ldots, p$, so that $b_{h-1}$ and $b_h$ are, respectively, the lower and upper limits of category $C_h$. These profiles can be considered as actions and, therefore, they are evaluated on all criteria in $G$. Consequently, it is possible, on the one hand, to compute the degree of outranking of $a$ over $b_h$ and, on the other hand, the degree of outranking of $b_h$ over $a$ using the Eq. (3.4). Consequently, $aSb_h$ iff $\sigma(a, b_h) \geqslant \lambda$ and $a$ is preferred to $b_h$ ($a \succ b_h$) iff $aSb_h$ but $not(b_hSa)$. On the basis of the comparison of each action $a$ with the reference profiles $b_h$ two different assignment procedures are defined in ELECTRE Tri-B:

- *pessimistic procedure*: $a$ is assigned to category $C_k$ iff $aSb_{k-1}$ but $not(aSb_h)$, for all $h = k, \ldots, p$. $b_{k-1}$ is then the first reference profile (starting from above) outranked from $a$;
- *optimistic procedure*: $a$ is assigned to category $C_k$ iff $b_k \succ a$ and $not(b_h \succ a)$ for all $h = 1, \ldots, k-1$. $b_k$ is then the first reference profile (starting from below) preferred to $a$.

As it is evident from the previous description, the implementation of the ELECTRE Tri-B method implies the knowledge of several parameters: reference profiles, indifference, preference and veto thresholds, weights of criteria and cutting level. Asking the DM to provide directly all these values is quite demanding and, therefore, in [37] an optimization problem based on indirect preferences provided by the DM is developed to get a set of parameters compatible with them. The indirect preference information is composed of assignment examples of some reference actions belonging to $A^* \subseteq A$ as well as information on the importance of criteria. All the parameters mentioned above are assumed to be unknown of the optimization problem except the veto thresholds which is assumed are provided by the DM in accordance with the analyst. The assignment is supposed to be performed using the pessimistic procedure.

The non-linear optimization problem to be solved is the following:

$$\left.\begin{array}{l}
\max\left(\alpha+\varepsilon\sum\limits_{a_k\in A^*}(x_k+y_k)\right),\ \text{subject to,}\\
\alpha\leqslant x_k, \forall a_k\in A^*,\\
\alpha\leqslant y_k, \forall a_k\in A^*,\\
\left.\begin{array}{l}
\dfrac{\sum\limits_{j=1}^{m} w_j\varphi_j(a_k,b_{h_k-1})}{\sum\limits_{j=1}^{m} w_j}-x_k=\lambda,\\
\dfrac{\sum\limits_{j=1}^{m} w_j\varphi_j(a_k,b_{h_k})}{\sum\limits_{j=1}^{m} w_j}+y_k=\lambda,
\end{array}\right\}\text{if } a_k\in A^* \text{ is assigned to } C_{h_k},\\
\lambda\in]0.5,1],\\
g_j(b_{h+1})\geqslant g_j(b_h)+p_j(b_h)+p_j(b_{h+1}),\ \forall g_j\in G,\ \forall h=1,\ldots,p-1,\\
p_j(b_h)\geqslant q_j(b_h),\ \forall g_j\in G,\ \forall h=1,\ldots,p-1,\\
w_j\geqslant 0,\ q_j(b_h)\geqslant 0,\ \forall g_j\in G,\ \forall h=1,\ldots,p-1,
\end{array}\right\}$$

where:

- $\varepsilon$ is a small positive number,
- $x_k$ and $y_k$ are two fictitious variables used to translate the assignment of $a_k$ to $C_{h_k}$ and, in particular, the fact that $\sigma(a,b_{h_k-1})\geqslant\lambda$ and $\sigma(a,b_{h_k})<\lambda$.

Because $\varphi_j(a_k,b_h)$ is non differentiable, it is approximated by the sigmoidal function

$$f(x)=\frac{1}{1+e^{\left[\frac{-5.55}{p_j(b_h)-q_j(b_h)}\cdot\left(g_j(a_k)-g_j(b_h)+\frac{p_j(b_h)+q_j(b_h)}{2}\right)\right]}}$$

being a function of the evaluation of reference profiles $(g_j(b_h))$, as well as indifference and preference thresholds $\big(q_j(b_h),p_j(b_h)\big)$.

If, solving the previous optimization problem, one gets $\alpha > 0$, then there exists at least one set of parameters compatible with the reference assignments provided by the DM, while, in the opposite case, it is possible to check which are the troublesome assignments and, therefore, revising them with the DM.

The previous procedure is implemented in the ELECTRE Tri Assistant software that is presented and described in detail in [38].

### 3.3.2 Searching for an Equivalence Between Decision Rules and Concordance-Discordance Preference Model in Multicriteria Choice Problems

Reference [26] proposes a procedure to induce weights and veto thresholds for defining a concordance relation on the basis of "if ..., then ..." decision rules obtained through the application of Dominance-based Rough Set Approach (DRSA) [24] on a set of examples of decisions represented in a Pairwise Comparison Table (PCT [23]). More precisely, the DM is proposed a certain number of pairs of actions $(a, b) \in A \times A$ and he is required to express his preferences by selecting one of the three following possible answers

- $a$ outranks $b$, that is $a$ is at least as good as $b$, denoted by $aSb$,
- $a$ does not outrank $b$, that is $a$ is not at least as good as $b$, denoted by $aS^cb$,
- $a$ and $b$ are not comparable.

The PCT has the form of a matrix with a row for each pair of actions $(a, b) \in A \times A$ for which the DM declared $aSb$ or $aS^cb$ and a column for each criterion $g_j \in G$ and a further column with the judgment provided by the DM. Each row of the PCT contains the following information related to a pair $(a, b) \in A \times A$:

- for each criterion $g_j \in G$, the value

$$T_j(a,b) = \begin{cases} 0 & \text{if } g_j(a) \geqslant g_j(b) - q_j(a), \\ g_j(a) - g_j(b) & \text{otherwise} \end{cases} \tag{3.5}$$

  with $q_j(a)$ being an indifference threshold. Observe that

  – $T_j(a, b) = 0$ means that $a$ outranks $b$ with respect to criterion $g_j$, that is $a$ is at least as good as $b$ with respect to $g_j$,
  – $T_j(a, b) < 0$ means that $a$ does not outrank $b$ on $g_j$, with the smaller $T_j(a, b)$ the more $g_j$ is against the outranking of $a$ over $b$,

- the overall preference comparison expressed by the DM, that is, $aSb$ or $aS^cb$.

By means of DRSA, the judgments $aSb$ and $aS^cb$ can be explained with a set of decision rules having the following syntax:

- $D_{\geqslant}-$rule: if $T_{j_1}(a,b) = 0$ and ... and $T_{j_p}(a,b) = 0$ and ... and $T_{j_{p+1}}(a,b) \geqslant t_{j_{p+1}}$ and ... $T_{j_q}(a,b) \geqslant t_{j_q}$, then $aSb$, with $P = \{g_{j_1}, \ldots, g_{j_p}\} \subseteq G$, $R = \{g_{j_{p+1}}, \ldots, g_{j_q}\} \subseteq G$, $P \cap R = \emptyset$,
- $D_{\leqslant}-$rule: if $T_{j_1}(a,b) \leqslant t_{j_1}$ and ... $T_{j_p}(a,b) \leqslant t_{j_p}$, then $aS^cb$, with $P = \{g_{j_1}, \ldots, g_{j_p}\} \subseteq G$.

The above $D_{\geqslant}$ rules and $D_{\leqslant}$ rules are interpreted in the terms of the outranking relation of the ELECTRE Iv method (see Sect. 3.2).

The decision rules suggest some constraints on the value of weights $w_j$, $j = 1, \ldots, m$, the cutting level $\lambda$ and the veto thresholds $v_j$, $j = 1, \ldots, m$. In particular

- from the $D_{\geqslant}-$rule "if $T_{j_1}(a,b) = 0$ and ... and $T_{j_p}(a,b) = 0$ and ... and $T_{j_{p+1}}(a,b) \geqslant t_{j_{p+1}}$ and ... and $T_{j_q}(a,b) \geqslant t_{j_q}$, then $aSb$," the following constraints can be obtained
  - $w_{j_1} + \ldots + w_{j_p} \geqslant \lambda$,
  - $v_{j_{p+1}} > -t_{j_{p+1}}, \ldots, v_{j_q} > -t_{j_q}$;
- from the $D_{\leqslant}-$rule "if $T_{j_1}(a,b) \leqslant t_{j_1}$ and ... $T_{j_p}(a,b) \leqslant t_{j_p}$, then $aS^cb$," the following constraints can be obtained
  - $\sum_{j:g_j \notin P} w_j \leqslant \lambda$, with $P = \{g_{j_1}, \ldots, g_{j_p}\}$,
  - $v_{j_{p+1}} \leqslant -t_{j_1}, \ldots, v_{j_q} \leqslant -t_{j_q}$.

The constraints obtained through the decision rules can be used to define the parameters of the considered ELECTRE method in an ordinal regression approach. The advantage of using the decision rules is related to their understandability due to the natural language in which they are expressed, on the one hand, and to the possibility to trace back to the DM's judgments from which the decision rules were induced, on the other hand. Taking into account the importance of an interaction with the DM in the co-construction of the decision model [43], the use of decision rules within an ordinal regression approach seems very beneficial.

### 3.3.3 Axiomatization of Utility, Outranking, and Decision Rule Preference Models for Multiple Criteria Classification Problems Under Partial Inconsistency with the Dominance Principle

The paper [50] presents a very general axiomatic foundation of multiple criteria sorting methods, showing how sorting approaches based on value functions [56], sorting approaches based on outranking methods and sorting approaches based on "if ..., then ..." decision rules [24] have a common theoretical basis and can be considered equivalent, in the sense that each of the three models can be reformulated in terms of the other two. The theoretical frame of this general result is the following.

Consider a product space $X = \prod_{j=1}^{m} X_j$, with $X_j$, $j = 1, \ldots, m$ being a finite or a countable set representing the evaluation scale of criterion $g_j$, that is the set of evaluations that can be taken by criterion $g_j$. Let $(x_j, z_{-j})$, $x_j \in X_j$, $z \in X$, denote one element of $X$ being equal to $z$ except to its $j-th$ component which is equal to $x_j$. Consider also the family of sets $\mathbf{Cl} = \{Cl_1, \ldots, Cl_n\}$ being a partition of $X$, that is $\emptyset \subset Cl_r \subset X, r = 1, \ldots, n$, $Cl_r \cap Cl_s = \emptyset$ for all $r, s = 1, \ldots, n$ and $\bigcup_{r=1}^{n} Cl_r = X$. The sets $Cl_r, r = 1 \ldots, n$, represent increasingly ordered classes so that for each $x \in Cl_r$ and $y \in Cl_s$ with $r > s$, $x$ has a better overall evaluation of $y$. The sets $Cl_t^{\geqslant} = \bigcup_{r=t}^{n} Cl_r$, $t = 1, \ldots, n$ are called upward union of classes, while the sets $Cl_t^{\leqslant} = \bigcup_{r=1}^{t} Cl_r$, $t = 1, \ldots, n$ are called downward union of classes. The basic result proposed by the article is the following Theorem.

**Theorem** The following four propositions are equivalent

1. (Basic axiom) for each $j = 1, \ldots, m$, for each $x_j, y_j \in X_j$, $a, b \in X$, $r, s \in \{1, \ldots, n\}$

$$\left[(x_j, a_{-j}) \in Cl_r \text{ and } (y_j, b_{-j}) \in Cl_s\right] \Rightarrow \left[(y_j, a_{-j}) \in Cl_r^{\geqslant} \text{ or } (x_j, b_{-j}) \in Cl_s^{\geqslant}\right],$$

2. (Value function-based sorting) there exist
   - functions $g_j : X_j \to \mathbb{R}$ for each $j = 1, \ldots, m$,
   - a function $U : \mathbb{R}^m \to \mathbb{R}$ non-decreasing in each argument,
   - $n - 1$ thresholds $z_2 < \ldots < z_n$

   such that, for each $x \in X$ and for each $t = 2, \ldots, n$

$$x \in Cl_t^{\geqslant} \Leftrightarrow U(g_1(x_1), \ldots, g_m(x_m)) \geqslant z_t,$$

3. (Outranking-based sorting) there exist
   - functions $g_j : X_j \to \mathbb{R}$ for each $j = 1, \ldots, m$,
   - a function $S : \mathbb{R}^{2m} \to \mathbb{R}$ non-decreasing in each first $m$ arguments and non-increasing in each other $m$ arguments,
   - $n - 1$ reference profiles $p^2, \ldots, p^n \in X$

   such that, for each $x \in X$ and for each $t = 2, \ldots, n$

$$x \in Cl_t^{\geqslant} \Leftrightarrow S(g_1(x_1), \ldots, g_m(x_m), g_1(p_1^t), \ldots, g_m(p_m^t)) \geqslant 0,$$

4. (Decision rule-based sorting) there exist
   - functions $g_j : X_j \to \mathbb{R}$ for each $j = 1, \ldots, m$,
   - a set of "decision rules" having a syntax
     "if $g_{j_1}(x) \geqslant r_{j_1}$ and $\ldots g_{j_p}(x) \geqslant r_{j_p}$, then $x \in Cl_t^{\geqslant}$"
     with $\{j_1, \ldots, j_p\} \subseteq \{1, \ldots, m\}$, $t = 2, \ldots, n$,
     such that for each $y \in Cl_t$, $t = 2, \ldots, n$, there is at least one decision rule for which $y \in Cl_t^{\geqslant}$ and there is no decision rule for which $y \in Cl_r^{\geqslant}$ with $r > t$.

The relation between the outranking-based sorting in point 3. and the ELECTRE Tri-B method (pessimistic assignment) can be seen by taking $S(g_1(x_1), \ldots, g_m(x_m),$
$g_1(p_1^t) \ldots, g_m(p_m^t)) = \sigma(x, p^t) - \lambda$ so that we get

$$x \in Cl_t^{\geqslant} \Leftrightarrow S(g_1(x_1), \ldots, g_m(x_m), g_1(p_1^t), \ldots, g_m(p_m^t)) \geqslant 0 \Leftrightarrow \sigma(x, p^t) - \lambda \geqslant 0.$$

Let us observe that an axiomatic foundation of ELECTRE Tri-B method was provided in [4] using an axiom that, interestingly, was proposed in the same paper [50] to characterize a sorting method based on the Sugeno integral [52]. Observe also that an axiomatic foundation of ELECTRE outranking relation was provided in [25] (see also Section 4.3 in [21]). Let us remind that a different axiomatic foundation of ELECTRE outranking relation was provided by Denis Bouyssou and Marc Pirlot in a series of papers [5–7] (for a comparison between the two axiomatic foundations see Section 4.3 in [21]).

### *3.3.4 Handling Effects of Reinforced Preference and Counter-Veto in Credibility of Outranking*

In [44] Bernard Roy and Roman Słowiński propose the use of two new thresholds, namely reinforced preference threshold and counter-veto threshold in outranking methods and, in particular in ELECTRE ones. The two thresholds, denoted by $rp_j$ and $cv_j$, respectively, are not defined for all criteria $g_j$ but only for some of them. They should be used if:

- the DM wants to give a greater weight to $g_j$ in the computation of $C(a, b)$ iff $g_j(a) - g_j(b) \geqslant rp_j$, that is, if the difference is particularly high. In this case, the weight of $g_j$ is not equal to $w_j$ but to $\omega_j w_j$ where $\omega_j > 1$,
- the DM wants to decrease the veto effect of some criterion $g_j$ even if $g_j(b) - g_j(a) > p_j$.

The two thresholds are such that $p_j < rp_j < v_j$ and, analogously, $p_j < cv_j < v_j$. However, no specific inequality between $rp_j$ and $cv_j$ exists.

To better understand how the concordance and credibility indices have to be modified to take into account the reinforced preference threshold and the counter-veto threshold, let us underline that the concordance index in Eq. (3.2) can be rewritten as follows:

$$C(a,b) = \frac{\sum\limits_{g_j \in C(aSb)} w_j + \sum\limits_{g_j \in C(bQa)} \eta_j w_j}{\sum\limits_{g_j \in G} w_j},$$

where $C(aSb) = \{g_j \in G : g_j(b) - g_j(a) \leqslant q_j, \}$, $C(bQa) = \{g_j \in G : q_j < g_j(b) - g_j(a) \leqslant p_j\}$ and $\eta_j = \frac{p_j - [g_j(b) - g_j(a)]}{p_j - q_j}$.

Then the consideration of the two thresholds mentioned above implies the following modifications in the definitions given above:

1. $C(a,b) = \frac{\sum\limits_{g_j \in C(aRPb)} \omega_j w_j + \sum\limits_{g_j \in C(aSb) - C(aRPb)} w_j + \sum\limits_{g_j \in C(bQa)} \eta_j w_j}{\sum\limits_{g_j \in C(aRPb)} \omega_j w_j + \sum\limits_{g_j \in G - C(aRPb)} w_j},$
   where $C(aRPb) = \{g_j \in G : g_j(a) - g_j(b) \geqslant rp_j\}$,
2. $d_j(a,b) = \begin{cases} 1 & \text{if } g_j(b) - g_j(a) \geqslant v_j \\ 0 & \text{if } g_j(b) - g_j(a) \leqslant cv_j \\ \frac{[g_j(b) - g_j(a)] - cv_j}{v_j - cv_j} & \text{if } cv_j < g_j(b) - g_j(a) < v_j \end{cases},$
3. $\sigma(a,b) = C(a,b) \left[ \prod\limits_{g_j \in G : d_j(a,b) > C(a,b)} \frac{1 - d_j(a,b)}{1 - C(a,b)} \right]^{\left(1 - \frac{k}{m}\right)},$
   where $k = |\{g_j \in G : g_j(b) - g_j(a) > cv_j\}|$.

### 3.3.5 *ELECTRE$^{GKMS}$: Robust Ordinal Regression for Outranking Methods*

As already shown in the introductory section, the application of all ELECTRE methods implies the knowledge of some parameters: discriminating thresholds, weights, and cutting level. To get them, a direct or an indirect preference information can be provided by the DM. In the first, the DM is asked to provide directly values of the parameters involved in the method, while, in the second, the DM has to provide some preference information in terms of comparison between actions or comparison between criteria in terms of their importance so that parameters compatible with these preferences can be inferred and, therefore, applied to get a final recommendation on the problem at hand. In recent years, the indirect technique is the preferred between the two because it requires less cognitive effort to the DM [33]. However, in this case, more than one set of parameters could be compatible with the preferences provided by the DM. Even if the application of the ELECTRE methods with each of these sets of compatible parameters would give the same recommendations on the actions used as reference examples from the DM, the same application would give different recommendations on the other actions. For such a reason, it is meaningful providing information on the problem at hand taking into account not only one but the whole set of instances of the assumed preference model compatible with the preferences given by the DM (*compatible models* in the following). This gave arise to the Robust Ordinal Regression (ROR) presented for the first time in [27]. ROR defines a necessary and a possible preference relation on $A$ on the basis of the whole set of compatible models. On the one hand, $a$ is necessarily preferred to $b$ iff $a$ is at least as good as $b$ for all compatible models, while, on the other hand, $a$ is possibly preferred to $b$ iff $a$ is at least as good as $b$ for at least one of them [10, 28].

In [30], ROR has been applied to the ELECTRE methods for the first time giving arise to the ELECTRE$^{GKMS}$ method. Going a bit more in detail, the DM has the opportunity to express the following pieces of preference information:

- $a$ outranks $b$ (we shall write $aS_{DM}b$) or $a$ does not outrank $b$ $\left(aS^{C}_{DM}b\right)$,
- $q_j \in \left[q_{j,*}, q^*_j\right]$ where $q_{j,*}$ and $q^*_j$ represent the minimum and the maximum value that the indifference threshold $q_j$ can get,
- $p_j \in \left[p_{j,*}, p^*_j\right]$ where $p_{j,*}$ and $p^*_j$ represent the minimum and the maximum value that the preference threshold $p_j$ can get,
- the difference between $g_j(a)$ and $g_j(b)$ is not significant for the DM ($a \sim_j b$),
- the difference between $g_j(a)$ and $g_j(b)$ is significant for the DM ($a \succ_j b$).

In particular, if for a criterion $g_j$, the DM is not able to provide the interval of variation of the discriminating thresholds $q_j$ and $p_j$, he is asked to give at least one pair of actions $(a, b)$ such that $a \sim_j b$ and at least one pair of actions $(c, d)$ such that $c \succ_j d$.

An important novelty of the ELECTRE$^{GKMS}$ is that the concordance index $C(a, b)$ is such that

$$C(a,b) = \sum_{j=1}^{m} \Psi_j(a,b),$$

where $\Psi_j(a, b)$ is not a piecewise function as $\varphi_j(a, b)$ (see (3.1)) but a non-increasing function of $g_j(b) - g_j(a)$ such that $\Psi_j(a, b) = w_j$ if $g_j(b) - g_j(a) \leqslant q_j$ and $\Psi_j(a, b) = 0$ if $g_j(b) - g_j(a) \geqslant p_j$.

The outranking relation used in the ELECTRE$^{GKMS}$ method is the one defined in the ELECTRE Iv method and briefly reviewed in Sect. 3.2. A compatible model is therefore a vector composed of $C(a, b)$, $\lambda$ and $\Psi_j(a, b)$, $q_j$, $p_j$, and $v_j$ for all

$g_j \in G$ and all $(a, b) \in A \times A$. To check for the existence of at least one compatible model a specific MILP problem is solved (see [30]).

Regarding the application of the ROR, from the output point of view, four different binary relations are computed:

- $aS^N b$ iff $a$ outranks $b$ for all compatible models,
- $aS^P b$ iff $a$ outranks $b$ for at least one compatible model,
- $aS^{CN} b$ iff $a$ does not outrank $b$ for all compatible models,
- $aS^{CP} b$ iff $a$ does not outrank $b$ for at least one compatible model.

Also in this case the four binary relations are computed by solving some specific MILP problems (see [30]).

### 3.3.6 Multiple Criteria Hierarchy Process with ELECTRE and PROMETHEE

In real world problems, all evaluation criteria are not at the same level but instead, they are structured in a hierarchical way. It is therefore possible to underline a root criterion being the main objective of the considered problem, some macro-criteria representing the main indicators used to deal with the decision problem, until the elementary criteria on which the actions are evaluated and being placed at the bottom of the hierarchy of criteria. To deal with such problems [8] presents the Multiple Criteria Hierarchy Process (MCHP). The main idea behind the MCHP introduction is that decision problems can be decomposed in several pieces that can be coped and dealt in a more detailed way. The introduction of the MCHP permits to provide and obtain information not only at comprehensive level, that is, considering all criteria simultaneously, but also at partial one, focusing, therefore, on some particular aspects of the considered problem.

In [9] the MCHP is applied to the ELECTRE methods defining in consequence a hierarchical ELECTRE method. To describe the hierarchical ELECTRE method, we shall only briefly recall the nomenclature used in the MCHP and that will be useful for the description of what the authors proposed in [9]:

- $g_{\mathbf{t}}$ denotes an elementary criterion, that is, a criterion at the bottom of the hierarchy and on which actions are evaluated;
- $EL$ is the set of the indices of the elementary criteria, while $LBO$ is the set of the indices of the criteria placed at the last but one level of the hierarchy;
- $g_{\mathbf{r}}$ is a generic criterion in the hierarchy; in particular, $g_{\mathbf{0}}$ denotes the root criterion;
- given criterion $g_{\mathbf{r}}$, $E(g_{\mathbf{r}}) \subseteq EL$ is the set of the indices of elementary criteria descending from $g_{\mathbf{r}}$, while $LBO(g_{\mathbf{r}})$ is the set of the indices of the criteria descending from $g_{\mathbf{r}}$ placed at the last but one level of the hierarchy.

The generalization of the ELECTRE methods to the MCHP is based on the definition of an outranking relation $S_{\mathbf{r}}$ for each non-elementary criterion $g_{\mathbf{r}}$ in the hierarchy. Since, as observed in Sect. 3.2, the definition of this relation is based on the computation of the concordance index between actions, we have to underline the main changes necessary to this implementation. At first, the partial concordance index $\varphi_{\mathbf{t}}(a,b)$ in Eq. (3.1) and the partial discordance index $d_{\mathbf{t}}(a,b)$ in Eq. (3.3) need to be defined for each elementary criterion $g_{\mathbf{t}}$ and for each ordered pair of actions $(a,b) \in A \times A$. Then, a partial concordance index with respect to $g_{\mathbf{r}}$ is computed as follows:

$$C_{\mathbf{r}}(a,b) = \sum_{\mathbf{t} \in E(g_{\mathbf{r}})} w_{\mathbf{t}} \varphi_{\mathbf{t}}(a,b).$$

With respect to the parameters involved in the application of the ELECTRE methods we have to underline the following points:

- the indifference, preference, and veto thresholds are defined (if the DM wishes to do it) for the elementary criteria,
- the weights $w_{\mathbf{t}}$ have to be defined for the elementary criteria only. Moreover, the weight of each non-elementary criterion $g_{\mathbf{r}}$, that is $W_{\mathbf{r}}$, is given by the sum of the weights of the elementary criteria descending from $g_{\mathbf{r}}$:

$$W_{\mathbf{r}} = \sum_{\mathbf{t} \in E(g_{\mathbf{r}})} w_{\mathbf{t}},$$

- a cutting level $\lambda_{\mathbf{l}}$ has to be defined for each non-elementary criterion placed at the last but one level of the hierarchy. Then, the cutting level $\lambda_{\mathbf{r}}$ of whichever non-elementary criterion $g_{\mathbf{r}}$ in the hierarchy is given by the sum of the cutting levels of the last but one level criteria descending from $g_{\mathbf{r}}$, that is,

$$\lambda_{\mathbf{r}} = \sum_{\mathbf{l} \in LBO(g_{\mathbf{r}})} \lambda_{\mathbf{l}}.$$

In conclusion, the technical parameters have to be defined for the elementary criteria and for the last but one level criteria only. After that, the ELECTRE I and ELECTRE III methods have been extended to the MCHP defining an outranking relation $S_{\mathbf{r}}$ for each non-elementary criterion in the hierarchy.

Analogously, the ELECTRE$^{GKMS}$ recalled in Sect. 3.3.5 has been extended to the MCHP providing, therefore, a necessary $S_{\mathbf{r}}^N$ and a possible $S_{\mathbf{r}}^P$ outranking relation for each non-elementary criterion $g_{\mathbf{r}}$ so that:

- $a S_{\mathbf{r}}^N b$ iff $a S_{\mathbf{r}} b$ for all compatible models,
- $a S_{\mathbf{r}}^P b$ iff $a S_{\mathbf{r}} b$ for at least one compatible model.

To compute $a S_{\mathbf{r}}^N b$ and $a S_{\mathbf{r}}^P b$, the LP problems presented in [30] need to be slightly modified to adapt them to the MCHP context (see [9]).

### *3.3.7 Multiple Criteria Hierarchy Process for ELECTRE Tri Methods*

Three are the main contributions in [12]:

1. A formal definition of the steps involved in the SRF method application;
2. An extension of the SRF method to get the weights of criteria in case they are structured in a hierarchical way;
3. The generalization of the ELECTRE Tri-B, ELECTRE Tri-C, and ELECTRE Tri-nC methods to the MCHP, taking also into account interacting effects between criteria.

As already underlined more than once before, the application of all ELECTRE methods (apart from the ELECTRE IV) implies the knowledge of the weights of criteria $w_j$ for each criterion $g_j$. To get them, several methods have been proposed in literature but the most applied is the SRF (Simos-Roy-Figueira) method [19] being a generalization of the Simos method [48, 49]. In the SRF method, the DM is provided with $m$ cards, one for each criterion, presenting their characteristics, and with several blank cards that should be used to increase the difference of importance between considered criteria. The DM is then asked to:

(Step 1) Rank order the criteria from the least important $L_1$ to the most important $L_v$ with the possibility of some ex-aequo. This means that criteria in $L_1$ are less important than criteria in $L_2$ and so on until criteria in $L_{v-1}$ being less important than criteria in $L_v$. In the following, by $w_s$ we shall denote the weight of a criterion in the set $L_s$,

(Step 2) Put (possibly) some blank cards between two successive subsets of criteria ($L_s$ and $L_{s+1}$) in order to increase the difference between the importance of criteria in $L_s$ and the importance of criteria in $L_{s+1}$. The greater the number of blank cards between two successive subsets of criteria, the higher the difference between the importance of criteria contained in these sets. Pay attention to the fact that if the DM does not put any blank card between two successive sets of criteria, this does not mean that criteria in these sets have the same importance but that this difference is minimal. In the following, by $e_s$ we shall denote the number of blank cards the DM put between $L_s$ and $L_{s+1}$ (see [15] for an extension of the SRF method in which the DM has the possibility to provide the number of blank cards not only between successive subsets of criteria but between whichever pair of subsets of criteria),

(Step 3) Provide the ratio $z$ between the weight of the most important criteria (those in $L_v$) and the weight of the least important one (those in $L_1$):

$$z = \frac{w_v}{w_1}$$

(see [1] for a recent improvement of the SRF method in which the $z$ value can be provided in a more intelligible way for the DM).

On the basis of the previous preference information, for each criterion $g_j$, the following non-normalized weight $w'_j$ can be computed:

$$w'_j = 1 + \frac{(z-1)\left[l(j) - 1 + \sum_{s=1}^{l(j)-1} e_s\right]}{v - 1 + \sum_{s=1}^{v-1} e_s},$$

where $l(j)$ represents the index of the subset to which $g_j$ belongs, that is, $g_j \in L_{l(j)}$ and $v$ is the number of levels.

The normalized weight of criterion $g_j$, denoted by $w_j$, is then obtained as

$$w_j = \frac{w'_j}{\sum_{i=1}^{m} w'_i}.$$

In case the criteria are structured in a hierarchical way, the SRF can be applied as well on the basis of what has been already presented above. As first step, the SRF is applied to the first level criteria $g_{\mathbf{1}}, \ldots, g_{\mathbf{n}}$, obtaining their weights $w_{\mathbf{1}}, \ldots, w_{\mathbf{n}}$. After that, for each non-elementary criterion $g_{\mathbf{r}}$, the DM has to apply the SRF method to the subset of criteria $\{g_{(\mathbf{r},1)}, \ldots, g_{(\mathbf{r},n(\mathbf{r}))}\}$ being composed of the subcriteria of $g_{\mathbf{r}}$ placed at the level immediately below it, obtaining their weights $w^*_{(\mathbf{r},1)}, \ldots, w^*_{(\mathbf{r},n(\mathbf{r}))}$ that, multiplied by $w_{\mathbf{r}}$, that is the weight of the criterion from which $g_{(\mathbf{r},1)}, \ldots, g_{(\mathbf{r},n(\mathbf{r}))}$ immediately descend, give the final weights $w_{(\mathbf{r},s)} = w^*_{(\mathbf{r},s)} \cdot w_{\mathbf{r}}$.

Another important contribution of the paper is due to the extension of the ELECTRE Tri methods to the hierarchical case considering the possible interacting effects between elementary criteria. Following [20], the concordance index $C(a, b)$ and, consequently, the partial concordance indices $C_{\mathbf{r}}(a, b)$ for each non-elementary criterion $g_{\mathbf{r}}$, can be redefined to take into account the mutual-weakening effects, the mutual-strengthening effects or the antagonistic effects between criteria in case the set of criteria is not mutually preferentially independent [35, 54].

### 3.3.8 A Robust Ranking Method Extending ELECTRE III to Hierarchy of Interacting Criteria, Imprecise Weights and Stochastic Analysis

As already presented in Sect. 3.3.7, the SRF method together with the generalization to the MCHP proposed by [12] requires a precise value regarding the number of blank cards between successive subsets of criteria as well as regarding the $z$ value. In [13] the authors extended the SRF method and the hierarchical SRF method to take into account an imprecise preference information provided by the DM. To apply the hierarchical and imprecise SRF method the DM is then asked to:

(Step 1) Rank order the criteria from the least important $L_1$ to the most important $L_v$ with the possibility of some ex-aequo;

(Step 2) Provide (possibly) an imprecise number of blank cards between two successive subsets of criteria $L_s$ and $L_{s+1}$, with $s = 1, \ldots, v-1$. Denoting by $e_s$ the number of blank cards to be put between $L_s$ and $L_{s+1}$, $e_s \in [low_s, upp_s]$ where $low_s$ and $upp_s$ represent the minimum and maximum number of blank cards that could be put between $L_s$ and $L_{s+1}$;

(Step 3) Provide an interval of possible values for the $z$ ratio, that is $z \in \left[z_{low}, z_{upp}\right]$.

Denoting by $w_s$ the weight of a criterion in $L_s$, the set of constraints translating the preferences given by the DM is the following:

$$E^{DM} \begin{cases} \left.\begin{array}{l} w_{s+1} \geqslant w_s + (low_s + 1) \cdot C \\ w_{s+1} \leqslant w_s + (upp_s + 1) \cdot C \end{array}\right\} \text{for all } s = 1, \ldots, v-1, \\ z_{low} \cdot w_1 \leqslant w_v, \\ w_v \leqslant z_{upp} \cdot w_1, \\ w_1 > 0, \\ C > 0, \\ \displaystyle\sum_{s=1}^{v} |L_s| \cdot w_s = 1. \end{cases}$$

Of course, the fact that the DM can provide a preference information in a simpler way is counterbalanced by the plurality of models (weights vectors in this case) that are compatible with the same preference. To take into account all of them, the Stochastic Multicriteria Acceptability Analysis (SMAA; [36, 39]) is applied. SMAA explores the whole space of compatible models providing recommendations to the considered problem in probabilistic terms. The application of SMAA starts with the sampling of several instances of the preference model compatible with the information provided by the DM (in this case, since the linear constraints in $E^{DM}$ define a convex set of vectors of weights, the Hit-And-Run method [51, 53] can be applied). After that, the hierarchical ELECTRE III method is applied for each of these vectors of weights. Then, for each non-elementary criterion $g_{\mathbf{r}}$, the probability with which an action $a$ is preferred, incomparable or indifferent to another action $b$ is provided to the DM.

### 3.3.9 ELECTRE-III-H: An Outranking-Based Decision Aiding Method for Hierarchically Structured Criteria

Another approach to deal with decision problems presenting a hierarchical structure of criteria using the ELECTRE III method has been proposed by [17] extending [16], namely the ELECTRE-III-H method. However, some differences can be observed between the way the hierarchy is dealt in this paper in comparison with [9].

While both methods build a partial preorder at the root level as well as at the macro-criteria level, the way this partial preorder at the intermediate level is built is different in the two methods. Indeed, in [9], as shown in Sect. 3.3.6, the partial preorder of the actions on an intermediate criterion $g_{\mathbf{r}}$ is built taking into account the evaluations of the actions on the elementary criteria descending from $g_{\mathbf{r}}$ only. In [17], instead, the procedure starts from the bottom building the partial preorder of the actions on criteria $g_{\mathbf{r}}$ having only elementary criteria as subcriteria using the classical ELECTRE III method. After that, since the actions are not evaluated on intermediate criteria $g_{\mathbf{r}}$, a concordance index $C_{\mathbf{r}}(a, b)$ and a discordance index $d_{\mathbf{r}}(a, b)$ are defined in an appropriate way on the basis of the partial preorder on $g_{\mathbf{r}}$ obtained above. That is, the value assigned to $C_{\mathbf{r}}(a, b)$ and $d_{\mathbf{r}}(a, b)$ depends on the fact that $a$ is preferred, indifferent or incomparable to $b$ on $g_{\mathbf{r}}$ or that $b$ is preferred to $a$ on $g_{\mathbf{r}}$ and from the difference between the number of actions preferred to $a$ and the number of actions preferred to $b$ on $g_{\mathbf{r}}$ in the same preorder.

The other difference is that, while in [9] the weights are defined for elementary criteria only and, then, the weight of an intermediate criterion $g_{\mathbf{r}}$ is equal to the sum of the weights of the elementary criteria descending from $g_{\mathbf{r}}$, in [17] for each intermediate criterion $g_{\mathbf{r}}$, the sum of the weights of the criteria directly descending from it is equal to one implying that a greater number of variables needs to be known to implement the method.

### 3.3.10 Other Contributions

As described in Sect. 3.3.5 ELECTRE$^{GKMS}$ implements the ROR concepts providing recommendations on the set of actions by taking into account the whole set of models compatible with the preference information provided by the DM. However, in some real world context, it is necessary to provide a single ranking or sorting of the considered actions. For such a reason, to summarize the results obtained by ROR taking into account all compatible models, [34] propose a procedure to compute the most representative set of outranking parameters based on the preference information given by the DM in indirect terms and the necessary and possible outranking relations obtained by the ROR application. The ELECTRE methods can then be used with the obtained most representative set of parameters to get a recommendation on the problem at hand.

In [29] the ROR is applied to group decisions, that is, to decision problems presenting a plurality of decision makers (DMs). In this case, the DMs share the same problem composed of the same actions, criteria, and evaluations of the actions on the considered criteria. However, each DM has his own preferences that are expressed in an indirect way. On the basis of this preference information translated in terms of outranking using the ELECTRE notation described in the previous sections, four different binary relations on $A$ are defined. Denoting by $D'$ the set of DMs, for each $(a, b) \in A \times A$ one has:

- $aS^{N,N}b$ iff $aS^{N}b$ for all $d_r \in D'$,
- $aS^{N,P}b$ iff $aS^{N}b$ for at least one $d_r \in D'$,
- $aS^{P,N}b$ iff $aS^{P}b$ for all $d_r \in D'$,
- $aS^{P,P}b$ iff $aS^{P}b$ for at least one $d_r \in D'$.

Of course, specific LP and MILP problems have to be defined to compute such outranking relations (see [29]).

The papers of Roman Słowiński propose also some applications of ELECTRE methods to classical problems in decision theory and finance. More in particular, [11] uses ELECTRE methods to handle decisions under uncertainty, proposing an application to the newsvendor problem, while [31], taking into account the Markowitz model, applies the ELECTRE methods to the classical financial portfolio selection problem.

Let us also remember two more papers proposing interesting real world applications of the ELECTRE methods. In [46], ELECTRE III is used to setting up the priority of water users according to some socio-economic criteria in the context of a multicriteria programming water supply system for rural areas. In [18], an MCDA approach based on the ELECTRE III method is applied to measure the multi-functionality of an agri-food value chain. To this aim, 9 different indicators are taken into account based on a literature review. Even if the proposed methodology can be applied to different agri-food value chains, in the paper it is used to measure the multi-functionality of the olive oil food value chain in five different European countries being the biggest oil producers, that is, France, Greece, Italy, Portugal, and Spain.

Finally, a review on the ELECTRE main characteristics and principles together with a brief description of most of the ELECTRE methods has been provided in [21]. Let us observe that a more recent review on ELECTRE methods and their applications to real world decision making problems has been presented by [22].

## 3.4 Conclusions

Roman Słowiński proposed several and diversified contributions to ELECTRE methods, ranging from theoretical investigations on the axiomatic basis to enrichment of the methodology with new approaches emerging in MCDA, such as ordinal regression and robust ordinal regression or consideration of hierarchy of criteria, passing through many innovative real world applications. We believe that we can look at the approach of Roman Słowiński to ELECTRE methods as the result of a constant tension to go beyond the mere passive application of formulas and algorithms of these methods, proposing ingenious extensions and advances that facilitate and enhance the effective application in real life decision aiding procedures. All of this is done remaining always faithful to the spirit with which the ELECTRE methods were proposed by Bernard Roy. We hope that the survey we have proposed in this paper can contribute to encourage other researchers and practitioners to proceed with the same spirit in the effective application and development of ELECTRE methods.

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# Chapter 4
# How Can Decision Sciences and MCDM Help Solve Challenging World Problems?

**Hannele Wallenius and Jyrki Wallenius**

**Abstract** The world is witnessing rapid changes, some positive, some negative. In this paper we overview several of the technology mega trends and other trends, which are of interest for the Decision Science/MCDM community. We discuss, what role our field could and should play in helping solve world problems.

## 4.1 Introduction

The world has been witnessing rapid advances of digital technology during the last decade. At the same time, the world is witnessing great challenges concerning the climate change and environment, and increasing world population. The recent COVID-19 pandemic has accelerated the adoption of digital technology on many fronts. Interestingly, results of recent research have made the world aware of the possibility that pandemics and climate change could be linked.[1] At the individual, corporate, and even society level, the implications of what we call mega trends are pervasive, posing both opportunities, challenges, and even threats, although it is not easy to predict all the implications. Many individuals, businesses, and even government leaders may not be aware of, let alone be prepared for the future

[1] Coronavirus, Climate Change, and the Environment: An Interview with Dr. Aaron Bernstein, Director of Harvard School of Public Health, 2019 (https://www.hsph.harvard.edu/c-change/subtopics/coronavirus-and-climate-change/).

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changes. One could argue that the world has always been changing, but what is novel now is the unprecedented pace at which changes take place all over the world. We live in an era, where there is great demand for scholarly research and collaboration[2] at supra-national levels. Wisdom is called for.

We will separate the technology mega trends from other mega trends.

In 2015, the World Economic Forum published a report Deep Shift: Technology Tipping Points and Societal Impact, covering the ongoing technology mega trends. The report groups the technological mega trends into the following six:

1. The Internet—world's access to the Internet will continue improving; people's interaction with it will become more ubiquitous
2. Further enhancements in computing power, communications technologies,[3] and data storage
3. The "Internet of Things"
4. Big data and Artificial Intelligence—the ability to access and analyze huge amounts of data; coupled with the "ability" of computers to make decisions based on this data
5. The sharing (or platform) economy and distributed trust (based on, for example, the block chain technology)
6. 3D-printing

The possibilities of the digital technology are almost unlimited, both in enhancing traditional industrial processes (robotics), and in generating novel digital services and business opportunities. The digital revolution has begun, although decades (centuries) are needed for its full potential to realize. One interesting cause of the Internet and social media is the increased transparency of societies, which helps to improve democracy. At the same time, we face severe problems with data privacy issues and cyber-security.

In addition to technology mega trends, the world is witnessing highly important other mega trends. These mega trends, unlike technology mega trends, are generally perceived as challenges or threats to humankind. Some of them are discussed in PwCForesight#megatrends and by the World Economic Forum:

1. Demographic and social change taking place in many countries (aging populations; decreasing fertility (Western world, but also China); urbanization (in particular, in Africa); refugee problem)
2. Increasing world population (in particular, in India and Africa): growing need for food, clean water, and cheap energy
3. Climate change, concern for the environment

[2] It is beyond the scope of our paper to discuss negotiations. However, providing analytical support for negotiators in the Howard Raiffa tradition is as timely as ever [20].

[3] Zoom and Teams represent the state-of-the-art communication technologies, which have been in common use by the world during pandemic. Both universities and businesses went to online teaching/work, when the pandemic started in March of 2020. It remains to be seen, to what extent distance work will remain an option for employees.

The above three mega trends force governments and businesses to operate more efficiently under resource scarcity. Moreover, the world is more than ever interdependent on each other. The wealthier nations need to provide humanitarian help to poor nations to help solve the world problems.[4]

With robots/Artificial Intelligence (AI) "outsmarting" many individuals (with time, perhaps most individuals), what do most people do in year 2121? Brechbuhl asks the good question, "What will happen to the sense of worth, place and contribution to society that human beings have derived from work throughout much of recorded history?" (World Economic Forum report).

We discuss technology mega trends 1, 4, and 5, and non-technology mega trend 3 from the above World Economic Forum's list, in more detail. What role can Decision Sciences/multi-criteria optimization, preference modeling, play? We argue that our profession should be able to provide tools, software, and ideas to capitalize on rising opportunities and tackle problems due to world mega trends. We provide concrete examples.

## 4.2 Internet Searches

E-commerce is going strong. It started with digital products, such as movies and music, and products which could be distributed electronically, such as airline tickets, but has gained new territory. People buy today to an increasing extent online, and even more so in the future. Besides travel and leisure industries, the clothing or fashion industry is almost driving the change. Online grocery shopping was relatively modest, until the world was hit by the COVID-19 pandemic. Online grocery shopping experienced a rapid growth. According to Shopper Insights (February 16th, 2021), online grocery shopping grew to 10% of all grocery shopping, implying a 5-times growth from pre-pandemic time.

The growth of e-commerce has many implications. Besides, it has generated many interesting research questions. One of the implications is the increased importance of logistics for delivering products which are not in digital format. Customers have traditionally taken care of the home delivery of products themselves. In electronic commerce the goods are delivered directly from warehouse to customers, which has a number of logistical consequences. For example, storage and warehousing can be centralized, and delivery size decreases. From an MCDM point of view, delivery times and cost become important, in addition to the price and quality of the product. An example of a challenging logistics problem is the COVID-19 Pfizer vaccinations which require (truly) cold storage. An important research question is, how to gain the consumer's trust when buying online. Because the consumer is unable to inspect the product physically, one has to trust the store

[4] Humanitarian logistics is a relatively new field close to Management Science/Operations Research.

that they will deliver good quality products.[5] Mechanisms to build trust include, but are not limited to showing videos or original photos of the product, giving shoppers as much information as possible, sharing customer reviews, offering generous return policy, and ranking high on Google searches. Behavioral science offers advice, how to build trust. Prisoner's dilemma, in particular, the iterated prisoner's dilemma game, teaches useful lessons, how to build trust. Regarding the clothing or fashion industry, it is interesting that two opposite strategies are followed by online stores regarding return policy. In a nutshell, when the product is relatively cheap, the stores often encourage the consumer to order several units (even ten), keep one, and return the rest, almost maximizing the returns. Some stores even provide free pick-up delivery for returned goods. For expensive pieces of clothing, stores have, for example, developed elaborate (visual) ways to help measure the size. Instead of just one traditional size (number), the online stores solicit several measurements (sleeve length, waist, etc.). Expensive fashion stores in a nutshell attempt to minimize the returns. The return policy has an impact on the consumer choice problem. In the former case, the choice process is two-staged: in the first (online) stage the consumer must decide which products are delivered to her home; in the second (on-site) stage the consumer narrows down the choice to one (typically).

Typically, when people buy online, they use some search engines, such as Google, to help them find the products or services which they are looking for. Quite commonly on top of the list provided by search engines emerge the cheapest products/services. A typical example is flight tickets between two cities. The search engines are not good in differentiating among offers (besides price). We realize that Google has a dominating market position, but our scholars could work with Google to develop better search engines. For an attempt, see Roy et al. [22]. Their idea was based on two consecutive searches, where the user marks the relevant documents (from the first search) with radio buttons and uses a multi-objective technology to conduct a second improved search.[6] The second improved search would provide "more" relevant documents/products/services.

We recently heard of a problem related to searches conducted in rare languages, such as Finnish (5.5 million people). Google was alerted to a problem associated with the search word "vaccination." Before Google corrected the situation, the postings which ranked highest represented anti-vaccine, non-scientific propaganda, although in Finland the anti-vaccination ideas have not gained any greater momentum.

[5] https://www.sellbrite.com/blog/how-to-build-trust-with-online-shoppers-an-actionable-guide/. Consumers have experienced problems with products delivered from some countries.

[6] The multi-objective technology refers to the "lambda problem" associated with the Zionts-Wallenius interactive multi-criteria optimization algorithm [28]. The lambda problem generates an improved set of attribute weights, which are used to find improved solutions.

### 4.2.1 Recommender Systems

To combat the information overload pertaining to both digital and non-digital products or services, many companies (and academics) have found it worthwhile to develop the so-called recommender systems. Such systems have predominantly been developed by computer scientists. A recommender system is a subclass of information filtering systems that seeks to predict the "rating" or "preference" that a user would give to an item (Wikipedia). Recommender systems are extensively used, for example, in choosing movies to watch, music to listen, news to follow, books to read, and restaurants to visit. In order for recommender systems to function, they need user data. This can be explicit (feedback in the form of product reviews), or implicit (judged based on past choices). For some companies like Netflix, Amazon Prime, and Hulu, their business model and its success depends on how good their recommendations are. Netflix even offered a million dollars in 2009 to anyone who could improve its system by 10%.

Recommender systems can be categorized into collaborative filtering approaches and content-based filtering approaches [21, 26]. There also exist hybrid recommender systems [24], who target to overcome the problems experienced with content-based and collaborative approaches.

Collaborative filtering approaches are based on the idea of building a model from a user's past behavior as well as other users' behavior (items previously purchased). The logic of incorporating other person's likings is that if other people found this item (or similar items) popular, so would you (as their peer)! A positive feature is that the recommendations are perceived to improve over time. However, there are problems providing recommendations to new users. Moreover, if an individual is "off the main street" with very specific likings, it is difficult to find good recommendations.

Content-based filtering approaches develop a set of characteristics that an item possesses, and build a profile of each user (that is, what they like) in order to recommend additional items with similar characteristics. This idea is close in spirit to MCDM, where the properties are called attributes or criteria. It is not uncommon that the importance of a characteristic is correlated with the number of times the specific characteristic is mentioned in the description of the product/service (text). Similarity of characteristics vectors is normally measured with the cosine of the angle of two vectors (algebraically, the dot product). We make two observations. Firstly, the number of times a characteristic is mentioned in the text does not necessarily correlate with its importance. Secondly, the dot product is not necessarily the best similarity measure between two vectors.

Recommender systems are based on the "similarity" logic. What you and others liked in the past; you will like more of the same in the future. The human desire for variety does not play a role in this logic. The assumption is that people lack curiosity and the desire to experiment. The underlying logic of recommender systems is even dangerous, when we apply it to filter news. If an individual is solely or largely dependent on reading news (in social media, as opposed to traditional media) recommended (filtered) by a system, the set of news offered becomes narrow, representing a very narrow world-view. In the world there are millions of people, whose world-view has become narrow, as a function of this. We think that the recommender systems should periodically suggest different types of products/services (or news), to broaden the person's horizon. You might like something which is out there, even though you have not tried it before. How to operationalize this idea, calls for research.

MCDM scholars should easily understand the logic underlying recommender systems. Both MCDM and recommender systems are about modeling user's preferences [13]. Preferences can be modeled with the help of value or utility functions [10]. Some other interesting preference modeling approaches have been described in Fürnkranz et al. [7] and Branke et al. [1, 2].

When purchasing relatively expensive goods (such as consumer durables), irrespective of whether a person uses recommender systems or not, she has to make a choice. Besides recommender systems, there is very little decision support available for consumers in online environments. Interestingly, the Internet is changing the concept, who a "decision maker" is, and what type of support she needs. Operations researchers have largely been in the business of supporting corporate leaders and managers. However, many of the hundreds of millions of consumers who buy online could benefit from some support when making purchasing decisions on the Internet. Such decision support must be targeted at masses; hence it must be simple. We think there is a need for developing, in addition to complicated algorithms and decision support tools, simple tools to be used by masses.

## 4.3 Big Data (and Artificial Intelligence)

According to the lead article in the Economist published on May 6th, 2017, the world's most valuable resource is no longer oil, but data. Data is being continuously generated from various sources, including cash registers, mobile phones, and Internet sites visited by hundreds of millions of people daily. Many corporations are realizing that they should better utilize data to their (strategic) advantage. Some talk about the monetization of data, meaning that there would be a market and monetary value for data. We feel that data is necessary, but data analytics is needed to realize the potential from data.

Typical advertising and marketing agencies or corporate departments do not know how to analyze big data, even though they realize its importance or potential. The need for people possessing analytics skills is high. What role does big data play in advertising? In a nutshell, big data can be used to help create targeted and personalized campaigns, which increase the efficiency of advertising or marketing. How is this done? Simply by gathering information and learning about user behavior. Many reward- and loyalty programs are based on the use of consumer data. Recommender systems use past purchases or searches to make new recommendations. An interesting phenomenon is the use of social media by ad agencies. It is easy to document and share experiences as customer or consumer in social media. It is not uncommon that thousands of people read these posted reviews and are influenced by them.

Another area where big data will find its uses is medicine and health care at large. Various monitoring instruments continuously generate data, so do human genome studies. They eventually lead to better preventive and actual care, and more accurate diagnostics. An interesting problem from the perspective of MCDM is, how to better incorporate patients' views on their own healthcare plans and treatment decisions. Is an average person without medical training able to express such preferences? We think many would, if somebody explains to them the likely consequences of each treatment option.

A more general level concern in healthcare is to make the system more efficient and more personalized. A good starting point would be if various health care providers could share data (with the patient's consent). Such a system would have to gain everybody's trust due to the sensitive nature of the information. Blockchain technology is being investigated for this. Healthcare decisions naturally have to deal with multiple criteria, and complex tradeoffs between cost, the quality of care, even potential loss of lives. Wojtek Michalowski's (University of Ottawa) work is a good example of the type of impactful work a person with an Operations Research/MCDM background can do in healthcare. He has collaborated with Ottawa Hospital.

Artificial Intelligence (AI) is a very important field today. AI is either rule-based or based on machine learning [11]. In many cases, the technology is there or "almost" there. A good example of the use of AI is in cyber-security to detect cyber-criminal activities. Another example is driver-less automobiles. However, many complex legal and ethical issues remain to be solved. AI must make complex moral choices as well. Work is also currently being conducted to incorporate emotions into "robots." We ask, whose moral choices and emotions should be programmed? The society's? Or the person who owns the robot. In case the individual's ethical choices differ from the society's, it would seem natural to impose the society's views. But the question is, does society have an answer (or a code book) for every conceivable situation. Certainly not today.

We personally would hesitate to delegate decision-making powers in important matters to "robots," no matter how "intelligent" they are. We feel that humans should be in control of their lives. AI is a good tool, but a dangerous master. Who guarantees that the AI-driven robots are friendly towards humankind?[7] If it is against the robot's objective, a human may not be able to stop the robot. Futurists commonly predict AI as posing one of the greatest threats to humankind. Having said that, certainly lower level routine decisions could be delegated to robots, if subjected to periodic review by humans.

[7] Physicist Hawkings, among other famous people, was concerned of this. See his last book: Brief answers to the big questions, 2018, Hodder and Stoughton.

## 4.4 The Platform Economy

Sharing economy is an umbrella term with a range of meanings, often used to describe economic activity involving online transactions. It was born from the open source community. We focus on the use of the term to describe sales transactions conducted via online market places called platforms [27]. An example of such a market place are online auctions, which have been around since late 1990s. Newer examples are the San Francisco based taxi companies, Uber and Lyft; and the online market for housing, Airbnb. Both Uber and Lyft were listed in the stock exchange in 2019, Airbnb the following year. Their respective estimated value was 75 billion USD (Uber), 30 billion USD (Lyft), and close to 100 billion USD (Airbnb).

Uber and Lyft realized that they did not need to own any vehicles, just a platform, where owners of cars and people in need of rides or deliveries can communicate. Uber is now operating globally. In case of Uber and Lyft, the drivers indicate to the platform the hours they are available. Customers then specify their needs for driving services from place A to B. The platform checks the availability of cars near A. The customer sees upfront the price.

Airbnb is an American company, which hosts an online marketplace and hospitality service, for people to lease or rent short-term lodging including vacation rentals, apartment rentals, homestays, or hotel rooms (Wikipedia). They have currently over 4 million listings. Airbnb does not own a single house, apartment, or condominium, but it provides a platform, where supply and demand for short-term housing meet. Its commission is typically between 10–13%. In case of Airbnb, people who want to rent out their homes or other rental property owned by them, post to the platform the dates their home is available for rental, and details about the rental property. Customers can specify the type of housing they are looking for, price range, number of bedrooms, beds, and baths. Customers can also use extra filters, for example, for handicapped people. Customers can then check out the locations and the units within their wishes, and make their choice. The matching seems to work out reasonably fine. In the lingo of MCDM, the choice problem is discreet, choosing one most preferred option from a relatively small set. To facilitate the choice (or make it more complicated), many sites post customer reviews. Practically all sites have pictures of the rentals. Note that the filters usually operate in the fixed constraints mode. MCDM literature seems to favor flexible constraints, however.

We worked with the platform economy concept already in late 1990s. We developed a multi-attribute reverse (or procurement) auction site, called NegotiAuction [23]. We realized that price-only auctions were too simplistic, and that auctions (transactions in general) need to include other aspects as well, such as quality and terms of delivery. Our NegotiAuction system was based on "pricing out" (or costing out) all other attributes besides cost. "Pricing out" is an old tool used by practicing decision analysts [10]. Today there exist many such commercial multi-attribute auction sites, for example, Perfect, Ariba, and CombineNet to name a few [18].

Another example of platform economy is crowdfunding and other peer-to-peer lending sites, where private people (instead of banks) lend money to people in need of money. Equity crowdfunding is a decade old, a rapidly growing alternative form of entrepreneurial finance, reaching a 1,5 billion USD volume in 2020, posing many interesting research questions [14, 15].

An example of a platform/recommender system is Voting Advice Applications (VAAs). They are commonly used in many European countries, but also elsewhere where we have multi-party elections. In many European countries (such as Finland and Holland) over half of the general population uses them, and they have a significant impact on elections. This was recognized by the previous Prime Minister of Finland, who wanted to commission a study of existing Voting Advice Applications and their importance in Finland. VAAs are platforms to help voters find suitable candidates to vote for in national, presidential, and regional elections. They are based on both the candidates and the voters answering a set of questions concerning political issues. The system (the algorithm) then finds the candidates and party, which are "closest" to the voter's political preferences. The development of such Voting Advice Applications involves solving many MCDM/behavioral decision-making problems. The questions must be discriminating and there cannot be too many of them. They must have proper Likert-scales to make distance measurement meaningful. What distance measure should one use? Are the questions of equal importance to voters or should importance weights be used? If yes, how are they determined? Are voters interested in voting for candidates who have a higher likelihood of becoming elected? Current VAAs are based on the so-called proximity voting model, where the voter looks for candidates who in the issue space would be "closest" to them. But political science literature also talks about the directional model and the discounting model [25]. Have the VAA developers too quickly locked in the proximity model? The voting problem is an important problem, in particular, in multi-party, multi-candidate elections, where voters have been shown to benefit from the use of such support. For additional details, see Pajala et al. [17].

Generally speaking, many MCDM scholars are equipped with the skills to develop online platforms. There is a growing market for them. In sharing economy platforms, some type of matching based on preferences is sought, where supply meets demand. The importance and popularity of platform economy has grown during COVID-19 pandemic [16].

## 4.5 Climate Change, Concern for Environment

The scientists are of the opinion that human-induced climate change is highly probable [5]. The concern for the environment is almost universal. Almost 200 countries have signed the Paris Accord, including the US, Russia, and China. Under Trump's administration, US withdrew from the agreement, but under Biden rejoined it. When making decisions, governments and corporations are increasingly forced to consider the impact of their decisions on the environment.

Environmental studies naturally require the decision makers to consider multiple criteria and complex tradeoffs between them, making environmental applications common [9]. Another case in point is flood risk management, an area, which is growing in importance due to the climate change [3]. From an MCDM scholar's perspective, many models which are being used by various environmental authorities may not be up-to-date, prompting our community to collaborate with the environmental authorities.

From the MCDM's perspective an under-researched topic is sustainable (or green or ethical) investing. Markowitz's original portfolio optimization model has two objectives, maximizing expected returns, and minimizing risk (or volatility). Now many investors (including institutional investors) also want to consider, how the listed companies perform in terms of a third dimension: sustainability (or Environmental, Social, and Corporate Governance, for short ESG). Dow Jones has for over 20 years provided a list of companies who perform well regarding ESG criteria. It lists the most sustainable companies from across 61 industries. The list is periodically updated and serves as a benchmark to guide investors interested in sustainable companies. If a company does not make it to the list, it may not be easy to find out, how they perform regarding ESG criteria. The company's website, however, usually provides some self-reported information about this. However, measuring sustainability is far from trivial. Sustainability is multi-dimensional, and measuring it calls for research. Our community should help. Interesting early work is reported in Hallerbach et al. [8]. In their work a framework for managing a portfolio of socially responsible investments is presented. Real-life data is used to illustrate, how multi-dimensional sustainability actually is. However, there are many open questions. To make matters more complicated, ESG criteria also include possible use of child labor or forced labor by oppressed people. How would we know if this happens, and how are such considerations factored in?

We would like to conclude this subsection by referring to interesting research, with a link to sustainability, conducted at the International Institute for Advanced Systems Analysis (IIASA) in Austria. The Population and Just Societies Program seeks to generate insights into current and future "population sizes, structures, and distributions, which are fundamental to understanding human impact on ecosystems and simultaneously, the impact of environmental changes on human wellbeing." The program's research agenda forms a key priority in the IIASA strategic plan by "identifying sustainable development challenges and exploring people-centric systems solutions for sustainable, resilient, just and equitable societies." From a methods point of view, decision support tools as well as scenario analysis are in frequent use in the program.

## 4.6 Where the Opportunities Lie?

The future for MCDM looks bright. Many of the world mega trends reinforce the role of MCDM by pointing out novel application areas, such as sustainable investing. The MCDM community needs to be prepared for seizing the opportunities.

The world is getting more complex. Despite the relative affluence of the world, resource constraints still prevail. Because of resource constraints, we cannot achieve everything we want, and tradeoffs must be made. The concepts of efficiency (or Pareto optimality) and tradeoff are at the core of MCDM. They are still as valid as ever.

Because of the increasing complexity of the world, heuristics offer a good practical option to optimization approaches. One good example is Evolutionary Multi-Objective Optimization (EMO), which consists of heuristic tools mimicking the survival-of-the-fittest ideas in nature [4]. Originally developed mainly for bi-objective problems, with the purpose of generating all approximately Pareto-optimal solutions, much recent research has focused on developing hybrid interactive-EMO/MCDM approaches for multiple-objective problems. Such hybrid approaches naturally interact with a decision maker and highlight the importance of the decision maker, and psychology.

The importance of preference modeling is highlighted because of recommender systems. MCDM scholars need to tailor their tools to be used as part of recommender systems.

The importance of psychology, or behavioral decision theory, is being rediscovered in decision-making, and more broadly in Operations Research [6]. Three Nobel Prizes in Economics have been awarded to decision psychologists, the first to Herbert Simon in 1978; the most recent being awarded to Richard Thaler (December, 2017), whose work builds on Daniel Kahneman and Amos Tversky.[8] We think that the more realistic our tools are from a behavioral perspective, the better are our chances to support individual decision makers. Hence there is a need for better incorporating decision psychologists' findings into our decision support tools (whether MCDM or EMO).

We also think that there is an increased awareness of the fact that situations vary and the needs of decision makers vary. In some cases, there is a need for more formal analysis than in other cases. Sometimes, quick-and-dirty calculations may be all that is needed. For an interesting study demonstrating that more is not always better when talking about decision support, we refer to Johanna Bragge's master's thesis (described in [12, pp. 118–119]).

The Internet is changing the concept, who a "decision maker" is, and what type of support she needs. Consumers purchasing online need decision support. Such support must be targeted at masses; hence the requirement for simplicity. In the Internet era, the classical economics problem of matching has gained in importance [19]. MCDM scholars can help by developing good matching algorithms.

[8] Daniel Kahneman received the Nobel Prize in Economics in 2002.

## 4.7 Final Word

We summarize our suggestions, where we see great potential for novel contributions from our field.

1. Develop better recommender systems and search engines.
2. Develop better matching algorithms for various situations.
3. Promote the use of big data in companies and the public sector (for example, health care).
4. Develop better measures (indices) for sustainable investing.
5. Develop decision support tools targeted at masses, instead of business leaders, to help make online purchases.

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# Chapter 5
# Preference Disaggregation Analysis: An Overview of Methodological Advances and Applications

**Michalis Doumpos, Evangelos Grigoroudis, Nikolaos F. Matsatsinis, and Constantin Zopounidis**

**Abstract** Preference disaggregation analysis has been widely used for constructing decision models in multi-criteria decision aid (MCDA). Disaggregation approaches adopt a data-driven scheme, in which preferential information is inferred from decision examples through regression-like approaches. This chapter provides an overview of the preference disaggregation paradigm and the existing methodologies covering different modeling forms and decision contexts. Moreover, recent applications are summarized along with developments in this field.

## 5.1 Introduction

The development of decision models is a challenging and time-consuming task for both decision-makers (DMs) and analysts. The development of a decision model requires the specification of technical parameters and the representation of the DM's preferences in a way that fits the structuring of the decision problem, and the judgment policy of the DM. Model development is an interactive process with the DM having an active role on various steps, such as the specification of the appropriate type of model, providing information about the preferential parameters involved, and the verification of the model's validity.

Among these issues, the specification of a model's parameters that represent the preferences of the DM is a crucial step in the development of decision models. Two types of interactive approaches can be considered in this context. The first requires

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the DM to provide direct information about his/her judgment policy and system of preferences. A typical example involves the specification of value trade-offs, which are used in value function models to define the weights of the criteria [1].

An alternative approach relies on indirect approaches to elicit preferential information from the DM. This is known as *preference disaggregation analysis* (PDA; [2]). In PDA, examples of decisions taken by the DM are analyzed to identify the underlying decision model, which can then be used in the actual decision instance under consideration. Such an approach can be quite convenient when the DM can easily provide decision examples, i.e., evaluations of representative alternatives, using past cases, or considering examples of alternatives that are well-understood and easy to evaluate. The inference of a decision model from such examples is performed through regression-like techniques. Once the results from the inference procedure are obtained, an interactive process can be initiated to identify and resolve the inconsistencies between the model's outputs and the preferences of the DM, thus leading to the final model.

The roots of the PDA paradigm can be traced back to the early works on the use of linear programming models for regression analysis and the analysis of preferences [3, 4]. A formal approach in the field of multicriteria decision aid (MCDA), PDA was first introduced with the development of the UTA method for constructing additive value function models by Jacquet-Lagrèze and Siskos [5]. Since then, PDA has evolved covering different types of MCDA models, various decision contexts, and application areas. The aim of this chapter is to provide an overview of the methodological advances and the applications of PDA approaches and to highlight emerging research trends.

The rest of the chapter is organized as follows. Section 5.2 describes the basic principles and the framework of PDA. Section 5.3 discusses the use of PDA approaches for the inference of different types of MCDA models, covering various developments in methodologies for eliciting preferential information from decision examples. Section 5.4 focuses on the issue of robustness of PDA approaches, whereas Sect. 5.5 provides an overview of the literature on applications of PDA methods and their implementation in decision support systems. Finally, Sect. 5.6 concludes the chapter and discusses some future research directions.

## 5.2 The General Framework of Preference Disaggregation Analysis

As mentioned in the introduction, PDA is a regression-like approach for inferring decision models from data. More formally, we assume a typical multicriteria decision problem involving a set $G = \{g_1, \ldots, g_n\}$ of $n$ evaluation criteria, expressed in maximization form, with $g_k(a)$ denoting the performance of alternative $a$ on criterion $k$. For the evaluation of a finite set of alternatives $A$, a decision model $f$ is employed. Depending on the decision problematic [6], the evaluation may result in the selection of the best alternatives (problematic $P_\alpha$), the sorting of the alternatives to predefined performance categories (problematic $P_\beta$), or the ranking of the alternatives from the best to the worst (problematic $P_\gamma$).

Regarding the decision model, it can be expressed various forms, leading to a distinction between:

- Functional models (e.g., value/utility functions; [5, 7])
- Relational models (outranking relations; [8, 9])
- Symbolic models (decision rules; [10])

Depending on the type of model that best fits the nature of the problem and the policy of the DM, one may need to specify various technical and preferential parameters. For instance, several types of models require information about the relative importance of the criteria. Henceforth, $f(\mathcal{P})$ will denote a decision model defined by the set of parameters $\mathcal{P}$.

In PDA approaches, the specification of the parameters in $\mathcal{P}$ is performed though the analysis of a set of decision examples $A_R = \{a_1, \ldots, a_m\}$, which is often referred to as the *reference set*. The instances in the reference set involve alternatives evaluated by the DM; these may be alternatives that the DM has evaluated in the past (i.e., past decisions), a subset of $A$ or fictitious cases that can be easily analyzed and evaluated by the DM [2]. The evaluations that the DM provides on the alternatives of $A_R$ are assumed to be representative of his/her system of preferences and judgment policy. Thus, the reference set implicitly incorporates all the information needed to describe the underlying decision model characterizing the DM.

The evaluations of the DM on the reference alternatives can be expressed in various forms. For instance, the DM may define a ranking of the alternatives, a classification, or may provide richer information in the form of defining pairwise relations among the alternatives. Denoting by $Y$ the evaluations of the DM on the reference set, PDA seeks to identify a set of parameters $\mathcal{P}^*$, such that the evaluations $\hat{Y}_{\mathcal{P}^*}$ derived with the corresponding model $f(\mathcal{P}^*)$ are as close as possible to the given evaluations $Y$, i.e., $\hat{Y}_{\mathcal{P}^*} \approx Y$. This leads to an optimization problem of the following general form:

$$\mathcal{P}^* = \arg\min_{\mathcal{P}} L\left(\hat{Y}_{\mathcal{P}}, Y\right), \tag{5.1}$$

where $L$ is a loss function for the differences between $\hat{Y}_{\mathcal{P}}$ and $Y$. The exact formulation of the above optimization problem (5.1) depends on the type of the considered decision model and the decision problematic. Examples of some well-known MCDA models are provided in the next section.

It is worth noting that the context of PDA is closely related to the framework of supervised machine learning (ML), which is also focused on learning models from data. However, ML adopts a data-driven, algorithmically oriented perspective, in which the role of the DM is rather limited, whereas PDA assumes that the DM has an active role in calibrating model, using the results from the optimization (5.1) as the starting point. Moreover, ML algorithms are mainly designed for large-scale problems, whereas MCDA problems usually involve a smaller number of alternatives. A detailed discussion of the connections between ML and PDA can be found in the works of Doumpos and Zopounidis [11, 12]. Despite the existing fundamental differences, the interactions and synergies between the two areas have gained interest among researchers working on these fields, leading to new developments for both domains, such as preference learning [13], as well as new approaches for implementing PDA methodologies [14–17].

## 5.3 Models, Formulations, and Methodological Advances

This section provides an outline of the main methodological approaches in PDA, categorized by the type of MCDA models in each approach. The presentation starts with approaches based on value functions, which have been the first to be considered in the PDA framework. Moreover, approaches for outranking relations and rule-based models are also presented.

### *5.3.1 Value Function Models*

Multi-attribute utility/value theory (MAUT/MAVT) adopts a normative approach to decision-making, relying on an axiomatic basis first formally described by Von Neumann and Morgenstern [18], who characterized the foundations of utility models for decision-making under uncertainty. Details about the principles of multi-attribute utility and value functions can be found in the comprehensive book of Keeney and Raiffa [1]. Henceforth, we shall refer to value functions assuming decision problems in a deterministic framework.

#### 5.3.1.1 Modeling Forms

Depending on the preferential independence conditions that describe the underlying system of preferences of the DM, different types of value functions can be defined. The most widely used form of value function is the additive one, which assumes mutual preferential independence of the criteria:

$$V\left(g_1, g_2, \ldots, g_n\right) = \sum_{k=1}^{n} v_k\left(g_k\right), \tag{5.2}$$

where $v_k(g_k)$ is the marginal value function of criterion $g_k$. The marginal value functions provide the means to evaluate the performance of the alternatives on the criteria through a common (value) scale; they are increasing for benefit criteria and decreasing for cost criteria. The marginal value functions can be scaled such that $v_k(g_{k_*}) = 0$ and $v_k\left(g_k^*\right) = w_k$, where $g_{k_*}$ and $g_k^*$ denote the least and most preferred levels of criterion $g_k$. Setting $u_k(g_k) = v_k(g_k)/w_k$, the value function (5.2) can be equivalently written as:

$$V\left(g_1, g_2, \ldots, g_n\right) = \sum_{k=1}^{n} w_k u_k\left(g_k\right). \tag{5.3}$$

The weights define the trade-offs among the criteria, and they are defined to be non-negative and summing up to 1. With this specification, the global value functions (5.2) and (5.3) range in [0, 1] with higher values corresponding to better-performing alternatives.

Although most studies on PDA approaches for value function models have assumed an additive form, non-additive functions have also been considered. For instance, Bugera et al. [19] presented an optimization formulation for quadratic value functions, whereas Liu et al. [20] presented an approach for inferring value function models augmented with components for handling the interactions among criteria. During the past two decades, value models in the form of Choquet integrals have attracted strong interest [21]. In this context, Angilella et al. [22] used the Choquet integral to represented a non-additive model in an ordinal regression framework, whereas Aggarwal and Fallah Tehrani [23] formulated a nonlinear optimization problem to derive preferential information with the Choquet integral through pairwise comparisons. Finally, Bresson et al. [24] presented an ML approach, based on a neural network, for learning hierarchical Choquet integrals from data.

#### 5.3.1.2 Ranking Problems

The construction of value functions in the context of PDA, involves the specification of the form of the marginal value functions. For problematic $P_\gamma$ (ranking problems), the reference alternatives $a_1, \ldots, a_m$ are ranked from the best (alternative $a_1$) to the worst one (alternative $a_m$). With the predefined ranking, the value function model should satisfy the following relations:

$$\begin{array}{l} V\left(a_i\right) > V\left(a_j\right) \ \forall a_i \succ a_j \\ V\left(a_i\right) = V\left(a_j\right) \ \forall a_i \sim a_j \end{array}, \tag{5.4}$$

where $\succ$ and $\sim$ denote the preference and indifference relations, respectively.

With conditions (5.4), the UTA method uses the following optimization problem to infer the value function model that represents the predefined ranking as consistently as possible:

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m} \sigma_i & \\
\text{Subject to :} & V(a_i) - V(a_{i+1}) + \sigma_i - \sigma_{i+1} \geq \delta & a_i \succ a_{i+1} \\
& V(a_i) - V(a_{i+1}) + \sigma_i - \sigma_{i+1} = 0 & a_i \sim a_{i+1} \\
& V(a_*) = 0,\ V(a^*) = 1 & \\
& \sigma_i \geq 0 & i = 1, \ldots, m
\end{array}
\quad . \tag{5.5}
$$

In this optimization model, $\sigma_i$ represents the error variable for alternative $a_i$ and $\delta$ is a user-defined small positive constant that imposes the strict inequality $V(a_i) > V(a_j)$ for pairs of alternatives for which the preference relation $a_i \succ a_j$ holds. The first two constraints represent the conditions (5.4) for the ranking model, whereas the third constraint scales the value function such that the global value of the least preferred alternative $a_* = [g_{1*}, g_{2*}, \ldots, g_{n*}]$ is equal to 0 and the global value of the most preferred alternative $a^* = \left[g_1^*, g_2^*, \ldots, g_n^*\right]$ is equal to 1.

The basic formulation (5.5) can be expressed in linear programming form, if a piece-wise modeling approach is adopted for the marginal value functions (for the details, see [5]).

The introduction of the UTA method has led to the development of various variants, which can be grouped into two main categories. The first involves refinements of the optimization model (5.5) and the formulation of the additive decision function to improve the quality of the model inference results. For instance, Siskos and Yannacopoulos [25] presented the UTASTAR method, which uses two error variables for each alternative instead of the single error assumed in formulation (5.5). Doumpos and Zopounidis [16] and Liu et al. [17] presented formulations based on the regularization principle of statistical learning theory, in order to control the complexity of the model. This issue was also addressed through a different approach by Doumpos et al. [26] and Sobrie et al. [27], who employed smooth marginal value functions instead of the piecewise linear setting used in the traditional UTA methods. Finally, Bous et al. [28] employed the concept of analytic center, as a means of improving the robustness of the inferred model A.

The second category of studies has considered different decision contexts, thus extending the range of applicability of UTA-based method. For instance, Siskos [29] presented a variant for decision problems under uncertainty and Patiniotakis et al. [30] considered a fuzzy setting. Moreover, value function models have also been considered in the context of non-monotonic preferences [17, 31–33] and nominal attributes [34], whereas Greco et al. [7] introduced the UTA$^{\text{GMS}}$, which enables the use of a set of value functions inferred from decision examples in ranking problems, extending a similar idea first proposed by Siskos [35]. This approach was later extended to group decision-making problems [36]. Finally, UTA-based methods have been combined with other MAVT approaches, such as the MACBETH method [37], to facilitate the decision aiding process and the inference of decision models from data [38, 39].

#### 5.3.1.3 Sorting/Classification Problems

A lot of research has also been done on using UTA-based approaches for classification and sorting problems (i.e., problematic $P_\beta$), which arise in various domains in business and engineering [40]. In this case, the objective is to infer a decision model that is compatible with a predefined assignment of a set of reference alternatives into categories $C_1, \ldots, C_q$, where $C_1$ corresponds to the class of best alternatives and $C_q$ includes the worst performing ones. The most straightforward way to formulate such problems is to assume a threshold-based classification rule, as in the UTADIS method [5, 41], such that:

$$t_\ell < V(a_i) < t_{\ell-1} \Longrightarrow a_i \in C_\ell. \tag{5.6}$$

Alternatively, an example-based approach can be employed, in which the assignment of an alternative to a set of predefined categories is derived through its comparison against reference alternatives that serve as representative examples from each class [42, 43].

Under the threshold-based setting, the inference of a value function model that minimizes the errors for the classification of the reference alternatives is formulated through the following optimization problem:

$$\begin{array}{lll} \min & \sum_{i=1}^{m}(\sigma_i+\varepsilon_i) & \\ \text{Subject to}: & V(a_i)-t_\ell+\sigma_i \geq \delta & \forall a_i \in C_\ell, \ell=1,\ldots,q-1 \\ & V(a_i)-t_{\ell-1}-\varepsilon_i \leq -\delta & \forall a_i \in C_\ell, \ell=2,\ldots,q \\ & V(a_*)=0,\ V(a^*)=1 & \\ & \sigma_i \geq 0 & i=1,\ldots,m \end{array}, \tag{5.7}$$

where $\sigma_i$ and $\varepsilon_i$ are error variables representing the violations of the classification rules (5.6) and $\delta$ is a user-defined, small positive constant used to impose the strict inequalities involved in the classification rules. An overview of the developments in PDA approaches for inferring decision models in multicriteria classification problems can be found in [44].

#### 5.3.1.4 MUSA Method

The MUSA (MUlticriteria Satisfaction Analysis) is a UTA-based ordinal regression analysis that has been developed for measuring and analyzing customer satisfaction [45, 46]. Similar to the PDA philosophy, the method aims to assess a set of marginal satisfaction functions in such a way that the global satisfaction criterion becomes as consistent as possible with customer's judgments. Although the MUSA method has been proposed in the context of customer satisfaction analysis, it may be applied in other decision-making problems where the main objective is to aggregate individual judgments into a collective value function.

Given ordinal global and partial judgments $Y$ and $X_k$, respectively, the MUSA method assesses global and partial value functions $Y^*$ and $X_k^*$, respectively, using the following ordinal regression analysis equation:

$$\overset{\sim *}{Y} = \sum_{k=1}^{n} w_k X_k^* + \sigma^+ - \sigma^-, \tag{5.8}$$

where $\overset{\sim *}{Y}$ is the estimation of the global value function $Y^*$, $\sigma^+$ and $\sigma^-$ are the overestimation and the underestimation errors, respectively, and the value functions $Y^*$ and $X_k^*$ are normalized in [0, 100]. It should be noted that the notations of ordinal regression are adopted in the MUSA method in order to emphasize the regression analysis orientation of the method. Thus, a criterion $g_k$ is considered as a monotone variable $X_k$ and its marginal value function is denoted as $X_k^*$.

In order to assure the linearity of the model and decrease its complexity, the following transformations proposed in the UTASTAR algorithm are used:

$$\begin{array}{ll} z_r = y^{*r+1} - y^{*r} & r = 1, \ldots, \alpha - 1 \\ t_{kq} = w_k x_k^{*q+1} - w_k x_k^{*q} & k = 1, \ldots, n, q = 1, \ldots, \alpha_k - 1 \end{array}, \tag{5.9}$$

where $y^{*r}$ is the value of the $y^r$ global ordinal scale, $x_k^{*q}$ is the value of the $x_k^q$ partial ordinal scale, and $\alpha$, $\alpha_i$ are the number of global and partial levels of the aforementioned ordinal scales.

Based on the above, the MUSA method is given by the following optimization problem:

$$\begin{array}{lll} \min & \sum_{i=1}^{m} \left(\sigma_i^+ + \sigma_i^-\right) & \\ \text{Subject to :} & \sum_{k=1}^{n} \sum_{q=1}^{x_k^i} t_{kq} - \sum_{r=1}^{y^i - 1} z_r - \sigma_i^+ + \sigma_i^- = 0 & i = 1, \ldots, m \\ & \sum_{m=1}^{\alpha-1} z_m = 100, \ \sum_{k=1}^{n} \sum_{q=1}^{\alpha_k - 1} t_{kq} = 100 & \\ & z_r, t_{kq}, \sigma_i^+, \sigma_i^- \geq 0 \forall r, k, q, i & \end{array} \tag{5.10}$$

where $y^i$ and $x_k^i$ are the $i$th level on which variables $Y$ and $X_k$ are estimated.

As already noted, the MUSA method has a regression analysis orientation. Contrary to the previous PDA approaches, the necessary basic input information is not based on a classical MCDA table, where alternatives are evaluated on a set of criteria. Rather, a set of $m$ individuals provide global and partial evaluations $y^i$ and $x_k^i$, respectively.

Grigoroudis and Siskos [45] give an analytical presentation of the MUSA method, while further developments and reviews can be found in Grigoroudis and Siskos [47] and Grigoroudis and Politis [48]. Alternative formulations of the previous PDA problems and extensions of the MUSA method can also be found in the recent MCDA literature. For example, in order to increase the stability of the results, Joao et al. [49] used a dummy variable regression technique with additional constraints and employed an alternative optimality criterion (least square approach). Similarly, additional constraints may be considered in the aforementioned optimization problem. For example, Grigoroudis and Politis [50] examined the constraints regarding special properties for the assessed average indices and additional customer preferences about the criteria importance and showed that the modeling of these additional information may improve the stability of the estimated results. A different extension refers to the MUSA-INT model, proposed by Angilella et al. [51], which takes into account positive and negative interactions among criteria. Other extensions include the incorporation of the six sigma analysis and the principles of Kano's model in the previous optimization model [52] and the development of the fuzzy MUSA method in order to produce fuzzy global and partial value functions [53].

## *5.3.2 Outranking Models*

In contrast to the functional decision models in MAUT/MAVT, outranking models adopt a relational approach. The foundations of outranking approaches can be traced to social choice theory [54] and the works of Bernard Roy, who first formalized them in the context of decision aiding, through the introduction of the ELECTRE method [55].

### 5.3.2.1 Brief Outline of Outranking Relations

In the context of outranking methods, the evaluation of a finite set of alternatives is performed through pairwise comparisons, involving the examination of relations of the form $a\ S\ b$, which is usually interpreted as "alternative $a$ is at least as good as alternative $b$." In the context of the ELECTRE methods, this is referred to as an outranking relation. Other outranking methods rely on a different type of binary relation. For instance, the PROMETHEE methods [56] use a preference relation indicating whether an alternative $a$ is preferred over another alternative $b$. It is worth noting that the binary relations considered by outranking methods are not necessarily complete and transitive.

Typically, outranking models have an elaborate structure and often require the specification of several parameters. As an example, we briefly refer to the fuzzy outranking relation used in methods like ELECTRE III (problematic $P_\gamma$) and ELECTRE TRI (problematic $P_\beta$). The comparison of alternative $a$ against $b$ through the examination of the relation $a\ S\ b$, starts with the calculation of the concordance index, which measures, on a 0–1 scale, the strength of the evidence in support of the considered relation:

$$C\left(a,b\right)=\sum_{k=1}^{n}w_{k}c_{k}\left(a,b\right), \tag{5.11}$$

where $w_1, w_2, \ldots, w_n$ are the weights of the criteria, and $c_k(a,b)$ is the partial concordance index, which measures the strength of the affirmation "alternative $a$ is at least as good as alternative $b$ on criterion $g_k$." The partial concordance index is defined by two parameters, namely the preference and indifference threshold, denoted by $p_k$ and $q_k$, respectively:

$$c_k\left(a,b\right)=\begin{cases}0 & \text{if } g_k(b)-g_{k(a)}\geq p_k\\ \frac{g_k(a)-g_k(b)+p_k}{p_k-q_k} & \text{if } q_k<g_k(b)-g_k(a)<p_k.\\ 1 & \text{if } q_k\geq g_k(b)-g_k(a)\end{cases} \tag{5.12}$$

Except for the concordance index, a discordance index is also defined for each decision criterion, measuring the strength of the evidence against the relation $a\ S\ b$:

$$D_k\left(a,b\right)=\begin{cases}0 & \text{if } g_k(a)\geq g_k(b)-p_k\\ \frac{g_k(a)-g_k(b)+p_k}{p_k-v_k} & \text{if } q_k<g_k(b)-g_k(a)<p_k\\ 1 & \text{if } g_k(a)\leq g_k(b)-v_k\end{cases}, \tag{5.13}$$

where $v_k$ is a veto threshold parameter defined by the DM.

The combination of the positive and negative evidence for the relation $a\ S\ b$, is performed through the credibility index $\sigma(a,b)$, which measures the overall credibility of the relation taking into account both the concordance and discordance indices:

$$\sigma\left(a,b\right)=\begin{cases}0 & \text{if } F=\varnothing\\ C\left(a,b\right)\prod_{g_k\in F}\frac{1-D_k(a,b)}{1-C(a,b)} & \text{if } F\neq\varnothing\end{cases}, \tag{5.14}$$

where $F=\{g_k\,|\,D_k(a,b)>C(a,b)\}$. In the second step of the analysis, given a set of $m$ alternatives, the credibility indices derived from all pairwise comparisons are used as inputs to algorithms for the derivation of the final evaluation results, i.e., the ranking or classification of the given alternatives.

#### 5.3.2.2 Inference Procedures

It is evident from the above brief description, that the complex structure of typical outranking models, requires the specification of several parameters (e.g., weights, indifference/preference/veto thresholds). PDA approaches can facilitate the construction of such models. However, the complexity of outranking methods poses some challenges, because following similar ideas as those presented in the previous subsection for value function models, leads to non-convex optimization problems, as first shown by Mousseau and Słowiński [9] for the ELECTRE TRI method.

To overcome this difficulty, two main approaches have been adopted. The first is based on using simplifications or variants of outranking models, that lead to trackable optimization formulations. For instance, Mousseau et al. [57] presented a linear programming formulation for inferring the weights of the criteria in an ELECTRE TRI model, with fixed indifference/preference/veto thresholds. Bisdorff et al. [58] also considered the elicitation of the criteria weights from a given set of outranking relations, through a mixed-integer linear programming model, focusing on the stability of the results. Regarding other parameters of outranking models, Dias and Mousseau [59] presented a mathematical programming formulation to infer the veto parameters in ELECTRE III/TRI, assuming the other parameters are given, whereas The and Mousseau [60] and Cailloux et al. [61], focused on sorting problems and presented formulations for inferring the category limits that discriminate the classes in the ELECTRE TRI method. Mathematical programming approaches have also employed for PROMETHEE methods [62–65], which have a simpler additive structure that allows the inference of the required preference parameters through optimization formulations that are computationally tractable. Other studies, such as those of Mousseau and Dias [66] as well as Sobrie et al. [67] presented variants of existing models, which enable the implementation of PDA procedures that reduce the computational burden of the inference procedure.

An alternative approach to cope with the complexity of PDA schemes for outranking models involves the use of heuristics and metaheuristics. Doumpos and Zopounidis [68], presented a heuristic algorithm to infer all parameters of an ELECTRE TRI model from classification data, whereas Belahcène et al. [69] presented a satisfiability problem formulation for the inference of a non-compensatory model from a reference set of classification assignments. Several metaheuristics have also been employed, such as reduced variable neighborhood search [70], the differential evolution algorithm [8], as well as single-objective and multi-objective genetic algorithms [71, 72]. Finally, it is worth noting that the inference of relational models based on the principles of the outranking theory, has also been considered through well-known machine learning approaches, such as kernel methods [73] and neural networks [74]. Such approaches enable the inference of outranking models from large data sets, thus extending the range of applicability of outranking MCDA methodologies in various data-intensive domains.

### *5.3.3 Rule-Based Models*

Except for functional and relational models, a third major class of decision models in MCDA involves symbolic models, typically expressed in the form of IF-THEN decision rules. The natural language form of decision rules makes them easy to comprehend by DMs. Such models have been widely used in machine learning. The most widely used rule-based approach in the context of MCDA is based on the rough set theory of Pawlak [75]. Rough sets have been initially introduced as a data mining/machine learning methodology to describe dependencies between attributes and to deal with inconsistent data in classification problems. The key idea of rough sets is that sets, or classes of observations, are often impossible to be described in an exact and accurate manner by data, due to inconsistencies, vagueness, and errors. In such cases, rough approximations can be useful, allowing the identification of the cases that are certainly members of a class, or the cases that may be members of a class.

Soon after its introduction, Pawlak and Słowiński [76] extended the rough sets theory to consider decision criteria and preference ordered classes, thus allowing the modeling of MCDA classification/sorting problems. While the traditional theory of rough sets relies on an indiscernibility relation for the specification of the rough approximations, in the MCDA framework, the dominance relation is used instead [77]. Since the introduction of the new dominance-based rough sets approach (DRSA) in the context of MCDA, significant research has been conducted on the use of this approach for preference modeling and decision aiding purposes. Among others, one can note extensions to choice and ranking problems [10, 78–80], group decision-making [81], and multi-objective optimization [82, 83]. Moreover, the interactions of RST with other related theories and methodologies have been considered, such as probabilistic/stochastic approaches [84] and fuzzy sets theory [85, 86], whereas Greco et al. [79] considered the integration of DRSA with disaggregation methods based on value functions for ranking problems. Dembczyński et al. [15] and considered the integration of MAVT with DRSA, presenting an approach to construct an additive value model from dominance-based rough approximations.

## 5.4 The Robustness Concern

This section focuses on the robustness of PDA methodologies, which has emerged as an important area of research for the analysis of the quality of the results obtained from PDA. We start with a brief outline of the robustness issue and its implications, and then proceed with an overview of the related literature, covering both methodological developments and experimental results.

### *5.4.1 The Issue of Robustness in PDA*

Robustness is an active research topic in MCDA and operations research in general. In the context of decision aiding, robustness concerns arise due to the uncertainties, fuzziness, vagueness, and errors that characterize the parameters and data of a decision problem. A comprehensive discussion of the robustness concern in decision aiding can be found in the works of Roy [87] and Vincke [88].

The robustness concern is particularly important in PDA, as it involves a type of an ex-ante analysis of the quality of the model inference procedure and the derived results. As explained in Sect. 5.2, the model inference procedure in PDA starts with a set $A_R$ of reference alternatives and the parameters of the model are specified through the optimal solution of the optimization problem (5.1). However, often there is no unique optimal solution to that problem, thus leading to a set of optimal solutions $\mathcal{X}^*$. Even through all solutions in $\mathcal{X}^*$ correspond to decision models that provide similar results (possibly identical) for the reference example in $A_R$, they may lead to very different results when applied to instances outside the reference set. Moreover, it should be noted that even in cases where problem (5.1) has a unique optimal solution, it may not be robust to changes in the problem data.

The robustness issue has important implications for the quality of decision aid provided through PDA approaches and the derived recommendations or findings. On the one hand, non-robust inferences about the parameters of the model may lead to incorrect interpretations about the characteristics of the problem and create confusion. For instance, if the solutions in $\mathcal{X}^*$ correspond to criteria weights with a large range, this may lead to doubts regarding the usefulness of the results with respect to the information that they provide about the important aspects of the problem. On the other hand, if minor changes in the problem data (e.g., addition or removal of reference alternatives) lead to major changes in the outputs of the model inference process, this also raises concerns about the validity of the results. Therefore, identifying and addressing such issues is of major importance for the successful implementation of PDA methodologies.

### *5.4.2 Methodological Approaches*

Four main areas of research can be identified in the literature on the robustness concern in PDA. The first stream of the literature focuses on approaches for the description and characterization of the set $\mathcal{X}^*$ of models that are compatible with the judgments of the DM as represented by the reference set. Jacquet-Lagrèze and Siskos [5], in the context of the UTA method, first noted that $\mathcal{X}^*$ is often non-singleton and proposed a simple heuristic post-optimality analysis approach to identify a set of characteristic solutions from $\mathcal{X}^*$, instead of relying on one. In their post-optimality analysis, the set $\mathcal{X}^*$ consists of all optimal or near-optimal solutions of problem (5.5).[1] The authors proposed the search of solutions in $\mathcal{X}^*$

[1] In case of an inconsistent referent set, the optimal objective function value of problem (5.5) is positive. In this case $\mathcal{X}^*$ may also include near optimal solutions, corresponding to objective function values similar to the optimal one (according to a tolerance parameter defined by the analyst and the DM).

that maximize and minimize the weights of the criteria in the additive model of the UTA method, thus identifying a range for the weights of the criteria in which the resulting decision model is consistent with the rank-order of the reference alternatives. Despite the simplicity of such post-optimality techniques, they provide a limited view of the complete set of models that are compatible with the DM's preferences. A more thorough analysis requires analytic approaches, which can be computationally intensive for large problems [89]. As an alternative, simulation methods have been considered [90, 91].

A second group of studies has examined improved optimization formulations to select a good representative model from the set $\mathcal{X}^*$. This line of research is closely related to the approaches discussed above, in the sense that having a description of the set $\mathcal{X}^*$, it is natural to ask what the most representative model among the existing candidates. In their post-optimality approach, Jacquet-Lagrèze and Siskos [5] use the average of the extreme solutions that define the range of the criteria weights. The reasoning behind this approach is that the optimization formulation (5.5) for the UTA method is expressed in linear programming form, which implies that $\mathcal{X}^*$ is a polyhedron. The extreme solutions derived through post-optimality analysis correspond to some representative vertices of the polyhedron, and their average approximates the barycenter of polyhedron. Such a solution is expected to be more robust to changes in the problem data (i.e., changes in the polyhedron $\mathcal{X}^*$) compared to other solutions that are closer to the boundaries of the polyhedron. However, given that the post-optimality approach of Jacquet-Lagrèze and Siskos [5] is a heuristic, other approaches have been proposed. Beuthe and Scannella [92] presented a comparison of various alternative post-optimality formulations for selecting the more representative model, whereas other studies have considered formulations based on the concepts of the analytic center [28] and the Chebyshev center [93]. The experimental analysis of Doumpos et al. [93] showed that such formulations are promising alternatives compared to barycenter solutions derived through post-optimality analysis. The selection of a representative model has also been considered in the context of an interactive decision aiding process [42].

The third stream on the literature about robustness in PDA has focused on deriving recommendations from a set of decision models, instead of one. This idea was first presented by Siskos [35] in the context of the UTA method, as a way to build fuzzy preference relations based on the results of post-optimality analysis. Later this approach was extended to consider not only the (limited) results of post-optimality procedures but all models compatible with the evaluations of the reference alternatives. This approach was first introduced for ranking problems with additive value models [7] but has been also been employed in various other contexts, such as for outranking relations [94], rule-based models [95], for sorting problems [96], as well as for problems with a hierarchical structure [97], and group decision-making [36].

A final group of studies focuses on the development of robustness measures. These measures may depend on the post-optimality analysis results, and especially on the form and the extent of the polyhedron of the linear programs (5.5), (5.7), or (5.10). In this context, stability measures are based on the variance observed in the post-optimality results. For example, Grigoroudis and Siskos [45] proposed an Average Stability Index (ASI), which is basically the average of the normalized standard deviation of the estimated values $v_k\left(g_k^*\right)$. In a similar context, instead of exploring only the previous extreme values, ASI may be calculated based on the investigation of every value $v_k(\cdot)$ during post-optimality analysis [47]. A general form for ASI may be assessed as the average value of the normalized standard deviation of the estimated preferential parameters:

$$\mathrm{ASI} = 1 - \frac{1}{S}\sum_{s=1}^{S}\sqrt{\frac{R\sum_{r=1}^{R} p_{rs}^2 - \left(\sum_{r=1}^{R} p_{rs}\right)^2}{R\sum_{r=1}^{R} p'^2_{rs} - \left(\sum_{r=1}^{R} p'_{rs}\right)^2}}, \tag{5.15}$$

where $S$ and $R$ are the number of parameters and the number of instances examined during the post-optimality analysis, respectively, $p_{rs}$ is the $r$th instance of the $s$th parameter and $p'_{rs}$ is the possible value of $p_{rs}$ that maximizes its variance during the post-optimality procedure. ASI ranges in [0, 1] and takes the value of 1 only in case of perfect robustness. A detailed discussion of the ASI is given by Siskos and Grigoroudis [47] and Matsatsinis et al. [98].

Robustness measures may be incorporated in a general interactive disaggregation and robustness control framework. Siskos and Grigoroudis [47] propose the following steps:

(a) An applied PDA method is used to infer a representative model based on $A_R$.
(b) Potential inconsistencies between the DM's preferences and the results of the PDA method are removed using interactive techniques [99, 100].
(c) A robustness measure (e.g., ASI) is estimated.
(d) If the robustness measure is considered satisfactory by the analyst, the model is proposed to the DM, it can be extrapolated in $A$, and the process is terminated. Otherwise, the process goes to step (e).
(e) If the robustness measure is not satisfactory, alternative rules of robustness analysis are examined. These rules may include the addition of new global preference judgments (e.g., pairwise comparisons, preference intensities new reference actions), the visualization of the observed value variations to support the DM in choosing his/her own model, the development of new preference relations on the $A$ during the extrapolation phase, etc. The process goes back to step (c).

An extension of the previous framework has been proposed by Siskos and Psarras [101] who developed an interactive bipolar robustness control. Their approach manages robustness in both phases/poles of the interactive decision support process: disaggregation phase and aggregation phase. Specifically, the robustness control process is initiated after the inference of the PDA model, where the aforementioned steps may be applied. In the reverse direction, the process moves from the disaggregation to the aggregation pole, where the PDA model is extrapolated to $A$ and the stability of results is evaluated through an appropriate robustness measure.

Matsatsinis et al. [98] propose several robustness measures on the aggregation pole: average range of the ranking (ARRI) (possible number of positions that an average action can occupy in the whole ranking), Ratio of the average range of the ranking (RARR) (ratio of the deviation in ARRI, with respect to the whole number of the alternatives under evaluation), and Statistical preference relations index (SPRI) (probability that an alternative gets ranked in a specific position based on results of random sampling techniques). In case of a satisfactory robustness the algorithm ends; otherwise, the analyst returns to the disaggregation pole and asks the DM for the additional preferential information. This approach can be applied not only to PDA methods but also to other MCDA approaches.

### *5.4.3 Experimental Studies*

Except for theoretical and methodological developments, the robustness properties of PDA approaches have also been examined through experimental studies. Most of these experimental studies employ simulated data sets, generated with predefined properties with respect to the DM's preference structure and the characteristics of the reference set (e.g., number of criteria or alternatives). Such simulation tests, provide a controlled environment to test PDA approaches against well-defined hypotheses.

For instance, Vetschera et al. [102] examined two sorting approaches, namely the weighted average model and a case-based model, with respect to their robustness and accuracy. Among others, they considered the effect that the number of reference alternatives has on the quality of the model inference results, as well as the effect that the complexity of the problem has (i.e., number of criteria and categories).

In a similar sorting context, Doumpos et al. [93] compared various approaches for inferring a representative additive value funcion from classification data. Their results confirmed that central solutions to the polyhedron $\mathcal{X}^*$ yield more robust decision models, with the discrepancies among different approaches becoming more noticeable when the size of $\mathcal{X}^*$ increases. Moreover, the number of reference alternatives and the complexity of the model (number of free parameters) were found to have a significant impact on robustness.

In contrast to the previous two studies, Kadziński et al. [103] examined the inference of decision models (additive functions) from holistic judgments provided in the form of pairwise comparisons and noticed that there is a tradeoff between the robustness of an additive model and its "expressiveness," which refers to the ability of the model to describe a given set of judgments by the DM. This implies, that increasing the complexity of a model in order to improve its expressiveness, may have a negative effect on robustness. The authors additionally considered different approaches for the specification of an additive model with piecewise linear marginal value functions in order to achieve a good balance between the two objectives (expressiveness and robustness).

Finally, Rangel-Valdez et al. [104] examined the inference of ELECTRE III models in the context of PDA using a genetic algorithm, focusing on the effect that noisy data have on the results and found that even if there is a moderate level of errors in the judgments of the DM, a PDA can still provide reliable results.

## 5.5 Overview of Applications and Decision Support Systems

The theoretical and methodological advances in the field of PDA have been accompanied by various applications in different domains of management and engineering. Table 5.1 presents an indicative list of applications, covering studies published since 2015. The studies in the table are categorized by the field of application and the types of models used (MAVT-multiattribute value theory, OR-outranking relations, DRSA - dominance-based rough sets approach).

The implementation of PDA approaches in practical applications has been greatly facilitated by the development of decision support systems (DSSs). In the literature,

**Table 5.1** Indicative list of recent applications of PDA approaches (since 2015)

| Field of application | Studies | Model types |
|---|---|---|
| Energy | [105–110] | MAVT |
| Strategy and quality management | [111–113] | MAVT |
| Finance | [114, 115] | MAVT |
| | [116, 117] | DRSA |
| | [118] | MAVT and OR |
| | [119] | OR |
| | [120] | TOPSIS |
| Education | [121, 122] | DRSA |
| | [20, 123] | MAVT |
| Manufacturing | [124, 125] | MAVT |
| Marketing, consumer behavior and customer satisfaction | [65] | OR |
| | [126–132] | MAVT |
| | [133] | DRSA |
| Project portfolio management | [134] | DRSA |
| | [135] | MAVT |
| Public administration | [136] | MAVT |
| Environmental management and sustainability | [137–142] | MAVT |
| | [143] | OR |
| | [144] | DRSA |
| Transportation | [145, 146] | MAVT |
| Medicine and healthcare | [147, 148] | DRSA |
| | [149–154] | MAVT |
| | [155] | OR |

several PDA-based DSSs have been presented, which can be categorized into two main groups:

- The first group involves systems developed as general-purpose tools that can be applied in various decision aiding contexts. Some examples of such systems are listed in Table 5.2. Most of the DSSs in this group have been developed following the standard principles of DSS technology, i.e., in the form of fully functional software, integrating a graphical user interface with database management, a model base, and reporting capabilities. Recently, however, efforts have been made to adopt an open-source approach, focusing on the implementation of PDA approaches and other MCDA methods through standardized protocols, thus allowing independent developers to contribute collectively to the development of MCDA general-purpose software and DSSs. A typical example of this approach is the DIVIZ system [156], which is based on the XMCDA data standard for MCDA [161]. Toolboxes and packages for open-source software such as R have also been developed [162].
- The second group includes DSSs that have been developed for specific fields. Such systems, except for PDA methods, enable the DMs to use a variety of other analytical and reporting tools that are tailored to the requirements of each application area. Some examples are shown in Table 5.3.

**Table 5.2** General-purpose DSSs implementing PDA methods

| DSS | Model types |
|---|---|
| DIVIZ [156] | Various |
| ELECTRE TRI Assistant [157] | Outranking |
| IRIS [158] | Outranking |
| MINORA [99] | MAVT |
| MIIDAS [39] | MAVT |
| RAVI [159] | MAVT |
| RACES [160] | MAVT |

**Table 5.3** Domain-specific DSSs implementing PDA methods

| DSS | Model types | Field |
|---|---|---|
| FINEVA [163] | MAVT | Finance |
| FINCLAS [164] | MAVT | Finance |
| MARKEX [165] | MAVT | Marketing |
| TELOS [166] | MAVT | Marketing |
| [137] | MAVT | Environmental planning |
| [167] | DRSA, TOPSIS | Territorial planning |
| [168] | MAVT | Finance |

## 5.6 Conclusions and Future Perspectives

The PDA paradigm is a powerful approach for constructing decision models in a multicriteria setting, allowing the DM to describe his/her system of preferences through holistic judgments and decision examples, rather than direct procedures, which can pose cognitive limitations. Over the past 40 years, this area of MCDA research has evolved rapidly, and now covers a wide variety of different types of decision models and contexts. Given that many decision problems in various fields become more and more data-driven, PDA tools could be particularly useful for preference modeling and decision aiding.

This chapter presented an overview of the developments and trends in this area, covering different types of decision models, highlighting new trends in model inference procedures, as well as applications and DSS implementations.

Given the active research on the PDA approach of MCDA, various interesting future research trends and directions can be noted. For instance, the recent trend in exploring the connections between PDA and related techniques from the field of artificial intelligence (AI) could extend the current PDA approaches to a broad range of applications where preference learning plays a crucial role. Moreover, this will allow to take advantage of developments in AI research for improving the existing methodologies for inferring decision models from data in a PDA context. Moreover, the investigation of behavioral issues for the successful development and application of PDA methods is also an interesting research area [169, 170]. Such issues may involve the role that potential biases and errors in holistic judgments may have in the model inference process, and the protocols needed to collect an adequate set of input data for PDA. Reporting and visualization could also be an interesting research area [159] to design improved interactive approaches that would facilitate the communication between analysts and DMs. Finally, further research is required on procedures and metrics for assessing and validating the results of PDA methodologies, with respect to their robustness and effectiveness.

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# Chapter 6
# Modeling and Learning of Hierarchical Decision Models: The Case of the Choquet Integral

**Eyke Hüllermeier and Christophe Labreuche**

**Abstract** In this paper, we elaborate on two important developments in the realm of multi-criteria decision aid, which have attracted increasing attention in the recent past: first, the idea of leveraging methods from preference learning for the data-driven (instead of human-centric) construction of decision models, and second, the use of hierarchical instead of "flat" decision models. In particular, we show the advantage of combining the two, that is, of learning hierarchical MCDA models from suitable training data. This approach is illustrated by means of a concrete example, namely the learning of tree-structured combinations of the Choquet integral as a versatile aggregation function.

## 6.1 Introduction

To support a decision maker in the task of choosing among a set of alternatives, ranking these alternatives from best to worse, or sorting them into preferential categories, various methodologies have been developed in the decision sciences. Multi-criteria decision aid (MCDA), for example, puts specific emphasis on decision problems in which alternatives are characterized by their values on multiple, possibly conflicting criteria [17, 18]. To this end, MCDA offers a wide variety of decision models, most of which aggregate the evaluations on individual (local) criteria into an overall assessment of an alternative.

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Such models can be as simple as a weighted average, but may also aggregate criteria in a more sophisticated manner. The (discrete) Choquet integral [8] can be mentioned as a specific though important example of a sophisticated model of that kind. Regardless of the type, the construction of models in MCDA is typically accomplished in the course of an interactive process, in which a decision analyst seeks to elicit the decision maker's preferences by asking informative questions, and to fit the model to these preferences as closely as possible [4].

There are (at least) two recent developments in the field, which, in the opinion of the authors, significantly increase the usefulness, performance capacity, effectiveness, and applicability of decision aid and automated decision making, all the more if being combined with each other. The first of these developments is the data-driven (instead of human-centric) construction of decision models, i.e., the use of methods from preference *learning* instead of preference *elicitation*. This development is fostered by the increasing availability of data and popularity of machine learning. Preference learning (PL) is geared toward the automated induction of models from large amounts of data [11]. Obviously, replacing the interaction with a single decision maker by a process of learning from data implies a number of differences between the settings of PL and MCDA and the underlying assumptions. For example, a model in PL typically refers to an entire population rather than to a single individual. Moreover, while user feedback in MCDA is assumed to be consistent, or inconsistencies can be repaired by the decision maker, PL generally tolerates noise in the observed data. Last but not least, PL puts very much emphasis on generalization performance and predictive accuracy.

The second development is the use of hierarchical instead of "flat" decision models. In the multi-criteria case, such models typically decompose criteria into sub-criteria in a recursive manner, and the evaluation of the former is then obtained through an aggregation of the evaluation of the latter. This allows for conquering complexity through abstraction and hierarchical structuring. Indeed, the complexity of flat models significantly increases with the number of criteria involved (unless strong independence assumptions are made), hampering both model construction and interpretation.

The goal of this paper is to elaborate on recent advances in hierarchical modeling and preference learning for MCDM, showing advantages in comparison to previous methods, and illustrating the learning of hierarchical MCDA models by means of a concrete example, namely the learning of tree-structured combinations of the Choquet integral as a versatile aggregation function.

## 6.2 Hierarchical Multi-Criteria Decision Models

We are interested in multi-criteria decision aid using hierarchical models and refer to models supporting this task as "hierarchical multi-criteria decision models" (HMCDM). This section provides the necessary background on MCDA, HMCDM, and the Choquet integral as a specifically interesting aggregation function.

### 6.2.1 Multi-Criteria Decision Models

We assume choice alternatives to be described in terms of a predefined set of attributes, i.e., by a feature vector $\boldsymbol{x} = (x_1, \ldots, x_m)$, where $x_i \in X_i$ and $\boldsymbol{x} \in X = X_1 \times \cdots \times X_m$. We denote by $M = \{1, \ldots, m\}$ the index set of the attributes. Moreover, we assume that each attribute $x_i$ is first evaluated by means of a *marginal utility function*, and thereby turned into a *local utility degree* $u_i = u_i(x_i) \in \mathbb{R}$. A *criterion* is a marginal utility function associated with an attribute. By abuse of notation, we also denote by $M$ the set of criteria. These marginal utility functions are supposed to represent partial preferences of the decision maker. Assume we are given domain knowledge in the form of a binary relation $\succsim_i$ on $X_i$, where $x_i \succsim_i x_i'$ suggests that the decision maker prefers $x_i$ at least as much as $x_i'$ ceteribus paribus. Then, the marginal utility function shall be consistent with $\succsim_i$:

$$\forall x_i, x_i' \in X_i \ : \quad u_i(x_i) \geq u_i(x_i') \ \Leftrightarrow \ x_i \succsim_i x_i'. \tag{6.1}$$

We denote by $\succ_i$ and $\sim_i$ the asymmetric and symmetric parts of $\succsim_i$, respectively.

In a second step, the local utility degrees $u_1, \ldots, u_m$ are aggregated into a *global utility* $U = A(u_1, \ldots, u_m)$, where $A : \mathbb{R}^m \to \mathbb{R}$ is an aggregation function. In MCDA, the overall utility is meant to represent preferences of the decision maker over alternatives in $X$, taking the form of a binary relation $\succsim$, where

$$\forall \boldsymbol{x}, \boldsymbol{x}' \in X : \quad \boldsymbol{x} \succsim \boldsymbol{x}' \ \Leftrightarrow \ U(\boldsymbol{x}) \geq U(\boldsymbol{x}') \, . \tag{6.2}$$

We then denote by $\succ$ and $\sim$ the asymmetric and symmetric parts of $\succsim$, respectively. The parameters of the models of this kind, i.e., that of the marginal utility functions $u_i$ as well as of the aggregation function $A$, can be specified in different ways—for example, as already mentioned, through preference elicitation [4]. In preference learning, where the data is mainly observational, one commonly assumes that attributes $\boldsymbol{x}$ are provided as training information together with a decision $y$ (for example, the rank assigned to an alternative). This decision is supposedly taken on the basis of the utility $U$, which, however, is a latent variable that is not observed directly. Because the data might not be consistent, and human decisions might be subject to additional influences that are not captured by the model, the dependence between $U$ and $y$ is modeled in terms of a probabilistic model $P$. Eventually, we thus obtain a model that can be written (in a somewhat simplified form) as follows:

$$y = P(U(\boldsymbol{x})) = P\Big(A\big(u_1(x_1), \ldots, u_m(x_m)\big)\Big).$$

The general structure of a multi-criteria decision model (MCDM) of that kind is shown in Fig. 6.1.

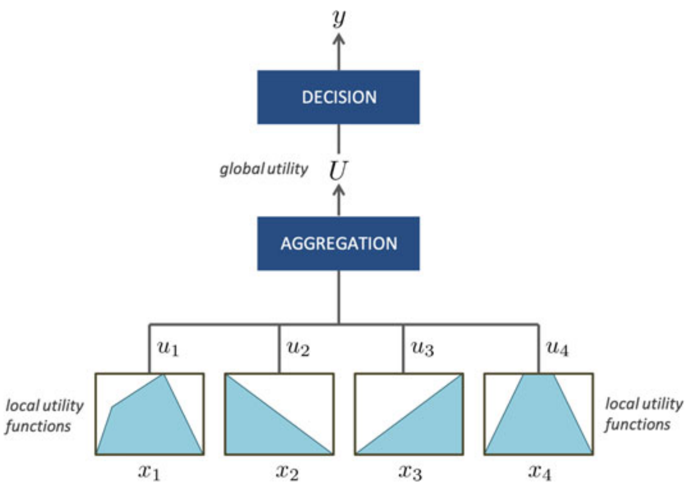



**Fig. 6.1** General structure of an MCDM model

### 6.2.2 Choquet Integral

Let us consider an aggregation function $A : \mathbb{R}^m \rightarrow \mathbb{R}$ on a set of criteria $M = \{1, \ldots, m\}$. The arguably simplest aggregation model is the weighted sum parameterized by weights $\boldsymbol{w} = (w_1, \ldots, w_m)$, which takes the form $A^{\boldsymbol{w}}(\mathbf{a}) = \sum_{i=1}^{m} w_i\, a_i$. This aggregation model is easy to understand but fails to account for subtle decision strategies, for example, involving interaction among criteria. To overcome the limitations of the weighted sum, a possible extension is to consider weights assigned to subsets of criteria, e.g., accounting for the fact that having several criteria met simultaneously might be preferred to the cumulative effect of having them satisfied separately.

A *fuzzy measure* (also called *capacity*) on a set $M$ is a set function $\mu : 2^M \rightarrow [0, 1]$ satisfying two properties [30]:

$$\text{Normalization: } \mu(\emptyset) = 0 \text{ and } \mu(M) = 1, \tag{6.3}$$

$$\text{Monotonicity: } A \subseteq B \subseteq M \Rightarrow \mu(A) \leq \mu(B). \tag{6.4}$$

The *(discrete) Choquet integral*, parameterized by a fuzzy measure $\mu$, is defined on utility vectors as follows [8]:

$$C_\mu(\boldsymbol{a}) = \sum_{i=1}^{m} (a_{\tau(i)} - a_{\tau(i-1)})\, \mu(\{\tau(i), \tau(i+1), \ldots, \tau(m)\})\,, \tag{6.5}$$

with $\tau$ being a permutation on $C$ such that $a_{\tau(1)} \leq a_{\tau(2)} \leq \cdots \leq a_{\tau(m)}$ and $a_{\tau(0)} = 0$ by definition. A Choquet integral thus involves $2^m - 2$ parameters $\mu(A)$, one for each nonempty subset $A \subset M$.

As an aggregation model, the Choquet integral can easily be interpreted through the mean importance of criteria [28] and the mean interaction among criteria [13, 21], which can be computed from the weights associated with any subset of criteria. The Choquet integral satisfies many properties [12], which contributes to the interpretability of this model. One can mention monotonicity with respect to their inputs: for all $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{R}^m$,

$$(\forall i \in C,\ a_i \geq a_i') \ \Rightarrow\ C_\mu(\boldsymbol{a}) \geq C_\mu(\boldsymbol{a}')\,. \tag{6.6}$$

Another important property is *idempotency* [12, 20]:

$$\forall \alpha \in \mathbb{R}: \quad C_\mu(\alpha, \ldots, \alpha) = \alpha\,. \tag{6.7}$$

It requires that an alternative having the same score $\alpha$ on all criteria should have exactly that score as its overall evaluation.

The drawback of the Choquet integral is its exponential (in $m = |M|$) number of parameters. Several models of intermediate complexity, in-between the weighted sum and the general Choquet integral, have been proposed as a compromise. An especially relevant one is the so-called *2-additive* Choquet integral. It is the subclass of Choquet integrals where interactions among criteria are limited to pairs, and takes the following form [14]:

$$C_\mu(\boldsymbol{a}) = \sum_{i=1}^{m} w_i u_i + \sum_{1 \leq i < i' \leq m} \left( w_{i,i'}^{\wedge} (u_i \wedge u_{i'}) + w_{i,i'}^{\vee} (u_i \vee u_{i'}) \right)\,, \tag{6.8}$$

where $\wedge$ and $\vee$ denote the min and max operators, respectively, and the parameters are the weights $w_i, w_{i,i'}^{\wedge}, w_{i,i'}^{\vee}$. The monotonicity conditions on the 2-additive Choquet integral are

$$\begin{aligned} &\forall i \in M: \quad w_i^j \geq 0,\\ &\forall \{i,i'\} \subseteq M: \quad w_{i,i'}^{j,\wedge} \geq 0,\\ &\forall \{i,i'\} \subseteq M: \quad w_{i,i'}^{j,\vee} \geq 0. \end{aligned} \tag{6.9}$$

The monotonicity property can thus be obtained through $m^2$ constraints. Likewise, the normalization of the 2-additive Choquet integral is obtained by the following condition:

$$\sum_{i=1}^{m} w_i + \sum_{1 \leq i < i' \leq m} \left( w_{i,i'}^{\wedge} + w_{i,i'}^{\vee} \right) = 1. \tag{6.10}$$

### 6.2.3 Hierarchical Multi-Criteria Decision Models

The hierarchical part of an HMCDM essentially concerns the aggregation function $A$, which is represented in terms of a hierarchical, tree-like structure. The organization of a set of criteria in the form of a tree has a long tradition in MCDA [24, 25], but can also be found in more recent approaches, such as the *multiple criteria hierarchy process* [9].

More specifically, we consider a rooted tree $\mathcal{T}$ composed of a set of nodes $N$ and a function $C$ returning the set of children $C(j)$ for every node $j \in N$. The leaves of the tree correspond to the criteria $M$, and the root of $\mathcal{T}$ is denoted by $r \in N \setminus M$. We also denote by $D(j)$ the set of leaves that are descendants of $j$; for instance, $D(r) = M$ and $D(j) = \{j\}$ for every $j \in M$.

We assume that a local aggregation function $A_j : \mathbb{R}^{|C(j)|} \to \mathbb{R}$ is defined for every aggregation node $j \in N \setminus M$. Then, for given marginal utilities $\boldsymbol{a}_M = (a_1, \ldots, a_M)$ on all $M$ criteria, the overall aggregation function is defined as $A(\boldsymbol{a}_M) = \alpha_r$, where the value $\alpha_j$ for a node $j$ is defined in a recursive manner as follows:

$$\alpha_j = \begin{cases} a_j & \text{if } j \in M \\ A_j\left(\bigcup_{i \in C(j)} \alpha_i\right) & \text{if } j \in N \setminus M. \end{cases}$$

When all aggregation functions are Choquet integrals, HMCDM is called *Hierarchical Choquet Integrals* (HCI) model.

## 6.3 Expressiveness of Hierarchical Models: The Case of Choquet Integrals

In this section, we elaborate on an important feature of hierarchical models, namely their expressivity: Using a hierarchical model, it is often possible to represent preferences that are not representable in terms of a flat model, or to represent the same preferences in a more compact form. We illustrate these advantages for the specific case of the Choquet integral as an aggregation function.

### 6.3.1 An Illustrative Example in an MCDA Setting

MCDA is primarily interested in representing the preferences $\succsim$ of a decision maker in terms of a decision model, like in (6.2). In the normative approach to decision science, axiomatic characterizations provide necessary and sufficient conditions under which $\succsim$ (or the overall utility $U$) is representable by some specific models [17]. Let us consider the case where this specific model is a Choquet integral. In the context of decision under uncertainty, all features are identical ($X_1 = \cdots = X_m$), and hence all marginal utility functions, too, yielding a model called Choquet Expected Utility model—CEU [26, 36]. In MCDA, we can mention different results: axiomatization of a single Choquet integral as an aggregation function [20], axiomatization of a single Choquet integral and its marginal utility functions as a utility model [19], and axiomatic representation of a preference relation by a single Choquet integral and its marginal utility functions [35]. Yet, we are not aware of any axiomatic characterization of a general hierarchical Choquet integral model.

On the experimental side, there are some works in psychology and cognitive science, analyzing the behavior of decision makers in practice and checking whether they are representable by the models studied in the normative approach. In this setting, preferential paradoxes are examples of simple preferences that a user would in general accept, and which can be represented by an elaborate model though not by a simpler one. Let us mention the Allais paradox [1] in the context of decision under risk and the Ellsberg paradox [10] in the context of decision under uncertainty. For instance, in Ellsberg's paradox, the user is asked to compare several lotteries, in which she picks up a ball from an urn containing balls of three colors and receives some amount of money depending on the color of the ball, where the number of balls of each color is only partly known. Here, users tend to be risk-averse and prefer those bets that minimize the worst possible gain—such preferences can be represented by CEU but not by the usual expected utility model.

In the context of MCDA, there are simple examples illustrating the interest of a Choquet integral compared to a weighted sum [12]. The aim of this section is to provide such an example for an HMCDM.

To this end, consider a simple example of the selection of a car on the basis of three criteria: (1) CAPEX (capital expenditure), the buying cost of the car, (2) OPEX (operational expenditure), including gas, maintenance and so on, and (3) safety rating from 1 (very bad) to 5 (very good). The preference relations $\succsim_1, \succsim_2, \succsim_3$ are clear as one wishes to minimize the value on the first two attributes and maximize the value on the third one. We consider that attributes 1 and 2 are bounded continuous domains—typically finite intervals, which perfectly makes sense for costs. Thus for each attribute $i \in M$, there exist a best and a worst element, denoted by $\mathbb{1}_i$ and $\mathbb{O}_i$, respectively, such that $\forall x_i \in X_i$, $\mathbb{1}_i \succsim_i x_i \succsim_i \mathbb{O}_i$. We assume that

$$u_i(\mathbb{1}_i) = 1 \quad \text{and} \quad u_i(\mathbb{O}_i) = 0\,. \tag{6.11}$$

The decision maker describes the following preferences in the form of rules:

**R1:** She is interested in alternatives that are good at both costs and safety. She is not happy if one of these concerns are not met.

**R2:** Criteria 1 and 2 compensate each other: Improving (significantly) on a cost criterion fully compensates a degradation on the other cost criterion.

We fix the value of attribute 3 and look at the preferences in the subspace $X_1 \times X_2$. Let us fix $c \in X_3$ with $\mathbb{1}_3 \succ_3 c \succ_3 \mathbb{O}_3$. Rule R1 says that the decision maker is extremely intolerant regarding costs and safety. In other words, when she considers an alternative, she systematically focuses on the weaker of the two aspects, completely discarding the stronger one. When safety is the weak point, then the relatively good values on costs do not matter, so that replacing the current values $x_1, x_2$ of costs by the best possible values $\mathbb{1}_1, \mathbb{1}_2$ would not help. Hence, such a situation clearly corresponds to an alternative in $X^c_\sim$ defined by

$$X^c_\sim := \{(x_1, x_2, c) \in X \ : \ (x_1, x_2, c) \sim (\mathbb{1}_1, \mathbb{1}_2, c)\}.$$

According to rule R1, the previous situation often occurs, so that $X^c_\sim$ is not reduced to a singleton set $\{(\mathbb{1}_1, \mathbb{1}_2, c)\}$. Hence,

$$\exists(\overline{x}_1, \overline{x}_2, c) \in X^c_\sim \ \text{ with } \ \overline{x}_1 \prec_1 \mathbb{1}_1 \text{ and } \overline{x}_2 \prec_2 \mathbb{1}_2\,. \tag{6.12}$$

As $X_1, X_2$ are closed intervals and by virtue of monotonicity, we obtain that $[\overline{x}_1, \mathbb{1}_1] \times [\overline{x}_2, \mathbb{1}_2] \times \{c\} \subseteq X^c_\sim$. Note that condition (6.12) is far from being classical, as most practical models are strictly monotone, which is not the case for (6.12).

Now, when costs are the weakest points, then we are in the following set:

$$X^c_\prec := \{(x_1, x_2, c) \in X \ : \ (x_1, x_2, c) \prec (\mathbb{1}_1, \mathbb{1}_2, c)\},$$

as replacing the cost values by the ideal costs $\mathbb{1}_1, \mathbb{1}_2$ would clearly improve the overall preference. By rule R1, this situation also often occurs, so that $X^c_\prec$ cannot be empty or reduced to a singleton set. More precisely,

$$\exists(\underline{x}_1, \underline{x}_2, c) \in X^c_\prec \ \text{ with } \ \underline{x}_1 \succ_1 \mathbb{O}_1 \text{ and } \underline{x}_2 \succ_2 \mathbb{O}_2. \tag{6.13}$$

By virtue of monotonicity, $[\mathbb{O}_1, \underline{x}_1] \times [\mathbb{O}_2, \overline{x}_2] \times \{c\} \subseteq X^c_\prec$.

Rule R2 implies that for any $(x_1, x_2, c_3) \in X$ with $x_1 \prec_1 \mathbb{1}_1$ and $x_2 \succ_2 \mathbb{O}_2$, there exists $x'_1 \succ_1 x_1$ and $x'_2 \prec_2 x_2$ such that $(x_1, x_2, c_3) \sim (x'_1, x'_2, c_3)$. In other words, one can fully compensate a degradation on attribute 2 by a (sufficient) improvement on attribute 1, and vice versa. Hence,

$$\begin{aligned}&(x_1, x_2, c_3) \in X^c_\prec \text{ with } x_1 \prec_1 \mathbb{1}_1 \text{ and } x_2 \succ_2 \mathbb{O}_2\\ &\quad \Rightarrow \ \exists x'_1 \succ_1 x_1 \text{ and } x'_2 \prec_2 x_2 \ : \ (x'_1, x'_2, c_3) \in X^c_\prec\,.\end{aligned} \tag{6.14}$$

Let us try to represent the previous preferences with a flat model

$$U(\boldsymbol{x}) = C_\mu(u_1(x_1), u_2(x_2), u_3(x_3)). \tag{6.15}$$

**Lemma 6.1** *The flat model (6.15) cannot represent the previous preferences.*

***Proof*** By (6.12), (6.13), and (6.14), for any $\mathbb{O}_3 \prec_3 c \prec_3 \mathbb{1}_3$, there exists $\mathbb{O}_1 \prec_1 x_1 \prec_1 x_1' \prec_1 x_1'' \prec_1 \mathbb{1}_1$ (i.e. $0 < u_1(x_1) < u_1(x_1') < u_1(x_1'') < 1$) and $\mathbb{O}_2 \prec_2 x_2 \prec_2 x_2' \prec_2 \mathbb{1}_2$ (i.e. $0 < u_2(x_2) < u_2(x_2') < 1$) such that

$$(x_1, x_2, c) \in X^c_{\prec}\ ,\ (x_1', x_2, c) \in X^c_{\sim}\ ,\ (x_1', x_2', c) \in X^c_{\prec} \text{ and } (x_1'', x_2', c) \in X^c_{\sim}.$$

These four points are represented in Fig. 6.2. We know that the Choquet integral is characterized by the separation frontiers $u_1(x_1) = u_2(x_2)$, $u_1(x_1) = u_3(c)$ and $u_2(x_2) = u_3(c)$—see Fig. 6.3. There are three segments between the successive points $(x_1, x_2, c)$, $(x_1', x_2, c)$, $(x_1', x_2', c)$, and $(x_1'', x_2', c)$ (see the blue segments in Fig. 6.3). One can readily see that these three segments cannot be cut by the three separation frontiers. For instance, if segment $[(x_1, x_2, c), (x_1', x_2, c)]$ is cut by $u_1(x_1) = u_3(c)$ and segment $[(x_1', x_2, c), (x_1', x_2', c)]$ is cut by $u_2(x_2) = u_3(c)$,

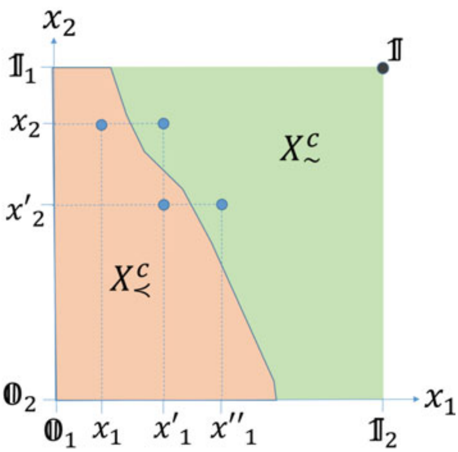



**Fig. 6.2** Representation of sets $X^c_{\sim}$ (in green), $X^c_{\prec}$ (in orange), and of the four points $(x_1, x_2, c)$, $(x_1', x_2, c)$, $(x_1', x_2', c)$ and $(x_1'', x_2', c)$

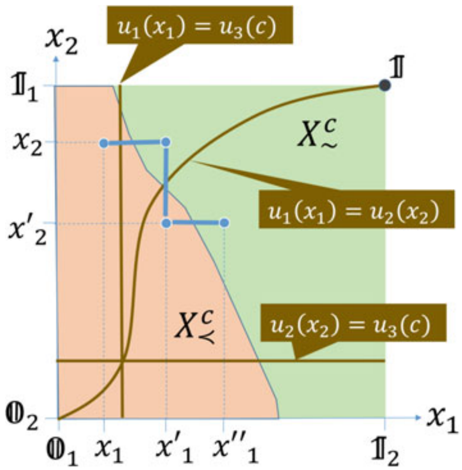



**Fig. 6.3** Representation of three separation frontiers (in brown)

then the last segment $[(x'_1, x'_2, c), (x''_1, x'_2, c)]$ cannot be cut by $u_1(x_1) = u_2(x_2)$. Likewise, if segment $[(x_1, x_2, c), (x'_1, x_2, c)]$ is cut by $u_1(x_1) = u_3(c)$ and segment $[(x'_1, x'_2, c), (x''_1, x'_2, c)]$ cannot be cut by $u_1(x_1) = u_2(x_2)$, then the last segment $[(x'_1, x_2, c), (x'_1, x'_2, c)]$ cannot be cut by $u_2(x_2) = u_3(c)$. We can proceed in the same way for all possible cases.

Hence, there must be one segment with two points (denoted by $\boldsymbol{y}$ and $\boldsymbol{y}'$) not separated by a frontier and belonging to $X^c_{\prec}$ and $X^c_{\sim}$, respectively. As segment $[\boldsymbol{y}, \boldsymbol{y}']$ is not separated by a frontier, their three components are ordered in the same way: $U(\boldsymbol{y}) = u_1(y_1)\ (\mu(S_1 \cup \{1\}) - \mu(S_1)) + u_2(y_2)\ (\mu(S_2 \cup \{2\}) - \mu(S_2)) + u_3(c)\ (\mu(S_3 \cup \{3\}) - \mu(S_3))$ and $U(\boldsymbol{y}') = u_1(y'_1)\ (\mu(S_1 \cup \{1\}) - \mu(S_1)) + u_2(y'_2)(\mu(S_2 \cup \{2\}) - \mu(S_2)) + u_3(c)(\mu(S_3 \cup \{3\}) - \mu(S_3))$, for some $S_1, S_2, S_3 \subseteq M$. Alternative $(\mathbb{1}_1, \mathbb{1}_2, c_3)$ is on the frontier $u_1(x_1) = u_2(x_2)$ and can thus take the same expression with sets $S, 1, S_2, S_3$: $U(\mathbb{1}_1, \mathbb{1}_2, c_3) = (\mu(S_1 \cup \{1\}) - \mu(S_1)) + (\mu(S_2 \cup \{2\}) - \mu(S_2)) + u_3(c)\ (\mu(S_3 \cup \{3\}) - \mu(S_3))$.

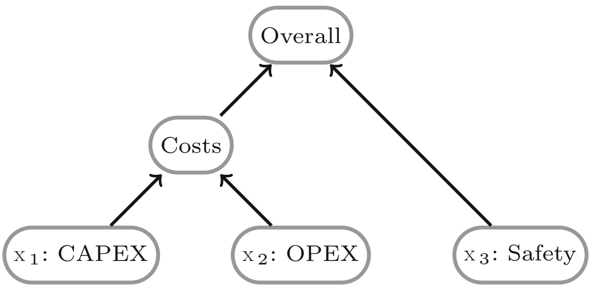
Moreover, as $\boldsymbol{y} \in X^c_{\prec}$ and $\boldsymbol{y}' \in X^c_{\sim}$, we have $\boldsymbol{y} \prec (\mathbb{1}_1, \mathbb{1}_2, c_3)$ and $\boldsymbol{y}' \sim (\mathbb{1}_1, \mathbb{1}_2, c_3)$ with $\mathbb{O}_3 \prec_3 c \prec_3 \mathbb{1}_3$. Hence,

$$\begin{aligned} & u_1(y_1)\ (\mu(S_1 \cup \{1\}) - \mu(S_1)) + u_2(y_2)\ (\mu(S_2 \cup \{2\}) - \mu(S_2)) \\ & < (\mu(S_1 \cup \{1\}) - \mu(S_1)) + (\mu(S_2 \cup \{2\}) - \mu(S_2))\ , \\ \text{and}\ \ & u_1(y'_1)\ (\mu(S_1 \cup \{1\}) - \mu(S_1)) + u_2(y'_2)\ (\mu(S_2 \cup \{2\}) - \mu(S_2)) \\ & = (\mu(S_1 \cup \{1\}) - \mu(S_1)) + (\mu(S_2 \cup \{2\}) - \mu(S_2)). \end{aligned}$$

As the marginal utility lies in [0, 1] and that $0 < u_1(y_1), u_2(y_2), u_1(y'_1), u_2(y'_2) < 1$, the first relation implies that $\mu(S_1 \cup \{1\}) > \mu(S_1)$ and $\mu(S_2 \cup \{2\}) > \mu(S_2)$, while the second one implies that $\mu(S_1 \cup \{1\}) = \mu(S_1))$ and $\mu(S_2 \cup \{2\}) = \mu(S_2)$. Hence we obtain a contradiction, and we have shown that the flat model (6.15) cannot represent rules R1 and R2. ■

We note that in R1, the two cost criteria 1 and 2 are taken together. The idea is thus to organize them in a subtree, like in Fig. 6.4.

**Fig. 6.4** Hierarchy of criteria in the example of cars



Let us try to represent the previous preferences with a hierarchical model given by tree in Fig. 6.4:

$$U(\boldsymbol{x}) = \left(\frac{u_1(x_1) + u_2(x_2)}{2}\right) \wedge u_3(x_3). \tag{6.16}$$

**Lemma 6.2** *The hierarchical model (6.16) can represent the previous preferences.*

***Proof*** We have

$$\begin{aligned} X^c_\sim &= \left\{(x_1, x_2, c) \in X \ : \ \left(\frac{u_1(x_1) + u_2(x_2)}{2}\right) \wedge u_3(c) = u_3(c)\right\} \\ &= \left\{(x_1, x_2, c) \in X \ : \ \frac{u_1(x_1) + u_2(x_2)}{2} \geq u_3(c)\right\} \end{aligned}$$

and

$$X^c_\prec = \left\{(x_1, x_2, c) \in X \ : \ \frac{u_1(x_1) + u_2(x_2)}{2} < u_3(c)\right\} .$$

Hence, (6.12) and (6.13) are clearly satisfied. Moreover, the expression $\frac{u_1(x_1)+u_2(x_2)}{2}$ is in essence compensatory in $x_1$ and $x_2$, so that (6.14) is also fulfilled. ■

## 6.3.2 *The Hierarchical Choquet Integral Model*

The weighted sum is an easily interpretable aggregation function, as the weights are directly understandable to the user. However, it cannot model interacting criteria. The impossibility to model interaction is due to the fact that this model adopts the same linear expression in the entire alternative space. A natural extension is to consider piecewise affine aggregation functions. Such a function is indeed locally interpretable, as it is locally characterized by simple weights. At the same time, the representation of interacting criteria is made possible by changing the weights from one (sub)domain to another one.

According to (6.5), a Choquet integral is a piecewise affine function, in which the separation frontiers between two affine parts take the form $a_i = a_{i'}$ for $i, i' \in C(j)$. This special form of separation is justified by the fact that the argument of a Choquet integral is a vector of commensurate elements, where a similar score on different criteria has the same meaning. In particular, an alternative having the same score on all criteria should have this value as the overall score. This property, called *idempotency*, thus justifies the frontiers of the form $a_i = a_{i'}$. This implies that interaction occurs across frontiers of the form $a_i = a_{i'}$. In order to have more complex forms of interactions, one might wish to have more complex separation frontiers between the linear parts.

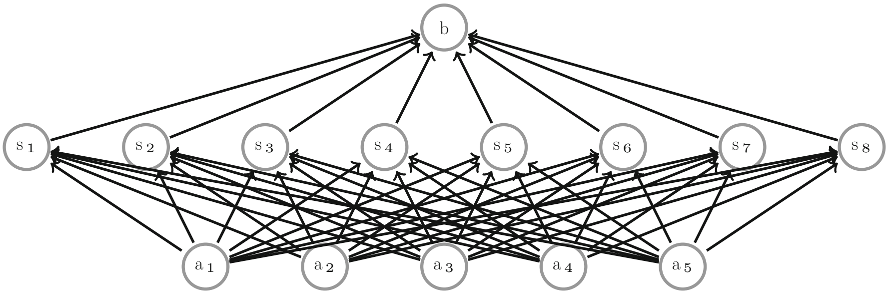



**Fig. 6.5** Graph of interconnected nodes of depth 2 with $m = 5$ and $r = 8$

It has been shown that any piecewise affine function[1] can be represented by a network of interconnected Choquet integrals [22]. More precisely, consider a piecewise affine function $f$ characterized by $r$ affine parts given by $r$ affine functions $f_1, \ldots, f_r$. Then, $f$ can be described by a network of depth 2, as in Fig. 6.5, where the leaves are the $m$ inputs $a_1, \ldots, a_m$, the next layer is composed of the $r$ affine functions $f_1, \ldots, f_r$, and the last layer is a Choquet integral with a fuzzy measure taking only values 0 and 1. This latter Choquet integral is a min-max function of its inputs and triggers the correct function $f_i$.

We note that adding more layers to the interconnected graph does not increase expressivity, and that the width of the second layer, which corresponds to the number of affine parts, can be arbitrarily large [22]. Fully connected graphical models of that kind will normally be difficult to interpret, however.

In spite of the expressivity of Choquet models with a single (hidden) layer, this could be a motivation to consider models with a deeper structure. In this regard, an interesting connection could be drawn to neural networks, where advantages of "deep" over "flat" structures have also been observed, both for representation and training [5]: Although it is true that neural networks with a single hidden layer exhibit universal approximation capabilities, the practical realization of this theoretical property may require an extremely large number of neurons in this layer. The same approximation quality might be achieved with a significantly smaller number of neurons if these are distributed on several layers and connected in a proper way, and this may also facilitate the learning process.

Here, we are specifically interested in the family of *hierarchical Choquet integrals* (HCI) [6]. Roughly speaking, these are models consisting of a tree of Choquet integrals, as described in Sect. 6.2.3. A similar conception of hierarchical Choquet models was put forward in [2, 3].

Apart from the compromise between flat and fully connected graphs, the HCI exhibits properties that are appealing from the perspective of learning and interpretation. An important example of such a property is *identifiability*. As already

[1] It shall be noted that piecewise affine functions are necessarily continuous.

mentioned, any fully connected graph of Choquet integrals can be transformed into an equivalent graph of depth two, showing that such a model is *not* identifiable. In other words, the same function $X \to \mathbb{R}$ can be represented in different ways. On the other side, as recently shown in [7], an HCI model is identifiable (under certain technical assumptions), which means there are no two distinct HCI models (with different hierarchies or the same hierarchy but different parameters for aggregation functions and marginal utility functions) yielding exactly the same function $X \to \mathbb{R}$. In particular, for a given HCI model, the separation frontier between two affine parts is of the form $\sum_{i \in A} w_i \ u_i(x_i) = \sum_{i \in B} w_i \ u_i(x_i)$, where $A, B \subseteq M$, $A \cap B = \emptyset$, $w_i > 0$ for all $i \in A \cup B$, and $A$ and $B$ correspond to descendants of two different subtrees. The idea is that one can recover the shape of the tree by looking at the expression of the separation frontiers between the affine parts. In the example of the tree in Fig. 6.4, we obtain separation frontiers of the form $w_1 \, u_1(x_1) = w_2 \, u_2(x_2)$ (as $x_1$ and $x_2$ belong to two different subtrees below Costs), and $w'_1 \, u_1(x_1) + w'_2 \, u_2(x_2) = w_3 \, u_3(x_3)$ (as $\{x_1, x_2\}$ and $x_3$ belong to two separate subtrees below node Overall).

## 6.4 Neur-HCI Framework to Learn a HMCDM

In this section, we illustrate the idea of hierarchical Choquet integral models by means of a concrete example of preference learning, namely a method called Neur-HCI, which makes use of a neural representation for learning such models from data [6].

### *6.4.1 Marginal Utility Functions*

A *marginal utility* $u_i : X_i \to [0, 1]$ is a function mapping the $i$-th attribute domain $X_i$ to the unit interval. As described in Sect. 6.2.1, one usually assumes an underlying binary preference relation $\succsim_i$ on each attribute, which represents domain knowledge. Obviously, $u_i$ should then be monotone with respect to this relation—see (6.1). For the sake of simplicity, we will assume here that $X_i \subseteq \mathbb{R}$ and that the marginal utility functions are non-decreasing. This means that when the attribute is discrete, we will assume that $X_i$ takes integer values $\{0, 1, 2, \ldots\}$, which are the labels of the attributes, ordered from the worst to the best according to $\succsim_i$. Several representations of non-decreasing marginal utility functions are conceivable:

– Piecewise smooth functions with fixed breakpoints: We can think of piecewise affine functions, which are quite common in MCDA (see for instance [16]). The use of splines has also been proposed: M-Splines and I-Splines in [23], and Cubic splines in [29]. As a drawback of such a model, the breakpoints are often fixed arbitrarily, e.g., equally spaced in the attribute domain. This is a limitation, as one does not know in advance which parts of the attribute domain may require a fine-grained modeling with a resolution higher than others, for example, because the marginal utility has a steep gradient.

- Polynomials: The marginal utility function can be considered as a polynomial of a fixed dimension [29], which avoids breakpoints. In order to learn non-negative and non-decreasing polynomials, one needs to use semidefinite programming, which is a limiting factor with regard to high dimensional problems.
- Sigmoids: A marginal utility function $u_i$ can be put in the form of a convex combination of sigmoid functions [31]:

$$u_i(x_i) = \sum_{k=0}^{p} \frac{r_i^k}{1 + e^{-(\eta_i^k x_i - \beta_i^k)}} , \tag{6.17}$$

where the hyper-parameter $p$ is the maximum number of sigmoids involved in the representation; $\beta_i^k$ and $\eta_i^k$ are the bias and precision parameters of the $k$-th sigmoid, respectively, and $r_i^k$ its weight. Each sigmoid is thus characterized by its inflection point $-\frac{\beta_i^k}{\eta_i^k}$, which is the point where the gradient of the function is the largest, and its slope. In the course of a learning process, this allows one to look for the parts of the attribute domain where the marginal utility has its largest gradient. Monotonicity of $u_i$ is simply ensured by assuming

$$\forall\, i \in M \;\; \forall\, k \in \{0, \ldots, p\} : \quad r_i^k \geq 0 \text{ and } \eta_i^k \geq 0 . \tag{6.18}$$

Finally, the normalization condition on the utility is

$$\forall\, i \in M : \quad \sum_{k=0}^{p} r_i^k = 1 . \tag{6.19}$$

In the sequel, we consider the sigmoid model due to its flexibility and the simplicity of the monotonicity conditions.

## 6.4.2 Learning Tasks

We describe three settings to train the parameters of a hierarchical model $U(\boldsymbol{x})$ for a known hierarchy, given some dataset $\mathcal{E}$.

*Binary Classification* In binary classification, $\mathcal{E} = \{(\boldsymbol{x}^{(j)}, y^{(j)}), j = 1, \ldots, q\}$, where $y^{(j)} \in \{0, 1\}$ is the binary label associated with alternative $\boldsymbol{x}^{(j)} \in X$, with 1 indicating a "good" evaluation and 0 a "bad" one. Generalizing logistic regression, we are interested in models of the form

$$\log\left(\frac{\pi(\boldsymbol{x})}{1 - \pi(\boldsymbol{x})}\right) = \lambda\, U_\theta(\boldsymbol{x}) + \beta , \tag{6.20}$$

where $\beta$ is a bias term (intercept), $\lambda$ a precision parameter that specifies how sharply the good and bad instances are separated from each other, $\theta$ the parameters of the utility function $U$, and $\pi(\boldsymbol{x}) = P(y = 1 \mid \boldsymbol{x})$ the probability of $\boldsymbol{x}$ belonging to class 1. This is equivalent to

$$\pi(\boldsymbol{x}) = \pi_{\beta,\lambda,\theta}(\boldsymbol{x}) = \frac{1}{1 + \exp(-\beta - \lambda\, U_\theta(\boldsymbol{x}))} \ . \tag{6.21}$$

The loss function is given by

$$\begin{aligned}\mathcal{L}(\beta, \lambda, \theta) &= \log P(\mathcal{E} \mid \beta, \lambda, \theta) = \log \prod_{j=1}^{q} P\big(y^{(j)} \mid \boldsymbol{x}^{(j)}, \beta, \lambda, \theta\big) \qquad (6.22)\\ &= \sum_{j=1}^{q} y^{(j)}\, \log \pi_{\beta,\lambda,\theta}(\boldsymbol{x}^{(j)}) + \sum_{j=1}^{q} (1 - y^{(j)})\, \log(1 - \pi_{\beta,\lambda,\theta}(\boldsymbol{x}^{(j)})) \ .\end{aligned}$$

This loss ought to be minimized through a proper choice of the parameters $\beta, \lambda, \theta$. We note that the loss can be augmented with a parsimony term on the marginal utility function or on the fuzzy measures (limiting the interaction terms).

*Regression* Another setting is when $\mathcal{E} = \{(\boldsymbol{x}^{(j)}, y^{(j)}), j = 1, \ldots, q\}$ and $y^{(j)} \in [0, 1]$ is a real-valued utility of the alternative $\boldsymbol{x}^{(j)}$. In this case, the model is trained using a standard regression criterion (e.g., mean squared error) :

$$\mathcal{L}(\theta) = \sum_{j=1}^{q} \Big(y^{(j)} - U_\theta(\boldsymbol{x}^{(j)})\Big)^2 .$$

*Ranking* In a third setting, the data might be given in the form of pairs $\mathcal{E} = \{(\boldsymbol{x}^{(j,1)}, \boldsymbol{x}^{(j,2)}), j = 1, \ldots, q\}$, indicating preferences $\boldsymbol{x}^{(j,1)} \succ \boldsymbol{x}^{(j,2)}$ between alternatives (expressed by an expert). A loss function could then be specified as follows:

$$\mathcal{L}(\theta) = \sum_{j=1}^{q} \Big(U(\boldsymbol{x}^{(j,2)}) - U(\boldsymbol{x}^{(j,1)}) - \eta\Big)_+ \ ,$$

with $\alpha_+ = \max(\alpha, 0)$ and $\eta$ a hyper-parameter enforcing a margin effect (strict preference $\boldsymbol{x}^{(j,1)} \succ \boldsymbol{x}^{(j,2)}$ is replaced by weak preference $U(\boldsymbol{x}^{(j,1)}) \geq U(\boldsymbol{x}^{(j,2)}) + \eta$). To prevent the learning algorithm from over-fitting the training data, the loss can be augmented by regularization terms on the utility function or on the Choquet integrals.

### *6.4.3 Learning and Optimization*

The problem of binary classification, i.e., the optimization of (6.22), has first been tackled in [31], modeling $U$ in terms of a single Choquet integral (a flat organization); moreover, the utility functions $u_i$ are excluded from the actual learning process and instead determined in a pre-processing step (essentially through standardization of the empirical distribution in the data). Referring to the use of the Choquet integral as a replacement of the linear utility in logistic regression, the authors call the approach "choquistic regression." It has furthermore been generalized to ordinal classification [33] and ranking [32]. An extension toward learning a flat model together with the marginal utility functions has been proposed in [34] under the name "choquistic utilitaristic regression."

Different optimization methods have been used in these approaches for fitting a "flat" Choquet integral. These methods turned out to be difficult to generalize to the case of a hierarchal model (HCI), for example, due to a lack of stability and modularity. In [6], it was therefore proposed to leverage recent advances in the learning of neural networks (NNs), which, of course, presumes a representation of an HCI model in terms of an NN-like structure.

To this end, we first of all represent marginal utility functions in terms of a "marginal utility module". According to (6.17), the function $u_i$ is a convex combination of $p$ sigmoids. This is represented by a hidden layer of the NN-structure composed of $p$ hidden neurons with sigmoidal activation function (cf. Fig. 6.6). Non-negativity of $\eta_i^k$ and $r_i^k$ is ensured by clipping, while the normalization condition $\sum_{k=0}^{p} r_i^k = 1$ is guaranteed through batch normalization (without creating any instability).

For an aggregation node $j \in N \setminus M$, the aggregation function is chosen as a 2-additive Choquet integral (6.8). It can be described in a NN-structure in the following way. We set $m_j = |C(j)|$. A 2-additive Choquet integral over an $m_j$-dimensional utility vector $\mathbf{a} = (a_1, \ldots, a_{m_j})$ is represented as a neural architecture with a single hidden layer comprising ${m_j}^2$ neurons (Fig. 6.7). We distinguish three types of neurons:

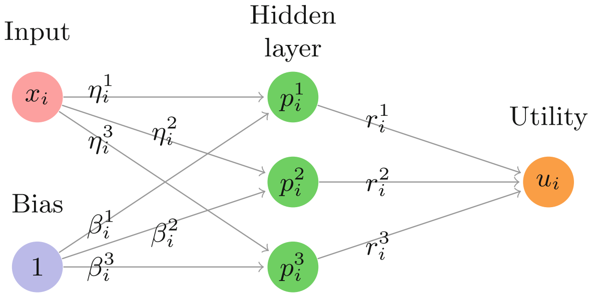



**Fig. 6.6** A marginal utility module with 3 hidden nodes ($p = 3$)

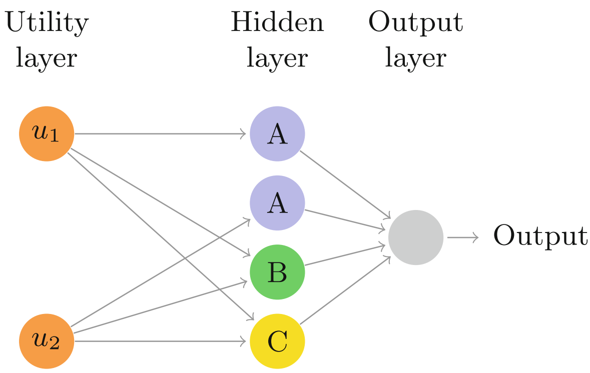



**Fig. 6.7** A 2-additive Choquet module with two inputs and hidden neurons of type $A$, $B$, and $C$

A: $m_j$ neurons with a single output $a_i$. In order to enforce monotonicity condition (6.9), one could apply clipping $w_i^j$ to 0 when it becomes negative during the learning process. However, this turned out to generate high instability. Hence, the non-negativity of $w_i^j$ is ensured by an appropriate change of variable: $w_i^j = \sigma(z_i^j)$, where $z_i^j$ is any real number, and $\sigma$ denotes the differentiable softplus function $\sigma(z) = \ln(1 + \exp(z))$. Neurons of type A then return $\sigma(z_i^j)\, a_i$.

B: $\frac{m_j(m_j-1)}{2}$ neurons, denoted $h^{\wedge}_{i,i'}$ with inputs $a_i$ and $a_{i'}$. As before, we write $w^{j,\wedge}_{i,i'} = \sigma(z^{j,\wedge}_{i,i'})$, for any real number $z^{j,\wedge}_{i,i'}$, in order to fulfill monotonicity condition (6.9). Neurons of type B then return $\sigma(z^{j,\wedge}_{i,i'})\,(a_i \wedge a_{i'})$. It corresponds to a weighted min-pooling of criteria $i$ and $i'$.

C: $\frac{m_j(m_j-1)}{2}$ neurons, denoted $h^{\vee}_{i,i'}$ with inputs $a_i$ and $a_{i'}$. As before, we write $w^{j,\vee}_{i,i'} = \sigma(z^{j,\vee}_{i,i'})$, for any real number $z^{j,\vee}_{i,i'}$, in order to fulfill monotonicity condition (6.9). Neurons of type C then return $\sigma(z^{j,\vee}_{i,i'})\,(a_i \vee a_{i'})$. It corresponds to a weighted max-pooling of criteria $i$ and $i'$.

Finally, the normalization constraint (6.10) is enforced through a batch normalization layer [15]. This module is called *Choquet module*.

The previous marginal utility and 2-additive Choquet integrals modules are combined following the tree structure of the hierarchical model. Consider Fig. 6.4 composed of

– Three attributes $x_1$ (CAPEX), $x_2$ (OPEX), and $x_3$ (Safety).
– Two aggregation nodes: 4 (Costs) and 5 (Overall).

The general neural network is depicted in Fig. 6.8. It combines application of the following modules: application of the marginal utility module from $x_1$ to $u_1$ (see upper red rectangle in Fig. 6.8); application of the marginal utility module from $x_2$ to $u_2$ (see middle red rectangle in Fig. 6.8); application of the marginal utility module from $x_3$ to $u_3$ (see lower red rectangle in Fig. 6.8); application of the 2-additive Choquet Integral module from $u_1, u_2$ to $u_4$ (see upper green rectangle in

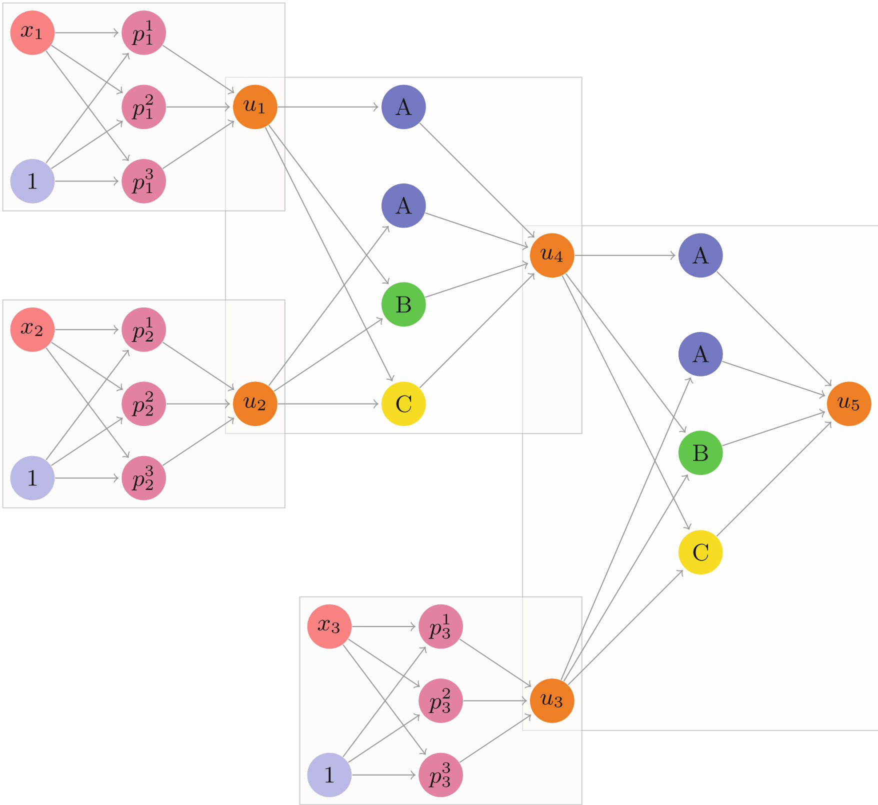



**Fig. 6.8** Example of the combination of the two modules on a tree with three marginal utility functions and two aggregation nodes

Fig. 6.8); application of the 2-additive Choquet Integral module from $u_4, u_3$ to $u_5$ (see green rectangle in the right hand side of Fig. 6.8).

With the process outlined above, any hierarchical organization of a set of criteria can be transformed into a neural network. One can readily see that the neural network rigorously encodes the expression of the marginal utility functions and the 2-additive Choquet integral, as well as their monotonicity and normalization conditions. The values of their parameters can be learned using standard back-propagation techniques for neural network training. This ensures that the solution to the learning process of the neural network returns a proper HCI model. Once the HCI model is learned, it can be used for the purpose of inference and prediction, for which the NN-structure is no longer needed.

## 6.5 Summary and Conclusion

We elaborated on the idea of leveraging machine learning methods to construct hierarchical MCDA models from suitable training data, and illustrated this idea by means of Neur-HCI, a method for learning tree-structured combinations of Choquet integrals. As stated repeatedly, we believe that this is an extremely interesting research direction, both from the perspective of MCDA and machine learning. First promising results have already been achieved, but many opportunities remain and a lot of work still needs to be done. For example, while the Choquet integral as an aggregation function in Neur-HCI covers the class of generalized averaging operators, it might be useful to also include conjunctive and disjunctive aggregation operators—in the literature, a model class of that kind has already been proposed under the name fuzzy pattern tree [27], and it would be interesting to see whether it is amenable to the neural representation and learning techniques of Neur-HCI.

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# Chapter 7
# Preference Learning Applied to Credit Rating: Applications and Perspectives

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**Abstract** To build a portfolio that satisfies their objectives, investors need information about possible losses when they choose among a diversity of financial assets with fixed or variable income. Although variable income is the main option for investors more tilted to accept higher risk, fixed income alternatives also expose more conservative investors to the default risk. Preference learning approaches have been deployed to help investors to better understand why some alternatives are riskier than others and to discriminate financial assets among different classes of risk. The objective of this chapter is to give an overview about the credit rating problem for a discrete set of financial assets and how preference learning techniques can assist in analyzing the relations between multiple attributes and the performance of the alternatives in a more transparent and objective way. The focus was on the country and corporate credit risk applications and presenting the perspectives to increment the preference models and methods related to this problem.

## 7.1 Introduction

Preference learning is a research field that emerged from the intersection of machine learning and multiple criteria decision aiding (MCDA) [28, 31, 42, 43, 52] considering the notion of "preferences" in economics and operations research along with the learning perspective. There are two preference learning approaches: direct and indirect. In the direct approach, an acceptable representation of the DM judgment policy is required, which can involve an elicitation of his/her preferences. Although more fluid and up-to-date applications in complex strategic problems can be performed when there is good cooperation between decision analysts and

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stakeholders [9], this can be a very costly process and may not apply to repetitive decisions, with small-time windows [37, 39, 67].

To deal with indirect and incomplete specification of preferences of a decision-maker (DM), three basic instances can be considered in multiple criteria decision aiding (MCDA): functions, outranking relations, and rule based [22, 67]. Research involving indirect preference elicitation includes diverse contexts: country [15, 51] and corporate [16, 27] credit risk, economic freedom [14], nanotechnology [40, 41], public health and medicine [32, 33, 50, 63], and many other applications [13, 22, 36]. See [23, 38] for an overview on preference disaggregation approaches.

Although in the multicriteria literature, the term sorting is already established as a process for ordinal classification, from a broad perspective, rating is a widely used term as remarked by Colorni and Tsoukiàs [10]. An everyday use of MCDA methods in Finance is observed in credit rating problems [13, 19, 27, 46, 68]. The credit assessment of companies and countries refers to analyzing the degree of risk that an entity is not capable of meeting its debt obligations. In different decisions and negotiation processes, it is essential to have a view of some parts' creditworthiness and credit risk. For example, banks need to understand the risk of lending financial resources to their customers; a particular company, entity of a supply chain, needs to use its credit to buy raw material from its suppliers and may sell its outputs to customers also on credit; investors should understand the credit risk of bonds' issuers, which companies may represent (i.e., corporate bonds, debentures) or even governments (sovereign bonds) [15, 16, 27]. While credit scoring analysis results in a numerical score for each borrower (indicating the probability of default), credit ratings are expressed within an ordinal qualitative scale [27]. In this chapter, we focus on the problem of rating.

This chapter is organized as follows: Sect. 7.2 presents how the rating problem can be modeled as multiple criteria sorting problem, Sect. 7.3 brings recent applications of preference learning models and methods in-country and corporate risk, and the conclusions and future perspectives of some preference learning models and methods applied to credit rating are presented in Sect. 7.4.

## 7.2 Credit Rating or Sorting with Multiple Criteria

The multidimensional characteristic of financial problems, which frequently involve various objectives that should be addressed in the decision process, makes them typical problems for multiple criteria applications [13, 68]. In this perspective, an analyst may include classical criteria (i.e., profitability, risk) and modern dimensions (social responsibility, liquidity, solvency). de Almeida-Filho et al. [13] grouped and analyzed the main criteria used in the literature regarding MCDA applications in finance; see Fig. 7.1.

The problem of rating companies or investment alternatives can be structured as an ordinal classification (sorting) problem. Sorting refers to allocating a set of alternatives in pre-defined classes that are ordered in terms of preferences [49].

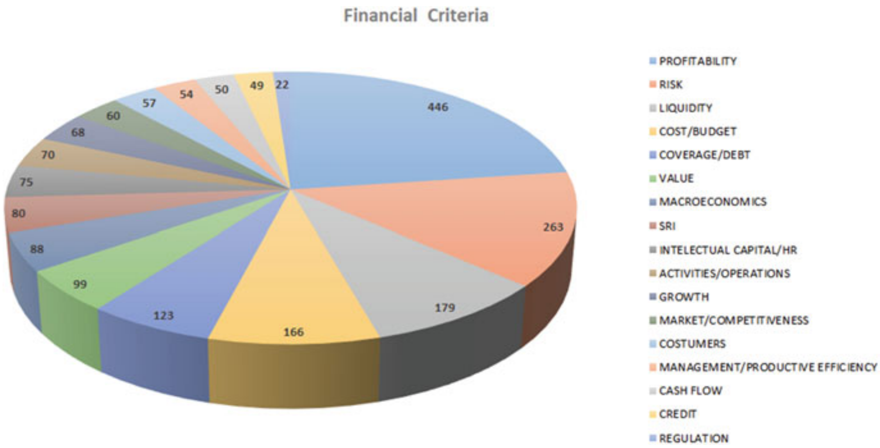


**Fig. 7.1** Financial criteria [13]

Several sorting methods have been used in the last decades, and new proposals and studies continue to emerge in the literature [30]. Let the problem be defined as follows.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of $m$ alternatives (actions, instances); $G = \{g_1, g_2, g_3, \ldots, g_n\}$ be a set of $n$ criteria (features); $W = [w_1, w_2, w_3, \ldots, w_n]$ be a vector of weights with $n$ elements, where $w_j$ is the weight of criterion $g_j$; and $G^+$ and $G^-$ be, respectively, the subsets of beneficial and cost criteria. Let $a_{i,j}$ be the performance of alternative $a_i$ regarding criterion $g_j$. Also, let $C = \{C_1, C_2, \ldots, C_q\}$ be a set of $q$ pre-defined ordered ratings (classes), where $C_1 \succeq C_2 \succeq, \ldots, \succeq C_q$.

In some MCDA preference learning approaches for sorting (PDTOPSIS-Sort [16], ELECTRE-TRI variations [29, 45], PROMETHEE variations [15]), the characteristics of each class are described with the use of profiles, defined in the center of the classes (characteristic profiles) or in the limit between consecutive classes (boundary profiles). So, let $P = \{P_1, P_2, \ldots, P_p\}$ represent a set of $p = q - 1$ boundary profiles or $p = q$ characteristic profiles, and let $P_{k,j}$ be the performance of profile $P_k$ regarding criterion $g_j$.

Ordered ratings (classes) can also be characterized by rules when adopting the dominance-based rough set approach (DRSA) [34], which is an evolution of the classical rough set approach proposed by Pawlak [48]. In DRSA, decision-makers' preferences can be represented by a set of *"If…, then..."* rules $R$, induced from the set of decision examples. The general form of a decision rule is presented below.

**IF** $a_{i,1}$ satisfies $r_{k,1}$ and $a_{i,2}$ satisfies $r_{k,2}$ and ... and $a_{i,n}$ satisfies $r_{k,n}$; **THEN** $i$ belongs to $Y_k$.

Where $r_{t,j}$ is $j$-th criteria threshold for rule $k \in R$, which defines the conditional part of the rule, and $Y_k = C_1$ or $C_2$ or ... or $C_q$ defines the decision part for rule $k$. To understand the types of rules in more detail, data reduction, preference discovery, rule induction algorithms, and classification issues, please refer to [3–5, 34, 54].

### *7.2.1 Country Risk*

The 2007 economic crisis reflected the lack of clarity in procedures adopted by the international credit rating agencies, which brought to the forefront a debate about the quality and the true role of the agencies that rate risk. The risk assessment process for these sovereign bonds is still considered subjective because of the lack of criteria-related information and transparency of the methodologies used by international credit rating agencies [15, 51].

Preference learning with indirect elicitation has been applied in the sovereign risk assessment context [1, 15, 24]. The literature also contains some applications of rough sets in sovereign risk assessment, both with the classical approach [35] and with a preference learning approach [8, 12, 51]. The developed preference learning works studied ways to support the DM in finding the criteria that most influence the risk classification, evaluation of model's classification accuracy, and how to increment the results' interpretability for different types of decision-makers (investors, governments, financial institutions, and others).

### *7.2.2 Corporate Credit Risk*

On several occasions, companies need to raise the financial amount necessary to finance their projects through instruments in the financial market. For example, a company can issue fixed-income securities that can be purchased by individual investors. Although those investments are composed of fixed income contracts, the investors are exposed to the default risk, i.e., the risk the company issuing the bond will not be able to comply with the agreed requirements and pay back the investors. This can happen, for example, when an issuing company goes bankrupt.

Multiple criteria methods have been applied in the literature to sort financial and non-financial companies in ordered classes in terms of credit risk or corporate performance [13, 23, 36, 44]. In these applications, a set of criteria, which usually include accounting indicators (financial ratios), is used to describe the objectives of the decision problem. This perspective can be used also to select qualified equities to compose a portfolio of investments, which may be obtained afterward through a multiobjective optimization approach [57–62]. Among the used MCDA methods, the applications for ratings include Electre variations [27, 55, 58], PROMETHEE variations [21], UTADIS [20, 26, 56, 64, 65], MHDIS [25, 47], Rough Set developments [18, 53], and PDTOPSIS-Sort [16].

## 7.3 Applications

Many different preference learning approaches to finance exist in the literature. Applications include a sovereign bonds risk assessment to bankruptcy prediction in companies. The literature reviews of [68] and [13] cover preference learning methods applied in finance, where credit risk is included. In this section, we present recent applications of preference learning models and methods in-country and corporate risk.

### *7.3.1 Sorting Sovereign Bonds with Two Preference Learning Approaches*

#### 7.3.1.1 Alternatives and Reference Set Construction

In the context of sorting sovereign bonds, investors want to discriminate "high risk" countries from "low risk" countries, so that they can allocate capital in a more rational way, according to their objectives (building a portfolio to obtain high return or to have less risk of losing capital over time). From time to time, the countries are sorted into pre-defined risk categories by risk agencies, such as Standard and Poor's, Moody's, and Fitch Ratings, defining a preference order among the risk categories. To build a more simplified model, which facilitates summarization of results and interpretability to obtain better insights, de Lima Silva et al. [15] proposed to group the agencies' ratings into three categories: "very low risk" ($C_1$), "low to moderate risk" ($C_2$), and "speculative" ($C_3$), where $C_1 \succeq C_2 \succeq C_3$. Table 7.1 shows the risk classes proposed by the authors and how they are associated with the ratings of the considered agencies.

When considering a preference disaggregation approach, the multicriteria decision aid method can learn preferences through a reference set of alternatives. The reference set is a data table composed of decision examples, thus in this problem, each decision example consists of a country's criteria (conditional attributes) and it's class (decision criteria). Among different ways to allocate a country for a certain risk class, one can choose to allocate according to the worst rating among those performed by Standard and poor's (S&P) and Moody's. An example of a reference set composed of six alternatives and a demonstration of this "rating to risk class" conversion is presented in Table 7.2.

In a recent study, Silva et al. [51] experimented with different reference sets to increment the performance of DRSA when considering the model of [15] for the country risk problem. The authors simulated different country sampling strategies, where the best strategy was to give more importance to the worst-rated countries from $C_1$, the best-rated countries from $C_3$, and a combination of the best- and worst-rated countries from $C_2$.

**Table 7.1** The three classes proposed in [15] and how they are associated with the ratings of standard and Poor's and Moody's

| Category | Moody's ratings | Standard and poor's ratings |
|---|---|---|
| $C_1$ | Aaa | AAA |
| | Aaa1 | AA+ |
| | Aaa2 | AA |
| | Aaa3 | AA− |
| | A1 | A+ |
| | A2 | A |
| | A3 | A− |
| $C_2$ | Baa1 | BBB+ |
| | Baa2 | BBB |
| | Baa3 | BBB− |
| $C_3$ | Ba1 | BB+ |
| | Ba2 | BB |
| | Ba3 | BB− |
| | B1 | B+ |
| | B2 | B |
| | B3 | B− |
| | Caa1 | CCC+ |
| | Caa2 | CCC |
| | Caa3 | CCC− |
| | Ca | CC |
| | C | C |
| | D | |

**Table 7.2** Converting ratings to risk classes according to the worst rating case in a reference set

| Alternative | Moody's | S&P | Final risk class |
|---|---|---|---|
| Country 1 | A1 | A+ | $C_1$ |
| Country 2 | Aa2 | AA | $C_1$ |
| Country 3 | Baa3 | A | $C_2$ |
| Country 4 | Baa2 | BBB+ | $C_2$ |
| Country 5 | Caa3 | CCC− | $C_3$ |
| Country 6 | Baa3 | BB− | $C_3$ |

#### 7.3.1.2 Conditional Criteria and Data Reduction

Since the procedure used by the agencies is extremely subjective and often questioned, objective indicators that measure the performance of a country in different sectors (economy and growth, poverty, environment, and others) must be incorporated in preference learning models to provide more transparent associations between countries' and agencies' ratings. The study performed by de Lima Silva et al. [15] found 9 real economic indicators among 18 collected from the World Bank database that potentially influence the ratings of Moody's and S&P. These 9 indicators and their associated preference directions are presented in Table 7.3.

**Table 7.3** Criteria available in the World Bank website that potentially influence agencies' ratings

| Criteria | Preference direction |
|---|---|
| GDP per capita (current US$) | Max |
| Exports of goods and services (% of GDP) | Max |
| Gross savings (% of GDP) | Max |
| Foreign direct investment, net inflows (BoP, current US$) | Max |
| GDP at market prices (current US$) | Max |
| Total reserves (includes gold, current US$) | Max |
| GNI per capita, Atlas method (current US$) | Max |
| Lending interest rate (%) (current US$) | Min |
| Reals interest rate (%) (current US$) | Min |

The use of reducts in [51] brought interesting results since fewer criteria were needed to classify new objects into the risk groups. Reducts can be used to identify the most relevant criteria and increase the number of countries that will be evaluated since not all indicator values are computed for all the countries listed in the World Bank database. Furthermore, less effort will be spent on result interpretation, since fewer variables are contained in the set of rules. Finally, while de Lima Silva et al. [15] sorted 36 countries using the criteria from Table 7.3, the adoption of reducts in [51] enabled more sovereign bonds to be sorted among risk classes over time.

#### 7.3.1.3 Preference Learning Approach

After the data table containing a set of classified decision examples and their evaluation on the considered criteria is settled, a preference learning method is applied to complete the credit risk evaluation process. de Lima Silva et al. [15] evaluated a PROMETHEE approach, which has been adapted to the sorting problematic with preference disaggregation. This PROMETHEE variation was proposed by Doumpos and Zopounidis[21] and to obtain the necessary parameters for the components of the sorting process, a linear programming model is used, modeling each preference function (for each criterion $j$) parts of a linear function.

The advantage of using an indirect elicitation approach is that when it is not possible to elicit DMs' preferences directly by the classical approaches, decision examples enable to reduce the cognitive effort in situations when the DM is not able to specify the preference model's parameters or, due to time constraints, is not able to participate in a costly elicitation process [45].

When using this PROMETHEE II variation, de Lima Silva et al. [15] indicated some advantages to support the investor decision-making process. The linear programming applied in the model indicated the weights of the considered criteria that best defined the assignments for the chosen reference set. Thus, it is possible to verify which criteria have more influence on the credit rating assignments. Also, it is possible to exploit the preference learning model to classify new countries into one of the three classes, using a relatively small reference set to infer the parameters of the preference model. This is interesting because the decision-maker only needs to choose a small subset of countries (i.e., 9 countries) to build the final model to start the risk evaluation process.

Due to the cognitive reduction advantages of indirect elicitation in country risk, Silva et al. [51] also investigated a rule-based approach that requires the DM to specify decision examples only. In DRSA, decision rules keep the ordinal character of input data rather than transforming it into numeric information. The interpretable set of rules enable the analyst to understand why some suggestions/decisions were made and the possibility to verify the importance of each attribute considered in the analysis [3, 6].

The method used by Silva et al. [51] enabled to check consistency between economic indicators and worst-case classifications of Moody's and S&P, and to reduce the data used in the analysis (this last feature was presented in the last paragraph of Sect. 7.3.1.2). In addition, the induced rules offer interpretability concerning the classifications of risk and therefore support decisions by providing more detail, depending on the context that the decision-maker is dealing with. An example of a rule that was induced by Silva et al. [51] is presented below.

**IF** Exports of goods and services $<=40.43\%$ and Total reserves $<=1.78e10$ and Real interest rate $>=8.42$, **THEN** credit risk is at most $C_3$ (Very low risk)".

This rule characterizes a set of countries classified by Moody's and S&P as low risk in the worst-case scenario. The authors also performed some comparisons and verified that DRSA obtained better performance results than MRSort [7] and UTADIS [17, 65, 66] in the best reference set encountered. The results are shown in Fig. 7.2.

ELECTRE-TRI is a non-compensatory method that uses the pessimistic classification rule, and its parameters are estimated based on the provided decision

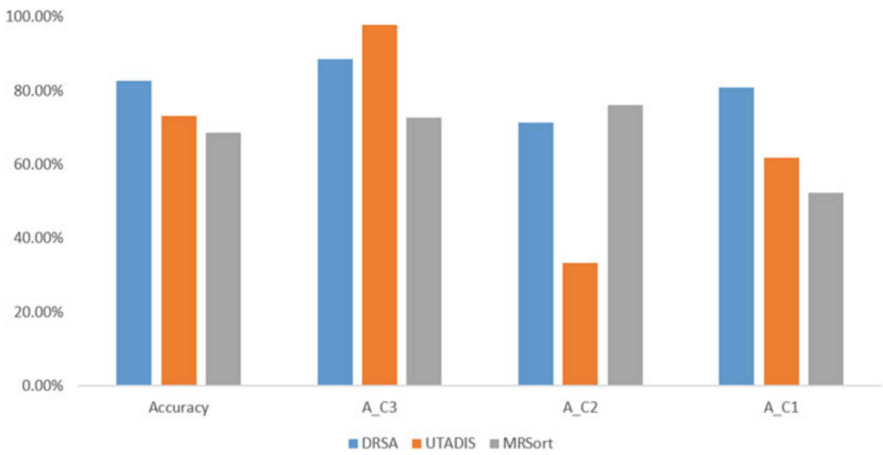


**Fig. 7.2** Comparison between DRSA, UTADIS, and MRSort

examples. A rough version of ELECTRE-TRI is implemented in MRSort and was used for the comparison. For each attribute, UTADIS elicits value functions from the provided reference set, and the assignment of each decision examples to the classes is performed based on an additive value function. DRSA presented better overall accuracy than UTADIS and MRSort. Despite the advantage of UTADIS on the $C_3$ accuracy (A_C3) performance, DRSA had a balanced performance, and thus obtained good results for all risk classes, especially in classes 1 and 3, as expected considering previous experiments.

## 7.3.2 *Sorting Brazilian Debentures with the PDTOPSIS-Sort Method*

de Lima Silva et al. [16] present a decision problem consisting of sorting a set of investment alternatives, represented by Brazilian debentures, which are a kind of corporate bond. The application is organized into seven steps, summarized below.

### 7.3.2.1 Problem Definition

The Problem is defined in the first step. Thus, the information about the decision-makers and their goals, the alternatives of the problem, and the criteria (representing the decision-makers' objectives) are identified. Also, the ratings are ordered. In Brazil, debentures represent the main available alternatives of corporate bonds to individual investors. Therefore, a set of 50 Brazilian debentures was selected as alternatives to the decision problem. The set includes both private and public companies from different sectors, such as energy, logistics, retail, mining, real estate, oil, and gas.

The sorting problem was defined with three predefined and ordered classes (ratings): $C_1$, $C_2$, and $C_3$. The rating $C_1$ represents investments with low risk, while the rating $C_3$ represents higher risk debentures. Therefore, these are ordered as follows: $C_1 \succeq C_2 \succeq C_3$, maintaining the idea that was structured in Table 7.2.

The criteria presented in Table 7.4 were defined similarly with the literature [21, 58] which includes profitability, activity, liquidity, and debt measures. The criteria financial ratios were calculated based on the companies' balance sheets regarding the year 2016.

### 7.3.2.2 Learning the Expert's Preferences

In the second step, decision examples were obtained from the decision-maker, a financial analyst of a large Brazilian investment bank. Based on this holistic information, the PDTOPSIS-Sort method uses a preference disaggregation approach

**Table 7.4** Set of criteria—corporate bonds rating

| gj | Criteria | Description |
|---|---|---|
| g1 | Return on assets | Earnings before interest and taxes divided by total assets |
| g2 | Return on equity | Net income divided by shareholders' equity |
| g3 | Net profit margin | Net income divided by sales |
| g4 | Asset turnover ratio | Sales divided by total assets |
| g5 | Acid liquidity ratio | Current assets minus inventories divided by current liabilities |
| g6 | Cash asset ratio | Cash plus cash equivalents divided by current liabilities |
| g7 | Working capital to current liabilities ratio | Current assets minus current liabilities divided by current liabilities |
| g8 | 1/solvency ratio | Shareholder's equity divided by total liabilities |
| g9 | Leverage ratio | Total assets divided by shareholder's equity |
| g10 | Interest coverage ratio | Earnings before interest and taxes divided by interest expenses |

to infer boundary profiles and weights for a TOPSIS-based sorting approach. The set of reference alternatives can be formed by a subset of the original alternatives of the problem, other alternatives that the DM feels confident to assign into the classes, or fictitious alternatives.

In the third step of PDTOPSIS-Sort, a domain is defined for the criteria space. The configuration of a domain in TOPSIS is important to prevent the occurrence of ranking reversals deriving from the inclusion (or exclusion) of alternatives in the initial set considered in the model [14, 16]. A way of defining the domain is through the definition of two fictitious alternatives ($a^*$ and $a^-$) that receive, respectively, the highest and lowest value considered for each criterion in the application.

The fourth step of the model regards the inference of the necessary parameters to perform the allocations of the alternatives into the ratings of risk. In PDTOPSIS-Sort, weights and boundary profiles are inferred with the use of a nonlinear optimization solver. The model works as a regression, searching for TOPSIS-Sort parameters that allocate the reference examples into the correct classes. For that, besides weights and the performance of the boundary profiles, error variables are defined and should be minimized in the objective function. Details of the model are available in [16].

It might be the case that the DM does not approve the set of parameters found during the last step. Then, a fifth step considered a validation of the inferred parameters. If the decision-maker wants, he/she can include new constraints to the optimization problem, for example, indicating a maximum or minimum weight for a criterion.

In the sixth step, the sorting process takes place. The alternatives are assigned to the risk classes according to their closeness coefficients and those coefficients found

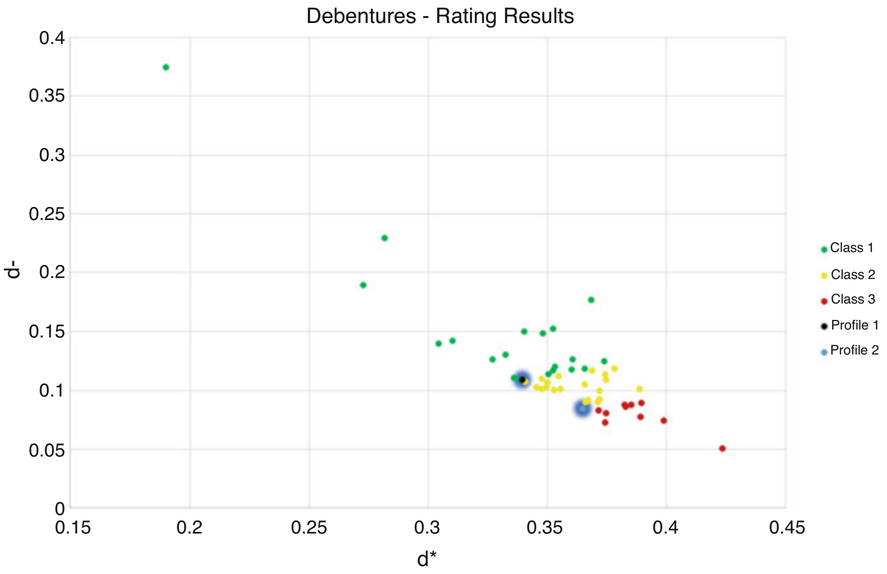


**Fig. 7.3** Rating Brazilian debentures. Adapted from [16]

for the boundary profiles. In TOPSIS, an alternative's closeness coefficient is based on its distance from an ideal solution and an anti-ideal solution [14].

Finally, in the seventh step, a sensitivity analysis is performed. After analyzing the impact of small parameters' variations, the financial analyst can decide to maintain the results of the ratings. Figure 7.3 illustrates the alternatives allocated to the different classes in terms of their distances to the ideal solution ($d^*$) and to the anti-ideal ($d^-$) solution. More details about the method and the sorting process can be found in [16].

## 7.4 Conclusion and Future Perspectives

The adoption of preference learning approaches is necessary to deal with complex decision problems when several and possibly conflicting objectives are considered. The applications presented in this chapter demonstrated how an analyst can structure different credit rating problems and how to apply preference learning methods to the developed models. When using an indirect preference elicitation method, as presented here, the decision-maker might focus more on the interpretability of the results provided by the model rather than on the elicitation of parameters, which is associated with more cognitive effort.

Preference learning methods can put forward the use of an objective and transparent methodology to sort sovereign bonds. Using the perspective obtained with preference learning sorting methods, it is possible to verify the consistency of agencies' ratings, consider a set of non-redundant attributes, and understand why sovereign bonds have a certain level of risk. Investors can use this kind of model to calibrate their decisions and protect them from the subjectivity incorporated by the mainly used rating systems.

The authors of sovereign credit risk studies that were analyzed in this chapter [15, 51] made some considerations on directions to improve the preference learning models in this problem. The first is to consider indicators from other sectors rather than Economy and Growth, such as those considered by Corrente et al. [11]. The use of economic variables resulted in good performance concerning the identification of speculative countries ($C_3$), but there was relatively poor performance when the considered methods tried to identify countries belonging to $C_1$ and $C_2$. The use of indicators from other sectors may help to improve the performance in the task of characterizing countries with less credit risk. Also, given the diversity of countries contained in $C_3$, it is interesting to investigate the subdivision of this class into two.

In the case of the corporate risk application [16], the authors identified that the method was able to assign the alternatives to the risk classes as expected. On the other hand, they highlight only objective criteria related to a single year were used, limiting the scope of the results. Criteria such as managers' quality and experience, sector risk, and the state of the economy could influence the default risk of the alternatives. Moreover, criteria regarding the changes on the indicators over a different year could be added and be more plausible to get an overall view of the companies' financial status.

In a world of great amounts of data and growing computers' capacity, preference learning methods have gained prominence and visibility. Various MCDA, statistical, and machine learning techniques have emerged to support managerial decision-making in different areas. In finance, the complexity of the problems involving large sets of data and multiple variables makes these techniques essential for business, and their use should continue to grow in the future.

Works involving preference learning methods and applications to real problems point to a new direction of investigation, which is the tradeoff between interpretability of the results and the classification task performance evaluation given by preference learning methods against machine/statistical learning methods [2, 13, 39, 51], and the combination of these approaches to build a better model [2, 50]. Consequently, future work involving preference learning approaches applied to credit rating should take this demand into consideration, performing comparisons with competitive machine learning algorithms or evaluating combinations of machine and preference learning approaches.

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# Chapter 8
# *U*Sort-nB and *U*Sort-nC: Two Multi-criteria Ordinal Classification Methods Using Interval Value Functions

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**Abstract** Two novel methods to address multi-criteria ordinal classification problems are presented here. The main characteristic of these methods is that interval value functions are used for the first time to model compensatory and transitive preferences of decision makers that hesitate about the precise value of criteria weights and criteria scores. The flexibility achieved by these interval value functions provide flexibility to the decision maker and robustness to the methods. One of the methods uses interval limiting actions to characterize boundaries between adjacent classes. The other method employs representative interval actions to characterize the classes. Each of the methods uses two procedures that are symmetric with respect to the transposition operation. Under certain conditions, we show that both methods fulfill the set of basic consistency properties established for this type of methods: Unicity, Independence, Homogeneity, Monotonicity, Conformity, and Stability. The requirements on the limiting boundaries are stronger than those on the representative actions.

## 8.1 Introduction

Generally speaking, the verb "to classify" relates to the action of assigning objects of a certain universe to predefined classes. Unlike nominal classification, in ordinal classification the decision maker (DM) is interested in assigning objects (maybe potential actions, decision alternatives) to elements of a set of ordered classes or categories. The multi-criteria ordinal classification problem is a fundamental instance of ordinal classification problems, which arises when the decision actions are described by multiple evaluation criteria. Many multi-criteria ordinal classifica-

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tion methods have been reported in the scientific literature. From the point of view of the underlying decision model, most of the methods are grouped in one of three main paradigms:

- Construction of value functions (the functional paradigm) (e.g., [18, 20, 21, 32]).
- Symbolic methods coming from the Artificial Intelligence paradigm (e.g., [3, 6, 16, 17]).
- The relational paradigm based on the construction of outranking relations (e.g., [1, 4, 5, 8, 14, 25, 30, 31]).

Regardless of the decision model used, classes should be characterized in some way. There are two basic forms:

(i) Using limiting actions that describe boundaries between adjacent classes (e.g. [4, 5, 11, 13, 24–26]);
(ii) Through decision examples (or reference actions) whose classification is (or maybe) known (e.g. [1, 2, 7, 9, 10, 13, 17–21, 24, 32]).

Some methods in (ii) use a single "central" (or representative) action to characterize each class (e.g., [1, 19, 24]). Other methods use information about a few characteristics or representative actions (e.g. [2]). Several approaches can handle many reference actions, which are not necessarily representative of their class (e.g., [7, 8, 20]).

Within the methods based on limiting actions, the most popular one is ELECTRE TRI- B, originally proposed by Yu [31] with the name ELECTRE TRI, and afterwards detailed by Roy and Bouyssou [26]. In this method, classes are characterized by their boundaries; a single limiting action (profile) is used to describe the boundary between adjacent classes; in each boundary, the limiting action belongs to the upper class. Actions to be assigned are compared with the limiting actions through an outranking (respectively, preference) relation, in the pseudo-conjunctive (resp. pseudo-disjunctive) procedure. The assignment is suggested according to the result of this comparison.

ELECTRE TRI-B evolved to ELECTRE TRI-nB [11]; in the later method, the boundaries are characterized by several limiting actions, what permits more informed assignments. ELECTRE TRI-B and ELECTRE TRI-nB fulfill several fundamental properties (conformity, monotonicity, stability, homogeneity, independence, and uniqueness), which were originally proposed by Roy and Bouyssou [26]. Similar properties are satisfied by other methods as ELECTRE TRI-C, ELECTRE TRI-nC, and ElectreSort [1, 2, 19]. The non-fulfillment of some of these properties is a serious drawback of many multi-criteria ordinal classification approaches.

ELECTRE TRI-B and its variants have been criticized from a theoretical point of view. Roy [27] and Bouyssou and Marchant [4] pointed out that the pseudo-conjunctive and pseudo-disjunctive assignment rules do not correspond via the transposition operation, which consists of inverting the direction of preferences on all criteria, also inverting the ordering of the classes. Bouyssou and Marchant [4] argued that the conclusions obtained after this operation should not be different from the original conclusions. The information provided by "$x$ outranks a limiting action $b$" has the same value as that obtained from "a limiting action $b$ outranks $x$"; the overall information should be considered for assigning $x$. So, if a multi-criteria ordinal classification procedure has an "image" method through the transposition operation, both methods should be used conjointly. Hence, on the one hand, the pseudo-conjunctive (respectively, pseudo-disjunctive) ELECTRE TRI-B (or ELECTRE TRI-nB) brings only partial information. On the other hand, the conjoint use of the pseudo-conjunctive and pseudo-disjunctive rules is not theoretically justified, because these procedures have no symmetry with respect to the transposition operation. Such a lack of symmetry is a consequence of considering the classes as closed from below [4].

The methods based on eliciting limiting profiles can be also criticized because defining these actions is often a difficult task, especially when the decision maker has only a vague idea about the boundary between two adjacent categories. The existence of such boundaries is often questioned in real-world problems (cf. [1]). Being the imprecise setting of limiting profiles perhaps the most important criticism to this kind of methods, a way to represent imperfect known criterion scores can be a real advantage.

As stated by Roy et al. [28] "…the definition of each criterion frequently comprises some part of arbitrariness, and the data used to build criteria are also very often imprecise, ill-determined, and uncertain." Also, to a certain extent, the elicitation of model's parameters often cannot avoid imprecision, ill-determination, and arbitrariness. This is, for instance, the case when the entity in charge of the decision is a group whose members have conflicting values, or when the DM represents a general view (e.g., the public), or a very hardly accessible entity (as a multinational CEO). Hence, the decision analyst (frequently in collaboration with the DM or his/her representative) should be prepared to handle imperfect knowledge of data from the above-mentioned sources.

Within this avenue of research, ELECTRE TRI-nB and ELECTRE TRI-nC were extended to the interval framework in [13]. In INTERCLASS-nB and INTERCLASS-nC, the imperfect information on criterion scores and model's parameters is modeled by interval numbers. This imperfect information (imprecision, ill-determination, and arbitrariness) can be naturally and easily characterized by interval numbers. Interval numbers represent a modeling alternative to sophisticated mathematical tools that can require significant cognitive efforts from the DM. An interval number is related to a magnitude whose precise value is unknown, but the range within which this value lies is well determined.

Often, the DM may feel more comfortable using an additive value function as preference model. In this context, interval numbers can also represent a natural and easy way to model imprecision and ill-determination in criterion weights and criterion scores (of both reference actions and actions that should be allocated to a class); therefore, interval-based generalizations of modeling preferences through additive functions seem plausible. This contribution presents two multi-criteria ordinal classification approaches that use an interval-based value function as model of the DM's preferences. One of the methods uses interval limiting actions to characterize boundaries between adjacent classes. The other method employs representative interval actions to characterize the classes. Each method is composed of two procedures, which correspond via the transposition operation. The essential properties of conformity, monotonicity, stability, homogeneity, independence, and uniqueness are fulfilled.

The structure of this contribution is the following: A required background on interval numbers and interval value functions is provided in Sect. 8.2. The ***U***Sort-nB method is detailed in Sect. 8.3. Section 8.4 presents the ***U***Sort-nC, including a simple example that illustrates the basic operation of the approach. Some conclusions are discussed in Sect. 8.5.

## 8.2 Some Background

According to Moore [23], an interval number is a range $\boldsymbol{E} = [\underline{E}, \overline{E}]$, where $\underline{E}$ denotes its lower limit and $\overline{E}$ its upper limit. Boldface italic letters will be used throughout the rest of the chapter to represent these numbers. Real numbers can be defined as particular interval numbers for which $\underline{E} = \overline{E}$ (degenerate interval numbers).

Some basic arithmetic operations with interval numbers are the following:

$$\boldsymbol{D} + \boldsymbol{E} = [\underline{D} + \underline{E}, \overline{D} + \overline{E}]$$

$$\boldsymbol{D} - \boldsymbol{E} = [\underline{D} - \overline{E}, \overline{D} - \underline{E}]$$

$$\boldsymbol{D} \times \boldsymbol{E} = [\min\{\underline{DE}, \underline{D}\overline{E}, \overline{D}\underline{E}, \overline{DE}\}, \max\{\underline{DE}, \underline{D}\overline{E}, \overline{D}\underline{E}, \overline{DE}\}]$$

Following [15], we call realization of the interval number $\boldsymbol{E}$ any real number $e$ in $[\underline{E}, \overline{E}]$. An order relation on interval numbers can be defined as: Let $e$ and $d$ be two unknown realizations of $\boldsymbol{E}$ and $\boldsymbol{D}$, respectively, we say that $\boldsymbol{E} > \boldsymbol{D}$ if the proposition "$e$ is greater than $d$" has more credibility than "$d$ is greater than $e$."

Shi et al. [29] introduced the possibility function:

$$P(\boldsymbol{E} \geq \boldsymbol{D}) = \begin{cases} 1 & \text{if } P_{ED} > 1, \\ P_{ED} & \text{if } 0 \leq P_{ED} \leq 1, \\ 0 & P_{ED} < 0. \end{cases} \tag{8.1}$$

where $\boldsymbol{E} = [\underline{E}, \overline{E}]$ and $\boldsymbol{D} = [\underline{D}, \overline{D}]$ are interval numbers and $P_{ED} = \frac{\overline{E} - \underline{D}}{(\overline{E} - \underline{E}) + (\overline{D} - \underline{D})}$.

When $\overline{E} = \underline{E}$ and $\overline{D} = \underline{D}$, then

$$P(\boldsymbol{E} \geq \boldsymbol{D}) = \begin{cases} 1 & \text{if } e \geq d, \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

In [12], the value $P(\boldsymbol{E} \geq \boldsymbol{D})$ is interpreted as the credibility of the statement "given two realizations from $\boldsymbol{E}$ and $\boldsymbol{D}$, $e$ and $d$, $e$ will be greater than or equal to $d$."

From the interpretation of the order relation $>$ on interval numbers, it is easy to prove that $\boldsymbol{E} > \boldsymbol{D} \Leftrightarrow e > d$ and $\boldsymbol{E} = \boldsymbol{D} \Leftrightarrow e = d$.

Some properties of the order relation and the possibility function follow (see [12]):

(i) $P(\boldsymbol{E} \geq \boldsymbol{D}) = \alpha_1 \geq 0.5$ and $P(\boldsymbol{D} \geq \boldsymbol{C}) = \alpha_2 \geq 0.5 \Rightarrow P(\boldsymbol{E} \geq \boldsymbol{C}) = \min\{\alpha_1, \alpha_2\}$ (transitivity).
(ii) $\boldsymbol{E} > \boldsymbol{D}$ and $\boldsymbol{D} > \boldsymbol{C} \Rightarrow \boldsymbol{E} > \boldsymbol{C}$.
(iii) If $e$ and $d$ are, respectively, the middle points of the intervals $\boldsymbol{E}$ and $\boldsymbol{D}$, then $\boldsymbol{E} > \boldsymbol{D} \Leftrightarrow e > d$ and $\boldsymbol{E} = \boldsymbol{D} \Leftrightarrow e = d$ (dictatorship of the middle point).
(iv) $P(\boldsymbol{E} \geq \boldsymbol{D}) = \alpha \Rightarrow P(\boldsymbol{D} \geq \boldsymbol{E}) = 1 - \alpha$ (negation).

Given a set $\boldsymbol{X}$ of interval numbers, $\geq$ is a weak order on $\boldsymbol{X}$. Using this order relation, the concepts of maximum and minimum can be defined as: $\boldsymbol{x}*$ is the maximum (respectively minimum) in $\boldsymbol{X}$ iff for all $\boldsymbol{y} \in \boldsymbol{X}$, $\boldsymbol{x}* \geq \boldsymbol{y}$ (resp. $\boldsymbol{x}* \leq \boldsymbol{y}$). These concepts have the same properties as in real numbers. If $\boldsymbol{X}$ is a finite set, the existence of maximum and minimum on $\boldsymbol{X}$ is guaranteed.

As higher the value of $\alpha$, the more reliable the strict order given by Eq. (8.3).

$$\boldsymbol{E} >_{\alpha} \boldsymbol{D} \Leftrightarrow P(\boldsymbol{E} \geq \boldsymbol{D}) \geq \alpha > 0.5. \tag{8.3}$$

Let $\boldsymbol{I}$ be the set of interval numbers. An interval function $\boldsymbol{f} : \boldsymbol{I} \rightarrow \boldsymbol{I}$ is said to be increasing iff for all $(\boldsymbol{E}, \boldsymbol{D}) \in \boldsymbol{I} \times \boldsymbol{I}$, $\boldsymbol{E} > \boldsymbol{D} \Rightarrow \boldsymbol{f}(\boldsymbol{E}) > \boldsymbol{f}(\boldsymbol{D})$. We will use a preference model given based on the following assumption:

**Assumption 8.1** The DM's multi-criteria preferences are compatible with an interval function $\boldsymbol{U}$. This means that for any pair of actions $(x, y)$, $\boldsymbol{U}(x) \geq \boldsymbol{U}(y)$ is an argument in favor of the statement "action $x$ is at least as good as $y$."

Under reasonable conditions, it is possible to model $\boldsymbol{U}$ through the additive form:

$$\boldsymbol{U}(x) = \sum \boldsymbol{w}_i \boldsymbol{u}_i(\boldsymbol{g}_i) \tag{8.4}$$

where the $\boldsymbol{u}_i$s are one-dimensional value functions, $\boldsymbol{g}_i$ denotes the $i$-th criterion score of action $x$, and $\boldsymbol{w}_i$ is the interval weight associated with the $i$-th criterion. The simplest form of (8.4) is the interval weighted-sum function in which $\boldsymbol{u}_i(\boldsymbol{g}_i) = \boldsymbol{g}_i$. This linear function was used by Liesio et al. [22] and Fliedner and Liesio [15] in robust project portfolio optimization.

**Definition 8.1** Action $x$ is said to be $\alpha$-preferred to action $y$ if $P(\boldsymbol{U}(x) \geq \boldsymbol{U}(y)) \geq \alpha > 0.5$.

This relation will be denoted as $x P_{\geq\alpha} y$. Based on Assumption 8.1, it is interpreted as "$x$ is preferred to $y$ with credibility greater than or equal to $\alpha$."

*Remark 8.1* From the above properties (i) and (iv) of the possibility function, $x P_{\geq\alpha} y$ is asymmetric and transitive on the decision set $A$.

## 8.3 An Ordinal Classification Method Based on Limiting Boundary Actions

### 8.3.1 *Description of the Method*

First of all, we present the requirements to the limiting profiles of an ordinal classification method based on Assumption 8.1, which satisfies the consistency properties from [26], being also symmetric with respect to the transposition operation.

**Condition 8.1 (Requirements on Limiting Profiles)** Set $\alpha > 0.5$. Consider a set of $M$ ordered and predefined classes $C = \{C_1, \ldots, C_k, \ldots, C_M\}, (M \geq 2)$ (ordered in the sense of increasing preference). The boundary between $C_k$ and $C_{k+1}$ is described by a set of limiting actions $B_k$, for $k = 1, \ldots, M-1$. Each $B_k$ is composed of two disjoint subsets $B_{Uk}$ and $B_{Lk}$ such that each $w \in B_{Uk}$ is in $C_{k+1}$, and each $z \in B_{Lk}$ is in $C_k$. Additionally:

(i) There is no pair $(w, z) \in B_k \times B_h (h > k)$ fulfilling $w P_{\geq\alpha} z$.
(ii) There is no pair $(w, z) \in B_{Uk} \times B_{Uk}$ fulfilling $z P_{\geq\alpha} w$.
(iii) There is no pair $(w, z) \in B_{Lk} \times B_{Lk}$ fulfilling $w P_{\geq\alpha} z$.
(iv) For each $w \in B_{Uk}$ there is $z \in B_{Lk}$ such that $w P_{\geq\alpha} z$.
(v) For each $w \in B_{Lk}$ there is $z \in B_{Uk}$ such that $z P_{\geq\alpha} w$.
(vi) For each $w \in B_{Uk}$ there is $z \in B_{k+1}$ such that $z P_{\geq\alpha} w$.
(vii) For each $w \in B_{Lk}$ there is $z \in B_{k-1}$ such that $w P_{\geq\alpha} z$.

**Definition 8.2 ($P_{\geq\alpha}$ Relation Between Actions and Boundaries)**

(a) $x P_{\geq\alpha} B_k \Leftrightarrow$ There is $w \in B_k$ such that $x P_{\geq\alpha} w$.
(b) $B_k P_{\geq\alpha} x \Leftrightarrow$ There is $w \in B_k$ such that $w P_{\geq\alpha} x$.

*Remark 8.2* Combining Definition 8.2.a, Conditions 8.1.iv and 8.1.vii, and Remark 8.1 (transitivity of $P_{\geq\alpha}$), it is easy to prove that $x P_{\geq\alpha} B_k \Rightarrow$ there is $w \in B_{Lk}$ such that $x P_{\geq\alpha} w$.

From Definition 8.2.b, Condition 8.1.v, and Remark 8.1 (transitivity of $P_{\geq\alpha}$), it follows that $B_k P_{\geq\alpha} x \Rightarrow$ there is $w \in B_{Uk}$ such that $w P_{\geq\alpha} x$.

**Proposition 8.1** *Under Condition 8.1, the following propositions are fulfilled:*

*(i)* $x P_{\geq\alpha} B_k \Rightarrow x P_{\geq\alpha} B_h$ *for* $k > h$.
*(ii)* $B_h P_{\geq\alpha} x \Rightarrow B_k P_{\geq\alpha} x$ *for* $k > h$.

***Proof*** Proposition 8.1.i: From Remark 8.2, there is $w \in B_{Lk}$ such that $x P_{\geq\alpha} w$. Then, Definition 8.2.a and Condition 8.1.vii imply $x P_{\geq\alpha} B_{k-1}$. Applying the same argument recursively, we have $x P_{\geq\alpha} B_h$ for $h < k$.

Proposition 8.1.ii: From Remark 8.2, there is $w \in B_{Uh}$ such that $w P_{\geq\alpha} x$. From Condition 8.1.vi and Remark 8.1, there is $z$ in $B_{h+1}$ fulfilling $z P_{\geq\alpha} x$. So, $B_{h+1} P_{\geq\alpha} x$ (Definition 8.2.b). The recursive application of this argument leads to $B_k P_{\geq\alpha} x$ for $k > h$.

The assignments of actions are obtained by the following rules:

**Definition 8.3 (Primal Assignment Procedure)** Set $B_k$, $k = 1, \ldots, M-1$ fulfilling Condition 8.1 and take $B_M$ and $B_0$ as the ideal and anti-ideal actions, respectively. Set $B_M P_{\geq\alpha} x$.

- Compare $B_k$ with $x$ for $k = 1, \ldots, M$.
- Let $B_k$ be the first boundary such that $B_k P_{\geq\alpha} x$.
- Take $C_k$ as a possible class to assign $x$.

**Definition 8.4 (Dual Assignment Procedure)** Set $B_k, k = 1, \ldots, M-1$, fulfilling Condition 8.1 and take $B_M$ and $B_0$ as the ideal and anti-ideal actions, respectively. Set $x P_{\geq\alpha} B_0$.

- Compare $x$ with $B_k$ for $k = M-1, \ldots, 0$.
- Let $B_k$ be the first boundary such that $x P_{\geq\alpha} B_k$.
- Take $C_{k+1}$ as a possible class to assign $x$.

*Remark 8.3* The above assignment rules correspond via the transposition operation. As was discussed in Introduction, they should be used conjointly. In the following, the conjoint method will be called ***U***Sort-nB.

**Proposition 8.2 (Relationship Between the Two Assignment Rules)** *Let $C_h$ and $C_{k'}$ be the assignments suggested by the primal and dual rules, respectively. Then, $h \geq k' - 1$.*

***Proof*** $x$ is assigned to $C_h$ by the primal procedure $\Rightarrow B_h P_{\geq\alpha} x \Rightarrow$ there is $w \in B_h$ such that $w P_{\geq\alpha} x$; $x$ is assigned to $C_{k'}$ by the dual procedure $\Rightarrow x P_{\geq\alpha} B_{k'-1} \Rightarrow$ there is $z \in B_{k'-1}$ such that $x P_{\geq\alpha} z$; Then, there is a pair $(w, z) \in B_h \times B_{k'-1}$ such that $w P_{\geq\alpha} z$; $h < k' - 1$ contradicts Condition 8.1.i. Hence, $h \geq k' - 1$.

### *8.3.2 Consistency Properties of the USort-nB Primal and Dual Procedures*

In this subsection, we analyze whether the ***U***Sort-nB rules satisfy the properties that were firstly proposed by Roy and Bouyssou [26] for ELECTRE TRI-B. They are fulfilled also by ELECTRE TRI-nC [2], ElectreSort [19], ELECTRE TRI-nB [11], and INTERCLASS-nB [13]; these properties have become a rational paradigm for multi-criteria ordinal classification methods.

In the framework of ***U***Sort-nB, we will use the same definitions of merging classes and splitting a class as in [11]. The reader is referred to such a paper for a formal definition.

**Definition 8.5 (Stability Property)** A method is considered stable under merging and splitting operations, if and only if:

(i) After performing a merging or a splitting operation, the actions belonging to a non-modified class previously to the change will keep their assignments after such a modification.
(ii) After performing a merging of two classes, the actions belonging to the merged classes (before merging) are still belonging to the new one.
(iii) After performing a splitting operation of a class, the actions belonging to the modified class (before splitting) are still belonging to one of the two new ones.

**Proposition 8.3 (Consistency Properties of *U*Sort-nB)** *Under Condition 8.1, the primal and dual procedures of USort-nB fulfill the following consistency properties:*

*i. Uniqueness: An action is classified only into a single class.*
*ii. Independence: When assigning an action to a certain class, the assignment does not depend on the assignment of other actions.*
*iii. Conformity:*

*(a) A limiting action $w \in B_{Lk}$ is assigned to $C_k$.*
*(b) A limiting action $w \in B_{Uk}$ is assigned to $C_{k+1}$.*

*iv. Monotonicity: If $x$ is assigned to $C_k$ and $yP_{\geq\alpha}x$, then $y$ is classified into $C_{k'}$ with $k' \geq k$.*
*v. Homogeneity: If two actions fulfill the same preference relation of Definition 8.2 with respect to the limiting boundaries, then they are assigned to the same class.*
*vi. Stability: The methods are stable according to Definition 8.5.*

***Proof*** The proofs of Uniqueness, Independence, and Homogeneity are trivial, determined by the form the assignment rules were designed (see Definitions 8.3 and 8.4).

*Conformity*
Dual procedure:
$w \in B_{Lk} \Rightarrow$ there is no $z \in B_k$ such that $wP_{\geq\alpha}z$ from Conditions 8.1.ii and 8.1.iii $\Rightarrow not(wP_{\geq\alpha}B_k)$ (Definition 8.2.a) $\Rightarrow not(wP_{\geq\alpha}B_h)(h \geq k)$ (Counter-reciprocal of Proposition 8.1.i).

There is $y \in B_{k-1}$ such that $wP_{\geq\alpha}y$ from Condition 8.1.vii. Hence, $wP_{\geq\alpha}B_{k-1}$ (Definition 8.2.a). Hence, $w$ is assigned to $C_k$ (Definition 8.4).

$w \in B_{Uk} \Rightarrow$ there is no $z \in B_{k+1}$ such that $wP_{\geq\alpha}z$ from Condition 8.1.i; $\Rightarrow not(wP_{\geq\alpha}B_{k+1})$ (Definition 8.2.a) $\Rightarrow not(wP_{\geq\alpha}B_h)(h \geq k+1)$ (Counter-reciprocal of Proposition 8.1.i).

There is $y \in B_k$ such that $w P_{\geq\alpha} y$ from Condition 8.1.iv. Hence, $w P_{\geq\alpha} B_k$ (Definition 8.2.a). Hence, $w$ is assigned to $C_{k+1}$ (Definition 8.4).

*Monotonicity*
Dual procedure:
$x$ is assigned to $C_k \Rightarrow$ there is $w$ in $B_{k-1}$ such that $z P_{\geq\alpha} w$. Since $y P_{\geq\alpha} x$, it follows that $y P_{\geq\alpha} w$ (transitivity of $P_{\geq\alpha}$). So, $y P_{\geq\alpha} B_k$ (Definition 8.2.a). Then, $y$ is assigned to $C_{k'}$ with $k' \geq k$ (Definition 8.4).

*Stability*
Dual procedure:

a. Merging operation between two adjacent classes.

  i. Consider that action $x$ was classified into class $C_h$, for all $h > k+1$ (i.e., $h - 1 > k$) before we proceed to a merging operation. Given the two propositions $not(x P_{\geq\alpha} B_h)$ and $x P_{\geq\alpha} B_{h-1}$ are verified, after removing set $B_k$, we obtain exactly the same situation; $x$ will be assigned to the same class as before once $B'_{h-1} = B_h$ and $C'_{h-1} = C_h$ for $h > k$.
  ii. Consider that action $x$ is classified into either $C_k$ or $C_{k+1}$ before we proceed to a merging operation. If before withdrawing $B_k$, $x$ was assigned to $C_{k+1}$, it naturally follows that $x P_{\geq\alpha} B_k$ and $not(x P_{\geq\alpha} B_h)$ for all $h > k$ (from Definition 8.4) and $x P_{\geq\alpha} B_{k-1}$ from Proposition 8.1.i. After we proceed to a merging operation, $x$ will be assigned to $C'_k$, which is the class that substitutes the two classes $C_k$ and $C_{k+1}$. If, before we proceed to a merging operation, action $x$ was classified into $C_k$, from Definition 8.4 we have both $x P_{\geq\alpha} B_{k-1}$ and $not(x P_{\geq\alpha} B_h)$, for all $h > k - 1$. After a merging operation, none of the previous conditions changes; from Definition 8.4, action $x$ is assigned to $C'_k$, which is the class that substitutes both $C_k$ and $C_{k+1}$.
  iii. Consider that action $x$ belongs to category $C_h$, for all $h < k$ before we proceed to a merging operation. It is clear that after withdrawing the set $B_k$, none of the previous conditions changes; according to Definition 8.4, the action $x$ is classified into the same class as previously.

b. Splitting a single category into two new consecutive ones.

  i. Consider that action $x$ was assigned to $C_h$, for all $h > k(h \geq k+1)$ before we proceed to a splitting operation. Then, from Definition 8.4, it naturally follows both conditions, $not(x P_{\geq\alpha} B'_{h+1})$ and $x P_{\geq\alpha} B'_h$, where $B'_h = B_{h-1}$ and $B'_{h+1} = B_h$. Hence, $x$ will be assigned to $C'_{h+1}$ (the same class $C_h$).
  ii. Consider that action $x$ is classified into $C_k$ before we proceed to a splitting operation. From Definition 8.4, we obtain both conditions, $not(x P_{\geq\alpha} B_k)$ and $x P_{\geq\alpha} B_{k-1}$. After inserting the set $B'_k$, according to the definition of the splitting operation, we obtain both conditions, $not(x P_{\geq\alpha} B'_{k+1})$ and $x P_{\geq\alpha} B'_{k-1}$. From Definition 8.4 and $x P_{\geq\alpha} B'_k$, it follows that $x$ will be assigned to $C'_{k+1}$. Otherwise, $x$ would be assigned to $C'_k$. Hence, $x$ will be assigned to one of the classes in which the old class $C_k$ was split.

iii. Consider now that action $x$ was classified into $C_h$, for all $h < k$ before we proceed to a splitting operation. According to Proposition 8.1.i, from $not(xP_{\geq\alpha}B_h)$, it follows that $not(xP_{\geq\alpha}B'_k)$. After splitting, none of the previous conditions changes, given that $B'_h = B_h$, for all $h < k$. Hence, after we proceed to a splitting operation, action $x$ will be classified into $C'_h$ (which is the same $C_h$), for all $h < k$.

The proofs for the primal rule are omitted. They can be justified by the equivalence through the transposition operation.

*Remark 8.4* It should be underlined that the ***U***Sort-nB method is not based on any particular form of function $\boldsymbol{U}$; the single requirement to this function is to be compatible with Assumption 8.1.

## 8.4 *U*Sort-nC: An Ordinal Classification Method Based on Representative Actions

### *8.4.1 Description of the Method*

We present below the requirements to the representative actions of an ordinal classification method based on Assumption 8.1, which satisfies the consistency properties from [26], being also symmetric with respect to the transposition operation.

**Condition 8.2 (Requirements to the Set of Reference Actions)** Set $\alpha > 0.5$. Consider a set of $M$ ordered and predefined classes $C = \{C_1, \ldots, C_k, \ldots, C_M\}$, $(M \geq 2)$ (ordered in the sense of increasing preference). Let $R_k = \{r_{k,j},\ j = 1, \ldots, card(R_k)\}$ denote the subset of reference actions introduced to characterize $C_k$, $k = 1, \ldots, M$. Let $\{r_0, R_1, \ldots, R_M, r_{M+1}\}$ be the set of all reference actions, in which $r_0$ and $r_{M+1}$ are the anti-ideal and ideal actions, respectively. The elements in $R_k$, $k = 1, \ldots, M$ must satisfy the following conditions:

i. For each $w \in R_k$ there is $z \in R_{k+1}$ such that $zP_{\geq\alpha}w$.
ii. For each $w \in R_k$ there is $z \in R_{k-1}$ such that $wP_{\geq\alpha}z$.
iii. For each $k$, $min_{R_{k+1}}\{\boldsymbol{U}(r_{k+1,j})\} > max_{R_k}\{\boldsymbol{U}(r_{k,j})\}$.

**Definition 8.6 ($P_{\geq\alpha}$ Relation Between Actions and Representative Subsets)**

(a) $xP_{\geq\alpha}R_k \Leftrightarrow$ There is $w \in R_k$ such that $xP_{\geq\alpha}w$ and $\boldsymbol{U}(x) > max_{R_k}\{\boldsymbol{U}(r_{k,j})\}$.
(b) $R_kP_{\geq\alpha}x \Leftrightarrow$ There is $w \in R_k$ such that $wP_{\geq\alpha}x$ and $min_{R_k}\{\boldsymbol{U}(r_{k,j})\} > \boldsymbol{U}(x)$.

*Remark 8.5*

(A) From Definition 8.6, it follows that the preference relation between actions and representative subsets is asymmetric.

(B) Combining Definition 8.6 and the transitivity property of the order relation on interval numbers, it follows that $y P_{\geq\alpha} x$ and $x P_{\geq\alpha} R_k \Rightarrow y P_{\geq\alpha} R_k$.

**Proposition 8.4** *Under Condition 8.2, the following propositions are fulfilled:*

*(a)* $x P_{\geq\alpha} R_k \Rightarrow x P_{\geq\alpha} R_h$ *for* $k > h$.
*(b)* $R_h P_{\geq\alpha} x \Rightarrow R_k P_{\geq\alpha} x$ *for* $k > h$.

The proofs follow from Definition 8.6, Conditions 8.2, and the transitivity property of the order relation on interval numbers.

**Definition 8.7 (Preference Non-closeness Measure)** The non-closeness measure between $x$ and $R_k$ is defined as:

$$\boldsymbol{nc}(x, R_k) = min_{R_k}\{\boldsymbol{U}(r_{k,j})\} - \boldsymbol{U}(x) \text{ if } min_{R_k}\{\boldsymbol{U}(r_{k,j})\} > \boldsymbol{U}(x)$$
$$\boldsymbol{nc}(x, R_k) = \boldsymbol{U}(x) - max_{R_k}\{\boldsymbol{U}(r_{k,j})\} \text{ if } \boldsymbol{U}(x) > max_{R_k}\{\boldsymbol{U}(r_{k,j})\}$$
$$\boldsymbol{nc}(x, R_k) = 0 \text{ if } min_{R_k}\{\boldsymbol{U}(r_{k,j})\} \leq \boldsymbol{U}(x) \leq max_{R_k}\{\boldsymbol{U}(r_{k,j})\}$$

Note that $\boldsymbol{nc}(x, R_k)$ strictly increases (respectively, decreases) with $\boldsymbol{U}(x)$ when $x P_{\geq\alpha} R_k$ (resp., $R_k P_{\geq\alpha} x$).

Under the previous definitions and requirements, the assignment rules are the following:

**Definition 8.8 (Primal Assignment Procedure)** Set $R_{M+1} P_{\geq\alpha} x$.

(a) Compare $x$ with $R_k$ for $k = 1, \ldots, M + 1$, until the first value, $k$, such that $R_k P_{\geq\alpha} x$.
(b) For $k = 1$, select $C_1$ as a possible class to assign action $x$.
(c) For $1 < k < M + 1$, if $\boldsymbol{nc}(x, R_k) < \boldsymbol{nc}(x, R_{k-1})$, then select $C_k$ as a possible class to assign $x$; otherwise, select $C_{k-1}$.
(d) For $k = M + 1$, select $C_M$ as possible class to assign $x$.

**Definition 8.9 (Dual Assignment Procedure)** Set $x P_{\geq\alpha} R_0$.

i. Compare $x$ with $R_k$ for $k = M, \ldots, 0$, until the first value $k$ such that $x P_{\geq\alpha} R_k$.
ii. For $k = M$, select CM as a possible class to assign action $x$.
iii. For $0 < k < M$, if $\boldsymbol{nc}(x, R_k) \leq \boldsymbol{nc}(x, R_{k+1})$, then select $C_k$ as a possible category to assign $x$; otherwise, select $C_{k+1}$.
iv. For $k = 0$, select $C_1$ as a possible class to assign $x$.

As in ELECTRE TRI-nC and the method described in Sect. 8.3, the above assignment rules correspond via the transposition operation, and should be used conjointly. In the following, the conjoint method will be called ***U***Sort-nC.

## 8.4.2 *Consistency Properties of USort-nC*

First of all, it is necessary to define the merging and splitting operation on the new context.

**Definition 8.10 (Merging and Splitting Operations in the Context of USort-nC)**

(a) Merging operation: Two adjacent classes, $C_k$ and $C_{k+1}$, will be merged to become a new one, $C'_k$, characterized by a new subset of reference actions, $R'_k = R_k \cup R_{k+1}$. The new set of classes is $C = \{C1, \ldots, C_{k-1}, C'_k, C_{k+2}, \ldots, C_M\}$, which (updating the subscripts) can be denoted as $\{C'_1, \ldots, C'_{k-1}, C'_k, C'_{k+1}, \ldots, C'_{M-1}\}$. The new set of reference actions is $R = \{R_1, \ldots, R_{k-1}, R'_k, R_{k+2}, \ldots, R_M\}$, which can be denoted as $R' = \{R'_1, \ldots, R'_{k-1}, R'_k, R'_{k+1}, \ldots, R'_{M-1}\}$.

The fulfillment of Condition 8.2 of the new reference set $R'$ is a direct consequence of the fulfillment of Condition 8.2 on the previous (before merging) reference set $R$.

(b) Splitting operation: $C_k$ is split into two new adjacent classes, $C'_k$ and $C''_k$, characterized by two new distinct subsets of reference actions, $R'_k$ and $R''_k$ respectively. The new set of classes is $C = \{C1, \ldots, C_{k-1}, C'_k, C''_k, C_{k+1}, \ldots, C_M\}$, which, (updating the subscripts), will be denoted as $\{C'_1, \ldots, C'_{k-1}, C'_k, C'_{k+1}, C'_{k+2}, \ldots, C'_{M+1}\}$. The new set of reference actions is $R = \{R_1, \ldots, R_{k-1}, R'_k, R''_k, R_{k+1}, \ldots, R_M\}$, which will be denoted as $R' = \{R'_1, \ldots, R'_{k-1}, R'_k, R'_{k+1}, R'_{k+2}, \ldots, R'_{M+1}\}$. $R'$ should fulfill Condition 8.2, and additionally:

$$\min_{r\in R'_k}\{\boldsymbol{U}(r)\} \le \min_{r\in R_k}\{\boldsymbol{U}(r)\} \le \min_{r\in R''_k}\{\boldsymbol{U}(r)\}$$

$$\max_{r\in R'_k}\{\boldsymbol{U}(r)\} \le \max_{r\in R_k}\{\boldsymbol{U}(r)\} \le \max_{r\in R''_k}\{\boldsymbol{U}(r)\}.$$

*Remark 8.6* Let $C_k$ and $C_{k+1}$ be the classes that were merged into the new class $C'_k$.

(A) From Condition 8.2 iii and Definition 8.10(a) we have:

- $\max_{r\in R'_k}\{\boldsymbol{U}(r)\} = \max_{r\in R_{k+1}}\{\boldsymbol{U}(r)\}$
- $\min_{r\in R'_k}\{\boldsymbol{U}(r)\} = \min_{r\in R_k}\{\boldsymbol{U}(r)\}$.

(B) It is easy to prove that:

- $x P_{\ge\alpha} R_k$ and $x P_{\ge\alpha} R_{k+1} \Rightarrow x P_{\ge\alpha} R'_k$
- $not(x P_{\ge\alpha} R_k)$ and $not(x P_{\ge\alpha} R_{k+1}) \Rightarrow not(x P_{\ge\alpha} R'_k)$.

**Proposition 8.5 (Consistency Properties of *U*Sort-nC)** *Under Condition 8.2, the primal and dual procedures of USort-nC fulfill the following consistency properties:*

*i. Uniqueness: An action is classified only into a single class.*
*ii. Independence: When assigning an action to a certain class, the assignment does not depend on the assignment of other actions.*
*iii. Conformity: A representative action $w \in R_k$ is assigned to $C_k$.*

iv. *Monotonicity: If* $x$ *is assigned to* $C_k$ *and* $yP_{\geq\alpha}x$*, then* $y$ *is classified into* $C'_k$ *with* $k' \geq k$*.*

v. *Homogeneity: If two actions fulfill the same preference relation of Definition 8.6 with respect to the subsets of representative actions, then they are assigned to the same class.*

vi. *Stability with respect to merging operation: After performing a merging operation, the actions belonging to a non-modified class previously to the change will keep their assignments after such a modification; the actions previously assigned to the merged categories are assigned to the new class.*

vii. *Stability with respect to splitting operation: After splitting a class into two new classes, the actions previously assigned to the modified class are assigned to one of the new classes; any action previously assigned to a non-adjacent class to the modified one will remain in the same category; any action previously assigned to an adjacent class to the modified one will either be assigned to the same category or to a new category.*

***Proof*** The proofs of Uniqueness, Independence, and Homogeneity are trivial, determined by the way the assignment rules were designed (see Definitions 8.8 and 8.9).

*Conformity*
Dual assignment rule:

If $w \in R_1$ or $w \in R_M$, the proof is trivial.

If $1 < k < M$:
$w \in R_k$ and Definition 8.6 $\Rightarrow$ $not(wP_{\geq\alpha}R_k) \Rightarrow not(wP_{\geq\alpha}R_h)$ for $k < h$ (Counter-reciprocal of Proposition 8.4.a).
$w \in R_k \Rightarrow$ there is $z \in R_{k-1}$ such that $wP_{\geq\alpha}z$ (Condition 8.2.ii); $wP_{\geq\alpha}z$ and Condition 8.2.iii $\Rightarrow wP_{\geq\alpha}R_{k-1}$.

$\boldsymbol{nc}(w, R_{k-1}) = \boldsymbol{U}(w) - \max_{R_{k-1}}\{\boldsymbol{U}(r_{k-1,j})\} > 0$ from Condition 8.2.iii
$\boldsymbol{nc}(w, R_k) = 0$ since $w \in R_k$;
$\boldsymbol{nc}(w, R_k) < \boldsymbol{nc}(w, R_{k-1})$; hence $w$ is assigned to $C_k$ according to Definition 8.9.

This completes the proof.

*Monotonicity*
Dual procedure:
Suppose that $x$ is assigned to $C_M$. From Definition 8.9, we have two possibilities:
$xP_{\geq\alpha}R_M$ or $(not(xP_{\geq\alpha}R_M), xP_{\geq\alpha}R_{M-1}$, and $\boldsymbol{nc}(x, R_{M-1}) > \boldsymbol{nc}(x, R_M))$;
$xP_{\geq\alpha}R_M$ and $yP_{\geq\alpha}x \Rightarrow yP_{\geq\alpha}R_M$ from Remark 8.5.B $\Rightarrow$ $y$ is assigned to $C_M$ (Definition 8.9).

Consider the case $not(xP_{\geq\alpha}R_M)$, $xP_{\geq\alpha}R_{M-1}$, and $\boldsymbol{nc}(x, R_{M-1}) > \boldsymbol{nc}(x, R_M)$; since $yP_{\geq\alpha}x$, we have $yP_{\geq\alpha}R_{M-1}$ (Remark 8.5.B) and perhaps also $yP_{\geq\alpha}R_M$; if $yP_{\geq\alpha}R_M$, $y$ is assigned to $C_M$ (Definition 8.9).

Suppose now that $not(yP_{\geq\alpha}R_M)$; since $yP_{\geq\alpha}x$, we have $\boldsymbol{nc}(y, R_{M-1}) > \boldsymbol{nc}(x, R_{M-1})$ and $\boldsymbol{nc}(x, R_M) > \boldsymbol{nc}(y, R_M)$ (Definition 8.7); hence, $\boldsymbol{nc}(x, R_{M-1}) > \boldsymbol{nc}(x, R_M) \Rightarrow \boldsymbol{nc}(y, R_{M-1}) > \boldsymbol{nc}(y, R_M)$;

$not(yP_{\geq\alpha}R_M)$, $yP_{\geq\alpha}R_{M-1}$ and $\boldsymbol{nc}(y, R_{M-1}) > \boldsymbol{nc}(y, R_M) \Rightarrow y$ is assigned to $C_M$ (Definition 8.9).

Let us consider the case where $x$ is assigned to $C_k$ ($k < M$). $k = 1$ is trivial; according to Definition 8.9, we have two possibilities:

$xP_{\geq\alpha}R_k$ and $\boldsymbol{nc}(x, R_k) \leq \boldsymbol{nc}(x, R_{k+1})$, or
$xP_{\geq\alpha}R_{k-1}$, $not(xP_{\geq\alpha}R_k)$, and $\boldsymbol{nc}(x, R_k) < \boldsymbol{nc}(x, R_{k-1})$

$xP_{\geq\alpha}R_k$ and $yP_{\geq\alpha}x \Rightarrow yP_{\geq\alpha}R_k$ (Remark 8.5.B); $y$ is assigned to $C'_k$ ($k' \geq k$) (Definition 8.9);
$xP_{\geq\alpha}R_{k-1}$ and $yP_{\geq\alpha}x \Rightarrow yP_{\geq\alpha}R_{k-1}$ (Remark 8.5.B) $\Rightarrow yP_{\geq\alpha}R_k$ or ($yP_{\geq\alpha}R_{k-1}$ and $not(yP_{\geq\alpha}R_k)$);
If $yP_{\geq\alpha}R_k$ then $y$ is assigned to $C'_k$ ($k' \geq k$) (Definition 8.9).
Suppose that $yP_{\geq\alpha}R_{k-1}$ and $not(yP_{\geq\alpha}R_k)$;
Since $yP_{\geq\alpha}x$, $\boldsymbol{nc}(x, R_k) > \boldsymbol{nc}(y, R_k)$ and $\boldsymbol{nc}(x, R_{k-1}) < \boldsymbol{nc}(y, R_{k-1})$ (Definition 8.7). Hence, $\boldsymbol{nc}(x, R_k) < \boldsymbol{nc}(x, R_{k-1}) \Rightarrow \boldsymbol{nc}(y, R_k) < \boldsymbol{nc}(y, R_{k-1})$. From Definition 8.9, it follows that $y$ is assigned to $C_k$. This completes the proof.

*Stability*

Dual procedure:

a. Merging operation between two adjacent classes ($C_k$ and $C_{k+1}$).

   i. Let $h > k + 1$ be the first value of $k$ such that $xP_{\geq\alpha}R_h$ before a merging operation. From Definition 8.9 it follows that the action $x$ will be assigned to the same class as previously.

   ii. Let $h = k + 1$ be the first value of $h$ such that $xP_{\geq\alpha}R_h$ before a merging operation. $xP_{\geq\alpha}R_{k+1} \Rightarrow \boldsymbol{nc}(x, R_{k+1}) = \boldsymbol{U}(x) - \max_{r\in R_{k+1}} \boldsymbol{U}(r)$ (from Definitions 8.6 and 8.7) and $xP_{\geq\alpha}R_k$ (Proposition 8.4); using Remark 8.6 we have $\boldsymbol{nc}(x, R_{k+1}) = \boldsymbol{nc}(x, R'_k)$ and $xP_{\geq\alpha}R'_k$. If $x$ was assigned to class $C_{k+2} \Rightarrow \boldsymbol{nc}(x, R_{k+2}) < \boldsymbol{nc}(x, R_{k+1}) = \boldsymbol{nc}(x, R'_k)$, then $x$ is assigned to $C_{k+2}$ (the same class). If $x$ was assigned to class $C_{k+1} \Rightarrow \boldsymbol{nc}(x, R_{k+2}) > \boldsymbol{nc}(x, R_{k+1}) = \boldsymbol{nc}(x, R'_k)$, then $x$ is assigned to $C'_k$ (the new class).

   iii. Let $h = k$ be the first value of $h$ such that $xP_{\geq\alpha}R_h$ before a merging operation. $xP_{\geq\alpha}R_k \Rightarrow x$ is assigned to class $C_k$ or $C_{k+1} \Rightarrow x$ is assigned to class $C'_k$ (the new class).

   iv. Let $h = k - 1$ be the first value of $h$ such that $xP_{\geq\alpha}R_h$ before a merging operation. We have $not(xP_{\geq\alpha}R'_k)$ (Remark 8.6) and $xP_{\geq\alpha}R_{k-1}$ (Proposition 8.4). If $x$ is assigned to class $C_k \Rightarrow \boldsymbol{nc}(x, R_k) < \boldsymbol{nc}(x, R_{k-1})$, we have three options: (1) $\boldsymbol{U}(x) > \max_{r\in R'_k}\{\boldsymbol{U}(r)\}$; by Remark 8.6 and Condition 8.2 we can affirm that $\boldsymbol{U}(x) > \max_{r\in R_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{nc}(x, R_k) = \boldsymbol{U}(x) - \max_{r\in R_k}\{\boldsymbol{U}(r)\}$ and $\boldsymbol{nc}(x, R'_k) = \boldsymbol{U}(x) - \max_{r\in R'_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{nc}(x, R'_k) < \boldsymbol{nc}(x, R_k) \Rightarrow \boldsymbol{nc}(x, R'_k) < \boldsymbol{nc}(x, R_{k-1})$; (2) $\max_{r\in R'_k}\{\boldsymbol{U}(r)\} \geq \boldsymbol{U}(x) \geq \min_{r\in R'_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{nc}(x, R'_k) = 0$ (Definition 8.7) $\Rightarrow \boldsymbol{nc}(x, R'_k) < \boldsymbol{nc}(x, R_{k-1})$; (3) $\boldsymbol{U}(x) < \min_{r\in R'_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{U}(x) < \min_{r\in R_k}\{\boldsymbol{U}(r)\}$ (Remark 8.6) $\Rightarrow \boldsymbol{nc}(x, R'_k) = \boldsymbol{nc}(x, R_k) \Rightarrow \boldsymbol{nc}(x, R'_k) < \boldsymbol{nc}(x, R_{k-1})$. In all cases $x$ is assigned to class $C'_k$ (the new class). If $x$ is assigned to class $C_{k-1} \Rightarrow \boldsymbol{nc}(x, R_{k-1}) \leq \boldsymbol{nc}(x, R_k)$, we know that $\boldsymbol{nc}(x, R_{k-1}) = \boldsymbol{U}(x) - \max_{r \in R_{k-1}}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{nc}(x, R_k) = \min_{r \in R_k}\{\boldsymbol{U}(r)\} - \boldsymbol{U}(x) \Rightarrow \boldsymbol{U}(x) < \min_{r \in R_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{U}(x) < \min_{r \in R'_k}\{\boldsymbol{U}(r)\}$ (Remark 8.6) $\Rightarrow \boldsymbol{nc}(x, R_k) = \boldsymbol{nc}(R'_k) \Rightarrow \boldsymbol{nc}(x, R_{k-1}) \leq \boldsymbol{nc}(x, R'_k) \Rightarrow x$ is assigned to class $C_{k-1}$ (the same class).

v. Let $h < k - 2$ be the first value of $h$ such that $x P_{\geq\alpha} R_h$ before a merging operation. From Definition 8.9 and Remark 8.6 it follows that the action $x$ will be assigned to the same class as previously.

b. Splitting a single category into two new consecutive ones ($C_k$ into $C'_k$ and $C''_k$).

i. Consider that action $x$ was assigned to $C_k$, before we proceed to a splitting operation. There are two options: (1) Let $h = k$ be the first value such that $x P_{\geq\alpha} R_h$ before a split operation; (2) Let $h = k - 1$ be the first value such that $x P_{\geq\alpha} R_h$ before a split operation. (1) We know that $\boldsymbol{nc}(x, R_k) \leq \boldsymbol{nc}(x, R_{k+1})$ also $x P_{\geq\alpha} R_{k-1}$ (Proposition 8.4). If $x P_{\geq\alpha} R''_k \Rightarrow \boldsymbol{nc}(x, R''_k) = \boldsymbol{U}(x) - \max_{r \in R''_k}\{\boldsymbol{U}(r)\} < \boldsymbol{U}(x) - \max_{r \in R_k}\{\boldsymbol{U}(r)\} = \boldsymbol{nc}(x, R_k)$ (Definition 8.10 (b)) $\Rightarrow \boldsymbol{nc}(x, R''_k) < \boldsymbol{nc}(x, R_{k+1})$. If $x P_{\geq\alpha} R'_k$ by Definition 8.9 it follows that $x$ is assigned to class $C'_k$ or $C''_k$. If $not(x P_{\geq\alpha} R'_k)$, since $x P_{\geq\alpha} R_k \Rightarrow \boldsymbol{U}(x) > \max_{r \in R_k} \boldsymbol{U}(r)$ also we know that $\max_{r \in R_k} \boldsymbol{U}(r) \geq \max_{r \in R'_k}\{\boldsymbol{U}(r)\} > \max_{r \in R_{k-1}}\{\boldsymbol{U}(r)\}$ (Definition 8.10(b) and Condition 8.2) $\Rightarrow \boldsymbol{nc}(x, R'_k) < \boldsymbol{nc}(x, R'_k) \Rightarrow x$ is assigned to class $C'_k$. (2) We know that $\boldsymbol{nc}(x, R_k) \leq \boldsymbol{nc}(x, R_{k-1})$. If $x P_{\geq\alpha} R_{k-1} \Rightarrow \boldsymbol{nc}(x, R_{k-1}) = \boldsymbol{U}(x) - \max_{r \in R_{k-1}}\{\boldsymbol{U}(r)\}$. $\boldsymbol{U}(x)$ has three options: $\boldsymbol{U}(x) > \max_{r \in R'_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{nc}(x, R'_k) < \boldsymbol{nc}(x, R_{k-1})$ (Condition 8.2 and Definition 8.7); $\boldsymbol{U}(x) < \min_{r \in R'_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{U}(x) < \min_{r \in R_k}\{\boldsymbol{U}(r)\}$ (Definition 8.10) $\Rightarrow \boldsymbol{nc}(x, R'_k) \leq \boldsymbol{nc}(x, R_k) \Rightarrow \boldsymbol{nc}(x, R'_k) \leq \boldsymbol{nc}(x, R_{k-1})$; $\min_{r \in R'_k}\{\boldsymbol{U}(r)\} \leq \boldsymbol{U}(x) \leq \max_{r \in R'_k}\{\boldsymbol{U}(r)\} \Rightarrow \boldsymbol{nc}(x, R'_k) = 0 \leq \boldsymbol{nc}(x, R_k) \Rightarrow \boldsymbol{nc}(x, R'_k) \leq \boldsymbol{nc}(x, R_{k-1})$. In all of the cases, $x$ is assigned to class $C'_k$.

ii. Consider that action $x$ was assigned to $C_h$, with $h > k+1$, before we proceed to a splitting operation. We have $x P_{\geq\alpha} R_m$ with $m > k$ (Definition 8.9) $\Rightarrow x$ will be assigned into the same class as previously.

iii. Consider that action $x$ was assigned to $C_h$, with $h < k-1$, before we proceed to a splitting operation. We have $x P_{\geq\alpha} R_m$ with $m < k - 1$ (Definition 8.9) $\Rightarrow x$ will be assigned into the same class as previously.

iv. Consider that action $x$ was assigned to $C_{k+1}$, before we proceed to a splitting operation. There are two options: (1) Let $h = k + 1$ be the first value $h$ such that $x P_{\geq\alpha} R_h$ before a splitting operation and $\boldsymbol{nc}(x, R_{k+1}) \leq \boldsymbol{nc}(x, R_{k+2}) \Rightarrow x$ will be assigned to the same class as previously (trivial); (2) Let $h = k$ be the first value $h$ such that $x P_{\geq\alpha} R_h$ before a splitting operation and $\boldsymbol{nc}(x, R_{k+1}) < \boldsymbol{nc}(x, R_k) \Rightarrow \boldsymbol{U}(x) > \max_{r \in R_k}\{\boldsymbol{U}(x)\} \Rightarrow \boldsymbol{U}(x) > \max_{r \in R'_k} \boldsymbol{U}(x)$ (Definition 8.10(b)) $\Rightarrow \boldsymbol{nc}(x, R'_k) < \boldsymbol{nc}(x, R_{k-1})$, we also know that $x P_{\geq\alpha} R_{k-1}$ (Proposition 8.4) then in the worst case where

$not(xP_{\geq\alpha}R'_k)$ and $not(xP_{\geq\alpha}R''_k)$, $x$ will be assigned to the class $C'_k$. $\Rightarrow x$ will be assigned to the same class as previously or some of the new classes.

v. Consider that action $x$ was assigned to $C_{k-1}$, before we proceed to a splitting operation. There are two options: (1) Let $h = k - 2$ be the first value $h$ such that $xP_{\geq\alpha}R_h$ before a splitting operation and $\boldsymbol{nc}(x, R_{k-1}) < \boldsymbol{nc}(x, R_{k-2}) \Rightarrow$ $x$ will be assigned to the same class as previously (trivial); (2) Let $h = k - 1$ be the first value of $h$ such that $xP_{\geq\alpha}R_h$ before a splitting operation and $\boldsymbol{nc}(x, R_{k-1}) \leq \boldsymbol{nc}(x, R_k) \Rightarrow$ since $\max_{r\in R_k}\{\boldsymbol{U}(x)\} > \max_{r\in R_{k-1}}\{\boldsymbol{U}(x)\} \Rightarrow$ $\boldsymbol{U}(x) < \min_{r\in R_k}\{\boldsymbol{U}(x)\} \leq \min_{r\in R''_k}\{\boldsymbol{U}(x)\}$ (Definitions 8.7 and 8.10(b)) $\Rightarrow not(xP_{\geq\alpha}R''_k) \Rightarrow$ by Definition 8.9, it follows that $x$ will be assigned to the same class as previously or some one of the new classes.

The proofs for the primal rule are omitted. They can be justified by the equivalence through the transposition operation.

### *8.4.3 An Illustrative Example*

For a certain DM, the global impact of Research and Development projects is determined by four criteria: economic impact, social impact, scientific impact, and improvement of the research team competence. Each criterion is evaluated in a quantitative scale ranging in [0, 5]; the classes are {Poor, Acceptable, Very Good}. The DM will fund projects assigned to "Very Good," and reject those classified into "Poor." Project assigned to the class Acceptable could be funded in the case of sufficient resources. The DM agrees with an interval weighted-sum model. (S)he assesses roughly a similar importance to all the criteria. But accepting some imprecision, the DM sets $w_i = [0.2, 0.3]$, $i = 1, \ldots, 4$.

Below table provides representative projects for each class. For simplicity, we will use a single action per class and real number scores.

| Project Id | Economic impact score | Social impact score | Scientific impact score | Improvement of research competence score | Overall Impact |
|---|---|---|---|---|---|
| $r_3$ | 4 | 4 | 4 | 5 | Very good |
| $r_2$ | 4 | 2 | 2 | 2 | Acceptable |
| $r_1$ | 1 | 1 | 1 | 3 | Poor |

Using the interval weighted-sum model and the basic operations on interval numbers, we have:

$\boldsymbol{U}(r_1) = [1.2, 1.8]$
$\boldsymbol{U}(r_2) = [2, 3]$
$\boldsymbol{U}(r_3) = [3.4, 5.1]$

The DM sets $\alpha = 0.9$. It is easy to see that the reference set fulfills Condition 8.2.

Let us analyze the assignment of the project $p_1$ with scores (3.5, 3.5, 3.5, 4.5). It follows that $\boldsymbol{U}(p_1) = [3, 4.5]$. Using the possibility function of Eq. (8.1) we have:

$$P(\boldsymbol{U}(p_1) \geq \boldsymbol{U}(r_1)) = P(\boldsymbol{U}(p_1) \geq \boldsymbol{U}(r_2)) = 1;\ P(\boldsymbol{U}(p_1) \geq \boldsymbol{U}(r_3)) \approx 0.344;$$
$$P(\boldsymbol{U}(r_1) \geq \boldsymbol{U}(p_1)) = P(\boldsymbol{U}(r_2) \geq \boldsymbol{U}(p_1)) = 0;\ P(\boldsymbol{U}(r_3) \geq \boldsymbol{U}(p_1)) \approx 0.656.$$

According to the primal assignment rule of ***U***Sort-nC (Definition 8.8), $not(R_3 P_{\geq\alpha} p_1) \Rightarrow p_1$ is assigned to the highest class (Very Good).

According to the dual assignment rule (Definition 8.9), descending from $k = M$, the first value of $k$ for which $x P_{\geq\alpha} R_k$ is $k = 2$; then, we should select the assignment between $C_2$ and $C_3$ by using the non-closeness measure (Definition 8.7).

$\boldsymbol{nc}(p_1, R_2) = [3, 4.5] - [2, 3] = [0, 2.5]$; $\boldsymbol{nc}(p_1, R_3) = [3.5, 5.1] - [3, 4.5] = [-1, 2.1]$.

Applying Eq. (8.1): $P(\boldsymbol{nc}(p_1, R_2) \geq \boldsymbol{nc}(p_1, R_3)) = 0.625$.

Hence, $\boldsymbol{nc}(p_1, R_2) > \boldsymbol{nc}(p_1, R_3)$, and $p_1$ is assigned to $C_3$ (Very Good) by the dual procedure (Definition 8.8).

The primal and dual assignments are coincident.

Note that the assignments do not change in a wide range of $\alpha$ values.

## 8.5 Concluding Remarks

Interval value functions can be used to model compensatory and transitive preferences from a decision maker who hesitates about the appropriate assessment of criterion weights, and criterion scores. To the present authors' knowledge, this contribution is the first in addressing multi-criteria ordinal classification problems through this kind of functions.

We have proposed here two novel methods. ***U***Sort-nB is inspired by ELECTRE TRI-nB, but modified to the functional paradigm and an interval framework. ***U***Sort-nC follows ELECTRE TRI-nC, but using interval numbers on the functional paradigm.

***U***Sort-nC (respectively, ***U***Sort-nB) uses characteristic (respectively, limiting) actions to describe the (resp., boundaries between adjacent) classes. In both methods, the criterion scores of reference actions may be interval numbers, what brings more flexibility for the DM. Unlike ELECTRE TRI-nB, the ***U***Sort-nB is composed of two procedures, which are symmetric with respect to the transposition operation, and should be used conjointly. The same happens with ***U***Sort-nC (as in ELECTRE TRI-nC).

Under certain requirements on the reference actions, both methods fulfill the set of consistency properties proposed by Roy and Bouyssou [26] for ELECTRE TRI-B and revisited by Almeida-Dias et al. [1, 2] for ELECTRE TRI-C and ELECTRE TRI-nC: Unicity, Independence, Homogeneity, Monotonicity, Conformity, and Stability. The requirements for ***U***Sort-nB are, to a great extent, stronger than those for ***U***Sort-nC (compare Conditions 8.1 and 8.2). Additionally, setting characteristic actions is easier than limiting actions. Hence, most DMs could feel ***U***Sort-nC more comfortable. More research is needed in order to characterize the effectiveness of the methods, thus deriving conclusions about their appropriateness.

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# Chapter 9
# Constructing an Outranking Relation from Semantic Criteria and Ordinal Criteria for the ELECTRE Method

**Aida Valls and Antonio Moreno**

**Abstract** This chapter reviews the work done at the ITAKA research group in the last decade on extending the ELECTRE multiple criteria decision aiding method. We present three different versions of the classic concordance and discordance indices, which are in the core of the construction of outranking relations. The first version is used when the elementary criteria have multi-valued linguistic scales, consisting on a set of terms from a domain ontology. Semantics is then used to interpret the linguistic words and the user preferences can be applied to calculate concordance and discordance values (ELECTRE-SEM). The other two are needed in order to use ELECTRE with a hierarchy of criteria for ranking (ELECTRE-III-H) and for sorting (ELECTRE-TRI-H). In that way, problems with a large set of elementary criteria can be represented with a tree structure distinguishing different groups of criteria at different levels. ELECTRE is applied to each node and the propagation of results is made using these new concordance and discordance indices. The chapter illustrates the application of these methods in two fields: environmental risk analysis and tourism recommendation systems. These works were developed in collaboration with many partners from different institutions. We want to highlight the ideas and support of Prof. Roman Slowinski and his research group.

## 9.1 Introduction

The well-known ELECTRE family of multiple criteria decision aiding methods (MCDA) was created by Prof. Bernard Roy [20]. This method originated a new MCDA approach based on preference relations for solving a decision problem. In particular, ELECTRE proposes the construction of a pairwise outranking relation as the core component of the MCDA methodology, as will be shown in Sect. 9.2.

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By building and exploiting an outranking relation, the ELECTRE method is able to establish different types of relations between the alternatives: indifference, preference, and incomparability. From the different versions, let us highlight two of them: ELECTRE-III generates a partial pre-order from an outranking matrix using a procedure called Distillation to solve ranking problems [9], and ELECTRE-TRI classifies a set of alternatives into a predefined set of categories using the outranking relations between the alternatives and the limiting or central profile of each category [23].

Outranking methods have been very successful because they are based on social choice models that copy the human natural reasoning procedure. Many applications can be found in diverse areas such as environmental analysis, project management, logistics, or health care [12].

An interesting advantage of ELECTRE is the fact that the operations are directly made on the original criteria scales, without requiring any normalization of transformation into an arbitrary common range of values. However, ELECTRE methods have been defined for numerical or ordinal data, but other kinds of values are common nowadays, among them linguistic terms. The need to associate a set of terms (i.e. tags) to the alternatives is present in many applications. These tags are usually nouns or adjectives that describe the content of an alternative. The decision maker may have different preferences (i.e. interests) on each term. The semantics (i.e. meaning) of the tags is important in order to correctly manage them in the decision process. In Sect. 9.3, we will explain how the preferences of the decision maker are stored in an ontology-based user profile. Then, we will use this information to adapt the ELECTRE procedure for constructing semantic-based outranking relations between the alternatives.

In complex decision aiding problems, it is sometimes appropriate to define a hierarchy of criteria. Some methods proposed this structure intrinsically in their approach, such as the AHP (Analytic Hierarchy Process) [21] and LSP (Logic Scoring of Preference) [7]. In other cases, a method initially working on a unique flat set of criteria is extended to deal with a criteria structure in form of a tree, as in Corrente [2]. In Del Vasto we also proposed a method to apply ELECTRE methods in a hierarchy of criteria like the one presented in Fig. 9.1.

The nodes of the hierarchy contain three different types of criteria: (1) The *overall criterion* in the root, representing the global goal of the decision problem to be solved, (2) *Intermediate criteria*, representing the decomposition of the global problem into partial sub-problems, and (3) *Elementary criteria*, representing the most specific criteria where the decision maker directly evaluates the alternatives. The number of levels is not limited and the length of the branches may be different. In Fig. 9.1, we can see 4 criteria in blue, that correspond to the root and intermediate criteria, and 7 elementary criteria in gray.

The main idea in our approach is to use the standard two-stage procedure of ELECTRE in each of the non-elementary nodes. When the descendants of a node are all elementary criteria, the classic definition of concordance and discordance indices may be used to build an outranking relation, as they are defined for numerical scales. However, if one of these nodes has a descendant that is a non-elementary node, it

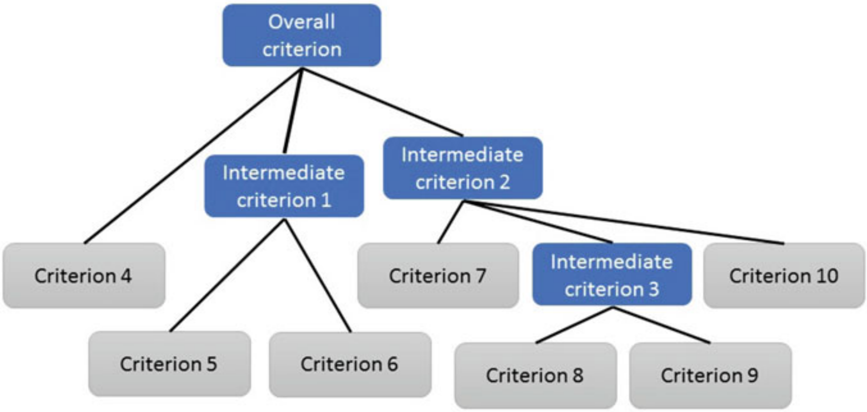


**Fig. 9.1** A hierarchical organization of criteria

will not have a numerical value for each alternative, but the output of the ELECTRE exploitation stage. Consequently, it is necessary to define new ways of building an outranking relation from the different outputs obtained in the exploitation stage of ELECTRE. In Sect. 9.4, we will explain how to calculate the concordance and discordance indices from a partial pre-order (obtained after a distillation procedure in the ELECTRE-III ranking method), and in Sect. 9.5, how to build those indices using the assignments made by the sorting procedure in ELECTRE-TRI.

The methods explained in this chapter have been implemented in a program called *ELECTRE-H Software Package*, with license at Universitat Rovira i Virgili, which will be presented in Sect. 9.6. The software includes also additional tools to visualize and store the results, to compare results with specific measures, and to help the decision maker in the definition of the thresholds in hierarchical criteria.

The chapter will finish with some examples of applications of these methods in the field of analysis of environmental risk, as well as in the personalized recommendation of touristic activities.

Several researchers and doctoral students participated in the definition of the methods that are explained in this chapter, in their implementation, and in the experimentation with different case studies. It is worth to highlight the joint work done with the Institute of Intelligent Decision Support Systems of Poznań University of Technology and, in particular, with Professor Roman Slowinski, who participated in several of the contributions presented in this chapter. The collaboration between our groups started in 2009 and still continues in the frame of different projects related to decision support.

## 9.2 An Introduction to the Classic Methodology for Constructing an Outranking Relation in ELECTRE

The aim of the outranking method is to build a binary outranking relation for any pair of alternatives in the set $A$. A credibility value on the outranking relation between $a, b \in A$ measures the strength of the statement "$a$ is at least as good as $b$" (denoted $aSb$). It is obtained by a pairwise comparison of the alternatives for each criterion in a set of criteria $G$. The value of this outranking relation for all possible pairs of options (i.e. alternatives) is calculated on the basis of two socially inspired rules: the *majority opinion* and the *right of veto*. The aim is to use this outranking relation to establish a realistic representation of four basic situations of preference: indifference, weak preference, strict preference, and incomparability. From these relations, we can select, rank, or classify the set of alternatives. In fact, there exist different versions of ELECTRE to solve different decision problems [8]: ELECTRE-I is for the selection of the best alternatives (choice), ELECTRE-II and ELECTRE-III for constructing a ranking, and ELECTRE-TRI for classifying alternatives into predefined categories.

In outranking methods, the values of an alternative $a$ for each criterion $g_j(a)$ are not useful by themselves, as they are always compared to the values of other alternatives. In this comparison procedure, three parameters are used. The two first ones are related to the discrimination power of a criterion and are used to model the uncertainty or flexibility in comparing two values. The third one is used to avoid compensations with other criteria. The three thresholds are:

- $q_j$: Indifference threshold for criterion $g_j$: This value marks the point where an alternative is strictly preferred to another.
- $p_j$: Preference threshold for criterion $g_j$: This value defines an interval to decide if the preference between two values is significant or not.
- $v_j$: Veto threshold for criterion $g_j$: This value sets the maximum negative difference we can allow before vetoing.

The following relation must always be fulfilled: $0 \leq q_j \leq p_j \leq v_j$.

Depending on the values of the two first thresholds, we can distinguish two types of criteria: True-criterion when $p = q = 0$ and Pseudo-criterion when $p \geq q \geq 0$.

Another important parameter in ELECTRE is the set of weights $W$, such that $\sum w_i = 1$. In ELECTRE, weights represent the voting power of each criterion when evaluating the outranking statement $aSb$. Thus, they indicate the relative voting importance with respect to other criteria.

With this information, ELECTRE constructs the pairwise outranking matrix by merging two indices obtained from the set of criteria. The most general formulation of those indices is the following:

- *Concordance index*: It measures if most criteria are in favor of the assertion $aSb$, taking into account their weights.

$$c(a,b) = \sum_{j=1}^{n} w_j c_j(a,b) \tag{9.1}$$

$$c_j(a,b) = \begin{cases} 1 & \text{if} \quad g_j(a) + q_j \geq g_j(b) \\ 0 & \text{if} \quad g_j(a) + p_j \leq g_j(b) \\ \frac{p_j - (g_j(a) - g_j(b))}{p_j - q_j} & otherwise \end{cases} \tag{9.2}$$

- *Discordance index*: It checks if any criterion can deny the assertion $aSb$. It is used to introduce the veto right for minorities.

$$d_j(a,b) = \begin{cases} 1 & \text{if} \quad g_j(b) > g_j(a) + v_j \\ 0 & \text{if} \quad g_j(b) \leq g_j(a) + p_j \\ \frac{g_j(b) - g_j(a) - p_j}{v_j - p_j} & otherwise \end{cases} \tag{9.3}$$

From them, the credibility of the outranking relation is calculated as follows:

$$\rho(aSb) = \begin{cases} c(a,b) & \text{if} \quad d_j(a,b) \leq c(a,b), \forall j \\ c(a,b) \prod_{j \in J(a,b)} \frac{1 - d_j(a,b)}{1 - c(a,b)} & otherwise \end{cases} \tag{9.4}$$

being $J(a,b)$ the set of criteria with a partial discordance value higher than the overall concordance.

In this formulation, the criteria are assumed to be numerical and its performance must be maximized (i.e. they are considered gain criteria).

## 9.3 Constructing an Outranking Relation from Semantic Criteria

In this section, we deal with a new type of criterion that does not use a numerical scale: the *semantic* criterion. This kind of criterion associates a list of words (i.e. tags) with each alternative. Thus, these attributes are multi-valued. In order to apply ELECTRE, first of all, a proper way of representing (i.e. storing) the decision maker preferences on the tags is needed. A method based on ontologies will be explained. After that, the information of the ontology will be used in the concordance and discordance measures when comparing two lists of tags corresponding to two different alternatives.

The tags are words (usually nouns or adjectives) with a concrete meaning. Knowing the semantics of the tags is important when comparing them, as two

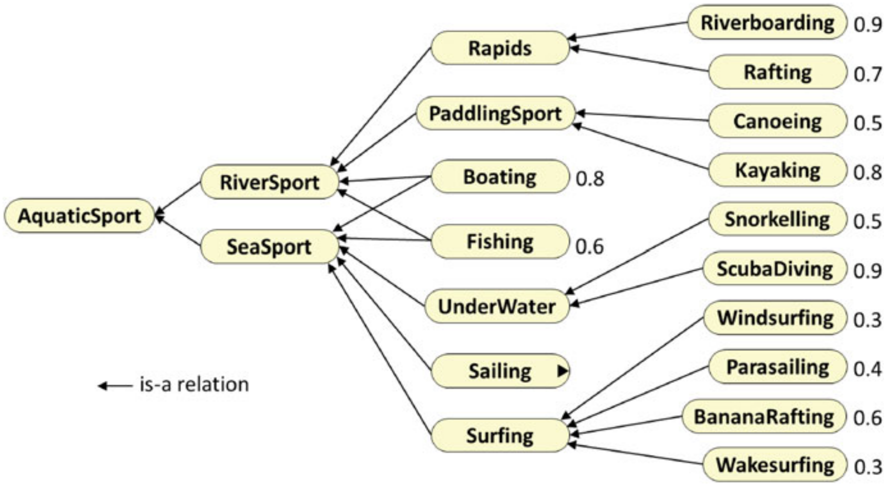


**Fig. 9.2** Preferences in an ontology for aquatic sports

different words can refer to similar concepts. In order to know the semantic relations between tags, many intelligent systems rely on a data structure called ontology [11]. An *ontology* is basically formed by a set of concepts (mapped into words), which are linked through taxonomic (i.e. is-a) and non-taxonomic relationships. It can also store properties of the concepts and specific instances of them. In a decision problem, if each attribute is referring to a different property, a different domain ontology should be associated with each semantic criterion.

An example of a portion of an ontology of aquatic sports is given in Fig. 9.2. We assume that the set of tags that will appear in the values of a semantic criterion will belong to the most specific terms of the ontology (i.e. the ones that do not have descendants).

We define a function that associates a preference score with a tag $t$, denoted as Tag Interest Score, $TIS(t)$. The tag interest score is a numerical score between 0 and 1, also shown in Fig. 9.2, that indicates the satisfaction degree of the user with the corresponding tag $t$ according to the decision maker's goals. We assume that $TIS(t)$ has to be maximized. This score can be directly given by the user. However, due to the large number of terms in real-world ontologies, a procedure to infer unknown preference scores can be used [17].

ELECTRE-SEM is a version of ELECTRE able to construct the outranking relation when semantic criteria are considered. In order to manage a different type of criteria, ELECTRE-SEM changes the first step of the ELECTRE method, because the exploitation procedure only takes into account the credibility values. In particular, appropriate ways of calculating the partial concordance index and partial discordance index in the case of a semantic criterion have to be defined [15].

A semantic criterion can be defined as a pseudo-criterion, with two discriminant thresholds (preference and indifference) as well as the veto threshold. This procedure follows the same principles than the classic ELECTRE method, but concordance and discordance indices are fuzzy functions defined in terms of the pairwise comparison of the Tag Interest Scores. First, we define how to measure the strength of the assertion $aSb$ in terms of one semantic criterion, called *Semantic Win Rate*, $SWR_j(a,b)$.

The Semantic Win Rate is a numerical value in [0..1] that indicates the degree of performance of the alternative $a$ with respect to the alternative $b$ on the semantic criterion $g_j$. It is based on the two sets of tags $g_j(a) = \{t_{1,a}, t_{2,a}, t_{3,a}, \ldots, t_{|g_j(a)|,a}\}$ and $g_j(b) = \{t_{1,b}, t_{2,b}, t_{3,b}, \ldots, t_{|g_j(b)|,b}\}$ (the values taken by the alternatives in the criterion), and it is calculated as follows:

$$SWR_j(a,b) = \frac{\sum_{t_{i,a}} \sum_{t_{k,b}} f(t_{i,a}, t_{k,b})}{|g_j(a)| \cdot |g_j(b)|} \tag{9.5}$$

where

$$f(x,y) = \begin{cases} 1 \text{ if } & TIS(x) \geq TIS(y) - q_j \\ 0 \text{ if } & TIS(x) < TIS(y) - q_j \end{cases} \tag{9.6}$$

Thus, $SWR_j(a,b)$ is the percentage of pairwise comparisons between the tags of $a$ and $b$ for the semantic criterion $g_j$ for which the user has a higher (or equal) preference for $a$ than for $b$.

We introduce here the possibility of using an indifference threshold $q_j$ similar to the one in standard ELECTRE, in order to define an interval of indistinguishability regarding the TIS range of values. In that way, if two scores are similar enough, they can be considered equally preferred by the decision maker.

*Example* Let us consider two lists of tags describing a touristic activity, with their associated TIS value:

- $a$: (boating 0.8, canoeing 0.5)
- $b$: (fishing 0.6, scuba-diving 0.9, windsurfing 0.3)

In the first look, we take $q_j = 0$ and then $SWR_j(a,b) = 3/6 = 0.5$ and $SWR_j(b,a) = 3/6 = 0.5$, so both options have a similar preference level for the decision maker, because in option $b$, although the user is not interested in windsurfing, the other two sports compensate that.

Let us now introduce some indifference on similar scores with $q_j = 0.1$. Now, $SWR_j(a,b) = 5/6 = 0.83$ and $SWR_j(b,a) = 3/6 = 0.5$. This means that scores 0.8 and 0.9 are considered to be in the same level of preference (for boating and scuba-diving). The same situation arises in the comparison of 0.5 and 0.6 (for canoeing and fishing). Thus, the semantic win rate (SWR) changes, being $a$ the best option as the score of windsurfing is penalizing option $b$, while the rest of sports have the same level of interest for the user.

Using the Semantic Win Rate value, the partial concordance and discordance indices are defined as follows:

$$c_j(a,b) = \begin{cases} 1 & \text{if} \quad SWR_j(a,b) \geq \mu_j \\ 0 & \text{if} \quad SWR_j(a,b) \leq p_j \\ \frac{SWR_j(a,b)-p_j}{\mu_j - p_j} & \text{otherwise} \end{cases} \tag{9.7}$$

$$d_j(a,b) = \begin{cases} 1 & \text{if} \quad SWR_j(a,b) \leq v_j \\ 0 & \text{if} \quad SWR_j(a,b) \geq p_j \\ \frac{p_j - SWR_j(a,b)}{p_j - v_j} & \text{otherwise} \end{cases} \tag{9.8}$$

As $SWR_j(a,b)$ is a percentage that represents the comparison of the performance of $a$ over $b$, the usual thresholds used in concordance and discordance must be revised and now they have the following meaning:

- $\mu_j$ is a strong threshold of the strength of $SWR_j(a,b)$ to consider maximum concordance with *aSb*.
- $p_j$ is a weak threshold of the strength of $SWR_j(a,b)$ where the user may still have some preference of $a$ with regard to $b$, thus still supporting the relation $aSb$ to a certain degree.
- $v_j$ is the veto threshold, which is a value threshold below which $SWR_j(a,b)$ is low enough to imply the full discordance with the outranking relation.

In this case, the role of the thresholds is analogous to the one of the numerical case: $p_j$ being the threshold that indicates if the value of the $SWR_j(a,b)$ is in favor of or against *aSb*, whereas $\mu_j$ and $v_j$ are used to determine the value of the concordance or discordance vote for a certain criterion. Notice that the following condition must hold: $0 \leq v_j \leq p_j \leq \mu_j$.

*Example* Following the same example in which $SWR_j(a,b) = 0.83$ and $SWR_j(b,a) = 0.5$, let us consider two scenarios:

- First case: $\mu(j) = 0.8$, $p_j = 0.6$, and $v_j = 0.4$.
  In this situation, $aSb$ holds because the $SWR_j(a,b)$ exceeds 0.8. So, $c_j(a,b) = 1$. On the contrary, when evaluating $bSa$, we have $c_j(b,a) = 0$ because $SWR_j(b,a) < 0.6$. Moreover, regarding the discordance index, we have $d_j(b,a) = (0.6 - 0.5)/(0.6 - 0.4) = 0.5$, which indicates a medium level of discordance with $b$ outranking $a$.
- Second case: $\mu(j) = 0.7$, $p_j = 0.4$, and $v_j = 0.3$.
  In this scenario, $c_j(a,b)$ continues to be 1 because of the large value of $SWR_j(a,b) = 0.83 > 0.7$. But now, we also have some level of concordance in $c_j(b,a) = 0.33$, as the threshold of preference is set at 0.4 and the semantic win rate is $SWR_j(b,a) = 0.5$, which is larger than 0.4. Obviously, the two discordance indices are now 0.

With this example, we can see that these thresholds permit to model the discrimination power between alternatives in a similar way to the ones used for numerical data. It is worth noting that the ELECTRE-SEM thresholds must be defined in terms of the SWR measure, which is a value of the relative preference of one alternative with respect to another (unlike the classical ELECTRE methods, where the thresholds are set in terms of differences between values in a numerical scale).

## 9.4 Constructing an Outranking Relation from a Partial Pre-order on a Set of Alternatives

After presenting the management of semantic criteria, we now move to the case of having a hierarchy of criteria. Each of the intermediate nodes of the tree indicates a decision point where information from the direct sub-criteria must be taken into account. In this section, we focus on the problem of ranking a set of alternatives using the ELECTRE-III distillation procedure. The output of this ranking method is a partial pre-order, such as the one shown in Fig. 9.3.

ELECTRE-III-H is a method proposed in [4] that applies, at different levels of a hierarchy of criteria, the ELECTRE-III process with pseudo-criteria and ranking by distillation. ELECTRE-III-H generates a partial pre-order at each of the intermediate nodes, as well as in the root node (overall criterion).

The procedure follows a bottom-up approach. For instance, in Fig. 9.1, we would start calculating a partial pre-order $O_1$ using criteria number 5 and 6, and another partial pre-order $O_3$ with criteria 8 and 9. Next, criterion 7, criterion 10, and $O_3$ (the output of ELECTRE in criterion 3) must be used to calculate the partial pre-order $O_2$ using ELECTRE-III in criterion 2. Finally, the partial pre-orders $O_1$ and $O_2$, and the elementary criterion 4 are used for calculating the overall result. To achieve this, in intermediate levels, ELECTRE-III-H must use the information of the partial pre-orders from lower levels, instead of the usual numerical scales. Therefore, in order to propagate the information of the pre-orders to an upper level, a proper way of calculating the partial concordance and discordance indexes was defined.

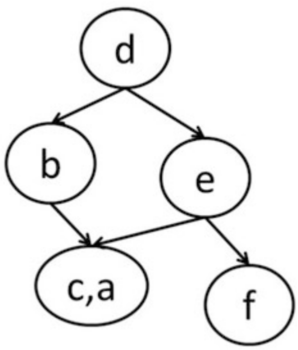


**Fig. 9.3** A partial pre-order $O_{example}$

**Table 9.1** Table of relations and rank order value for $O_{example}$

| | a | b | c | d | e | f | $\Gamma$ |
|---|---|---|---|---|---|---|---|
| a | $I$ | $P^-$ | $I$ | $P^-$ | $P^-$ | $R$ | 3 |
| b | $P$ | $I$ | $P$ | $P^-$ | $R$ | $R$ | 1 |
| c | $I$ | $P^-$ | $I$ | $P^-$ | $P^-$ | $R$ | 3 |
| d | $P$ | $P$ | $P$ | $I$ | $P$ | $P$ | 0 |
| e | $P$ | $R$ | $P$ | $P^-$ | $I$ | $P$ | 1 |
| f | $R$ | $R$ | $R$ | $P^-$ | $P^-$ | $I$ | 2 |

To calculate the concordance and discordance indices from a partial pre-order $O_j$, two main concepts are proposed in [4]: first, a numerical indicator related with the order expressed in $O_j$, and second, the relation existing in $O_j$ between any pair of compared alternatives. An example of a partial pre-order among 6 alternatives is given in Fig. 9.3. Table 9.1 illustrates these two pieces of information for this example. The four types of relations in the partial pre-order are denoted as preference ($P$), inverse preference ($P-$), indifference ($I$), and incomparability ($R$).

The Rank Order Value $\Gamma_j(a)$, for an alternative $a$ in the $O_j$ partial pre-order, is defined as the number of alternatives that are preferred to alternative $a$ in this partial pre-order. This value must be minimized, as the lower the rank order value, the better. It is shown in the last column of Table 9.1.

Depending on the relation type between a pair of alternatives $(a, b)$, the calculation of the partial concordance and the partial discordance is different. Three situations are distinguished:

- Preference ($P$) and Indifference ($I$): We establish $c_j(aPb) = 1$ and $c_j(aIb) = 1$, whereas $d_j(aPb) = 0$ and $d_j(aIb) = 0$, because in both cases the relation is indicating support to the statement "$a$ is equal or better than $b$" ($aSb$).
- Inverse preference ($P^-$): In this relation, $aSb$ is not supported, but some tolerance is allowed depending on the discrimination thresholds:

$$c_j(aP^-b) = \begin{cases} 1 & \text{if} \quad \Gamma_j(a) - \Gamma_j(b) \le q_j \\ 0 & \text{if} \quad \Gamma_j(a) - \Gamma_j(b) > p_j \\ \frac{p_j - (\Gamma_j(a) - \Gamma_j(b))}{p_j - q_j} & \quad otherwise \end{cases} \tag{9.9}$$

$$d_j(aP^-b) = \begin{cases} 1 & \text{if} \quad \Gamma_j(a) - \Gamma_j(b) > v_j \\ 0 & \text{if} \quad \Gamma_j(a) - \Gamma_j(b) \le p_j \\ \frac{\Gamma_j(a) - \Gamma_j(b) - p_j}{v_j - p_j} & \quad otherwise \end{cases} \tag{9.10}$$

- Incomparability ($R$): This relation type means there is not a direct relation between alternatives $a$ and $b$. Thus, it is equally probable that it turns into $aPb$, $aP^-b$, or $aIb$. In 2/3 of these cases $aSb$ is supported, so we can assume this value for the partial concordance. Hence, the remaining 1/3 is assumed for the partial discordance. Moreover, an $\alpha$ factor is introduced to increase or decrease the partial concordance or discordance based on the magnitude of the difference between the alternatives ($n$ is the number of alternatives).

$$\text{if}\,\Gamma(a) - \Gamma(b) \leq p_j \text{then} \begin{cases} c_j(aRb) = 2/3 + \frac{\alpha(\Gamma_j(b)-\Gamma_j(a)-q_j)}{(p_j-q_j)+(n-2)} \\ d_j(aRb) = 0 \end{cases} \tag{9.11}$$

$$\text{if}\,\Gamma(a) - \Gamma(b) > p_j \text{then} \begin{cases} c_j(aRb) = 0 \\ d_j(aRb) = 1/3 + \frac{\alpha(\Gamma_j(a)-\Gamma_j(b)-v_j)}{(v_j-p_j)+(n-2)} \end{cases} \tag{9.12}$$

The thresholds used in these equations of ELECTRE-TRI-H must be defined in terms of the new *Rank Order Value* indicator, so they are based on the positions of the alternatives in the partial pre-orders.

In the example of Table 9.1, the partial concordance and discordance values of the first alternative $a$ with respect to the others are given in Table 9.2. The example considers $q_j = 0$, $p_j = 2$, $v_j = 3$, and $\alpha = 0.3$. Notice that for the relation $aSd$ we obtain a discordance of 1, while the discordance is zero for $aSe$. This is due to the veto threshold that establishes that a difference of 3 units in the rank order value will activate the veto, whereas a difference of just 2 units corresponds to the point of non-concordance and non-discordance, fixed by the preference threshold.

Therefore, having these equations to calculate a partial discordance index and a partial concordance index, we can later make the rest of the ELECTRE credibility calculations in the usual way (with Eqs. 9.1 and 9.4). It is then straightforward to apply any of the standard exploitation stages for choice or ranking. In Sect. 9.7, some applications of this method are presented.

## 9.5 Constructing an Outranking Relation from an Assignment of Alternatives to Categories

In this section, we focus on sorting problems, where a set of alternatives must be assigned to some predefined categories or labels. The categories have an order from the worst to the best, indicating several degrees of interest or suitability of the alternatives for a certain user. ELECTRE-TRI is a sorting method based on the notion of outranking. From its several versions, we will focus on ELECTRE-TRI-B-H, which is a sorting method that works with a hierarchy of criteria, such as the one seen in the previous section.

In [5], we propose extending ELECTRE-TRI-B to handle assignments of alternatives to several levels of the hierarchy. This method is able to manage different sets

**Table 9.2** Partial concordance and discordance of $a$ with respect to the rest of alternatives in $O_{example}$

| | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| Conc | 1 | 0 | 1 | 0 | 0 | 0.62 |
| Disc | 0 | 0 | 0 | 1 | 0 | 0 |

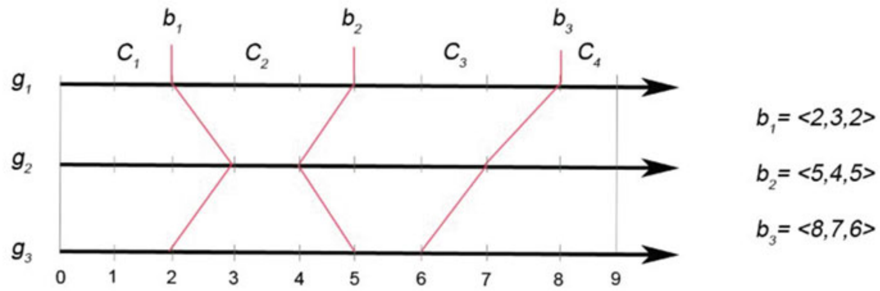


**Fig. 9.4** Graphical representation of profile limits in ELECTRE-TRI-B

of categories at each intermediate criteria and at the root criterion, giving to the user the possibility of using the most appropriate labels at each node. ELECTRE-TRI-B considers a set of fictitious profiles $B = \{B_1, ..B_k\}$ that determine the boundaries between the $k + 1$ categories (see Fig. 9.4).

The aim of ELECTRE-TRI-B is to compare alternatives to profile limits to build a valued outranking relation $S$, where $aSb_h$ means "$a$ is at least as good as the profile $b_h$." This outranking relation is calculated as explained in the previous sections from elementary criteria (numerical or semantic). From this outranking relation, ELECTRE-TRI-B assigns the alternatives in $A$ to the predefined categories $C$.

The first step in the sorting stage consists in transforming the valued credibility values into a crisp outranking relation by applying a cutting level $\lambda$-cut, which is considered as the smallest value of the credibility index $\rho$ to consider that the outranking relation holds [19]. Next, each alternative (independently from the others) is assigned using two logic operations:

- The conjunctive rule (pessimistic), in which an alternative can be assigned to a category when its evaluation is at least as good as the lower limit of this category. The alternative is then assigned to the highest category $C_h$ that fulfills this condition.
- The disjunctive rule (optimistic), in which an alternative can be assigned to a category if it has, on at least one criterion, an evaluation at least as good as the lower limit of this category. The alternative is then assigned to the lowest category $C_h$ that fulfills this condition.

Classical ELECTRE-TRI-B only considers a set of elementary criteria and makes a unique assignment of the alternatives to the categories of an overall criterion. In the hierarchical version ELECTRE-TRI-B-H [5], the method makes an assignment of the alternatives to a category at each of the non-elementary criteria. That is, the assignment procedure is not only computed at the root, but also at each of the intermediate sub-criteria. Consequently, when going upwards in the hierarchy, the information to be considered in an upper level may come from a previous assignment into categories and not from a numerical or semantic criterion. In this regard, ELECTRE-TRI-B-H introduces a new procedure to define profile limits in

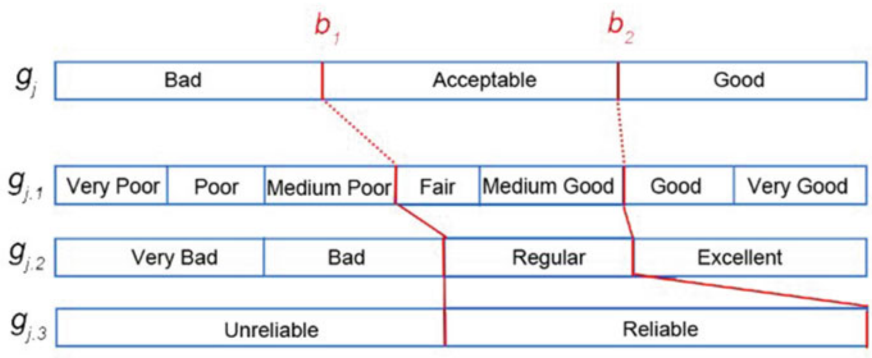


**Fig. 9.5** Graphical representation of profiles in categories in several criteria

terms of different sets of categories from criteria at lower levels in the hierarchy. Moreover, it also redefines the classical outranking construction step in terms of assignment of alternatives to categories, instead of numerical ratings.

Let us introduce some notation of ELECTRE-TRI-B-H:

- For each non-elementary criterion $g_j$, $B^j = \{ b_1^j, b_2^j,.., b_{k_j}^j \}$ is the finite set of indices of the profiles defining $k_j + 1$ categories on this criterion.
- For each non-elementary criterion $g_j$, $C^j = \{ C_1^j, C_2^j,.., C_{k_j+1}^j\}$ is the finite set of ascending predefined categories on criterion $g_j$, being $b_h^j$ the upper limit of the category $C_h^j$ and the lower limit of $C_{h+1}^j$, $h = 1, 2, \ldots, k_j$.

The user gives rules to define the boundary profiles separating the categories of the node $g_j$ with respect to the categories of its direct descendants $g_{j.d}$. These rules establish a mapping between the assignments established at the lowest level (to $g_{j.d}$) and the assignments that must be established at the current node $g_j$. Then, the vector profiles can be automatically obtained from these rules, which have the following format:

$$\textbf{if } \Psi_{j.1} \text{ and } \Psi_{j.2} \ldots \text{ and } \Psi_{j.|D_j|} \textbf{ then } (g_j = C_{h+1}^j),$$

where $\Psi_{j.d}$ is a disjunctive condition such as $(C_{h'}^{j.d}$ or $C_{h''}^{j.d}$ or $\ldots)$, in which the subset of categories is ascending and the categories are consecutive.

The user must give an assignment mapping rule for each category in $C^j$. All rules must fulfill two conditions: (1) the maximum value $\max(\Psi_{j.d})$ in rule $i$ must be smaller or equal than the minimum value $\min(\Psi_{j.d})$ in rule $i + 1$; and (2) all the categories $C^{j.d}$ of the descendent criterion must appear in at least one rule condition $\Psi_{j.d}$.

For example, consider the following graphical representation of boundary profiles on a parent criterion with 3 sub-criteria (Fig. 9.5).

In this example we would have the following set of mapping rules:

*Rule 1:* **if** $g_{j.1}$ = (Very Poor or Poor or Medium Poor) and $g_{j.2}$ = (Very Bad or Bad) and $g_{j.3}$ = Unreliable **then** $g_j$ = Bad
*Rule 2:* **if** $g_{j.1}$ = (Fair or Medium Good) and $g_{j.2}$ = Regular and $g_{j.3}$ = Reliable **then** $g_j$ = Acceptable
*Rule 3:* **if** $g_{j.1}$ = (Good or Very Good) and $g_{j.2}$ = Excellent and $g_{j.3}$ = Reliable **then** $g_j$ = Good

From the mapping assignment rule for a category $C^j_{h+1}$, which is of the form **if** $\Psi_{j.1}$ and $\Psi_{j.2}$ ...and $\Psi_{j.|D_j|}$ **then** $(g_j = C^j_{h+1})$, the profile limit $b^j_h$ can be represented as a vector with the lowest category for each condition $min(\Psi_{j.d})$. In that way, the profile establishes the lowest value of a criterion $g_{j.d}$ that should be assigned to $C^j_{h+1}$.

$$b^j_h = \langle \min(\Psi_{j.1}), \min(\Psi_{j.2}), \ldots \min(\Psi_{j.|D_j|}) \rangle$$

For the previous example in Fig. 9.5, the profile limits are

$$b^j_1 = \langle C^{j.1}_4, C^{j.2}_3, C^{j.3}_2 \rangle = \langle\, Fair, Regular, Reliable \rangle$$
$$b^j_2 = \langle C^{j.1}_6, C^{j.2}_4, C^{j.3}_2 \rangle = \langle\, Good, Excellent, Reliable \rangle$$

Once the boundary profiles are known, we need a numerical value that can be used to compare alternatives and profiles. We proposed in [5] the definition of the *Category Improvement Value* as a function of the form $\Phi_j : A \cup B^j \longrightarrow N$ that determines how many categories an alternative (or profile) may improve to get the best performance value for criterion $g_j$ (Eq. 9.13). Then, the indifference $q_j(b^j_h)$, preference $p_j(b^j_h)$, and veto $v_j(b^j_h)$ thresholds referring to criteria are defined in terms of the difference between the category improvement values of the alternatives assignment to $C^{j.d}$.

$$\Phi_j(x) = \begin{cases} k_{j.d} + 1 - i & \text{if } x \in A \text{ and } x \in C^{j.d}_i \\ k_{j.d} + 1 - h' + 1 & \text{if } x = b^j_h \in B^j \text{ and } b^j_h = \langle \ldots, C^{j.d}_{h'}, \ldots \rangle \end{cases} \tag{9.13}$$

The category improvement value permits to calculate the partial concordance indices $c_j$ in the different nodes of the hierarchy. Having their corresponding relative weights $w_j$, the overall concordance can be calculated with the classical equations of ELECTRE. Similarly, the partial concordance indices obtained for each different type of criterion are merged when the credibility matrix is calculated. Finally, the credibility matrix is exploited with either the pessimistic or optimistic rules mentioned before.

## 9.6 The ELECTRE-H Software Package: Tools for Assisting the Decision Maker in the Determination of Thresholds

Researchers of the ITAKA research group have built a software tool that supports the methods explained in this chapter. It is called *ELECTRE-H Software Package* and has the intellectual protection and license for Universitat Rovira i Virgili. It is commercialized by URV through the InnoGet platform.[1] Figure 9.6 shows the main window of ELECTRE-H, with data of a typical example of selecting a car using inside and outside criteria. The hierarchy of criteria is displayed in the left panel, while the right part has all the tools for the configuration of the parameters and the selection of the exploitation procedure.

The *ELECTRE-H Software Package* provides some tools to help the user to find the most suitable values for the thresholds at intermediate levels of the hierarchy of criteria. As the first assistance, the interface provides a list of possible policies for the discrimination and veto thresholds, which are related to the attitude of the decision maker toward uncertainty and risk. Each policy has some values for the thresholds in terms of percentages (see Table 9.3). These percentages are applied to the original scales of elementary criteria to find the values for each threshold. Moreover, they can also be applied at intermediate nodes, depending on the nature of the method. For ranking using ELECTRE-III-H, the Rank Order Values are used, while for sorting with ELECTRE-TRI-H, the percentages are applied to the Category Improvement Values. These percentages have been established on the basis of previous works [10]. In practical applications, we could observe the difficulty for the decision makers to define thresholds on the Rank Order or Category Improvement values; for this reason, these predefined options may be very useful to the users [22].

The results are displayed in different windows, both as a list or using graphical tools (Fig. 9.7). These results can be exported to Excel and JPG files. To analyze the results, the system includes a second block of functionalities that permits to explore the space of possible values starting from a given configuration. Some graphics display the results obtained using different parameters, in order to facilitate the study of different cases. A comparative analysis of a pair of rankings may also be performed to identify easily the changes produced by different values on the thresholds, so that the user can discover which ones represent his/her preferences.

To assist the user, the software provides two indices: (1) the *rank acceptability index* $A_j(a)$ that measures the frequency with which an alternative $a$ is placed at the $k$-th quartile in the ranking computed for criterion $g_j$; and (2) the *relation index* $R_j^r(a, b)$ that calculates the frequency of the binary relations $r = \{P, I, R, P^-\}$ for a pair of alternatives $(a, b)$. In both cases, the system constructs a stacked bar chart. The number of bars can be given by the user, as well as the order of the bars (see Figs. 9.8 and 9.9). The bottom panel of the window displays some data in text

[1] INNOGET platform: https://www.innoget.com/technology-offers.

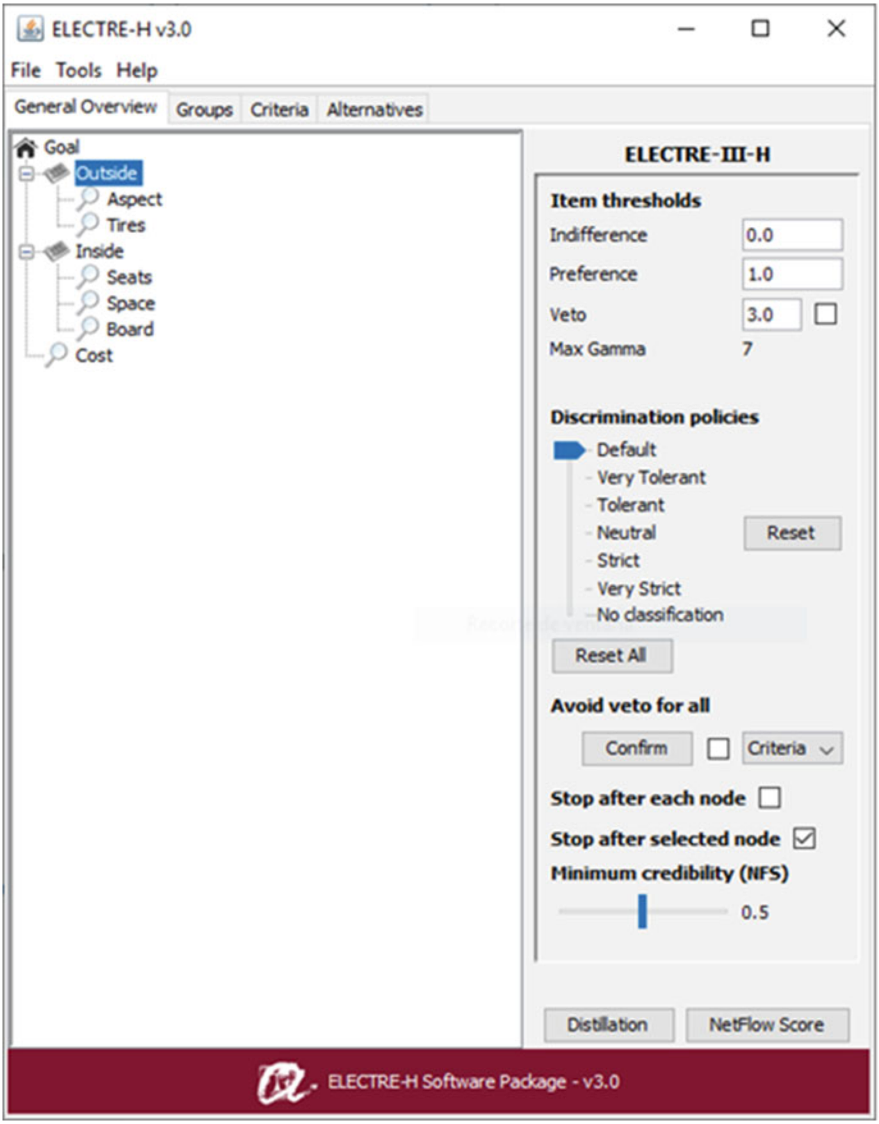


**Fig. 9.6** Main window of ELECTRE-III-H software package

**Table 9.3** Percentages for calculating the thresholds on 5 basic policies

| Policy | Indifference (%) | Preference (%) | Veto (%) |
|---|---|---|---|
| Very strict | 0 | 0 | 10 |
| Strict | 0 | 5 | 20 |
| Neutral | 5 | 10 | 30 |
| Tolerant | 10 | 20 | 40 |
| Very tolerant | 15 | 25 | 50 |

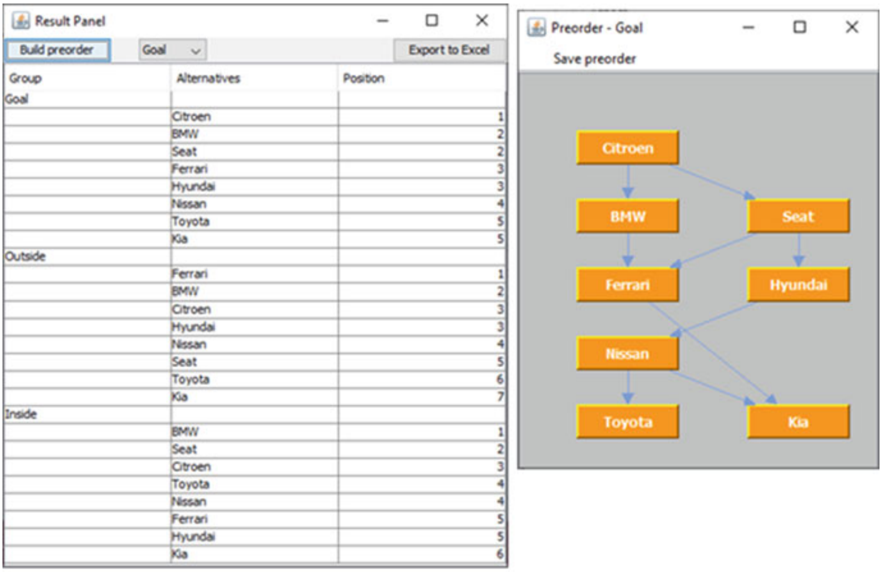


**Fig. 9.7** Results displayed by ELECTRE-H software package for ranking with distillation

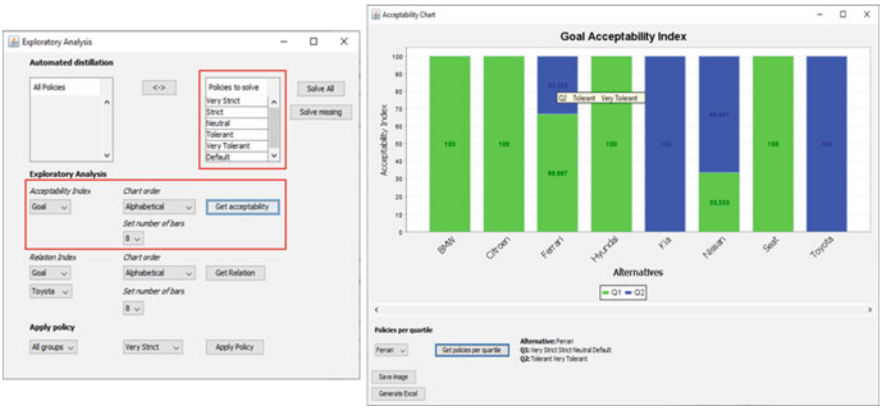


**Fig. 9.8** Configuration and display of the acceptability index

mode. The decision maker selects the alternative to display and detailed information regarding this alternative is printed. The chart can be saved as an image file and exported to csv format.

The *ELECTRE-H Software Package* also includes some other analysis tools like the calculation of correlations between rankings, distances between partial preorders, and distances between classification assignments. For semantic criteria, special management and visualization of the terms and ontologies are provided as well.

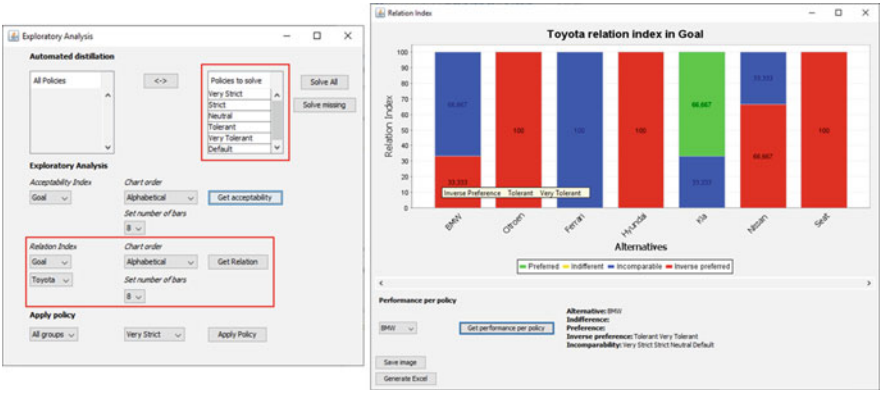


Fig. 9.9 Configuration and display of the relation index

## 9.7 Examples of Applications

The semantic and hierarchical versions of ELECTRE explained in this chapter have been used in several applications by the authors, in collaboration with other entities. This section will briefly present applications in two different fields: environmental risk analysis and tourism. For the first, four case studies are briefly summarized to illustrate how the methods presented can be used. Regarding tourism case studies, we first present an application for touristic destination managers and secondly, the use of ELECTRE methods in a personalized recommender system is outlined.

### *9.7.1 Environmental Analysis*

Environmental analysis is a discipline that has been using MCDA methods for a long time. The complexity and number of the indicators makes it suitable for a hierarchical approach. In 2015–16, under a large research project, ELECTRE-III-H was used to analyze and compare several future scenarios of Mediterranean small rivers due to the climate change. Global warm is expected to produce a drought in small rivers whose yield is very dependent on the seasonal yearly rains. The case of 3 rivers in Tarragona (Catalonia, Spain) was used as case study, named Francolí, Gaià, and Ebre. Environmental experts defined a hierarchy of criteria with three main groups: economic criteria, ecological impact, and water supply criteria. The complete hierarchy can be seen in Fig. 9.10. The values of elementary indicators were obtained from simulation studies, and three different analyses were done for different time spans (2040, 2070, 2100). For each period, three scenarios were considered regarding the strength of the effects of global warm on the rivers. The alternatives were a set of possible actions of water supply management, which


**Fig. 9.10** Hierarchy of criteria for evaluating water supply actions

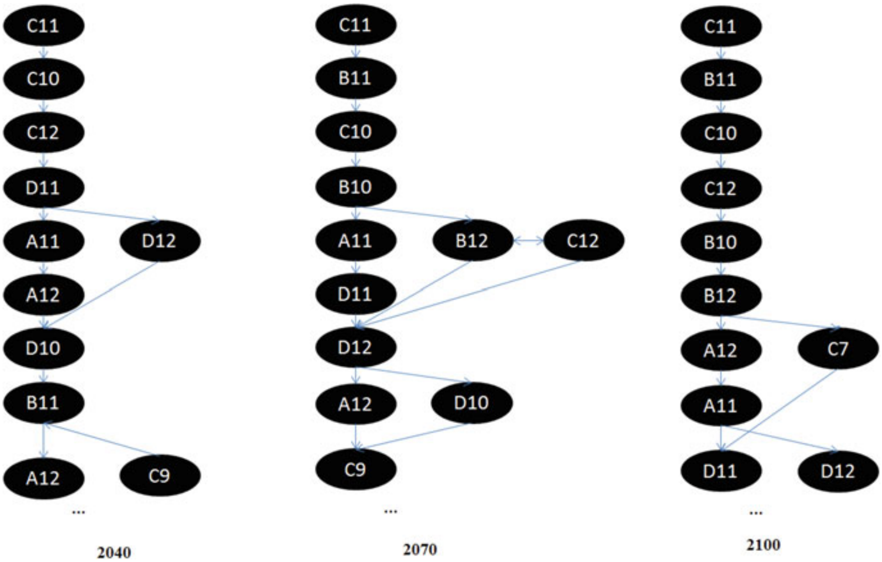


**Fig. 9.11** Example of ranking of water supply actions

ranged from using only the water from nature ("A"), to a high use of alternative resources ("D"), such as desalination plants or reclaimed water.

As illustration of the results obtained, Fig. 9.11 shows the partial pre-orders obtained at the root node for a balanced configuration (equal weights in sub-criteria) and an optimistic scenario with low reduction of water yield in the rivers. Options with "C" make a moderate use alternative resources, being C11 and C10 the best ones for the different time spans. B11 uses less alternative resources, which seems good at a further future but not for until 2040. The options that rely only on water from the nature (A11 and A12) appear always between the 5th and 8th position in these rankings, from a total of 48 possible actions. More details can be found in [6, 14].

A similar approach based on comparison of multiple scenarios was used in [10]. In this case, different irrigation scheduling actions were compared. The adoption of well-scheduled irrigation models is required in countries with poor water resources in order to prevent water loss while keeping high productivities in agricultural crops. The hierarchy defined has three levels with three main areas: crop productivity indicators, production of different kinds of water resources, and economical criteria. The case study with ELECTRE-III-H was made in Tunisia, in the Mornag Area.

Another field of application that is common nowadays is the study of natural energy production, to progressively substitute fossil fuels. In [16], semantic variables were included in a study to evaluate several energy production plants. It included 5 renewable technologies (solar, wind, hydro-power, geothermal, and biopower) and 4 non-renewable (natural gas, coal gasification, nuclear, and pulverized coal). In this case, there was no hierarchy, a unique set with 3 numerical and 2 semantic criteria were used. The semantic criteria were: Waste by-products, which indicates the different contaminating substances produced by a certain energy generation type, and Pollution Damage, which indicates the different kind of pollution effects produced. Each type of plant was tagged with an average of 4 waste by-products and 3 pollution damages. Figure 9.12 shows the rank positions of the 9 different types of plants obtained with ELECTRE-SEM. Position 1 is for geothermal plants or for biopower and wind (which are in a tie for all cases). These two first positions change depending on the values of the preference threshold $p$ and the activation or not of veto $v$. Nuclear and pulverized coal plants are the worst performing, followed by the other non-renewable plants. Therefore, the conclusion is that a change to renewable energies is highly recommended.

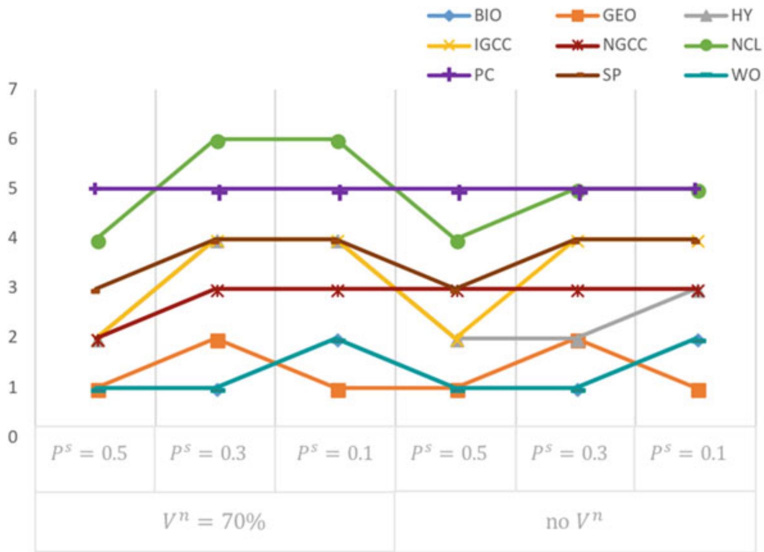


**Fig. 9.12** Ranking obtained with ELECTRE-SEM for energy production plants

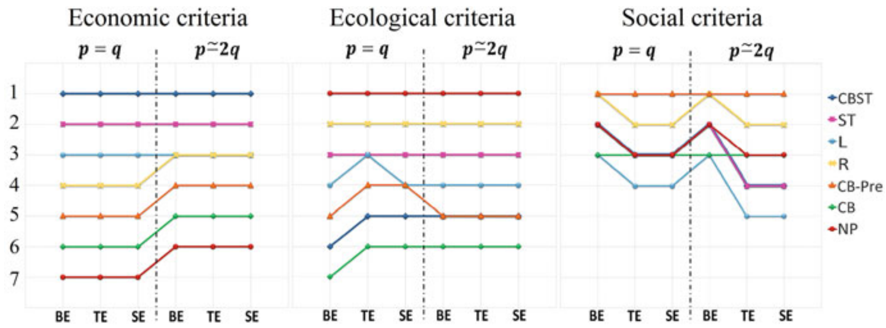


**Fig. 9.13** Rankings with ELECTRE-II-H on 3 intermediate criteria for several wind power plants sites

Focusing on wind power plants, another study for finding the best possible location for a wing farm was conducted in [1]. Seven projects with different locations for the construction of wind turbines were evaluated, being one of them the possibility of not building any plant. Given public concern about the impact of this wind farm, the options considered were based on combining information from participatory processes, interviews, and a review of the projects in the regions of study. The assessment and selection of the best site was made with ELECTRE-III-H, with a hierarchy of 3 levels, with 3 intermediate criteria (economic, ecological, and social) and 9 elementary criteria. Thresholds at elementary level were defined by experts, and for intermediate level we considered different configurations and conducted a robustness analysis. Figure 9.13 shows the rank positions for the three intermediate criteria. We can see that the different sites obtain different positions depending on each of them, having a lot of ties in case of social criteria. When merging that rankings, the final result considered as best the location in R area, followed closely by ST option. Both of them are in good positions in the three partial-pre-orders.

## *9.7.2 Tourism*

Tourism is a sector with great influence in the economy of many countries, especially in the Mediterranean area. Improving the experience of the visitors is a growing interest of destination management offices. Tarragona province and Catalonia attract tourists from many countries during the year, with larger number in Spring and Summer. In this section, we will introduce two different works with ELECTRE methods.

Promoting a destination in the website is of major importance to attract new tourists. In [3], the content and design of 10 different destinations were analyzed using ELECTRE-III-H. Experts in brand communication in tourism defined a

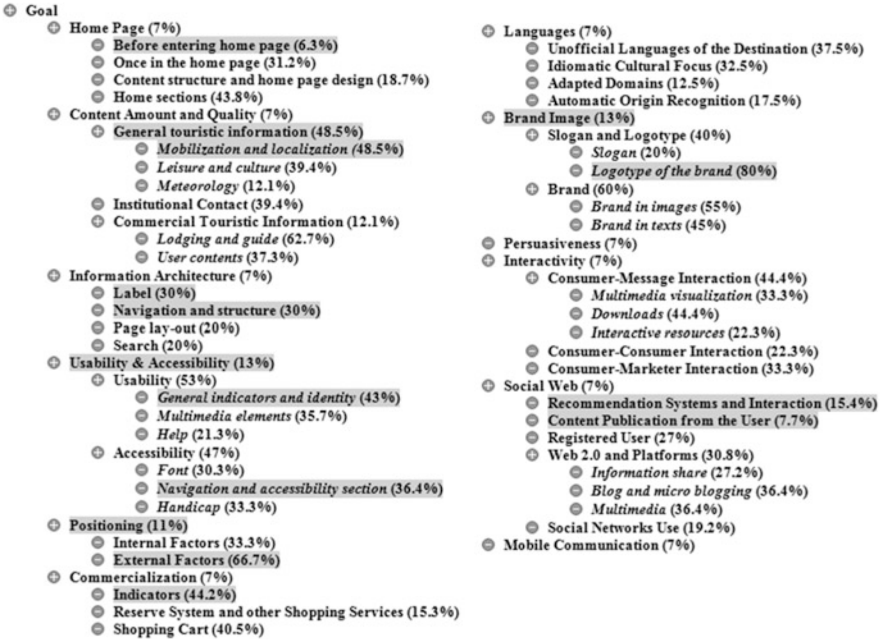


**Fig. 9.14** Hierarchy of criteria to evaluate a touristic destination website

large hierarchy of criteria (see Fig. 9.14), setting also weights and veto thresholds. The model includes different aspects, such as the architecture of the website, the usability, the brand image, or social web tools among others. It is difficult to find a website that performs well in all of them. A complete study was done on the different intermediate criteria, revealing the good and bad points of each of the websites. The relative position of a destination website against its competitors was highly appreciated by the decision makers, as they could know which aspects should be improved in order to get a better position than another website. The graphical tools and the statistical comparative analysis provided in ELECTRE-H Software Package were very useful to generate reports.

Once the tourist has decided to come to a certain destination, the managers must make his/her stay the best experience. The final feeling of the tourist may influence other tourists (word-of-mouth effect) as well as increase the possibility of coming again in the near future. The experience of the visitor is good if he/she is able to make activities that fit with his/her interests. Recommender systems are tools designed for modeling the user's preferences in a personalized profile and then provide a set of proposals according to this profile. The SigTur system was built in collaboration with experts in tourism for recommendation of activities in Tarragona and Costa Daurada area in Catalonia as part of a large research project.

About 900 activities were collected in a database and described using several criteria, both numerical and semantic. In [18], the set of criteria and the recom-

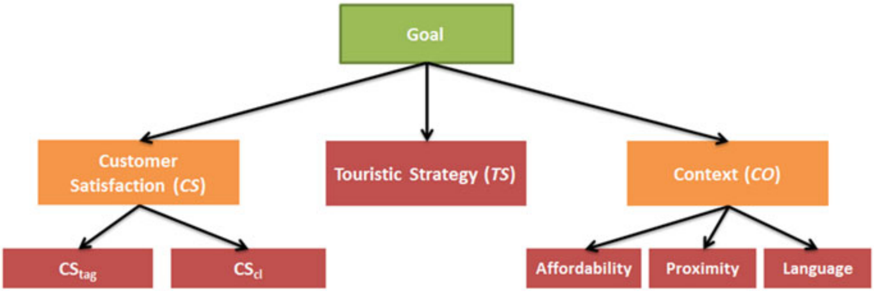


**Fig. 9.15** Hierarchy of criteria in the touristic recommender system

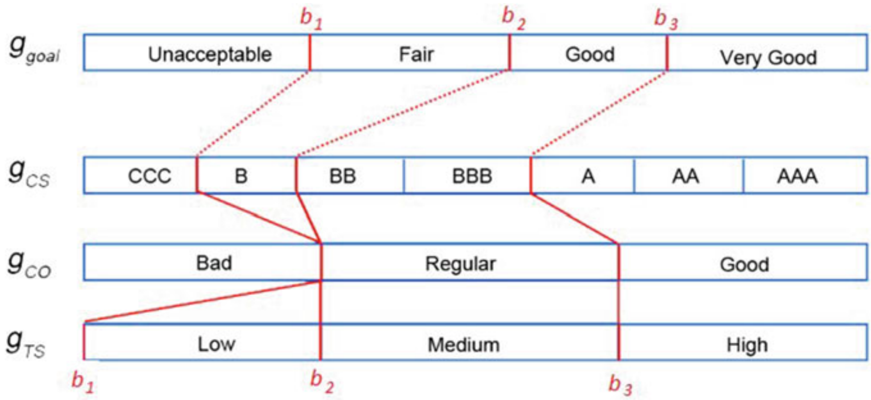


**Fig. 9.16** Mapping of categories for building the boundary profiles in ELECTRE-TRI-H

mendation procedures are explained. A combination of different methods was made using both content-based and collaborative-based models. First, ELECTRE-TRI-H method was used as the semantic information was transformed into a numerical score by means of aggregation operators [5]. The hierarchy had 2 intermediate nodes (customer satisfaction and context) and 6 elementary criteria (Fig. 9.15), and different categories were used in the different nodes. The representation of the boundary profiles is shown in Fig. 9.16. The final assignment was made in 4 categories as shown in Fig. 9.17. Changing the values of the thresholds, we could configure the system to be more or less strict in the assignment.

In a second study about recommendation of touristic activities [15, 17], ELECTRE-SEM formulations were applied directly to the semantic variables and next, ELECTRE-III was then applied to obtain a ranking instead of a sorting of the activities. A study of the thresholds influence on the partial pre-orders was made in [15], showing that both discrimination thresholds and veto for semantic variables had a similar effect to the one produced in the usual numerical criteria. These results validate the methods explained in this chapter.

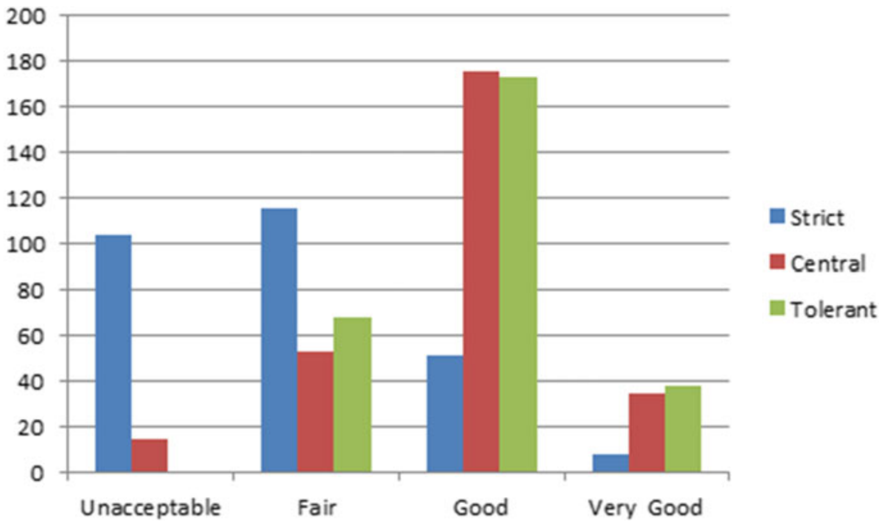


**Fig. 9.17** Number of activities in each of the final categories using ELECTRE-TRI-H

## 9.8 Summary and Conclusions

This chapter has made an overview of several contributions on the modification of the indices of concordance and discordance to handle non-numerical criteria. On the one hand, we have presented the case of semantic data expressed by means of linguistic terms from a domain ontology. On the other hand, the proposal of a hierarchical structure of criteria in ELECTRE has motivated the definition of indices for partial pre-orders and for category assignments. The definition of a rank order value and a category improvement value has been used as main component for the new formulation of partial concordance and partial discordance. This method allows to propagate the results of ELECTRE-III and ELECTRE-TRI-B in a bottom-up procedure.

A software tool provides these methods in a user-friendly interface. In addition, this software has a set of options that assist the decision maker in the definition of the parameters of the proposed methods. Different graphical representations of the results can be obtained and exported to files. This software has been used in different applications, which have been outlined in the last section, providing references for further details for the interested readers.

The ELECTRE family of methods for decision aiding is well recognized in the research field, and they have been used to solve problems in many different fields. With the current trend of increasing the computerization of the processes and the advent of intelligent support technologies, this kind of methods should gain relevance as they provide an operational procedure for intelligent decision support. Therefore, extending the capabilities of ELECTRE opens the possibility of using this method in more complex problems. In particular, the decomposition into a hierarchy of intermediate goals enables the management of large sets of criteria in a more comprehensible way. It is remarkable the fact that the proposed method has the possibility of obtaining partial solutions regarding a certain subset of criteria, which facilitates the analysis of the different dimensions of the problem separately. With the aim of including the use of other kinds of data, the semantic approach presented here opens the door to consider knowledge in linguistic terms. Tagging of objects is common nowadays in social networks or other online systems, which may be prone to integrate some kind of personalized recommendation algorithms (see [13] for a primer step in this direction).

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# Chapter 10
# Robust Ordinal Regression for Multiple Criteria Decision Aiding

**Miłosz Kadziński**

**Abstract** We review the Multiple Criteria Decision Aiding (MCDA) methods in the stream of Robust Ordinal Regression (ROR). They incorporate indirect preference information in the form of decision examples and verify the consequences of applying all compatible instances of an assumed preference model. We focus on four aspects distinguishing the ROR approaches: considered problem typologies, forms of accepted preference information, employed decision models, and types of delivered outcomes quantifying the robustness of results. We also discuss significant extensions of ROR. The most prevailing ones include Stochastic Ordinal Regression, active learning strategies, algorithms for generating dedicated explanations of the decision outcomes, and procedures for answering questions regarding the stability of results. Finally, we list selected real-world applications of ROR in various fields. The review confirms the position of ROR as one of the essential methodological streams in MCDA in the last decades.

## 10.1 Introduction

Multiple Criteria Decision Aiding (MCDA) concerns decision problems involving a set of alternatives (e.g., actions, solutions, or objects) evaluated on a consistent family of criteria [44]. The only objective information that comes from a formulation of such problems is the dominance relation established in the set of alternatives. Since the criteria represent heterogeneous viewpoints, this relation is usually too poor to compare all alternatives completely. As a result, the decision problem is far from being solved.

This situation can be addressed by eliciting preference information from a Decision Maker (DM) or a group of DMs. Such information represents a value system of the DMs, forming an input for learning a more or less explicit preference

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model [87, 94]. Each formalized decision model requires the specification of some parameter values. For example, when using a value-based model, these parameters are related to the formulation of marginal value functions [76]; in case of outranking relation, these can be weights and comparison thresholds [86], whereas for decision rules—the parameters are involved in some logical conditions validating a given decision [95]. A suitably parameterized model is then applied to aggregate the vector evaluations or performances of alternatives and induce a preference relation. Its proper exploitation leads to a recommendation in terms of ranking (i.e., ordering alternatives from the most to the least preferred), sorting (i.e., assignment of alternatives to pre-defined, ordered decision classes), or choice (i.e., identifying the most preferred subset of alternatives) [16].

In MCDA, many methods have been proposed to determine or learn the values of preference model parameters. This can be attained by asking the DM for these values directly or indirectly [47]. The experience from using decision aiding methods indicates that direct specification of the model's form or precise values of such parameters is too demanding in terms of the cognitive effort [21]. Thus, more and more focus is put on incorporating indirect preferences. They take the form of some example holistic decisions concerning a small subset of reference alternatives or desired requirements that the final results should satisfy. The psychologists confirm that DMs are used to providing preference information in such intuitive forms, which are consistent with their experience and knowledge. Consequently, indirect preference information is considered more user friendly. Moreover, decision support based on indirect preference information is gaining importance in many fields of science. For example, in revealed preference theory, one analyzes choices made by consumers, whereas learning by examples is one of the most critical trends in Machine Learning and Artificial Intelligence [37].

In MCDA, one of the most prevailing preference learning paradigms is ordinal regression. Its original meaning was given by Srinivasan and Shocker [97], who proposed a model incorporating goal programming to explain a set of pairwise comparisons in terms of numerical variables. A similar interpretation has been then followed in the stream of papers devoted to the UTA-like methods (see [46] and [92]). For example, in UTA, a Linear Programming (LP) model is used to explain a complete pre-order of alternatives by a sum of monotone additive value functions. In general, ordinal regression disaggregates preferences to construct preference model instances that are compatible with the DM's holistic judgments [47]. In this way, it emphasizes the discovery of intentions as an interpretation of decision examples. Even though this paradigm was originally coupled with a value-based preference model, it has been subsequently generalized to outranking-[82] and rule-based [36] models.

Indirect preference information is affected by natural imperfections because the preference structure may not be well defined in the DM's mind and the DMs are often not fully aware of the adopted multiple criteria model. Specifically, the incompleteness of preference information results in the ambiguous definition of a preference model, i.e., the existence of its multiple instances compatible with the DM's preferences [37]. The traditional MCDA methods apply some arbitrary rules for the selection of a single, compatible instance of the preference model [56]. However, the conclusions that can be derived from applying each of these instances on the set of alternatives may differ substantially. As a result, the delivered recommendation strongly depends on which particular instance is chosen. Investigating the stability of results given all preference model instances compatible with imperfect preferences has been a focus of methods such as ARIADNE [90], Preference Programming [91], VIP [31], IRIS [32], and UTADIS-KU [77].

Robust Ordinal Regression (ROR) combines preference disaggregation [47] and robustness analysis [87]. It incorporates decision examples concerning a subset of reference alternatives and verifies the consequences of applying all compatible instances of an assumed preference model on the set of all alternatives. By doing so, it examines the ambiguity in representing the DM's indirect preference information. The first method proposed in the ROR stream was UTA$^{GMS}$ [37]. It asks the DM for pairwise comparisons, considers all compatible additive value functions with general non-decreasing marginal functions, and quantifies the results through the necessary and possible preference relations. The former holds for a pair of alternatives in case all compatible value functions confirm the preference. In contrast, the latter is instantiated when being supported by at least one compatible value function.

This chapter reviews the methodological developments in the stream of ROR [21]. We first focus only on the works co-authored by Professor Roman Słowiński while referring to four aspects. First, we discuss various problem typologies, which are tackled by ROR. These range from choice [39], ranking [37], and sorting [38] through interactive [33] and evolutionary [11] multiple objective optimization to group decision problems [41] and decision under uncertainty [66]. Second, we discuss different forms of indirect and incomplete preference information (see [21] and [63]). These differ depending on the tackled problem, accounted perspective, the scope of considered criteria, and certainty level. Third, we review preference models to which ROR was applied, including value-[34], outranking-[49], and rule-based [64] models. Some of them assume the preferential independence of criteria and monotonicity of per-criterion preferences, whereas others admit interactions [3, 43] and non-monotonicity [73]. Fourth, we present different types of outcomes that quantify the robustness of results given the multiplicity of compatible preference model instances [21, 75]. These include necessary [37], possible [38], extreme [55], and representative [40] results as well as various certainty levels implied by the imprecision of performances [29] and multiplicity of DMs [41].

Then, we discuss significant extensions of ROR as well as related methodological and application-driven developments. On the one hand, we refer to various types of decision models [2] and robust results [52]. On the other hand, we mention algorithms that support active questioning of the DMs [17], provide dedicated explanations of the decision outcomes [61], and investigate the scenarios of performance modification needed to reach a certain target [19]. We also refer to some investigations of the properties of models [68, 96] and methods [50] as well as selected real-world applications in various fields [4, 81, 100].

## 10.2 Review of Core Methods in Robust Ordinal Regression

In this section, we review the core methods in the stream of ROR. To honor Professor Roman Słowiński on the occasion of his 70th birthday, we focus only on his co-authored works. Instead of discussing the proposed approaches one after another, we generalize the review by referring to the four important features. These include different problem typologies, forms of indirect and incomplete preference information, decision models considered in the preference disaggregation approach, and robust outcomes.

### *10.2.1 Problem Typologies Considered in the ROR Methods*

This section concerns different problem typologies that have been tackled explicitly by ROR methods. We refer to different types of decision-making problems, characteristics of criteria used to assess alternatives, and sources of multiple dimensions in decision aiding.

When it comes to types of decision problems, ROR has been originally applied to *multiple criteria ranking* to quantify the robustness of ordering the alternatives from the most to the least preferred. Some of these approaches (e.g., UTA$^{GMS}$ [37], GRIP [34], RUTA [58], UTA$^{GMS}$-INT [43], MUSA-INT [3], SSOR [24], or PROMETHEE$^{GKS}$ [55]) deliver the robust relation, which imposes an incomplete order admitting incomparability. Others provide means for establishing a complete order (e.g., Extreme Ranking Analysis (ERA) [55]), incorporate some procedures that aggregate the robust results to deliver a score for each alternative (e.g., ELECTRE$^{GKMS}$ [39] and DRSA-ROR-RANK [64]), or select a single representative model instance that emphasizes the outcomes obtained for all such instances (e.g., UTA-REPR [56] or PROMETHEE-REPR [57]). In this way, one can obtain a complete or partial ranking depending on the type of recommendation delivered by the respective non-robust method (e.g., UTA [46] or PROMETHEE II [15]).

As far as *multiple criteria choice* is concerned, a few ROR methods are specifically oriented toward selecting the most preferred subset of alternatives. These include ELECTRE$^{GKMS}$ [39] and DRSA-ROR-RANK [64], which construct some robust relations that are subsequently exploited to offer a choice-oriented recommendation. Other approaches systematically reduce a set of potentially optimal alternatives that are ranked at the top for at least one compatible preference model instance. A dedicated method proceeding in this way is Interactive Robust Cone Contraction (IRCC) [54], whereas the potential of ERA in such a context has been discussed in [18].

The *sorting* methods in the ROR stream support the assignment of alternatives to pre-defined ordered classes. Approaches such as UTADIS$^{GMS}$ [38], ROR-UTADIS [63], DIS-CARD [60], UTADIS-NM [73], and DRSA-ROR-SORT [62] quantify the robustness of recommendation given multiplicity of compatible sorting models, hence revealing potential imprecision in the recommended assignments. On the contrary, UTADIS-REPR [40] builds a single sorting model delivering a precise assignment on top of all compatible ones. Among these methods, ROR-UTADIS and DIS-CARD allow dealing with so-called constrained sorting problems that require control over the number of alternatives assigned to each decision class.

The methods mentioned above deal with a flat structure of criteria. However, ROR has also been generalized to Multiple Criteria Hierarchy Process (MCHP), which accounts for criteria organized in levels, *hierarchically*, from general to detailed ones. MCHP-UTA [20], MCHP-ELECTRE, MCHP-PROMETHEE [22], and MCHP-Choquet [5] deal with ranking problems, whereas MCHP-UTADIS [28] is applicable in the context of sorting. Also, the vast majority of ROR methods deal with deterministic performances of alternatives on particular criteria. The sole exception in this regard is ROR-Imprecise [29] that deals with *uncertain performances* in the form of $n$-point intervals.

The prevailing assumption in ROR is that the criteria are *monotonic*, and the order of preference is defined as constantly non-decreasing or non-increasing with respect to the per-criterion performances of alternatives. On the contrary, UTADIS-NM [73] tolerates an unknown order that needs to be discovered based on the DM's preference information. The latter method admits *non-monotonicity* implied by potential changes in the preference directions in different regions of the evaluation scale.

Multiple criteria are not the sole source of conflicting evaluations in decision problems considered in ROR. On the one hand, in *group decision problems*, one needs to account for preference expressed by multiple DMs. Such a setting has been considered in the context of ranking in UTA$^{GMS}$-GROUP [41] or UTA-REPR-GROUP [59] and sorting in UTADIS$^{GMS}$-GROUP [41]. These approaches assume that all DMs play the same role in the committee. On the other hand, in the *decision problems under uncertainty*, multiple possible states of the world imply various consequences of the actions (probabilities of outcomes). Such a setting has been considered in [66] and [26] by drawing analogies to multiple criteria sorting or ranking and choice, respectively.

ROR has also been applied to *Multiple Objective Optimization (MOO)* dealing with problems involving several objectives to be optimized simultaneously. Various ROR methods have a potential for use in *Interactive MOO*, but it has been directly demonstrated for UTA$^{GMS}$ and GRIP in [33] and IRCC in [54]. In these approaches, two alternative stages are performed. In the first stage, a representative sample of solutions from the Pareto optimal set is generated. In the second stage, the DM's preference information concerning some solutions from the sample is used to derive a model that is subsequently applied on the whole Pareto optimal set. The procedure cycles until a satisfactory solution is selected or the DM comes to the conclusion that there is no such solution for the current problem setting.

The ROR-inspired concepts have also been incorporated into *Evolutionary MOO (EMO)*, which is prevailing in decision contexts where an entire Pareto front needs to be approximated. Methods such as NEMO-I [11], NEMO-II [12], NEMO-II-Ch [13], CDEMO, and DCEMO [74] exploit the set of preference model instances compatible with the progressively supplied preference information concerning the solutions from the current population. In this way, evolutionary optimization can be focused on the DM's most preferred region of the Pareto front.

### *10.2.2 Preference Information Elicited in the ROR Methods*

The type of admitted preference information and elements of responses obtained by the DMs greatly impact the consistency between the value system of the stakeholders, the evolution of the decision process, and the recommendation of a specific decision. ROR methods accept *indirect, incomplete, and imprecise preference information* in the form of example decisions and requirements on the desired results. In this way, they avoid direct specification of the values of preference model parameters. Some approaches tolerate only a single type of indirect preference information, whereas others accommodate various forms of preferences coming from the DM, hence increasing the flexibility of the interactive procedure. In what follows, we review different types of preference information considered in ROR.

A primary type of indirect preference information considered in multiple criteria ranking and choice and MOO problems is *pairwise comparisons* of reference alternatives. For example, in UTA$^{GMS}$ [37], these comparisons refer to the weak or strict preference and indifference relations (e.g., alternative $a$ is preferred to alternative $b$; alternatives $c$ and $d$ are indifferent), whereas in ELECTRE$^{GKMS}$ [39] and DRSA-ROR-RANK [64]–they build on the outranking and non-outranking relations. In PROMETHEE$^{GKS}$ [55], pairwise comparisons can refer to the *level of construction and exploitation* of the preference model. The former refers to the direct comparison of strengths of arguments in favor of one alternative over another and vice versa, whereas the latter refers to the final positions (ranks) of alternatives. The above pairwise comparisons are *certain*, indicating that some relations should certainly hold for a given pair of alternatives. In SSOR [24], one may additionally account for *uncertain pairwise comparisons* pointing out that although the preference of one alternative over another is not certain, it is more credible than inverse preference.

Other types of indirect preference information applicable in the context of ranking and choice include:

– *Comparisons of intensities of preference* for different pairs of reference alternatives either comprehensively, on all criteria, or partially, on a particular criterion (e.g., the intensity of preference of alternative $e$ over alternative $f$ is greater than for the comparison of alternatives $g$ and $h$) postulated in GRIP [34].
– *Imprecise rank-related requirements* (e.g., alternative $i$ should be ranked in the top 5; alternative $j$ should be placed in the lower half of the ranking) postulated in RUTA [58].

– *Constraints on the desired comprehensive values* attained by the reference alternatives (e.g., the comprehensive value of alternative $k$ should be greater than 0.7) postulated in RUTA [58].

When it comes to multiple criteria sorting, ROR methods account for the following four types of indirect and imprecise preference information:

– *Precise or imprecise assignment examples* for a subset of reference alternatives (e.g., alternative $a$ should be assigned to the best class; alternative $b$ should not be assigned to the worst class) considered in, e.g., UTADIS$^{\text{GMS}}$ [38] and DIS-CARD [60].
– *Assignment-based pairwise comparisons* for reference alternatives (e.g., alternative $c$ should be assigned to a class at least as good as the class of alternative $d$; there is a difference of at least two classes between alternatives $e$ and $f$) used in ROR-UTADIS [63].
– *Imprecise desired class cardinalities* (e.g., we wish to accept at most 10 candidates; we need to reject at least 30 applications) used in DIS-CARD [60] and ROR-UTADIS [63].
– Constraints on the desired comprehensive values of alternatives assigned to a given class or class range (e.g., alternatives assigned to a class at most medium should have value not greater than 0.4) considered in ROR-UTADIS [63].

When criteria are organized in a *hierarchy*, MCHP allows the DM to express *preference information comprehensively and partially*, considering a sub-criterion at an intermediate level of the hierarchy. Such capability has been demonstrated in the context of pairwise comparisons (e.g., MCHP-UTA [20] and MCHP-ELECTRE [22]), preference intensities (e.g., MCHP-UTA and MCHP-Choquet [5]), and assignment examples (e.g., MCHP-UTADIS [28]).

Since ROR methods are intended to support the DMs in an interactive process, they permit an incremental specification of indirect preferences. In this perspective, one may account for different *confidence levels* assigned to preference information (e.g., “absolutely sure” or “mild”). Such an option has been considered for pairwise comparisons [37] and assignment examples [38], but it can be generalized to other types of preferences.

### *10.2.3 Preference Models Employed in the ROR Methods*

The role of a preference model in decision aiding is two-fold. On the one hand, it represents the DM’s value system through some mathematical formalisms. On the other hand, it allows conducting a comprehensive evaluation of alternatives in line with the DM’s preference information. In this way, the model’s indication can be used to derive a recommendation that is likely to be accepted by the DM. A general aim of ROR methods is to learn (i.e., to reconstruct faithfully) the DM’s preferences by the preference model. In what follows, we review different types of preference information considered in ROR.

ROR has been applied to all *three significant families of preference models*: scoring functions, binary relations, and decision rules. In this regard, it can deal with qualitative or quantitative performance scales while admitting null, partial, or full compensation between criteria. Moreover, some of these models incorporate the weights of criteria, admit specification of pairwise comparison thresholds, or account for interactions between criteria. Overall, one can choose a model best suited for a particular decision-making problem from various options offering different capabilities.

When it comes to *scoring functions*, the prevailing model in ROR is an additive value function (e.g., UTA$^{GMS}$ [37], GRIP [34], ERA, [55], and RUTA‘[58]). This model requires a rather strong assumption about mutual independence in the sense of preference. It has been relaxed in UTA$^{GMS}$-INT [43] and MUSA-INT [3], where a value function is augmented by two types of components corresponding to “bonus” or “penalty” values of positively and negatively interacting pairs of criteria, respectively. Interactions can also be represented by the Choquet integral through the use of a set of non-additive weights called capacities (see, e.g., MCHP-Choquet [5]). Other scoring functions used in ROR for representing the DM’s holistic judgments include Achievement Scalarizing Functions (ASFs) [54] and L-norms [74]. Depending on the selected parameter $\alpha$, the latter ones admit various compensation levels, from full for a weighted sum to none for the Chebyshev function.

As far as *binary relations* are concerned, ROR has been applied to an outranking relation that holds for a pair of alternatives if one of them is at least as good as another. In ELECTRE$^{GKMS}$C [39] and PROMETHEE$^{GKS}$ [55], the parameters that are inferred from the DM’s preference information are criteria weights, whereas in ELECTRE$^{GKMS}$—these are additionally credibility level and veto thresholds. A definition of an outranking model requires knowing additional parameters (e.g., indifference and preference thresholds) whose values cannot be inferred from the DM’s comprehensive judgments. In this regard, ROR methods tolerate imprecision in the specification of their values.

Finally, the *decision rules* aggregate the performance on different criteria using “if …, then …” statements that handle interactions between criteria. Such a model has been used in DRSA-ROR-RANK [64] and DRSA-ROR-SORT [62] that consider all minimal sets of minimal decision rules compatible with the DM’s indirect preference information. An important remark is that when using binary relations or decision rules, only proper exploitation of the constructed model can lead to a comprehensive assessment in choice (e.g., graph kernel methods), ranking (e.g., Net Flow Score procedures), or sorting (e.g., disjunctive or conjunctive assignment procedures or a rule-based sorting scheme).

The score- or value-based compatible model instances are constructed in ROR using *LP* [37, 39]. The basic mathematical constraints correspond to monotonicity, normalization, and translation of preference information into model parameters. When the DM’s preferences are consistent with an assumed model, there usually exist *infinitely many model instances*. Hence, one cannot investigate them one by one. In turn, to check the validity or determine some robust result, one needs to solve some dedicated LP models that verify the truth of some smartly formulated hypothesis in the set of all compatible model instances. An alternative procedure based on the *Segmenting Description approach* has been discussed in [72].

On the contrary, when using a rule-based model, *the number of all minimal sets of rules is finite*. They are constructed using the algorithms for inducing all minimal rules, constructing all minimal covers of preference information pieces by rules, and computing a Cartesian product of rule sets for different preference relations or class unions in case of, respectively, ranking and sorting [62, 64]. The robust results summarize the outcomes obtained for each individual set of rules.

### *10.2.4 Decision Outcomes Provided in the ROR Methods*

Robustness analysis takes into account internal and external uncertainties observed in the actual decision situations [51]. ROR is focused on investigating the robustness of the provided conclusions, i.e., the stability of decision outcomes given the multiplicity of compatible preference model instances [87]. In particular, it investigates which results are valid for all or the most plausible sets of model parameters. In what follows, we review different types of results that are considered in ROR. For each type, we provide examples concerning various problem typologies. These results are often perceived as a communication and reflection tool that allows for looking more thoroughly into the problem by exploring, interpreting, or testing scenarios [88]. Some methods focus on a single decision outcome, whereas others conduct diversified robustness analysis for the delivered results.

The basic types of results considered in ROR are *the necessary and possible outcomes* [37]. The necessary results can be considered as robust with respect to the preference information due to the fact that they are confirmed for all compatible preference model instances [37]. In turn, the possible outcomes hold if they are supported by at least one compatible preference model instance. In this way, they reveal all parts of the recommendation, which are admissible given the incompleteness of the DM's preference information [38]. The necessity and possibility of decision recommendation have been considered:

- In case of ranking and choice problems—preference relations (e.g., weak preference in UTA$^{GMS}$ [37] or outranking and non-outranking in ELECTRE$^{GKMS}$ [39]) and intensities of preference (e.g., GRIP [34]) for, respectively, pairs and quadruples of alternatives (e.g., alternative $a$ is preferred to alternative $b$ for at least one compatible preference model instance; alternative $c$ is preferred to alternative $d$ at least as much as $e$ to $f$ for all compatible preference model instances).
- In case of sorting problems—assignments to decision classes (e.g., UTADIS$^{GMS}$ [38]) or class unions (e.g., MCHP-UTADIS [28]) for individual alternatives

and assignment-based preference relation (e.g., ROR-UTADIS [63]) for pairs of alternatives (e.g., alternative $a$ is always (necessarily) assigned to class good; the possible evaluation of alternative $b$ as bad or medium depends on the selected model instance; alternative $c$ is always assigned to a class at least as good as alternative $d$).

When verifying the robustness of some measure related to the performance of some alternative or characteristic of decision outcomes, it is more reasonable to account for the *extreme results*. They reflect the least and the most advantageous scenario or the pessimistic and optimistic attainments in the set of all compatible preference model instances. The extreme results have been considered in the context of:

- Ranks and comprehensive scores (e.g., ERA [55] and RUTA [58]) attained by each alternative (e.g., alternative $a$ is ranked third and sixth in the most and the least advantageous cases, respectively; the comprehensive values observed for alternative $b$ in the set of all compatible preference model instances are within the range $[0.4, 0.7]$).
- Class cardinalities (e.g., DIS-CARD [60] and ROR-UTADIS [63]), i.e., the maximal and minimal numbers of alternatives, which are simultaneously assigned to a given class for some compatible model instance (e.g., at least five and at most ten alternatives are assigned to class medium).

Note that alternatives that are ranked first in the most advantageous scenario are called *potentially optimal* [18].

The space between the necessary and the possible, as well as the difference between extreme outcomes, can often be quite large. For this reason, some ROR methods postulate the selection of a single *representative preference model instance* that builds on the results attained with all compatible instances. A general idea underlying such a "one for all, all for one" rule consists of (a) emphasizing the robust outcomes unanimously confirmed by all compatible instances and (b) leveling or neglecting these parts of the decision recommendation that are ambiguous given indications of all compatible instances [40]. This idea has been implemented in case of selecting representative value functions (e.g., UTA-REPR [56], UTADIS-REPR [40], and UTA-REPR-GROUP [59]), outranking models (e.g., ELECTRE-REPR and PROMETHEE-REPR [57]), and decision rules (e.g., DRSA-ROR-SORT [62] and DRSA-ROR-RANK [64]). Such an instance can be used to derive precise recommendation that is supported by the outcomes observed for all compatible instances, hence serving as their univocal representative.

The complexity of a decision problem may increase due to accounting for multiple dimensions that do not correspond to evaluation criteria. In such cases, ROR introduces *additional levels of certainty* to quantify the robustness of results. In group decision problems, the second level is related to the support given to some outcome in the set of DMs. In particular, the necessary-necessary result indicates some conclusion confirmed by all preference model instances compatible with the preferences of all DMs. In contrast, the possible-possible outcome needs to be supported by at least one compatible preference model instance for at least one DM (see, e.g., UTA$^{GMS}$-GROUP and UTADIS$^{GMS}$-GROUP [41]). When considering imprecise performances [29], the necessary and possible relations are additionally judged in terms of being *strong* or *weak*. The former holds for all possible realizations of performances, whereas the latter is instantiated when confirmed by at least one admissible performance setting. Finally, the concepts of the necessary and the possible can be adjusted to other sources of potential ambiguity than the multiplicity of compatible preference model instances. For example, in SSOR [24], one considers probability distributions over the space of consistent preference model instances inferred based on certain and uncertain preferences. In this context, one accounts for the *probabilistically* necessary and possible outcomes that need to be confirmed by, respectively, all or at least one compatible probability distribution.

The results derived by ROR can be used by the DM to learn about her/his preferences. In this way, mutual learning of the model and the DM can be implemented [21]. It is supported by the characteristics of delivered outcomes that provoke the DM to provide additional preferences as well as the convergence of results with the progressive specification of preference information. Specifically, the necessary consequences are enriched, the possible outcomes become more sparse, and the difference between extreme results becomes smaller [17]. Such an interactive process involving alternating stages of preference elicitation and DM's analysis of a recommendation is called *constructive preference learning* [21]. It is continued until the DM accepts the recommendation judging it as sufficiently convincing and decisive or opts for changing the problem setting.

## 10.3 Review of Other Developments Related to Robust Ordinal Regression

In this section, we review three types of developments related to ROR. These include *other ROR methods* that have not been co-authored by Professor Roman Słowiński. We also discuss *approaches that can be coupled with ROR* to support the preference elicitation process, deliver complementary results, or provide dedicated explanations. Finally, we list some selected *real-world studies* in which ROR was applied.

### *10.3.1 Robust Ordinal Regression methods*

When it comes to *other ROR methods*, they have been designed to support different problem typologies. The most notable ranking methods include ROR-DISTANCE [101], Non-additive ROR (NAROR) [2] and its extensions [10, 25, 99], and ROR applied to Bipolar PROMETHEE [23]. ROR-DISTANCE is a TOPSIS-like approach that computes distances from the ideal and anti-ideal alternatives in the space of a comprehensive value. In turn, NAROR incorporates a preference model in the form of the Choquet integral or one of its generalizations. The problem of selecting representative capacities in NAROR has been considered in [1] and [45]. In [23], PROMETHEE$^{GKS}$ has been extended to the case of interacting criteria on a bipolar scale by suitably adapting the bipolar Choquet integral to PROMETHEE. A recent ranking method couples ROR with dimensionality reduction techniques [8].

Some ROR-like sorting methods were developed even before the concept of Robust Ordinal Regression was formalized. These include significant extensions of UTADIS [77, 79] and ELECTRE TRI-B [32, 78]. Other ROR approaches that can handle ordinal classification problems include ROR-ELECTRE-DIS, ROR-PROMETHEE-DIS [49], ROR-ELECTRE TRI-C [65], and ROR-ELECTRE-TRI-B [75].

ROR has also been incorporated into the family of EMO algorithms, called NEMO-GROUP [53], designed for dealing with MOO group decision problems. Let us emphasize that the concepts originally proposed in ROR have also been referred to in other approaches that are not based on indirect preference information, in turn, admitting imprecision in the specification of values of preference model parameters (see, e.g., [27] and [71]).

### *10.3.2 Decision Aiding Methods Related to ROR*

ROR is useful to provide information on which particular outcomes occur with all, some, or no compatible preference model instances. However, it is not appropriate to estimate the probability of the results, which are possible though not necessary. To allow this, one has proposed *Stochastic Ordinal Regression (SOR)* for multiple criteria ranking [51] and sorting [52]. Its essence consists of using the Monte Carlo simulations to estimate the probability of different outcomes based on the sufficiently numerous samples of all compatible preference model instances. The outcomes of SOR are materialized with the acceptability indices quantifying the shares of all compatible preference model instances that confirm a particular result [80]. For example, in ranking problems, these are rank and pairwise outranking (winning) acceptability indices. Although the introduced SOR approaches have been originally designed for dealing with the basic forms of incomplete preference information (i.e., pairwise comparison or assignment examples) and a preference model in the form of an additive value function, they can be easily adapted to other types of preferences, models, and problem typologies (see, e.g., [5, 35, 75]). Moreover, the idea of coupling exact and stochastic robustness analysis conducted with, respectively, mathematical programming and the Monte Carlo simulations has been subsequently implemented in the context of other methods (see, e.g., [5–7]). Subsequently, SOR has been extended to Bayesian Ordinal Regression (BOR) to derive a posterior distribution over a set of all potential value functions instead of assuming a distribution exogenously given [89].

ROR, SOR, and selection of a representative preference model instance based on the analysis of robust outcomes represent three major approaches for dealing with the indetermination of the DM's preference model. A new stream of research has been initiated in [50] and [98]. It aims to *construct a recommendation by directly exploiting the outcomes of robustness analysis* without singling out a specific preference model instance. In [50], one has presented a few dozens of scoring procedures for transforming the results of robustness analysis to a univocal recommendation, whereas in [98], one has proposed some mathematical programming models for constructing a complete ranking based on the stochastic results.

All preference disaggregation methods have been conceived with some specific intentions on how they should perform and when they might be useful. Therefore, it is relevant to check whether they conform to what was expected from them. Such *theoretical and experimental-oriented perspectives* have been adopted in the context of ROR. For example, Spliet and Tervonen [96] formulated the necessary and sufficient conditions for the necessary inferences based on a set of preference model instances compatible with the DM's pairwise comparisons. Moreover, their experimental results indicated that general additive value models were unlikely to be useful by themselves for decision support in preference disaggregation context.

To capture a trade-off between the generality of the preference models used in ROR and their ability to reproduce preference information provided by the DM, one has assessed their expressiveness [68]. Furthermore, Kadziński and Michalski [50] investigated the ability of ROR-like ranking methods to comprehensively restore a preference model of the DM based on incomplete preference information. Other works focused on verifying the robustness of recommendation obtained with the use of different models and methods [50, 68, 96]. Such analyses provide insights on the usefulness of decision aiding tools in different contexts and the amount of preference information needed from the DM to restore his/her views faithfully.

To support the preference elicitation process in the context of ROR and SOR, one has proposed a variety of *active learning strategies*. These can be seen as heuristics for selecting the next pairwise or assignment-based question, aiming at minimizing the number of questions to be asked to the DM until deriving a sufficiently conclusive recommendation. Such strategies build on the outcomes of robustness analysis and estimate the information gain offered by each candidate question in terms of quantifying the uncertainty in recommendation attained with all compatible preference model instances. There exist questioning strategies that have been specifically designed for supporting the elicitation of holistic judgments in the context of ranking [17], choice [14, 18], and sorting [48]. The extensive experiments involving simulated and real-world DMs indicate that they can vastly reduce the number of interactions with the DM.

Each MCDA method should guarantee that the DM learns about the problem but also that (s)he is convinced about the relative advantage of the indicated solution. The latter requires some *dedicated explanations* to justify that the recommendation is logical, valid, and correct. The algorithms for generating dedicated explanations for ROR create arguments about the validity of results and the role of particular criteria (see [61, 72]). In this perspective, they compute preference reducts and constructs, which denote, respectively, the minimal subset of preference information pieces implying the truth of some outcome or the maximal subset of such pieces admitting the validity of some currently non-observable result. Moreover, Kadziński et al. [72] introduce the concept of holistic preference criteria reduct corresponding to a minimal subset of criteria sufficient for reproducing the DM's preferences. In turn, Greco et al. [42] show how DRSA provides a valuable interpretation of the preference relations in ROR in terms of decision rules. Finally, Belahcene et al. [9] consider the problem of explaining the necessary inferences by means of sequences of preference swaps, i.e., trade-offs on a subset of criteria, assuming the other ones remain unchanged.

The last methodological development related to ROR has been designed for answering questions regarding the stability of results. Knowledge about the necessary, possible, and extreme consequences of the provided preference information may stimulate the DM to wonder how the improvement or deterioration of some performances influences the sort of an alternative in the obtained recommendation. The framework of *Post Factum Analysis* [19, 67] considers the following example questions: "what improvement on all or some performances of a given alternative should be made so that it achieves a better result in the recommendation obtained with a set of compatible preference model instances?" or "what is the margin of safety in some or all performances of a given alternative, within which it can maintain some rank or class assignment as in the obtained robust recommendation?". To determine such improvements or deteriorations, PFA solves dedicated optimization problems.

### *10.3.3 Real-World Applications*

ROR and other related methods discussed in this section have been applied to *real-world problems* in various fields. They range from environmental management and medicine through nanotechnology and urban planning to logistics and strategic decision making. The example ranking applications include Swiss water infrastructure decision with UTA$^{GMS}$ [100], e-government benchmarking in the European Union [93] and pharmaceutical strategy determination [81] with ERA, and siting an urban waste landfill with NAROR [4]. Moreover, the problem of organization of result lists of medical evidence with UTA$^{GMS}$-GROUP and UTA-GROUP-REPR was considered in [84], whereas life cycle assessment was conducted with dimensionality reduced ROR in [8]. In [30], one demonstrated the potential of ROR to rank universities in the presence of hierarchical and interacting criteria.

When it comes to sorting, Kadziński et al. [70] considered a green chemistry-based classification model inspired by UTADIS$^{GMS}$ and SOR. In turn, Oppio et al. [83] accounted for sorting Urban Development Agreements in Italy with the robust decision rules. Furthermore, Palha et al. [85] applied ROR-UTADIS for classifying activities to be outsourced in the civil construction of a brewery in Brazil. The potential of NEMO algorithms for solving multiple objective optimization problems related to green supply chain design has been demonstrated in [69].

## 10.4 Summary

Robust Ordinal Regression has been one of the prevailing methodological streams in MCDA. Its most peculiar feature consists of considering all preference model instances compatible with the indirect preference information. This stream was initiated with the ranking method, called UTA$^{GMS}$, that constructs all additive value functions compatible with the DM's pairwise comparisons and delivers the necessary and possible preference relations.

In the last decade, ROR has been vastly developed in different directions. First, there are methods that are suitable not only for ranking but also for choice, sorting, group decision, multiple objective optimization, and decision under risk and uncertainty. Second, the types of admitted indirect preference information have been significantly extended. In particular, one may employ pairwise comparisons that are suitable for diverse problem typologies, have various interpretations, and refer to different levels of certainty, confidence, or hierarchy of criteria. However, the scope of indirect preference information pieces accounted in ROR is much broader, including, e.g., rank-related requirements or desired class cardinalities. Third, ROR has been applied to the three families of models: scoring functions, binary relations, and decision rules. As a result, one can choose a model best suited for a particular decision-making problem from various options. They offer different capabilities with respect to compensation levels, representing interactions between criteria, handling different performance scales, or tolerating potential non-monotonicity of per-criterion preferences. Fourth, a versatile robustness analysis can be conducted to offer the necessary, possible, extreme, and representative results. These outcomes quantify the stability of delivered recommendations given different perspectives that are relevant for the considered problem. For example, in the case of sorting problems, we may focus on the individual alternatives, pairs of alternatives, and decision classes.

ROR has been extended in different ways. As far as the stage of preference elicitation is concerned, dedicated active learning strategies select the next question to the DM while minimizing the number of questions to be asked until deriving a sufficiently conclusive recommendation. When it comes to the robustness analysis, the most prevailing methodological advances deliver stochastic acceptability indices computed with the Monte Carlo simulations (Stochastic Ordinal Regression) or construct a univocal recommendation by directly exploiting the robust outcomes. Furthermore, analysis of the recommended decision has been supported with the algorithms for generating dedicated explanations or answering questions regarding the stability of results. Finally, ROR and related approaches have been applied to real-world problems in environmental management, medicine, nanotechnology, urban and territorial planning, logistics, and strategic decision-making.

We envisage a few critical directions for future research. One of them concerns the specialization of ROR to specific real-life applications. This may require adaptation of ROR to less typical problems such as multiple criteria clustering or sorting with partially ordered classes or standard problem types involving uncertain performances, potentially non-monotonic criteria, or multiple DMs playing different roles in the committee. These aspects have so far received no or limited attention. Much effort should also be assigned to the development of user-friendly open-source software.

A much-neglected aspect in MCDA concerns performing extensive computational studies to assess the properties of preference models and methods that decide upon their usefulness for constructive preference learning. Such experimental verification involving simulated and real-world DMs should indicate which models are more suitable for use with different levels of incompleteness in preference information and which methods provide more credible recommendations when the DM's preference information is scarce.

Finally, an appealing research direction concerns the adaptation of ROR to handling extensive preference data. This should enhance the application of MCDA in big data environments, which are naturally affected by the high level of inconsistency in the available preference information.

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# Chapter 11
# What Is Legitimate Decision Support?

**Yves Meinard and Alexis Tsoukiàs**

**Abstract** Decision support is the science and associated practice that consist in providing recommendations to decision-makers facing problems, based on the available theoretical knowledge and empirical data. Although this activity is often seen as being mainly concerned with solving mathematical problems and conceiving algorithms, it is essentially an empirical and socially framed activity, where interactions between clients and analysts, and between them and concerned third parties, play a crucial role. Since the 80s, two concepts have structured the literature devoted to analyzing this aspect of decision support: validity and legitimacy. Whereas validity is focused on the interactions between the client and the analyst, legitimacy refers to the broader picture: the organizational context, the overall problem situation, the environment, culture, and history. Despite its unmistakable importance, this concept has not received the attention it deserves in the literature in operational research and decision support. The present chapter aims at filling this gap. For that purpose, we review the literature in other disciplines (mainly philosophy and political science) that is demonstrably relevant to elaborate a concept of legitimacy useful in decision support contexts. Based on this review, we propose a general theory of legitimacy, adapted to decision support contexts, encompassing the relevant contributions we found in the literature. According to this general theory, a legitimate decision support intervention is one for which the decision support provider produces a justification that satisfies two conditions: (i) it effectively convinces the decision support provider's interlocutors (effectiveness condition) and (ii) it is organized around the active elicitation of as many and as diverse counter-arguments as possible (truthfulness condition). Despite its conceptual simplicity, legitimacy, understood in this sense, is a very exacting requirement, opening ambitious research avenues that we delineate.

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## 11.1 Introduction

Although the term "decision" might at first sight seem to refer a punctual event, in fact most decisions are made through a set of cognitive activities that the decision-maker performs: a decision *process*. Decision support is the science and associated practice that consist in providing recommendations to clients (possibly decision-makers) facing problems, based on the available theoretical knowledge and empirical data. Just like decisions or decision-making, decision support is a process, rather than a punctual event. What do we do when, as decision analysts, we engage in such processes [46]? From an analyst's perspective, the answer is that we *manipulate* information to provide recommendations. To formulate this idea, we purportedly use the ambiguous term "manipulate," because it conveniently conveys the idea that this task is double-edged. Indeed, depending on the context, "manipulate" can be either a neutral term, synonym for "compute" or "handle," or be attached with negative connotations, and mean something more akin to "distort" or "falsify." When dealing with information in decision support processes, we are always in the gray zone between these two senses of "manipulate." On the one hand, we are mostly guided by a willingness to help the decision-maker, and in that sense we are not here to cheat or deceive her/him. But, on the other hand, when we work with information, using our data analysis technologies, our algorithms, and theoretical and computational devices, we unavoidably make choices that are to some extent arbitrary, and about which the decision-maker does not have a say—because we do not give her/him the opportunity, and/or because she/he does not have the required technical skills to make a cogent decision in this domain.

Usually, the information we "manipulate" in that sense consists of empirical observations, data collected in different forms and circumstances, as well as information about the subjectivity of the decision-maker, such as her/his values, beliefs, and intentions. To these, we may add norms, regulations, standards, which apply independently of the precise problem situation at hand, as well as culture, history, practices, and habits that frame the space and time within which the decision support process is conducted. Although it is clear that the boundaries between these concepts might be blurry, for the sake of simplicity, we may categorize these different types of pieces of information in four categories:

- Objective data
- Preference statements
- Constraints that can be called "hard," in the sense that they are untouchable at the scale of the decision support process, such as laws and regulations, budgets constraints, and so on
- Cultural, or "soft" constraints, which are to some extent binding, but can nonetheless be somewhat slackened, such as habits and customs

An extensive literature in Decision Sciences addresses the cognitive effort that gathering, computing, and analyzing these information entail for clients and analysts developing decision support models (see [18, 48]), and associated biases plaguing decision-aiding processes. An important outcome of these reflections is the development of "user friendly" methods (such as rule-based decision support models; see [17, 43]) and preference learning techniques (see [11, 16, 29]).

Beyond these contributions, since the 80s, the bulk of academic discussions of the appropriateness of such "manipulations" have been mainly developed around two key concepts: validity and legitimacy.

Concerning the former, Landry ([23]; see also [24]) famously introduced four types of validity checks: conceptual, logical, experimental, and operational. Seen from a decision support process perspective (focused on interactions between a client and an analyst), they can be regrouped into two categories:

1. To be valid, decision support should be meaningful for the analyst, in the sense that it should respect accepted axioms, theorems, and properties. For example, the "manipulation" should respect meaning invariance (for more about the concept of meaningfulness in measurement theory, see [37–39]).
2. To be valid, decision support should be meaningful for the client, in the sense that it should reply to her/his questionings, it should be useful in terms of advice on what to do (or not to do), and it should be felt as owned by the client and usable within the decision process within which it has been requested.

However, as already noted by Landry himself in the 90s, although necessary, validity is not enough to ensure that the "manipulation" we produce and the recommendation that follows will be effectively used, applied, and appreciated and will have an impact in the real world. This is because validity refers to interactions between client and analyst but ignores the larger picture: the organizational context, the overall problem situation, the environment, culture, and history. Besides, more often than not, beyond the decision-maker and stakeholders identified in the decision process, decisions also affect other stakeholders who can appreciate or not the decision, react to it by modifying their behavior, and *in fine* influence how the whole decision process for which the decision support has been asked is conducted.

Although many analysts, especially in academic contexts, often pursue research interests when "manipulating" data, in decision support processes, we should keep in mind that such "manipulations" of information are not aimed at supporting the analyst, but the client. In other terms, whatever "manipulation" we do is going to be available for use by others, within contexts that we know and control only partially, producing consequences that most of the times will affect the client and the stakeholders involved in our recommendation. Hence, while many academic decision analysts tend to see their activity as mainly concerned with solving mathematical problems or conceiving algorithms, in fact supporting the decision-making of others essentially is an empirical and socially defined activity [2, 10, 28, 40, 41]. This predicament means that, as analysts involved in decision support processes, we have to make sure that the "manipulations" we engage in can make sense for the decision-maker in light of the context in which we support her/his decision-making.

The concepts of validity and legitimacy are complementary in the sense that the latter encapsulates all the above-mentioned aspects of decision support interactions that the former, focused as it is on the interaction between client and analyst, fails to capture. When practicing decision support for their clients, analysts need not only check the validity of the information "manipulation" they perform, and the validity of the corresponding recommendation. They should also pay due attention to their legitimacy.

In this chapter, we propose and explain our vision of what the legitimacy requirement amounts to. This vision is aimed at clarifying debates on the desirable features of decision support processes, among other things by providing answers to the above questions.

For that purpose, we begin by showing that the issue of the legitimacy of decision support is both important and neglected in the current literature. We then proceed by reviewing the recent literature on legitimacy. Without claiming to be exhaustive, we will try to identify the main theoretical options exposed in the literature. Based on this review, we then propose a general theory of the legitimacy of decision support, designed to encompass the various visions presented in the preceding section, in such a way as to overcome their limitations and make the most of their strengths. This general theory is, to a large extent, based on preliminary discussions published in [26] and [27]. Equipped with this general theory, we will then be in a position to address a major, if relatively neglected question: the one of the challenges facing the quest for legitimacy in decision support contexts. Our aim in this section is to explore reasons why, despite all the major reasons we have to take legitimacy seriously (recalled above), in some cases the quest for legitimacy can prove extremely difficult, if not impossible to achieve. The exploration of theses hurdles on the path to legitimacy will enable us to identify a series of challenges for future research on means and approaches to construct the legitimacy of decision support.

## 11.2 The Legitimacy of Decision Support: An Important But Neglected Topic

The issue of the legitimacy of decision support is of unmistakable prominence when decision support activities are involved in policy making, policy design, or policy evaluation [22, 26]. In such contexts, decision support is expected to improve or strengthen policies or parts thereof. The latter are activities that typically limit the liberties of some individuals and/or groups and distribute financial or regulatory advantages among individuals and/or groups. This can, and often does, arouse questionings and disagreements. Besides, often enough, policies pertaining to different sectors compete with one another to capture public finances and public support (e.g., environmental policies vs. economic policies), and various such policies are involved in or criticized by the competing policy agendas of different political parties and/or candidates to elections in democracies. The corresponding debates around the well-foundedness of policies are often framed in the terminology of "legitimacy." The legitimacy of the decision support activities involved in the elaboration and implementation of the policies at issue is unavoidably raised as part of these debates.

The concept of legitimacy is, however, relevant to decision support well beyond political contexts [31]. Even when decision support is deployed in private companies to address issues without any link with public policies, questions echoing the ones mentioned above unavoidably emerge. Indeed, the typical decisions for which decision support is requested in private firms, such as possible changes in strategy, reorganizations of the workforce or workflow, or other organizational issues, typically have differential implications for various individuals within the organization: some individuals will gain prominence and/or responsibility, at the expense of others, which can raise debates and disputes. Although the latter do not have, in private firms, the same importance as in political democratic arenas, still they can endanger the stability of the organization, which should accordingly pay attention to legitimacy.

This is all the more true when the decisions made in private firms have implications for public policies or, more generally, for the public. This is the case, for example, when a private firm decides to use a certain type of data or algorithm, which can involve the infringement of privacy or raise other stakes of public interest. Even beyond private organizations, as soon as issues of public interest can be involved in or impacted by decisions, the legitimacy of the decision-maker and her/his decision and, consequentially, the legitimacy of the decision support she/he benefits from are raised, even in the archetypal case of a single, self-standing decision-maker. In the following, we provide two short examples allowing to show the difference between constructing a valid model and providing a legitimate model for the decision process where the model is expected to be used.

*Example 1 (Organizational Legitimization)* The second author has been involved in the past in "providing" decision support to a large Italian company facing the problem of massive software acquisitions that needed to be framed by a general policy assisted by a rigorous evaluation model. The case study is reported in [32]. When the whole study was completed, the final deliverable was almost dismissed by the General IT manager of the company because he considered it was too complicated for his staff to be effectively used. The project has been saved when the project manager revealed that the whole procedure was implemented on a spreadsheet (it has been largely used in the subsequent years). The reasoning of the IT manager was simple: if it runs on a spreadsheet, it fits our organizational knowledge. Otherwise, it is an academic exercise. On our side, the reason for using the spreadsheet implementation was only rapid prototyping; actually the project manager was pushing for a very sophisticated (although user-friendly) implementation. This is a typical case where a model certainly valid (for both the client and the analyst) risked to fail an organizational legitimacy check. It has been saved by chance.

*Example 2 (Society Legitimization)* In the late 60s, early 70s, the NYFD commissioned to the RAND corporation a large study concerning the location of the fire-fighters stations in order to improve the efficiency of the whole service and reduce the dramatic increase of casualties due to late intervention of fire brigades (see [49]). The project has been technically successful but rose an extensive number of controversies both political and with the trade unions, resulting in reducing drastically the effectiveness of the suggested solutions (for a nice discussion, see [19]). This is a typical case where neglecting the social complexity of the problem at hand may produce "valid" models that turn to be socially unacceptable and thus not legitimated.

Despite this unmistakable importance, and associated major theoretical and practical implications, the issue of the legitimacy of decision support has not received the attention it deserves in the literature on decision science, operational research, and management. [25] in operational research and [44] in management sciences are notable, if now a bit old, exceptions. Despite their undeniable contribution (to be discussed below), they cannot compensate for the overall scarcity of discussions on this topic in this literature. By contrast, the literature on this topic is immense in a wide range of domains, from economics (e.g., [47]) to philosophy (e.g., [21]) and political sciences (e.g., [4]). For lack of thorough, recent contributions to these debates in the specialized literature, the concrete meaning and implications of this large body of literature for the specific case of decision support are currently unclear.

Our aim here is to bridge this gap in the literature and, in so doing, hopefully, to bring our contribution to larger, interdisciplinary debates on the concept of legitimacy, from both theoretical and practical points of view.

## 11.3 Visions of Legitimacy

[25] notoriously emphasized the importance for models used in operational research practice to be legitimate, and they pointed the need to clearly make a difference between the validity of a model and its legitimacy. According to these authors, "legitimisation encompasses two complementary and often unconscious activities. The first one is a comparison of concrete actions, situations, or states of affairs with a set of abstract entities comprising values, norms or symbolic reference systems, which will be referred to as the 'code' henceforth. The second activity is a judgment as to the conformity of these concrete actions, situations, or states of affairs with the corresponding code." However, they do not clearly explain what they take this "code" to be. Fortunately, a rich and profuse literature is available to clarify this issue and overcome the limitations of [25]'s seminal effort.

Discussions on legitimacy in the literature appear, at first sight, to be highly complex and dispersed, diversely focused as they can be on sources of legitimacy, criteria of legitimacy, means to ensure legitimacy, proofs of legitimacy, and so on. In this section, our aim is to draw a map of the main theories of legitimacy developed and used in the literature, to clarify this complex theoretical landscape.

The question of the legitimacy of a given decision support activity can be raised from two, complementary points of view: positive and normative. The positive approach asks an empirical question: what are the criteria that people use, *as a matter of fact*, when deciding whether they take something to be legitimate or illegitimate? The normative question approach asks: what are the criteria that *should* be used to decide if something is legitimate or not?

As scientists, we might think that the normative question is not for us, but for moralists or preachers, to answer, and that the positive question is the only one that can be addressed in a scientific context such as the one of decision support. But this would be a mistake, for two associated reasons.

First, as famously explained by [34], the frontier between normative and positive is always blurred, since many issues, theories, or approaches that are typically seen as entirely positive in fact have a normative anchorage, and most issues that are typically seen as entirely normative are based on, or influenced by, positive data. Confining ourselves to the positive question is accordingly impossible in practice. In concrete terms, this impossibility stems from the fact that, when designing a scientific project to answer the empirical question above and when analyzing the data obtained, we unavoidably take stances on issues that pertain to the normative approach. This is the case, for example, when making decisions on how questions will be formulated to survey individuals, or on how behavior will be monitored and interpreted.

The second related reason is that, most of the time, there is no such thing as a “fact of the matter” when it comes to what people take to be legitimate or illegitimate. People might fail to have an opinion on what they take to be legitimate or not. They might start asking themselves the question and forming an opinion upon our asking them. They might change their mind if we give them pieces of information, perhaps even if this information is irrelevant. On issues such as legitimacy, the picture according to which people always already have a well-formed, stable vision, independent of the scientist and the scientific protocol that strive to capture this independent “fact of the matter,” is accordingly untenable.

The crude vision according to which normative questions are for preachers or moralists to address is therefore entirely irrelevant when issues such as those surrounding legitimacy are raised. Normative philosophy is, for that matter, to a large extent devoted to address normative questions in a rational way, rather than through preaching. In contemporary normative philosophy, Rawls [35] and Habermas [20] are the most prominent authors who have championed this rationalist approach to normative philosophy, in the wake of Kant’s philosophy of practical reason. Such philosophical approaches to normative questions do not evacuate positive questions: they strive to take advantage of studies of positive questions to enrich normative reflection, and conceive of the latter as relevant to improve the way positive questions are addressed. In the remainder of this chapter, we will endorse a similar approach. We will take into account both normative and positive approaches to legitimacy, and we will strive to use both as complementary approaches liable to enrich one another.

Beyond the normative/positive dichotomy of points of view, visions of legitimacy in the literature are classically divided into two broad categories: theories of output legitimacy, also called substantive theories [47], and theories of input legitimacy, also called procedural theories [4]. Substantive theories claim that the legitimacy of a policy or decision depends on the state of affairs that it brings about. By contrast, procedural theories hold that the legitimacy of a decision is determined by the decision-making process through which the decision was made. A toy example of discordance between substantive and procedural visions of legitimacy can be given by the following scenario: imagine that, through a democratic decision-making process such as a majority vote, a minority group in society is denied some basic rights, such as access to education and health insurance. A plausible procedural theory of legitimacy might claim that the decision is legitimate, because it was made through a legitimate process (majority vote). A plausible, dissenting substantive theory might claim that such a policy that ends up arbitrarily depriving some people from some basic rights cannot be legitimate, because the state of affairs in which a minority is oppressed is illegitimate.

This substantive/procedural dichotomy is useful to clarify some debates on legitimacy, since numerous theories can easily be classified along the lines of this dichotomy. Among prominent visions of legitimacy that can be classified in this way, take for example a vision according to which the essence of legitimacy is due process or legality. In this vision, a decision is legitimate if it rigorously abides by all the relevant regulatory rules. This first vision clearly falls in the procedural category. Similarly, a vision according to which the effective participation of citizens is the crux of legitimacy is another example of a plausible procedural theory. By contrast, a theory claiming that a policy is legitimate if it ensures that all the people affected see their welfare increased falls in the substantive category. The same goes for theories of so-called higher goods, such as the one championed by [45]. Among theories clearly falling into one or the other category, a diversity of concrete criteria through which legitimacy is ascertained can then emerge: procedural theories can champion criteria of fairness, impartiality, responsibility, while substantive theories will use criteria of equality, efficiency, or effectiveness.

For all its usefulness for clarification purposes, the substantive/procedural dichotomy has, however, its limits. Indeed, numerous theories of legitimacy mix procedural and substantive aspects. This intimate mix of substantive and procedural aspects can even be found within some basic concepts that can hardly be avoided in discussions on legitimacy. This is the case, for example, of the concept of right. On the one hand, the picture of who enjoys a given right and who is deprived from it in a given population is, in a sense, a state of affair. In that sense, a theory that would hold that this right is the crux of legitimacy would be called substantive (this is what we have done in our example above). But, on the other hand, a right is procedural, in the sense that a right specifies what people who enjoy it or are deprived from it can do. This is particularly evident when the right at issue is a right to vote or to partake in a decision, but this is true of rights in general. The theory holding that a right is the crux of legitimacy is, in that sense, also procedural. The same logic applies at least to some values, as illustrated, for example, by [6]'s theory of democracy. According to this author, the essence of democratic legitimacy is a set of "core values" that can materialize in both procedures (hence, the procedural aspect of his theory, which refers to voting and parliamentary procedures) and substantive judgments (made by judicial courts, such as the Supreme Court in the United States).

Another weakness of the substantive/procedural dichotomy is revealed by "epistemic" theories [15]. These theories focus on procedures to decide if a decision is legitimate or not. But they do not take procedures to be the core of legitimacy, which they locate in substantive features. They focus on procedures because they take them to be the most reliable means we have to make sure that the substantive features of interest are and/or will be brought about.

## 11.4 The Legitimacy of Decision Support: A General Theory

At this stage, we hence see that, although there is a large diversity of visions of legitimacy, a series of concepts (the normative/positive and substantive/procedural dichotomies, the notion of epistemic approaches, etc.) can be put to use to clarify the complex picture that this large diversity of visions draws. We have also seen that these various concepts have their limits. But they can be used as complementary tools, whose limited relevance should be assessed on a case-by-case basis when using them, to clarify debates on legitimacy.

Now that we have this complex landscape and a set of conceptual tools to navigate it, the question should be asked: is it possible to elaborate a unique, central theory of legitimacy, possibly useful to think through concrete issues, such as the ones associated with decision support activities?

We argue that the literature in normative philosophy on deliberative democracy [9, 12, 21, 36] provides the key to overcome the diversity of visions of legitimacy, so as to develop a unique, encompassing theory. Just like our overview of legitimacy witnesses a diversity of theories, theories of deliberative democracy are concerned with situations in which a diversity of ethical views co-exist and are championed by a diversity of people and/or groups composing a society. In this context primarily characterized by *pluralism*, theories of deliberative democracy are concerned to identify means to make collectively acceptable decisions, without hoping to identify decisions that will perfectly match any one of the diverse points of view that are concerned. The key concept through which theories of deliberative democracy claim to escape chaos is *justification*. According to theories of deliberative democracy, decisions can be collectively made in a pluralist setting if they can be justified to all the diversity of concerned actors or groups.

This idea raises numerous questions that fall beyond our scope in this chapter, such as: how can one be sure that it will be possible to justify a given policy to all those concerned? How should we identify who are those people that are called "concerned"? etc. We leave aside these questions here because our point is not to champion the theory of deliberative democracy, but to assess if the reference to justification, which is used by this theory to address pluralism, can be used in our case to address the diversity of visions of legitimacy.

Beyond the similitude in context (deliberative democracy faces a diversity of ethical views, and we face a diversity of visions of legitimacy), the idea to use the same concept of justification stems from the way deliberative democracy uses this concept, which appears relevant to our case as well. Indeed, the crux of the usage of the concept of justification in deliberative democracy consists in taking the various ethical views composing pluralism as a reservoir of *building blocks for candidate justifications*. Seen from these lenses, any given ethical theory contains or can lead to the formulation of justifications for some decisions but not for others. These justifications will, typically, be accepted by people endorsing this ethical view, but probably not by people endorsing other ethical views. In a deliberative dynamics, such disagreements should lead to the formulation of new justifications, less directly anchored in any given ethical view, and therefore liable to enable agreement among a diversity of people endorsing different ethical views. This approach to diversity and pluralism suggests that, in our case of a diversity of visions of legitimacy, the various visions can also be seen as reservoirs of building blocks for justifications.

In this general approach, legitimacy is a matter of justification, and the various visions of legitimacy found in the literature are partial justifications that can complement and enrich one another in various situations, as required by the context. Various applications of this or that substantive theory, or this or that procedural theory, understood in a normative or a positive interpretation should hence be seen as elements that can be combined to produce justifications. Some combinations will prove incoherent, others irrelevant, beside the point, unnecessarily intricate, and so on. But some combinations might constitute convenient justifications in some cases.

However, this usage of the concept of justification creates, at this stage, an important problem, heralded by the fact that, in the last sentence, we have had to add an adjective ("convenient") to qualify justifications. This need to qualify justifications stems from the fact that the term "justification" is, as it stands, ambiguous. Indeed, what, precisely, is a justification? A basic idea conveyed by this term, which we posit is shared by all the users of the term, is that a justification is an argumentative discourse. But our usage of the concept of justification in our general theory cannot be limited to this basic idea. Indeed, it would not make sense for us to claim that articulating an argumentative discourse that would be incoherent or nonsensical or beside the point is enough to yield legitimacy. Hence, the need to *qualify* the kinds of argumentative discourses that are relevant for our purposes.

We argue that two complementary qualifications are needed to equip ourselves with a relevant notion of justification: the first one has to do with *effectiveness*, and the second one with *truthfulness*. To explain the meaning of these two qualifications, let us focus on our core setting of interest: decision support activities involving decision support providers (typically, decision analysts or experts), decision-makers, and concerned stakeholders.

If the decision support provider is concerned to entrench the legitimacy of her/his intervention, according to our general theory, she/he will elaborate and voice a justification. But if no one understands her/his argument, or if it fails to convince anyone, it is clear enough that her/his justification will have failed to yield legitimacy. We therefore need to add an *effectiveness* requirement to the meaning we give to the concept of justification within our theory of legitimacy: what is needed, as a matter of justification, is an argumentative discourse *that manages to convince the relevant public*.

But this reference to *effectiveness* immediately raises two problems.

The first problem is just as ancient as philosophical reflections on speech and its ambivalent relation to rationality [8]. This problem is that, if we focus uniquely on *effectiveness*, we will end up with a wholly manipulative concept of justification (in the negatively connoted sense of the term). This would lead to an absurd approach in which the more manipulative (still in the negatively connoted sense of the term) the decision support provider is, the more legitimate her/his intervention is (this echoes [25]'s claim that "legitimisation cannot be mystification" and that we "should not confuse manipulation and legitimisation"). Therefore, we need another qualification, designed to ensure that *effectiveness* does not stem from mystification, but from rational persuasion [33]. Because the point of this qualification is to ensure that effectiveness does not reflect mystification, but rather a faithful account of relevant facts and theories, let us talk about a *truthfulness* qualification.

The second problem, which can be called the problem of "the targets of justification," is that, if we accept to abide by an effectiveness requirement, the question unavoidable arises: effective *for whom*? Should the justification be convincing for the decision-maker and only for her/him? Should it also include those actors who are tightly involved in the decision-making process, such as members of a steering committee monitoring the process when one such committee exists? Should the circle of interlocutors to be convinced include all concerned stakeholders, or all the people who see themselves as potentially impacted by the decision to be made, or all the people who can take a stance on the issue even though they cannot be directly impacted? The theory and practice of deliberative democracy and participation face notorious difficulties to answer such questions. Participatory practices usually informally choose the stakeholders who are asked to participate, and despite academic calls to formalize stakeholders' recruitment [30], there is currently no largely accepted technology available for that purpose. This lack of practical solutions reflects a theoretical difficulty, which is unmistakable in the main theoretical works on deliberative democracy. [36]'s idea to solve this problem was that justifications should be acceptable to all "reasonable" citizens, and he claimed that the precise content of this requirement should be clarified by reasonable citizens themselves. However, as [14] has shown (and as anyone should have expected), this purported solution does not work, since there is an "impervious" plurality of groups that might call themselves "reasonable." As opposed to Rawls's (untenable) refusal to clarify what "reasonable" means, [15] claims that philosophers of deliberative democracy should acknowledge that a “true” theory of who is reasonable and who is not is needed. But Estlund does not explain how this “truth” is to be discovered. Rawls’s and Estlund’s theoretical stances, which are the two options developed in the theoretical literature, therefore fail to solve the problem of the targets of justification.

We argue that these two problems can be solved by designing a truthfulness qualification fit for purpose.

Identifying means to ensure that a justification is truthful rather than manipulative is a notoriously difficult question. Here, we propose to take advantage of [27]’s approach, introduced in the context of a reflection on the justification of norms underlying decision support, to solve this problem. This approach proposes that, when developing a justification, one should actively seek as many counter-arguments as possible, including by soliciting the interventions of outsiders and people marginalized from the decision support process, and then enrich one’s justification by defending it against all these counter-arguments. The underlying idea is that mystifying arguments typically stress convenient aspects of the matter, while silencing inconvenient aspects. A powerful counter-manipulative tactic is therefore to track aspects that presumably mystifying discourses tend to silence. By organizing one’s justifications around an active search for counter-arguments, one therefore puts oneself in a position in which being mystifying is by design extremely difficult. By the same token, the “target of the justification” problem is solved. Indeed, if the search for counter-arguments is thorough enough, and if the defense against all these counter-arguments is effective, the justification will by definition be convincing *to all*.

At this stage, a natural rejoinder might be to claim that the idea of “actively seeking as many counter-arguments as possible” is exceedingly vague and easily manipulable: if the decision support provider concerned to produce a justification takes, say, five minutes to seek counter-arguments, is it enough? And how “active” should she/he be? The notion of “active search” might appear much too indeterminate. But, as [27] argue, this indeterminacy would be a serious flaw of the theory only if the latter had the pretension to achieve an “absolute” justification, taking into account all the possible counter-arguments, from absolutely all sides. Achieving such an “absolute” justification is, in any case, impossible, since the universe of counter-arguments is infinite, and there even exists an infinity of counter-arguments that have not yet been discovered. As opposed to this unreachable “absolute justification,” the truly worthwhile pursuit is the search for the best locally achievable justification, while keeping in mind that justifications are always provisional: new counter-arguments can emerge and ruin a hitherto convenient justification, and alternative decision support interventions can be launched and be supported by justifications that overcome the former one.

To sum up, our general theory of the legitimacy of decision support interventions, which we claim encompasses all the other theories reviewed above, is the following: “*A legitimate decision support intervention is one for which the decision support provider (or, for that matter, anyone else), produces an unavoidably provisional justification that satisfies two conditions: (i) it effectively convinces the decision* *support provider's interlocutors (effectiveness condition) and (ii) it is organised around the active elicitation of as many and as diverse counterarguments as possible (truthfulness condition)*".

In the following, we provide a short final example in order to show how our theory of legitimacy would apply in a recent real-world case study.

*Example 3 (The Legitimacy of Using Predictive Justice Tools)* In the past few years, there were extensive discussions about the use, abuse, and misuse of predictive justice devices. The best known controversy is the "COMPASS" case[1] (for a nice discussion, see [1]): it concerns the fact that, behind a device computing a "score" that is used in order to assist a decision-maker (a judge in this case) in making a decision, there are "hidden" hypotheses and assumptions. In that precise case, these hidden assumptions refer to manipulations that can be considered to be "racial discrimination" when the software computes the score for people from different racial origins.

A standard way to analyze such a case consists in striving to show that, because the tool manipulates data on racial origin in a certain way, it is unfair. However, fairness is a concept with several different formal definitions, and it turns out that many such definitions are incompatible. Fairness, as a general and vague concept, can be used both to defend the use of data on racial origins and to dismiss it as discriminatory. This standard way to discuss the case is therefore inconclusive, because the concept that is supposed to play the key role in criticizing the tool turns out to be ambiguous.

This case easily lends itself to an alternative approach, along the lines suggested by our theory of legitimacy. Instead of focusing on fairness, our approach suggests that the problem with the tool is not that it is unfair, but that the vision of fairness it presupposes has been imposed without discussion, in an opaque way, without any justification or explanation whatsoever. From the point of view of our theory, the tool is hence illegitimate because relevant criticisms that can be raised against it are swept under the carpet. Implementing our approach to legitimacy here would consist in actively searching for such criticisms (truthfulness conditions), and setting out to convince decision-makers, concerned stakeholders, and other interlocutors that the decisions made, and the associated authorizations and bans, are meaningful and relevant (effectiveness condition)—even if this means, in the end, that some of the procedural features currently structuring the process, or the final decision itself, will have to be adjusted to become more legitimate.

[1] https://www.propublica.org/article/machine-bias-risk-assessments-in-criminal-sentencing.

## 11.5 Hurdles on the Road to Legitimacy

The general theory of legitimacy presented in the former section is simple enough in its formulation. Its basic components (the production of argumentative discourses constituting justifications of decision support interventions, the test of the extent to which these justifications are convincing, and the active search for counter-arguments) are activities in which anyone can engage, more or less successfully. Many practicing decision support providers certainly already engage in these activities, informally and to some limited extent, in their everyday decision support interventions. However, to go beyond such informal, unchecked practices, there is a need to organize this legitimization endeavor in a systematic and formal way. As we will show in this section, despite the *prima facie* simplicity of the activity that consists in producing the kind of justifications structuring the above general theory of legitimacy, its formal systematization is deeply challenging. In this section, we will present what we take to be the two most important hurdles complicating the accomplishment of the legitimization task. These hurdles represent as many avenues for future research on the legitimacy of decision support interventions.

The first, and most evident, challenge is to elaborate operational methods and tools to support the various steps of the production of justifications. This involves elaborating and deploying technologies to search for counter-arguments, including the search for and elicitation of neglected and/or marginalized sources of counter-arguments. Because marginalized sources of arguments typically use means of expression that are different from the mainstream ones, integrating them in argumentative discourses will also generate difficulties. Besides, in cases in which relevant counter-arguments will be numerous and complex (which seems bound to be the general case, except for the most trivial applications), a risk will be that argumentative discourses including all those counter-arguments could become too complex, long, and convoluted to be understandable. There is hence a real challenge to construct readable and accessible argumentative architectures based on such a complex and profuse material. Informal [20, 33] and formal [3, 5, 13] approaches to argumentation theory will certainly prove useful to address these operational challenges. However, as explained by [7], as they stand, these approaches are ill-equipped to address the empirical dimensions of these challenges. This is because this literature does not explore how decision support providers can organize their interactions with decision-makers and other interlocutors, so as to assess how convincing various arguments are, without mystifying them (see [7] and [7, 27], for a deeper exploration of this first research frontier).

A second, perhaps even more difficult hurdle on the road to legitimacy, refers to what one might call "mediation" in justifications of decision support interventions: that is, the intervention of third parties in interactions between producers and receptors of justifications. Two kinds of mediation play a prominent role in many decision support interactions:

- *Representation.* Decision support interactions only rarely involve the direct participation of all the actors potentially concerned by the decision at issue.

Decisions are rather typically made in small circles, including the formal decision-maker(s) and the decision aid provider. Over the last decades, the inclusion of stakeholders in these circles has been increasingly championed (see, e.g., [42]), and the participation of stakeholders in decision-making is now commonplace, through various organizational devices, such as steering committees. In such settings, vast groups of stakeholders are typically represented by a tiny sample of "representatives," including elected representatives, trade unionists, agents working for institutions allegedly representing various issues of public interest, or simply individuals who see themselves and are seen by others as "typical" of a larger group of concerned people.

- *The "nesting" of decision support interactions.* In typical decision support interactions, decision support providers and experts with whom they interact often take advantage of, use or refer to various kinds of outcomes of antecedent or parallel decision support interactions: reports produced when trying to solve similar problems in other contexts, tools such as software or databases constructed in other contexts, methodological reports elaborated on the basis of a series of similar missions, scientific publications, etc. In so doing, decision support providers take the role of decision-makers supported by other decision support providers, who are themselves, for the same reason, decision-makers benefiting from antecedent or parallel decision support interactions. These various decision support interactions can take different forms, and, typically, the higher up we climb the hierarchy, the less interactive the "interaction" will be. For example, as authors of this chapter, we are decision-makers supported by, among others, Pythagoras and Aristotle, as great figures in our intellectual formation. But our "interaction" with them is much less interactive than the one we have with clients for whom we work as decision support providers. Anyways, through references and the usage of tools, decision support interactions are all nested in an infinite series of other, more or less clearly defined and formalized, decision support interactions.

Both aspects of mediation substantially complicate the task to produce legitimizing justifications:

- Representation raises the questions: If we manage to produce a justification that convinces representatives, can we admit without further ado that it is enough? Should not we rather strive to convince those people who are supposed to be represented by the representatives? What if we manage to convince representatives but not represented persons, or the other way round?
- The nesting of decision support interactions raises the questions: how far should we go when we decide on the aspects of the decision support interaction that we should justify? Evidently enough, we cannot set ourselves the requirement to justify each and every aspect of the decision support interaction, since this would mean, for example, that we would have to justify all the aspects of the foundations of the mathematical theories on which our theories and tools are based. But then, how are we to make a choice between the various aspects that could be justified?

At this stage, we do not claim to be able to answer these difficult but unavoidable questions. They constitute major agendas for further research.

## 11.6 Conclusions

Supporting the decision activities of clients (potentially decision-makers) can certainly be characterized by the use of formal and abstract models. However, pragmatically it is more complex than a simple application of such models. Although the topic of model validity has been discussed in the literature and can be based on some formal requirements (such as meaningfulness), there is a problem of model and more generally of decision support legitimacy.

In this chapter, we show that, with the notable exception of some seminal contributions, this topic is essentially neglected and barely developed. Renewing a tradition of discussions that used to animate the meetings of the EURO MCDA Working Group, this chapter aims at suggesting a new perspective on decision support legitimacy.

We have proposed a general theory of the legitimacy of decision support processes and used examples to illustrate the importance of the topic and the application of our theory. At the end of the day, supporting our clients within their decision processes consists in convincing:

– Ourselves that we appropriately used our models and methods
– Our clients that what we suggest and advise makes sense for them
– Any other potential stakeholder, about the potential impact of this advice

Our contribution is far from being exhaustive. Our topic has multiple theoretical and practical extensions and research pathways that could not be explored here. We hope that our broad community will pursue our effort by exploring these other aspects of the topic. Among the questions that researchers should address in this future effort, the most prominent ones are perhaps:

- Which are or can be considered to be legitimate sources of information?
- What does it mean to perform a legitimate information manipulation?
- Who is expected to release a "patent of legitimacy" within a decision support process?

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# Chapter 12
# MR-Sort with Partial Information to Decide Whether to Invest in Innovation Projects

**Philippe Fortemps and Marc Pirlot**

**Abstract** More often than not, writing a funding application for an innovation project requires information that is not yet available. To avoid rejecting incomplete applications a priori, we propose here a variant of the MR-Sort method for dealing with partial information.

It consists of identifying the criteria whose assessment is available and those whose assessment is missing. Depending on the mindset of the decision-maker, a bipolar hierarchy of ordered classes is run through to identify the class to be recommended on the basis of the state of information.

This new proposal keeps the simplicity of use and expressiveness of the original MR-Sort method. It can be applied in contexts where the information is not always initially complete and can be acquired progressively.

## 12.1 Introduction

Today, innovation is a key factor in the development of the economy, both regionally and globally [12, 24, 27, 43]. All sectors of activity are affected: ranging from agriculture [11] to motorsport [40], not to mention climate change, which is a terrible source of innovation challenges [14, 51]. About a century ago, Schumpeter had already identified innovation as a crucial parameter for accelerating economic growth [47]. For him, technological development and innovative entrepreneurs are the real drivers of this growth.

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Innovation can be defined as the application of new ideas and/or emerging technologies to produce products, services or processes that improve the daily lives of customers, the efficiency of industrial production or the performance of a company [41].

A delicate stage in an innovation project is obtaining early funding. At this point, potential investors must be convinced to support a project that is not yet fully defined. Indeed, not only is it an innovation project which, by its very nature, is characterized by many uncertainties as to its evolution. But, at the early stages of development, the project owner cannot even provide the full information that is usually required to build a financing file.

The purpose of this chapter is to propose a method of decision support for investors: faced with an innovation project whose parameters are only partially known, can one already decide to invest in it or to reject it, or should one ask for more information?

It will therefore be a decision support method, capable of handling multiple criteria and assigning submitted projects to ordered classes (from "to reject" to "to fund"). It will work with incomplete information and adapt to different investor profiles and different investors, varying from a banker to a public investment agency. Ultimately, it would be interesting to provide a rationale for a class assignment recommendation, both to convince investors of the relevance of the recommendation and to allow the project owner to focus on the information that would allow a quicker decision.

In the rest of this chapter, we present the methods that exist in the scientific literature and that approach the stated problem (Sect. 12.2). This state of the art identifies the points of attention from which we can build our methodological proposal (Sect. 12.3). Then, we return to the context of investment projects to apply this methodological proposal (Sect. 12.4). Finally, we outline the possibilities for extending the method, both conceptually and through another potential application context (Sect. 12.5).

## 12.2 Multiple Criteria Sorting When Evaluations Are Missing

Methods for assigning objects into ordered classes on the basis of their evaluations w.r.t. several criteria have been proposed since the 1990s. A crucial feature distinguishes such methods in the field of classification. The objects assignment respects the sense of preference on the various criteria. In other terms, an object that is at least as good as another on all criteria cannot be assigned to a worse class. This property is often called "respect of dominance" or "monotonicity".

ELECTRE TRI [46, 53] is among the first and most important methods for sorting objects into ordered classes. It belongs to the family of multi-criteria methods based on an *outranking* relation. The general idea behind outranking is that an object $a$ outranks another object $b$ if there are enough reasons (based on the evaluations of both $a$ and $b$) for saying that $a$ is at least as good as $b$ and there is no aspect on which $a$ is *unacceptably* worse than $b$. In the original version of ELECTRE TRI, classes are defined using special objects called *boundary profiles*. For assigning an object to a class, ELECTRE TRI relies on the set of boundary profiles outranked by the object (respectively, outranking the object). Since its inception, ELECTRE TRI has been very successful. It has been applied in many contexts. Methods for learning its parameters on the basis of assignment examples have been developed (e.g., [38]). Numerous variants of ELECTRE TRI have been proposed, as, e.g., one using reference actions (or objects) instead of boundary profiles [1], another using several boundary profiles associated with each class [17].

Even though we shall not use them, we have to mention that another family of sorting methods in ordered classes has developed within the framework of multi-attribute value functions (MAUT); see, for instance, [21, 29, 54].

When using ELECTRE TRI or one of its variant for sorting objects into ordered classes, usually, the evaluations of the objects are supposed to be known, even though, sometimes, imprecisely. In some contexts, in particular when dealing with innovative projects, some evaluations may simply be lacking. "Project selection decision makers frequently have much less information to evaluate possible innovation projects than they would wish" [32, p. 286].

There are not so many papers that have specifically dealt with missing evaluations in MCDA and in particular in sorting methods. Moreover, most of these papers address this question in the context of *learning* a sorting model on the basis of assignment examples. In such a case, the parameters of the learned model are imprecisely known, since, in general, many models equivalently fit the data. For instance, the *robust approach* to sorting enables to compute the possible and necessary assignments of any object. The assignment of an object to a class is *possible* if there is a model that assigns the object to the class in the family of models equivalently fitting the data. The assignment of an object to a class is necessary if this class is the only possible one. For examples of such an approach, see [21] for sorting by means of additive value functions, [22] for a robust approach to learn outranking relations and [30] for sorting relying on the dominance-based rough sets approach. The case of missing data has been addressed explicitly in the framework of the rough set approach [20].

In this work, we consider that the model has been elicited in an interactive process involving the decision-maker. It is assumed that its parameters are precisely known. The idea of possible and necessary assignments can be adapted in a straightforward manner. A project, with missing evaluations, is possibly (respectively, necessarily) assigned to a class if the model assigns it to this class for some value (respectively, all values) of the missing evaluations.

Actually, missing evaluations can be viewed as a limit case of uncertain or imprecise evaluations, in which the uncertainty or the imprecision of some evaluations is total. Therefore, most methods proposed to deal with imprecise or uncertain data (see, e.g., [16]) can be adapted to missing data. Assuming that the sorting model is known without imprecision, the following proposals can be derived from methods dealing with imprecise or uncertain data.

1. Replace the missing value(s) by a default value, for instance, by the criterion range midpoint or the average of the other alternatives evaluations on the same criterion or the most frequent value taken by objects on the criterion. In the context of methods based on pairwise comparisons, it has been suggested to assume that both objects have the same evaluations on the criteria on which the evaluation of one of (or both) the compared object(s) is lacking [9]. Note that an early experimental comparison of methods for handling missing values in data mining (i.e., classification) has determined that replacing the missing values by the most frequent one was the worst out of ten explored strategies [23]. In our case, it makes little sense to replace a missing evaluation by a single default evaluation, whatever it is.
2. Simulation. Assume a distribution of probability on the range of any criterion on which an evaluation is lacking. Contrary to the case of uncertain evaluations, this distribution should not convey any information on the missing value. Therefore, it should be a uniform distribution on the range of the criterion. Using a given assignment model for objects with missing evaluations replaced by uniformly distributed evaluations generates a probability distribution on the possible assignment classes for the object. The class with the highest probability could then be recommended (applying a maximum likelihood principle). This approach is in the spirit of the SMAA method proposed for dealing with imprecisely known model's parameters [50].
3. The robust approach was introduced above. The concepts of possible and necessary assignments originate from fuzzy logic, more specifically, possibility theory [15]. Again, in case of incomplete data, it is difficult to consider modelling lack of evaluation by anything else but a fuzzy number whose membership function is equal to 1 on the whole range of the related criterion. In such a case, the degree of possibility of the assignment of an object to a class is 0 or 1. The approach proposed in [10], which represents imprecise evaluations by a nested family of intervals, the smaller the less plausible, supposes a certain form of information on the evaluation. The latter is often not available in case of missing data.

All these proposals hardly fit our needs. Both simulation and the robust approach (items 2 and 3 above), in general, lead to several possible assignments for an object with lacking evaluations. The set of classes we have in mind is ordered in a bipolar way. In the middle, we have a neutral class the meaning of which will soon be presented. The classes above the neutral one contain projects for which a substantial set of evaluations pleads in favour of accepting them. Symmetrically, the classes below the neutral one contain projects for which a substantial set of evaluations pleads in favour of rejecting them. The neutral class in the middle collects projects whose status is undetermined: neither validated nor non-validated.[1]

This is reminiscent of the bipolar three-valued (or $(2k + 1)$-valued) logic developed in [4, 5] and used for expressing the credibility of outranking. This three-valued logic is implemented for sorting into ordered classes by using a simple version of ELECTRE TRI in [36]. The authors build two outranking relations, a pessimistic one and an optimistic one.[2] The pessimistic outranking relation is built by replacing the missing values by the range lower bound of the corresponding criteria and using a pessimistic concordance threshold (no veto is considered). The optimistic outranking relation is built by replacing the missing values by the range upper bound and using an optimistic concordance threshold. An upper (respectively, lower) bound class for the assignment of an object is determined by using the optimistic (respectively, pessimistic) outranking relation. Note that there is no special neutral class in this approach and no symmetry between the "good" and the "bad" classes. Also, the assignment rule is the classical pseudo-conjunctive one. In the next section, we describe our own proposal, which bears similarities with [36] but proposes assignment rules adapted to a bipolar ordered set of classes.

## 12.3 MR-Sort with Partial Information

Among the various methods for assigning objects to ordered classes, we consider MR-Sort [34], a simplified variant of ELECTRE TRI [39, 46, 48, 53]. The MR-Sort rule will be first briefly described in the usual case (Sect. 12.3.1). Then, we will propose the model for incomplete information. Section 12.3.2 describes the proposed principle, while Sects. 12.3.3 and 12.3.4 introduce the needed notation. Section 12.3.5 implements the principle with a cautious mindset, while Sect. 12.3.6 performs the same task in a more audacious mindset.

### *12.3.1 A Short Reminder of MR-Sort*

Let $X$ be the set of objects to be assigned into $p$ ordered classes $C_h$, with $h = 1, \ldots, p$, $C_1$ and $C_p$ being, respectively, the worst and the best class. A set

[1] Such a bipolar interpretation of the classes was at the root of trichotomic segmentation methods. The works [37, 45] have proposed procedures for assigning objects into three classes: the definitely good, the definitely bad and a class in-between gathering the objects for which a clear decision cannot be made. The nTOMIC method [35] also has a bipolar set of classes. Such methods can be viewed as forerunners of ELECTRE TRI.

[2] These pessimistic and optimistic outranking relations should not be confused with the pessimistic and optimistic assignment rules used in the ELECTRE TRI method; the latter are now, respectively, called pseudo-conjunctive and pseudo-disjunctive (see [18, pp. 170-171]).

$F$ of $n$ criteria (or ordered attributes) is used to evaluate the objects. Therefore, each object $x \in X$ can be described by a vector $(x_1, \ldots, x_j, \ldots, x_n)$, where $x_j$ is the evaluation of $x$ on the $j$-th criterion (or attribute) $K_j$. The classes are delimited by boundary profiles described using the same set of $n$ criteria. More precisely, each class $C_h$ has a lower boundary profile $b_{h-1}$ and an upper boundary profile $b_h$. Thus described, the lower boundary profile of $C_{h+1}$ is identical to the upper boundary profile of $C_h$, for $0 < h < p$. It is also assumed that the profiles respect the following dominance condition: $b_{h,j} \succcurlyeq_j b_{h-1,j}$ for $h \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$, where $\succcurlyeq_j$ is the preference order on the scale of criterion $j$. The lower boundary profile of $C_1$ and the upper boundary profile of $C_p$ are fictive; they are defined such that $b_0$ is dominated by every object and $b_{p+1}$ dominates every object.

To assign the appropriate class to an object $x$, MR-Sort progressively compares $x$ to each profile and determines the class based on the best profile outranked by $x$ (this corresponds to the pessimistic [46], now called pseudo-conjunctive [18], assignment rule of ELECTRE TRI). This is done by calculating the weight of the coalition of criteria for which $x$ is at least as good as each boundary profile $b_h$ :

$$\sigma(x, b_h) = \sum_{j : x_j \succcurlyeq_j b_{h,j}} w_j, \tag{12.1}$$

where $w_j$ is the non-negative weight of criterion $j$. Generally, the set of weights is chosen to sum up to one: $\sum_{j \in F} w_j = 1$.

We say that $x$ outranks the considered profile $b_h$ (denoted $x S b_h$), if this coalition is considered sufficient:

$$x \; S \; b_h \qquad \Longleftrightarrow \qquad \sigma(x, b_h) \geq \lambda, \tag{12.2}$$

where $\lambda$ is a majority threshold, with $\lambda \in ]1/2, 1]$.

Finally, $x$ is assigned into class $C_h$ (denoted $x \mapsto C_h$) if and only if $x$ outranks the lower boundary profile of $C_h$ and does not outrank its upper boundary profile, i.e.,

$$x \; S \; b_{h-1} \quad \text{and} \quad x \; \not S \; b_h \tag{12.3}$$

or equivalently

$$\sum_{j : x_j \succcurlyeq_j b_{h-1,j}} w_j \geq \lambda \text{ and } \sum_{j : x_j \succcurlyeq_j b_{h,j}} w_j < \lambda. \tag{12.4}$$

The outranking relation $S$ here is a concordance relation [6, 7, 46]. A version of MR-Sort with veto has been considered in [49].

### 12.3.2 Our Proposal

The idea of our proposal is quite simple. We assume first that the parameters of the sorting model are precisely known. On the other hand, some objects that we want to classify are partially described: some evaluations of an object $x$ are known while others are missing. The problem is to guess the class to be assigned to $x$. This should be done by taking into account the information available at that moment, but considering that additional information may arrive.

For example, when considering using a model like MR-Sort for the classification of investment projects, one can assume that the boundary profiles are well known, through interacting with the decision maker. However, the same is not necessarily true for the projects themselves, as project owners may only provide incomplete information. In other words, the projects may not be fully assessed. For a given project, some criteria may already have been evaluated and others not.

To determine the appropriate class for $x$, we start by provisionally assigning it into a neutral class. First, we consider the possibility of moving up towards good classes. If $x$ can move up even without assuming the missing evaluations to be favourable to $x$, we are certain that the actual class of $x$ is better than the neutral class: $x$ can thus surely move up. If $x$ did not succeed in moving up (i.e., $x$ is still in the neutral class), then we consider moving down towards the bad classes. If $x$ has to move down even though assuming the missing evaluations to be favourable to $x$, we are certain that the actual class of $x$ is worse than the neutral class: $x$ can thus surely move down. If $x$ did move neither up nor down, it has to remain in the neutral class, waiting for additional information. Finally, the neutral class contains both candidates about which there is not enough information and medium candidates, i.e., neither good nor bad.

### 12.3.3 The Bipolar Ordered Set of Classes

As in MR-Sort, we work with classes $C_h$ ordered from worst to best. Within this set, one class assumes a particular role that of the neutral class. We denote it $C_0$. Thus, the classes above $C_0$ will be numbered from $C_1$ to $C_p$ and the classes below from $C_{-q}$ to $C_{-1}$.

The set of ordered classes need not be symmetrical with respect to the neutral class $C_0$. In other words, the number of upper classes $p$ may be different from the number of lower classes $q$, depending on the needs of the decision-maker in a given situation. Indeed, according to the latter, it may be appropriate to bring more nuance in one direction of the decision than in the other.

Classes are separated by fully described boundary profiles using the $n$ criteria. More precisely, each class $C_h$ ($h$ ranging from $-q$ to $p$) is delimited by a lower boundary profile $b_h$ and an upper boundary profile $b_{h+1}$. Thus, for example, the class $C_0$ is bounded by $b_0$ (which separates it from $C_{-1}$) and by $b_1$ (which separates it from $C_1$). As with ELECTRE TRI and MR-Sort, profiles respect dominance, i.e., $b_{h,j} \succsim_j b_{h-1,j}$ for $h \in \{1, \dots, p\}$ and $j \in \{1, \dots, n\}$, where $\succsim_j$ is the preference order on the scale of criterion $j$.

## 12.3.4 The Coalition Weights

The idea of a sufficient majority coalition of criteria must be redefined with respect to the available and missing evaluations. We propose to redefine the coalition score $\sigma$ on the basis of criteria that are fully informed. Let us also define an indicator $\tau$ determining the weight of the criteria for which some information is missing.

$$\sigma(a, b) = \sum_{\substack{j: a_j \text{ and } b_j \text{ are known} \\ \text{and } a_j \succsim_j b_j}} w_j \tag{12.5}$$

$$\tau(a, b) = \sum_{j: a_j \text{ or } b_j \text{ is unknown}} w_j. \tag{12.6}$$

If we consider a partially evaluated project $a$ and a fully known profile $b$, the coalition score $\sigma(a, b)$ defined in Eq. (12.5) gives the total weight of the current coalition in favour of $a$ as compared to the profile $b$, based on the available information. Whatever information may arrive later on $a$, the score could only increase, since additional criteria could enter the coalition in favour of $a$ and no criterion could be removed from this coalition. On the other hand, in the same situation, $\tau(a, b)$ provides the total weight of the criteria that could still be involved. But, as the information is not yet available, one cannot certify that it will be in favour of $a$. This represents potential room for improvement.

In other words, $\sigma(a, b)$ provides the currently certain coalition weight (i.e., the coalition that is certainly in favour of $a$ when compared to profile $b$), while $\sigma(a, b) + \tau(a, b)$ represents the maximum coalition weight currently possible (i.e., the maximum coalition that is potentially in favour of $a$ in front of profile $b$), where "currently" means "on the basis of the currently available information". We leave the obvious proof of the following proposition to the reader.

**Proposition 12.1**

$$\sigma(a, b) + \tau(a, b) + \sigma(b, a) = 1 + \iota(a, b) \tag{12.7}$$

$$\tau(a, b) = \tau(b, a), \tag{12.8}$$

*where*

$$\iota(a, b) = \sum_{\substack{j: a_j \text{ and } b_j \text{ are known} \\ \text{and } a_j \sim_j b_j}} w_j. \tag{12.9}$$

### 12.3.5 The Conservative Mindset for Sorting

The conservative mindset is compatible with the pseudo-conjunctive logic, according to which an object can be assigned to a class if it is at least as good as the lower boundary profile of that class, on a sufficient majority of criteria. This majority can be determined on the basis of the scores $\sigma$ and $\tau$ introduced above.

Therefore, we can construct two outranking relations, the first one $S^F$ favourable to $x$ and the second $S^D$ adverse to $x$. On the one hand, $x$ certainly outranks profile $b$ (denoted by $xS^Db$) if its certain score $\sigma(x, b)$ is already sufficient (i.e., at least as large as the majority threshold $\lambda$); such an outranking will be maintained, even if all the unknown information were to turn out to be adverse to $x$. On the other hand, $x$ potentially outranks $b$ (denoted by $xS^Fb$) if its potential score $\sigma(x, b) + \tau(x, b)$ is sufficient; such an outranking can subsequently be revoked by new information.

$$xS^Db \iff \sigma(x, b) \geq \lambda \tag{12.10}$$

$$xS^Fb \iff \sigma(x, b) + \tau(x, b) \geq \lambda, \tag{12.11}$$

or equivalently,

$$x \not S^Db \iff \sigma(x, b) < \lambda \tag{12.12}$$

$$x \not S^Fb \iff \sigma(x, b) + \tau(x, b) < \lambda. \tag{12.13}$$

In other words, $S^F$ is called *favourable outranking*, because it is assumed that $\tau$ will turn to be in favour of $x$. $S^D$ is called *unfavourable outranking*, because the opposite assumption is made: $\tau$ will rather play in disfavour of $x$ (i.e., at least not in favour of $x$).

**Proposition 12.2** *If $x$ outranks a profile $b$ with $\tau$ being unfavourable, then $x$ potentially outranks $b$ with $\tau$ being favourable:*

$$xS^Db \implies xS^Fb, \tag{12.14}$$

*or equivalently,*

$$x \not S^Fb \implies x \not S^Db. \tag{12.15}$$

***Proof*** Indeed, $\sigma(x, b) + \tau(x, b) \geq \sigma(x, b)$, since $\tau(x, b)$ cannot be negative. □

In accordance with our proposal (see Sect. 12.3.2), the assignment of $x$ to a class on the basis of the information available at a given moment starts by positioning $x$ in the neutral class $C_0$ (see Algorithm 1). As long as $x$ manages to outrank the boundary profile $b_h$, based on the unfavourable relation $S^D$, for $h$ ranging from 1 to $p$, $x$ is moved up to the higher class $C_h$. When this ascent attempt ends up, either $x$ has reached a class $C_h$ with $h > 0$ or $x$ has remained in the class $C_0$. In the latter

**Algorithm 1** Assignment based on outranking: $\mapsto_S$

**Require:** A set of profiles $b_h, h : -q+1, \ldots, 0, \ldots, p$
An object $x$ to be assigned into some class

- ▷ *$h$ is the running index of the current class assigned to $x$* ◁
- $h \leftarrow 0$ ▷ *$x$ is initially placed into $C_0$*
- **while** $h < p$ and $x S^D b_{h+1}$ **do** ▷ *While $x$ is allowed to move up*
  - $h \leftarrow h+1$
- **if** $h = 0$ **then** ▷ *If $x$ is still in $C_0$*
  - **while** $h > -q$ and $x \not S^F b_h$ **do** ▷ *While $x$ has to move down*
    - $h \leftarrow h-1$
- $h^* \leftarrow h$ ▷ *$h^*$ is index of the assigned class*
- **output** $x \mapsto_S C_{h^*}$

case, we consider descending: as long as $x$ does not outrank the boundary profile $b_h$, based on the favourable relation $S^F$, for $h$ going from 0 to $-(q-1)$, we make $x$ descend in the lower class $C_{h-1}$. At the end of this descent attempt, either $x$ has reached a class $C_h$ with $h < 0$ or $x$ has remained in $C_0$.

The operation of this algorithm can also be described on the basis of the two extreme classes reachable on the basis of $S^F$ and $S^D$. Indeed, the relation $S^D$ leads to a lower approximation $\underline{h}$, since it considers that all the new pieces of information can be to the disadvantage of $x$. On the other hand, the relation $S^F$ leads to an upper approximation $\overline{h}$. We have

$$x \mapsto_{S^D} C_{\underline{h}} \iff x S^D b_{\underline{h}} \text{ and } x \not S^D b_{\underline{h}+1}, \tag{12.16}$$

$$x \mapsto_{S^F} C_{\overline{h}} \iff x S^F b_{\overline{h}} \text{ and } x \not S^F b_{\overline{h}+1}. \tag{12.17}$$

If we note $C_{h^*}$ the class determined by Algorithm 1, we have $\underline{h} \leq h^* \leq \overline{h}$. It is then easy to show the following.

**Proposition 12.3** *If the unfavourable outranking relationship does not allow $x$ to move into a positive class and the favourable outranking relationship does not force $x$ into a negative class, then the algorithm keeps $x$ in class $C_0$. More precisely,*

- *If $0 < \underline{h}$, then $h^* = \underline{h}$*
- *If $\overline{h} < 0$, then $h^* = \overline{h}$*
- *If $\underline{h} \leq 0 \leq \overline{h}$, then $h^* = 0$*

### *12.3.6 The Audacious Mindset for Sorting*

The procedure described above may appear a bit conservative. Indeed, assigning $x$ into a class $C_h$ requires that this candidate be able to outrank the $b_h$ profile in a pseudo-conjunctive way (i.e., w.r.t. a majority of criteria).

A more audacious decision-maker would like to place $x$ in a $C_h$ class, as long as the $b_h$ profile does not prevent her from doing so. Such a logic corresponds to the pseudo-disjunctive assignment rule in ELECTRE TRI.

Traditionally, the strict outranking relation $P$ is defined as the asymmetric part of the outranking relation $S$: $aPb \iff aSb$ and $b \not{S} a$. In our situation, we need to remember that $\tau(x, b)$ can either be credited in favour of $x$ or in disfavour of it, i.e., in favour of $b$.

A profile $b$ strictly outranks $x$, in a context favourable to $x$, if the certain coalition $\sigma(b, x)$ of $b$ compared to $x$ is sufficient while the enriched coalition $\sigma(x, b) + \tau(x, b)$ of $x$ compared to $b$ is not. We have

$$bP^F x \iff \sigma(b, x) \geq \lambda \text{ and } \sigma(x, b) + \tau(x, b) < \lambda; \quad (12.18)$$

$$bP^D x \iff \sigma(b, x) + \tau(b, x) \geq \lambda \text{ and } \sigma(x, b) < \lambda, \quad (12.19)$$

or equivalently,

$$b \not{P}^F x \iff \sigma(b, x) < \lambda \text{ or } \sigma(x, b) + \tau(x, b) \geq \lambda; \quad (12.20)$$

$$b \not{P}^D x \iff \sigma(b, x) + \tau(b, x) < \lambda \text{ or } \sigma(x, b) \geq \lambda. \quad (12.21)$$

**Proposition 12.4** *If a profile $b$ strictly outranks an object $x$ within a context favourable to $x$, then $b$ strictly outranks $x$ in a context unfavourable to the latter, i.e.,*

$$bP^F x \implies bP^D x. \quad (12.22)$$

*Therefore,*

$$b \not{P}^D x \implies b \not{P}^F x. \quad (12.23)$$

***Proof*** Indeed, $\sigma(b, x) + \tau(b, x) \geq \sigma(b, x)$ and $\sigma(x, b) \leq \sigma(x, b) + \tau(x, b)$, since $\tau(b, x) = \tau(x, b)$ cannot be negative. □

Again, based on our proposal (see Sect. 12.3.2), we derive another algorithm assigning $x$ into a class, using the strict outranking relations $P^D$ and $P^F$ (see Algorithm 2).

Applying the pseudo-disjunctive rule with $P^D$ and $P^F$ would lead to assigning $x$ to two classes defined as follows:

$$x \mapsto_{P^D} C_{\underline{h'}} \iff b_{\underline{h'}} \not{P}^D x \text{ and } b_{\underline{h'}+1} P^D x \quad (12.24)$$

$$x \mapsto_{P^F} C_{\overline{h'}} \iff b_{\overline{h'}} \not{P}^F x \text{ and } b_{\overline{h'}+1} P^F x. \quad (12.25)$$

If we note $C_{h'^*}$ the class determined by Algorithm 2, then $\underline{h'} \leq h'^* \leq \overline{h'}$. A proposition similar to Proposition 12.3 can be written regarding Algorithm 2.

**Algorithm 2** Assignment based on strict outranking: $\mapsto_P$

**Require:** A set of profiles $b_h, h : -q+1, \dots, 0, \dots, p$
An object $x$ to be assigned into some class

▷ *$h$ is the running index of the current class assigned to $x$* ◁
$h \leftarrow 0$ ▷ *$x$ is initially placed in $C_0$*
**while** $h < p$ and $b_{h+1} \not{P}^D x$ **do** ▷ *While $x$ is allowed to move up*
  $h \leftarrow h + 1$
**if** $h = 0$ **then** ▷ *If $x$ is still in $C_0$*
  **while** $h > -q$ and $b_h P^F x$ **do** ▷ *While $x$ has to move down*
    $h \leftarrow h - 1$
$h'^* \leftarrow h$ ▷ *$h'^*$ is index of the assigned class*
**output** $x \mapsto_P C_{h'^*}$

**Proposition 12.5** *Applying Algorithm 2 for assigning $x$, we have*

- *If $0 < \underline{h'}$, then $h'^* = \underline{h'}$*
- *If $\overline{h'} < 0$, then $h'^* = \overline{h'}$*
- *If $\underline{h'} \leq 0 \leq \overline{h'}$, then $h'^* = 0$*

Finally, one can show that the pseudo-disjunctive assignment is indeed more audacious than the pseudo-conjunctive assignment.

**Proposition 12.6** *The class provided by the audacious mindset is at least as good as that provided by the cautious mindset.*

- *If $x \mapsto_S C_{h^*}$ (see Algorithm 1) and $x \mapsto_P C_{h'^*}$ (see Algorithm 2), then $h'^* \geq h^*$.*

***Proof*** We know that $\underline{h} \leq \underline{h'}$ and $\overline{h} \leq \overline{h'}$ (see, for instance, [8, p. 382]). Using Propositions 12.3 and 12.5, we get the following:

- If $\underline{h} > 0$, then $h^* = \underline{h} \leq \underline{h'} = h'^*$;
- If $\overline{h'} < 0$, then $h'^* = \overline{h'} \geq \overline{h} = h^*$;
- If $\overline{h'} \geq 0$ and $\underline{h} \leq 0$, we have:
  - if $\underline{h'} \leq 0$, then $h'^* = 0$ and

$$\text{if } \overline{h} \geq 0, \text{ then } h^* = 0 = h'^*;$$

$$\text{if } \overline{h} < 0, \text{ then } h^* = \overline{h} < h'^*;$$

  - if $\underline{h'} > 0$, then $h'^* = \underline{h'} > 0$ and

$$\text{if } \overline{h} \geq 0, \text{ then } h^* = 0 < h'^*;$$

$$\text{if } \overline{h} < 0, \text{ then } h^* = \overline{h} < h'^*.$$

□

## 12.4 Application to the Evaluation of Innovation Projects

Doblin's framework [31] distinguishes 10 aspects that can give rise to innovation. Each of these aspects can be measured by different criteria. An exhaustive list of these criteria is beyond the scope of the current article (see [26, 28, 33]). We present a few of them, before reporting on the application of the approach with a potential investor.

The TRL (Technology Readiness Level) is a well-known maturity indicator [25]. Proposed by NASA in the 1970s, it characterizes the state of the technology needed for the project on a scale ranging from 1 (basic principles observed) to 9 (actual system proven in an operational environment). Similarly, maturity indicators have been constructed for an innovation project (IRL—Innovation Readiness Level), for industrialization processes (MRL—Manufacturing Readiness Level) and for marketing conditions (CRL—Customer Readiness Level or BRL—Business Readiness Level). In addition to the maturity indicators, there are also more traditional criteria such as NPV (Net Present Value), ROI (Return On Investment), market potential etc.

In a recent study conducted by one of our students [44], decision-makers (DMs) were able to express their views on how to proceed. Their daily task is to evaluate innovation projects, with a view to potentially investing financial or human resources. Among the investors, different profiles can be distinguished: bankers, public investors, entrepreneurship stimulation agencies etc. Of course, a banker takes a more cautious approach than a public investor. Therefore, the conservative mindset (Sect. 12.3.5) is more in line with the former, while the latter is more likely to adopt the audacious mindset (Sect. 12.3.6).

We will describe here the way of thinking of a banker $B$, as it is quite representative. Other DMs operate similarly, except for the choice of mindset (conservative vs audacious), criteria and boundary profiles, criteria weights $w_j$ and coalition threshold ($\lambda$). They may also have another hierarchy of ordered classes, but each has at least 3 classes: reject, neutral and fund.

To decide on financing, the banker bases her decision on three criteria

$K_1$ Reliability of the project owner (PO): this DM assesses the reliability of the PO on the basis of her banking history: has she ever been bankrupt? Is she currently repaying loans (private or professional)?

$K_2$ Her knowledge of the sector of economic activities: it is based on the self-assessment by the PO, but also on the assessment by the banker of the PO and her professional network.

$K_3$ Viability of the project: as seen by the banker, it is the ability of the PO to repay. On the one hand, it is evaluated on the basis of the ratio between the initial financial contribution of the PO and the total capital needed for the project. On the other hand, it is also assessed using the project margin rate; indeed, this margin rate reflects what remains available to companies to remunerate capital and invest. The margin rate of the project is compared to the mean margin rate of the targeted sector of economic activities.

**Fig. 12.1** Merging two indicators for project owner's reliability $K_1$

| | | Past bankruptcies | | | | |
|---|---|---|---|---|---|---|
| | | ? | 0 | 1 | 2 | 3+ |
| Active credits | ? | ? | G | G | M | B |
| | 0 | G | G | G | M | B |
| | 1 | G | G | M | M | B |
| | 2 | M | M | M | B | B |
| | 3+ | B | B | B | B | B |

**Fig. 12.2** Merging two indicators for project viability $K_3$: PO's contribution is related to the overall needed budget; the margin rate of the project is compared to that of the sector of economic activities

| | | PO's contribution | | | |
|---|---|---|---|---|---|
| | | ? | $\leq 5\%$ | $5-10\%$ | $\geq 10\%$ |
| Margin rate | ? | ? | B | M | M |
| | $\leq 85\%$ | B | B | B | B |
| | $85-100\%$ | M | B | M | G |
| | $\geq 100\%$ | M | B | G | G |

While one of the criteria ($K_2$) can be assessed quite simply (market knowledge is described by an indicator with 4 values: "Bad", "Medium", "Good" and "Very Good"), each of the other two criteria is described by two indicators.

Let us consider first the reliability of the project owner $K_1$. It is assessed on the basis of the number of bankruptcies already suffered by the PO and the number of her active credits. The number of bankruptcies and the number of credits can be equal to 0, 1, 2 or strictly greater than 2. Of course, information about them may also not be available. Usually, both pieces of information are available or missing at the same time (Fig. 12.1). If both pieces of information are missing, then the criterion is considered as missing. If only one of the two pieces of information is available, the criterion is assessed on the basis of the latter. If one of the two indicators is bad (i.e., greater than or equal to 3) or if the sum of the two indicators is too high (i.e., greater than or equal to 4), the criterion will have a "Bad" value. If the sum of the two indicators is low (i.e., less than 1), the criterion will receive the value "Good". In all other cases, it will have a "Medium" value.

Let us now consider the viability of the project $K_3$, as it is done by the banker (see Fig. 12.2). The first indicator is the percentage of the overall project budget brought in by the PO. Ideally, this percentage should be at least 10%. If it is less than 5%, this is a problem. The second indicator describes whether the margin rate expected for the project is sufficiently high w.r.t. the mean margin rate of the sector. As soon as one of the indicators is bad, the criterion has value "Bad". A "Good" value is only achieved by the criterion if at least one of the two indicators is good, and the other is at least medium. In the other cases, the criterion will have value "Medium", except in the case where the two missing indicators lead to a "Missing" value of the criterion.

In the above-mentioned study [44], it has been found that the majority of the criteria for all the DMs are the result of a similar merging of two indicators. In each case, a simple table can be used to deduce the value of the criterion based on the relevant indicators.

The banker considers that her three criteria are important, but that the first one is crucial. It was therefore decided to set the following weights: $w_1 = 0.5$, $w_2 = 0.25$ and $w_3 = 0.25$ with a majority threshold $\lambda = 0.7$. Thus, criterion 1 is necessary to pass the majority threshold; there are two minimum sufficient coalitions: $(K_1, K_2)$ and $(K_1, K_3)$. Indeed, these are the minimum combinations of criteria whose sum of weights exceeds the majority threshold. To divide the projects into 3 classes, $C_{-1}$ (to be rejected), $C_0$ (neutral) and $C_1$ (to be financed), it expresses two boundary profiles: $b_0 = [M, M, M]$ and $b_1 = [G, G, G]$.

Let us now consider the treatment of a project on which there is no information, within the conservative mindset (Sect. 12.3.5). The algorithm described in Algorithm 1 provides the following results: $h^* = 0$. Indeed, this situation is characterized by a maximum imprecision on the class to be assigned $\underline{h} = -1$ and $\overline{h} = 1$, all the criteria being "Unknown". In other words, in the absence of any information, the banker cannot decide anything, except to ask for more information.

If the banker learns that the PO has already filed for bankruptcy and still has an active loan, $K_1$ takes the value Medium. And in this case, the value provided by the algorithm remains the medium class $h^* = 0$. However, we can already know that it will be impossible to reach class $C_1$. Indeed, the uncertainty is reduced and $\overline{h} = 0$. It remains possible to reject the project or to have a medium project. If, in addition, it is found that the PO has no knowledge of the market ($K_2$ takes value 0) and does not contribute enough to the initial project budget (less than or equal to 5% of the overall budget), the banker will make the decision of refusing the project. Indeed, in this case, $h_* = -1$.

For another project, it is enough to know that the PO has already completed at least 3 bankruptcies or has at least 3 active credits. The banker will never finance this project, whatever the quality of the project w.r.t. the other criteria. Indeed, with this fragmentary information, she already knows that $K_1$ takes a zero value. And this is enough to definitely assign the project into class $C_{-1}$ (to be rejected).

## 12.5 Conclusion and Discussion

The adaptation of MR-Sort to incomplete evaluations leads to a simple and intuitive methodology. The available information is fully exploited, and no assumptions (e.g., of statistical nature) are made about what values the missing evaluation would take on at any given time. This approach also makes it possible to represent two types of decision-maker behaviours, either conservative (i.e., pseudo-conjunctive) or audacious (i.e., pseudo-disjunctive).

In the context of an innovation project, this allows decisions to be made early in the development of a project, without the need to collect all pieces of information.

And when there is still a lack of clarity about the decision, it helps to point out the information that needs to be collected. Indeed, the simplicity of the model makes it clear which available or missing evaluations drive the decision. In some cases, this may even allow the PO to review her project and identify on which criteria it should be improved.

Of course, there are many other contexts where this methodology can be useful, as Bernard Roy had already identified [45].

The most sensitive context is probably that of medical diagnosis assistance. For example, in the time of an epidemic, when a patient comes to the hospital or to the doctor's office, there is initially no information about her health status. It is therefore impossible to categorize her as healthy or sick. Of course, this discussion is multiplied when we consider not only one possible pathology, but all known pathologies.

To consider a priori that the patient is healthy is to ignore her request to be heard on the symptoms or discomforts. It also increases the risk of spreading a potential contagious disease or the risk of aggravation for the individual. On the other hand, considering her a priori as a sick individual can be harmful for the patient and costly for the healthcare system. It is therefore prudent to consider the patient a priori in a neutral class, requiring more information.

Each time a piece of information is obtained (for example, the identification of a symptom or the result of an examination), it enriches the decision. When the doctor has accumulated enough information, for or against good health, she can make a diagnosis. In the meantime, the range of decision imprecision can be gradually reduced.

In this chapter, we present an approach with classes that are separated by simple boundary profiles, each of which is described by a single vector. However, in the spirit of [17], it is possible to extend our proposal by describing each profile by several vectors. This increases the expressiveness of the model. However, the determination of these different vectors is not an easy task for the decision-maker. It therefore seems appropriate to try first the simple model presented here, before considering multiple profiles to delimit the ordered classes.

The neutral class has a very special meaning in our model, since it gathers objects for which a decision is impossible. This can happen because of a lack of information or, conversely, because of an excess of contradictory information. It would therefore be interesting to study this class further using the quadrivalent logic of Belnap [2, 3, 52]. The latter has been extended by [42] in a continuous way for preference modelling from possibly incomplete or conflicting sources of information. Their model admits an infinite number of truth values in the convex hull of four reference values {T—true, F—false, K—contradictory, U—unknown}. Ignatius of Loyola's (1548) advice on "how to make a good choice" [13] inspired another extension of Belnap's logic, to handle positive and negative arguments for the ranking of a finite set of alternatives [19]. By giving the arguments an interpretation in terms of necessity measures of truthfulness and falsity (see Possibility Theory [15]), the authors handle the arguments in an ordinal way. Both extensions could be useful to analyse the central class and to distinguish between objects that are unknown and objects that have contradictory data. Objects in the neutral class with a high value of $\tau$ fall under the logic value U (unknown), while those with a low value of $\tau$ fall rather under the logic value K (contradictory).

Our approach has quite strong links with the proposal of [36]. In the latter study, missing assessments are replaced with either the minimum or the maximum value of the domain of the criterion concerned. There are therefore two versions for each object, an optimistic version and a pessimistic version, leading to the assignment into two classes. These two classes define a class interval for the object with missing values. In our approach, when $x$ benefits from the weight $\tau$ of the criteria without information, it amounts to considering an optimistic version of $x$. Otherwise, we consider a pessimistic version of $x$. We also obtain a class interval $[\underline{h}, \overline{h}]$. However, our approach differs first by the internal mechanism, which is based on the weights of the criteria and does not require to determine the extreme values of the criteria. Furthermore, we make a recommendation $h^*$, which does not ignore the class interval but allows to clarify the situation. Finally, our approach can work both in a conservative (pseudo-conjunctive) mindset and in an audacious (pseudo-disjunctive) one. It can therefore be adapted to decision-makers with different reasoning.

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# Chapter 13
# Meta-Rankings of Journals Publishing Multiple Criteria Decision Aiding Research: Benefit-of-Doubt Composite Indicators for Heterogeneous Qualitative Scales

**Luis C. Dias and Miłosz Kadziński**

**Abstract** This chapter examines the possibility of deriving journal rankings based on multiple ratings from expert panels using different qualitative scales. Such comprehensive rankings are constructed using Benefit-of-Doubt (BoD) composite indicators derived from Data Envelopment Analysis (DEA). Specifically, we consider a value-based efficiency model. An analogous BoD composite indicator can be built considering an additive aggregation model with incomplete information. To illustrate the proposed approach, 50 journals publishing Multiple Criteria Decision Aiding (MCDA) research are ranked based on ratings provided by panels from different geographies (Australia, Brazil, China, Finland, Poland, and the United Kingdom). We construct various rankings depending on whether each alternative minimizes its maximal regret or maximizes its absolute comprehensive value. Moreover, we contrast the elitist and anti-elitist perspectives that emphasize the best classifications or assume the difference between levels decreases as one moves toward better assessments. The results confirm high consistency in the orders of MCDA-related journals irrespective of the modeling options. In particular, the European Journal of Operational Research always appears at the top.

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## 13.1 Introduction

This chapter addresses the problem of assessing a set of alternatives evaluated on heterogeneous qualitative scales. As a particular case, but without loss of generality, we consider the problem of deriving a comprehensive score for academic journals based on qualitative appreciations from different evaluation panels. Specifically, given the context of this book and the scientific area of the chapter's authors, the journals considered are those most open to Multiple Criteria Decision Aiding/Analysis (MCDA) [3, 8, 15] research.

The assessment of scientific journals is widely used in academia for evaluating the journals' impact, prestige, and position in a given field. The results of such an assessment are relevant for scholars to decide where to publish. Indeed, publishing in highly rated journals allows researchers to reach a better audience and boost their academic careers. Such outcomes are also important for librarians to decide which journals to subscribe to and for editors and publishers to define their policies. Moreover, they are commonly accounted for when evaluating the performance of researchers for hiring, tenure, promotion, and resource allocation decisions, as well as when evaluating research centers and universities [4].

There exist different ways of assessing the quality of journals. In particular, several metrics have been proposed for this purpose, serving as a proxy of journals' influence on the academic community. The most notable ones include Impact Factor used by Web of Science, Scopus-driven CiteScore, h5-index employed by Google Scholar, and SCImago Journal Score. They represent mostly citation-based measures based on the analysis of all papers published in a given journal over a certain period, possibly enriched with a perception of the importance of journals where such citations come from. Another approach for evaluating the quality of journals has been through surveys of academic leaders or committee votes. Nowadays, it is common to combine hard objective data (bibliometric indicators) and more subjective panel opinions that can be interpreted as revealed and stated preferences, respectively [19]. Other authors have proposed inferring journal quality by indirect evidence, such as from examining past promotion decisions [2].

Often, the journal assessments take the form of qualitative ratings (e.g., A, B, C, etc.) that provide an ordinal sorting or classification of the journal set (even if these are often called journal rankings). Such ratings have been developed by higher education and research institutions to evaluate their own outputs, but also by larger associations (e.g., The Association of Business Schools in the United Kingdom [22]) or individual researchers (e.g., [16], as one of the first rating proposals). In a broader Operations Research and Management Science field encompassing MCDA, one may cite the proposals of Olson [23] (a survey of faculty members), Donohue and Fox [12] (composite indicator), as well as Cheang et al. [5] and Xu et al. [29] (the PageRank method).

Let us remark that there exist some well-document studies on the negative consequences of using journal rankings [20] as well as manifestos for judging the contents rather than the outlet of a publication, such as the Leiden Manifesto [17] or the San Francisco Declaration on Research Assessment [28]. In this chapter, we do not wish to enter the debate about the advantages and disadvantages of different approaches for deriving the journal ratings [4, 12, 21, 22] nor the issues associated with the potential side effects of these assessments [20, 24]. In turn, we are interested in approaches that combine different qualitative ratings.

Combining different rankings or ratings has been addressed using MCDA and Data Envelopment Analysis (DEA) models. When it comes to the MCDA approaches, they aggregate the rankings or ratings, considering each one to be a different criterion. In an early MCDA type of approach for this purpose, Donohue and Fox [12] aggregate the ranking positions of decision and management science journals through a simple weighted sum, using a weighting vector chosen by the authors. More recently, Yuan et al. [30] combine different ratings using a weighted sum after transforming each rating level into a percentile score. The same authors apply TOPSIS to aggregate bibliometric indicators and select the weights for these two methods using a mathematical program that minimizes the differences between them. Other researchers see the problem of ranking or rating aggregation as a computation of a consensus ranking. In particular, Theu$\beta$l et al. [26] and Aledo et al. [1] use mathematical optimization to obtain a consensus ranking that minimizes the sum of its difference to the individual rankings. The use of social choice aggregation methods such as the Borda count has also been envisaged and compared with the previous approach [1]. Consensus rankings and the Borda count assume all rankings have the same weight.

Some DEA models have also been proposed to aggregate different journal quality indicators. Rather than assuming each rating has the same weight, DEA allows each journal to select the weights that make it compare to its peers under the best possible light. The weights are therefore determined by the DEA models in a data-driven way rather than by an evaluator. This type of reasoning has been originally applied to aggregate quality scores (cardinal information) of operations management journals [13] (a more recent DEA assessment of operations management journals based on cardinal bibliometric information is [25]). In our context of aggregating qualitative ratings, Tüselmann et al. [27] aggregates ten qualitative scales plus a cardinal one (journal impact factor) using a DEA model. The DEA model they used allows each journal the freedom to select the weight of each ranking and the worth (a cardinal value) associated with each level in the qualitative scales. When doing so, they assumed the “convexity constraint” (henceforth named “elitist”), implying that the difference between the neighboring levels increases as one moves in the direction of better levels.

In this chapter, we propose using DEA for deriving a composite indicator of journal quality based on various ordinal ratings. In this way, we overcome the absence of an undisputable vector of weights by implementing the Benefit-of-Doubt (BoD) perspective [7]. That is, the weighting problem is handled for each journal individually, and the weights are endogenously determined by looking at the journal’s strengths and weaknesses relative to other considered journals. We adopt such a data-oriented model to perform relative (efficiency) assessment of journals similarly to [27] and the idea of minimizing the maximum regret in additive aggregation models under incomplete (imprecise, partial) information [10]. In addition, we complement it with an absolute BoD assessment that seeks to maximize the comprehensive score of each journal, independently of the remaining ones, allowing each journal to select the best parameter values maximizing its value [10]. When performing the analysis, we consider a few minimal constraints on the weights and obtain results with and without the *elitist* constraint that emphasizes the best classifications. We contrast them with an *anti-elitist* perspective where the difference between levels decreases as one moves in the direction of better levels. The proposed family of models is applied for evaluating 50 journals publishing MCDA research. A comprehensive ranking is derived from the ratings provided by expert panels from six different geographies: Australia, Brazil, China, Finland, Poland, and the United Kingdom.

## 13.2 Benefit-of-Doubt Perspectives for Heterogeneous Qualitative Scales

The BoD approach has been proposed as a potential solution to the problem of weighting multiple indicators to obtain a composite indicator when no consensus exists about the weights to be used [7]. According to the BoD principle, instead of assuming equal weights or eliciting weights from experts or stakeholders, each alternative can be evaluated by the weights it would choose, i.e., with the weighting vector that makes it look as good as possible. The BoD DEA model [7] consists in a relative evaluation, in which each alternative is compared to its peers that define a Pareto–Koopmans efficiency frontier. In this work, we further exploit the BoD perspective considering three different models. Two of them perform a relative evaluation analogous to DEA, whereas a third one performs an absolute evaluation. We show that the latter is equivalent to the previous approaches when considering an ideal alternative.

The BoD models are typically concerned about weighting the indicators. However, they can also be adapted to set the value differences between performance levels when the indicators are provided on a qualitative scale. In our adaptation, besides choosing the weights, each alternative can also select such value differences. This is attained in the way that benefits each alternative the most, in the spirit of BoD. Thus, the different levels are not assumed to be equidistant.

Allowing complete freedom for each alternative to select values and weights would enable choices that would be hardly acceptable, such as considering only one indicator (putting all the weight for that indicator and null weights for the other indicators) or considering that different levels have no difference in value. Thus, we incorporated two reasonable constraints in all models. The first one is that no indicator alone can have more weight than all other indicators together (a non-dictatorship condition, see [11]). The other one is that a better level must have strictly more value than a lower level. Rather than assuming the qualitative levels are equidistant in terms of value, we present three variants:

- A *free* variant allows any strictly increasing values for the successively better levels.
- An *elitist* variant places a higher value difference for consecutive better levels than for lower levels, thus overvaluing the top-ranked journals, as assumed by Tüselmann et al. [27].
- An *anti-elitist* variant does the reverse, placing a higher value difference for lower levels than for higher levels, thus particularly penalizing the journals with the lowest level (in our case, journals not recognized by the indicator).

### *13.2.1 Notation and Common Elements*

The notation used in this chapter and the common elements in the forthcoming models are following:

- $n$ is the number of indicators (the qualitative journal panels).
- $t$ is the number of alternatives (the journals being assessed).
- $r_{p,1}, \ldots, r_{p,L(p)}$ denote the qualitative scale levels used by indicator $p$, with $L(p)$ being the best level on this indicator. Some journals have not been recognized by all the panels behind the ratings used in this work. We assume that level 1 (the worst level) corresponds to not being recognized by the panel.
- $a_1, \ldots, a_t$ denotes the list of alternatives being evaluated.
- $s_p(a_i) \in \{r_{p,1}, \ldots, r_{p,L(p)}\}$ denotes the assessment of alternative $a_i$ on indicator $p$.
- $v_{p,1}, \ldots, v_{p,L(p)}$ denote the quantitative scores associated with $r_{p,1}, \ldots, r_{p,L(p)}$. Each score (using a typical variable transformation) corresponds to the quantitative value of the qualitative level on the indicator multiplied by the weight associated with this indicator.
- A comprehensive value of alternative $a_i$ is given by

$$V(a_i) = \sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l} \tag{13.1}$$

with

$$m_{i,p,l} = \begin{cases} 1, \text{ if } s_p(a_i) = r_{p,l} \ (a_i \text{ is rated at level } l \text{ on indicator } p), \\ 0, \quad \text{otherwise.} \end{cases} \tag{13.2}$$

Note: The weight of each factor $p$ is implicitly defined as $\frac{v_{p,L(p)}}{\sum_{q=1}^{n} v_{q,L(q)}}$.

The models presented in the following subsections will consider the quantitative scores associated with the qualitative levels, $v_{1,1}, \ldots, v_{1,L(1)}, \ldots, v_{n,1}, \ldots, v_{n,L(n)}$, as decision variables, which need to comply with the following constraints. First, a normalization is needed. Otherwise, the value functions are not bounded. Therefore, we consider

$$\sum_{p=1}^{n} v_{p,1} = 0, \tag{13.3}$$

$$\sum_{p=1}^{n} v_{p,L(p)} = 100. \tag{13.4}$$

Then, we avoid that one indicator has more weight than all other ones considered jointly, and we ensure a minimum weight for each indicator by assuming

$$v_{1,L(1)}, \ldots, v_{n,L(n)} \geq \frac{100}{2(n-1)}. \tag{13.5}$$

Given constraints (13.4) and (13.5), even if $n-1$ indicators are assigned the minimum possible weights, the remaining indicator can have a maximum weight of $100 - (n-1)\frac{100}{2(n-1)} = 50$ (out of 100).

Other constraints are needed to bound the differences between consecutive levels. They depend on the variant considered:

- For the *free* variant, one only requires that two consecutive levels cannot have the same value, demanding a minimum value increase of $\epsilon$ (in this work, we consider $\epsilon = 1$ on a value scale 0–100), i.e.,

$$v_{p,l} - v_{p,l-1} \geq \epsilon, \forall p \in \{1, \ldots, n\}, \forall l \in \{2, \ldots, L(p)\}. \tag{13.6}$$

- For the *elitist* variant, when moving toward better levels the value differences cannot decrease, i.e.,

$$\begin{cases} v_{p,2} - v_{p,1} \geq \epsilon \\ v_{p,l} - v_{p,l-1} \geq v_{p,l-1} - v_{p,l-2}, \forall p \in \{1, \ldots, n\}, \forall l \in \{3, \ldots, L(p)\}. \end{cases} \tag{13.7}$$

- For the *anti-elitist* variant, when moving toward better levels the value differences cannot increase, i.e.,

$$\begin{cases} v_{p,L(p)} - v_{p,L(p)-1} \geq \epsilon \\ v_{p,l} - v_{p,l-1} \leq v_{p,l-1} - v_{p,l-2}, \forall p \in \{1, \ldots, n\}, \forall l \in \{3, \ldots, L(p)\}. \end{cases} \tag{13.8}$$

The models discussed in the following subsections generalize the Value-Based DEA model [14] to the case of qualitative levels. Thus, they are based on the concept of value difference for an undetermined (free to choose) value function, rather than based on an outputs-to-inputs ratio as other DEA models that are also able to cope with qualitative and ordinal scales [6, 9].

### *13.2.2 Model 1: BoD Minimize Regret*

The first BoD model finds the level scores that minimize the maximum regret [10, 14]. The regret is interpreted as the difference between the value of the alternative under evaluation and the best alternative when considering the value function that favors the former the most. For each alternative $a_i$, it is possible to find a value function (defined by $v_{1,1}, \ldots, v_{1,L(1)}, \ldots, v_{n,1}, \ldots, v_{n,L(n)}$) minimizing its regret using Linear Programming (LP):

**M1**:
VARIABLES: $v_{1,1}, \ldots, v_{1,L(1)}, \ldots, v_{n,1}, \ldots, v_{n,L(n)}, \tau$
MINIMIZE $\tau$
SUBJECT TO:

$$\sum_{p=1}^{n}\sum_{l=1}^{L(p)} m_{i,p,l}v_{p,l} + \tau \geq \sum_{p=1}^{n}\sum_{l=1}^{L(p)} m_{j,p,l}v_{p,l}, \text{ for } j = 1, \ldots, t,$$

Constraints (13.3)–(13.5),

Constraints (13.6), (13.7), or (13.8) (depending on the variant),

$$\tau \geq 0.$$

At the optimum of M1, if $\tau^* = 0$, then alternative $a_i$ under evaluation is deemed to be efficient, i.e., there is a value function such that $V(a_i) = \sum_{p=1}^{n}\sum_{l=1}^{L(p)} m_{i,p,l}v_{p,l}$ is at least as good as the value of all other alternatives. Otherwise, $a_i$ is deemed inefficient, and the value difference (i.e., the regret) $\tau^*$ is a measure for its inefficiency. The binding constraints in the first set, i.e., constraints such that $V(a_i) + \tau = V(a_j)$ identify the alternatives $a_j$ for which the difference is minimal, which are the efficient peers for the optimal value function chosen by alternative $a_i$.

### 13.2.3 Model 2: BoD Minimize Regret without Trade-Offs

The second model (M2) limits the set of alternatives with which the alternative under evaluation, $a_i$, is compared. Namely, it considers only potential peers that dominate it (in a multiple criteria sense). That is, each alternative only compares itself with alternatives that are not rated worse on any indicator. This corresponds to a No-Trade-offs perspective, according to which being worse on some indicator cannot be compensated by being better on another one. Hence M2 is similar to M1, except that fewer alternatives are considered in the first constraint group:

**M2**:
VARIABLES: $v_{1,1}, \ldots, v_{1,L(1)}, \ldots, v_{n,1}, \ldots, v_{n,L(n)}, \sigma$
MINIMIZE $\sigma$
SUBJECT TO:

$$\sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l} + \sigma \geq \sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{j,p,l} v_{p,l},$$

$$\text{for } j \in \{1, \ldots, t : s_p(a_j) \succeq s_p(a_i), \forall p\},$$

Constraints (13.3)–(13.5),

Constraints (13.6), (13.7), or (13.8) (depending on the variant),

$\sigma \geq 0.$

**Proposition 1** *Let us denote the optimal solutions of models M1 and M2 for alternative $a_i$ by $\tau^*$ and $\sigma^*$, respectively. Then, $\sigma^* \leq \tau^*$.*

***Proof*** According to the LP theory, since the objective of both M1 and M2 is minimization and the feasible solutions for M1 are a subset of the feasible solutions for M2, the optimal value of the more constrained problem (M1) cannot be less than the optimal value of the less constrained model (M2).

### 13.2.4 Model 3: BoD Maximum Value

The third model (M3) takes the BoD perspective to find the value function that maximizes the absolute value of each alternative, regardless of the other alternatives. Again, an LP formulation yields the desired result:

**M3**:
VARIABLES: $v_{1,1}, \ldots, v_{1,L(1)}, \ldots, v_{n,1}, \ldots, v_{n,L(n)}$
MAXIMIZE $v = \sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l}$
SUBJECT TO:

$$\text{Constraints (13.3)–(13.5)},$$

$$\text{Constraints (13.6), (13.7), or (13.8) (depending on the variant).}$$

This corresponds to a BoD evaluation of each alternative independent of the other alternatives (i.e., absolute rather than relative). Yet, it corresponds to the previous models if all alternatives are compared to an ideal alternative having the best level on all indicators.

**Proposition 2** *Let us denote the optimal solutions of models M1 and M3 for alternative $a_i$ by $\tau^*$ and $\nu^*$, respectively. Then, $\tau^* \leq 100 - \nu^*$.*

***Proof*** Considering $\mu = 100 - \nu$, M3 is equivalent to

MAXIMIZE 100-$\mu$
SUBJECT TO:

$$100 - \mu = \sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l},$$

$$\text{Constraints (13.3)–(13.5)},$$

$$\text{Constraints (13.6), (13.7), or (13.8) (depending on the variant)},$$

$$\mu \geq 0.$$

which is equivalent to

MINIMIZE $\mu$
SUBJECT TO:

$$\sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l} + \mu = 100,$$

$$\text{Constraints (13.3)–(13.5)},$$

$$\text{Constraints (13.6), (13.7), or (13.8) (depending on the variant)},$$

$$\mu \geq 0.$$

According to the LP theory, since the objective is minimization and the feasible solutions for the above LP, being equivalent to M3, are a subset of the feasible solutions for M1, the optimal value of the more constrained problem (the above LP) cannot be less than the optimal value or the less constrained model.

**Proposition 3** *If one alternative (possibly fictitious) has the best level on every factor, then the three models are equivalent, in the sense that $\tau^* = \sigma^* = 100 - \nu^*$.*

***Proof*** Suppose $a_I$ is an ideal alternative such that $s_p(a_I) = r_{p,L(p)}, \forall p$. Then, it is a potential peer for all other alternatives in M1 and M2, as $s_p(a_I) \succeq s_p(a_i), \forall p$. Moreover, per constraint (13.4), $V(a_I) = \sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{I,p,l} v_{p,l} = \sum_{p=1}^{n} \sum_{l=1}^{L(p)} v_{p,L(p)} = 100$. Hence, the value of this alternative is maximum and $V(a_I) \geq V(a_j), j = 1, \ldots, t$. In these conditions, both M1 and M2 can be simplified as

MINIMIZE $\sigma$

SUBJECT TO:

$$\sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l} + \sigma = 100,$$

Constraints (13.3)–(13.5),

Constraints (13.6), (13.7), or (13.8) (depending on the variant),

$$\sigma \geq 0.$$

Therefore, minimizing $\sigma$ (or $\tau$) amounts to minimizing $100 - \sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l}$, and the solution that minimizes $\sigma$ maximizes $\sum_{p=1}^{n} \sum_{l=1}^{L(p)} m_{i,p,l} v_{p,l}$, and vice versa.

## 13.3 Selection of MCDA-Related Journals and Rankings

This work is focused on journals publishing MCDA research (including multiple objective programming). The set of journals to be assessed (see Table 13.1) resulted from the following process:

- First, we searched for journals in the Clarivate Web of Science database, in the Operations Research & Management Science (ORMS) category. For each of these journals, we divided the number of articles published since 2017 that mention "multi-criteria," "multi-attribute," or "multi-objective" (also considering variants without the hyphen) in the *Topic* search field (which includes title, abstract, authors' keywords, and automatically generated keywords) by the total number of articles published in that journal in the same period. The obtained fraction of papers relevant to MCDA serves as an approximate indication of the relative weight of the MCDA area in the journal. It might exclude articles on MCDA mentioning specific methods (e.g., AHP or ELECTRE) but not "multi-criteria," "multi-attribute," or "multi-objective." We retained only 41 journals with at least 5% of their articles on MCDA.
- To increase the diversity of journals, we looked for the journals in other Web of Science categories publishing a high number of MCDA articles in relative terms (as a ratio to their total number of articles) or in absolute terms. We retained

five journals with 20% or more of their articles mentioning MCDA (IEEE Transactions on Evolutionary Computation, Swarm and Evolutionary Computation, International Journal of Intelligent Systems, Evolutionary Computation, and Complex & Intelligent Systems) and three other journals that published over 500 articles mentioning MCDA in this period (IEEE Access, Journal of Cleaner Production, and Applied Soft Computing).
- Finally, to reach a round number of 50 journals, we picked the Journal of Multi-Criteria Decision Analysis since it is explicitly devoted to the MCDA area.

Concerning the set of panel-based journal ratings, we aimed at having a diverse set in terms of geographies, focus, and number of levels. The selected ratings were the following (based on the versions available in May 2021):

- ABDC—a rating from the Australian Business Deans Council mainly focused on fields relevant to Business/Management, foreseeing four levels: C, B, A, A* (from worst to best).
- AJG (Academic Journal Guide)—a rating from the Chartered Association of Business Schools, based in the United Kingdom. It is mainly focused on fields relevant to Business/Management, foreseeing five levels: 1, 2, 3, 4, 4* (from worst to best).
- POL—a rating from the Polish Ministry of Education and Science, covering all areas. It foresees six levels labeled as 20, 40, 70, 100, 140, 200 (from worst to best), which we deal with qualitatively.
- FMS (Federation of Management Societies of China)—a rating developed jointly by the Chinese Society of Optimization, Overall Planning and Economical Mathematics, the Society of Management Science and Engineering of China, and the Systems Engineering Society of China, focused on Management Science. It foresees four levels: D, C, B, A (from worst to best).
- JUFO (Julkaisufoorumi)—a forum by the Federation of Finnish Learned Societies, covering all areas. It foresees three levels: 1, 2, 3 (from worst to best).
- Qualis—a rating from CAPES, a government agency linked to the Brazilian Ministry of Education, covering all areas. It foresees eight levels: C, B5 to B1, A2, A1 (from worst to best). As journals can have various ratings in different areas, we considered the best level.

The levels assigned to particular journals combine revealed and stated preferences. For example, AJG is based upon peer review, editorial and expert judgments, and statistical information related to citations, whereas POL levels are proposed by the expert panels considering the prestige and bibliometric indicators of journals.

Table 13.1 lists the ratings for the 50 journals selected for this study. For example, the European Journal of Operational Research attains the best levels in terms of ABCD, AJG, FMS, and Qualis, and the second-best levels according to the remaining two ratings. In turn, the variability of ratings attained by Applied Soft Computing is significantly greater with, e.g., the best level in Brazil and Poland, but having the worst level (not recognized) in two other ratings.

**Table 13.1** Ratings of 50 journals relevant to the MCDA research

| Id | Journal name | ABDC | AJG | POL | FMS | JUFO | Qualis |
|---|---|---|---|---|---|---|---|
| $a_1$ | 4OR | B$(r_{1,3})$ | 2$(r_{2,3})$ | 70$(r_{3,4})$ | C$(r_{4,3})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_2$ | Annals of Operations Research | A$(r_{1,4})$ | 3$(r_{2,4})$ | 70$(r_{3,4})$ | B$(r_{4,4})$ | 1$(r_{5,2})$ | A1$(r_{6,9})$ |
| $a_3$ | Applied Soft Computing | C$(r_{1,2})$ | $-(r_{2,1})$ | 200$(r_{3,7})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | A1$(r_{6,9})$ |
| $a_4$ | Asia Pacific Journal of Operational Research | $-(r_{1,1})$ | $-(r_{2,1})$ | 40$(r_{3,3})$ | C$(r_{4,3})$ | 1$(r_{5,2})$ | B2$(r_{6,6})$ |
| $a_5$ | Central European Journal of Operations Research | C$(r_{1,2})$ | 1$(r_{2,2})$ | 40$(r_{3,3})$ | D$(r_{4,2})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_6$ | Complex and Intelligent Systems | $-(r_{1,1})$ | $-(r_{2,1})$ | 20$(r_{3,2})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_7$ | Computers and Operations Research | A$(r_{1,4})$ | 3$(r_{2,4})$ | 140$(r_{3,6})$ | B$(r_{4,4})$ | 1$(r_{5,2})$ | A1$(r_{6,9})$ |
| $a_8$ | Concurrent Engineering Research and Applications | $-(r_{1,1})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_9$ | Engineering Economist | C$(r_{1,2})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{10}$ | Engineering Optimization | $-(r_{1,1})$ | 2$(r_{2,3})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{11}$ | European Journal of Industrial Engineering | $-(r_{1,1})$ | 2$(r_{2,3})$ | 40$(r_{3,3})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{12}$ | European Journal of Operational Research | A*$(r_{1,5})$ | 4$(r_{2,5})$ | 140$(r_{3,6})$ | A$(r_{4,5})$ | 2$(r_{5,3})$ | A1$(r_{6,9})$ |
| $a_{13}$ | Evolutionary Computation | $-(r_{1,1})$ | 3$(r_{2,4})$ | 100$(r_{3,5})$ | C$(r_{4,3})$ | 2$(r_{5,3})$ | A1$(r_{6,9})$ |
| $a_{14}$ | Expert Systems with Applications | C$(r_{1,2})$ | 3$(r_{2,4})$ | 140$(r_{3,6})$ | C$(r_{4,3})$ | 1$(r_{5,2})$ | A1$(r_{6,9})$ |
| $a_{15}$ | Fuzzy Optimization and Decision Making | $-(r_{1,1})$ | 3$(r_{2,4})$ | 100$(r_{3,5})$ | C$(r_{4,3})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_{16}$ | IEEE Access | $-(r_{1,1})$ | $-(r_{2,1})$ | 100$(r_{3,5})$ | $-(r_{4,1})$ | 2$(r_{5,3})$ | A2$(r_{6,8})$ |
| $a_{17}$ | IEEE Transactions on Evolutionary Computation | $-(r_{1,1})$ | 4$(r_{2,5})$ | 200$(r_{3,7})$ | B$(r_{4,4})$ | 3$(r_{5,4})$ | A1$(r_{6,9})$ |
| $a_{18}$ | IMA Journal of Management Mathematics | $-(r_{1,1})$ | 2$(r_{2,3})$ | 40$(r_{3,3})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{19}$ | INFOR | B$(r_{1,3})$ | 1$(r_{2,2})$ | 20$(r_{3,2})$ | D$(r_{4,2})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{20}$ | INFORMS Journal on Computing | B$(r_{1,3})$ | 3$(r_{2,4})$ | 100$(r_{3,5})$ | A$(r_{4,5})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{21}$ | Int. J. of Computer Integrated Manufacturing | B$(r_{1,3})$ | 2$(r_{2,3})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{22}$ | Int. J. of Industrial Engineering Computations | $-(r_{1,1})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | $-(r_{5,1})$ | B3$(r_{6,5})$ |
| $a_{23}$ | Int. J. of Information Technology and Decision Making | C$(r_{1,2})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | B$(r_{4,4})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_{24}$ | Int. J. of Intelligent Systems | $-(r_{1,1})$ | $-(r_{2,1})$ | 100$(r_{3,5})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a_{25}$ | Int. J. of Production Economics | A$(r_{1,4})$ | 3$(r_{2,4})$ | 140$(r_{3,6})$ | B$(r_{4,4})$ | 2$(r_{5,3})$ | A1$(r_{6,9})$ |
| $a_{26}$ | Int. J. of Production Research | A$(r_{1,4})$ | 3$(r_{2,4})$ | 100$(r_{3,5})$ | B$(r_{4,4})$ | 1$(r_{5,2})$ | A1$(r_{6,9})$ |
| $a_{27}$ | Int. Transactions in Operational Research | B$(r_{1,3})$ | 1$(r_{2,2})$ | 100$(r_{3,5})$ | C$(r_{4,3})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{28}$ | Journal of Cleaner Production | A$(r_{1,4})$ | 2$(r_{2,3})$ | 140$(r_{3,6})$ | C$(r_{4,3})$ | 2$(r_{5,3})$ | A1$(r_{6,9})$ |
| $a_{29}$ | Journal of Global Optimization | $-(r_{1,1})$ | $-(r_{2,1})$ | 100$(r_{3,5})$ | $-(r_{4,1})$ | 2$(r_{5,3})$ | A2$(r_{6,8})$ |
| $a_{30}$ | Journal of Industrial and Management Optimization | $-(r_{1,1})$ | 1$(r_{2,2})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_{31}$ | Journal of Manufacturing Systems | B$(r_{1,3})$ | 1$(r_{2,2})$ | 140$(r_{3,6})$ | D$(r_{4,2})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{32}$ | Journal of Multi-Criteria Decision Analysis | B$(r_{1,3})$ | 1$(r_{2,2})$ | 40$(r_{3,3})$ | D$(r_{4,2})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_{33}$ | Journal of Optimization Theory and Applications | A$(r_{1,4})$ | 3$(r_{2,4})$ | 70$(r_{3,4})$ | B$(r_{4,4})$ | 1$(r_{5,2})$ | A1$(r_{6,9})$ |
| $a_{34}$ | Journal of Systems Science and Systems Engineering | $-(r_{1,1})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | C$(r_{4,3})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_{35}$ | Journal of the Operational Research Society | A$(r_{1,4})$ | 3$(r_{2,4})$ | 70$(r_{3,4})$ | B$(r_{4,4})$ | 2$(r_{5,3})$ | A1$(r_{6,9})$ |
| $a_{36}$ | Memetic Computing | $-(r_{1,1})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{37}$ | Military Operations Research | $-(r_{1,1})$ | $-(r_{2,1})$ | 20$(r_{3,2})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_{38}$ | Omega | A$(r_{1,4})$ | 3$(r_{2,4})$ | 140$(r_{3,6})$ | B$(r_{4,4})$ | 2$(r_{5,3})$ | A1$(r_{6,9})$ |
| $a_{39}$ | Operational Research | C$(r_{1,2})$ | 1$(r_{2,2})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B2$(r_{6,6})$ |
| $a_{40}$ | Optimization | B$(r_{1,3})$ | 1$(r_{2,2})$ | 70$(r_{3,4})$ | D$(r_{4,2})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{41}$ | Optimization and Engineering | $-(r_{1,1})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{42}$ | OR spectrum | B$(r_{1,3})$ | 3$(r_{2,4})$ | 100$(r_{3,5})$ | C$(r_{4,3})$ | 1$(r_{5,2})$ | A2$(r_{6,8})$ |
| $a_{43}$ | Pacific Journal of Optimization | $-(r_{1,1})$ | 1$(r_{2,2})$ | 40$(r_{3,3})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B2$(r_{6,6})$ |
| $a_{44}$ | Proc. of The Institution of Mechanical Eng. Part O | $-(r_{1,1})$ | $-(r_{2,1})$ | 70$(r_{3,4})$ | B$(r_{4,4})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{45}$ | RAIRO Operations Research | $-(r_{1,1})$ | 1$(r_{2,2})$ | 40$(r_{3,3})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{46}$ | Socio Economic Planning Sciences | C$(r_{1,2})$ | 2$(r_{2,3})$ | 100$(r_{3,5})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | A1$(r_{6,9})$ |
| $a_{47}$ | Studies in Informatics and Control | $-(r_{1,1})$ | $-(r_{2,1})$ | 40$(r_{3,3})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |
| $a_{48}$ | Swarm and Evolutionary Computation | $-(r_{1,1})$ | $-(r_{2,1})$ | 140$(r_{3,6})$ | $-(r_{4,1})$ | 2$(r_{5,3})$ | A1$(r_{6,9})$ |
| $a_{49}$ | Systems Engineering | $-(r_{1,1})$ | $-(r_{2,1})$ | 40$(r_{3,3})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | $-(r_{6,1})$ |
| $a_{50}$ | TOP | $-(r_{1,1})$ | 1$(r_{2,2})$ | 70$(r_{3,4})$ | $-(r_{4,1})$ | 1$(r_{5,2})$ | B1$(r_{6,7})$ |

Figure 13.1 presents the frequencies of these ratings. The distributions differ vastly from one rating to another. For example, half of the journals are not included in the ABCD rating list, the numbers of journals with levels C, B, or A are relatively similar, and there is only one journal assigned the best level. On the contrary, all journals are included in the POL list; very few are assigned the second and seventh levels, and 18 out of 50 are assigned an intermediate, fourth level.

## 13.4 Results for the Different Models and Variants

Models M1, M2, and M3 were run three times per journal, corresponding to the three variants of constraints: *free*, i.e., any increasing value function (13.6), an *elitist* value function (13.7), or an *anti-elitist* value function (13.8). The results are depicted in Figs. 13.2 and 13.3. In these figures, the bars represent the BoD absolute value of each journal for M3 ($\nu^*$) and the complement to 100 of their BoD relative efficiency (maximum regret) for M1 ($100 - \tau^*$) and M2 ($100 - \sigma^*$). In all cases, the larger the bar, the better is the journal. This allows comparing models, variants, and journals.

### *13.4.1 Model Comparison*

Since $\sigma^* \leq \tau^*$ (Proposition 1) and $\tau^* \leq 100 - \nu^*$ (Proposition 2), we find necessarily $\nu^* \leq 100 - \tau^* \leq 100 - \sigma^*$ for each journal, regardless of the variant considered. For example, when considering an *elitist* variant for Annals of Operations Research ($a_2$), $\tau^*$, $\sigma^*$, and $\nu^*$ derived from models M1, M2, and M3 are equal to 8, 7.5, and 80.4, respectively, corresponding to bar lengths 92, 92.5, and 80.4 in Fig. 13.2.

This is in accordance with the logic of minimizing the maximum regret in relation to its peers (minimizing the difference to the best), considering the set of peers defined for each model. The most penalizing case (M3) occurs when considering the peer is an ideal alternative with the best performance on all indicators. An intermediate case occurs when considering the peers are all other alternatives, which might even be worse on some indicators but compensate for that by being better on other indicators (i.e., allowing trade-offs). The least penalizing case occurs when considering the peers are only those alternatives that dominate the alternative under evaluation.

When evaluating the 50 journals, M1 and M2 yield the same results for all journals under the *free* variant, and differ for, respectively, 1 ($a_3$) and 12 (e.g., $a_1$, $a_2$, and $a_3$) journals when considering the *anti-elitist* and *elitist* settings. The results are fully consistent for 91.3% of the cases. Moreover, the observed differences are relatively small (not exceeding 1 on a scale between 0 and 100) except for one journal.

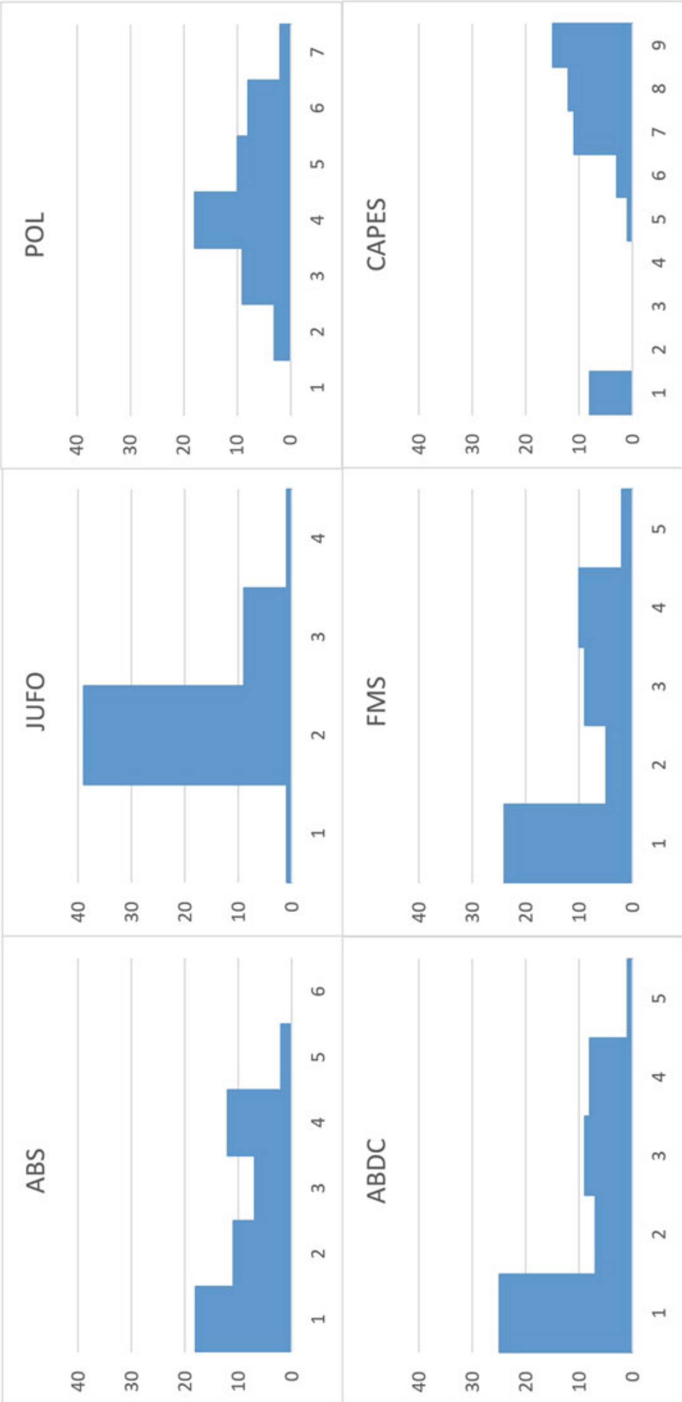


**Fig. 13.1** Frequency of levels for each rating (1 denotes the worst level, corresponding to being absent from the list)

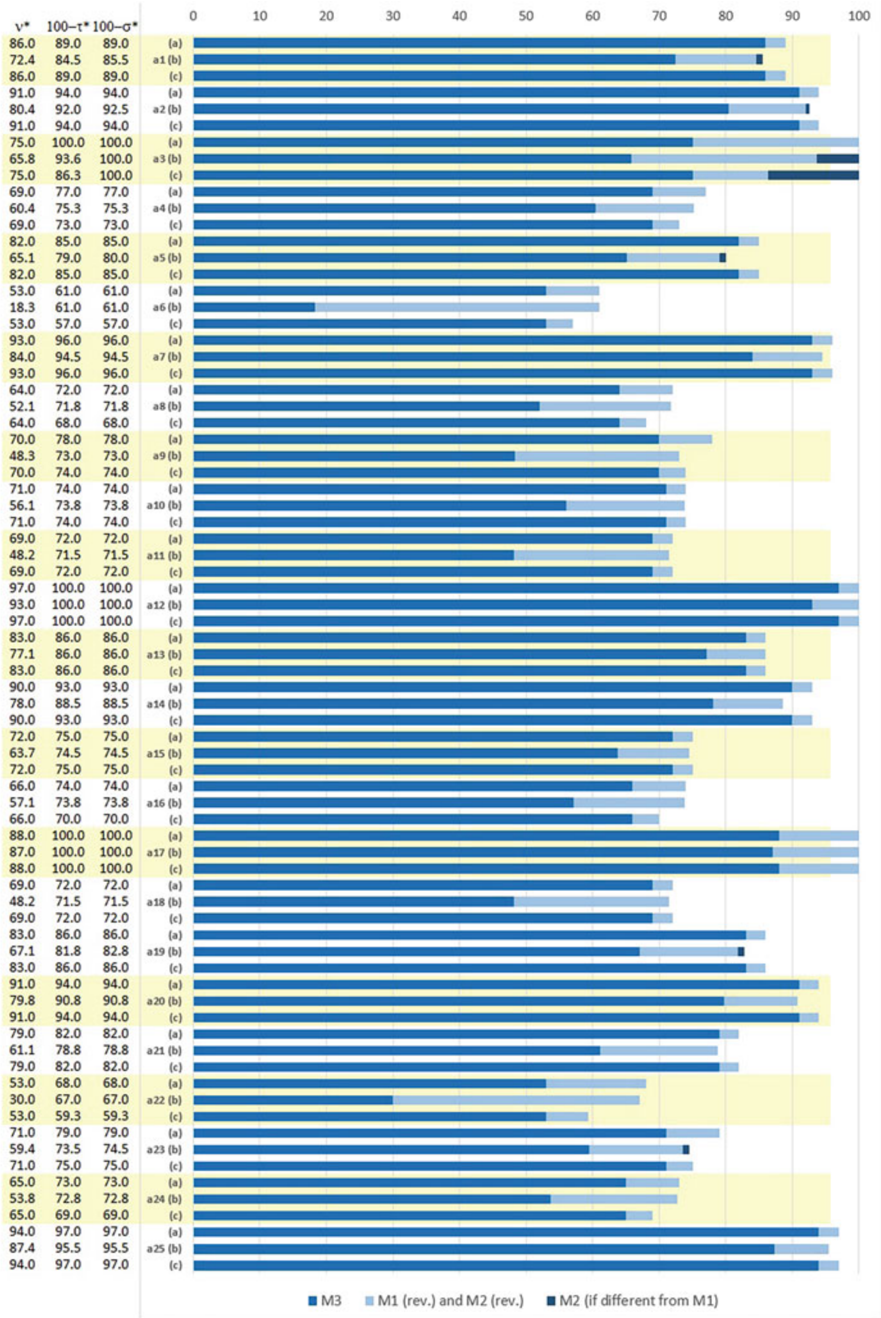


**Fig. 13.2** Results for journals $a_1$–$a_{25}$ for the three variants: (a) *free*; (b) *elitist*; (c) *anti-elitist* (results for M1 and M2 are reversed subtracting from 100 (the more, the better))

| Journal | Variant | ν* | 100−τ* | 100−σ* |
|---|---|---|---|---|
| a26 | (a) | 92.0 | 95.0 | 95.0 |
| a26 | (b) | 82.2 | 93.0 | 93.5 |
| a26 | (c) | 92.0 | 95.0 | 95.0 |
| a27 | (a) | 87.0 | 90.0 | 90.0 |
| a27 | (b) | 73.5 | 86.3 | 86.8 |
| a27 | (c) | 87.0 | 90.0 | 90.0 |
| a28 | (a) | 92.0 | 95.0 | 95.0 |
| a28 | (b) | 84.4 | 93.5 | 93.5 |
| a28 | (c) | 92.0 | 95.0 | 95.0 |
| a29 | (a) | 66.0 | 74.0 | 74.0 |
| a29 | (b) | 57.1 | 73.8 | 73.8 |
| a29 | (c) | 66.0 | 70.0 | 70.0 |
| a30 | (a) | 61.0 | 64.0 | 64.0 |
| a30 | (b) | 27.3 | 64.0 | 64.0 |
| a30 | (c) | 61.0 | 64.0 | 64.0 |
| a31 | (a) | 87.0 | 90.0 | 90.0 |
| a31 | (b) | 74.3 | 86.8 | 86.8 |
| a31 | (c) | 87.0 | 90.0 | 90.0 |
| a32 | (a) | 75.0 | 78.0 | 78.0 |
| a32 | (b) | 60.1 | 74.0 | 75.0 |
| a32 | (c) | 75.0 | 78.0 | 78.0 |
| a33 | (a) | 91.0 | 94.0 | 94.0 |
| a33 | (b) | 80.4 | 92.0 | 92.5 |
| a33 | (c) | 91.0 | 94.0 | 94.0 |
| a34 | (a) | 63.0 | 71.0 | 71.0 |
| a34 | (b) | 55.9 | 70.0 | 70.0 |
| a34 | (c) | 63.0 | 67.0 | 67.0 |
| a35 | (a) | 92.0 | 95.0 | 95.0 |
| a35 | (b) | 83.8 | 93.5 | 93.5 |
| a35 | (c) | 92.0 | 95.0 | 95.0 |
| a36 | (a) | 63.0 | 71.0 | 71.0 |
| a36 | (b) | 45.8 | 70.5 | 70.5 |
| a36 | (c) | 63.0 | 67.0 | 67.0 |
| a37 | (a) | 53.0 | 61.0 | 61.0 |
| a37 | (b) | 18.3 | 61.0 | 61.0 |
| a37 | (c) | 53.0 | 57.0 | 57.0 |
| a38 | (a) | 94.0 | 97.0 | 97.0 |
| a38 | (b) | 87.4 | 95.5 | 95.5 |
| a38 | (c) | 94.0 | 97.0 | 97.0 |
| a39 | (a) | 75.0 | 78.0 | 78.0 |
| a39 | (b) | 44.1 | 72.8 | 72.8 |
| a39 | (c) | 75.0 | 78.0 | 78.0 |
| a40 | (a) | 84.0 | 87.0 | 87.0 |
| a40 | (b) | 69.4 | 82.5 | 83.5 |
| a40 | (c) | 84.0 | 87.0 | 87.0 |
| a41 | (a) | 64.0 | 72.0 | 72.0 |
| a41 | (b) | 52.1 | 71.8 | 71.8 |
| a41 | (c) | 64.0 | 68.0 | 68.0 |
| a42 | (a) | 89.0 | 92.0 | 92.0 |
| a42 | (b) | 77.5 | 88.3 | 88.8 |
| a42 | (c) | 89.0 | 92.0 | 92.0 |
| a43 | (a) | 67.0 | 70.0 | 70.0 |
| a43 | (b) | 39.9 | 69.3 | 69.3 |
| a43 | (c) | 67.0 | 70.0 | 70.0 |
| a44 | (a) | 72.0 | 80.0 | 80.0 |
| a44 | (b) | 64.4 | 78.5 | 78.5 |
| a44 | (c) | 72.0 | 76.0 | 76.0 |
| a45 | (a) | 68.0 | 71.0 | 71.0 |
| a45 | (b) | 46.2 | 70.5 | 70.5 |
| a45 | (c) | 68.0 | 71.0 | 71.0 |
| a46 | (a) | 80.0 | 83.0 | 83.0 |
| a46 | (b) | 66.5 | 78.5 | 78.5 |
| a46 | (c) | 80.0 | 83.0 | 83.0 |
| a47 | (a) | 62.0 | 70.0 | 70.0 |
| a47 | (b) | 44.2 | 69.5 | 69.5 |
| a47 | (c) | 62.0 | 66.0 | 66.0 |
| a48 | (a) | 68.0 | 76.0 | 76.0 |
| a48 | (b) | 65.0 | 76.0 | 76.0 |
| a48 | (c) | 68.0 | 72.0 | 72.0 |
| a49 | (a) | 54.0 | 62.0 | 62.0 |
| a49 | (b) | 20.0 | 62.0 | 62.0 |
| a49 | (c) | 54.0 | 58.0 | 58.0 |
| a50 | (a) | 69.0 | 72.0 | 72.0 |
| a50 | (b) | 47.8 | 71.5 | 71.5 |
| a50 | (c) | 69.0 | 72.0 | 72.0 |

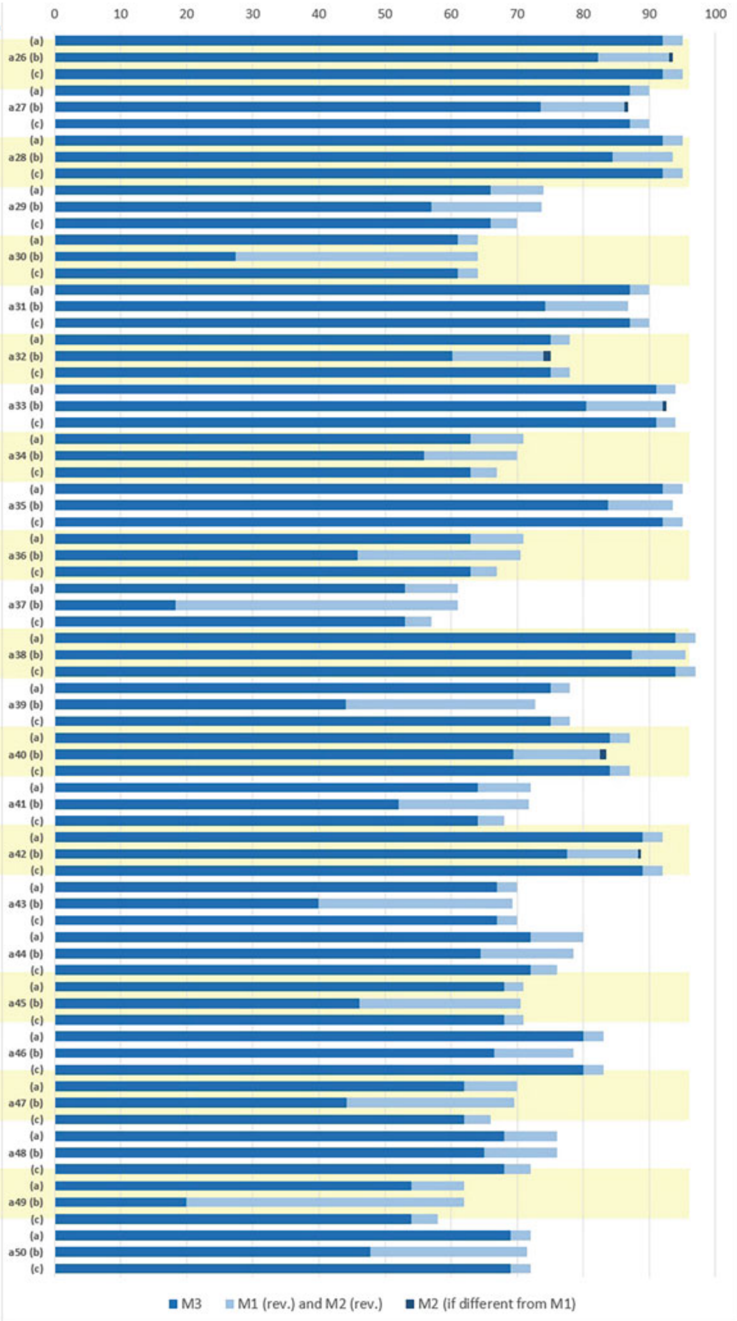


**Fig. 13.3** Results for journals $a_{26}$–$a_{50}$ for the three variants: (a) *free*; (b) *elitist*; (c) *anti-elitist* (results for M1 and M2 are reversed subtracting from 100 (the more, the better))

This exception is $a_3$ (Applied Soft Computing) for the *elitist* variant (a difference of 6.4) and, even more markedly, for the *anti-elitist* variant (a difference of 13.7). For both variants, $a_3$ is considered efficient by model M2 ($\sigma^* = 0$), and this happens because no other journal dominates it. Indeed, it is one of two journals attaining the top level of the POL rating, and the only other journal attaining the same level for POL, $a_{17}$, is beaten by $a_3$ for not being recognized by the ABDC list, whereas $a_3$ is one level above (C). Hence, in M2, $a_3$ is compared only with itself. In M3, the advantage of $a_3$ in the ABDC rating is clearly outweighed by its disadvantages to $a_{17}$ in AJG (4 levels worse), FMS (3 levels worse), and JUFO (2 levels worse), taking into account the non-dictatorial weight constraint (preventing it from placing null weight in the ratings where it is worse) and the value function constraints. Comparing $a_3$ with another strong peer, $a_{12}$, the former is one level better in POL, but it is four levels worse in AJG and FMS, three levels worse in ABDC, and one level worse in JUFO. Again the constraints do not allow an advantageous trade-off for $a_3$. In the *elitist* variant, it will place as much weight as it can in POL, and it is allowed to place a significant value difference between the top two levels in that ranking, thereby minimizing the maximum regret to only 6.4. However, the *anti-elitist* variant does not allow placing such a high difference between these two levels, and the BoD regret increases to 13.7. In the *free* variant, which allows all the freedom to set the values corresponding to different levels, $a_3$ manages to have a null BoD regret and match the performance of $a_{12}$ by placing a large value difference between the top two levels in POL but not doing the same for other ratings.

Overall, the correlation coefficients between the quality measures derived using different models are very high. They range from 0.932 for the comparison of results attained for models M1 and M3 under the *elitist* variant to 1.0 for models M1 and M2 under the *free* variant.

### *13.4.2 Variant Comparison*

The statistics depicted in Fig. 13.4 indicate that for the considered journals, the *anti-elitist* constraints did not lead to major changes compared to the *free* variant and were the same for M3. In contrast, the *elitist* variant tends to provide worse scores to the journals. While for M1 and M2 the average differences with respect to *free* and *anti-elitist* variants are not greater than 0.2 and 1.9, respectively, they are more substantial for the third model.

In particular, the average difference between the scores for the *elitist* and *free* variants for M3 is 14.6. The largest differences ($\geq$30) are observed for journals $a_6$, $a_{30}$, $a_{37}$, $a_{39}$, and $a_{49}$. They have relatively low levels on all indicators. Hence they maximize their values in the *free* variant by placing as much value as possible for these levels. In the *elitist* variant, they have less freedom to do the same. Interestingly, using the relative efficiency models M1 and M2, the scores for the *elitist* and *free* variants are the same for most of the previous journals. Their optimal choice in the *free* variant is to select $a_{12}$ as a peer and try to decrease a

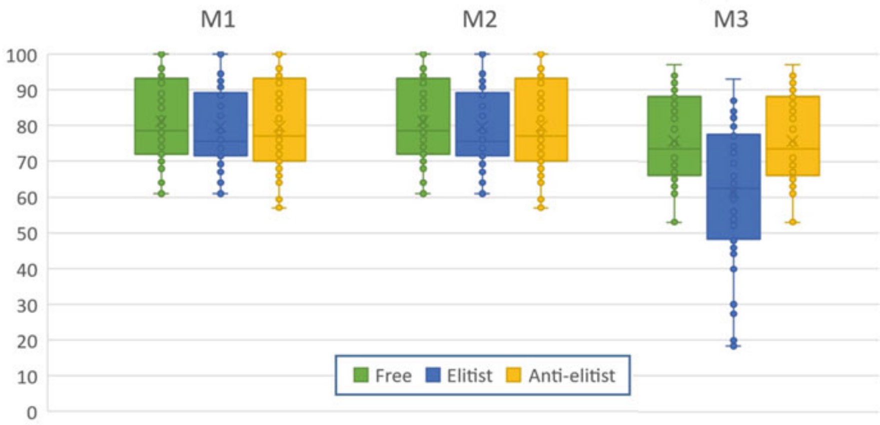


**Fig. 13.4** Box-plots of results for the three variants (results for M1 and M2 are reversed subtracting from 100 (the more, the better))

comprehensive value of the latter as much as possible. They achieve this by selecting the same value function as the *elitist* model in a way that places a large difference of value between the top level and the second-best level on the two indicators where $a_{12}$ does not have the top rating. The exception is $a_{39}$, which also wants to use an elitist value function in those indicators, but would benefit from an *anti-elitist* value function to minimize the difference value to $a_{12}$ in other indicators. Therefore, for this journal, the *elitist* variant provides a worse score than the *free* variant also for models M1 and M2.

In turn, the least differences ($\leq 4$) between the scores obtained for the *elitist* variant and the *free* and *anti-elitist* variants using M3 are observed for journals $a_{12}$, $a_{17}$, and $a_{48}$. Among them, $a_{12}$ and $a_{17}$ attain the best levels for 4 out of 6 ratings.

The correlation coefficients between the quality measures derived using different variants are very high. The lowest values (0.935) are observed for the comparison of results attained under the *elitist* variant and the *anti-elitist* or *free* ones for M3. In turn, a perfect correlation is observed between the latter two variants.

### *13.4.3 Journal Comparison*

Regardless of the model and variant considered, the European Journal of Operational Research ($a_{12}$), coordinated by Editor-in-Chief Roman Slowinski, always appears at the top, sometimes tied with IEEE Transactions on Evolutionary Computation ($a_{17}$) and Applied Soft Computing ($a_3$). The rankings of the journals that reach the top 10 for some model or variant are depicted in Table 13.2.

The BoD standing of $a_3$ and $a_{17}$ varies markedly across models and variants. Journal $a_{17}$ is able to reach the top rank ex aequo in relative efficiency models M1

**Table 13.2** Ranking of journals in the top 10 for different models (M1, M2, and M3) and variants: (a) *free*; (b) *elitist*; (c) *anti-elitist*

| | | M1 | | | M2 | | | M3 | |
|---|---|---|---|---|---|---|---|---|---|
| Id | Journal | (a) | (b) | (c) | (a) | (b) | (c) | (a) | (b)=(c) |
| $a_{12}$ | European Journal of Operational Research | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a_{17}$ | IEEE Transactions on Evolutionary Computation | 1 | 1 | 1 | 1 | 1 | 1 | 13 | 4 |
| $a_3$ | Applied Soft Computing | 1 | 6 | 18 | 1 | 1 | 1 | 23 | 21 |
| $a_{25}$ | Int. J. of Production Economics | 4 | 3 | 3 | 4 | 4 | 4 | 2 | 2 |
| $a_{38}$ | Omega | 4 | 3 | 3 | 4 | 4 | 4 | 2 | 2 |
| $a_7$ | Computers and Operations Research | 6 | 5 | 5 | 6 | 6 | 6 | 4 | 6 |
| $a_{28}$ | Journal of Cleaner Production | 7 | 7 | 6 | 7 | 7 | 7 | 5 | 5 |
| $a_{35}$ | Journal of the Operational Research Society | 7 | 7 | 6 | 7 | 7 | 7 | 5 | 7 |
| $a_{26}$ | Int. J. of Production Research | 7 | 9 | 6 | 7 | 7 | 7 | 5 | 8 |
| $a_2$ | Annals of Operations Research | 10 | 10 | 9 | 10 | 10 | 10 | 8 | 9 |
| $a_{33}$ | Journal of Optimization Theory and Applications | 10 | 10 | 9 | 10 | 10 | 10 | 8 | 9 |
| $a_{20}$ | INFORMS Journal on Computing | 10 | 12 | 9 | 10 | 12 | 10 | 8 | 11 |

and M2, benefiting from being the only journal with the top rating from JUFO. It is, therefore, able to place as much value as possible for that level (50), and ensure a large advantage over its peers even in the *anti-elitist* variant where the second-best level in that indicator can remain relatively distant with a value of 33.3. In model M3, however, it is penalized for not being recognized in the ABDC list and for not reaching the top level in the FMS rating. Journal $a_3$ is able to reach the top rank ex aequo in relative efficiency model M2 (as explained in Sect. 13.4.1 it has no competitors in this model) and the *free* variant of M1, but drops many positions for the remaining variants of M1 (particularly the *anti-elitist*) and M3. This results from not being recognized in the AJG and FMS lists and also for its C rating in the ABCD list. Given these shortcomings, it is noteworthy how $a_3$ can still be efficient in M1's *free* variant. This is achieved by matching the value of its peers $a_{12}$ and $a_{17}$ using an elitist value function for AJG (forcing its peers to lose value for not attaining the top level) and also for POL (to gain a relatively large advantage over $a_{12}$), but uses an anti-elitist value function for ABDC able to gain a relatively large advantage over $a_{17}$ while not losing much to $a_{12}$, and an anti-elitist value function for JUFO to minimize its disadvantage versus these two peers. As this freedom is not allowed in the *elitist* and *anti-elitist* variants, $a_3$ can no longer reach the top position in M1.

Journals $a_3$ and $a_{17}$ are among six for which the difference between the extreme ranks is at least ten. The remaining ones are: $a_9$ (ranks between 26 and 37), $a_{34}$ (33–43), and $a_{39}$ (23–44), and $a_{48}$ (23–35). The reasons for this are very different. For example, $a_{48}$ attains very favorable levels given three ratings, while not being recognized by the remaining three rankings. This allows it to attain an advantageous ranking under the elitist setting, while decreasing its performance in the remaining two scenarios. On the contrary, $a_{39}$ attains rather low ratings in all rankings, but it is not recognized only in a single list. As a result, it can perform well under the *anti-elitist* setting, while being punished under the *elitist* one.

The stability of ranks attained by other top-ten journals listed in Table 13.2 is significantly greater. In particular, the International Journal of Production Economics ($a_{25}$) and Omega ($a_{38}$) journals are tied and solidly placed among the top five for all models and variants, and even reach the second position for M3. Next, Computers and Operations Research is ranked between fourth and sixth. The ranks attained by the bottom-ranked journals are also stable. Journals $a_6$, $a_{22}$, $a_{30}$, $a_{37}$, and $a_{49}$ are placed as the worst five according to all models and variants. The underlying reasons are the same for all these journals. They are not recognized in three or four input ratings while attaining relatively low levels in the remaining ones.

Overall, the rankings obtained for different models and variants are very similar. For example, Kendall's $\tau$ between the journal orders for the *free* and *anti-elitist* variants is equal to 0.92, 0.94, and 1.0 for models M1, M2, and M3, respectively. The lowest rank correlation value (0.82) is observed when comparing the ranks attained for the *elitist* and the other two variants for M3.

## 13.5 Summary

We proposed a family of Linear Programming models for constructing a univocal ranking based on multiple ratings using different qualitative scales. The underlying idea consisted in adapting value-based efficiency analysis for deriving Benefit-of-Doubt composite indicators. The basic models minimized the maximal regret of each alternative to the best one. In this regard, we accounted for all alternatives as potential benchmarks or limited the latter set to these dominating a specific alternative under consideration. However, we also accounted for the maximization of an absolute comprehensive value score. All models were adapted to tolerate the elitist, anti-elitist, and unconstrained perspectives when moving toward better qualitative levels. Under all settings, the considered alternative was free to set the weights and values corresponding to different levels that made it compare to its peers under the best possible light.

We used the elaborated models to evaluate 50 journals publishing MCDA research. To have a diverse set of panel-based journal ratings, we considered the ones established in Australia, Brazil, China, Finland, Poland, and the United Kingdom. The obtained rankings were consistent to a large extent when considering both the quantitative scores and the respective ranks of journals. The greatest similarities were observed for the results derived from the minimization of the maximal regret with or without trade-offs as well as for *free* and *anti-elitist* variants for all models. In turn, slight differences with respect to the remaining settings were noted for the models maximizing an absolute comprehensive score and/or assuming the elitist perspective emphasizing the best classifications.

A common feature of all rankings was that the European Journal of Operational Research always appeared at the top. Thanks to its most favorable levels in four journal ratings and two second-based levels in the remaining two ratings, the minimal regret for EJOR could be nullified, whereas its maximal comprehensive score was close to a perfect 100. Moreover, for all journals but IEEE Transactions on Evolutionary Computation and Applied Soft Computing, EJOR served as a peer in a relative evaluation.

We envisage the following future research directions. First, we aim at extending the proposed models with robustness analysis that would quantify the stability of rankings attained for all feasible weight vectors and other parameter values [18]. This would contrast with the perspective adopted in this chapter that allowed each journal to choose a value function from which it could benefit the most. Second, it would be interesting to incorporate expert knowledge when defining the constraints on the value differences between the neighboring levels. The points assigned to these levels in the Polish rating or the level names in Brazil (one C level, five B levels, and 2 A levels) suggest that the underlying marginal value functions should be neither linear nor fully convex or concave, and their character may differ from one rating to another. Third, it is possible to consider other ratings, preferably from the world's regions that were not represented in the conducted study. Note, however, that such ratings are the most popular in Europe. In fact, we also analyzed France's

CNRS Section 37 and Denmark's BFI, but they were not included in the data set due to their remarkable similarity to AJG and JUFO, respectively. Fourth, it would be interesting to consider other (possibly broader) journal categories. As a final remark, let us emphasize that even though in this chapter we focused on the analysis of ratings of academic journals, the proposed models can be applied for evaluating any set of alternatives evaluated on heterogeneous qualitative scales.

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The authors would like to thank Professor Roman Slowinski for his dedication serving as Coordinating Editor of the European Journal of Operational Research since 1999. Under his guidance, EJOR has grown to be a premier journal in Operations Research and Management Science, and MCDA has risen as one of EJOR's most essential areas. His work, occupying tens of hours each week, is greatly appreciated by the entire MCDA and OR communities.

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# Chapter 14
# Interactive Multicriteria Methodology Based on a Synergy of PROMETHEE II and Robust Simos Methods: Application to the Evaluation of E-government in Europe

**Eleftherios Siskos and Yannis Siskos**

**Abstract** This chapter proposes a multicriteria evaluation methodology, which is based on a synergy of the outranking method PROMETHEE II and the Robust Simos method for the elicitation of criteria importance weights. The evaluation system operates via a robustness control algorithm, called "Bipolar Robustness Control", which measures and progressively improves the robustness of both the evaluation model and the ranking results. The new framework is then implemented to evaluate and rank 22 developed European countries, on the basis of their e-government readiness, considering the knowledge and the preferential data of a senior expert. The net outranking flow, given by the PROMETHEE II method, indicates the degree of superiority or inferiority of a country, compared to the average e-government performance in Europe.

## 14.1 Introduction

The PROMETHEE outranking methods originally developed by Brans and Vincke [1] are popular multicriteria methods that are producing outranking relationships between the actions/alternatives $a \in A$ of a decision-making problem (see [2, 3], for instance). Just like in the case of the ELECTRE family methods, the $n$ criteria that frame the decision problem, are supposed to be weighted by a Decision Maker (DM), through the elicitation of importance weights $p_1, p_2, \ldots, p_n$, which express the relative magnitude of them and always sum up to one.

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$$\sum\nolimits_{j=1}^{n} p_j = 1. \tag{14.1}$$

The importance weights are assessed and quantified, using another auxiliary method, in conjunction with the PROMETHEE method, which requires them as input. Such methods and techniques include direct assessment procedures, indirect methods that require pairwise comparisons, elicitation of importance ranges, ranking of the criteria etc. (see [4], for more details).

This paper proposes an interactive multicriteria evaluation procedure, which is based on the synergy of the PROMETHEE II method for assigning a global score to the set of actions under evaluation, and a procedure named "Robust Simos Method" [5] for the elicitation of the criteria importance weights. This synergy aims to provide the decision-maker (DM) with a simple, transparent and easily comprehensible elicitation protocol for the criteria weights and assuring at the same time that these are stable enough to produce robust ranking results.

The implementation of the synergistic framework of the two aforementioned Multicriteria Decision Aid (MCDA) methods is coordinated and guided by a robustness control algorithm, called "Bipolar Robustness Control". This algorithm measures, assesses and progressively improves the robustness of the criteria weights, calculated by the Simos procedure, and the robustness of the evaluation results, given by the PROMETHEE II method. In the same spirit, Corrente et al. [6] propose the application of the Stochastic Multicriteria Acceptability Analysis (SMAA) to the family of PROMETHEE methods, in order to explore the whole set of parameters, compatible with some preference information provided by the Decision Maker (DM). Corrente et al. [7] also propose a ranking framework, extending the ELECTRE III to prioritize interacting criteria, allowing the handling of imprecise data and defining the space of feasible weighting solution. The robustness analysis in that case is achieved through a stochastic analysis of a sampled set of possible weights.

In the end, the proposed robustness control framework is implemented, in conjunction with the synergy of the PROMETHEE II and the Robust Simos method, to evaluate and rank 22 developed European countries, on the basis of their e-government performance. The whole interactive assessment procedure considers the knowledge and the preferential information, provided by a senior expert in the field of e-governance.

The rest of the paper is organized as follows: Section 14.2 presents the proposed synergy of PROMETHEE II—Robust Simos procedure. Section 14.3 outlines the bipolar robustness control algorithm, and additionally proposes a set of robustness indicators, assigned to the poles of the system. Subsequently, the implementation of the methodological framework to the e-government evaluation problem is presented in two parts: the first elicitation of criteria weights (Sect. 14.4), and the progressive implementation of robustness control, until the final evaluation of the European countries, is achieved (Sect. 14.5). Section 14.6 summarizes the achieved outcomes and concludes the paper.

## 14.2 The Synergy of PROMETHEE II and Robust Simos Multicriteria Evaluation Methods

### *14.2.1 A Brief Presentation of the PROMETHEE Methods*

The PROMETHEE methods make use of the notion of *generalized criterion*. Hence they model the value that the DM or stakeholder/evaluator is attributing to the score difference $[g_j(a) - g_j(b)]$ on a criterion $g_j$ for a pair of actions $(a, b)$, where $g_j(a)$ is the performance of action $a$ on criterion $g_j$. This is performed through the modelling of a *preference function* $P_j(a, b)$, defined as follows:

$$P_j\ (a, b) = F_j\left[d_j\ (a, b)\right] \forall a, b \in A, \tag{14.2}$$

where

$$d_j\ (a, b) = g_j(a) - g_j(b). \tag{14.3}$$

$$0 \leq P_j\ (a, b) \leq 1. \tag{14.4}$$

$$P_j\ (a, b) = 0, \quad \text{when} \quad g_j(a) - g_j(b) \leq 0. \tag{14.5}$$

When $g_j(a) - g_j(b) > 0$, the non-negative value that the preference function $P_j$ exhibits, is dependent on the type of generalized criterion that is selected by the analyst (see Appendix 1 for the description of the 6 different types of preference functions). The specific parameters that eventually need to be provided by the DM for the specification of these functions are:

$q$: indifference threshold
$p$: strict preference threshold
$s$: median threshold between $q$ and $p$

#### 14.2.1.1 A Multicriteria Pairwise Outranking Indicator

For every pair of actions $(a, b)$, the following weighted preference indicator is defined:

$$\pi\ (a, b) = \sum\nolimits_{j=1}^{n} p_j P_j\ (a, b). \tag{14.6}$$

It expresses the global outranking intensity of action $a$ over action $b$.

#### 14.2.1.2 Outranking Flows

For each action *a* the following outranking indicators are defined, in relation to the rest of the alternatives $x \in A$ under evaluation:

Positive outranking flow:

$$\varphi^{+}(\alpha) = \frac{1}{n-1} \sum_{x \in A} \pi(\alpha, x). \tag{14.7}$$

Negative outranking flow:

$$\varphi^{-}(\alpha) = \frac{1}{n-1} \sum_{x \in A} \pi(x, \alpha). \tag{14.8}$$

Net outranking flow:

$$\varphi(\alpha) = \varphi^{+}(\alpha) - \varphi^{-}(\alpha). \tag{14.9}$$

#### 14.2.1.3 PROMETHEE II Ranking Procedure

For the specific case of the PROMETHEE II method, the actions under evaluation are rank-ordered, based on their net outranking flows $\varphi$. It should be noted that PROMETHEE I, on the contrary, creates two rankings according to the flows $\varphi^{+}$ and $\varphi^{-}$, and gives a partial ranking of the alternatives, through the merging of the aforementioned two.

### *14.2.2 Elicitation of Criteria Weights Through the Simos Methods*

#### 14.2.2.1 The Original Method of Cards

The Simos Method or method of cards (see [8–10]) consists of the following three information extraction steps:

1. The DM is given a set of cards with the name of one criterion on each (*n* cards, each corresponding to a specific criterion of a family *F*). A number of white cards are also provided to the DM.
2. The DM is asked to rank the cards/criteria from the least to the most important, by arranging the cards in an ascending order. If multiple criteria have the same importance, she/he should build a subset by holding the corresponding cards together with a clip.

3. The DM is finally asked to introduce white cards between two successive cards (or subsets of ex aequo criteria) if she/he deems that the difference between them is more extensive. The greater the difference between the weights of the criteria (or the subsets of criteria), the greater the number of white cards that should be placed between them. Specifically, if $u$ denotes the difference in the magnitude between two successive criteria cards, then one white card means a difference of *two times u*, two white cards mean a difference of *three times u* etc.

The information provided by the DM is exploited by the Simos method for the determination of the weights, according to the following algorithm:

(i) Ranking of the subsets of ex aequo from the least important to the most important, considering also the white cards
(ii) Assignment of a position to each criterion/card and to each white card
(iii) Calculation of the non-normalized weights, and
(iv) Determination of the normalized weights

The least qualified card is given *Position 1*, while the most qualified one receives *Position n*. The non-normalized weight of each rank/subset is determined by dividing the sum of positions of a rank, by the total number of criteria belonging to it. The non-normalized weights are then divided by the total sum of positions of the criteria in each rank (excluding the white cards), in order to normalize them. The obtained values are rounded off to the lower or higher nearest integer value. The reader is referred to the Sect. 14.4.3 for the actual implementation of the original Simos method.

Figueira and Roy [11] proposed a revised version of the Simos method. In addition to the three-step data collection process, the new procedure introduces a fourth step, which demands from the DM to state *"how many times the last criterion is more important than the first one in the ranking"* (ratio $z$). This ratio is applied, in order to define a fixed interval between the weights of criteria or their subsets. This interval is denoted by $u = \frac{z-1}{e}$, where $e$ is the number of different weighting classes (namely single cards, subsets of ex aequo cards, and white cards).

Very recently, Corrente et al. [12] presented an improved version of the deck of cards method to render the construction of ratio and interval scales more "accurate" compared to the ones built in the original version. The improvement comes from the fact that a richer and finer preference information is provided by the DM, which permits a more accurate modelling of the strength of preference between different levels of a scale.

#### 14.2.2.2 The Robust Simos Method

Siskos and Tsotsolas [5] proved that the information provided by a DM, in the context of the original or revised Simos method, is not sufficient to ensure the attainment of a unique set of weights. On the contrary, there is an infinite number of weighting vectors, compatible with the decision-maker, which are elements of a hyper-polyhedron ***P***. This polyhedron is framed by the set of linear relations, imposed by the DM's comparison of the criteria, with respect to their relative importance. Tsotsolas et al. [13] focused on the presentation of the weights assessment through prioritization method (WAP), constituting a specific integrated implementation of the Robust Simos Method. WAP method on the one hand enriches the preferential information used, in a friendly and comprehensive for the DM way, while on the other hand, it leads to the estimation of weighting vectors with higher robustness.

Therefore, it is expected that the existence of multiple weighting vectors in outranking methods, such as the ELECTRE and PROMETHEE methods, will cause distortions in the evaluation of the actions of the problem. These distortions, translated to potential significant changes in the final ranking, cannot guarantee the proposition of robust recommendations to the DM.

To this end, the control and progressive improvement of the stability of the criteria weights are obtained through a set of robustness measures (see Sect. 14.3), which are integrated into a single methodological framework, called "Robust Simos Method". This algorithmic process is expected to include an initialization phase, which consists of the transformation of the DM's hierarchy of criteria into a $n$-dimensional convex polyhedron ***P***, $n$ being the number of criteria. This is followed by the bipolar robustness control procedure, which is shown in the next Section. A numerical investigation of this methodology on the e-government evaluation problem is extensively presented in Sects. 14.4 and 14.5.

## 14.3 Bipolar Robustness Control and Decision Support

### *14.3.1 Methodological Framework of Bipolar Robustness Control*

The bipolar robustness control, presented in this section includes a set of robustness measures and indices (see Sects. 14.3.2 and 14.3.3 below). These can support analysts to apply the Simos method and ensure high robustness, in both the elicitation of the weights *(Disaggregation Pole)* and the evaluation of the results, given by the implementation of the outranking method *(Aggregation Pole)*.

The lack of stability in one of the poles drives the analyst to reinforce the Simos' initial criteria ranking with new preferential data, such as the addition of the ratio $z$, thresholds and upper levels for certain criteria weights, bilateral comparisons of alternatives, and more. The flowchart of the bipolar robustness control algorithm, in conjunction with the Simos method, eliciting the criteria weights and the evaluation results, is presented in the flowchart of Fig. 14.1.

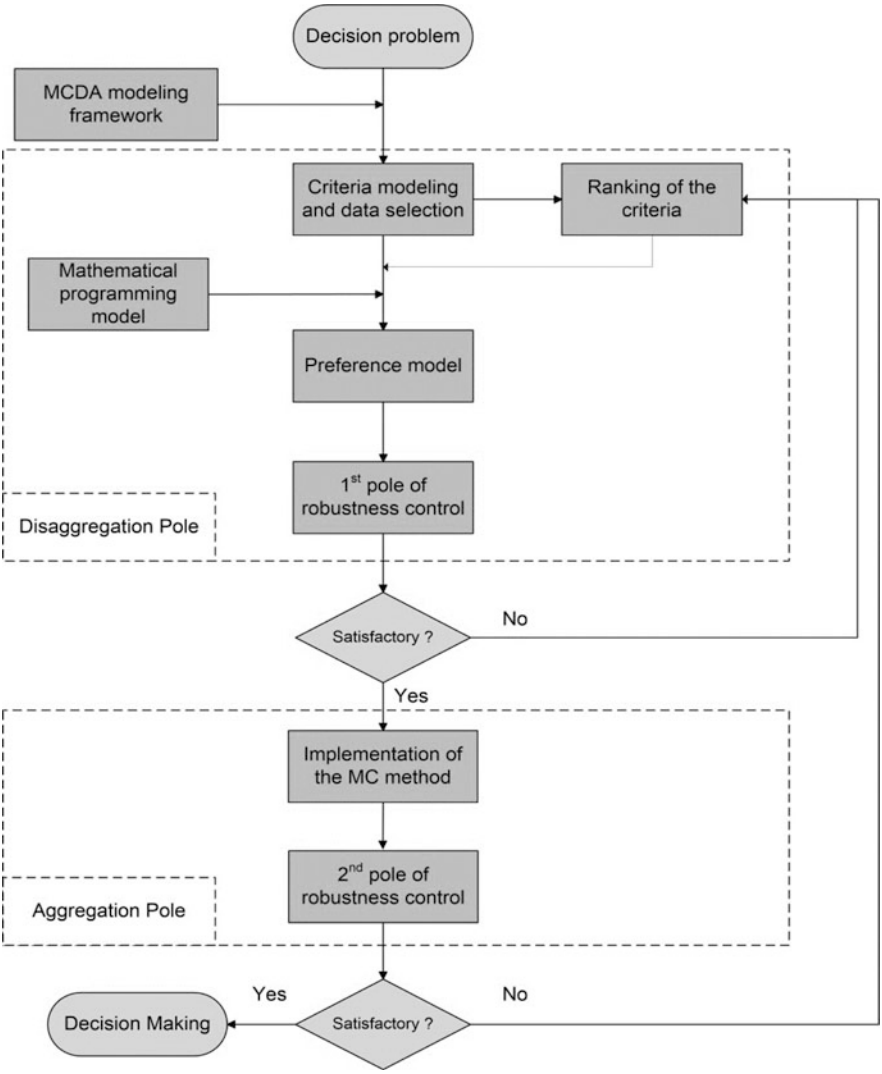



**Fig. 14.1** Bipolar robustness control flowchart for the synergy of Simos with an outranking method

In case the level of robustness in one of the two control points is not considered satisfactory, the analyst can return to the Disaggregation Pole and enrich the preferential information input, with the help of the DM, in an attempt to improve the robustness of both the model and the evaluation results.

### 14.3.2 Robustness Measures in the Disaggregation Pole

The measures proposed below aim at measuring the stability of the weighting coefficients, achieved through the Simos method, prior to their use in a given outranking method. These measures and indices are as follows:

*Variation range of weights* for each criterion separately, by means of the execution of $2n$ linear programmes ($max - min$ approach):

$$\min p_j \& p_j, \text{ for every } j = 1, 2, \ldots, n. \tag{14.10}$$

*s.t.*

$$\boldsymbol{p} \in \boldsymbol{P}. \tag{14.11}$$

**Average Weighting Vector (Barycenter)** This weighting solution is considered as the most representative weighting solution of polyhedron $\boldsymbol{P}$ and may be obtained by different techniques. One of them consists in listing and averaging all weighting solutions of the $2n$ linear programmes (14.10)–(14.11). Another technique consists of finding and recording all the vertices of the polyhedron $\boldsymbol{P}$, by using the Manas-Nedoma [14] analytical algorithm, which traverses all the vertices of the Hamiltonian path, and calculating a new average weighting vector, which represents the barycenter of $\boldsymbol{P}$ (see also [15]).

For the acquisition of a representative weighting solution, it is also possible to implement a random weight sampling algorithm/technique to produce and analyze statistically a great number of weighting sets from the polyhedron. A relevant technique is the stochastic multiobjective acceptability analysis (SMAA) initiated by Lahdelma et al. [16]. The analyst is capable of computing an average weighting solution, which could also be considered as a representative solution of $\boldsymbol{P}$. Other related techniques have been proposed by Greco et al. [15], Tervonen et al. [17] and Corrente et al. [6, 7].

**Average Range of the Preferential Parameters (ARP)** This index reveals the potential range of an average preferential parameter of the model, after considering the preference information, extracted by the DM. The calculation of the *ARP* index requires the a priori implementation of the max − min approach and is defined as follows:

$$\mathrm{ARP} = \frac{1}{m} \sum_{i=1}^{m} \left[ \max_i \left( p_{ij} \right) - \min_i \left( p_{ij} \right) \right], \tag{14.12}$$

where $m$ is the number of weighting instances of the system, considered during the max − min LPs procedure, $n$ is the number of criteria, and $p_{ij}$ is the weight of $i$th criterion for the $j$th instance. This index ranges in [0, 1] and receives lower values, when the robustness of a model increases. *ARP* receives the perfect value of 0, when a unique preference model reflects the preference statements of the DM.

**Average Stability Index (ASI)** For each of the above robustness techniques, the robustness measure *ASI* (Average Stability Index) can be also calculated. It reflects the mean value of the normalized standard deviation of the estimated weights:

$$\mathrm{ASI} = 1 - \frac{1}{n}\sum\nolimits_{i=1}^{n} \frac{\sqrt{m\sum\nolimits_{j=1}^{m} p_{ij}^2 - \left(\sum\nolimits_{j=1}^{m} p_{ij}\right)^2}}{\frac{m}{n}\sqrt{m-1}}. \tag{14.13}$$

*ASI* takes values in the interval [0, 1]. The robustness of the weights is considered acceptable (transition to the second robustness pole), when the value of the index approaches 1, i.e. around 0.95 [18].

### 14.3.3 Robustness Measures in the Aggregation Pole and Decision Support

Siskos and Tsotsolas [5] recommend that, when a decision analyst is willing to use an MCDA framework, coupled with a weighting elicitation procedure, which does not uniquely define the preferential parameters, one or more of the following activities should be implemented. Such procedures include the Simos method and calculate the criteria weights for the selection, ranking or sorting of a set of actions $A$.

1. Build on $A$ two distinct outranking relations, the necessary outranking ($aS^N b \iff aSb$, i.e. action $a$ outranks action $b$, for every weighting vector $\mathbf{p} \in \boldsymbol{P}$), and the possible outranking, $aS^P b$, meaning that there exists at least one weighting vector $\mathbf{p} \in \boldsymbol{P}$ for which $aSb$ (see [19, 20], for definitions and properties of these outranking relations, under the notion of *Robust Ordinal Regression*).
2. Define the maximum and minimum possible ranking positions for every action $a \in A$, through the use of mixed integer linear programming techniques (see [21]).
3. Perform random samplings in $\boldsymbol{P}$, in order to compute statistical measures, associated to outranking relations between all the actions in $A$ and to possible ranking positions for each action separately (see [22]). These statistical indices indicate either the probability that an action $a$ outranks action $b$, or the probability that action $a$ maintains its initial position, given by a representative ranking.

Additional robustness measures, which complement the aforementioned activities, are proposed below.

**Average Range of the Ranking (*A*RRI) and Ratio of the Average Range of the Ranking (*R*ARR)** The calculation of these indices prerequisites the implementation of the Extreme Ranking Analysis technique, proposed by Kadzinski et al. [21]. Specifically, *ARRI* depicts the possible number of positions that an average action can occupy in the complete ranking, while *RARR* reflects the ratio of the aforementioned deviation, in association with the total number of alternatives under evaluation. The optimal values of *ARRI* and *RARR* are 1% and 0%, respectively, and they are calculated using the following formulae:

$$\text{ARRI} = \frac{1}{N} \sum_{k=1}^{N} \left( \left| R_*(k) - R^*(k) \right| + 1 \right). \tag{14.14}$$

$$\text{RARR} = \frac{\text{ARRI} - 1}{N - 1} \cdot 100\%. \tag{14.15}$$

$R_*(k)$ and $R^*(k)$ are the worst and best possible ranking positions, respectively for the $k$th action and $N$ is the total number of actions under evaluation.

**Statistical Preference Indices** These indices offer a comprehensive way to examine the stability of the ranking positions, achieved by the whole entity of actions. Their calculation prerequisites the implementation of methods that generate a statistically adequate number of weighting vectors, within the polyhedron $\boldsymbol{P}$, such as the Manas-Nedoma algorithm, the SMAA technique, the Hit and Run algorithm [17, 23]. Then, an equal number of different rankings is calculated, as the number of different weighting vectors, generated by the weighting generation method.

Building on these multiple rankings, the statistical preference indices calculate the separate probabilities, that each action occupies a single ranking position in the final ranking, and constitute a measure to give a clear insight into the robustness of the results.

## 14.4 Implementation to the E-government Evaluation: Phase A'—First Elicitation of the Criteria Weights

### *14.4.1 E-government Evaluation Importance and Criteria Modelling*

Currently, with the global health crisis of 2020, caused by the COVID-19 pandemic, the availability, acceptance and impact of e-government are increasingly attracting attention [24]. Government measures taken by all the countries worldwide and the restraining of citizens from reaching public services, in the form of lockdown, stress the need for an empowerment and deepening of e-government services, provided by the national and local authorities to citizens and businesses. Respectively, Sá et al. [25] performed a literature review on the evaluation of the quality of local e-government services.

On the other hand, the rapid development, transformation and modernization of e-government creates an urgent need for the continuous evaluation of the performance of digitalized services in the world [26]. Towards this direction, e-government benchmarks are used to assess the progress in the area of service digitalization made by an individual country over a period of time, and to compare its growth against other countries [27]. Such benchmarks can have a significant practical impact, both political and potentially economic [28] and can influence the development of e-government services [29, 30]. The results of benchmarking and ranking studies, particularly global projects, conducted on a fixed chronical base by international organizations, attract considerable interest from a variety of observers, including governments and policy makers [31]. Such major e-government evaluations worldwide include, among others, the UN E-Government Survey [32] and the European Commission e-Government benchmark [33].

Researchers and practitioners evaluate the performance of e-government on the basis of four complementary perspectives, including readiness assessment, availability assessment, demand assessment, and impact assessment [34, 35]. The readiness evaluation examines the maturity of the e-government environment by evaluating the awareness, willingness, and preparedness of e-government stakeholders and identifying the respective enabling factors [29, 36]. Benchmarking indices and indicators for the readiness assessment are generally quantitative in nature, and collectively form a framework for ranking. To maximize the acceptability of results, rankings should be based on well understood and supported frameworks and indices, and sound computational procedures [37].

Towards this direction, Siskos et al. [38] outlined an MCDA methodological framework in order to evaluate and rank 22 European countries, based on their e-government performance. The evaluation framework models a consistent family of eight evaluation criteria, which are built on the following four points of view:

1. *Infrastructures* (two criteria: access to the web; broadband internet connection).
2. *Investments* (one criterion: % GDP on Information & Communications Technologies and Research & Development).
3. *E-processes* (two criteria: online sophistication; e-participation), and.
4. *User's attitude* against e-processes (three criteria: citizens' online interaction with authorities; businesses' online interaction with authorities; user's experience).

A summary of this evaluation system and the corresponding evaluation scales is presented in Table 14.1 (see [38] for a detailed presentation of the evaluation system). The evaluation of the 22 European countries on the eight criteria is given in Appendix 2. The data refer to the year 2017.

### *14.4.2 Initialization of the PROMETHEE II Method*

The application of PROMETHEE II for the evaluation of e-government requires the specification of the preference functions for all the criteria, as well as the estimation

**Table 14.1** Indices, criteria ranges, and preference thresholds

| Criterion | Index | Preference threshold |
|---|---|---|
| $g_1$-Access to the web | % population | 5% |
| $g_2$-Broadband internet connection | % population | 5% |
| $g_3$-% of GDP on ICT and R&D | % GDP | 0.1% |
| $g_4$-Online Sophistication | % | 10% |
| $g_5$-E-participation | Index [0–1] | 0.1 |
| $g_6$-Citizens' online interaction with authorities | % citizens | 10% |
| $g_7$-Businesses' online interaction with authorities | % businesses | 5% |
| $g_8$-User's experience | % index | 10% |

of the criteria importance weights. The decision-maker (DM) is a senior expert in the field of e-governance and the digitalization and interoperability of services.

After a dialogue between the analyst and the DM, for the determination of the preference function for each criterion, according to the typology of PROMETHEE II (see Appendix 1), the DM-evaluator concluded that, since all the criteria data stem from external but reliable sources, any difference in the evaluation between two countries is meaningful. However, he did question the preferential reliability of the so-called "small differences". To this end, he agreed on the adoption of a $p$ preference threshold for each criterion, and specifically the Type 3 function $P(p)$ for all eight criteria, based on formula (14.16). All eight preference thresholds $p_j$, $j = 1$, 2, .., 8, as given by the DM, are presented in the last column of Table 14.1.

$$P(d) = \begin{cases} 0 & d \leq 0 \\ \frac{d}{p} & 0 \leq d \leq p \\ 1 & d > p \end{cases} . \tag{14.16}$$

The Type 3 preference function (or V-shape) indicates that there is always a preference between two countries on any criterion, when there exists a difference in their values. This preference increases linearly, as long as the difference between the two countries in a criterion is less than the threshold value $p$, and takes the value of 1, when that difference reaches or exceeds the value of $p$.

In this way, the calculations of the tables $P_j(a, b)$ are performed for each criterion $g_j$ and for all the countries, as dictated by the formulae (14.2)–(14.5).

### 14.4.3 A′ Phase of the Robust Simos Method

After the calculation of the PROMETHEE preference tables, the elicitation of the weights of the criteria is performed, through the Robust Simos Method (RSM), in synergy with the bipolar robustness control algorithm and in full interaction with the DM. To this end, the analyst asks the DM to give a complete ranking of the criteria, based on their importance, in order to proceed to the application of the Simos method. The rank order of the criteria, in increasing importance and with the inclusion of white cards, is given in Table 14.2.

The results obtained after the implementation of the original Simos method are shown in Table 14.3. Particularly, the calculated criteria weights are given in the last two columns of the Table. At this point, the analyst could deliberately set the weight of the two least important criteria to the value: $p_5 = p_7 = 3.5$ (instead of 4.0), so that the sum of all the weights sharply reaches the value of 100.

Contrary to the original Simos method, the subsequent application of the Robust Simos Method (RSM) and the implementation of its underlying mathematical relations gives a clear overview of the first polyhedron $\boldsymbol{P}$. The polyhedron is framed by the linear constraints, imposed by the Simos ranking, see below, and it has been assumed that the minimum preference difference between two consecutive criteria or subset of criteria, $\delta$, is 0.01. The weights of the two white cards are denoted below as $w_1$ and $w_2$.

**Table 14.2** Ranking of the criteria as given by the DM (Increasing importance)

| **Criteria Ranking (increasing importance)** |
|---|
| $g_5, g_7$ |
| $g_8$ |
| White Card I |
| $g_4$ |
| $g_1$ |
| $g_3$ |
| $g_6$ |
| White Card II |
| $g_2$ |

**Table 14.3** Calculation of the e-government criteria weights with the original Simos algorithm

| Class | No of cards | Position | Non-normalized weight | Normalized weight | Total |
|---|---|---|---|---|---|
| {5, 7} | 2 | 1,2 | $\frac{1+2}{2} = 1.5$ | $\frac{1.5}{42} \times 100 = 3.6 \to 4$ | $2 \times 4 = 8$ |
| {8} | 1 | 3 | 3 | $\frac{3}{42} \times 100 = 7.1 \to 7$ | $1 \times 7 = 7$ |
| *White Card I* | 1 | (4) | – | – | – |
| {4} | 1 | 5 | 5 | $\frac{5}{42} \times 100 = 11.9 \to 12$ | $1 \times 12 = 12$ |
| {1} | 1 | 6 | 6 | $\frac{6}{42} \times 100 = 14.3 \to 14$ | $1 \times 14 = 14$ |
| {3} | 1 | 7 | 7 | $\frac{7}{42} \times 100 = 16.7 \to 17$ | $1 \times 17 = 17$ |
| {6} | 1 | 8 | 8 | $\frac{8}{42} \times 100 = 19.0 \to 19$ | $1 \times 19 = 19$ |
| *White Card II* | 1 | (9) | – | – | – |
| {2} | 1 | 10 | 10 | $\frac{10}{55} \times 100 = 23.8 \to 24$ | $1 \times 24 = 24$ |
| Total | 10 | 42 | | | ~100 |

**1st Polyhedron *P***

| | | | |
|---|---|---|---|
| 1. $p_5 - p_7 = 0$ | 2. $p_8 - p_7 \geq 0.01$ | 3. $w_1 - p_8 \geq 0.01$ | 4. $p_4 - w_1 \geq 0.01$ |
| 5. $p_1 - p_4 \geq 0.01$ | 6. $p_3 - p_1 \geq 0.01$ | 7. $p_6 - p_3 \geq 0.01$ | 8. $w_2 - p_6 \geq 0.01$ |
| 9. $p_2 - w_2 \geq 0.01$ | | | |
| 10. $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = 1$ | | | |
| 11. $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, w_1, w_2 \geq 0$ | | | |

The robustness control procedure that follows the framework of the RSM is initially implemented in the Disaggregation Pole, through the use of the max − min technique. After solving the relevant 16 linear programming problems (14.10)–(14.11), i.e. 8 max and 8 min, the results of Table 14.4 are obtained.

Figure 14.2 depicts the range of the eight criteria weights in the first polyhedron ***P***. Observing this diagram, it is immediately apparent that the weights are practically uncontrollable, since they exhibit very high variance within the feasible range [0, 1]. An extreme example is $p_2$, the fluctuation range of which exceeds 0.60 (60% of the maximum possible fluctuation).

Consequently, the analyst noticed without difficulty the inadequate stability of the model at this stage, which is also confirmed by the relatively low value of 0.901, obtained by the optimistic ASI index. On the other hand, the Average Range of Parameters (ARP) index elevates to the unsatisfactory value of 23.5%.

Therefore, the analyst considers that at this phase, it is impossible to calculate a representative model and thus proceed to the Aggregation Pole. The algorithm

**Table 14.4** Results of the A′ Phase max − min procedure

| Solution type | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ |
|---|---|---|---|---|---|---|---|---|
| min $p_1$ | 0.04 | 0.323 | 0.293 | 0.03 | 0 | 0.303 | 0 | 0.01 |
| max $p_1$ | 0.223 | 0.263 | 0.233 | 0.03 | 0 | 0.243 | 0 | 0.01 |
| min $p_2$ | 0.131 | 0.171 | 0.141 | 0.121 | 0.091 | 0.151 | 0.091 | 0.101 |
| max $p_2$ | 0.04 | 0.81 | 0.05 | 0.03 | 0 | 0.06 | 0 | 0.01 |
| min $p_3$ | 0.04 | 0.445 | 0.05 | 0.03 | 0 | 0.425 | 0 | 0.01 |
| max $p_3$ | 0.04 | 0.323 | 0.293 | 0.03 | 0 | 0.303 | 0 | 0.01 |
| min $p_4$ | 0.223 | 0.263 | 0.233 | 0.03 | 0 | 0.243 | 0 | 0.01 |
| max $p_4$ | 0.186 | 0.226 | 0.196 | 0.176 | 0 | 0.206 | 0 | 0.01 |
| min $p_5$ | 0.162 | 0.202 | 0.172 | 0.152 | 0 | 0.182 | 0 | 0.132 |
| max $p_5$ | 0.131 | 0.171 | 0.141 | 0.121 | 0.091 | 0.151 | 0.091 | 0.101 |
| min $p_6$ | 0.04 | 0.81 | 0.05 | 0.03 | 0 | 0.06 | 0 | 0.01 |
| max $p_6$ | 0.04 | 0.445 | 0.05 | 0.03 | 0 | 0.425 | 0 | 0.01 |
| min $p_7$ | 0.162 | 0.202 | 0.172 | 0.152 | 0 | 0.182 | 0 | 0.132 |
| max $p_7$ | 0.131 | 0.171 | 0.141 | 0.121 | 0.091 | 0.151 | 0.091 | 0.101 |
| min $p_8$ | 0.186 | 0.226 | 0.196 | 0.176 | 0 | 0.206 | 0 | 0.01 |
| max $p_8$ | 0.162 | 0.202 | 0.172 | 0.152 | 0 | 0.182 | 0 | 0.132 |
| Barycenter | 0.121 | 0.328 | 0.161 | 0.088 | 0.017 | 0.217 | 0.017 | 0.050 |

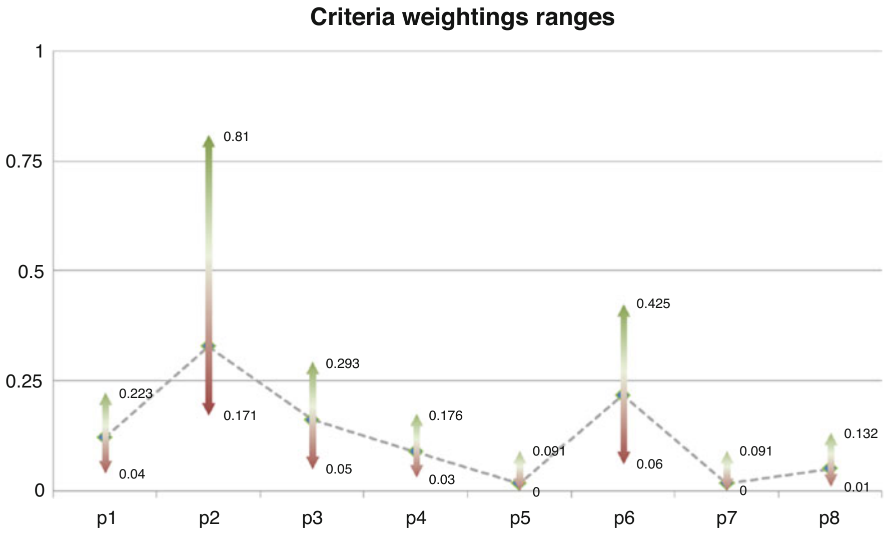



**Fig. 14.2** Criteria weights ranges (A′ Phase)

returns to the starting point of the Disaggregation Pole, with a view to acquiring additional preferential information by the DM and determine therefore more accurately the decision model.

## 14.5 Implementation to the E-government Evaluation: B′ and C′ Phases

### *14.5.1 B′ Phase of Robustness Control*

At the start of the B′ Phase of Robustness Control, the analyst considers it appropriate to ask the DM for further information, in order to improve the robustness of the model. Specifically, after observing, in the diagram of Fig. 14.2, that the weights of the two least significant criteria, $g_5$ and $g_7$, are possible to even be zero, which is unreasonable, he asks the DM to set minimum importance thresholds for them.

Indeed, the DM responds and sets the minimum possible weight for these two criteria at 3%. Thus, the following two new inequalities emerge, which are incorporated to the existing system of constraints:

$$p_5, p_7 \geq 0.03. \tag{14.17}$$

The new polyhedral space, in which the weights of the eight criteria are confined, is described below:

**Second Polyhedron *P***

| | | | |
|---|---|---|---|
| 1. $p_5 - p_7 = 0$ | 2. $p_8 - p_7 \geq 0.01$ | 3. $w_1 - p_8 \geq 0.01$ | 4. $p_4 - w_1 \geq 0.01$ |
| 5. $p_1 - p_4 \geq 0.01$ | 6. $p_3 - p_1 \geq 0.01$ | 7. $p_6 - p_3 \geq 0.01$ | 8. $w_2 - p_6 \geq 0.01$ |
| 9 $p_2 - w_2 \geq 0.01$ | 10. $p_5 \geq 0.03$ | 11. $p_7 \geq 0.03$ | |
| 12. $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = 1$ | | | |
| 13. $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, w_1, w_2 \geq 0$ | | | |

Having this new preference information included in the system of constraints, the analyst proceeds to the robustness control of the model in the Disaggregation Pole, using the max − min technique. The solution of the 16 corresponding linear programming problems (14.10)–(14.11) produces the results for the criteria weights that are shown in Table 14.5. The ranges of the criteria weights are graphically shown in the diagram of Fig. 14.3.

By comparing the two diagrams of the A′ and B′ Phase, a substantial reduction of the criteria weights ranges can be noticed. This improvement in the robustness of the model parameters between the two phases justifies the importance of the additional piece of information, provided by the DM. Simultaneously, the ASI increased to 0.937, while the ARP index decreased to 15.8%.

Based on the new findings, the analyst deems that it is now possible to build a tentative representative model, which marks the transition to the Aggregation

**Table 14.5** Results of the B′ Phase max − min procedure

| Solution type | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ |
|---|---|---|---|---|---|---|---|---|
| min $p_1$ | 0.07 | 0.273 | 0.243 | 0.06 | 0.03 | 0.253 | 0.03 | 0.04 |
| max $p_1$ | 0.193 | 0.233 | 0.203 | 0.06 | 0.03 | 0.213 | 0.03 | 0.04 |
| min $p_2$ | 0.131 | 0.171 | 0.141 | 0.121 | 0.091 | 0.151 | 0.091 | 0.101 |
| max $p_2$ | 0.07 | 0.60 | 0.08 | 0.06 | 0.03 | 0.09 | 0.03 | 0.04 |
| min $p_3$ | 0.07 | 0.355 | 0.08 | 0.06 | 0.03 | 0.335 | 0.03 | 0.04 |
| max $p_3$ | 0.07 | 0.273 | 0.243 | 0.06 | 0.03 | 0.253 | 0.03 | 0.04 |
| min $p_4$ | 0.193 | 0.233 | 0.203 | 0.06 | 0.03 | 0.213 | 0.03 | 0.04 |
| max $p_4$ | 0.168 | 0.208 | 0.178 | 0.158 | 0.03 | 0.188 | 0.03 | 0.04 |
| min $p_5$ | 0.152 | 0.192 | 0.162 | 0.142 | 0.03 | 0.172 | 0.03 | 0.122 |
| max $p_5$ | 0.131 | 0.171 | 0.141 | 0.121 | 0.091 | 0.151 | 0.091 | 0.101 |
| min $p_6$ | 0.07 | 0.60 | 0.08 | 0.06 | 0.03 | 0.09 | 0.03 | 0.04 |
| max $p_6$ | 0.07 | 0.355 | 0.08 | 0.06 | 0.03 | 0.335 | 0.03 | 0.04 |
| min $p_7$ | 0.152 | 0.192 | 0.162 | 0.142 | 0.03 | 0.172 | 0.03 | 0.122 |
| max $p_7$ | 0.131 | 0.171 | 0.141 | 0.121 | 0.091 | 0.151 | 0.091 | 0.101 |
| min $p_8$ | 0.168 | 0.208 | 0.178 | 0.158 | 0.03 | 0.188 | 0.03 | 0.04 |
| max $p_8$ | 0.152 | 0.192 | 0.162 | 0.142 | 0.03 | 0.172 | 0.03 | 0.122 |
| Barycenter | 0.124 | 0.277 | 0.155 | 0.099 | 0.041 | 0.195 | 0.041 | 0.067 |

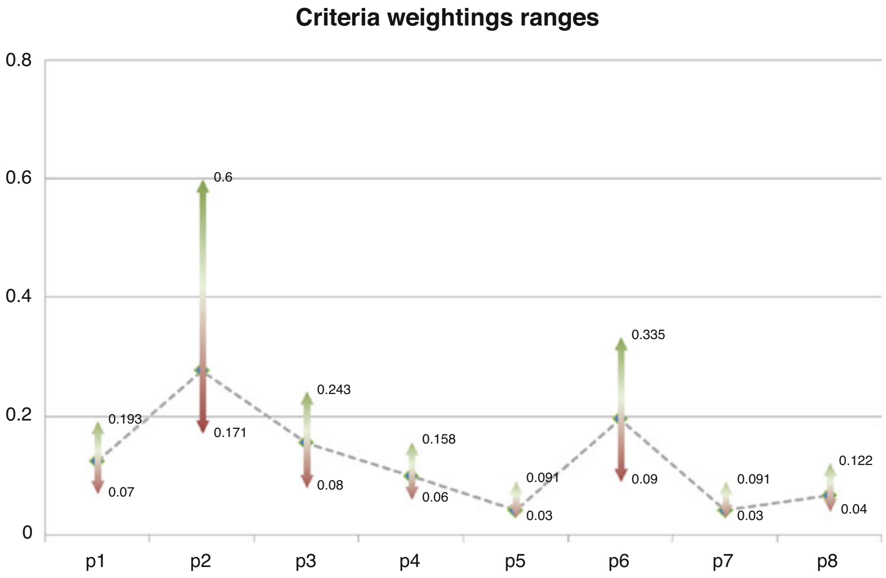



**Fig. 14.3** Criteria weights ranges (B′ Phase)

Pole. The specification of a representative model, using the barycentric weighting vector of Table 14.5, makes it possible to rank the 22 countries. The e-government ranking is obtained by applying the PROMETHEE II method and calculating the Net Outranking Flows $\varphi(a)$ of each country, according to the relations (14.7)–(14.9). In the end, the 22 countries are rank-ordered in a descending net outranking flow, as shown in Table 14.6.

At this point, the application of the model on the real country data allows the implementation of certain Aggregation Pole tools and indicators to assess the robustness of the results. In particular, the Extreme Ranking Analysis is applied, in order to calculate for each country the number of different positions it can achieve in the complete ranking.

The algorithm of Extreme Ranking Analysis was modelled and executed in the GAMS platform, and the occurring results are presented in the diagram of Fig. 14.4.

The visualization of the results of the Extreme Ranking Analysis advocate, that for the majority of the countries under evaluation, the ranking ranges are quite extensive. The robustness index ARRI gives the value of 3.0, which means that an average country has 3 possible ranking positions. Accordingly, the RARR index receives the non-acceptable value of 9.7%.

The aforementioned observations lead the analyst to the conclusion that the evaluation results are still not exhibiting satisfactory levels of robustness. He consequently decides to return to the Disaggregation Pole and repeat the whole procedure, as indicated by the bipolar robustness control methodological framework.

**Table 14.6** E-government ranking (representative evaluation model—B′ Phase)

| Rank | Country | Net Outranking flow |
|---|---|---|
| 1 | FI | 0.784 |
| 2 | SE | 0.682 |
| 3 | DN | 0.652 |
| 4 | NL | 0.631 |
| 5 | NO | 0.421 |
| 6 | FR | 0.296 |
| 7 | AT | 0.231 |
| 8 | EE | 0.197 |
| 9 | GE | 0.153 |
| 10 | UK | 0.141 |
| 11 | IR | 0.095 |
| 12 | SLO | 0.016 |
| 13 | BE | −0.048 |
| 14 | ES | −0.062 |
| 15 | PT | −0.227 |
| 16 | CZ | −0.409 |
| 17 | SLK | −0.435 |
| 18 | IT | −0.481 |
| 19 | HU | −0.549 |
| 20 | PO | −0.599 |
| 21 | HR | −0.744 |
| 22 | GR | −0.744 |

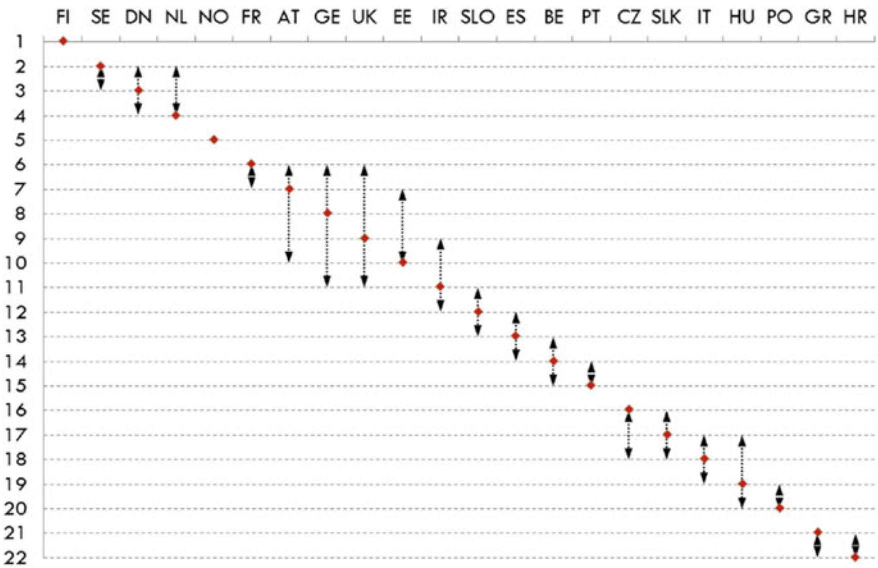



**Fig. 14.4** Extreme ranking analysis of the European countries for the B′ Phase evaluation model

### 14.5.2 C′ Phase of Robustness Control

As part of the C′ Phase of Robustness Control, the analyst revisits again the communication protocol with the DM, according to the procedures defined in the Disaggregation Pole, in order to improve the stability of the model. At this stage, he asks the DM to set a range for the ratio of significance ($z$, see Revised Simos above) between the most important and the least important criterion. Specifically, he asks the DM approximately how many times more important is the mostly appreciated criterion $g_2$ compared to the least appreciated criteria $g_5$ and $g_7$. The DM, after careful reasoning, specifies that the criterion $g_2$ is 5 to 5.5 times more important than the criteria $g_5$ and $g_7$.

This statement is mathematically translated to the double inequality (14.18), which again is added to the existing system of constraints.

$$5p_5 \leq p_2 \leq 5.5p_5. \tag{14.18}$$

The resulting new polyhedral space, within which the weights of the eight criteria are confined, is described below:

**Third Polyhedron *P***

| | | | |
|---|---|---|---|
| 1. $p_5 - p_7 = 0$ | 2. $p_8 - p_7 \geq 0.01$ | 3. $w_1 - p_8 \geq 0.01$ | 4. $p_4 - w_1 \geq 0.01$ |
| 5. $p_1 - p_4 \geq 0.01$ | 6. $p_3 - p_1 \geq 0.01$ | 7. $p_6 - p_3 \geq 0.01$ | 8. $w_2 - p_6 \geq 0.01$ |
| 9. $p_2 - w_2 \geq 0.01$ | 10. $p_5, p_7 \geq 0.03$ | 11. $5p_5 \leq p_2$ | 12. $p_2 \leq 5.5p_5$ |
| 13. $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = 1$ | | | |
| 14. $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, w_1, w_2 \geq 0$ | | | |

The analyst, upon acquiring the new information, proceeds to examine the robustness of the model in the Disaggregation Pole by reapplying the max − min technique. The results that occurred after solving the relevant 16 linear programming problems, 8 max and 8 min, are presented in Table 14.7. The ranges of the 8 criteria weights are additionally visualized in Fig. 14.5.

It is now apparent, from the diagrams above, that the ranges of the criteria weights have been significantly reduced, compared to the results of the previous two Phases. It is therefore evident that this contraction of the variability of the weights provides a solid basis for the derivation of a more reliable representative model.

The improvement of robustness of the results is likewise reflected in the ASI, which increased to 0.964. The value of the index is even closer to the value of one, which indicates the increasing robustness of the decision model. At the same time, the ARP index fell below 10%, specifically to 9.3%.

The considerable increase in the robustness of the model clearly enables the analyst to build a representative model and apply it to the dataset of the 22 countries. The construction of a representative evaluation model, using the barycentric weighting vector of Table 14.7, makes it possible to evaluate and rank the 22 countries.

**Table 14.7** Results of the C′ Phase max − min procedure

| Solution type | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ |
|---|---|---|---|---|---|---|---|---|
| min $p_1$ | 0.085 | 0.248 | 0.218 | 0.075 | 0.045 | 0.228 | 0.045 | 0.055 |
| max $p_1$ | 0.182 | 0.222 | 0.192 | 0.07 | 0.04 | 0.202 | 0.04 | 0.05 |
| min $p_2$ | 0.149 | 0.189 | 0.159 | 0.139 | 0.038 | 0.169 | 0.038 | 0.119 |
| max $p_2$ | 0.105 | 0.356 | 0.115 | 0.095 | 0.065 | 0.125 | 0.065 | 0.075 |
| min $p_3$ | 0.092 | 0.288 | 0.102 | 0.082 | 0.052 | 0.268 | 0.052 | 0.062 |
| max $p_3$ | 0.085 | 0.248 | 0.218 | 0.075 | 0.045 | 0.228 | 0.045 | 0.055 |
| min $p_4$ | 0.182 | 0.222 | 0.192 | 0.07 | 0.04 | 0.202 | 0.04 | 0.05 |
| max $p_4$ | 0.164 | 0.204 | 0.174 | 0.154 | 0.037 | 0.184 | 0.037 | 0.047 |
| min $p_5$ | 0.15 | 0.19 | 0.16 | 0.14 | 0.035 | 0.17 | 0.035 | 0.12 |
| max $p_5$ | 0.108 | 0.338 | 0.118 | 0.098 | 0.068 | 0.128 | 0.068 | 0.078 |
| min $p_6$ | 0.105 | 0.356 | 0.115 | 0.095 | 0.065 | 0.125 | 0.065 | 0.075 |
| max $p_6$ | 0.092 | 0.288 | 0.102 | 0.082 | 0.052 | 0.268 | 0.052 | 0.062 |
| min $p_7$ | 0.15 | 0.19 | 0.16 | 0.14 | 0.035 | 0.17 | 0.035 | 0.12 |
| max $p_7$ | 0.108 | 0.338 | 0.118 | 0.098 | 0.068 | 0.128 | 0.068 | 0.078 |
| min $p_8$ | 0.164 | 0.204 | 0.174 | 0.154 | 0.037 | 0.184 | 0.037 | 0.047 |
| max $p_8$ | 0.15 | 0.19 | 0.16 | 0.14 | 0.035 | 0.17 | 0.035 | 0.12 |
| Barycenter | 0.129 | 0.254 | 0.155 | 0.107 | 0.047 | 0.184 | 0.047 | 0.076 |

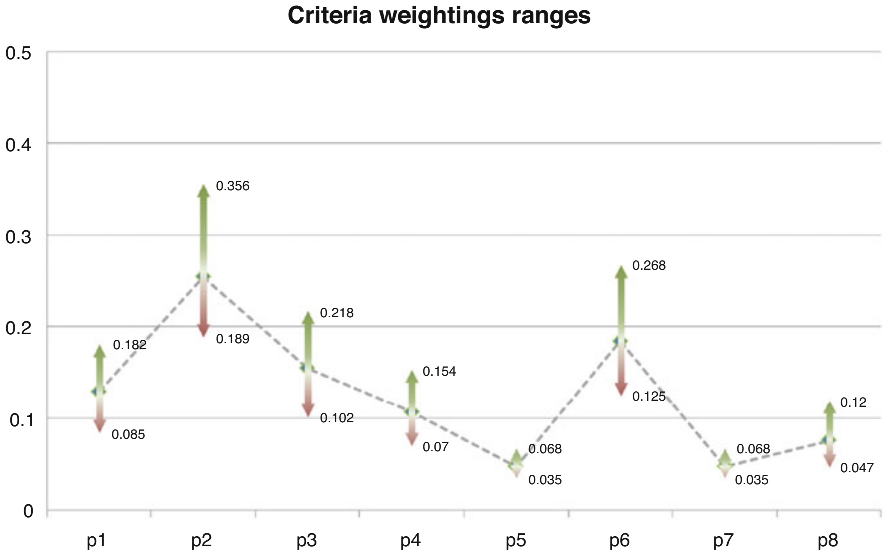



**Fig. 14.5** Criteria weights ranges (C′ Phase)

The ranking of the countries, resulting from the application of PROMETHEE II, is presented in Table 14.8, in descending Net Outranking Flows.

After applying the representative model to the country data, the analyst decides to implement certain robustness control techniques of the Aggregation Pole to validate

**Table 14.8** E-government ranking (representative evaluation model—C′ Phase)

| Rank | Country | Net Outranking flow |
|---|---|---|
| 1 | FI | 0.806 |
| 2 | SE | 0.717 |
| 3 | DN | 0.689 |
| 4 | NL | 0.666 |
| 5 | NO | 0.450 |
| 6 | FR | 0.270 |
| 7 | AT | 0.230 |
| 8 | GE | 0.211 |
| 9 | EE | 0.180 |
| 10 | UK | 0.170 |
| 11 | IR | 0.070 |
| 12 | SLO | 0.032 |
| 13 | BE | 0.003 |
| 14 | ES | −0.085 |
| 15 | PT | −0.282 |
| 16 | CZ | −0.424 |
| 17 | SLK | −0.446 |
| 18 | IT | −0.524 |
| 19 | HU | −0.549 |
| 20 | PO | −0.633 |
| 21 | HR | −0.763 |
| 22 | GR | −0.787 |

the robustness of the results. Similar to Phase B′, the Extreme Ranking Analysis gives a clear picture of the robustness of the final ranking of the 22 countries. The obtained results are shown in Fig. 14.6.

The application of the Extreme Ranking Analysis revealed a notable reduction in the number of potential ranking positions of each country. In particular, the ARRI decreased to 2.3, while RARR dwindled to 6.2%. These indicators show that an average country in the ranking can occupy 2.3 positions in the ranking, while the overall ranking exhibits a total volatility of 6.2%.

At the end of the C′ Phase, the analyst, in agreement with the DM, considers that the results of the bipolar robustness control procedure are sufficiently satisfactory and can support a reliable ranking of the 22 European countries. The decision model, developed on the basis of the Simos Method elicitation protocol, is assumed as sufficiently robust too. Both the DM and the analyst deem that it can adequately perform the evaluation and ranking of additional countries and/or the same, when new data on the indicators are made available. They consequently decide to terminate the algorithmic process of robustness control, and mark the e-government ranking of C′ Phase as final.

Finally, the evolution of the values of the robustness indices, through the three-phase decision support procedure, is illustrated in Table 14.9. The parentheses show the percentagewise improvement of the indicators from one Phase to the next.

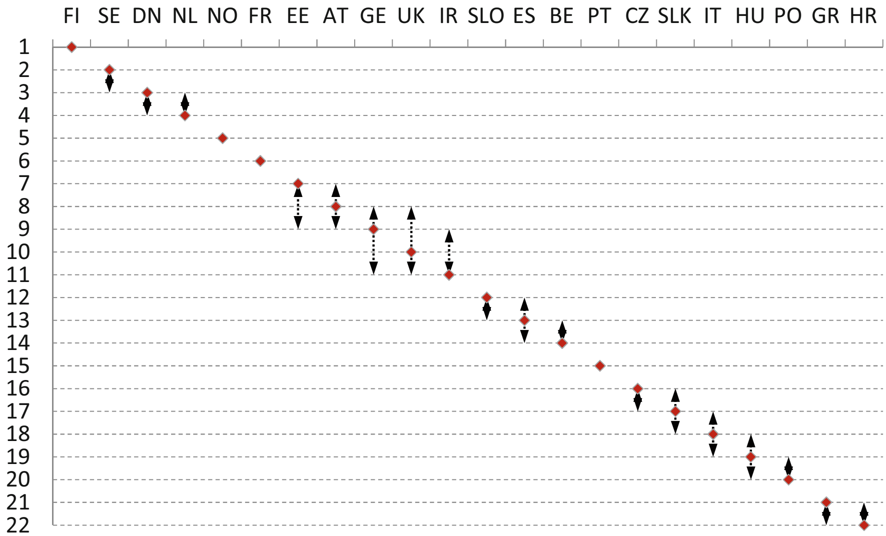



**Fig. 14.6** Extreme ranking analysis of the European countries for the C′ Phase evaluation model

**Table 14.9** Evolution of the robustness indices through the implementation of the Robust Simos Method

| Index | A′ phase | B′ phase | C′ phase |
|---|---|---|---|
| ASI | 0.906 | 0.937 (+3.4%) | 0.964 (+2.9%) |
| ARP | 23.5% | 15.8% (−32.7%) | 9.3% (−41.1%) |
| ARRI | – | 3.0 | 2.3 (−23.3%) |
| RARR | – | 9.7% | 6.2% (−36.1%) |

## 14.6 Conclusion

This paper proposes and outlines an interactive synergy of the two complementary multiple criteria methods "PROMETHEE and Robust Simos", when used to evaluate a predefined set of actions. Towards this directions, special effort is made to achieve robust results, i.e. results that are resistant to the possible fluctuations of preference and other data. This has been ensured through the proposition of an algorithmic procedure, called bipolar robustness control, which measures and manages the robustness of the evaluation results by controlling the quality of the input data. The proposed framework is successfully applied to the problem of the evaluation of the e-government performance of 22 European countries. The net outranking flows, achieved through the application of the PROMETHEE II method, can be interpreted as a degree of superiority (positive flow) or inferiority (negative flow) of each country, compared to the e-government performance of the average European country.

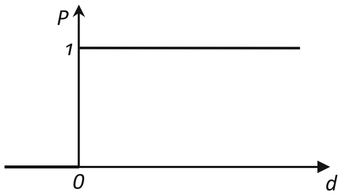
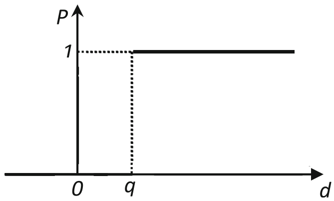
This evaluation procedure can identify structural gaps of performance, as well as insufficiencies in specific countries, which need to be accounted by Europe as a whole, in order to pursue a mutual path of continuous improvement and development. On the other hand, the benchmarking and dynamic nature of the evaluation can provide valuable insights to policy makers, to assess the progress made by a specific country and adjust the strategical e-government empowerment framework likewise.

From a methodological point of view, the benchmarking and ranking tools, presented in this paper, can also serve to different evaluation problems and complex decision-making procedures, which concern the allocation of resources for efficient policymaking. Such problems include for instance the ranking of institutions, the evaluation of technological and human development projects for clean energy and climate change mitigation, the ranking of global energy security and poverty and the evaluation of urban planning solutions.

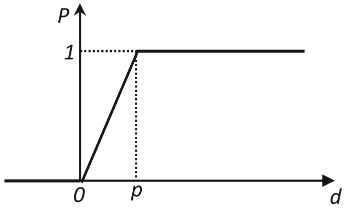
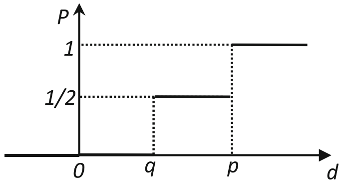
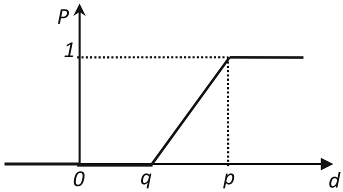
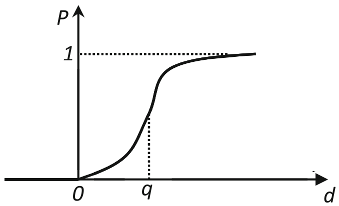
The proposed modelling of e-government evaluation, in particular, could set the basis for a more inclusive global or European assessment, incorporating, more explicit aspects of e-governance, such as extensive e-participation, as well as the interoperability of services and websites. Such a framework can account for new technological developments, in the field of maturity of digital services and public acceptance, which are gradually becoming largely applicable.

# Appendices

## *Appendix 1: Typology of PROMETHE's Generalized Criteria: Preference Function P(d),d: Evaluation Difference [2]*

| Generalized criterion | Definition | Parameters |
|---|---|---|
| *Type 1:*<br>Uniform<br> | $P(d) = \begin{cases} 0 & d \le 0 \\ 1 & d > 0 \end{cases}$ | – |
| *Type 2:*<br>U-shape<br> | $P(d) = \begin{cases} 0 & d \le q \\ 1 & d > q \end{cases}$ | ***q*** |

(continued)

| Generalized criterion | Definition | Parameters |
|---|---|---|
| *Type 3:*<br>V-shape criterion<br><br> | $P(d) = \begin{cases} 0 & d \leq 0 \\ \frac{d}{p} & 0 \leq d \leq p \\ 1 & d > p \end{cases}$ | $\boldsymbol{p}$ |
| *Type 4:*<br>Level<br><br> | $P(d) = \begin{cases} 0 & d \leq q \\ \frac{1}{2} & q < d \leq p \\ 1 & d > p \end{cases}$ | $\boldsymbol{p}, \boldsymbol{q}$ |
| *Type 5:*<br>V-shape with indifference criterion<br><br> | $P(d) = \begin{cases} 0 & d \leq q \\ \frac{d-q}{p-q} & q < d \leq p \\ 1 & d > p \end{cases}$ | $\boldsymbol{p}, \boldsymbol{q}$ |
| *Type 6:*<br>Gaussian criterion<br><br> | $P(d) = \begin{cases} 0 & d \leq 0 \\ 1 - e^{\frac{-d^2}{2s^2}} & d > 0 \end{cases}$ | $\boldsymbol{s}$ |

## *Appendix 2: Performance of the European Countries on the Eight Evaluation Criteria*

| Country | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ |
|---|---|---|---|---|---|---|---|---|
| Belgium | 90.00 | 67.75 | 2.28 | 63.00 | 0.25 | 36.00 | 74.33 | 58.00 |
| Czech Republic | 88.00 | 63.76 | 1.91 | 56.00 | 0.25 | 21.33 | 87.67 | 44.50 |
| Denmark | 96.00 | 72.78 | 3.05 | 85.00 | 0.55 | 65.33 | 89.33 | 65.00 |
| Germany | 93.50 | 71.26 | 2.94 | 67.00 | 0.70 | 33.33 | 58.67 | 45.50 |
| Estonia | 89.50 | 69.50 | 1.74 | 87.00 | 0.76 | 35.00 | 80.00 | 77.50 |
| Ireland | 90.00 | 66.17 | 1.58 | 87.00 | 0.65 | 43.00 | 88.00 | 62.00 |
| Greece | 77.50 | 59.62 | 0.78 | 46.00 | 0.80 | 27.67 | 78.33 | 41.00 |
| Spain | 86.00 | 66.42 | 1.24 | 91.00 | 0.78 | 36.33 | 69.00 | 72.50 |
| France | 91.00 | 68.51 | 2.23 | 75.00 | 0.96 | 44.00 | 89.00 | 68.50 |
| Croatia | 82.00 | 61.67 | 0.81 | 53.00 | 0.33 | 19.33 | 81.00 | 48.00 |
| Italy | 85.50 | 63.35 | 1.25 | 77.00 | 0.78 | 15.67 | 69.67 | 60.50 |
| Hungary | 81.50 | 61.96 | 1.41 | 45.00 | 0.45 | 34.33 | 82.33 | 35.50 |
| Holland | 98.00 | 77.40 | 1.98 | 82.00 | 1.00 | 57.67 | 80.67 | 65.50 |
| Austria | 89.50 | 67.41 | 2.81 | 86.00 | 0.63 | 40.33 | 80.67 | 70.50 |
| Poland | 84.00 | 62.50 | 0.87 | 76.00 | 0.49 | 17.33 | 81.67 | 51.00 |
| Portugal | 81.00 | 63.63 | 1.36 | 96.00 | 0.65 | 30.67 | 81.00 | 74.00 |
| Slovenia | 87.50 | 67.39 | 2.59 | 68.00 | 0.39 | 37.00 | 85.00 | 63.00 |
| Slovakia | 88.00 | 63.53 | 0.83 | 72.00 | 0.63 | 33.00 | 80.67 | 30.00 |
| Finland | 95.00 | 75.46 | 3.32 | 86.00 | 0.71 | 64.00 | 91.33 | 71.00 |
| Sweden | 94.00 | 74.94 | 3.21 | 83.00 | 0.61 | 60.33 | 90.67 | 68.50 |
| Norway | 95.00 | 71.42 | 1.69 | 78.00 | 0.69 | 64.33 | 84.33 | 63.50 |
| United Kingdom | 92.50 | 72.33 | 1.63 | 74.00 | 0.96 | 35.00 | 75.00 | 51.00 |

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# Chapter 15
# The Use of Decision Maker's Preferences in Multiobjective Metaheuristics

**Juergen Branke, Andrzej Jaszkiewicz, and Piotr Zielniewicz**

**Abstract** In multiobjective optimization, there is usually not a single optimal solution as result, but a set of so-called Pareto-optimal solutions with different trade-offs between the objectives. Metaheuristics, in particular evolutionary algorithms, have been very successful in solving such problems, because they can simultaneously search for multiple Pareto-optimal solution. However, at some point a decision maker has to be involved to pick the final solution. This chapter reviews and summarizes the research of Prof. Roman Słowiński, members of his Laboratory, and his main collaborators related to the use of the decision maker's preferences in metaheuristic and evolutionary algorithms.

## 15.1 Introduction

Metaheuristics, including evolutionary algorithms, are very successful tools for approximate continuous and combinatorial optimization [10, 11]. Although single objective metaheuristics may be applied in multiobjective context, for example for optimization of parametrized scalarizing functions, the full potential of this class of algorithms is exhibited in multiobjective metaheuristics (MOMHs), which aim at generation of multiple approximately Pareto-optimal solutions in a single run [22, 23]. This stream of research started with the proposition of the Vector Evaluated Genetic Algorithm by Schaffer [24] and became an extremely active field of research and application. In particular, evolutionary multiobjective optimization algorithms (EMOs) and other population-based algorithms can take advantage of processing a population of solutions that could spread over the whole or over a sub-

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region of the Pareto front [9, 22]. However, in most applications, the decision maker (DM) is only interested in a single solution, for example a project manager needs to select a single schedule for a project taking into account multiple objectives. So, at some point the DM's preferences with respect to the relative importance of the different criteria need to be revealed. Depending on the moment of collecting the preference information with respect to the optimization process, multiobjective methods can be classified as either having a priori, a posteriori, or progressive (interactive) articulation of preferences [16, 25]. A priori elicitation of preferences may require single objective optimization (e.g. optimization of a utility function), which may be performed with single objective metaheuristics. In the case of interactive and a posteriori approaches, MOMHs and EMOs are more natural choices.

Taking into account the interest of Roman Słowiński in DM preference elicitation, modelling, and exploitation, it is natural that his research and the research of some members of his team has been focused on the use of DM preferences in the class of MOMH and EMO algorithms. So this chapter highlights the contributions he and his team made to the field. Since he started working on this topic, the combination of ideas from multi-criteria decision analysis and multiobjective metaheuristics has quickly become an active research field over recent years, with a regular Dagstuhl seminar (https://www.dagstuhl.de) on the topic, which also resulted in a book co-edited by Roman Słowiński [1].

## 15.2 Two-Stage Approach for Interactive Analysis of Multiobjective Combinatorial Optimization Problems

In 1997 two members of the Roman Słowiński's team, Piotr Czyżak and Andrzej Jaszkiewicz, proposed Pareto simulated annealing (PSA) [7], one of the first multiobjective version of SA algorithm. PSA uses a population of solutions that repel each other in order to spread over the whole Pareto front. Combining this proposition with the interest of Roman Słowiński in preference modelling, project scheduling, and optimization under uncertainty, he and his co-authors proposed a methodology for interactive analysis of multiple criteria fuzzy project scheduling problems with a two-stage approach [14, 15]. Since project scheduling is a multiobjective combinatorial optimization problem, PSA has been applied to generation of a large set of approximately Pareto-optimal solutions from the point of view of the following objectives:

- The project completion time
- The total project cost
- The resource smoothness rate expressed as the average deviation from the average resource usage expressed in resource units

The first two objectives were taking fuzzy values, which required an adaptation of PSA for comparison of fuzzy numbers. In addition a fuzzy scheduling procedure using weak and strong comparison rules for fuzzy numbers was applied. Since the number of approximately Pareto-optimal solutions could be very large, the DM may still require a support in selecting the best compromise solution. The authors proposed the use of Light Beam Search (LBS) method [20] in the second stage for interactive analysis of large set of alternatives. LBS enables an interactive analysis through presentation of samples of solutions to the DM in each iteration. It uses a local preference model in the form of an outranking relation to define the neighbourhood of the current solution from which the sample is generated. The method has been originally proposed by Jaszkiewicz and Słowiński for multiobjective nonlinear optimization; however, it was later adapted to the analysis of large sets of predefined alternatives/solutions [19]. The DM can control the search by either modifying the aspiration and reservation points or by shifting the current point to a selected better point from its neighbourhood. The proposed two-stage approach has been applied to an agricultural project scheduling problem and it was one of the first approaches combining multiobjective metaheuristics and the use of DM's preferences.

## 15.3 Preference-Based Evaluation of Multiobjective Metaheuristics

In 1998 Andrzej Jaszkiewicz from Roman Słowiński's Laboratory and Michael Hansen from Technical University of Denmark published a technical report [13], which was one of the first papers on evaluation of multiobjective metaheuristics and evolutionary algorithms. Although this paper has never been published as a journal or conference publication (the topic was probably too new for reviewers at that time), it became one of the most cited papers from Roman Słowiński's Laboratory, and many results were later re-confirmed in other papers (see e.g. [31]). What is important, motivated by the works of Roman Słowiński, Hansen and Jaszkiewicz looked at the problem of measuring the quality of approximations to the non-dominated set from the perspective of the DM's preferences. They started with the observation that in the case of a multiobjective optimization problem, the overall goal of the decision-maker (DM) is to select the single solution, which is the most consistent with his or her preferences. Generating an approximation to the non-dominated set is only a first phase in solving the problem. In the second phase, the DM selects the best compromise solution from the approximation, possibly supported by an interactive procedure. Therefore, the DM may consider approximation A as being better than approximation B if he or she can find a better compromise solution in A than in B. They assumed, however, that the DM's preferences are not known a priori. In fact, the use of heuristics generation of approximations to the full non-dominated set is justified only in this case.

Nevertheless, one may be able to make some general assumptions about possible DM's preferences. Using these assumptions, one can state that an approximation A outperforms (is better than) B if, for some possible preferences held by the DM, the DM may find a better compromise solution in A than may be found in B, and for other possible preferences, the solutions found in A will be not worse than those found in B. This led to the definitions of weak, strong, and complete outperformance relations w.r.t. a set of possible preferences and the concept of compatibility of quality indicators with these relations. The most general and widely accepted assumption about DM's preferences is the compatibility with a dominance relation, which means that the DM always prefers a dominating solution over the dominated one. This led to the concept of the compatibility of quality indicators with the dominance relation, which was the basis for the general acceptance of the hypervolume indicator that guarantees this compatibility [31].

Andrzej Jaszkiewicz continued the work on evaluation of multiobjective metaheuristics and evolutionary algorithms in [17], where he analysed the efficiency of multiobjective metaheuristics vs. their single objective counterparts applied to optimization of weighted scalarizing functions from (among others) the point of view of interactive analysis. The question asked was, what is the relative computational efficiency of the scenario where a multiobjective metaheuristic is run first and then the DM uses an interactive method to analyse the large set of potentially Pareto-optimal solutions (see Sect. 15.2) vs. the scenario where a single objective metaheuristics is interactively used to optimize scalarizing functions with different weights to generate on-line solutions presented to the DM. The different ways of using interactive analysis together with metaheuristics and evolutionary algorithms were further summarized in [18].

## 15.4 Evolutionary Multiobjective Optimization Algorithms Based on Robust Ordinal Regression

In contrast to traditional MCDA methods that rely on the selection of a single compatible instance of the preference model, the methods based on the Robust Ordinal Regression (ROR) construct recommendations taking into account all instances of the preference model compatible with the preference information provided by the DM. Since these different instances of the preference model may favour different solutions, the result may still be a range of Pareto-optimal solutions. Since finding a set of Pareto optimal solutions is precisely the strength of multiobjective evolutionary algorithms, a combination of the two methods makes a lot of sense.

In [2–5] Roman Słowiński with his co-workers presented an original family of algorithms for solving multiobjective optimization problems, called NEMO (Necessary-preference-enhanced Evolutionary Multiobjective Optimizer), which combines an evolutionary multiobjective optimization with an interactive procedure based on robust ordinal regression. The motivation for suggesting this approach was the observation that with the increase of the number of criteria, many EMO algorithms have difficulty to find a good representation of the Pareto front, and the conviction that focusing on a small area of the Pareto front will allow to find an interesting solution more quickly and reach the proper region of the Pareto front with more accuracy.

The preference model used in almost all NEMO algorithms has the form of an additive value function, which is compatible with the preference information provided by the DM in the form of holistic pairwise comparisons of some non-dominated solutions from the population. The marginal value functions can be linear, piecewise linear, or general monotonic.

The basic NEMO-I algorithm proposed in [2] is based on the 2-criteria selection mechanism known from the NSGA-II algorithm [8], in which the dominance relation used to rank solutions in the population is replaced by the necessary preference relation, and the crowding distance calculated in the criteria space is replaced by the crowding distance calculated in the value space.

The necessary preference relation between two solutions occurs if and only if the first one is at least as good as the second for all instances of the preference model compatible with the preference information provided by the DM [12]. This relation can be considered as robust with respect to the preference information. To determine the occurrence of the necessary preference relation between two solutions, it is necessary to solve one or two corresponding linear programming problems. The crowding distance in the value space is calculated using a single, representative value function compatible with the preference information given by the DM. For this purpose, the NEMO-I algorithm uses the most discriminating value function, i.e. value function that maximizes the difference of scores between solutions related by preference in the necessary ranking.

However, the study of the NEMO-I algorithm revealed two of its practical drawbacks. The first one was a long time of calculation (especially in the case of large problems) due to its computational complexity. In every iteration of the evolutionary algorithm, it is necessary to solve $O(n^2)$ linear programming problems (with $n$ being the population size) in order to rank solutions in the population from the best to the worst using the necessary preference relation. The second drawback was the preference model based on general-monotonic marginal value functions, which turned out to be too flexible. As a consequence of this flexibility, a lot of preference information is required to learn a useful model and obtain satisfactory results.

In [4] the same authors proposed a new variant of the algorithm, called NEMO-0, devoid of the above-mentioned drawbacks. The main assumption of the NEMO-0 algorithm is to rank solutions in the population using a single value function compatible with the preference information provided by the DM. The selection mechanism implemented in the algorithm is based on two steps. First, the solutions in the population are ranked by the dominance relation, as in the NSGA-II method. Next, all solutions located in the same dominance front are sorted using a representative value function compatible with the preference information obtained from the DM. In the discussed paper, the authors implemented and tested 5 different representative functions.

In [5] Roman Słowiński and his co-workers presented another effective concept of ranking solutions in the population during the selection process. This variant of the algorithm, called NEMO-II, focuses on finding potential preferred solutions, i.e. those that are best for at least one value function compatible with the preference information provided by the DM. For each solution in the population, the algorithm solves the corresponding linear programming problem. The linear computational complexity of this algorithm positions it between the NEMO-I algorithm (square complexity) and the NEMO-0 algorithm (constant complexity).

Moreover, for the first time in the context of evolutionary multiobjective optimization, the authors proposed the non-additive DM's preference model having the form of the Choquet integral (NEMO-II-Ch), which allows interaction between the criteria. The Choquet integral is based on the concept of capacity (fuzzy measure) that assigns a weight to each subset of criteria rather than to each single criterion. As there exists a trade-off between the flexibility of the value function model and the complexity of learning a faithful model of DM's preferences, the NEMO-II-Ch algorithm starts the interactive process with a simple linear model but then switches to the Choquet integral when a richer DM's preference information becomes available and can no longer be represented using the linear model. Conducted research confirmed the high efficiency of the NEMO-II-Ch algorithm with respect to its variants based on an additive preference model with linear and piecewise-linear marginal value functions. Yet another preference model, namely preference cones following the $L_\alpha$ norm from a pre-defined reference point, has been proposed in [27].

In [21], Kadzinski and Tomczyk extend NEMO-0 and NEMO-II to a group decision setting, either searching for the solution(s) that provide the best total utility or maximize the minimum utility among the DMs.

The EMOSOR algorithm by Milosz Kadzinski and co-authors [26] follows a similar idea as NEMO-II but rather than solving LPs to determine whether a solution may be preferable according to a compatible value function, it randomly generates a set of compatible value functions and then uses this set to rank solutions. This is not only computationally more efficient but also allows to derive a more detailed ranking of the solution. Different options for ranking solutions are examined in the paper, such as a ranking of solutions according to the percentage of generated value functions that prefer a particular solution. As a further advantage, such information can be used in the preference elicitation step to decide which pair of solutions should be shown to the DM for ranking. The idea there is to gather as much information as possible in the sense of maximally reducing the number of potentially optimal solutions remaining in the population (i.e. narrowing down as much as possible the region of interest).

While the above methods are adaptations of NSGA-II [8] as the baseline EMO algorithm, similar ideas have also been applied to MOEA/D [30]. MOEA/D co-evolves a number of sub-populations, each optimizing a different scalarized value function. However the subpopulations share information by using each others' individuals for crossover or by individuals migrating from one sub-population to another. The scalarized value functions in MOEA/D are usually carefully chosen to ensure an even spread of solutions along the Pareto front. In [27, 29], scalarized value functions are chosen randomly, but only among those compatible with the DM's preference information. Preference information is once again based on pairwise comparisons, and as more and more pairwise preference information is elicited, the set of compatible scalarized value functions shrinks, and consequently the sub-populations focus increasingly on the most desirable (according to the DM's preferences) part of the Pareto frontier. As an additional feature, the algorithm in [29] dynamically alters the sizes of the co-evolved sub-populations.

In [28] the preference elicitation step in interactive evolutionary multiobjective optimization is focused. It examines strategies for selecting pairs of solutions to be shown to the DM, proposes a mechanism to decide when the DM should be questioned, and compares different types of indirect preference elicitation mechanisms such as pairwise comparisons, best-of-k judgments, or complete orders of a small subset of solutions.

In [6] the Dominance-based Rough Set Approach to model DM preferences is used. In this approach, the DM is asked to classify sets of solutions into good and bad, from which preference information is derived in the form or "if … then …" decision rules, which can then be used to guide the evolutionary algorithm. The main motivation is the explainability of the learned preference model.

## 15.5 Conclusion

Professor Roman Słowiński was one of the pioneers in bridging the gap between multiple-criteria decision analysis focused on elicitation, modelling, and exploitation of the DM's preferences, as well as multiobjective metaheuristics focused on computation of the (approximately) Pareto-optimal solutions for hard optimization problems. Originally researchers from each of these fields knew little about activities and achievements of the other, and, in fact, poorly understood the importance of the other field. This situation changed, among others, due to a series of already mentioned Dagstuhl seminars (https://www.dagstuhl.de) with participation of top researchers from both fields. Roman Słowiński was one of the organizers and very active member of these seminars. It resulted in the growing interest of the research community in the use of preferences in multiobjective metaheuristics. For example, Fig. 15.1 shows the number of publications (according to Scopus) with terms "(interactive OR preference) AND multi-objective AND evolutionary algorithm" in the title, abstract, or keywords, between years 2000 and 2020. Roman Słowiński and his research group have substantially contributed to the growth of this field and shaped it in many ways.

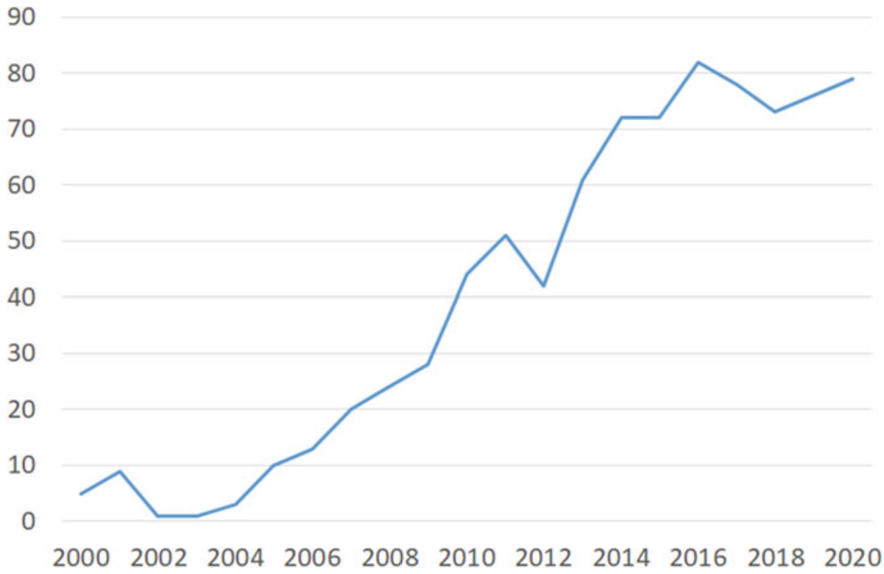


**Fig. 15.1** Numbers of publication on the use of preferences and interactive analysis in multiobjective evolutionary algorithms

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# Chapter 16
# Decomposition and Coordination for Many-Objective Optimization

**Margaret M. Wiecek and Philip J. de Castro**

**Abstract** Many-objective programs have often been reported in the literature as mathematical optimization tools to model and solve decision-making problems in many areas of human activity. However, in the presence of many objectives, the solution of the overall problem in its entirety may be challenging or even impossible, and decomposition of the problem into subproblems with a smaller number of criteria becomes appealing provided solving the subproblems can be coordinated and related to solving the initial problem. Based on an overview of selected decomposition and coordination methods, we put forward a list of steps that may be followed with certain desirable properties for a decomposition–coordination technique to effectively support the decision-maker and allow for an interactive tradeoff analysis. We support this process with theoretical results pertaining to a quasi-separable many-objective program and illustrate this process on an engineering design example problem.

## 16.1 Introduction

Multicriteria decision-making (MCDM) has become an important modeling and methodological tool to successfully support decision-making processes in business, management, and engineering in the presence of multiple and conflicting objectives (criteria or goals) or multiple decision-makers (DMs) [27]. Since the 1970s, this research field has been rapidly growing and gaining numerous studies performed around the world by academics and practitioners of diverse backgrounds [3]. Multiobjective optimization, as part of MCDM, addresses decision-making for systems whose performance results from the decisions selected within a feasible set and is evaluated by vector-valued functions representing multiple and potentially conflicting objectives. Due to possible conflicts, a unique optimal solution (decision)

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yielding the best system performance does not generally exist. Rather, within the set of feasible solutions, there exists a set of *efficient solutions* that offer tradeoff options for system performance. Solving a multiobjective optimization problem (MOP) is understood as finding the set of efficient solutions, its representative subset, or a preferred efficient solution. Each of these tasks can be challenging in its own way. Computing the efficient set may not be accomplished with a polynomial time algorithm [15], while deciding whether efficient solutions are representative or preferred may be subjective [4]. Furthermore, factors such as the type of mathematical model or the number of objective functions also profoundly affect the difficulty of solving MOPs.

MOPs with objectives modeled by linear and/or quadratic functions and feasible sets modeled by linear functions might be considered easy because they can be solved for their complete efficient sets [37, 48]. However, if these MOPs have, say, four or more criteria, the selection of preferred solutions may not be straightforward for the DM due to the difficulty of comparing tradeoff decisions in four or higher dimensions. On the other hand, a tradeoff analysis for a biobjective optimization problem (BOP) may be simple, but computation of efficient solutions may not be if the BOP is a global optimization problem.

MOPs with four or more objectives have often been reported in the literature and attracted interest of many researchers. Such works include 4-objective problems in engineering [22] and radiotherapy [56], 9-objective control problem [9], 7-objective classification problem [39], and 10-objective calibration problem in ecology [46]. In [49], the number of objectives is classified as small (2–3), moderate (4–20), and large (up to hundreds), and a wide range of applications from aerodynamic design to medical decision-making and land use planning are presented. MOP models proposed more recently in chemical engineering [47], pharmaceutical industry [44], autonomous ground systems [33], and emission reduction [11] also have many objectives.

Two decades ago, MOPs with four or more criteria were renamed into many-objective optimization problems to recognize the computational and decision-making challenges that are not typical when the number of objectives is lower [21]. The evolutionary multicriteria optimization (EMO) community has been first to adopt this term and has actively undertaken studies on such MOPs. Procedures based on evolutionary algorithms have been proposed to interactively lead the DM to the most desirable part of the solution set (see, e.g., [5, 6]). At the Workshop on Many-Criteria Optimization and Decision Analysis (MACODA), EMO experts continued these efforts [53].

In the presence of many objectives, not all functions may be of interest to the DM or not all objectives may be in conflict with each other. It is of interest to make the original MOP simpler by removing unnecessary objective functions, while the solution set remains unchanged. The concept of redundant (or, also called later, nonessential) objective functions is first introduced and studied in [23] and followed in [40, 41]. An objective function is said to be *redundant* if the efficient set is unchanged when that function is removed. On the other hand, the concept of representative criteria is introduced in [51]. A collection of criteria is called *representative* provided all criteria not in the collection can be represented as a conical combination of the criteria in the collection.

If a reduction of the objectives is not possible, the solution of the overall problem in its entirety may be challenging or even impossible, and decomposition of the MOP into a set of MOPs with a smaller number of criteria (sub-MOPs) becomes appealing provided solving the sub-MOPs can be coordinated and related to solving the original MOP. In general, MOPs can be decomposed into single-objective optimization problems (SOPs) or sub-MOPs. The former is typically accomplished by techniques that convert the vector-valued objective function into a scalar-valued function by means of scalarizing parameters [52] and has become a successful decomposition tool in EMO [57, 58]. To achieve improved performance, EMO algorithms making use of the scalarizations have been enhanced, for example, with a customized dominance relation [8] or a capability to learn the characteristics of the estimated solution set [54]. For an earlier extended survey and two recent brief overviews of exact decomposition methods of MOPs into SOPs and sub-MOPs and their subsequent coordination, the reader is referred to [18] and [14, 38], respectively.

In this chapter, being motivated by real-life decision-making problems with many objectives, we focus on decomposing the MOP into sub-MOPs and their coordination to accomplish two goals. If computation of the overall solution set is possible, the goal is to enhance capability of making tradeoff decisions by working in lower dimensional spaces. Otherwise, if computation of the overall solution set is not possible, the goal is to enable its construction and to facilitate decision-making in a similar way. We collect and put forward a list of steps that should be followed and properties that ought to be satisfied by a decomposition–coordination (DC) technique to effectively support the DM and allow for an interactive tradeoff analysis.

This chapter is structured as follows. In Sect. 16.2, a problem statement and conceptual foundations for decomposition are given. An overview of selected DC techniques is presented in Sect. 16.3. Based on this literature review, a comprehensive process and guidance for developing a DC technique are proposed in Sect. 16.4 and applied in Sect. 16.5 to an engineering design problem. This chapter is concluded in Sect. 16.6.

## 16.2 Foundations for Decomposition

Let $\mathbb{R}^n$ and $\mathbb{R}^p$ denote Euclidean vector spaces as the *decision* and *objective* space, respectively. Let $u, v \in \mathbb{R}^p$. We write $u < v$ if $u_i < v_i$ for each $i = 1, \ldots, p$, $u \leq v$ if $u_i \leq v_i$ for each $i = 1, \ldots, p$ with at least one $i$ such that $u_i < v_i$, and $u \leqq v$ if $u_i \leq v_i$ for each $i = 1, \ldots, p$. Furthermore, let $\mathbb{R}^p_{\geqq} = \{u \in \mathbb{R}^p : u \geqq 0\}$.

Define a vector-valued function $f(x) = \left[f_1(x)\ f_2(x)\ \ldots\ f_p(x)\right]$ with $f_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, 2, \ldots, p$. MOPs assume the following form:

$$\begin{aligned} \min\ & f(x) = \left[f_1(x)\ f_2(x)\ \ldots\ f_p(x)\right] \\ \text{s.t. } & x \in X, \end{aligned} \tag{MOP}$$

where $X \subseteq \mathbb{R}^n$ is a feasible set that may include equality and inequality constraints, as well as bounds on the decision variables. The usual way of addressing optimality in multiobjective optimization is with the notion of *efficient solutions* defined with the Pareto partial order in $\mathbb{R}^p$.

**Definition 16.1** Let $x^* \in X$. We say that $x^*$ is *(weakly) efficient* to (MOP) if there is no feasible $x$ such that $f(x)(<) \leq f(x^*)$. We say that $f(x^*)$ is a *(weak) Pareto point or outcome*.

For any problem of the form (MOP), let $E(X, f)$ ($E_w(X, f)$) denote the set of all efficient (weakly efficient) solutions in $X$.

Since it is of interest to decompose (MOP) into multiobjective subproblems, (MOP) is referred to as the all-in-one (AiO) MOP, the set $X$ is called the AiO-feasible set, and the efficient solutions are referred to as AiO-efficient. In [14], these solutions are defined in a more general context and are called *superior*. For decomposition, Pareto efficiency is not the only useful notion of optimality, and generalizations such as $\varepsilon$-*efficient* and $(1+\delta)$-*efficient* solutions are helpful.

**Definition 16.2** Let $x^* \in X$.

1. Let $\varepsilon \in \mathbb{R}^p_{\geqq}$. We say that $x^*$ is (*weakly*) $\varepsilon$-*efficient* to (MOP) if there is no feasible $x$ such that $f(x)(<) \leq f(x^*) - \varepsilon$.
2. Let $\delta \in \mathbb{R}_{\geq}$. We say that $x^*$ is (*weakly*) $(1+\delta)$-*efficient* to (MOP) if there is no feasible $x$ such that $(1+\delta) \cdot f(x)(<) \leq f(x^*)$.

Similar to [20], we decompose (MOP) into a number, say $M$, of subproblems with criteria taken from (MOP). Let $I_k \subset \{1, 2, \ldots, p\}$ for $1 \leq k \leq M$ with $\bigcup_{k=1}^{M} I_k = \{1, 2, \ldots, p\}$, and consider two cases where $\bigcap_{k=1}^{M} I_k = \emptyset$ or $\bigcap_{k=1}^{M} I_k \neq \emptyset$ depending on the context of the problem. Then the $k^{\text{th}}$ subproblem of (MOP) contains $p_k$ functions selected from (MOP), with $\sum_{k=1}^{M} p_k = M$, and assumes the form

$$\begin{aligned} \min\ & F_k(x) = \left[s_{k_1}(f_{k_1}(x)) \cdots s_{k_t}(f_{k_t}(x))\right] \\ \text{s.t. } & x \in X_k \subseteq X. \end{aligned} \tag{MOP$_k$}$$

In (MOP$_k$), the original scalar-valued functions assigned to this subproblem make up $k_t$ scalar and/or vector-valued functions, $1 \leq k_t \leq p_k$, $f_{k_j} : \mathbb{R}^n \rightarrow \mathbb{R}^{k_j}$ for $j = 1, \ldots, t$, $\sum_{j=1}^{t} k_j = p_k$, that are scalarized by means of some real-valued functions $s_{k_j} : \mathbb{R}^{k_j} \rightarrow \mathbb{R}$ for $j = 1, \ldots, t$. Note that if the desired decomposition requires individual functions from (MOP) to be in ($\text{MOP}_k$), the identity map is selected as the scalarization function. In general, the use of scalarizing functions may control the number of objectives in the subproblem. The feasible set $X_k$ is a restriction of the original feasible set that is relevant to ($\text{MOP}_k$).

Indeed, while we may in general decompose (MOP) in any way we choose, it is often most useful to decompose an (MOP) into a collection of BOPs in order to make visualization and analysis of the subproblems much simpler for DMs. Furthermore, it is important to note that often times efficient solutions for one subproblem may not perform efficiently in other subproblems. To compensate, a method of subproblem *coordination* enables DMs to discover (weakly) efficient solutions for the AiO that are “sufficiently good” across each subproblem.

## 16.3 State-of-the-Art in Multiobjective Decomposition and Coordination

We provide an overview of recent studies on multiobjective decomposition and coordination (DC) and decision-making and examine the collected works with respect to three aspects: (1) the type of the AiO-MOP considered and the solution concepts used; (2) the proposed coordination approach to obtain AiO-efficient solutions and the resulting decision-making capability; (3) the applications of the model.

### *16.3.1 Models and Solution Concepts*

The AiO-MOP models differ with respect to the type of the objective functions, location of the decision variables in these functions, type of the constraints, and the related structure of the feasible set. Two types of objective functions are considered. In a majority of studies, objectives are vector-valued functions of decision variables, while in [25, 42], they are composite vector-valued functions of decision variables.

Decision variables are located in some or all objective functions, which makes such variables, respectively, local or global [12, 14, 20, 24, 25, 28, 50]. Since the constraints may be associated with specific subproblems as well as carry local or global variables, they may also be either local or global. The latter are also referred to as linking. The interchangeable role of global variables and linking constraints is discussed in [14]. The feasible sets of the subproblems are called independent when they are built by its own (and not linking) constraints carrying local and global variables. The MOP is called *decomposable* if its objectives and variables assume a block-diagonal structure [50]. To model interaction between subsystems, linking variables as functions of local and global variables are introduced in [25, 28]. In [38], MOPs with linking variables are called *interwoven*. A general graph-based model of the AiO-MOP is developed in [14] to capture the overall general complexity.

The MOP models are extended with additional concepts to perform coordination. To guarantee feasibility, one needs to distinguish between the AiO-feasibility and subproblems' feasibility. For MOPs with linking constraints, *consistent* solutions are defined as the solutions that satisfy these constraints. If a solution is feasible and consistent, it is called *valid* [14]. To take care of efficiency, for subproblems with global variables, the concept of $\varepsilon$-efficient solutions to subproblems is used [20]. It relaxes the subproblem efficiency and allows to reach AiO-efficient solutions. For subsystems with local, global, and linking variables, the concepts of duplicated global and linking variables, consistency constraints, and coordination constraints for subproblems are used to allow communication between the sub-MOPs when they are solved [12, 24, 25, 28]. For the same type of problems, new concepts of efficiency such as *individually*, *cooperative*, and *mutually* efficient solutions are also proposed [38]. If AiO-efficient solutions cannot be achieved through decomposition, they are replaced with $(1+\delta)$-AiO-efficient solutions relaxing the AiO-efficiency or compromising solutions computed as the median among the subproblems' efficient sets [14]. For the latter, however, the validity of the compromise solution is an issue.

### *16.3.2 Coordination and Decision-Making*

The general conceptual goal of a DC method is to construct the AiO-efficient solutions by computing the efficient solutions of the subproblems. Since the available methods offer different levels of accomplishing this goal, we put them in the following classes:

1. *Complete coordination*: There exists a set of subproblems such that a subset of their feasible solutions is equal to the AiO-efficient set [14, 25, 42].
2. *Relaxed coordination*: For each AiO-efficient solution, there exists a set of subproblems whose feasible solutions can produce that AiO-efficient solution [20].
3. *Partial coordination*: For at least one AiO-efficient solution, there exists a set of subproblems whose feasible solutions can produce that AiO-efficient solution [12, 14, 24, 25, 28].
4. *Approximate coordination*: For at least one AiO-efficient solution, there exists a set of subproblems whose feasible solutions can approximate that AiO-efficient solution [14, 50].

The word "method" is used rather liberally because some coordination methods come as mathematical propositions that guarantee one of the four types above and may imply implementable procedures, while other approaches are presented in the form of algorithms whose outputs ensure the types. It is expected but not guaranteed that the subsystems' feasible solutions used in the coordination are also subsystems' efficient.

The decision-making goal of a DC method to support the DM in choosing an AiO preferred efficient solution is typically achieved interactively by examining performances of efficient solutions in smaller dimensional spaces and checking how the efficient solutions in one subproblem affect the performance of the other subproblems. This interactive process typically consists of two repetitive stages: (1) computation of subproblems' efficient solutions and (2) coordination of these solutions and other information leading to construction of a preferred AiO-efficient solution.

In [19, 20, 28, 50], sub-MOPs are scalarized to be solved for their preferred efficient solutions. Despite the scalarization of the AiO-MOP in [12, 13], only the scalarized subproblems are actually solved due to the use of the block coordinate descent (BCD) method.

The coordination is based on two types of information. First, this information is subproblem-specific and includes, e.g., the current values of global or linking variables, the value of $\epsilon$ that quantifies how the performance of one subproblem can be decayed to improve the performance of another subproblem [19, 20]. Second, other entities such as the dual variables associated with consistency constraints participate in coordination. The update of the coordination variables is done through Lagrangian relaxation integrated with the BCD method [12, 13] or using subgradient optimization [28].

There is no coordination in [50] because the approximated AiO-MOP assumes a block-diagonal structure and the preferred subproblems' efficient solutions are concatenated to produce the preferred AiO-efficient solution.

Due to the two-stage strategy, decision-making is supported by two types of tradeoff information: *tradeoffs within* a subproblem between efficient solutions to each sub-MOP, and *tradeoffs between* subproblems between efficient solutions to different sub-MOPs. For each subproblem, tradeoffs within are naturally available. Tradeoffs between problems are quantified by the dual variables associated with coordination constraints of auxiliary SOPs associated with sub-MOPs [19, 20], or the parameters used in the scalarizations and the coordination dual variables [13, 28].

### *16.3.3 Applications*

Although the applications of DC methods could be reviewed according to the real-life domains in which they have been used, we review them in the context of how the decomposition has been performed.

Engineering design is an important area of application for DC methods because many engineering design problems are quite complex and never solved as AiO. They are naturally decomposed because the teams of involved designers work in *different disciplines* and rarely as one team. In bilevel vehicle design [13], the battery designers optimize heat distribution in the battery depending on its dimensions, while the vehicle designers optimize the vehicle performance with respect to several criteria and depending on the placement of the battery in the underhood. The best performing battery design does not necessarily guarantee the best overall vehicle design. In [28], the teams are involved in the design of the vehicle suspension system and separately build its subsystems. In both cases, coordination and evaluation of tradeoffs between the design subproblems become indispensable. The coordination plays the role of negotiation among the teams that operate within their field of expertise while collaborating to reach a compromise as the AiO solution.

The AiO-MOP can be decomposed into subproblems for which it is easy to conduct *tradeoff evaluation* because of the physical meaning of the objectives. This is the case in the design of the layout of a set of vehicle components for a medium-sized truck for use in both civilian and military environments [17]. The coordination ensures that the specific and critical design objectives are achieved in the final truck configuration. The AiO-MOP can also be decomposed so that the tradeoffs within are examined for each pair of criteria. In [20], a three-objective MOP representing the design of a seat/head restraint system to optimize vehicle safety is decomposed in this way. The coordination performed on three BOPs makes use of tradeoffs for three pairs of criteria and yields an AiO preferred solution that is $\epsilon$-efficient to each subproblem.

Another major application of DC schemes is in optimization under *uncertainty* because the AiO-MOP can be decomposed with respect to uncertainty realizations. The design of truss topology is decomposed with respect to uncertain loading conditions, while the coordination makes sure that the truss performs well in all conditions [19]. In the portfolio management model, return and risk are optimized under uncertainty represented by multiple scenarios that are coordinated to yield not only an efficient but also robust portfolio [20].

Based upon this section's discussion, in the next section we present the components a DC method shall have to effectually support decision-making in the presence of many criteria.

## 16.4 Developing a Decomposition–Coordination Technique

We propose that a DC method should consist of the following five elements: (1) method to decompose the AiO-MOP into subproblems with a smaller number of criteria; (2) theory relating the efficient solutions of the subproblems to the AiO-efficient solution and vice versa; (3) theory addressing a coordination approach in one of the classes identified in Sect. 16.3.2; (4) method for computing efficient solutions; (5) capability to engage the DM in an interactive decision process. The first three elements are rather theoretical, while the last two are related to implementation and actual use of the DC method.

Although we present these elements on a specific mathematical model, they represent a DC method for a broad class of AiO MOPs. Let $n_0, n_1, n_2 \in \mathbb{N}$ and $n = n_0 + n_1 + n_2$. Let $x_0 \in \mathbb{R}^{n_0}, x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$ and $x = (x_0, x_1, x_2) \in \mathbb{R}^n$. Let $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{p_1}$ and $f_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{p_2}$, where $p = p_1 + p_2$. We observe a quasi-separable AiO-MOP as formulated in [24].

$$\begin{aligned} &\min f(x_0, x_1, x_2) = \left[f_1(x_0, x_1)\ f_2(x_0, x_2)\right] \\ &\text{s.t. } (x_0, x_1, x_2) \in X = \left\{(x_0, x_1, x_2) \in \mathbb{R}^n \;\middle|\; \begin{matrix} g_1(x_0, x_1) \leqq 0 \\ g_2(x_0, x_2) \leqq 0 \end{matrix}\right\}. \end{aligned} \tag{QS}$$

Note that (QS) has a global variable, $x_0$, and two local variables, $x_1, x_2$, and that there are no global constraints linking all three variables together.

### 16.4.1 Types of Decomposition

Decomposition of AiO-MOPs into subproblems with a smaller number of objectives may be conducted with respect to the following aspects:

1. Disciplines of science and engineering: The subproblems may be associated with particular disciplines such as control theory, mechanics, mixed-integer optimization, and evolutionary optimization [13, 28].
2. Physical components: The subproblems may model particular physical subsystems or components of the overall system represented by the AiO-MOP [13].
3. Mathematical model: The placement of the variables in the objective and constraint functions in the AiO-MOP mathematical model may dictate a decomposition scheme [50].
4. Tradeoffs: The DM's ability to assess tradeoffs between specific objectives in the AiO-MOP may determine the placement of specific criteria in specific subproblems [17, 20].
5. Performance: The subproblems may model the scenarios in which the overall system performs [19, 20].

Decomposition may require scalarization if there is a need to keep several criteria together or if it is desired that the subproblems be biobjective. Since scalarization methods provide the associated tradeoff information, the scalarization choice will affect the decision-making process.

For (QS), we apply a decomposition with respect to the mathematical model. Considering the problem structure, we set the number of subproblems $M = 2$, and respecting the local variables, we duplicate the global variable to also make it local to each subproblem. Letting $z_1 = x_0$ and $z_2 = x_0$ and $X_1 = \{(z_1, x_1) \in \mathbb{R}^{n_0+n_1} \mid g_1(z_1, x_1) \leqq 0\}$, $X_2 = \{(z_2, x_2) \in \mathbb{R}^{n_0+n_2} \mid g_2(z_2, x_2) \leqq 0\}$, we have the following subproblems when (QS) is decomposed.

$$\begin{aligned} &\min f_1(z_1, x_1) \\ &\text{s.t. } (z_1, x_1) \in X_1 \end{aligned} \tag{MOP$_1$}$$

$$\begin{aligned} &\min f_2(z_2, x_2) \\ &\text{s.t. } (z_2, x_2) \in X_2. \end{aligned} \tag{MOP$_2$}$$

Each subproblem carries the original objectives (with some of them possibly scalarized) and can be solved independently of the other one.

### 16.4.2 Efficient Solutions of Subproblems Up to AiO

Using the context of (QS), we address the relationship "up" from the subproblems to the overall problem.

**Proposition 16.1 (Proposition 6.7 in [24])** *Let $(\hat{z}_i, \hat{x}_i) \in E(X_i, f_i)$ for $i = 1, 2$ such that $\hat{z}_1 = \hat{z}_2 = \hat{x}_0$. Then $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \in E(X, f)$.*

**Proposition 16.2** *Let $(\hat{z}_i, \hat{x}_i) \in E(X_i, f_i, \varepsilon_i)$ with $\varepsilon_i \in \mathbb{R}^{p_i}_{\geqq}$ for $i = 1, 2$ such that $\hat{z}_1 = \hat{z}_2 = \hat{x}_0$. Then $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \in E(X, f, \varepsilon)$ with $\varepsilon^T = \left[\varepsilon_1^T \ \varepsilon_2^T\right]$.*

***Proof*** Let $(\hat{z}_i, \hat{x}_i) \in E(X_i, f_i, \varepsilon_i)$ with $\varepsilon_i \in \mathbb{R}^{p_i}_{\geqq}$ for $i = 1, 2$ such that $\hat{z}_1 = \hat{z}_2 = \hat{x}_0$, and suppose, toward a contradiction, that $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \notin E(X, f, \varepsilon)$ with $\varepsilon^T = \left[\varepsilon_1^T \ \varepsilon_2^T\right]$. Thus, there exists $(x_0, x_1, x_2) \in X$ such that $\begin{bmatrix} f_1(x_0, x_1) \\ f_2(x_0, x_2) \end{bmatrix} \leq \begin{bmatrix} f_1(\hat{x}_0, \hat{x}_1) - \varepsilon_1 \\ f_2(\hat{x}_0, \hat{x}_2) - \varepsilon_2 \end{bmatrix}$. We have three cases to consider. Case 1 has that $f_1(x_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1) - \varepsilon_1$ and $f_2(x_0, x_2) \leqq f_2(\hat{x}_0, \hat{x}_2) - \varepsilon_2$. Note that $f_1(x_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1) - \varepsilon_1 = f_1(\hat{z}_1, \hat{x}_1) - \varepsilon_1$, which contradicts the $\varepsilon_1$-efficiency of $(\hat{z}_1, \hat{x}_1)$ in (MOP$_1$). We have an analogous argument for Case 2, $f_1(x_0, x_1) \leqq f_1(\hat{x}_0, \hat{x}_1) - \varepsilon_1$ and $f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2) - \varepsilon_2$, and for Case 3, $f_1(x_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1) - \varepsilon_1$ and $f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2) - \varepsilon_2$. Thus, we must have that $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \in E(X, f, \varepsilon)$. □

**Corollary 16.1** *Without loss of generality, let $(\hat{z}_1, \hat{x}_1) \in E(X_1, f_1)$ and $(\hat{z}_2, \hat{x}_2) \in E(X_2, f_2, \varepsilon_2)$ for some $\varepsilon_2 \in \mathbb{R}^{p_2}_{\geqq}$ such that $\hat{z}_1 = \hat{z}_2 = \hat{x}_0$. Then $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \in E(X, f, \varepsilon)$ with $\varepsilon^T = \left[0^T \ \varepsilon_2^T\right] \in \mathbb{R}^p_{\geqq}$.*

***Proof*** Note that $(\hat{z}_1, \hat{x}_1) \in E(X_1, f_1)$ implies that $(\hat{z}_1, \hat{x}_1)$ is $\varepsilon_1 = 0$-efficient. Thus, with $\varepsilon^T = \left[0^T \ \varepsilon_2^T\right]$, apply Proposition 16.2. □

It is important to note that Propositions 16.1 and 16.2 do not themselves guarantee the possibility of the existence of efficient solutions to each subproblem that have a global variable with a common value. In fact, it is not difficult to show that there are problems of the form (QS) whose subproblems do not have efficient solutions in which a global variable assumes the same value. Thus, an area for further research is to propose a method recognizing this issue and possibly relaxing this requirement.

### 16.4.3 Efficient Solutions of AiO Down to Subproblems

Staying in the context of (QS), we address the relationship "down" between the overall problem and the subproblems.

**Proposition 16.3** *If* $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \in E(X, f)(E_w(X, f))$, *then* $(\hat{x}_0, \hat{x}_i) \in E(X_i, f_i)(E_w(X_i, f_i))$ *for at least one* $i = 1, 2$, *where for all* $(z_i, x_i) \in X_i$, $z_i = \hat{x}_0$, $i = 1, 2$.

***Proof*** We prove the above for efficiency as the proof for weak efficiency proceeds analogously. Let $\hat{x} = (\hat{x}_0, \hat{x}_1, \hat{x}_2) \in E(X, f)$. This means there is no $x = (x_0, x_1, x_2) \in X$ such that $\begin{bmatrix} f_1(x_0, x_1) \\ f_2(x_0, x_2) \end{bmatrix} \leq \begin{bmatrix} f_1(\hat{x}_0, \hat{x}_1) \\ f_2(\hat{x}_0, \hat{x}_2) \end{bmatrix}$. Ergo, there is no $(\hat{x}_0, x_1) \in X_1$, or there is no $(\hat{x}_0, x_2) \in X_2$ such that

$$\begin{cases} f_1(\hat{x}_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1) \\ f_2(\hat{x}_0, x_2) \leqq f_2(\hat{x}_0, \hat{x}_2) \end{cases} \text{ or } \begin{cases} f_1(\hat{x}_0, x_1) \leqq f_1(\hat{x}_0, \hat{x}_1) \\ f_2(\hat{x}_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2) \end{cases} \text{ or } \begin{cases} f_1(\hat{x}_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1) \\ f_2(\hat{x}_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2). \end{cases}$$

Thus, we must have that $(\hat{x}_0, \hat{x}_1) \in E(X_1, f_1)$ or $(\hat{x}_0, \hat{x}_2) \in E(X_2, f_2)$. □

### 16.4.4 Coordinating Subproblems

We extend the coordination designed for subsystems with only global variables in [20] to the case of local and global variables as required by subproblems ($\text{MOP}_1$) and ($\text{MOP}_2$) in (QS). We formulate coordination problems that allow the subproblems to "communicate with each other" meaning that the performance of an efficient solution to one subproblem can be checked in the other subproblems.

For $i = 1, 2,$, we choose a reference subproblem ($\text{MOP}_i$), a reference efficient solution $(x_0^*, x_i^*)$ based upon its preferred performance in the outcome space of ($\text{MOP}_i$), and a relaxation parameter $\varepsilon_i \in \mathbb{R}^{p_i}_{\geqq}$. Note that $\varepsilon_i$ is chosen as a relaxation in the outcome space and thus relaxes the performance of the reference Pareto point. We then form the coordination problem by adding an additional constraint to ($\text{MOP}_j$) for $j \neq i$.

$$\begin{aligned} &\min f_j(x_0, x_j) \\ &\text{s.t. } f_i(x_0, x_i) \leqq f_i(x_0^*, x_i^*) + \varepsilon_i \\ &(x_0, x_i, x_j) \in X, \end{aligned} \tag{$\text{COP}_j$}$$

where the natural ordering is maintained in $(x_0, x_i, x_j)$ based upon the values of $i, j$. For ease of notation, we let $X(\varepsilon_i) = \{(x_0, x_i, x_j) \in X \mid f_i(x_0, x_i) \leqq f_i(x_0^*, x_i^*) + \varepsilon_i\}$, i.e., the feasible region for ($\text{COP}_j$).

**Proposition 16.4** *Let $\hat{x} = (\hat{x}_0, \hat{x}_1, \hat{x}_2) \in X(\varepsilon_i)$, for $i = 1, 2$, and let $j \neq i$.*

1. *If $\hat{x} \in E_w(X(\varepsilon_i), f_j)$, then $\hat{x} \in E_w(X, f)$.*
2. *If $\hat{x} \in E(X(\varepsilon_i), f_j)$ and $f_j$ is injective, then $\hat{x} \in E(X, f)$.*
3. *If $\hat{x} \in E(X, f)$, then $\hat{x} \in E(X(\varepsilon_i), f_j)$ with $(x_0^*, x_i^*) = (\hat{x}_0, \hat{x}_i)$ and $\varepsilon_i = 0$ for $i = 1, 2$ and $j \neq i$.*

***Proof*** Without loss of generality, for parts 1. and 2., we let $i = 1$ and $j = 2$, that is, we assume that (MOP$_1$) is the reference problem with a preferred efficient solution $(x_0^*, x_1^*)$ and relaxation parameter $\varepsilon_1 \in \mathbb{R}^{p_1}_{\geqq}$. Thus, we consider the coordination problem (COP$_2$):

1. Let $\hat{x} \in E_w(X(\varepsilon_1), f_2)$. Toward a contradiction, suppose that $\hat{x} \notin E_w(X, f)$. Thus, there exists $x \in X$ such that $\begin{bmatrix} f_1(x_0, x_1) \\ f_2(x_0, x_2) \end{bmatrix} < \begin{bmatrix} f_1(\hat{x}_0, \hat{x}_1) \\ f_2(\hat{x}_0, \hat{x}_2) \end{bmatrix}$. In particular, we have that $f_1(x_0, x_1) < f_1(\hat{x}_0, \hat{x}_1) \leqq f_1(x_0^*, x_1^*) + \varepsilon_1$. Therefore, as $(x_0, x_1, x_2) \in X$ and $f_1(x_0, x_1) < f_1(x_0^*, x_1*) + \varepsilon_1$, then $(x_0, x_1, x_2) \in X(\varepsilon_1)$. But we also have that $f_2(x_0, x_2) < f_2(\hat{x}_0, \hat{x}_2)$, which contradicts the weak efficiency of $(\hat{x}_0, \hat{x}_1, \hat{x}_2)$ in (COP$_2$). Thus, we must have that $\hat{x} \in E_w(X, f)$.
2. Let $\hat{x} \in E(X(\varepsilon_1), f_2)$ and $f_2$ be injective. Note that for $f_2$ to be injective, it is sufficient for at least one function in $f_2$ to be injective. Suppose that $\hat{x} \notin E(x, f)$. Then, there exists $x \in X$ such that $\begin{bmatrix} f_1(x_0, x_1) \\ f_2(x_0, x_2) \end{bmatrix} \leq \begin{bmatrix} f_1(\hat{x}_0, \hat{x}_1) \\ f_2(\hat{x}_0, \hat{x}_2) \end{bmatrix}$. Note that $f_1(x_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1) \leqq f_1(x_0^*, x_1^*) + \varepsilon_1$, and thus $(x_0, x_1, x_2) \in X(\varepsilon_1)$. We have three cases:

   a. $f_1(x_0, x_1) \leqq f_1(\hat{x}_0, \hat{x}_1)$ and $f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2)$. This contradicts the efficiency of $\hat{x}$ in (COP$_2$).
   b. $f_1(x_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1)$ and $f_2(x_0, x_2) \leqq f_2(\hat{x}_0, \hat{x}_2)$. By assumption, $f(x) \leq f(\hat{x})$ and so $x \neq \hat{x}$. Since $f_2$ is injective, there is at least one scalar-valued function $f_{2j}$ in $f_2$ that is injective. Thus, since $x \neq \hat{x}$, $f_{2j}(x_0, x_2) \neq f_{2j}(\hat{x}_0, \hat{x}_2)$, and therefore $f_2(x_0, x_2) \neq f_2(\hat{x}_0, \hat{x}_2)$. Thus, the only way for $f_2(x_0, x_2) \leqq f_2(\hat{x}_0, \hat{x}_2)$ is $f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2)$, which again contradicts the efficiency of $\hat{x}$ in (COP$_2$).
   c. $f_1(x_0, x_1) \leq f_1(\hat{x}_0, \hat{x}_1)$ and $f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2)$. Again, $f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2)$ contradicts the efficiency of $\hat{x}$ in (COP$_2$).

   Thus, we must have that $\hat{x} \in E(X, f)$.
3. Let $\hat{x} = (\hat{x}_0, \hat{x}_1, \hat{x}_2) \in E(X, f)$. Note that $(\hat{x}_0, \hat{x}_1, \hat{x}_2)$ is feasible for (COP$_2$) with $(\hat{x}_0, \hat{x}_1) = (x_0^*, x_1^*)$ and $\varepsilon_1 = 0$, that is, $(\hat{x}_0, \hat{x}_1, \hat{x}_2) \in X(\varepsilon_1 = 0)$ and $f_1(\hat{x}_0, \hat{x}_1) = f_1(x_0^*, x_1^*)$. Toward a contradiction, suppose that $\hat{x}$ is not efficient for (COP$_2$) with $(\hat{x}_0, \hat{x}_1) = (x_0^*, x_1^*)$ and $\varepsilon_1 = 0$. Therefore, there exists $x = (x_0, x_1, x_2) \in X(\varepsilon_1)$ such that $f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2)$. We thus have $\begin{bmatrix} f_1(x_0, x_1) \leqq f_1(\hat{x}_0, \hat{x}_1) \\ f_2(x_0, x_2) \leq f_2(\hat{x}_0, \hat{x}_2) \end{bmatrix}$, which contradicts that $\hat{x} \in E(X, f)$. Thus, $\hat{x}$ is efficient for (COP$_2$).

□

Proposition 16.4 gives guidance about the type of coordination between ($MOP_1$) and ($MOP_2$) so that AiO-efficient solutions of (QS) can be achieved. Based on part 3 of this proposition, for every AiO-efficient solution, one can construct coordination subproblems ($COP_1$) and ($COP_2$) that, under certain conditions, produce that AiO-efficient solution. In view of the classification proposed in Sect. 16.3.2, this coordination is relaxed.

### *16.4.5 Computation of Efficient Solutions*

A DC method must make use of multiobjective solvers that are needed to perform computation of the efficient solutions of the subproblems or other auxiliary MOPs such as the proposed coordination subproblems. The choice of solver depends on several factors among which the first consideration is the availability of the subproblems' mathematical representation.

If the representation is available, then the computation depends on this representation's structure and also on the computational goal. The structure pertains to the type of the objective and constraint functions (convex and/or nonconvex) and the variables (continuous/discrete/mixed). The computational goal may be to compute an efficient solution, several efficient solutions, or an approximation of the Pareto set of the subproblems. Since the literature on the available computational methods is vast, we refer the reader to the reviews on exact methods and evolutionary algorithms [16, 34, 52].

When the mathematical representation is not available or the functions are expensive to compute, simulation- or approximation-based solvers are needed and have recently been proposed [2, 10, 43, 50].

More information about a variety of solvers can also be found in the International Society on MCDM website [35].

### *16.4.6 Interactive Process*

If the goal of a DC method is to identify a preferred AiO-efficient solution, then this process ought to be conducted in an interactive fashion to allow DM's progressive exploration of the AiO-Pareto set, while their preferences are being adjusted.

The exploration should be integrated with coordination as it is done in the coordination problem ($COP_j$) in which the relaxation parameter controls the region being searched. This region may be additionally sampled for specific solutions as it is done, e.g., in the Light Beam Search [32, 36].

The modeling of DM's preferences ought to make use of the tradeoff information directly resulting from coordination (cf. [20]), but it may also rely on other approaches to preference modeling such as "if-then" decision rules [26] or many others as reviewed in [4, 55].

## 16.5 Example

In this section, we apply the six steps of the DC method outlined in the previous section to an example problem in engineering design. We consider the design of a speed reducer that is formulated in [29] as an MOP that has been used by EMO researchers to demonstrate applications of genetic algorithms in hierarchical optimization [30, 31, 45]. In [24], this design problem illustrates the (QS) model.

There are seven design variables to consider: the width of the gear face, $x_{01}$, the size of the teeth module, $x_{02}$, the number of pinion teeth, $x_{03}$, the shaft 1 and shaft 2 length between bearings 1 and 2, $x_{11}$ and $x_{21}$, respectively, and the diameters of shaft 1 and shaft 2, $x_{12}$ and $x_{22}$, respectively. There are three objective functions to minimize including the sum of the volumes of the reducer parts, $V_1, \ldots, V_7$, the stress in shaft 1, $S_1$, and the stress in shaft 2, $S_2$. The model includes inequality constraints, $\mathscr{G}_k \leqq 0, k = 1, \ldots, 11$, on the dimensions, deflections and stresses, and bounds on the design variables. For the complete mathematical formulation, refer to [24, 29].

**Decomposition** This design problem may be decomposed with respect to the following aspects:

1. Physical components: The reducer consists of two shafts that imply decomposition into two subsystems. This physical structure is also reflected in the mathematical model.
2. Mathematical model: There are local design variables related to each shaft and bearing, and global design variables. Since the volume objective function is additive, it can be split into two parts. Consequently, the model can be decomposed into two subproblems.
3. Tradeoffs: There are two types of objectives: volume and stress. Analyzing the performance of the designs in the objective space, the DM may want to trade volume with volume, stress with stress, or volume with stress. Each of these cases implies a different decomposition.

Since the mathematical model is closely related to the physical structure of the reducer, the decomposition naturally results from both these aspects. The original model is decomposed into the following two subproblems:

$$\min \begin{bmatrix} f_{11} = V_1 + V_2 + V_4 + V_6 \\ f_{12} = S_1 \end{bmatrix} \tag{MOP$_1$}$$

$$\text{s.t. } [\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_5, \mathscr{G}_6, \mathscr{G}_7] \leqq 0$$

$$\begin{bmatrix} 2.6 \\ 0.7 \\ 17 \end{bmatrix} \leqq \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} \leqq \begin{bmatrix} 3.6 \\ 0.8 \\ 28 \end{bmatrix}$$

$$[\mathscr{G}_3, \mathscr{G}_8, \mathscr{G}_{10}] \leqq 0$$

$$\begin{bmatrix} 7.3 \\ 2.9 \end{bmatrix} \leqq \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \leqq \begin{bmatrix} 8.3 \\ 3.9 \end{bmatrix}$$

$$\min \begin{bmatrix} f_{21} = V_3 + V_5 + V_7 \\ f_{22} = S_2 \end{bmatrix} \tag{MOP$_2$}$$

$$\text{s.t. } [\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_5, \mathscr{G}_6, \mathscr{G}_7] \leqq 0$$

$$\begin{bmatrix} 2.6 \\ 0.7 \\ 17 \end{bmatrix} \leqq \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} \leqq \begin{bmatrix} 3.6 \\ 0.8 \\ 28 \end{bmatrix}$$

$$[\mathscr{G}_4, \mathscr{G}_9, \mathscr{G}_{11}] \leqq 0$$

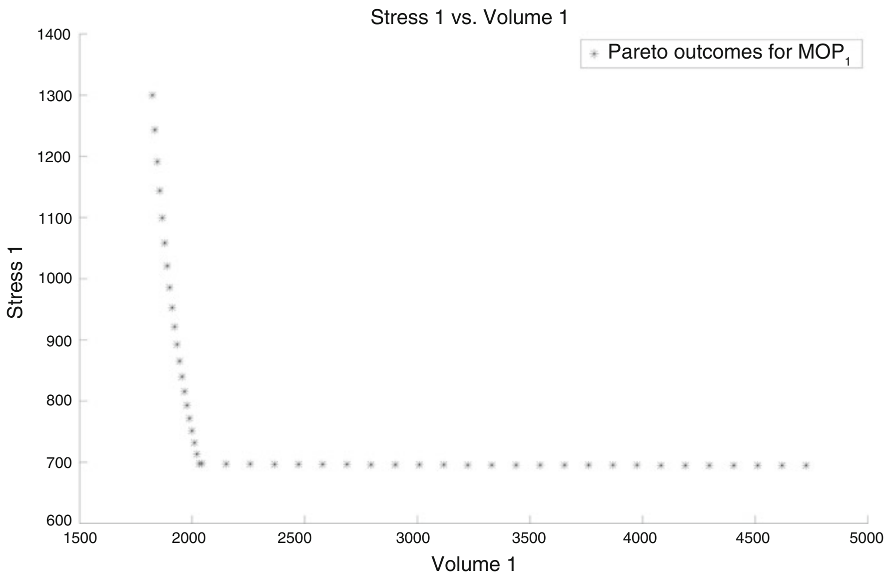
$$\begin{bmatrix} 7.3 \\ 5.0 \end{bmatrix} \leqq \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \leqq \begin{bmatrix} 8.3 \\ 5.5 \end{bmatrix}.$$

Note that $\text{MOP}_1$ and $\text{MOP}_2$ imply the four-objective AiO-MOP with the objective function $f = (f_{11}, f_{21}, f_{12}, f_{22})$ rather than the original triobjective MOP with the objective function $(f_{11} + f_{21}, f_{12}, f_{22})$. Therefore in the subsequent discussion, the four-objective problem is considered as the (QS) model.

**Efficient Solutions of Subproblems up to AiO** The theory presented in Sect. 16.4.2 is immediately applicable to (QS). If there are efficient solutions to ($\text{MOP}_1$) and ($\text{MOP}_2$) that have common global variables, then, based on Proposition 16.1, they are also efficient solutions to (QS). Figure 16.1a and b depict the Pareto sets for ($\text{MOP}_1$) and ($\text{MOP}_2$). Interestingly, these sets assume the form of convex curves despite the nonlinearity of the objective and constraint functions in the mathematical model. Also, all points depicted in Fig. 16.1a are Pareto, which may not be immediately visible due to the scaling on the Volume 1 axis.

**Efficient Solutions of AiO Down to Subproblems** The theory presented in Sect. 16.4.3 is not applicable to (QS). Based on Proposition 16.3, an efficient solution to (QS) does not provide an efficient solution to at least one of the subproblems. However, this issue is only of theoretical importance since (QS) will not be directly solved for its efficient solutions.

**Coordinating Subproblems** Suppose the DM is able to assess the Pareto set of ($\text{MOP}_1$) rather than of ($\text{MOP}_2$) and chooses a preferred Pareto outcome from the former, which makes ($\text{MOP}_1$) the reference subproblem. Furthermore, assume the DM chooses the "knee" of that Pareto set as the region of preferred performance with respect to the volume and stress and selects $x^* = (x^*_{01}, x^*_{02}, x^*_{03}, x^*_{11}, x^*_{12}) = (3.5, 0.7, 17, 7.3, 3.8977)$ as a preferred efficient solution to ($\text{MOP}_1$). The resulting



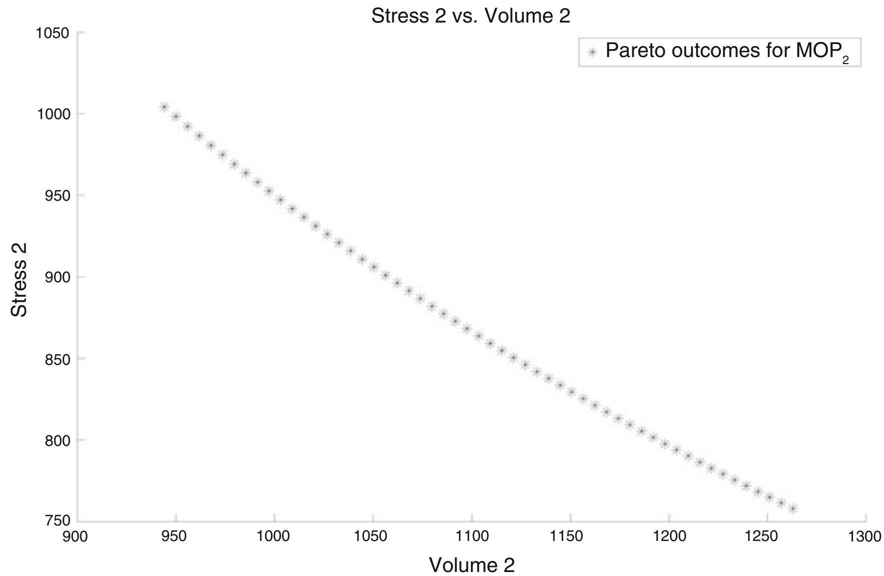




**Fig. 16.1** Pareto points for (**a**) ($MOP_1$). (**b**) ($MOP_2$)

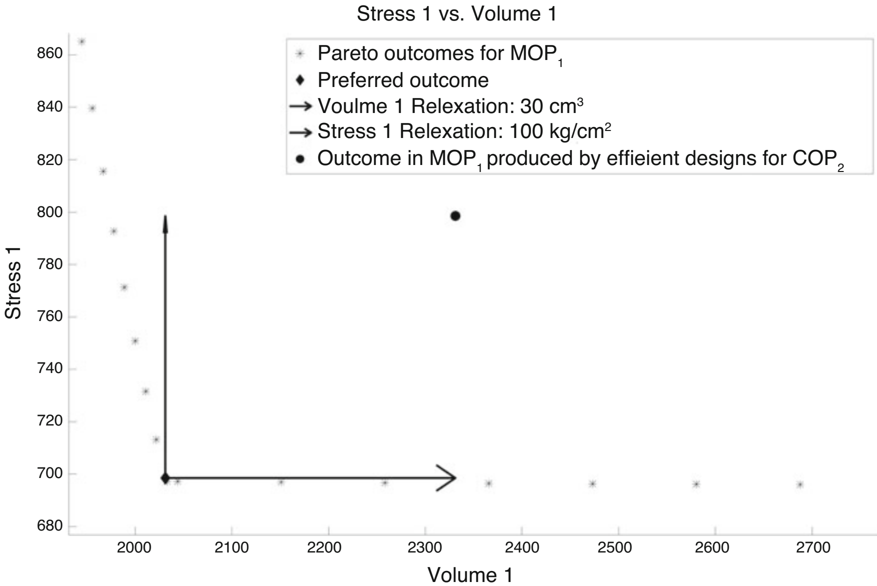



**Fig. 16.2** Coordination in (MOP$_1$).

Pareto outcome $(f_{11}(x^*) = 2031.1\,\text{cm}^3, f_{12}(x^*) = 698.5\,\text{kg/cm}^2)$ gives its performance in (MOP$_1$) and is plotted as a diamond in Fig. 16.2.

The goal now is to examine the performance of $x^*$ in (MOP$_2$). If this performance is satisfactory, an AiO-efficient solution should be constructed for (QS) and the process would end. Otherwise, to achieve a better performance in (MOP$_2$), either the performance of $x^*$ should be relaxed in (MOP$_1$) or another preferred efficient point to (MOP$_1$) is to be selected. To follow this strategy, the DM formulates the coordination problem (COP$_j$) for $j = 2$. In (COP$_2$), the parameters $\varepsilon_{11}$ and $\varepsilon_{12}$ model the relaxation of the volume and stress, respectively, that the DM could agree on if necessary. However, to check the performance of $x^*$ in (MOP$_2$), no relaxation is needed and (COP$_2$) is solved with $\varepsilon = (0, 0)$.

$$\min \begin{bmatrix} f_{21} = V_3 + V_5 + V_7 \\ f_{22} = S_2 \end{bmatrix} \tag{COP$_2$}$$

$$\text{s.t.} \begin{bmatrix} f_{11}(x) \\ f_{12}(x) \end{bmatrix} \leqq \begin{bmatrix} f_{11}(x^*) = 2031.1 \\ f_{12}(x^*) = 698.5 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \end{bmatrix}$$

$$[\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6, \mathcal{G}_7, \mathcal{G}_8, \mathcal{G}_9, \mathcal{G}_{10}, \mathcal{G}_{11}] \leqq 0$$

$$\begin{bmatrix} 2.6 \\ 0.7 \\ 17 \end{bmatrix} \leqq \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} \leqq \begin{bmatrix} 3.6 \\ 0.8 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} 7.3 \\ 2.9 \end{bmatrix} \leqq \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \leqq \begin{bmatrix} 8.3 \\ 3.9 \end{bmatrix}$$

$$\begin{bmatrix} 7.3 \\ 5.0 \end{bmatrix} \leqq \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \leqq \begin{bmatrix} 8.3 \\ 5.5 \end{bmatrix}.$$

Because ($COP_2$) is a biobjective optimization problem whose objective space is that of ($MOP_2$), its Pareto set is depicted together with the Pareto set of ($MOP_2$) in Fig. 16.3a and b. Each figure shows a different portion of the same Pareto sets. The stars are the computed Pareto outcomes for ($MOP_2$), while the triangles are the computed (weak) Pareto outcomes for ($COP_2$) with no relaxation. These triangles indicate the performance of $x^*$ in ($MOP_2$). Because the values of volume and stress at the triangle points are located higher than those at the star points, the DM may decide that the performance of $x^*$ in ($MOP_2$) is rather poor and there is no balance in how subproblems ($MOP_1$) and ($MOP_2$) behave.

Hoping to find better performing solutions for ($MOP_2$), the DM decides to sacrifice the performance (2031.1 cm$^3$, 698.5 kg/cm$^2$) of $x^*$ by relaxing the volume by 300 cm$^3$ and stress by 100 kg/cm$^2$. In Fig. 16.2, the relaxation is marked with the two arrows. ($COP_2$) is solved with ($\varepsilon_{11} = 300\,\text{cm}^3$ and $\varepsilon_{12} = 100\,\text{kg/cm}^2$), and the resulting Pareto points are plotted as squares in Fig. 16.3a and b. These square points are located much closer to the Pareto set of ($MOP_2$) indicating the relaxation in ($MOP_1$) improved the performance in ($MOP_2$).

Note that ($COP_2$) is solved for all local and global variables, ($x_{01}, x_{02}, x_{03}, x_{11}, x_{12}, x_{21}, x_{22}$), and therefore, its efficient points can be mapped into the objective space of ($MOP_1$). Applying this mapping, the unique outcome(2331.14 cm$^3$, 798.514 kg/cm$^2$) is obtained and plotted as a round dot in Fig. 16.2. This uniqueness, as a special feature of the overall mathematical model, is beyond the scope of this chapter.

At this stage of coordination, the DM may choose between (1) the performance of (2031.1 cm$^3$, 698.5 kg/cm$^2$) for ($MOP_1$) and a preferred Pareto point selected from the triangle-dotted Pareto set, and (2) the performance of (2331.14 cm$^3$, 798.514 kg/cm$^2$) and a preferred Pareto point selected from the square-dotted Pareto set (see Fig. 16.3a and b). Alternatively, the DM may resolve ($COP_2$) for different relaxations or a different preferred efficient point for ($MOP_1$).

**Computation of Efficient Solutions** In this chapter, the epsilon-constraint method was used to compute the efficient solutions of three biobjective problems, ($MOP_1$), ($MOP_2$), and ($COP_2$) [7]. Interestingly, as can be seen in Fig. 16.3b, this method found efficient and weakly efficient solutions to ($COP_2$) for $\varepsilon = (300\,\text{cm}^3, 100\,\text{kg/cm}^2)$. By Proposition 16.4.1, all efficient and weakly efficient solutions to ($COP_2$) are weakly efficient for (QS). Therefore, all outcomes depicted in Fig. 16.3a and b are weak Pareto for (QS), and their pre-images are weakly efficient for (QS).

**Interactive Process** A complete interactive process would use the presented coordination and additional preference information to help the DM choose a final

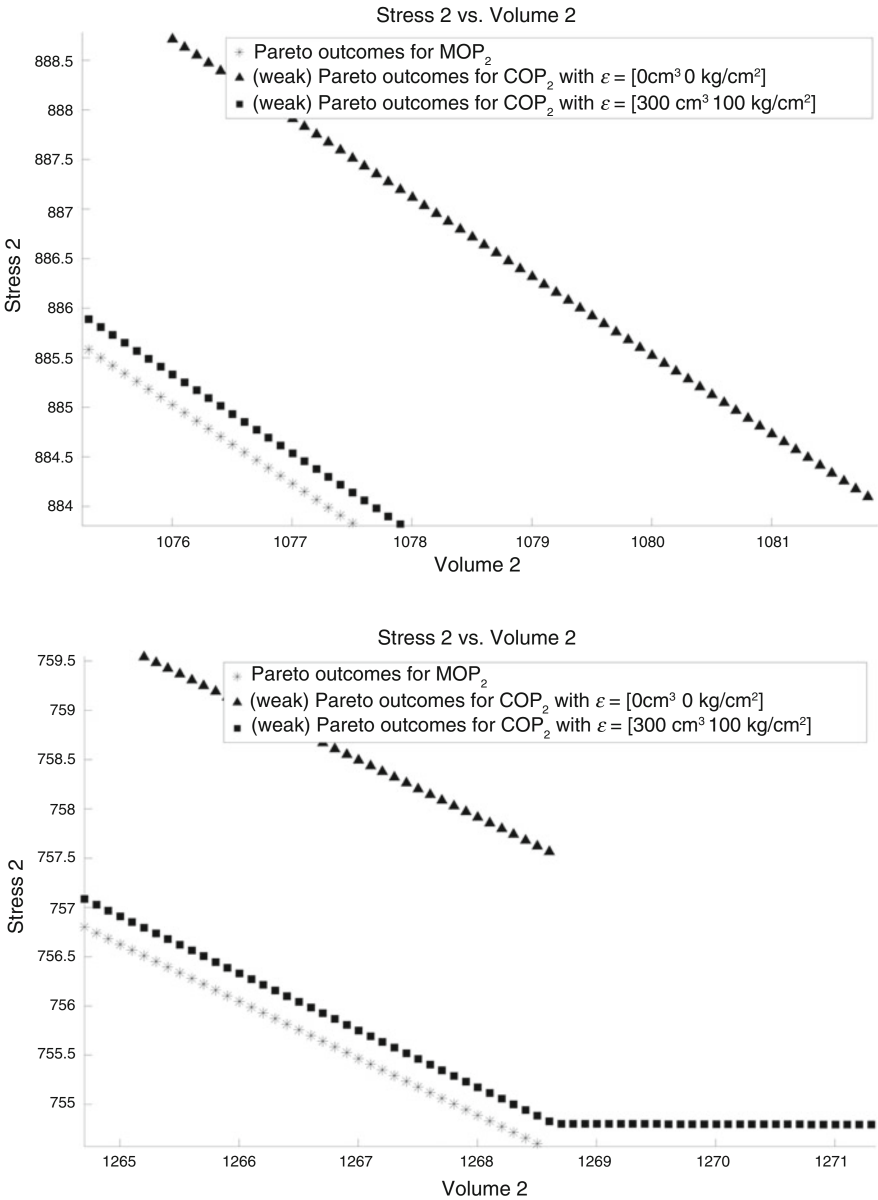



**Fig. 16.3** (**a**) Middle section of Pareto points for ($MOP_2$). (**b**) Endpoint of Pareto points for ($MOP_2$)

AiO-efficient solution. Similar to [19, 20], the preference information may come from the tradeoff values between volume and stress available at different Pareto points within each subproblem and tradeoff values between the two subproblems.

However, data-driven interactive multiobjective optimization, which emerges as a new research direction in MCDM, gives a promise of providing even better decision-making tools [1].

## 16.6 Conclusions

We summarized prior studies on DC approaches in many-objective optimization and proposed a six-step strategy for designing a DC method that will effectively simplify and support the decision-making. We presented this strategy in the context of a quasi-separable MOP and applied to an engineering design example.

Further research can go in several directions. For the quasi-separable case, the tradeoff information provided by the coordination problem shall be revealed, and an interactive decision process making use of this information shall be developed. This case can be extended to more than two subsystems. More generally, decomposition of MOPs of different structures could be examined, and "limits of decomposability" could be determined for MOPs that do not lend themselves to decomposition. Since uncertainty often accompanies conflict, DC methods recognizing this important aspect of real-life modeling are desirable. Based on the literature review and our research experience, we believe that engineering design is an area that can significantly benefit from DC methods. Therefore, developing and implementing DC methods for specific engineering design applications will be the best way to prove the value of DC-based MCDM.

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# Chapter 17
# Fuzzy Linear Programming with General Necessity Measures

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**Abstract** In this chapter, a robust treatment of possibilistic linear programming problems with linear membership functions is studied. After necessity measures and their representations and properties are reviewed, necessity fractile optimization models are introduced as optimization models with robust constraints. Those problems are reduced to a semi-infinite linear programming problem. Conditions on functions associated with the necessity measures are investigated so that the problems are reduced to simpler problems. It is revealed that the problem is reduced simply to a linear programming problem or solved by a more efficient method when functions associated with necessity measures are convex and concave. Applying the results, we show that necessity fractile optimization problems with many famous implication functions are reduced to linear programming problems or solved rather easily by the proposed solution procedure.

## 17.1 Introduction

The parameters, coefficients and right-hand side values, of mathematical programming problems are assumed to be specified as real numbers. In real-world problems, we may face cases when those parameters cannot be specified as real numbers because of environmental fluctuation and/or the lack of knowledge. Moreover, we may have cases when we cannot describe our goals and constraints with exact values.

Fuzzy and possibilistic programming approaches are proposed to mathematical programming problems with ambiguity and vagueness [8, 21]. By those approaches, we obtain reasonable solutions under conflicting soft constraints and goals, robust solutions under hard and soft constraints, hopeful solutions of attaining high-level goals, and so on. Fuzzy programming approaches were formulated by proposing

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treatments of inequality constraints whose coefficients and right-hand side values are fuzzy numbers [23–25]. Tanaka and Asai [25] proposed the nonnegativity index of a fuzzy number and applied it to the treatments of the inequality constraints of fuzzy linear programming problems. As the nonnegativity index takes a positive value when the center of the fuzzy number is positive, and the unity when the lower bound of the support of the fuzzy number is non-negative, this treatment can be considered a strongly robust one. Słowiński [23, 24] proposed two indices: optimistic and pessimistic indices. The optimistic index shows to what extent the constraint is potentially satisfied as it compares the upper bound of the level set of a fuzzy number to be larger is not less than the lower bound of the level set of the other fuzzy number. The pessimistic index is defined by the difference of the upper bounds of the level sets of fuzzy numbers. To control the required levels of constraint satisfaction, the minimally required differences are assumed to be given as real numbers including negative values. Namely, this treatment can be seen as a weakly robust one. Similar approaches were proposed in the literature [11, 20, 26].

After the possibility theory [4, 30], possibilistic programming approaches [3, 7, 12] were proposed. It has been shown that many fuzzy programming approaches can be seen as variations of possibilistic programming approaches [7, 12]. In possibilistic programming approaches, possibility and necessity measures are used to reduce the problems to the conventional programming problems. Many results demonstrate that possibilistic linear programming problems preserve the linearity in the reduced problems when possibility and necessity measures are defined, respectively, by minimum operation and Dienes implication function. However, cases with the other conjunction and implication functions have not yet been considerably investigated, while several alternative approaches [11, 21] have been proposed in the calculation of linear functions with fuzzy coefficients. Inuiguchi [6] has shown that the necessity fractile optimization models of possibilistic linear programming problems with soft constraints can be reduced to semi-infinite linear programming problems even when necessity measures are not defined by Dienes implication function. Tanaka and Asai's approach [25] for fuzzy programming problems is equivalent to a possibilistic programming approach using the necessity measure defined by Dienes implication function. Słowiński's approach [23, 24] for fuzzy programming problems can have a close relation to a possibilistic programming approach using the possibility measure and the necessity measures defined by Gödel and reciprocal Gödel implication functions.

In recent years, the theoretical and methodological contributions in fuzzy optimization have shifted mainly to nonlinear programming problems [2, 17, 18, 27] and optimization over fuzzy relational constraints [15, 16, 19, 28, 29] as fuzzy linear programming problems have been investigated deeply. However, approaches developed in fuzzy linear programming problems are useful in other types of mathematical programming problems. Indeed, the fuzzy linear programming techniques are applied to many real-world programming problems [1, 5, 22].

In this chapter, we further study fuzzy and possibilistic linear programming problems as there still exist some open problems. Namely, the introduction of various implication functions into fuzzy linear programming problems has not yet studied considerably, while it increases the representability of decision-maker's request on the robustness of constraints and goals. Therefore, we study the possibilistic linear programming approach using general necessity measures. We assume that all fuzzy coefficients as well as fuzzy constraints have linear membership functions and that necessity measures are defined by modifier functions based on the approach proposed by Inuiguchi et al. [9, 13]. As the results are more or less complex due to the treatment of general cases, we concentrate the necessity fractile model among various models in possibilistic programming approaches [8]. This model treated in this chapter can be seen as a robust optimization approach.

This chapter is organized as follows. In the next section, necessity measures are reviewed. Some properties and representations of necessity measures are briefly described. The possibilistic linear programming problem treated in this chapter is explained in Sect. 17.3. The reduction to a semi-infinite linear programming problem is shown. Moreover, the differences of inclusion relations equivalent to necessity fractile constraints defined by famous implication functions are illustrated. In Sect. 17.4, results in cases where functions associated with necessity measures are convex and concave are shown. Similar results in cases where modifier functions defining necessity measures are convex and concave are described in Sect. 17.5. In Sect. 17.6, the results in Sects. 17.4 and 17.5 are applied to R-, reciprocal R-, and S-implication functions as well as to famous implication functions. It is shown that the possibilistic linear programming problems with necessity measures defined by many famous implication functions are reduced to linear programming problems or solved rather easily by the proposed solution procedure.

## 17.2 Necessity Measures

Necessity measure [9, 13] of fuzzy event $B$ under fuzzy set $A$ is defined by

$$N_A(B) = \inf_{u \in U} I(\mu_A(u), \mu_B(u)), \tag{17.1}$$

where $\mu_A$ and $\mu_B$ are the membership functions of $A$ and $B$. $I : [0, 1] \times [0, 1] \to [0, 1]$ is an implication function satisfying the following properties:

(I0) $I$ is upper semi-continuous.

(I1) $I(0, 0) = I(0, 1) = I(1, 1) = 1$ and $I(1, 0) = 0$.

(I2) $I(a, b) \le I(c, d)$, for $0 \le c \le a \le 1$ and $0 \le b \le d \le 1$.

The relation of the necessity measure to the inclusion relation can be found in the following equivalence [6]:

$$
\begin{aligned}
N_A(B) \geq h &\Leftrightarrow \inf_{u \in U} I(\mu_A(u), \mu_B(u)) \geq h \\
&\Leftrightarrow (\forall u \in U, \forall k \in [0,1];\ \mu_A(u) \geq k \Rightarrow \mu_B(u) \geq \theta(k,h)) \\
&\Leftrightarrow \forall k \in [0,1];\ [A]_k \subseteq [B]_{\theta^i(k,h)},
\end{aligned} \tag{17.2}
$$

where $\theta(k, h) = \inf\{s \in [0, 1] \mid I(k, s) \geq h\}$ and $[A]_k$ is a $k$-level set of a fuzzy set $A \subseteq U$, i.e., $[A]_k = \{u \in U \mid \mu_A(u) \geq k\}$. In what follows, we use a strong $k$-level set $(A)_k$ of a fuzzy set $A \subseteq U$ defined by $(A)_k = \{u \in U \mid \mu_A(u) > k\}$.

Although a necessity measure is defined by implication function $I$, it is not an easy task to select suitable $I$ depending on the situation. Then Inuiguchi and Tanino [9] and Inuiguchi et al. [13] proposed a method for selecting a necessity measure suitable for the decision-maker's requirement. By this method, a necessity measure is specified based on the decision-maker's satisfaction degrees to several inclusion relations between two fuzzy sets from weak to strong ones. The transition of the inclusion relation from weak to strong can be expressed by two modifier-generating functions $g^m,\ g^M : [0, 1] \times [0, 1] \to [0, 1]$ satisfying:

(g1) $g^m(a, \cdot)$ and $g^M(a, \cdot)$ are lower and upper semi-continuous for all $a \in [0, 1]$, respectively.
(g2) $g^m(1, h) = g^M(1, h) = 1$ and $g^m(0, h) = g^M(0, h) = 0$ for all $h > 0$.
(g3) $g^m(a, 0) = 0$ and $g^M(a, 0) = 1$ for all $a \in [0, 1]$.
(g4) $h_1 \geq h_2$ implies $g^m(a, h_1) \geq g^m(a, h_2)$ and $g^M(a, h_1) \leq g^M(a, h_2)$ for all $a \in [0, 1]$.
(g5) $a \geq b$ implies $g^m(a, h) \geq g^m(b, h)$ and $g^M(a, h) \geq g^M(b, h)$ for all $h \in [0, 1]$.
(g6) $g^m(a, 1) > 0$ and $g^M(a, 1) < 1$ for all $a \in (0, 1)$.

$g^m$ is called an inner modifier-generating function, while $g^M$ is called an outer modifier-generating function.

Then, a necessity measure is defined by

$$N_A(B) \geq h \Leftrightarrow m_h(A) \subseteq M_h(B), \tag{17.3}$$

where $m_h(A)$ and $M_h(B)$ are defined by

$$\mu_{m_h(A)}(u) = g^m(\mu_A(u), h), \quad \mu_{M_h(B)}(u) = g^M(\mu_B(u), h). \tag{17.4}$$

To put it differently, a necessity measure is defined by

$$N_A(B) = \sup\{h \in [0, 1] \mid m_h(A) \subseteq M_h(B)\}. \tag{17.5}$$

Inuiguchi and Tanino [9] and Inuiguchi et al. [13] showed that the necessity measures defined by modifier-generating functions can be defined by (17.1) with the following implication function:

$$I(a, b) = \sup_h \{h \in [0, 1] \mid g^m(a, h) \le g^M(b, h)\}. \tag{17.6}$$

When we define $g^m(a, h) = a$ and $g^M(a, h) = a$, $\forall a \in [0, 1]$ for some $h \in (0, 1]$, $N_A(B) \ge h$ is equivalent to the normal inclusion relation between fuzzy sets $A$ and $B$, i.e., $A \subseteq B$. Then the condition $N_A(B) \ge h$ for general modifier-generating functions can be seen as a generalization of the inclusion relation between $A$ and $B$, and $N_A(B)$ can be regarded as the degree of inclusion.

Using modifier-generating functions $g^m$ and $g^M$, we define a necessity measure $N_A(B)$ by giving the equivalent condition of $N_A(B) \ge h$ ($h \in (0, 1]$) by an inclusion relation between a modified fuzzy set $A$ and a modified fuzzy set $B$, i.e., $m_h(A) \subseteq M_h(B)$. Then, a necessity measure can be specified by giving modifier-generating functions $g^m(\cdot, h)$ and $g^M(\cdot, h)$ that determine how $A$ and $B$ are contracted/expanded and expanded/contracted according to degree $h$, respectively. The specification of modifier-generating functions would be easier than that of implication function directly.

## 17.3 Possibilistic Linear Programming

We consider the following possibilistic linear programming problem:

$$\begin{aligned} &\text{maximize } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ &\text{subject to } \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \precsim_i b_i,\ i = 1, 2, \ldots, m, \\ &\qquad\qquad \boldsymbol{x} \ge \boldsymbol{0}, \end{aligned} \tag{17.7}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is a decision vector. $b_i$, $i = 1, 2, \ldots, m$ are constants. Components $c_j$ of $\boldsymbol{c}$ and $a_{ij}$ of $\boldsymbol{a}_i$ are not known exactly, but the possible ranges of those values are known as trapezoidal fuzzy numbers $C_j$ and $A_{ij}$, respectively. A trapezoidal fuzzy number $C$ is characterized by a quadruple $(c^{\mathrm{L}}, c^{\mathrm{R}}, \gamma^{\mathrm{L}}, \gamma^{\mathrm{R}})$, where $c^{\mathrm{L}}$ and $c^{\mathrm{R}}$ are the lower and upper bounds of the most plausible interval for $C$, while $\gamma^{\mathrm{L}}$ and $\gamma^{\mathrm{R}}$ are the left and right spreads so that $c^{\mathrm{L}} - \gamma^{\mathrm{L}}$ and $c^{\mathrm{R}} + \gamma^{\mathrm{R}}$ show the lower and upper bounds of the least plausible interval. More concretely, the membership function $\mu_C$ of $C$ is defined by

$$\mu_C(r) = \max\left(0, \min\left(1 - \frac{c^{\mathrm{L}} - r}{\gamma^{\mathrm{L}}}, 1 - \frac{r - c^{\mathrm{r}}}{\gamma^{\mathrm{R}}}, 1\right)\right). \tag{17.8}$$

We assume $C_j$ and $A_{ij}$ are trapezoidal fuzzy numbers characterized by $(c_j^{\mathrm{L}}, c_j^{\mathrm{R}}, \gamma_j^{\mathrm{L}}, \gamma_j^{\mathrm{R}})$ and $(a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{R}}, \alpha_{ij}^{\mathrm{L}}, \alpha_{ij}^{\mathrm{R}})$. The notation $\precsim_i$ is a fuzzified inequality so that $\precsim_i b_i$ corresponds to a fuzzy set $B_i$ with verbal expression "a set of real numbers which are roughly smaller than $b_i$." We assume that the membership function $\mu_{B_i}$ of $B_i$ is defined by

$$\mu_{B_i}(r) = \max\left(0, \min\left(1 - \frac{r - b_i}{\beta_i^{\mathrm{R}}}, 1\right)\right), \tag{17.9}$$

where $\beta_i^{\mathrm{R}}$ is a spread showing the tolerance.

The possibilistic linear programming problem (17.7) is a fuzzy linear programming problem as the coefficients are specified by fuzzy numbers showing their possible ranges. In the possibilistic linear programming problem, each coefficient is considered as an uncertain variable taking a value in the given fuzzy number and treated by the possibility theory [4, 30].

As in the literature [8, 21], because $\boldsymbol{x} \geq \boldsymbol{0}$, $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ and $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}$ are restricted by trapezoidal fuzzy numbers $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}$ and $\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}$ characterized by $(\boldsymbol{c}^{\mathrm{L}^{\mathrm{T}}}\boldsymbol{x}, \boldsymbol{c}^{\mathrm{R}^{\mathrm{T}}}\boldsymbol{x}, \boldsymbol{\gamma}^{\mathrm{L}^{\mathrm{T}}}\boldsymbol{x}, \boldsymbol{\gamma}^{\mathrm{R}^{\mathrm{T}}}\boldsymbol{x})$ and $(\boldsymbol{a}_i^{\mathrm{L}^{\mathrm{T}}}\boldsymbol{x}, \boldsymbol{a}_i^{\mathrm{R}^{\mathrm{T}}}\boldsymbol{x}, \boldsymbol{\alpha}_i^{\mathrm{L}^{\mathrm{T}}}\boldsymbol{x}, \ \boldsymbol{\alpha}_i^{\mathrm{R}^{\mathrm{T}}}\boldsymbol{x})$, respectively, where we define $\boldsymbol{c}^{\mathrm{L}} = (c_1^{\mathrm{L}}, c_2^{\mathrm{L}}, \ldots, c_n^{\mathrm{L}})^{\mathrm{T}}$, $\boldsymbol{c}^{\mathrm{R}} = (c_1^{\mathrm{R}}, c_2^{\mathrm{R}}, \ldots, c_n^{\mathrm{R}})^{\mathrm{T}}$, $\boldsymbol{\gamma}^{\mathrm{L}} = (\gamma_1^{\mathrm{L}}, \gamma_2^{\mathrm{L}}, \ldots, \gamma_n^{\mathrm{L}})^{\mathrm{T}}$, $\boldsymbol{\gamma}^{\mathrm{R}} = (\gamma_1^{\mathrm{R}}, \gamma_2^{\mathrm{R}}, \ldots, \gamma_n^{\mathrm{R}})^{\mathrm{T}}$, $\boldsymbol{a}_i^{\mathrm{L}} = (a_{i1}^{\mathrm{L}}, a_{i2}^{\mathrm{L}}, \ldots, a_{in}^{\mathrm{L}})^{\mathrm{T}}$, $\boldsymbol{a}_i^{\mathrm{R}} = (a_{i1}^{\mathrm{R}}, a_{i2}^{\mathrm{R}}, \ldots, a_{in}^{\mathrm{R}})^{\mathrm{T}}$, $\boldsymbol{\alpha}_i^{\mathrm{L}} = (\alpha_{i1}^{\mathrm{L}}, \alpha_{i2}^{\mathrm{L}}, \ldots, \alpha_{in}^{\mathrm{L}})^{\mathrm{T}}$, and $\boldsymbol{\alpha}_i^{\mathrm{R}} = (\alpha_{i1}^{\mathrm{R}}, \alpha_{i2}^{\mathrm{R}}, \ldots, \alpha_{in}^{\mathrm{R}})^{\mathrm{T}}$.

Using a necessity measure $N^i$ defined by an implication function $I^i$, in this chapter, we formulate Problem (17.7) as a necessity fractile optimization model (see Inuiguchi and Ramík [8]):

$$\begin{aligned}
&\text{maximize } q,\\
&\text{subject to } N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q, +\infty)) \geq h^0,\\
&\qquad\qquad N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i, \ i = 1, 2, \ldots, m,\\
&\qquad\qquad \boldsymbol{x} \geq \boldsymbol{0},
\end{aligned} \tag{17.10}$$

where $q$ is an auxiliary variable. $h^0 \in (0, 1]$ and $h^i \in (0, 1]$, $i = 1, 2, \ldots, m$ are certainty levels of goal achievement and constraint satisfactions specified by the decision- maker. Constraints $N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q, +\infty)) \geq h^0$ and $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ are called necessity fractile constraints.

In Fig. 17.1, the difference between $\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ and $N_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ is shown. In the right figure, $\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ is not satisfied, but $I_i(\mu_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(r), \mu_{B_i}(r)) \geq 0.8$ with $I_i(a, b) = 1 - a + ab$ (Reichenbach implication function) is satisfied for all $r \in \mathbf{R}$. Namely, $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq 0.8$ is satisfied.

Table 17.1 shows the relations between several implication functions and their transitions of the inclusion relations required by constraint $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ as $h^i$ increases. The definitions of implication functions shown in Table 17.1 are given later in Table 17.3 of Sect. 17.6. From Table 17.1, we observe the difference of constraint $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ by the implication function defining the necessity measure.

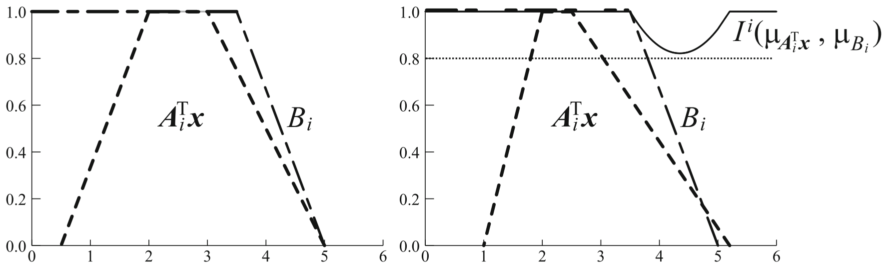


**Fig. 17.1** $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ and $N_{A_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$

**Table 17.1** Implication functions and transitions of inequality relations expressed by $N^i_{A_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$

| Implication | $N^i_{A_i^{\mathrm{T}}\boldsymbol{x}}(B_i) > 0$ | $N^i_{A_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq 0.5$ | $N^i_{A_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq 1$ |
|---|---|---|---|
| Dienes | $[A_i^{\mathrm{T}}\boldsymbol{x}]_1 \subseteq (B_i)_0$ | $(A_i^{\mathrm{T}}\boldsymbol{x})_{0.5} \subseteq [B_i]_{0.5}$ | $(A_i^{\mathrm{T}}\boldsymbol{x})_0 \subseteq [B_i]_1$ |
| Gödel | $(A_i^{\mathrm{T}}\boldsymbol{x})_0 \subseteq (B_i)_0$ | $[A_i^{\mathrm{T}}\boldsymbol{x}]_k \subseteq [B_i]_k, \forall k \in (0, 0.5)$ | $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ |
| Reciprocal Gödel | $[A_i^{\mathrm{T}}\boldsymbol{x}]_1 \subseteq [B_i]_1$ | $[A_i^{\mathrm{T}}\boldsymbol{x}]_k \subseteq [B_i]_k, \forall k \in (0.5, 1]$ | $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ |
| Łukasiewicz | $[A_i^{\mathrm{T}}\boldsymbol{x}]_1 \subseteq (B_i)_0$ | $[A_i^{\mathrm{T}}\boldsymbol{x}]_{0.5+k} \subseteq [B_i]_k, \forall k \in (0, 0.5]$ | $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ |
| Goguen | $(A_i^{\mathrm{T}}\boldsymbol{x})_0 \subseteq (B_i)_0$ | $[A_i^{\mathrm{T}}\boldsymbol{x}]_{2k} \subseteq [B_i]_k, \forall k \in (0, 0.5]$ | $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ |
| Reciprocal Goguen | $[A_i^{\mathrm{T}}\boldsymbol{x}]_1 \subseteq [B_i]_1$ | $[A_i^{\mathrm{T}}\boldsymbol{x}]_{k+0.5} \subseteq [B_i]_{2k}, \forall k \in (0, 0.5]$ | $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ |
| Reichenbach | $[A_i^{\mathrm{T}}\boldsymbol{x}]_1 \subseteq (B_i)_0$ | $[A_i^{\mathrm{T}}\boldsymbol{x}]_{0.5^{1-k}} \subseteq [B_i]_{1-0.5^k}, \forall k \in (0, 1]$ | $(A_i^{\mathrm{T}}\boldsymbol{x})_0 \subseteq [B_i]_1$ |
| Fodor | $[A_i^{\mathrm{T}}\boldsymbol{x}]_1 \subseteq (B_i)_0$ | $(A_i^{\mathrm{T}}\boldsymbol{x})_{0.5} \subseteq [B_i]_{0.5}$ | $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ |
| Inuiguchi [10] | $[A_i^{\mathrm{T}}\boldsymbol{x}]_1 \subseteq (B_i)_0$ | $A_i^{\mathrm{T}}\boldsymbol{x} \subseteq B_i$ | $(A_i^{\mathrm{T}}\boldsymbol{x})_0 \subseteq [B_i]_1$ |

This difference implies the significance of the selection of implication function $I^i$ defining necessity measure $N^i$. A more detailed transition of the inclusion relations required by the necessity fractile constraint can be useful for selecting the implication function defining the necessity measure. Modifier functions $g^m$ and $g^M$ corresponding to necessity measure $N^i$ are useful for seeing the transition of the inclusion relations. The decision-maker can choose an implication function $I^i$ and degree $h^i$ considering his requirement on the robustness of the constraint.

We first see that Problem (17.10) is reduced to a semi-infinite linear programming problem. Let $c_j^{\mathrm{L}}(h) = \inf[C_j]_h$, $c_j^{\mathrm{R}}(h) = \sup[C_j]_h$, $a_{ij}^{\mathrm{L}}(h) = \inf[A_{ij}]_h$, $a_{ij}^{\mathrm{R}}(h) = \sup[A_{ij}]_h$, and $b_i^{\mathrm{R}}(h) = \sup[B_i]_h$. Then, from (17.2) and $\boldsymbol{x} \geq \boldsymbol{0}$, we obtain

$$N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q, +\infty)) \geq h^0 \Leftrightarrow \left[\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}\right]_k \subseteq [[q, +\infty)]_{\theta^0(k,h^0)}$$

$$\Leftrightarrow \inf\left[\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}\right]_k \geq q, \ \forall k \in [0, 1], \ \theta^0(k, h^0) > 0,$$

$$\Leftrightarrow \sum_{j=1}^{n} c_j^{\mathrm{L}}(k)x_j \geq q,\ \forall k \in [0,1],\ \theta^0(k,h^0) > 0, \tag{17.11}$$

$$N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i \Leftrightarrow \left[\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}\right]_k \subseteq [B_i]_{\theta^i(k,h^i)}$$

$$\Leftrightarrow \sup\left[\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}\right]_k \leq \sup\,[B_i]_{\theta^i(k,h^i)}\,,\ \forall k \in [0,1],\ \theta^i(k,h^i) > 0,$$

$$\Leftrightarrow \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k)x_j \leq b_i^{\mathrm{R}}(\theta^i(k,h^i)),\ \forall k \in [0,1],\ \theta^i(k,h^i) > 0, \tag{17.12}$$

where $\theta^i(k,h) = \inf\{s \in [0,1] \mid I_i(k,s) \geq h\}$, $i = 0,1,\ldots,m$. We note that the constraints $N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q,+\infty)) \geq h^0$ and $N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([q,+\infty)) \geq h^0$ are vanished when $\theta^0(k,h^0) = 0$ and $\theta^i(k,h^i) = 0$, respectively.

From (17.11) and (17.12), Problem (17.10) is reduced to the following semi-infinite linear programming problem:

$$\begin{aligned}
&\text{maximize } t,\\
&\text{subject to } \sum_{j=1}^{n} c_j^{\mathrm{L}}(k)x_j \geq q,\ \forall k \in [0,1],\ \theta^0(k,h^0) > 0,\\
&\qquad \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k)x_j \leq b_i^{\mathrm{R}}(\theta^i(k,h^i)),\ \forall k \in [0,1],\ \theta^i(k,h^i) > 0,\\
&\qquad i = 1,2,\ldots,m,\\
&\qquad \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned} \tag{17.13}$$

Then, selecting finitely many numbers of $k \in [0,1]$, Problem (17.10) is approximately reduced to a linear programming problem. We note that, as fuzzy numbers are trapezoidal, we obtain

$$c_j^{\mathrm{L}}(k) = \begin{cases} c_j^{\mathrm{L}} - (1-k)\gamma_j^{\mathrm{L}}, & \text{if } k \in (0,1],\\ -\infty, & \text{if } k = 0, \end{cases} \quad j = 1,2,\ldots,n, \tag{17.14}$$

$$a_{ij}^{\mathrm{R}}(k) = \begin{cases} a_{ij}^{\mathrm{R}} + (1-k)\alpha_{ij}^{\mathrm{R}}, & \text{if } k \in (0,1],\\ +\infty, & \text{if } k = 0, \end{cases} \quad i = 1,2,\ldots,m,\ j = 1,2,\ldots,n, \tag{17.15}$$

$$b_j^{\mathrm{R}}(k) = \begin{cases} b_j^{\mathrm{R}} + (1-k)\beta_j^{\mathrm{R}}, & \text{if } k \in (0,1],\\ +\infty, & \text{if } k = 0, \end{cases} \quad j = 1,2,\ldots,n. \tag{17.16}$$

## 17.4 Case Where $\theta^i(\cdot, h^i)$'s Are Convex and Concave

As fuzzy parameters $C_j$, $A_{ij}$, and $B_i$ have linear membership functions, we obtain simpler reduced problems when $\theta^i(\cdot, h^i)$'s are convex and concave.

Let us consider a case where $\theta^i(\cdot, h^i) : [0, 1] \to [0, 1]$, $i = 1, 2, \ldots, m$, are convex in range (0, 1) (see Fig. 17.2). From (17.9), we obtain $b_i^{\mathrm{R}}(\theta^i(k, h^i)) = b_i + (1 - \theta^i(k, h^i))\beta_i$, $i = 1, 2, \ldots, m$. Therefore, the convexity of $\theta^i(\cdot, h^i)$ implies the concavity of $b_i^{\mathrm{R}}(\theta^i(\cdot, h^i))$. Under the concavity of $b_i^{\mathrm{R}}(\theta^i(\cdot, h^i))$, the following equivalence is valid:

$$N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i \Leftrightarrow \sum_{j=1}^n a_{ij}^{\mathrm{R}}(k)x_j \leq b_i^{\mathrm{R}}(\theta^i(k, h^i)),\ \forall k \in [0, 1],\ \theta^i(k, h^i) > 0$$
$$\Leftrightarrow \begin{cases} \displaystyle\sum_{j=1}^n \bar{a}_{ij}^{\mathrm{R}}(\bar{k}^i)x_j \leq \bar{b}_i^{\mathrm{R}}(\theta^i(\bar{k}^i, h^i)), \\ \displaystyle\sum_{j=1}^n a_{ij}^{\mathrm{R}}(\hat{k}^i)x_j \leq b_i^{\mathrm{R}}(\theta^i(\hat{k}^i, h^i)), \end{cases} \tag{17.17}$$

where we define $\bar{k}^i = \inf\{k \in [0, 1] \mid \theta^i(k, h^i) > 0\}$, $\hat{k}^i = \inf\{k \in [0, 1] \mid \theta^i(k, h^i) = \theta^i(1, h)\}$, $\bar{a}_{ij}^{\mathrm{R}}(k) = \sup(A_{ij})_k$, and $\bar{b}_i^{\mathrm{R}}(k) = \sup(B_i)_k$. As $A_{ij}$ and $B_i$ are linear membership functions, we obtain

$$\bar{a}_{ij}^{\mathrm{R}}(k) = \begin{cases} a_{ij}^{\mathrm{R}} + (1-k)\alpha_{ij}^{\mathrm{R}}, & \text{if } k \in [0, 1), \\ -\infty, & \text{if } k = 1, \end{cases} \quad i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n, \tag{17.18}$$

$$\bar{b}_i^{\mathrm{R}}(k) = \begin{cases} b_i^{\mathrm{R}} + (1-k)\beta_i^{\mathrm{R}}, & \text{if } k \in [0, 1), \\ -\infty, & \text{if } k = 1, \end{cases} \quad i = 1, 2, \ldots, m. \tag{17.19}$$

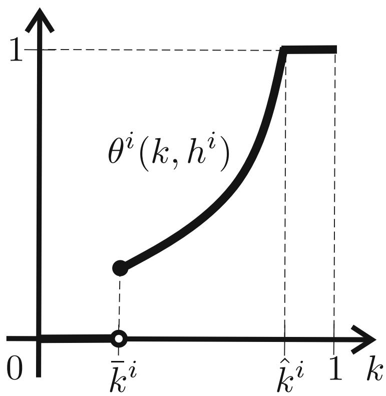


**Fig. 17.2** $\theta^i(\cdot, h^i)$ convex in range (0, 1)

From (17.17), Problem (17.13) can be reduced to the following linear programming problem:

$$\begin{aligned} &\text{maximize } \sum_{j=1}^{n} c_j^{\mathrm{L}}(\bar{k}^0)x_j \\ &\text{subject to } \sum_{j=1}^{n} \bar{a}_{ij}^{\mathrm{R}}(\bar{k}^i)x_j \le \bar{b}_i^{\mathrm{R}}(\theta^i(\bar{k}^i, h^i)),\ i = 1, 2, \ldots, m, \\ &\qquad\qquad \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(\hat{k}^i)x_j \le b_i^{\mathrm{R}}(\theta^i(\hat{k}^i, h^i)),\ i = 1, 2, \ldots, m, \\ &\qquad\qquad \boldsymbol{x} \ge \mathbf{0}. \end{aligned} \tag{17.20}$$

Now, let us consider a case where $\theta^i(\cdot, h^i) : [0, 1] \to [0, 1]$, $i = 0, 1, 2, \ldots, m$, are concave in the range (0, 1). In this case, $b_i^{\mathrm{R}}(\theta^i(\cdot, h^i))$ becomes convex. Because $a_{ij}^{\mathrm{R}}(k) = a_{ij}^{\mathrm{R}} + (1 - k)\alpha_{ij}^{\mathrm{R}}$, i.e., linear with respect to $k > 0$, given $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \ge \mathbf{0}$, there exists $k_i^*$ such that

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k)x_j \le b_i^{\mathrm{R}}(\theta^i(k, h^i)),\ \forall k \in [0, 1],\ \theta^i(k, h^i) > 0 \\ &\Leftrightarrow \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^*)x_j \le b_i^{\mathrm{R}}(\theta^i(k_i^*, h^i)). \end{aligned} \tag{17.21}$$

Utilizing the convexity of $b_i^{\mathrm{R}}(\theta^i(\cdot, h^i))$, Problem (17.13) can be solved by the following relaxation procedure [6] together with a bisection method, where $k_i^*$ is approximately calculated for each candidate solution $\boldsymbol{x}^*$:

S0. Specify $\varepsilon$ by a sufficiently small positive number.
S1. Let $z_i = 0$ and $k_i^{z_i} = 0.5\bar{k}^i + 0.5\hat{k}^i,\ i = 1, 2, \ldots, m$.
S2. Solve the following linear programming problem:

$$\begin{aligned} &\text{maximize } \sum_{j=1}^{n} c_j^{\mathrm{L}}(\bar{k}^0)x_j, \\ &\text{subject to } \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^l)x_j \le b_i^{\mathrm{R}}(\theta^i(k_i^l, h^i)),\ l = 0, 1, \ldots, z_i,\ i = 1, 2, \ldots, m, \\ &\qquad\qquad \boldsymbol{x} \ge \mathbf{0}. \end{aligned} \tag{17.22}$$

Let $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ be the obtained optimal solution.

S3. For $i = 1, 2, \ldots, m$, check the existence of $k_i^* \in (0, 1]$ such that $\sum_{j=1}^n a_{ij}^{\mathrm{R}}(k_i^*)x_j^* > b_i^{\mathrm{R}}(\theta^i(k_i^*, h^i))$ by a bisection method and update $z_i = z_i + 1$ with defining $k_i^{z_i} = k_i^*$ if $k_i^*$ exists.

S4. If at least one $z_i$ is increased, return S2. Otherwise, $\boldsymbol{x}^*$ is an optimal solution to Problem (17.10).

The bisection method in S3 can be performed as follows (see Fig. 17.3):

B0. Let $\lambda_i = -\sum_{j=1}^n \alpha_{ij}^{\mathrm{R}} x_j^*$, and let $\delta$ be a positive small number.

B1. Define $\tilde{k}_i = k_i^q$ with $q = \arg\min_{l=0,1,\ldots,z_i}\left(b_i^{\mathrm{R}}(\theta^i(k_i^l, h^i)) - \sum_{j=1}^n a_{ij}^{\mathrm{R}}(k_i^l)x_j^*\right)$.
Set $\overline{\kappa}_i = \min\left(\{k_i^l \mid k_i^l > k_i^q,\ i = 0, 1, \ldots, z_i\} \cup \{\hat{k}^i\}\right)$ and
$\underline{\kappa}_i = \max\left(\{k_i^l \mid k_i^l < k_i^q,\ i = 0, 1, \ldots, z_i\} \cup \{\bar{k}^i\}\right)$.

B2. If $b_i^{\mathrm{R}}(\theta^i(\tilde{k}_i + \delta, h^i)) < b_i^{\mathrm{R}}(\theta^i(\tilde{k}_i, h^i)) + \lambda_i\delta$, update $\underline{\kappa}_i = \tilde{k}_i$.

B3. If $b_i^{\mathrm{R}}(\theta^i(\tilde{k}_i - \delta, h^i)) < b_i^{\mathrm{R}}(\theta^i(\tilde{k}_i, h^i)) - \lambda_i\delta$, update $\overline{\kappa}_i = \tilde{k}_i$.

B4. If $\overline{\kappa}_i - \underline{\kappa}_i > \varepsilon$ and $\min(\overline{\kappa}_i - \tilde{k}_i, \tilde{k}_i - \underline{\kappa}_i) = 0$, update $\tilde{k}_i = 0.5\overline{\kappa}_i + 0.5\underline{\kappa}_i$ and return B2.

B5. If $\sum_{j=1}^n a_{ij}^{\mathrm{R}}(k_i^*)x_j^* > b_i^{\mathrm{R}}(\theta^i(k_i^*, h^i))$, terminate the procedure with $k_i^* = \tilde{k}_i$. Otherwise, there is no $k_i^*$ such that $\sum_{j=1}^n a_{ij}^{\mathrm{R}}(k_i^*)x_j^* > b_i^{\mathrm{R}}(\theta^i(k_i^*, h^i))$.

We note that Problem (17.10) can be solved by the relaxation procedure with a bisection method described above when $\theta^i(\cdot, h^i)$ associated with each constraint $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ is convex of concave in the range $(0, 1)$, although we considered only the case when all $\theta^i(\cdot, h^i)$ are concave.

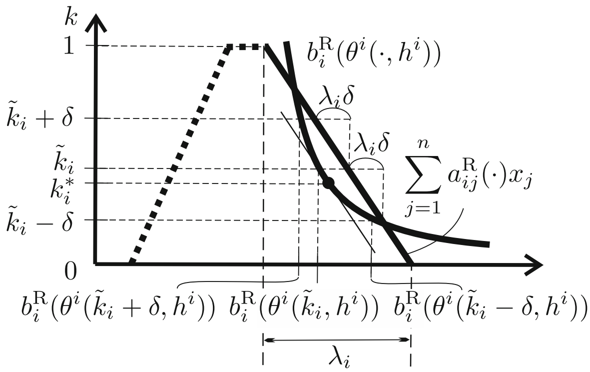


**Fig. 17.3** Figure for the explanation of the bisection procedure composed of B0 to B5

## 17.5 Similar Results for Necessity Measures Defined by Modifier-Generating Functions

Now we describe the case when necessity measure $N^i$ is defined by modifier-generating functions $g_i^m$ and $g_i^M$. In this case, constraints $N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([t,+\infty)) \geq h^0$ and $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ can be rewritten as

$$m^0_{h^0}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}) \subseteq [t,+\infty), \quad m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}) \subseteq M^i_{h^i}(B_i), \tag{17.23}$$

where $m^i_{h^i}$ and $M^i_{h^i}$ are defined by $g_i^m$, $g_i^M$, and $h^i$.

Let $\bar{q}_i^m = \inf\{g_i^m(k,h^i) \mid g_i^m(k,h^i) > 0,\ k \in [0,1]\}$ and $\hat{q}_i^M = \sup\{g_i^M(k,h^i) \mid g_i^M(k,h^i) < 1,\ k \in [0,1]\}$. Then we have

$$N^0_{\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}}([t,+\infty)) \geq h^0 \Leftrightarrow [m^0_{h^0}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x})]_{\bar{q}_0^m} \subseteq [t,+\infty). \tag{17.24}$$

As in the analysis with $\theta^i(\cdot,h^i)$'s, the second inclusion relation of (17.23) is reduced to semi-infinite linear inequalities. In what follows, we investigate cases where the second inclusion relation of (17.23) is treated in some easier ways.

First, let us consider the case where $g_i^{\mathrm{m}}(\cdot,h^i)$ is convex in range $(0,1)$ and $g_i^{\mathrm{M}}(\cdot,h^i)$ is concave in range $(0,1)$. Then, we have the following equivalence (see Fig. 17.4a):

$$N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i \Leftrightarrow \left\{\begin{array}{l} \left\{\begin{array}{ll} (m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}))_0 \subseteq (M^i_{h^i}(B_i))_0, & \text{if } \bar{q}_i^m = 0, \\ {[m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})]_{\bar{q}_i^m}} \subseteq (M^i_{h^i}(B_i))_{\bar{q}_i^m}, & \text{if } \bar{q}_i^m > 0, \end{array}\right. \\ \left\{\begin{array}{ll} {[m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})]_1} \subseteq [M^i_{h^i}(B_i)]_1, & \text{if } \hat{q}_i^M = 1, \\ {[m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})]_{\hat{q}_i^M}} \subseteq (M^i_{h^i}(B_i))_{\hat{q}_i^M}, & \text{if } \hat{q}_i^M < 1. \end{array}\right. \end{array}\right. \tag{17.25}$$

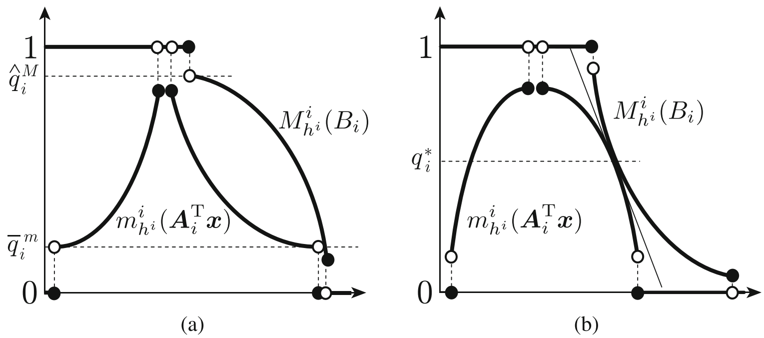


**Fig. 17.4** Properties of $g_i^m$ and $g_i^M$ $m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})$ and $M^i_{h^i}(B_i)$: (**a**) convex $g_i^m$ and concave $g_i^M$ and (**b**) concave $g_i^m$ and convex $g_i^M$

Namely, as fuzzy sets $A_{ij}$ and $B_i$ have linear membership functions, the convexity of $g_i^m(\cdot, h^i)$ and the concavity of $g_i^M(\cdot, h^i)$ in the range (0, 1) make the membership functions of $m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})$ and $M^i_{h^i}(B_i)$ convex and concave in the range (0, 1), respectively. From these properties of the membership functions of $m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})$ and $M^i_{h^i}(B_i)$, constraint $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ is equivalent to two inclusion constraints of level sets of $m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})$ and $M^i_{h^i}(B_i)$.

Let us define functions $k_i^m : [0, 1] \to [0, 1]$ and $k_i^M : [0, 1] \to [0, 1]$ by

$$k_i^m(a) = \begin{cases} \sup\{k \in [0,1] \mid g_i^m(k, h^i) \leq a\}, \text{ if } a \neq 1, \\ \sup\{k \in [0,1] \mid g_i^m(k, h^i) < 1\}, \text{ if } a = 1, \end{cases} \tag{17.26}$$

$$k_i^M(a) = \begin{cases} \inf\{k \in [0,1] \mid g_i^M(k, h^i) \geq a\}, \text{ if } a \neq 0, \\ \inf\{k \in [0,1] \mid g_i^M(k, h^i) > 0\}, \text{ if } a = 0. \end{cases} \tag{17.27}$$

Then, as fuzzy sets $A_{ij}$ and $B_i$ have linear membership functions, we have

$$\begin{aligned}
[m^0_{h^0}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x})]_{\bar{q}_0} \subseteq [t, +\infty)] &\Leftrightarrow [\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}]_{k_0^m(\bar{q}_0^m)} \subseteq [t, +\infty)] \\
&\Leftrightarrow c_j^{\mathrm{L}}(k_0^m(\bar{q}_0^m)) \geq t, && (17.28) \\
(m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}))_0 \subseteq (M^i_{h^i}(B_i))_0 &\Leftrightarrow (\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})_{k_i^m(0)} \subseteq (B_i)_{k_i^M(0)} \\
&\Leftrightarrow \sum_{j=1}^n \bar{a}_{ij}^{\mathrm{R}}(k_i^m(0))x_j \leq \bar{b}_i^{\mathrm{R}}(k_i^M(0)), && (17.29) \\
[m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})]_{\bar{q}_i^m} \subseteq \mathrm{cl}(M^i_{h^i}(B_i))_{\bar{q}_i^m} &\Leftrightarrow [\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}]_{k_i^m(\bar{q}_i^m)} \subseteq \mathrm{cl}(B_i)_{k_i^M(\bar{q}_i^m)} \\
&\Leftrightarrow \sum_{j=1}^n a_{ij}^{\mathrm{R}}(k_i^m(\bar{q}_i^m))x_j \leq \bar{b}_i^{\mathrm{R}}(k_i^M(\bar{q}_i^m)), && (17.30) \\
[m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})]_1 \subseteq [M^i_{h^i}(B_i)]_1 &\Leftrightarrow [\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}]_{k_i^m(1)} \subseteq [B_i]_{k_i^M(1)} \\
&\Leftrightarrow \sum_{j=1}^n a_{ij}^{\mathrm{R}}(k_i^m(1))x_j \leq b_i^{\mathrm{R}}(k_i^M(1)), && (17.31) \\
[m^i_{h^i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})]_{\hat{q}_i^M} \subseteq \mathrm{cl}(M^i_{h^i}(B_i))_{\hat{q}_i^M} &\Leftrightarrow [\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}]_{k_i^m(\hat{q}_i^M)} \subseteq \mathrm{cl}(B_i)_{k_i^M(\hat{q}_i^M)} \\
&\Leftrightarrow \sum_{j=1}^n a_{ij}(k_i^m(\hat{q}_i^M))x_j \leq \bar{b}_i^{\mathrm{R}}(k_i^M(\hat{q}_i^M)), && (17.32)
\end{aligned}$$

where cl$D$ is the closure of a set of $D \subseteq \mathbf{R}$.

As a result, applying the necessity fractile optimization model, Problem (17.7) is reduced to the following linear programming problem:

$$
\begin{aligned}
&\text{maximize} \begin{cases} \sum_{j=1}^{n} \bar{c}_j^{\mathrm{L}}(k_0^m(0))x_j, & \text{if } \bar{q}_0^m = 0, \\ \sum_{j=1}^{n} c_j^{\mathrm{L}}(k_0^m(\bar{q}_0^m))x_j, & \text{if } \bar{q}_0^m > 0, \end{cases} \\
&\text{subject to} \\
&\left.\begin{cases} \sum_{j=1}^{n} \bar{a}_{ij}^{\mathrm{R}}(k_i^m(0))x_j \le \bar{b}_i^{\mathrm{R}}(k_i^M(0)), & \text{if } \bar{q}_i^m = 0, \\ \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^m(\bar{q}_i^m))x_j \le \bar{b}_i^{\mathrm{R}}(k_i^M(\bar{q}_i^m)), & \text{if } \bar{q}_i^m > 0, \end{cases}\right\} i = 1, 2, \ldots, m, \\
&\left.\begin{cases} \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^m(1))x_j \le b_i^{\mathrm{R}}(k_i^M(1)), & \text{if } \hat{q}_i^M = 1, \\ \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^m(\hat{q}_i^M))x_j \le \bar{b}_i^{\mathrm{R}}(k_i^M(\hat{q}_i^M)), & \text{if } \hat{q}_i^M < 1, \end{cases}\right\} i = 1, 2, \ldots, m, \\
&\boldsymbol{x} \ge \boldsymbol{0}.
\end{aligned}
\tag{17.33}
$$

Now let us consider the case where $g_i^m(\cdot, h^i) : [0, 1] \to [0, 1]$ and $g_i^M(\cdot, h^i) : [0, 1] \to [0, 1]$ are concave and convex in range $(0, 1)$. Because $A_{ij}$ and $B_i$ have linear membership functions, given $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \ge \boldsymbol{0}$, there exists $q_i^*$ such that (see Fig. 17.4b)

$$
\begin{aligned}
m_{h^i}^i(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}) \subseteq M_{h^i}^i(B_i) &\Leftrightarrow [m_{h^i}^i(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x})]_{q_i^*} \subseteq \mathrm{cl}(M_{h^i}^i(B_i))_{q_i^*} \\
&\Leftrightarrow \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^m(q_i^*))x_j \le \bar{b}_i^{\mathrm{R}}(k_i^M(q_i*)).
\end{aligned}
\tag{17.34}
$$

Problem (17.10) can be solved by a relaxation procedure together with a bisection method. In this procedure, we explore an approximately optimal solution $\boldsymbol{x}$ by the relaxation procedure with searching $q_i^*$'s corresponding to tentative solutions generated in the procedure. Let $\bar{q}_i^M = \inf\{g_i^M(k, h^i) \mid g_i^M(k, h^i) > 0,\ k \in [0, 1]\}$ and $\hat{q}_i^m = \sup\{g_i^m(k, h^i) \mid g_i^m(k, h^i) < 1,\ k \in [0, 1]\}$. Then the procedure can be written as follows:

T0. Let $\varepsilon$ be a sufficiently small positive number. Let $\bar{q}_i = \max(\bar{q}_i^m, \bar{q}_i^M)$ and $\hat{q}_i = \min(\hat{q}_i^m, \hat{q}_i^M)$.

T1. Let $z_i = 0$ and $q_i^{z_i} = 0.5\bar{q}^i + 0.5\hat{q}^i,\ i = 0, 1, \ldots, m$.

T2. Solve the following linear programming problem:

$$\begin{aligned}
&\text{maximize} \sum_{j=1}^{n} c_j^{\mathrm{L}}(k_0^m(\bar{q}_0^m))x_j,\\
&\text{subject to} \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^m(q_i^l))x_j \le \bar{b}_i^{\mathrm{R}}(k_i^M(q_i^l)),\ l=0,1,\dots,z_i,\ i=1,2,\dots,m,\\
&\qquad\qquad \boldsymbol{x} \ge \mathbf{0}.
\end{aligned} \tag{17.35}$$

Let $\boldsymbol{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^{\mathrm{T}}$ be the obtained optimal solution.

T3. For $i = 1, 2, \dots, m$, check the existence of $q_i^* \in (0, 1]$ such that $\sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^m(q_i^*))x_j^* > \bar{b}_i^{\mathrm{R}}(k_i^M(q_i^*)) + \varepsilon$ by a bisection method and update $z_i = z_i + 1$ with defining $q_i^{z_i} = q_i^*$ if $q_i^*$ exists.

T4. If at least one $z_i$ is increased, then return S2. Otherwise, $\boldsymbol{x}^*$ is an optimal solution to Problem (17.10).

The bisection method in T3 can be performed as follows:

C0. Define $\varphi_i(q) = \bar{b}_i^{\mathrm{R}}(k_i^M(q)) - \sum_{j=1}^{n} a_{ij}^{\mathrm{R}}(k_i^m(q))x_j$, and let $\hat{q}_i = \arg\min_{l=0,1,\dots,z_i} \varphi(q_i^l)$. Let $UB = \min\left(\{q_i^l \mid q_i^l > \check{q}_i,\ l = 0, 1, \dots, z_i\} \cup \{\hat{q}_i\}\right)$ and $LB = \max\left(\{q_i^l \mid q_i^l < \check{q}_i,\ l = 0, 1, \dots, z_i\} \cup \{\bar{q}_i\}\right)$.

C1. If $\hat{q}_i = UB$ and $\varphi_i(\hat{q}_i) < \varphi_i(\hat{q}_i - \varepsilon)$, then terminate this procedure. In this case, if $\varphi_i(1) < 0$, then $q_i^* = 1$; otherwise, $q_i^*$ does not exist.

C2. If $\bar{q}_i = LB$ and $\varphi_i(\bar{q}_i) < \varphi_i(\bar{q}_i + \varepsilon)$, then terminate this procedure. In this case, if $\varphi_i(\bar{q}_i) < 0$, then $q_i^* = \bar{q}_i$; otherwise, $q_i^*$ does not exist.

C3. If $UB - LB \le \varepsilon$, then terminate this procedure. In this case, if $\varphi_i(\tilde{q}_i) < 0$, then $q_i^* = \tilde{q}_i$; otherwise, $q_i^*$ does not exist.

C4. Let $\tilde{q}_i = 0.5\check{q}_i + 0.5UB$. If $\varphi(\tilde{q}_i) < \varphi(\check{q}_i)$, then set $LB = \check{q}_i$ and $\check{q}_i = \tilde{q}_i$ and return to C3. Otherwise, set $UB = \tilde{q}_i$.

C5. Let $\tilde{q}_i = 0.5\check{q}_i + 0.5UB$. If $\varphi(\tilde{q}_i) < \varphi(\check{q}_i)$, then set $UB = \check{q}_i$ and $\check{q}_i = \tilde{q}_i$. Otherwise, set $LB = \tilde{q}_i$. Return to C3.

We note that Problem (17.10) can be solved by the relaxation procedure with a bisection method described in the previous section and this section when each constraint $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \ge h^i$ is reduced to two linear inequalities ($\theta(\cdot, h^i)$ is convex in the range (0, 1) or $g^m(\cdot, h^i)$ and $g^M(\cdot, h^i)$ are convex and concave in the range (0, 1), respectively) or treated by the relaxation procedure with a bisection method ($\theta(\cdot, h^i)$ is concave in the range (0, 1) or $g^m(\cdot, h^i)$ and $g^M(\cdot, h^i)$ are concave and convex in the range (0, 1), respectively), although we considered cases only when all $I^i$ have the same property in the previous section and this section.

## 17.6 Results Applied to Various Implication Functions

### *17.6.1 R-, Reciprocal R-, and S-Implication Functions*

In this section, we investigate Problem (17.10) with necessity measures defined by R-, reciprocal R-, and S-implication functions. R-, reciprocal R-, and S-implication functions are constructed from a t-norm $t : [0, 1] \times [0, 1] \to [0, 1]$ satisfying: (t1) $t(a, 1) = a, \forall a \in [0, 1]$, (t2) $t(a, b) = t(b, a), \forall a, b \in [0, 1]$, (t3) $t(a, t(b, c)) = t(t(a, b), c), \forall a, b, c \in [0, 1]$, and (t4) $t(a, b) \leq t(c, d), \forall a, b, c \in [0, 1]; a \leq c,\ b \leq d$ and a strong negation $n : [0, 1] \to [0, 1]$ satisfying (n1) $n(0) = 1$ and (n2) $n(n(a)) = a, \forall a \in [0, 1]$. Namely, under given t-norm $t$ and strong negation $n$, R-implication function $I^{\mathrm{R}}[t]$, reciprocal R-implication function $I^{\mathrm{r-R}}[t, n]$, and S-implication function $I^{\mathrm{S}}[t, n]$ are defined by

$$I^{\mathrm{R}}[t](a, b) = \sup\{s \in [0, 1] \mid t(a, s) \leq b\}, \tag{17.36}$$

$$I^{\mathrm{r-R}}[t, n](a, b) = \sup\{s \in [0, 1] \mid t(n(b), s) \leq n(a)\}, \tag{17.37}$$

$$I^{\mathrm{S}}[t, n](a, b) = n(t(a, n(b))). \tag{17.38}$$

A t-norm $t$ is said to be Archimedian if it satisfies $t(a, a) < a, \forall a \in (0, 1)$. It is known that any continuous Archimedian t-norm $t$ can be generated from a strictly decreasing and continuous function $f : [0, 1] \to [0, +\infty) \cup \{+\infty\}$ with $f(1) = 0$ as

$$t(a, b) = f^*(f(a) + f(b)), \tag{17.39}$$

where $f^* : [0, +\infty) \cup \{+\infty\} \to [0, 1]$ is a pseudo-inverse of $f$ defined by

$$f^*(r) = \sup\{h \in [0, 1] \mid f(h) \geq r\} = \begin{cases} f^{-1}(r), & \text{if } r < f(0), \\ 0, & \text{if } r \geq f(0). \end{cases} \tag{17.40}$$

Such a function $f$ is called an additive generator of t-norm $t$.

### *17.6.2 Results in R-Implication Functions*

When implication function $I$ is an R-implication function $I^{\mathrm{R}}[t]$, for any $k, h \in [0, 1]$, we have

$$\theta(k, h) = t(k, h). \tag{17.41}$$

Therefore, if necessity measure $N^i$ of Problem (17.10) is defined by an R-implication function $I^i$ with t-norm $t^i$, i.e., $I^i = I^{\mathrm{R}}[t^i]$ such that $t^i(\cdot, h^i)$ is convex in range

(0, 1), the constraint $N^i_{A_i^{\mathrm{T}}x}(B_i) \geq h^i$ can be reduced to a system of two linear inequalities shown in (17.17).

Moreover, if a necessity measure $N^i$ is defined by an R-implication function $I^i$ with t-norm $t_i$ such that $t_i(\cdot, h^i)$ is concave in range (0, 1), the relaxation procedure together with a bisection method described in Sect. 17.4 is applicable for $N^i_{A_i^{\mathrm{T}}x}(B_i) \geq h^i$.

For many famous t-norms such as minimum operation, arithmetic product, bounded product, and so on, $t(\cdot, h^i)$ becomes convex in range (0, 1). However, for Schweizer–Sklar t-norms [14] $t_\eta^{\mathrm{SS}}$ with parameter $\eta > 1$, Hamacher t-norms [14] $t_\eta^{\mathrm{H}}$ with parameter $\eta < 1$, and so on, $t(\cdot, h^i)$ becomes concave in range (0, 1), where $t_\eta^{\mathrm{SS}}$ and $t_\eta^{\mathrm{H}}$ are defined by

$$t_\eta^{\mathrm{SS}}(a, b) = \begin{cases} \max(0, a^\eta + b^\eta - 1)^{\frac{1}{\eta}}, & \text{if } \eta \in (-\infty, 0), \cup(0, +\infty), \\ \min(a, b), & \text{if } \eta = -\infty, \\ ab, & \text{if } \eta = 0, \\ \begin{cases} \min(a, b), & \text{if } \max(a, b) = 1, \\ 0, & \text{otherwise}, \end{cases} & \eta = +\infty, \end{cases} \tag{17.42}$$

$$t_\eta^{\mathrm{H}}(a, b) = \begin{cases} \begin{cases} \min(a, b), & \text{if } \max(a, b) = 1, \\ 0, & \text{otherwise}, \end{cases} & \eta = +\infty, \\ 0, & \text{if } \eta = a = b = 0, \\ \dfrac{ab}{\eta + (1 - \eta)(a + b - ab)}, & \text{otherwise}. \end{cases} \tag{17.43}$$

### 17.6.3 Results in Reciprocal R-Implication Functions

When implication function $I$ is a reciprocal R-implication function $I^{\mathrm{r-R}}[t, n]$, we have $\theta(k, h) = \inf\{s \in [0, 1] \mid t(n(s), h) \leq n(k)\}$. Then when a reciprocal R-implication function $I^{\mathrm{r-R}}[t, n]$ is defined by $n(h) = 1 - h$ and t-norm $t$ such that $t(\cdot, h^i)$ is convex in range (0, 1), we can prove that $\theta(\cdot, h^i)$ is convex in range (0, 1) as follows: for any $k_1, k_2 \in [0, 1]$, any $\lambda \in [0, 1]$, we have

$$\begin{aligned} &\theta(\lambda k_1 + (1 - \lambda)k_2, h^i) \\ &= \inf\{s \in [0, 1] \mid t(1 - s, h^i) \leq 1 - (\lambda k_1 + (1 - \lambda)k_2)\} \\ &= \inf\{\lambda s_1 + (1 - \lambda)s_2 \in [0, 1] \mid s_1, s_2 \in [0, 1] \\ &\qquad t(\lambda(1 - s_1) + (1 - \lambda)(1 - s_2), h^i) \leq \lambda(1 - k_1) + (1 - \lambda)(1 - k_2)\} \\ &\leq \inf\{\lambda s_1 + (1 - \lambda)s_2 \in [0, 1] \mid s_1, s_2 \in [0, 1] \\ &\qquad \lambda t(1 - s_1, h^i) + (1 - \lambda)t(1 - s_2, h^i) \leq \lambda(1 - k_1) + (1 - \lambda)(1 - k_2)\} \\ &\leq \lambda\theta(k_1, h^i) + (1 - \lambda)\theta(k_2, h). \end{aligned}$$

Therefore, if necessity measure $N^i$ of Problem (17.10) is defined by reciprocal R-implication function $I^i = I[t, n]$ with $n(h) = 1-h$ and t-norm $t$ such that $t(\cdot, h^i)$ is convex in range (0, 1), the constraint $N^i_{A_i^{\mathrm{T}}x}(B_i) \geq h^i$ can be reduced to a system of two linear inequalities shown in (17.17).

### 17.6.4 Results in S-Implication Functions

When implication function $I^i$ is an S-implication $I^{\mathrm{S}}[t, n]$, we have

$$\theta(k, h^i) = n\left(\sup\{s \in [0,1] \mid t(k, s) \leq n(h^i)\}\right). \tag{17.44}$$

Let us define a set $BS(h) \subseteq [0, 1] \times [0, 1]$ and a function $\psi^{BS(h)} : [0, 1] \to [0, 1]$ by

$$BS(h) = \{(k, s) \in [0,1] \times [0,1] \mid t(k, s) \leq h\}, \tag{17.45}$$

$$\psi^{BS(h)}(k) = \sup\{s \in [0,1] \mid t(k, s) \leq h\} = \sup\{s \mid (k, s) \in BS(h)\}. \tag{17.46}$$

It is easily shown that $\psi^{BS(h)}$ is concave if $BS(h)$ is a convex set. Then we obtain that if t-norm $t$ is quasi-convex and $n$ is convex, $\theta(\cdot, h^i)$ becomes convex. Therefore, if necessity measure $N^i$ of Problem (17.10) is defined by S-implication $I^i = I^{\mathrm{S}}[t, n]$ with a convex strong negation $n$ and a quasi-convex t-norm $t$, the constraint $N^i_{A_i^{\mathrm{T}}x}(B_i) \geq h^i$ can be reduced to a system of two linear inequalities shown in (17.17).

When t-norm $t$ is an Archimedian t-norm having the additive generator $f$ with (17.39), S-implication $I^{\mathrm{S}}[t, n]$ can be defined by (17.6) with the following modifier-generating functions (see Inuiguchi and Tanino [9] and Inuiguchi et al. [13]):

$$g^m(a, h) = \max\left(0, 1 - \frac{f(a)}{f(n(h))}\right),$$

$$g^M(a, b) = \min\left(1, \frac{f(n(a))}{f(n(h))}\right). \tag{17.47}$$

If $I^i$ is S-implication function $I^{\mathrm{S}}[t, n]$ with respect to a continuous Archimedian t-norm $t$ such that the additive generator $f$ is concave and $n$ is convex, from (17.47) and strict decreasingness of $f$, $g^m(\cdot, h)$ and $g^M(\cdot, h)$ are convex and concave in the range (0, 1). Then in this case, the constraint $N^i_{A_i^{\mathrm{T}}x}(B_i) \geq h^i$ can be reduced to a system of two linear inequalities.

Moreover, if a necessity measure $N^i$ is defined by an S-implication $I^i = I^{\mathrm{S}}[t, n]$ with respect to a continuous Archimedian t-norm such that the additive generator $f$ is convex and $n$ is concave, from (17.47) and strict decreasingness of $f$, $g^m(\cdot, h)$

and $g^M(\cdot, h)$ are concave and convex in the range $(0, 1)$. Therefore, the relaxation procedure with a bisection method described in the previous section is applicable for the constraint $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$.

### 17.6.5 Obtained Results Applied to Famous Implication Functions

The obtained results are arranged in Table 17.2. In Table 17.2, the convexity and concavity of $\theta(\cdot, h^i)$, $g^m(\cdot, h^i)$, $g^M(\cdot, h^i)$, and $t(\cdot, h^i)$ are restricted in the range $(0, 1)$. RP$^{\mathrm{BM}}$-applicable means that the relaxation procedure together with a bisection method is applicable. As shown in Table 17.2, some convexity and/or concavity of some functions related to implication functions are required for solving Problem (17.10) in a simpler way. These conditions may be seen as being strong.

However, if we apply the obtained results to famous and useful implication functions, we obtain Table 17.3. The implication functions can be found in [9, 10, 13]. In Table 17.3, column "reduc." shows the reduced constraints. "Linear" stands for linear inequalities, while "Relx." stands for relaxation procedure with a bisection method. As shown in Table 17.3, constraint $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ with respect to many of famous implication functions $I^i$ except Reichenbach implication function is reduced to two linear inequality conditions. When necessity measure $N^i$ is defined by Reichenbach implication function, if other necessity fractile constraints are reduced to linear inequalities or treated by the relaxation procedure with a bisection method, Problem (17.10) can be solved by the relaxation procedure with a bisection method described in previous sections.

Therefore, for many famous implication functions, necessity measures can be treated without great loss of linearity when fuzzy numbers $C_j$, $A_{ij}$ and fuzzy constraints $B_i$ have linear membership functions.

**Table 17.2** Implication functions and the conditions for reducing $N^i_{\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$

| Implication function $I^i$ | Linear inequalities | RP$^{\mathrm{BP}}$-applicable |
|---|---|---|
| General $I$ | $\theta(\cdot, h^i)$ is convex. | $\theta(\cdot, h^i)$ is concave. |
| By $g^m$ and $g^M$ | $g^m(\cdot, h^i)$ is convex and $g^M(\cdot, h^i)$ is concave. | $g^m(\cdot, h^i)$ is concave and $g^M(\cdot, h^i)$ is convex. |
| R-implication $I^{\mathrm{R}}[t]$ | $t(\cdot, h^i)$ is convex. | $t(\cdot, h^i)$ is concave. |
| Reciprocal R-implication $I^{\mathrm{r-R}}[t, n]$ | $t(\cdot, h^i)$ is convex and $n(h) = 1 - h$. | — |
| S-Implication $I^{\mathrm{S}}[t, n]$ | $t$ is quasi-convex and $n$ is convex. | — |
| S-Implication $I^{\mathrm{S}}[t, n]$ with $t(a, b) = f^*(f(a) + f(b))$ | $f$ is concave and $n$ is convex. | $f$ is convex and $n$ is concave. |

**Table 17.3** Reduction of $N^i_{A_i^{\mathrm{T}}\boldsymbol{x}}(B_i) \geq h^i$ with famous implication functions

| $I(a,b)$ | $\theta(a,b)$ | $g^m(a,h)$ | $g^M(a,h)$ | Reduc. |
|---|---|---|---|---|
| Dienes: $\max(1-a,b)$ | $\begin{cases} 0, & \text{if } a+b \leq 1 \\ b, & \text{if } a+b > 1 \end{cases}$ | $\begin{cases} 0, & \text{if } a+h \leq 1 \\ b, & \text{if } a+h > 1 \end{cases}$ | $\begin{cases} 0, & \text{if } a < h \\ 1, & \text{if } a \geq h \end{cases}$ | Linear |
| Gödel: $\begin{cases} b, & \text{if } a > b \\ 1, & \text{if } a \leq b \end{cases}$ | $\min(a,b)$ | $\begin{cases} 0, & \text{if } h = 0 \\ a, & \text{if } h > 0 \end{cases}$ | $\begin{cases} 0, & \text{if } a < h \\ 1, & \text{if } a \geq h \end{cases}$ | Linear |
| Reciprocal Gödel : $\begin{cases} 1-a, & \text{if } a > b \\ 1, & \text{if } a \leq b \end{cases}$ | $\begin{cases} 0, & \text{if } a+b \leq 1 \\ a, & \text{if } a+b > 1 \end{cases}$ | $\begin{cases} 0, & \text{if } a+h \leq 1 \\ a, & \text{if } a+h > 1 \end{cases}$ | $\begin{cases} a, & \text{if } h > 0 \\ 1, & \text{if } h = 0 \end{cases}$ | Linear |
| Łukasiewicz: $\min(1-a+b,1)$ | $\max(a+b-1,0)$ | $\begin{cases} 0, & \text{if } a+h \leq 1 \\ \frac{a+h-1}{h}, & \text{if } a+h > 1 \end{cases}$ | $\begin{cases} \frac{a}{h}, & \text{if } a < h \\ 1, & \text{if } a \geq h \end{cases}$ | Linear |
| Goguen: $\begin{cases} \frac{b}{a}, & \text{if } a > b \\ 1, & \text{if } a \leq b \end{cases}$ | $ab$ | $\begin{cases} 0, & \text{if } h = 0 \\ a, & \text{if } h > 0 \end{cases}$ | $\begin{cases} \frac{a}{h}, & \text{if } a < h \\ 1, & \text{if } a \geq h \end{cases}$ | Linear |
| Reciprocal Goguen: $\begin{cases} \frac{1-a}{1-b}, & \text{if } a > b \\ 1, & \text{if } a \leq b \end{cases}$ | $\begin{cases} 0, & \text{if } a+b \leq 1 \\ \frac{a+b-1}{b}, & \text{if } a+b > 1 \end{cases}$ | $\begin{cases} 0, & \text{if } a+h \leq 1 \\ \frac{a+h-1}{h}, & \text{if } a+h > 1 \end{cases}$ | $\begin{cases} a, & \text{if } h > 0 \\ 1, & \text{if } h = 0 \end{cases}$ | Linear |
| Reichenbach: $1-a+ab$ | $\begin{cases} 0, & \text{if } a+b \leq 1 \\ \frac{a+b-1}{a}, & \text{if } a+b > 1 \end{cases}$ | $\begin{cases} 0, & \text{if } a+h \leq 1 \\ 1-\frac{\ln a}{\ln(1-h)}, & \text{if } a+h > 1 \end{cases}$ | $\begin{cases} \frac{\ln(1-a)}{\ln(1-h)}, & \text{if } a < h \\ 1, & \text{if } a \geq h \end{cases}$ | Relax. |
| Fodor: $\begin{cases} \max(1-a,b), & \text{if } a > b \\ 1, & \text{if } a \leq b \end{cases}$ | $\begin{cases} \min(a,b) & \text{if } a+b > 1 \\ 0, & \text{if } a+b \leq 1 \end{cases}$ | $\begin{cases} a, & \text{if } a+h > 1 \\ 0, & \text{if } a+h \leq 1 \end{cases}$ | $\begin{cases} a, & \text{if } a < h \\ 1, & \text{if } a \geq h \end{cases}$ | Linear |
| Inuiguchi: $\begin{cases} \frac{1-a+b}{2}, & \text{if } a > 0 \,\&\, b < 1 \\ 1, & \text{otherwise} \end{cases}$ | $(*1)$ | $(*2)$ | $(*3)$ | Linear |

$(*1)$ $\begin{cases} 0, & \text{if } a = 0 \quad \text{or } a+2b \leq 1 \\ \min(a+2b-1,1), & \text{otherwise} \end{cases}$

$(*2)$ $\begin{cases} 0, & \text{if } a = 0 \quad \text{or } (a \neq 1 \,\&\, h = 0) \\ \max\left(\frac{a+2h-1}{2h}, 0\right), & \text{if } h \in (0, 0.5) \\ \min\left(\frac{a}{2-2h}, 1\right), & \text{if } h \in [0.5, 1) \\ 1, & \text{otherwise} \end{cases}$

$(*3)$ $\begin{cases} 0, & \text{if } a = 0 \quad \text{or } (a \neq 1 \,\&\, h = 1) \\ \min\left(\frac{a}{2h}, 1\right), & \text{if } h \in (0, 0.5) \\ \max\left(\frac{a-2b+1}{2-2h}, 0\right), & \text{if } h \in [0.5, 1) \\ 1, & \text{otherwise} \end{cases}$

## 17.7 Concluding Remarks

We have investigated in the necessity fractile optimization models of possibilistic linear programming problems with trapezoidal fuzzy numbers. We consider the models with general necessity measures defined by various implication functions. In general, the model is reduced to a semi-infinite linear programming problem that can be solved approximately by a linear programming technique with selecting many constraints from the semi-infinite constraints.

We showed that the model can be reduced to a usual linear programming problem or solved by a relaxation procedure with a bisection method when functions related to the implication function have convexity and/or concavity. Utilizing the obtained results, we demonstrated that the model can be reduced to a usual linear programming problem when many famous implication functions are used for defining necessity measures. To see the significance of the selection of implication function, differences of the equivalent conditions to the necessity fractile constraints by the implication functions are observed.

The studies on necessity measure optimization models would be one of the future topics derived from the results of this chapter. Moreover, the study on the specification of necessity measures suitable for decision-maker's requirements would be one of the important topics.

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# Chapter 18
# Dominance-Based Rough Set Approach: Basic Ideas and Main Trends

**Jerzy Błaszczyński, Salvatore Greco, Benedetto Matarazzo, and Marcin Szeląg**

**Abstract** Dominance-based Rough Set Approach (DRSA) has been proposed as a machine learning and knowledge discovery methodology to handle Multiple Criteria Decision Aiding (MCDA). Due to its capacity of asking the decision maker (DM) for simple preference information and supplying easily understandable and explainable recommendations, DRSA gained much interest during the years and it is now one of the most appreciated MCDA approaches. In fact, it has been applied also beyond MCDA domain, as a general knowledge discovery and data mining methodology for the analysis of monotonic (and also non-monotonic) data. In this contribution, we recall the basic principles and the main concepts of DRSA, with a general overview of its developments and software. We present also a historical reconstruction of the genesis of this methodology, with a specific focus on the contribution of Roman Słowiński.

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## 18.1 Introduction

Among the many merits of Roman Słowiński in his so long and so rich scientific carrier, we have to consider his pioneering approach to the use of artificial intelligence methodologies to decision support, and, in particular, to Multiple Criteria Decision Aiding (MCDA) (for an updated state of the art, see [48]). In this perspective, the proposal and the development of the Dominance-based Rough Set Approach (DRSA) is a cornerstone in the domain. The DRSA basic idea of a decision support procedure, based on a decision model expressed in natural language and obtained from simple preference information in terms of exemplary decisions, has attracted the interest of experts and it is now considered one of the three main approaches to MCDA, together with the classical Multiple Attribute Utility Theory (MAUT) [58] and the outranking approach [75]. In fact, DRSA is not a mere application to MCDA of concepts and tools already proposed and developed in the domain of artificial intelligence, knowledge discovery, data mining, and machine learning. Indeed, consideration of preference orders typical for MCDA problems required a reformulation of many important concepts and methodologies, so that DRSA became a methodology viable and interesting per se also in these domains. Consequently, after more or less 25 years from the proposal of DRSA, we try to present a first assessment taking into consideration the basic ideas and the main developments.

This paper is organized as follows. Next section presents some historical notes. Section 18.3 recalls the basic concepts of DRSA, while sect. 18.4 describes the main developments. Section 18.5 presents some available software. Section 18.6 collects conclusions.

## 18.2 Some Historical Notes on the Dominance-Based Rough Set Approach

In the beginning of the 1980s, Roman Słowiński entered in contact with two very relevant figures of researchers in two quite different domains: Bernard Roy and Zdzisław Pawlak. Bernard Roy, one of the pioneers of the Operation Research in Europe, was the founder of the European School of MCDA. Zdzisław Pawlak, one of the founders of the computer science and artificial intelligence, proposed the Rough Set Theory as a mathematical tool for data analysis and knowledge discovery. Roman Słowiński, who had already given fundamental contributions in scheduling theory, was enthusiastically interested in both MCDA and rough set theory as witnessed, among the others, by a translation in Polish [74] of the book in which Bernard Roy systematically presented the basis of MCDA [73] and by the organization of the first rough set international conference held in 1992 in Poznań (the proceedings of the conference are collected in [80]). Since his first contributions in the domain, Roman Słowiński was interested in the use of rough set theory for decision support. In particular, he proposed the first application in real world problems related to application of rough set theory to medical diagnosis [82] and edited the first handbook thought as a synthesis of experience with rough sets [79]. Moreover, he realized that very interesting developments could be obtained in rough set theory by applying concepts proposed in MCDA, as the construction of similarity relations using indifference thresholds [78], inspired by analogous concepts in outranking methods [77]. Pursuing this research line, very soon Roman Słowiński matured the conviction that beyond the simple application in rough set theory of some specific concepts originating in MCDA, there was the space for a whole extension of rough set theory that could become a relevant model for MCDA [71, 72]. In May 1994 Roman Słowiński went to Catania to give some seminars on MCDA and rough set theory. On that occasion, discussing with Benedetto Matarazzo and Salvatore Greco, a very interesting idea came out: what could be obtained approximating sets of pairs of objects (binary relations) rather than "standard" sets of single objects? The information to be processed with rough set theory had to be the preference on single criteria and the overall preference, in order to express the overall preference in terms of preferences on the single criteria. This idea was called rough set analysis of a Pairwise Comparison Table (PCT). In October of the same year Roman Słowiński, Salvatore Greco, and Benedetto Matarazzo met again at the 40th meeting of the EURO working group on Multiple Criteria Decision Aiding, held in Paris and Bordeaux. The program of the meeting was split into two parts, with the morning of October 6 in Paris and the following day in Bordeaux, with a transfer by train in the afternoon of October 6. The travel between Paris and Bordeaux was a very good occasion for a long discussion on the intuition that had come out in Catania. The three researchers remained in contact continuing to work on PCT. On the first days of May 1995, Salvatore Greco stayed one week in Poznań and during that week the first paper on PCT was completed [29]. In the same year the new rough set model was presented at an international conference, more precisely at the 12th International Conference on Multiple Criteria Decision Making held on June 19–23 in Hagen [21]. This was the first presentation of PCT at an MCDA conference. The year after, Roman Słowiński, Salvatore Greco, and Benedetto Matarazzo participated in the Fourth International Workshop on Rough Sets, Fuzzy Sets, and Machine Discovery, held during November 6–8, 1996, in Tokyo. This was the first exposition of the PCT to a rough set conference. On that occasion, Hirotaka Nakayama invited the three researchers to Kobe. Again during the travels in train between Tokyo and Kobe and back, there were very rich and constructive discussions about the new rough set model. Both researchers in MCDA and in rough set theory were interested in the idea. However, very soon it appeared clear that the PCT was not able to represent all the salient aspects of MCDA. Indeed, classical rough set theory is based on indiscernibility relation, so what is taken into account is only if the considered objects have the same description or not. In MCDA, and, more in general, in decision support, there is something more to be considered than the equal or different description. For example, in deciding about cars, a maximum speed of 200 km/h is greater and consequently preferred to a maximum speed of 180 km/h. However, under the lens of indiscernibility

200 km/h is only different from 180 km/h. How to take into account that, beyond the difference, there is the preference of 200 km/h over 180 km/h? For a certain time, this was not clear and there was even a point in which Roman Słowiński, Salvatore Greco, and Benedetto Matarazzo took into consideration to abandon the research. In the same period, Constantin Zopounidis invited Roman Słowiński to write a contribution for a book that he was editing [102]. Roman Słowiński proposed Salvatore Greco and Benedetto Matarazzo to write this contribution by applying their approach to rough set theory to bankruptcy evaluation, using the data that had been analyzed with classical rough set theory in [17]. Working on these data, the idea that would permit to extend the rough set theory to MCDA was finally found. The key concept was the substitution of the indiscernibility relation used in classical rough set theory with the dominance relation on which MCDA is rooted. On the basis of this intuition, definitions and results of classical rough set theory can be reformulated in the MCDA perspective. This idea quickly proved to be very successful in improving the results obtained using classical rough set theory, and the contribution in the book edited by Constantin Zopounidis [102] became the first application of the new extension of rough set theory called Dominance-based Rough Set Approach (DRSA). The new idea gave new impetus to the research, so that a systematic analysis of the basic concepts of DRSA and of its extension was pursued. At those times, Roman Słowiński received the invitation to write a chapter on fuzzy sets applied to MCDA, for a book proposing a state of the art on MCDA [25]. To write that contribution, in the spring 1998 Salvatore Greco stayed two months in Poznań to cooperate with Roman Słowiński, remaining constantly in contact with Benedetto Matarazzo. The final result was a contribution in the book in which rough set theory applied to MCDA, rather than fuzzy set theory, was presented [34]. The material in that chapter was continuously revised and improved until it became the basic paper in which DRSA is presented [40]. This paper has become one of the most read and cited of the European Journal of Operational Research. The research on DRSA took several directions. Some years after, on December 2006, Roman Słowiński together with Juergen Branke, Kaylamon Deb, and Kaisa Miettinen organized a Dagstuhl seminar with the aim of opening a discussion between researchers interested in interactive multiobjective optimization and evolutionary multiobjective optimization (the presentations of that seminar are collected in [11]). In that seminar, Roman Słowiński and Salvatore Greco presented the result of a new research together with Benedetto Matarazzo. The idea was the application of DRSA to guide the search of the most preferred solution in an interactive multiobjective optimization problem [46]. Again, the proposal was well accepted by the experts in multiobjective optimization, because through the DRSA the preference information asked to the DM is very simple and intuitive (the classification of some solutions as good or not) and the preference model supplied by DRSA in terms of "if …, then …" decision rules is very understandable for the DM and easy to be managed in the optimization algorithm. Recently, Roman Słowiński has come back to the multiobjective optimization through DRSA in a new research conducted again with Salvatore Greco and Benedetto Matarazzo with the addition of Salvatore Corrente. The new research [12] aims at using DRSA in an

evolutionary multiobjective optimization algorithm. The use of DRSA is justified taking into account also its good properties from the point of view of decision psychology [65].

## 18.3 Basic Concepts of the Dominance-Based Rough Set Approach

In this section, we want to present the basic ideas of DRSA [40] in comparison with the original Indiscernibility-based Rough Set Approach (IRSA) proposed by Zdzisław Pawlak [69, 70]. Both IRSA and DRSA consider a universe $U$ being a finite set of *objects*, a finite set of *attributes* $Q = \{q_1, q_2, \ldots, q_m\}$, each one of them having a value set $V_{q_i}$, $i = 1, \ldots, m$, and an information function $f : U \times Q \rightarrow V$, with $V = \bigcup_{q \in Q} V_q$, such that $f(x, q) \in V_q$ for each $q \in Q$.

Every set of attributes $P \subseteq Q$, $P \neq \emptyset$, defines an *indiscernibility relation* on $U$, denoted by $I_P$:

$$I_P = \big\{(x, y) \in U \times U : f(x, q) = f(y, q), \text{ for all } q \in P\big\}.$$

If $(x, y) \in I_P$, denoted also $xI_Py$, we say that the objects $x$ and $y$ are $P$-indiscernible. The indiscernibility relation $I_P$ is an equivalence relation on $U$ assigning to each object $x \in U$ its equivalence class

$$I_P(x) = \{y \in U : yI_Px\}.$$

The family of all the equivalence classes of relation $I_P$ is denoted by $U/I_P$. The equivalence classes of relation $I_P$ are called the *P-elementary sets* or *granules of knowledge* encoded by $P$.

Using the indiscernibility relation $I_P$, to any set $X \subseteq U$ may be associated the *P-lower approximation*

$$\underline{P}(X) = \big\{x \in U \colon I_P(x) \subseteq X\big\}$$

and the *P-upper approximation*

$$\overline{P}(X) = \big\{x \in U \colon I_P(x) \cap X \neq \emptyset\big\}.$$

Intuitively, an object $x$ belongs to $\underline{P}(X)$ if it is *certainly* contained in $X$, in the sense that all the objects that are indiscernible with it also belong to $X$. Instead, an object $x$ belongs to $\overline{P}(X)$ if it is *possibly* contained in $X$, in the sense that there is at least one object indiscernible with $x$ that belongs to $X$.

Often the set of attributes $Q$ is divided into the set of *condition* attributes $C \neq \emptyset$ and the set of *decision* attributes $D \neq \emptyset$, such that $C \cup D = Q$ and $C \cap D = \emptyset$.

The indiscernibility relation $I_S$ with respect to a set of decision attributes $S \subseteq D$, $S \neq \emptyset$, induces a partition of $U$, so that the lower and the upper approximation of each equivalence class $I_S(x)$, $x \in U$, with respect to $P \subseteq C$, $P \neq \emptyset$, can be computed, with the aim of discovering dependencies between condition attributes from $P$ and decision attributes from $S$. Indeed, if for $x, z \in U$, $P \subseteq C$ and $S \subseteq D$, $x \in \underline{P}(I_S(z))$, then $I_P(x) \subseteq I_S(z)$, that is, for all $y \in U$, if $f(y, q) = f(x, q)$ for all $q \in P$, then $f(y, q) = f(z, q)$ for all $q \in S$. This can be interpreted in the sense that the objects from $U$ suggest the following *certain* decision rule:

$$\boldsymbol{\rho}_c = \text{“if } f(y, q) = f(x, q) \text{ for all } q \in P, \text{ then } f(y, q) = f(z, q) \text{ for all } q \in S\text{”.}$$

Analogously, if for $x, z \in U$, $P \subseteq C$ and $S \subseteq D$, $x \in \overline{P}(I_S(z))$, then $I_P(x) \cap I_S(z) \neq \emptyset$, that is, there is at least one $y \in U$, such that $f(y, q) = f(x, q)$ for all $q \in P$ and $f(y, q) = f(z, q)$ for all $q \in S$. This can be interpreted in the sense that the objects from $U$ suggest the following *possible* decision rule:

$$\boldsymbol{\rho}_p = \text{“if } f(y, q) = f(x, q) \text{ for all } q \in P, \text{ then it is possible that } f(y, q) = f(z, q) \text{ for all } q \in S\text{”.}$$

The certain and possible decision rules induced from universe $U$ can also be applied to classify objects not belonging to the universe $U$, with an easily understandable explanation expressed in natural language. In fact, in the presence of a certain decision rule $\boldsymbol{\rho}_c$ (a possible decision rule $\boldsymbol{\rho}_p$), if there is an object $w \notin U$ such that $f(w, q) = f(x, q)$ for all $q \in P$, the objects from $U$ suggest that one has to (could) expect $f(w, q) = f(z, q)$ for all $q \in S$, with the certain decision rule $\boldsymbol{\rho}_c$ (the possible decision rule $\boldsymbol{\rho}_p$) that can be seen as an explanation.

Very often there is a single decision attribute $d$, that is, $D = \{d\}$. In this case, the equivalence classes $I_d(x)$, $x \in U$, can be identified with a set of decision classes $\mathbf{Cl} = \{Cl_1, \dots, Cl_n\}$. In this context, the lower and the upper approximation $\underline{P}(Cl_t)$ and $\overline{P}(Cl_t)$ of each decision class $Cl_t \in \mathbf{Cl}$ with respect to a set of condition attributes $P \subseteq C$, $P \neq \emptyset$, can be obtained.

The classical IRSA has been recognized as a mathematical theory useful in tasks considered in knowledge discovery and data mining. It has been widely investigated from the theoretical perspective. It has also been applied in the analysis of many real world problems. However, as explained in the historical notes of the previous section, it cannot deal with preferences and, more in general, with data exhibiting monotonic relationships. In order to handle this problem, Roman Słowiński with Salvatore Greco and Benedetto Matarazzo proposed to substitute the indiscernibility relation with a dominance relation. Suppose that to each attribute $q \in Q$ there is associated a preference relation $\succeq_q$, such that, without loss of generality, for all $x, y \in U$, $x$ is at least as good as $y$ with respect to attribute $q$, denoted by $x \succeq_q y$, if $f(x, q) \geq f(y, q)$. Given $x, y \in U$ and $P \subseteq Q$, $x$ *dominates* $y$ with respect to $P$, denoted by $x D_P y$, if $f(x, q) \geq f(y, q)$ for all $q \in P$. The $P$-dominance $D_P$ is a reflexive and transitive binary relation, i.e., it is a preorder.

Given a set of criteria $P \subseteq Q$, $P \neq \emptyset$, and $x \in U$, the granules of knowledge used for approximation in DRSA are the *P-dominating set* $D_P^+(x)$ (also called *positive dominance cone*) and the *P-dominated set* $D_P^-(x)$ (also called *negative dominance cone*) defined as follows:

$$D_P^+(x) = \{y \in U : yD_P x\}, \quad D_P^-(x) = \{y \in U : xD_P y\}.$$

Given $P \subseteq Q$, $P \neq \emptyset$, and $X \subseteq U$, the $P$-upward-lower and the $P$-upward-upper approximation of $X$ are defined as follows:

$$\underline{P}^+(X) = \left\{x \in U : D_P^+(x) \subseteq X\right\},$$

$$\overline{P}^+(X) = \left\{x \in U : D_P^-(x) \cap X \neq \emptyset\right\}.$$

Analogously, the $P$-downward-lower and the $P$-downward-upper approximation of $X$ are defined as follows:

$$\underline{P}^-(X) = \left\{x \in U : D_P^-(x) \subseteq X\right\},$$

$$\overline{P}^-(X) = \left\{x \in U : D_P^+(x) \cap X \neq \emptyset\right\}.$$

Also within DRSA one can consider a division of the set of attributes $Q$ into the set of condition attributes $C \neq \emptyset$ and the set of decision attributes $D \neq \emptyset$, so that, taken $P \subseteq C$ and $S \subseteq D$, $P \neq \emptyset$ and $S \neq \emptyset$, and $x \in U$, the $P$-lower and the $P$-upper approximation of the $S$-dominating set $D_S^+(x)$ and the $S$-dominated set $D_S^-(x)$ can be computed, with the aim of discovering dependencies between condition attributes from $P$ and decision attributes from $S$. Indeed, for example, if for $x, z \in U$, $P \subseteq C$ and $S \subseteq D$, $x \in \underline{P}^+(D_S^+(z))$, then $D_P^+(x) \subseteq D_S^+(z)$, that is, for all $y \in U$, if $f(y,q) \geq f(x,q)$ for all $q \in P$, then $f(y,q) \geq f(z,q)$ for all $q \in S$. This can be interpreted in the sense that the objects from $U$ suggest the following *certain* decision rule:

$$\boldsymbol{\rho}_c^+ = \text{“if } f(y,q) \geq f(x,q) \text{ for all } q \in P, \text{ then } f(y,q) \geq f(z,q) \text{ for all } q \in S\text{”.}$$

Analogously, if for $x, z \in U$, $P \subseteq C$ and $S \subseteq D$, $P \neq \emptyset$ and $S \neq \emptyset$, $y \in \overline{P}^+(D_S^+(z))$, then $D_P^-(y) \cap D_S^+(z) \neq \emptyset$, that is, there is at least one $x \in U$, such that $f(x,q) \leq f(y,q)$ for all $q \in P$ and $f(x,q) \geq f(z,q)$ for all $q \in S$. This can be interpreted in the sense that the objects from $U$ suggest the following *possible* decision rule:

$$\boldsymbol{\rho}_p^+ = \text{“if } f(y,q) \geq f(x,q) \text{ for all } q \in P, \text{ then it is possible that } f(y,q) \geq f(z,q) \text{ for all } q \in S\text{”.}$$

In this context, one has to consider a semantic correlation [41] between condition and decision attributes for which an improvement on a condition attribute should not worsen a decision attribute, if the values of the other condition attributes remain unchanged. For example, semantic correlation implies that if the evaluation of a student on a given subject, let us say history, improves, the overall evaluation should not decrease if for all other subjects the evaluations remain the same. Here, history is a condition attribute and the overall evaluation is a decision attribute. Considering semantic correlation, taken $S \subseteq D$, $S \neq \emptyset$, dominating sets $D_S^+(x)$ admit upward approximations and dominated sets $D_S^-(x)$ admit downward approximations, and, consequently, the following rough approximations can be considered:

$$\underline{P}^+(D_S^+(x)) = \left\{y \in U : D_P^+(y) \subseteq D_S^+(x)\right\},$$

$$\overline{P}^+(D_S^+(x)) = \left\{y \in U : D_P^-(y) \cap D_S^+(x) \neq \emptyset\right\},$$

$$\underline{P}^-(D_S^-(x)) = \left\{y \in U : D_P^-(y) \subseteq D_S^-(x)\right\},$$

$$\overline{P}^-(D_S^-(x)) = \left\{y \in U : D_P^+(y) \cap D_S^-(x) \neq \emptyset\right\}.$$

As was the case for IRSA, also for DRSA, usually, $D = \{d\}$, so that a single decision attribute $d$ is considered. In this case, the decision classes $\{Cl_1, \ldots, Cl_n\}$ can be preferentially ordered so that for all $x, y \in U$, with $x \in Cl_{t_1}$ and $y \in Cl_{t_2}$, if $t_1 \geq t_2$, then $x \succeq_d y$ and, equivalently, $f(x, d) \geq f(y, d)$. Consequently, the dominating and the dominated sets of the decision attribute $d$, $D_d^+(x)$ and $D_d^-(x)$, $x \in U$, can be formulated in terms of upward and downward unions of decision classes $Cl_t^{\geq}$ and $Cl_t^{\leq}$, $t = 1, \ldots, n$, defined as

$$Cl_t^{\geq} = \bigcup_{s \geq t} Cl_s, \quad Cl_t^{\leq} = \bigcup_{s \leq t} Cl_s.$$

In fact, for all $x \in U$, if $x \in Cl_t$, then

$$D_d^+(x) = \{y \in U : f(y, d) \geq f(x, d)\} = Cl_t^{\geq},$$

$$D_d^-(x) = \{y \in U : f(y, d) \leq f(x, d)\} = Cl_t^{\leq}.$$

In this context, taking into consideration semantic correlation, the upward lower and upper approximation $\underline{P}^+(Cl_t^{\geq})$ and $\overline{P}^+(Cl_t^{\geq})$ of each upward union of decision classes $Cl_t^{\geq}$ with respect to condition attributes $P \subseteq C$, $P \neq \emptyset$, can be obtained. Analogously, the downward lower and upper approximation $\underline{P}^-(Cl_t^{\leq})$ and $\overline{P}^-(Cl_t^{\leq})$ of each downward union of decision classes $Cl_t^{\leq}$ with respect to condition attributes $P \subseteq C$, $P \neq \emptyset$, can be computed.

## 18.4 Developments of DRSA

In this section we discuss some of the many developments and extensions of DRSA, related both to application of DRSA to other types of problems than just ordinal classification, considered in the basic version of DRSA presented in the previous section, and to adaptation of DRSA to different characteristics of analyzed data.

### *18.4.1 DRSA to Multicriteria Choice and Ranking*

Choice and ranking problems are one of the main problems considered in MCDA [73]. To address them using rough sets, Greco, Matarazzo and Słowiński proposed the concept of a pairwise comparison table (PCT) [26, 29] in which binary relations, that is sets of pairs of objects, are approximated, rather than sets of single objects as in the basic IRSA and DRSA. Although this way it became possible to take into account the preferences over particular criteria and the overall preference between two objects, still PCT was analyzed using IRSA, which could not take into account all types of inconsistency observed in pairwise comparisons. This deficiency was overcome when Greco, Matarazzo, and Słowiński proposed to employ dominance relation while processing PCT [27, 28].

Over the years, different ways of application of DRSA to multicriteria choice and ranking have been proposed in the literature. In the following, we discuss five such approaches in the order of their appearance, denoting them by Greek letters $\alpha - \varepsilon$. All these approaches involve five key steps:

($s_1$) Elicitation of preference information in terms of pairwise comparisons of some reference objects
($s_2$) Rough approximation of comprehensive relations implied by the pairwise comparisons, using the DRSA or a Variable Consistency DRSA (VC-DRSA; see Sect. 18.4.7), to handle possible inconsistencies observed in the PCT
($s_3$) Induction of decision rules from rough approximations of considered comprehensive relations
($s_4$) Application of induced decision rules on set $M \subseteq U$ of objects to be ranked
($s_5$) Exploitation of the resulting preference structure on $M$ to get a final ranking of objects (total preorder)

This final ranking is the recommendation presented to the DM when dealing with multicriteria ranking problem. In case of multicriteria choice, the recommendation is the object or the set of objects ranked as the best.

A common assumption of approaches $\alpha - \delta$ is that, for each cardinal criterion $q_i \in C$ (i.e., criterion with a cardinal scale, for which it is meaningful to consider intensity of preference), there is given a set of *graded preference relations* $T_i = \{P_i^h, h \in H_i\}$, where $H_i$ is a finite set of integer numbers ("grades of intensity of preference") (see, e.g., [40]). Relations $P_i^h$ are binary relations over $U$, such that for $a, b \in U$:

- If $aP_i^h b$ and $h > 0$, then object $a$ is preferred to object $b$ by degree $h$ w.r.t. criterion $q_i$.
- If $aP_i^h b$ and $h < 0$, then object $a$ is not preferred to object $b$ by degree $h$ w.r.t. criterion $q_i$.
- If $aP_i^h b$ and $h = 0$, then object $a$ is similar (asymmetrically indifferent) to object $b$ w.r.t. criterion $q_i$.

In approach $\varepsilon$, however, it is argued that the above modeling of binary relations $P_i^h$, involving determination of several thresholds for each cardinal criterion, may be considered impractical. In fact, it relates to discretization of the scale on which the strength of preference is expressed. Therefore, in approach $\varepsilon$, instead of grades of preference intensity, one puts in a PCT for each cardinal criterion $q_i \in C$ just differences of evaluations $f(a, q_i) - f(b, q_i)$.

The first application of DRSA to multicriteria choice and ranking, denoted by $\alpha$, was proposed in [27, 28, 30, 31, 33] and reminded in [34, 85]. It is characterized by the following steps:

($s_1^\alpha$) The pairwise comparisons of reference objects are expressed in terms of outranking and non-outranking relations; given a pair of objects $(a, b) \in U \times U$, a DM may: (i) state that object $a$ is comprehensively at least as good as object $b$ (or, in other words, $a$ outranks $b$), denoted by $aSb$, (ii) state that object $a$ is comprehensively not at least as good as object $b$ (or, in other words, $a$ does not outrank $b$), denoted by $aS^c b$, or (iii) abstain from any judgment.

($s_2^\alpha$) Relations $S$ and $S^c$ are approximated using *graded dominance relations* (called in the following *single-graded dominance relations*) w.r.t. the set of criteria $C$.

($s_3^\alpha$) The approximations of $S$ and $S^c$ are used to generate four types of single-graded decision rules (i.e., concerning the same grade of preference w.r.t. each criterion present in the rule condition part), denoted by $D_{++}$, $D_{-+}$, $D_{+-}$, $D_{--}$; if a pair $(a, b) \in U \times U$ is covered by at least one rule of the first two types, it is concluded that $aSb$, while if it is covered by at least one rule of the last two types, the conclusion is $aS^c b$.

($s_4^\alpha$) The application of induced rules on set $M \subseteq U$ yields four outranking relations called *true outranking relation*, *false outranking relation*, *contradictory outranking relation*, and *unknown outranking relation*, which together constitute the so-called four-valued outranking [95, 96].

($s_5^\alpha$) The final ranking of objects from set $M \subseteq U$ is obtained using their so-called net flow scores; the net flow score of an object $a \in M$, denoted by $S^{NF}(a)$, is calculated as the sum of:

(i) The number of objects $b \in M$ such that the induced rules suggest $aSb$
(ii) The number of objects $b \in M$ such that the induced rules suggest $bS^c a$

diminished by the sum of:

(iii) The number of objects $b \in M$ such that the induced rules suggest $bSa$
(iv) The number of objects $b \in M$ such that the induced rules suggest $aS^cb$

It is worth noting that the first approach presented in [27, 28, 30, 31, 33] does not account for ordinal criteria (i.e., criteria with ordinal scale, for which consideration of intensity of preference is not meaningful). Moreover, the single-graded dominance relation is lacking in precision [34] as it assumes a common grade of intensity of preference for all considered (cardinal) criteria.

The second application of DRSA to multicriteria choice and ranking, denoted by $\beta$, was presented in [34, 36, 40, 83–87]. It comprises the following steps:

($s_1^\beta$) $\equiv$ ($s_1^\alpha$).
($s_2^\beta$) Relations $S$ and $S^c$ are approximated using the dominance relation that accounts for both cardinal and ordinal criteria; w.r.t. cardinal criteria, the *multigraded dominance relation* is considered.
($s_3^\beta$) The approximations of $S$ and $S^c$ are used to generate three types of decision rules (that can use different grades of preference w.r.t. each cardinal criterion present in the rule condition part), denoted by $D_\geq$, $D_\leq$, and $D_{\geq\leq}$; if a pair of objects $(a, b) \in U \times U$ is covered by at least one rule of the first type, it is concluded that $aSb$, while if it is covered by at least one rule of the second type, the conclusion is $aS^cb$.
($s_4^\beta$) $\equiv$ ($s_4^\alpha$).
($s_5^\beta$) $\equiv$ ($s_5^\alpha$).

It is worth noting that definitions of lower approximations applied in approaches $\alpha$ and $\beta$ appear to be too restrictive in practical applications. In consequence, lower approximations of $S$ and $S^c$ are often small or even empty, preventing a good generalization of pairwise comparisons in terms of decision rules.

The third application of DRSA to multicriteria choice and ranking presented in [42, 83], denoted by $\gamma$, is characterized by the following steps:

($s_1^\gamma$) $\equiv$ ($s_1^\beta$).
($s_2^\gamma$) Relations $S$ and $S^c$ are approximated using the dominance relation that accounts for both cardinal and ordinal criteria; w.r.t. cardinal criteria, the multigraded dominance relation is considered; contrary to step ($s_2^\beta$), the approximations of $S$ and $S^c$ are calculated using a PCT-oriented adaptation of the VC-DRSA proposed originally in [38] w.r.t. the multicriteria classification problems; as this VC-DRSA measures consistency of decision examples using rough membership measure $\mu$, it will be denoted by $\mu$-VC-DRSA.
($s_3^\gamma$) The lower approximations of $S$ and $S^c$ are used to generate two types of *probabilistic decision rules* (that can use different grades of preference w.r.t. each cardinal criterion present in the rule condition part), denoted by $D_\geq$ and $D_\leq$; if a pair of objects $(a, b) \in U \times U$ is covered by at least one rule of the

first type, it is concluded that $aSb$, while if it is covered by at least one rule of the second type, the conclusion is $aS^cb$.

($s_4^\gamma$) $\equiv$ ($s_4^\beta$).

($s_5^\gamma$) $\equiv$ ($s_5^\beta$).

The fourth application of DRSA to multicriteria choice and ranking, denoted by $\delta$, was introduced in [22]. It is characterized by the following steps:

($s_1^\delta$) The pairwise comparisons of reference objects are expressed in terms of *comprehensive graded preference relations* $\succ^h$, $h \in [-1, 1]$; given a pair of objects $(a, b) \in U \times U$, a DM may: (i) state that object $a$ is comprehensively preferred to object $b$ in grade $h$, i.e., $a \succ^h b$ with $h > 0$, (ii) state that object $a$ is comprehensively *not* preferred to object $b$ in grade $h$, i.e., $a \succ^h b$ with $h < 0$, (iii) state that object $a$ is comprehensively indifferent to object $b$, i.e., $a \succ^0 b$, or (iv) abstain from any judgment.

($s_2^\delta$) *Upward cumulated preference relations* (upward unions of comprehensive graded preference relations) $\succ^{\geq h}$ and *downward cumulated preference relations* (downward unions of comprehensive graded preference relations) $\succ^{\leq h}$ are approximated using the dominance relation that accounts for both cardinal and ordinal criteria; w.r.t. cardinal criteria, the multigraded dominance relation is considered; analogously to step ($s_2^\gamma$), the approximations of $\succ^{\geq h}$ and $\succ^{\leq h}$ are calculated using a PCT-oriented adaptation of $\mu$-VC-DRSA proposed in [38].

($s_3^\delta$) The lower approximations of $\succ^{\geq h}$ and $\succ^{\leq h}$ are used to generate two types of *probabilistic decision rules* (that can use different grades of preference w.r.t. each cardinal criterion present in the rule condition part), denoted by $D_{\geq}$ and $D_{\leq}$; each induced rule is additionally characterized by the attained *confidence level*; if a pair of objects $(a, b) \in U \times U$ is covered by at least one rule of the first type, it is concluded that $a \succ^{\geq h} b$, while if it is covered by at least one rule of the second type, the conclusion is $a \succ^{\leq h} b$.

($s_4^\delta$) The application of induced rules on set $M \subseteq U$ yields a graded fuzzy preference relation (of level 2) over $M$; this relation is graded because of different grades of preference, but it is also fuzzy because of different confidence levels of rules matching pairs of objects from $M \times M$.

($s_5^\delta$) The final ranking of objects from set $M \subseteq U$ is obtained by exploitation of the preference structure on $M$ using either the Weighted Fuzzy Net Flow Score (WFNFS) procedure or a Lexicographic-fuzzy Net Flow Score procedure.

It is worth noting that elicitation of preferences in terms of comprehensive graded preference relations $\succ^h$ requires, in general, a greater cognitive effort of a DM. Moreover, it makes exploitation of the preference structure resulting from application of induced decision rules more complex.

It is also important to note that the application of variable consistency model of DRSA considered in approaches $\gamma$ and $\delta$, relying on rough membership consistency measure $\mu$, leads to the situation when calculated lower approximations of considered comprehensive relations lack several desirable monotonicity properties [4].

The fifth application of DRSA to multicriteria choice and ranking, denoted by $\varepsilon$, was introduced in [91, 92] and extended in [89]. In this approach, one takes into account the nature of the set of criteria $C$ and acts accordingly, both during composition of a PCT and when constructing the preference structure resulting from application of decision rules. In [91], a typical MCDA assumption is adopted that set $C$ is a *consistent family of criteria* [76], i.e., it satisfies the properties of *completeness* (all relevant criteria are considered), *monotonicity* (the better the evaluation of an object on considered criteria, the more it is preferable to another object), and *non-redundancy* (there is no criterion that could be removed without violating one of the previous two properties). Let us call this variant by $\varepsilon_c$. In [92], the above assumption is dropped, which is typical for Preference Learning [24] methods. Let us call this variant by $\varepsilon_{nc}$ (for not necessarily consistent set of criteria).

Approach $\varepsilon$ consists of the following steps:

($s_1^\varepsilon$) The preference information is elicited as in step ($s_1^\gamma$); moreover, relation $S$ is enriched with some additional pairs of objects—pairs $(x, y)$ such that $x$ dominates $y$ in variant $\varepsilon_c$, and pairs $(x, x)$ in variant $\varepsilon_{nc}$.

($s_2^\varepsilon$) One proceeds as in step ($s_2^\gamma$), but a PCT-oriented adaptation of the VC-DRSA proposed in [3, 4] is used instead, employing consistency measure $\varepsilon$ (denoted by $\varepsilon$-VC-DRSA).

($s_3^\varepsilon$) One proceeds as in step ($s_3^\gamma$), but inducing probabilistic decision rules using VC-DomLEM minimal cover algorithm [5], employing rule consistency measure $\widehat{\varepsilon}_T$, where $T \in \{S, S^c\}$ [89].

($s_4^\varepsilon$) Application of induced rules, both suggesting assignment to relation $S$ and $S^c$, on set $M \subseteq U$; this way one gets a preference structure on $M$, composed of relations $\mathbb{S}$ and $\mathbb{S}^c$ over $M$; both relations can be either crisp (when it is only checked if there exists at least one rule suggesting assignment of pair $(a, b) \in M \times M$ to relation $S/S^c$) or valued (when it is checked what is the *strength of the strongest rule* suggesting assignment of $(a, b)$ to $S/S^c$), and they are enriched with additional pairs of objects depending on the variant $\varepsilon_c$ or $\varepsilon_{nc}$ [89]; moreover, when constructing valued relations, rule strength can depend on rule's consistency only [89] or also on rule's coverage factor [89, 92].

($s_5^\varepsilon$) Crisp/valued preference structure composed of crisp/valued relations $\mathbb{S}$ and $\mathbb{S}^c$ over $M \subseteq U$ is exploited by transforming it to valued relation $\mathscr{R}$ defined as:

$$\mathscr{R}(a, b) = \frac{\mathbb{S}(a, b) + (1 - \mathbb{S}^c(a, b))}{2}, \tag{18.1}$$

and by applying the well-known Net Flow Rule (NFR) [9, 10] ranking method, possessing desirable properties [89], to produce the final ranking of objects from set $M$ (a weak order).

The efficiency of approach $\varepsilon$ was proved experimentally in [89]. Moreover, it is implemented in two computer programs: ruleRank and RUDE (see Sect. 18.5.5).

### 18.4.2 Case-Based Reasoning Using Dominance-Based Decision Rules

Around the year 2005, Greco, Matarazzo, and Słowiński noticed potential application of DRSA to case-based reasoning (CBR) (see, e.g., [59]). In CBR, also called similarity-based reasoning, the more similar are the causes, the more similar one expects the effects. This monotonic relationship can be employed by DRSA, when applied to similarity-based classification defined as follows. There are given a finite set of objects $U$ (called *universe of discourse*, or *case base*) and a finite family of pre-defined decision classes **Cl**. An object $y \in U$ (a "case") is described in terms of features $f_1, \ldots, f_m \in F$. For each feature $f_i \in F$, there is given a *marginal similarity function* $\sigma_{f_i} : U \times U \to [0, 1]$, such that the value $\sigma_{f_i}(y, x)$ expresses the similarity of object $y \in U$ to object $x \in U$ w.r.t. feature $f_i$; the minimal requirement that function $\sigma_{f_i}$ must satisfy is the following: for all $x, y \in U$, $\sigma_{f_i}(y, x) = 1$ if and only if $y$ and $x$ have the same value of feature $f_i$. Moreover, for each object $y \in U$, there is given an information concerning (normalized) *credibility* of its membership to each of the considered classes. To admit a graded credibility, each decision class $Cl_t \in$ **Cl**, $t \in \{1, \ldots, n\}$, is modeled as a fuzzy set in $U$ [101], characterized by membership function $\mu_{Cl_t} : U \to [0, 1]$. Thus, each object $y \in U$ can belong to different decision classes with different degrees of membership. The aim of decision aiding is to present to a DM a recommendation concerning a new object $z$, in terms of a degree of membership of this object to particular classes.

In similarity-based classification, the key issue is the aggregation of marginal similarities of objects into their comprehensive similarity. Typically, this aggregation is performed using some real-valued aggregation function (involving operators, like weighted $L_p$ norm, min, etc.) (see, e.g., [19]), which is always arbitrary to some extent. This motivated Greco, Matarazzo, and Słowiński to propose an approach that measures comprehensive similarity in a (more) meaningful way, avoiding the use of an aggregation function. An approach of this type, employing an adaptation of DRSA, was proposed for the first time in [43], and improved in [45, 47]. In the proposed approach, comprehensive similarity is represented by decision rules concisely characterizing classification examples. These rules are based on the general monotonic relationship ($mr_1$): "the more similar is object $y$ to object $x$ w.r.t. the considered features, the greater the membership of $y$ to a given decision class $Cl_t$," where $Cl_t \in$ **Cl**. This enabled to obtain a meaningful similarity measure, which is, moreover, resistant to irrelevant (or noisy) features because each decision rule, being a partial dominance cone in a similarity space, may involve conditions concerning only a subset of features. As the induced rules employ only ordinal properties of marginal similarity functions, the considered approach is also invariant to ordinally equivalent marginal similarity functions.

Few years later, a new monotonic relationship ($mr_2$) was formulated in [90]: "the more similar is object $y$ to object $x$ with respect to the considered features, the closer is $y$ to $x$ in terms of the membership to a given decision class $Cl_t$," where $Cl_t \in$ **Cl**. As observed in that paper, it is reasonable to consider ($mr_1$)

only if the membership of reference object $x$ to considered class $Cl_t \in \mathbf{Cl}$ takes a maximum value. On the other hand, ($mr_2$) does not require any assumption about the membership value of reference objects and can be considered as more general. Additionally, the authors of [90] extended the previous approach also by proposing the way of induction of decision rules using VC-DomLEM algorithm [5], and by indicating a suitable way of application of these rules according to [2]. Induced rules underline general monotonic relationship between comprehensive closeness of objects and their marginal similarities. An example of obtained decision rule is the following: "*if* similarity of flower $y$ to flower $x$ w.r.t. petal length is at least 0.7, *and* similarity of flower $y$ to flower $x$ with respect to sepal width is at least 0.8, *then* the membership of $y$ to class $setosa$ is between 0.7 and 0.9."

The approach started in 2011 was further extended in [89], by further formalizing the adaptation of DRSA, and by revising the rule classification scheme described in [2]. Finally, the complete approach was presented at a rough set conference in [93], where the description of revised rule classification scheme was much simplified.

### *18.4.3 Adaptations of DRSA to Handle Missing Attribute Values*

Shortly after the introduction of DRSA, Greco, Matarazzo, and Słowiński considered two extensions of DRSA that enabled analysis of classification data with missing attribute values [32, 35]. Also other authors considered this problem [7, 15, 16, 18, 55, 64, 94, 97]. This research, apart from [18], is well summarized in [94], where all the approaches are given an id, and their properties are thoroughly examined with respect to a list of 11 desirable properties.

An adaptation of DRSA to handle missing attribute values involves an adjusted definition of dominance relation that accounts for missing values. In some approaches, it also involves the change of the definition of rough approximation, to take into account the lack of some classical properties of redefined dominance relation (e.g., lack of transitivity). In [94], the authors denoted each adaptation identified in the literature by DRSA-$mv_j$, and respective adjusted dominance relation by $D_j$, where $j$ stands for the version id. In the following, we will use $j$ in the superscript, to allow a subset of attributes $P \subseteq C$ in the subscript. The authors of [94] pointed out, after [81, 97], that the following generalized definitions of $P$-lower and $P$-upper approximations of unions of decision classes $Cl_t^{\geq}, Cl_t^{\leq}$, $t = 1, \ldots n$, should be employed:

$$\begin{array}{ll} \underline{P}(Cl_t^{\geq}) = \{x \in U : d_P^{j+}(x) \subseteq Cl_t^{\geq}\} & \underline{P}(Cl_t^{\leq}) = \{x \in U : D_P^{j-}(x) \subseteq Cl_t^{\leq}\} \\ \overline{P}(Cl_t^{\geq}) = \bigcup\limits_{x \in Cl_t^{\geq}} D_P^{j+}(x) & \overline{P}(Cl_t^{\leq}) = \bigcup\limits_{x \in Cl_t^{\leq}} d_P^{j-}(x), \end{array} \tag{18.2}$$

where

$$d_P^{j+}(x) = \{y \in U : x\, d_P^j\, y\} \qquad d_P^{j-}(x) = \{z \in U : z\, d_P^j\, x\}. \tag{18.3}$$

In the above definitions, $D_P^j$ denotes adapted $P$-dominance relation ($x D_P^j y$ means $x$ $P$-dominates $y$), while $d_P^j$ denotes adapted $P$-inverse-dominance relation ($x d_P^j y$ means $x$ is $P$-dominated by $y$). These dominance relations are defined as:

$$y D_P^j x \Leftrightarrow y \succeq_q^j x \text{ for each } q \in P \tag{18.4}$$

$$z\, d_P^j\, x \Leftrightarrow z \preceq_q^j x \text{ for each } q \in P, \tag{18.5}$$

where $x, y, z \in U$ and versions of DRSA-$mv_j$ differ by definitions of relations $\succeq_q^j$ and $\preceq_q^j$. Remark that definition (18.2) can be applied even when relations $D_P^j$ and $d_P^j$ are only assumed to be reflexive. In particular, it fits the case when $y D_P^j x$ is not equivalent to $x d_P^j y$.

In [94], the analysis of properties of different adaptations of DRSA was presented. It involves the following adaptations: [32, 35]—introduce DRSA-$mv_1$ and DRSA-$mv_2$, [55]—presents DRSA-$mv_{2.5}$, [97]—proposes DRSA-$mv_{1.5}$, [15, 16]—introduce the concept of a lower-end dominance relation used in DRSA-$mv_4$, and the concept of an upper-end dominance relation resulting in DRSA-$mv_5$, [64]—presents DRSA-$mv_6$, and [7]—defines DRSA-$mv_3$. The analysis resulted in a conclusion that the only non-dominated approaches are DRSA-$mv_{1.5}$, DRSA-$mv_2$, DRSA-$mv_4$, and DRSA-$mv_5$.

It is important to note that taking into account the semantics of missing values considered, e.g., in [52, 53] (and in [88]), it can be said that DRSA-$mv_{1.5}$ treats missing values as "lost" ("absent") values, while DRSA-$mv_2$ treats missing values as "do not care" values. These approaches are defined as:

$$y \succeq_q^{1.5} x \Leftrightarrow y \succeq_q x, \text{ or } f(y,q) = ? \qquad z \preceq_q^{1.5} x \Leftrightarrow x \succeq_q z, \text{ or } f(z,q) = ? \tag{18.6}$$

$$y \succeq_q^{2} x \Leftrightarrow y \succeq_q x, \text{ or } f(y,q) = *, \text{ or } f(x,q) = * \qquad z \preceq_q^{2} x \Leftrightarrow x \succeq_q z, \text{ or } f(z,q) = *, \text{ or } f(x,q) = *, \tag{18.7}$$

where $x, y, z \in U$, $q \in C$, ? denotes a "lost" missing value, and $*$ denotes a "do not care" missing value.

In [18], Du and Hu proposed the so-called characteristic-based dominance relation with "do not care" values and "lost" values coexisting. It relates to the characteristic relation considered in IRSA [50, 51]. Let us denote this approach by

DRSA-$mv_{1.5\&2}$. Then

$$y \succeq_q^{1.5\&2} x \quad \Leftrightarrow \quad f(y,q) = ?, \text{ or } y \succeq_q x, \text{ or } f(y,q) = *, \text{ or } f(x,q) = *. \tag{18.8}$$

Although the idea of considering "do not care" values and "lost" values simultaneously is worth considering, the surprising part of [18] is that the authors do not employ $P$-inverse-dominance relation $d_P^{1.5\&2}$, $P \subseteq C$, determined as in (18.5) with relation $\preceq_q^{1.5\&2}$ defined as:

$$z \preceq_q^{1.5\&2} x \quad \Leftrightarrow \quad f(z,q) = ?, \text{ or } x \succeq_q z, \text{ or } f(z,q) = *, \text{ or } f(x,q) = *, \tag{18.9}$$

where $x, y, z \in U, q \in P$. Instead, Du and Hu consider that $x d_P^{1.5\&2} y \Leftrightarrow y D_P^{1.5\&2} x$. In our opinion, this is not a proper realization of the idea outlined by Słowiński and Vanderpooten [81] (even reminded in [18]) who claimed that for set $X \subseteq U$ and reflexive binary relation $R$ over $U$, the lower approximation of set $X$ with respect to $R$ should be calculated as:

$$\underline{R}(X) = \{x \in U : R^{-1}(x) \subseteq U\}, \tag{18.10}$$

where $R^{-1}(x) = \{y \in U : xRy\}$.

### 18.4.4 Extensions of DRSA for Interval Evaluations on Criteria

One of the ways of handling imprecision in object's evaluation is to use interval evaluations. Interval evaluations may also occur when replacing missing values with a range of possible evaluations. Yet another scenario concerns the case of hierarchical set of attributes, when a range of decisions obtained at some lower level of attribute hierarchy is translated to an interval of evaluations for an upper level criterion [13]. Motivated by these observations, Dembczyński, Greco, and Słowiński proposed some extensions of the classical DRSA that permit to analyze data with interval evaluations on criteria [13–16], and also with interval assignments to decision classes [14–16]. All these extensions involved the so-called P-possible dominance relation $\overline{D}_P$. Assuming, without loss of generality, that each criterion is of gain type, $P$-possible dominance relation is defined as:

$$x \overline{D}_P y \Leftrightarrow u(x,q) \geq l(y,q) \text{ for all } q \in P, \tag{18.11}$$

where $P \subseteq C$, and $u(x,q)$, $l(y,q)$ denote, respectively, upper limit of the interval for object $x$ on attribute $q$ and lower limit of the interval for object $y$ on attribute $q$.

In [15, 16], the authors additionally considered two other $P$-dominance relations, $P \subseteq C$. Assuming, without loss of generality, that each criterion is of gain type, they are defined as follows:

- *$P$-lower-end dominance relation $D^l_P$:*

$$xD^l_P y \Leftrightarrow l(x,q) \geq l(y,q) \text{ for all } q \in P, \tag{18.12}$$

- *$P$-upper-end dominance relation $D^u_P$:*

$$xD^u_P y \Leftrightarrow u(x,q) \geq u(y,q) \text{ for all } q \in P. \tag{18.13}$$

The above papers also took into account interval assignments to decision classes. Suppose $d \in Q$ is a decision criterion. Then, the following rough approximations of upward and downward unions of decision classes are considered:

$$\underline{Cl_t^{\geq}} = \{y \in U : l(y,d) \geq t\}. \tag{18.14}$$

$$\underline{Cl_t^{\leq}} = \{y \in U : u(y,d) \leq t\}. \tag{18.15}$$

$$\overline{Cl_t^{\geq}} = \{y \in U : u(y,d) \geq t\}. \tag{18.16}$$

$$\overline{Cl_t^{\leq}} = \{y \in U : l(y,d) \leq t\}. \tag{18.17}$$

Finally, the above rough approximations are the basis for second-order dominance-based rough approximations [15, 16]:

$$\underline{P}(\underline{Cl_t^{\geq}}) = \{x \in U : \overline{D_P^+}(x) \subseteq \underline{Cl_t^{\geq}}\}. \tag{18.18}$$

$$\underline{P}(\underline{Cl_t^{\leq}}) = \{x \in U : \overline{D_P^-}(x) \subseteq \underline{Cl_t^{\leq}}\}. \tag{18.19}$$

$$\overline{P}(\overline{Cl_t^{\geq}}) = \{x \in U : \overline{D_P^-}(x) \cap \overline{Cl_t^{\geq}} \neq \emptyset\}. \tag{18.20}$$

$$\overline{P}(\overline{Cl_t^{\leq}}) = \{x \in U : \overline{D_P^+}(x) \cap \overline{Cl_t^{\leq}} \neq \emptyset\}. \tag{18.21}$$

DRSA with interval evaluations has also been considered in [98], where $P$-possible dominance relation has been used for condition criteria. In that paper, the authors also consider intervals with one or two missing limits. Assuming that each $q \in C$ is of gain type, a missing lower limit is then replaced with the minimal value in the value set of the respective criterion, while missing upper limit is replaced with the maximal value in the value set of the respective criterion.

### 18.4.5 Extending DRSA to Address Hierarchical Structure of Attributes

In hierarchical classification/sorting problems, where the set of attributes has a hierarchical structure, the main difficulty is the propagation of inconsistencies along the tree structure, i.e., taking into consideration at each node of the tree the inconsistent information from lower level nodes. In [13], the inconsistencies are propagated bottom-up, in the form of subsets of possible attribute values (for hierarchical regular attributes), and in the form of intervals of possible values (for hierarchical criteria). Decision rules are induced at each node of the tree. The classification/sorting of new objects is also done from the bottom to the top of the hierarchy, to make the final decision in the root node of the tree.

The extension of DRSA proposed in [13] assumes that object $y$ $P$-dominates object $x$, $P \subseteq C$, if:

- For each regular attribute $q \in P$, the subset of attribute values for object $y$ has a non-empty intersection with the subset of attribute values for object $x$.
- For each criterion $q \in P$, $u(x, q) \geq l(y, q)$, where $u(x, q)$, $l(y, q)$ denote, respectively, upper limit of the interval for object $x$ on attribute $q$ and lower limit of the interval for object $y$ on attribute $q$.

### 18.4.6 Extensions of DRSA for Non-ordinal Data

Błaszczyński, Greco, and Słowiński proposed an approach to induction of laws from data, which makes use of the concept of monotonic relationships between values of condition and decision attributes, without assuming its direction a priori and allowing local monotonicity relationships in subregions of the evaluation space [6]. This approach is able to discover local and global monotonicity relationships existing in data. The relationships are represented by monotonic decision rules. To enable the discovery of monotonic rules, a non-invasive transformation of the input data was proposed. The proposed transformation should be applied to all non-ordinal attributes. Moreover, after transformation of input data, DRSA is applied to structure data into consistent and inconsistent parts.

For the purpose of the illustration, we may assume, without loss of generality, that the value sets of both decision attribute (class labels) and condition attributes are number-coded. As in non-ordinal classification problems, the natural complete ordering of classes $Cl_1$, $Cl_2$, ..., $Cl_n$ induced by number-coded class labels is not entering, in general, into some monotonic relationships with value sets of condition attributes, we have to consider $n$ binary ordinal classification problems with two sets of objects: class $Cl_t$ and its complement $\neg Cl_t$, $t = 1, \ldots, n$, which are number-coded by 1 and 0, respectively. This means that in the $t$-th ordinal binary classification problem, set $Cl_t$ is interpreted by DRSA as union $Cl_1^{\geq}$ and set $\neg Cl_t$ as union $Cl_0^{\leq}$, $t = 1, \ldots, n$. Ordinal classification problems can be handled by DRSA without altering the original number codes of the class labels.

The transformation of each non-ordinal condition attribute from $C$ is made individually, depending on its type: numerical (number-coded) or nominal. Each numerical (number-coded) attribute $q_i$ is represented, in the transformed form, as a pair of ordinal attributes $q_i'$, and $q_i''$. In other words, evaluation of each object $x \in U$ by numerical attribute $q_i$ is repeated twice, and the first evaluation $f(x, q_i)$ is renamed to $f(x, q_i')$, while the second evaluation $f(x, q_i)$ is renamed to $f(x, q_i'')$. Then, the first attribute $q_i'$ is set to have positive monotonic relationship with (possibly transformed) decision attribute, while the second attribute $q_i''$ is set to have negative monotonic relationship with the decision attribute.

Each nominal attribute $q_j$ with value set composed of $k$ distinct values is binarized, such that the presence or absence of the $l$-th value of this attribute is coded by a new ordinal attribute $q_{jl}$ taking value 1 or 0, respectively, $l = 1, 2, \ldots, k$. Then the binary attribute $q_{jl}$ is represented, in the transformed form, by a pair of ordinal attributes $q_{jl}'$, and $q_{jl}''$. Again, the first ordinal attribute in that pair has positive monotonic relationship with (possibly transformed) decision attribute, while the second attribute in that pair has negative monotonic relationship.

The proposed approach provides framework for analysis of heterogeneous classification data. It has been shown experimentally in [6] that the monotonic rules induced from transformed data, together with a specific classification scheme, have at least as good predictive ability as other well-known predictors.

### *18.4.7 Parametric, Decision Theoretic, and Stochastic DRSA*

Greco, Matarazzo, Słowiński, and Stefanowski identified the need to relax the definition of the lower approximation of union of classes and, in consequence, to admit to the lower approximation some inconsistent objects (i.e., objects which would not be admitted to lower approximations in the classical DRSA) for which there is enough evidence for their membership to the union of classes [38]. In this way, lower approximations are defined assuming an acceptable value of a measure expressing evidence of membership to the set. Following this idea, the evidence for this membership may be estimated by different types of consistency or precision measures, and lower approximations of unions of decision classes may be defined in different ways resulting in different approaches to this kind of relaxation, called generally parametric DRSA, including: Variable Consistency DRSA (VC-DRSA) [1, 3, 4, 38, 39, 42], and Variable Precision DRSA (VP-DRSA) [47, 56]. For example, in case of VC-DRSA, lower approximation is defined as a subset of the approximated set. In consequence, given an upward (downward) union of classes, objects that do not belong to this union are never included in its lower approximation, even if they dominate (are dominated by) an object from the considered union. On the other hand, in case of VP-DRSA, a lower approximation is not a subset of the approximated set. All objects belonging to dominance cones of objects from a lower approximation of a union of classes are also included in the lower approximation of the union.

The fact that precision measure can be interpreted as a conditional probability stimulated development of the decision-theoretic rough set model (DTRSM) [99, 100]. DTRSM for DRSA has also been studied in [44, 66]. DTRSM connects definition of approximations with conditional risk minimization in Bayesian decision theory. In that situation, the states correspond to the decision classes, and assignment of an object to positive, negative, and boundary regions is decided on the basis of its condition attribute values. Assuming an acceptable loss of the classification accuracy and estimating conditional probability of the considered decision classes, an optimal Bayes decision rule is obtained. In this way, calculation of approximations can be seen as a classification problem.

Another type of approach to treatment of inconsistent objects within DRSA has also been considered in [60, 61]. This approach originates from statistical learning and statistical decision theory and, in contrast to the previously mentioned approaches, involves relabeling of objects (i.e., change of class to which object belongs to the more probable one). It uses the notion of stochastic dominance and maximum likelihood estimators of probability of object belonging to union of classes. Stochastic lower approximations of unions of classes are composed of objects for which values of estimators are higher than a given threshold.

It is worth mentioning that statistical interpretations of VC-DRSA and VP-DRSA were proposed in [62], by connecting lower approximations with minimizers of empirical risk functions. In result, it has been demonstrated that families of monotonic classifiers and the hinge loss function serve as a foundation for characterization of the parametric DRSA with consistency measures having desirable monotonicity properties.

### *18.4.8 Decision Rules Induction*

In DRSA, induction of decision rules is subsequent to computation of rough approximations. In this context, computation of approximations can be viewed as a kind of preprocessing of data. Objects identified as sufficiently consistent are a good basis for induction of decision rules. The purpose of induction of decision rules is to discover strong relationships between description of these objects and their membership to a union of classes. If the rules are intended to be used in classification, then the goal of the induction procedure is to find a preferably small set of rules with high predictive accuracy. Decision rules should be short and accurate. Shorter decision rules are easier to understand. Shorter rules also allow to avoid *overfitting* the training data. Overfitting occurs when the learned model fits training data perfectly but is not performing well on new data.

In DRSA, for a given class $Cl_t$, $t \in \{1, \ldots, n\}$, we consider decision rules of the type:

$$\textit{if } f(y, q_{i_1}) \succeq r_{i_1} \wedge \ldots \wedge f(y, q_{i_p}) \succeq r_{i_p} \wedge f(y, q_{i_{p+1}}) = r_{i_{p+1}} \wedge \ldots \wedge f(y, q_{i_z}) = r_{i_z} \\ \textit{then } y \in Cl_t^{\geq}, \quad (18.22)$$

$$\textit{if } f(y, q_{i_1}) \preceq r_{i_1} \wedge \ldots \wedge f(y, q_{i_p}) \preceq r_{i_p} \wedge f(y, q_{i_{p+1}}) = r_{i_{p+1}} \wedge \ldots \wedge f(y, q_{i_z}) = r_{i_z} \\ \textit{then } y \in Cl_t^{\leq}, \quad (18.23)$$

where $q_j$, $j \in \{i_1, i_2, \ldots, i_p\}$ denotes a criterion and $q_k$, $k \in \{i_{p+1}, i_{p+2}, \ldots, i_z\}$ denotes a regular attribute. Moreover, $r_j \in V_j$, $j = \{i_1, i_2, \ldots, i_p, i_{p+1}, i_{p+2}, \ldots, i_z\}$, denotes a value from the domain of criterion / regular attribute $q_j$. We use symbols $\succeq$ and $\preceq$ to indicate weak preference w.r.t. single criterion and inverse weak preference, respectively. If $q_j \in C$ is a gain (cost) criterion, then elementary condition $f(y, q_j) \succeq r_j$ means that the value of covered object $y$ on criterion $q_j$ is not smaller (not greater) than value $r_j$. Elementary conditions for regular attributes are of type $f(y, q_j) = r_j$.

The most important decision rule induction algorithms in DRSA are inspired by LEM2 algorithm [49], proposed by Grzymała-Busse. The applied heuristic strategy of rule induction in these algorithms is called sequential covering [54] or separate and conquer [23, 67, 68]. It constructs a rule that covers a subset of training objects, removes the covered objects from the training set, and iteratively learns another rule that covers some of the remaining objects, until no uncovered objects remain. The first of such algorithms was proposed by Greco, Matarazzo, Słowiński, and Stefanowski and is called DomLEM [37]. VC-DomLEM, proposed by Błaszczyński, Szeląg, and Słowiński [5], is an adaptation of the same concept for parametric DRSA. This algorithm heuristically searches for rules that satisfy constraint with respect to chosen rule consistency measure.

## 18.5 Available Software

In this section, we describe some computational libraries and applications that implement DRSA and its extensions for rough set analysis of ordinal data.

### 18.5.1 ruleLearn

ruleLearn[1] is an open-source computation library written in Java and hosted on GitHub (project started in Dec 2017). Currently it implements DRSA and VC-DRSA (see Sect. 18.4.7) and allows to handle missing attribute values according to DRSA-$mv_{1.5}$ and DRSA-$mv_2$ (see Sect. 18.4.3). It also offers access to the VC-DomLEM rule induction algorithm (see Sect. 18.4.8) and different classification strategies. ruleLearn is utilized by other programs described in this section: RuLeStudio and RuleVisualization.

### 18.5.2 RuLeStudio

RuLeStudio[2] is an open-source client–server web application supporting data analysis using DRSA and VC-DRSA. It requires Java 8+ to run and employs ruleLearn API. This application was finished in 2020 [20]. It supports application of (VC-)DRSA to analysis of ordinal data, possibly containing missing values (handled according to DRSA-$mv_{1.5}$ or DRSA-$mv_2$). The application permits to consider a certain number of features such as data editor, presentation of dominance cones (both with respect to dominance and inverse-dominance relation), certain/possible rule generation according to VC-DomLEM algorithm, analysis of different characteristics of induced rules, application of rules on test objects using several classification strategies, cross-validation, and presentation of misclassification matrix.

### 18.5.3 RuleVisualization

RuleVisualization[3] is an open-source client–server web application for visualization and exploration of decision rules. It requires Java 8+ to run and employs ruleLearn API. This application was finished in 2019 [63]. This program can read rules typical for DRSA, induced, e.g., by RuLeStudio, and offers different visualizations of these rules (including the attributes used by rules), their filtering, sorting, matching to some test objects, and presentation of graphs showing co-occurrence of attributes and semantic/coverage similarity among rules.

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[1] https://github.com/ruleLearn/rulelearn.

[2] https://github.com/dominieq/rule-studio.

[3] http://www.cs.put.poznan.pl/mszelag/Software/RuleVisualization/RuleVisualization.html.

### 18.5.4 jMAF

jMAF[4] is a rough set data analysis software written in Java. It makes use of java Rough Set (jRS) library. jMAF and jRS library implement methods of data analysis provided by the DRSA and VC-DRSA, allowing to calculate lower approximations, induce decision rules, and apply the rules to classify objects. More details with regard to the analysis that this software package enables can be found in [8].

### 18.5.5 RuleRank Ultimate Desktop Edition

RuleRank Ultimate Desktop Edition (RUDE)[5] is a decision support tool for application of DRSA to multicriteria choice and ranking. It requires Java 10+ and JavaFX to run. This application was finished in 2018 [57]. RUDE employs approach denoted in Sect. 18.4.1 by $\varepsilon$, allowing setting different parameters characteristic for that approach and visualization of: PCT, decision rules induced from that PCT, preference graph resulting from application of these rules on a set of objects to be ranked, and final ranking.

## 18.6 Conclusions

We presented the main ideas and developments of the Dominance-based Rough Set Approach (DRSA), recalling the basic principles with a general overview of its developments and software. At the end of this survey, we have to conclude that DRSA continues to be appealing for experts in Multiple Criteria Decision Aiding (MCDA) on one hand, and artificial intelligence and machine learning on the other hand, for its most salient characteristic consisting in its capacity of asking the decision maker for simple preference information and supplying easily understandable and explainable recommendations. It is not difficult to see that the DRSA properties of being simple and easily understandable have to be acknowledged to the contribution of Roman Słowiński following the lines traced by Bernard Roy with respect to MCDA and Zdzisław Pawlak with respect to rough set theory.

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[4] http://www.cs.put.poznan.pl/jblaszczynski/Site/jRS.html.

[5] http://www.cs.put.poznan.pl/mszelag/Software/ruleRank/ruleRank.html.

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# Chapter 19
# Rule Set Complexity for Mining Incomplete Data Using Probabilistic Approximations Based on Generalized Maximal Consistent Blocks

**Patrick G. Clark, Jerzy W. Grzymala-Busse, Zdzislaw S. Hippe, and Teresa Mroczek**

**Abstract** In this chapter, missing attribute values in incomplete data sets have two possible interpretations: lost values and "do not care" conditions. Lost values are currently unavailable, e.g., they were erased, while "do not care" conditions are replaceable by any specified attribute value. For data mining, we use two kinds of probabilistic approximations, global and saturated. Both probabilistic approximations are constructed from maximal consistent blocks. Thus, since we use two kinds of missing attribute values and two kinds of probabilistic approximations, we use four different data mining methods. We have shown, in our previous study, that pairwise differences in an error rate, calculated by ten-fold cross validation between those four methods, are statistically insignificant (5% level of significance). Hence, we explore another problem: when the rule set complexity is the smallest. We show that the difference between using both kinds of probabilistic approximations is, in general, insignificant. However, we should explore both interpretations of missing attribute values, "do not care" conditions and lost values, since there are significant differences.

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## 19.1 Introduction

Incomplete data sets are affected by missing attribute values. In this chapter, we consider two possible interpretations of missing attribute values: lost values and "do not care" conditions. Lost values are currently unavailable, e.g., they were erased, while "do not care" conditions are replaceable by any specified attribute value. A lost value is denoted by "?," and a "do not care" condition is denoted by "*."

For rule induction, we use probabilistic approximations, a generalization of the idea of lower and upper approximations known in rough set theory. A probabilistic approximation of the concept $X$ is associated with a probability $\alpha$; if $\alpha = 1$, the probabilistic approximation becomes the lower approximation of $X$; if $\alpha$ is a small positive number, e.g., 0.001, the probabilistic approximation is reduced to the upper approximation of $X$. Usually, probabilistic approximations are applied to completely specified data sets [19, 22–29], and such approximations are generalized to incomplete data sets, using characteristic sets, in [15, 16].

Additionally, we may use another parameter, denoted by $\beta$, for rule induction. In our previous study [9], where $\alpha$ was constant and $\beta$ varied, it was shown that pairwise differences in an error rate, evaluated by ten-fold cross validation between these four methods of data mining, are statistically insignificant (5% level of significance). Hence, we explore another setup: $\alpha$ varies and $\beta$ is constant.

Recently, two new types of approximations were introduced, global probabilistic approximations in [5] and saturated probabilistic approximations in [7]. Results of experiments on an error rate, evaluated by ten-fold cross validation, using characteristic sets were presented in [7, 10, 11] and using maximal consistent blocks in [3, 4]. In these experiments, global and saturated probabilistic approximations were explored using data sets with lost values and "do not care" conditions. Results show that among these four methods there is no universally best method. If so, the next problem is when the rule sets are the simplest.

Thus, the main objective of this chapter is a comparison of the four methods of data mining from the rule complexity viewpoint, taking into account the number of rules and the total number of conditions in rule sets. Some experiments exploring complexity of rule sets were presented in [6, 8]. In these papers, we kept $\alpha = 0.5$ and $\beta$ varied between 0.001 and 1. In this chapter, we decided to use a new setup: $\alpha$ varies between 0.001 and 1, while $\beta = 0.5$.

Rule induction was conducted using a new version of the Modified Learning from Examples Module, version 2 (MLEM2) [2, 14]. The MLEM2 algorithm is a component of the Learning from Examples using Rough Sets (LERS) data mining system [1, 13, 14].

Our main result is that, in general, the total number of rules is the smallest when missing attribute values are interpreted as "do not care" conditions. A choice between global and saturated probabilistic approximations is not relevant to rule set complexity.

## 19.2 Incomplete Data

We assume that the input data sets are presented in the form of a decision table. An example of the decision table is shown in Table 19.1. Rows of the decision table represent cases, while columns are labeled by variables. The set of all cases will be denoted by $U$. In Table 19.1, $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Independent variables are called attributes, and a dependent variable is called a decision and is denoted by $d$. The set of all attributes will be denoted by $A$. In Table 19.1, $A = \{$*Temperature*, *Headache*, *Cough*$\}$ and $d$ is *Flu*. The value for a case $x$ and an attribute $a$ will be denoted by $a(x)$. For example, *Temperature*(1) = *high*.

The set $X$ of all cases defined by the same value of the decision $d$ is called a *concept*. For example, a concept associated with the value *yes* of the decision *Flu* is the set $\{1, 2, 3, 4\}$.

A *block* of the attribute–value pair $(a, v)$, denoted by $[(a, v)]$, is the set $\{x \in U \mid a(x) = v\}$ [12]. For incomplete decision tables, the definition of a block of an attribute–value pair is modified in the following way:

- If for an attribute $a$ and a case $x$ we have $a(x) = ?$, the case $x$ should not be included in any blocks $[(a, v)]$ for all values $v$ of attribute $a$.
- If for an attribute $a$ and a case $x$ we have $a(x) = *$, the case $x$ should be included in blocks $[(a, v)]$ for all specified values $v$ of attribute $a$.

For the data set from Table 19.1, the blocks of attribute–value pairs are

$$
\begin{array}{ll}
[(\text{Temperature, normal})] = \{5, 6, 8\}, & [(\text{Headache, yes})] = \{2, 4, 8\}, \\
[(\text{Temperature, high})] = \{1, 5\}, & [(\text{Cough, no})] = \{1, 3, 5, 7, 8\}, \\
[(\text{Temperature, very-high})] = \{2, 4, 5\}, & [(\text{Cough, yes})] = \{1, 3, 6, 7\}, \\
[(\text{Headache, no})] = \{2, 3, 5, 7, 8\}. &
\end{array}
$$

For a case $x \in U$ and $B \subseteq A$, the *characteristic set* $K_B(x)$ is defined as the intersection of the sets $K(x, a)$, for all $a \in B$, where the set $K(x, a)$ is defined in the following way:

**Table 19.1** A decision table

| | Attributes | | | Decision |
|---|---|---|---|---|
| Case | Temperature | Headache | Cough | Flu |
| 1 | high | ? | * | yes |
| 2 | very-high | * | ? | yes |
| 3 | ? | no | * | yes |
| 4 | very-high | yes | ? | yes |
| 5 | * | no | no | no |
| 6 | normal | ? | yes | no |
| 7 | ? | no | * | no |
| 8 | normal | * | no | no |

- If $a(x)$ is specified, then $K(x, a)$ is the block $[(a, a(x))]$ of attribute $a$ and its value $a(x)$.
- If $a(x) = ?$ or $a(x) = *$, then $K(x, a) = U$.

For Table 19.1 and $B = A$,

$$K_A(1) = \{1, 5\}, \quad K_A(5) = \{3, 5, 7, 8\},$$
$$K_A(2) = \{2, 4, 5\}, \quad K_A(6) = \{6\},$$
$$K_A(3) = \{2, 3, 5, 7, 8\}, \quad K_A(7) = \{2, 3, 5, 7, 8\}, \text{ and}$$
$$K_A(4) = \{2, 4\}, \quad K_A(8) = \{5, 8\}.$$

A binary relation $R(B)$ on $U$, defined for $x, y \in U$ in the following way:

$$(x, y) \in R(B) \; if \; and \; only \; if \; y \in K_B(x),$$

will be called the *characteristic relation*. In our example, $R(A)$ = {(1, 1), (1, 5), (2, 2), (2, 4), (2, 5), (3, 2), (3, 3), (3, 5), (3, 7), (3, 8), (4, 2), (4, 4), (5, 3), (5, 5), (5, 7), (5, 8), (6, 6), (7, 2), (7, 3), (7, 5), (7, 7), (7, 8), (8, 5), (8, 8)}.

We quote some definitions from [3]. Let $X$ be a subset of $U$. The set $X$ is $B$-*consistent* if $(x, y) \in R(B)$ for any $x, y \in X$. If there does not exist a $B$-consistent subset $Y$ of $U$ such that $X$ is a proper subset of $Y$, the set $X$ is called a *generalized maximal* $B$*-consistent block*. The set of all generalized maximal $B$-consistent blocks will be denoted by $\mathscr{C}(B)$. In our example, $\mathscr{C}(A)$ = {{1}, {2, 4}, {3, 5, 7}, {5, 8}, {6}}.

Let $B \subseteq A$ and $Y \in \mathscr{C}(B)$. The set of all generalized maximal $B$-consistent blocks that include an element $x$ of the set $U$, i.e., the set

$$\{Y | Y \in \mathscr{C}(B), x \in Y, \}$$

will be denoted by $\mathscr{C}_B(x)$.

For data sets in which all missing attribute values are “do not care” conditions, an idea of a maximal consistent block of $B$ was defined in [21]. Note that in our definition, the generalized maximal consistent blocks of $B$ are defined for arbitrary interpretations of missing attribute values. For Table 19.1, the generalized maximal $A$-consistent blocks $\mathscr{C}_A(x)$ are

$$\mathscr{C}_A(1) = \{\{1\}\}, \quad \mathscr{C}_A(5) = \{\{5, 8\}, \{3, 5, 7\}\},$$
$$\mathscr{C}_A(2) = \{\{2, 4\}\}, \quad \mathscr{C}_A(6) = \{\{6\}\},$$
$$\mathscr{C}_A(3) = \{\{3, 5, 7\}\}, \quad \mathscr{C}_A(7) = \{\{3, 5, 7\}\},$$
$$\mathscr{C}_A(4) = \{\{2, 4\}\}, \quad \mathscr{C}_A(8) = \{\{5, 8\}\}.$$

## 19.3 Probabilistic Approximations

In this section, we will discuss two types of probabilistic approximations: global and saturated.

### 19.3.1 Global Probabilistic Approximations

A special case of the global probabilistic approximation, limited only to lower and upper approximations and to characteristic sets, was introduced in [17, 18]. A general definition of the global probabilistic approximation was introduced in [8].

A $B$-*global probabilistic approximation* of the concept $X$, with the parameter $\alpha$ and denoted by $appr^{global}_{\alpha,B}(X)$, is defined as follows:

$$\cup\{Y \mid Y \in \mathscr{C}_x(B),\ x \in X,\ Pr(X|Y) \geq \alpha\}.$$

Obviously, for given sets $B$ and $X$ and the parameter $\alpha$, there exist many $B$-global probabilistic approximations of $X$. Additionally, an algorithm for computing $B$-global probabilistic approximations is of exponential computational complexity. So, we decided to use a heuristic version of the definition of $B$-global probabilistic approximation, called the MLEM2 $B$-global probabilistic approximation of the concept $X$, associated with a parameter $\alpha$ and denoted by $appr^{mlem2}_{\alpha,B}(X)$ [5]. This definition is based on the rule induction algorithm MLEM2. The approximation $appr^{mlem2}_{\alpha,B}(X)$ is a union of the generalized maximal consistent blocks $Y \in \mathscr{C}(B)$, the most relevant to the concept $X$, i.e., with $|X \cap Y|$ as large as possible and with $Pr(X|Y) \geq \alpha$. If more than one generalized maximal consistent block $Y$ satisfies both conditions, the generalized maximal consistent block $Y$ with the largest $Pr(X|Y) \geq \alpha$ is selected. If this criterion ends up with a tie, a generalized maximal consistent block $Y$ is picked up heuristically, as the first on the list [5].

Special MLEM2 $B$-global probabilistic approximations, with $B = A$, are called *global probabilistic approximations* associated with the parameter $\alpha$ and are denoted by $appr^{mlem2}_{\alpha}(X)$.

Let $E_\alpha(X)$ be the set of all eligible generalized maximal consistent blocks defined as follows:

$$\{Y \mid Y \subseteq \mathscr{C}(A),\ Pr(X|Y) \geq \alpha\}.$$

A heuristic version of the MLEM2 global probabilistic approximation is presented below.

**MLEM2 global probabilistic approximation algorithm**
**input**: a set $X$ (a concept), a set $E_\alpha(X)$,
**output**: a set $T$ ( a global probabilistic approximation $appr^{mlem2}_{\alpha}(X)$) of $X$

**begin**
  $G := X$;
  $T := \emptyset$;
  $\mathscr{Y} := E_\alpha(X)$;
  **while** $G \neq \emptyset$ **and** $\mathscr{Y} \neq \emptyset$
    **begin**
      select a generalized maximal consistent block $Y \in \mathscr{Y}$
      such that $|X \cap Y|$ is maximum;
      if a tie occurs, select $Y \in \mathscr{Y}$
      with the smallest cardinality;
      if another tie occurs, select the first $Y \in \mathscr{Y}$;
      $T := T \cup Y$;
      $G := G - T$;
      $\mathscr{Y} := \mathscr{Y} - Y$
    **end**
**end**

For Table 19.1, all distinct global probabilistic approximations are
$appr_1^{mlem2}(\{1, 2, 3, 4\}) = \{1, 2, 4\}$,

$appr_{0.333}^{mlem2}(\{1, 2, 3, 4\}) = \{1, 2, 3, 4, 5, 7\}$,

$appr_1^{mlem2}(\{5, 6, 7, 8\}) = \{5, 6, 8\}$, and

$appr_{0.667}^{mlem2}(\{5, 6, 7, 8\}) = \{3, 5, 6, 7, 8\}$.

### *19.3.2 Saturated Probabilistic Approximations*

Saturated probabilistic approximations are unions of generalized maximal consistent blocks while giving higher priority to generalized maximal consistent blocks with larger conditional probability $Pr(X|Y)$. Additionally, if the approximation covers all cases from the concept $X$, we stop adding generalized maximal consistent blocks.

Let $X$ be a concept and let $x \in U$. Let us compute all conditional probabilities $Pr(X|Z)$, where $Z \in \{Y \mid Y \subseteq \mathscr{C}(A), Pr(X|Y) \geq \alpha\}$. Then we sort the set

$$\{Pr(X|Y) \mid Y \subseteq \mathscr{C}(A)\}$$

in descending order. Let us denote the sorted list of such conditional probabilities by $\alpha_1, \alpha_2, \ldots, \alpha_n$. For any $i = 1, 2, \ldots, n$, the set $E_i(X)$ is defined as follows:

$$\{Y \mid Y \subseteq \mathscr{C}(A), Pr(X|Y) = \alpha_i\}.$$

If we want to compute a saturated probabilistic approximation, denoted by $appr_{\alpha}^{saturated}(X)$, for some $\alpha$, $0 < \alpha \leq 1$, we need to identify the index $m$ such that

$$\alpha_m \geq \alpha > \alpha_{m+1},$$

where $m \in \{1, 2, \ldots, n\}$ and $\alpha_{n+1} = 0$. The saturated probabilistic approximation $appr_{\alpha_m}^{saturated}(X)$ is computed using the following algorithm:

**Saturated probabilistic approximation algorithm**
**input**: a set $X$ (a concept), a set $E_i(X)$ for $i = 1, 2, \ldots, n$, index $m$
**output**: a set $T$ (a saturated probabilistic approximation $appr_{\alpha_m}^{saturated}(X)$)
of $X$
**begin**
$T := \emptyset$;
$\mathscr{Y}_i(X) := E_i(X)$ for all $i = 1, 2, \ldots, m$;
**for** $j = 1, 2, \ldots, m$ **do**
**while** $\mathscr{Y}_j(X) \neq \emptyset$
**begin**
select a generalized maximal consistent
block $Y \in \mathscr{Y}_j(X)$
such that $|X \cap Y|$ is maximum;
if a tie occurs, select the first $Y$;
$\mathscr{Y}_j(X) := \mathscr{Y}_j(X) - Y$;
**if** $(Y - T) \cap X \neq \emptyset$
**then** $T := T \cup Y$;
**if** $X \subseteq T$ **then exit**
**end**
**end**

For Table 19.1, all distinct saturated probabilistic approximations are the same as global probabilistic approximations.

### 19.3.3 Rule Induction

For given global and saturated probabilistic approximations associated with a parameter $\alpha$, rule sets are induced using the rule induction algorithm based on another parameter, also interpreted as a probability, and denoted by $\beta$. This algorithm also uses MLEM2 principles [20].

**MLEM2 rule induction algorithm**
**input**: a set $Y$ (an approximation of $X$) and a parameter $\beta$,
**output**: a set $\mathcal{T}$ (a rule set),

**begin**
  $G := Y$; $D := Y$; $\mathcal{T} := \emptyset$; $\mathcal{J} := \emptyset$;
  **while** $G \neq \emptyset$ **begin**
    $T := \emptyset$; $T_s := \emptyset$; $T_n := \emptyset$;
    $T(G) := \{t \mid [t] \cap G \neq \emptyset\}$;
    **while** ($T = \emptyset$ **or** $[T] \not\subseteq D$) **and** $T(G) \neq \emptyset$ **begin**
      select a pair $t = (a_t, v_t) \in T(G)$ with
      maximum of $|[t] \cap G|$; if a tie occurs,
      select a pair $t \in T(G)$ with the smallest
      cardinality of $[t]$; if another tie occurs,
      select the first pair;
      $T := T \cup \{t\}$;
      $G := [t] \cap G$;
      $T(G) := \{t \mid [t] \cap G \neq \emptyset\}$;
      **if** $a_t$ is symbolic {let $V_{a_t}$ be the domain of $a_t$}
       **then** $T_s := T_s \cup \{(a_t, v) | v \in V_{a_t}\}$
       **else** {$a_t$ is numerical, let $t = (a_t, u..v)$}
       and $T_n := T_n \cup \{(a_t, x..y) \mid$ disjoint $x..y$
       and $u..v\} \cup \{(a_t, x..y \mid x..y \supseteq u..v\}$;
      $T(G) := T(G) - (T_s \cup T_n)$;
    **end** {while};
      **if** $Pr(X \mid [T]) \geq \beta$ **then**
        **begin**
          $D := D \cup [T]$;
          $\mathcal{T} := \mathcal{T} \cup \{T\}$;
        **end** {then}
      **else** $\mathcal{J} := \mathcal{J} \cup \{T\}$;
      $G := D - \cup_{S \in \mathcal{T} \cup \mathcal{J}}[S]$;
  **end** {while};
  **for** each $T \in \mathcal{T}$ **do**
    **for** each numerical attribute $a_t$ with $(a_t, u..v) \in T$ **do**
     **while** ($T$ contains at least two different
     pairs $(a_t, u..v)$ and $(a_t, x..y)$ with
     the same numerical attribute $a_t$)
      replace these two pairs with a new pair
      ($a_t$, common part of $(u..v)$ and $(x..y)$);
    **for** each $t \in T$ **do**
     **if** $[T - \{t\}] \subseteq D$ **then** $T := T - \{t\}$;
  **for** each $T \in \mathcal{T}$ **do**
   **if** $\cup_{S \in (\mathcal{T} - \{T\})}[S] = \cup_{S \in \mathcal{T}}[S]$ **then** $\mathcal{T} := \mathcal{T} - \{T\}$;
**end** {procedure}.

For example, for Table 19.1 and $\alpha = \beta = 0.5$, using the global probabilistic approximations, the MLEM2 rule induction algorithm induces the following rules:

(Temperature, very-high) & (Headache, yes) → (Flu, yes)
(Temperature, high) & (Cough, yes) → (Flu, yes)
(Headache, no) & (Cough, no) → (Flu, no)
and
(Temperature, normal) → (Flu, no)

## 19.4 Experiments

In our experiments, we used eight data sets taken from *Machine Learning Repository* at the University of California at Irvine. For every data set, a new record was created by randomly replacing 35% of the existing specified attribute values by *lost values*. Data sets with "do not care" conditions were created by replacing "?"s with "∗"s.

In our experiments, the parameter $\alpha$ varied between 0.001 and 1, while the parameter $\beta$ was equal to 0.5. For a data set, the rule set was induced, and the number of rules as well as the total number of conditions was recorded. Results of our experiments are presented in Figs. 19.1, 19.2, 19.3, 19.4, 19.5, 19.6, 19.7, 19.8, 19.9, 19.10, 19.11, 19.12, 19.13, 19.14, 19.15, and 19.16, where "Global" denotes a MLEM2 global probabilistic approximation, "Saturated" denotes a saturated probabilistic approximation, "?" denotes lost values, and "∗" denotes "do not care" conditions. In our experiments, four methods for mining incomplete data sets were used, since we combined two interpretations of missing attribute values: lost and "do not care" conditions with two versions of probabilistic approximations: global and saturated.

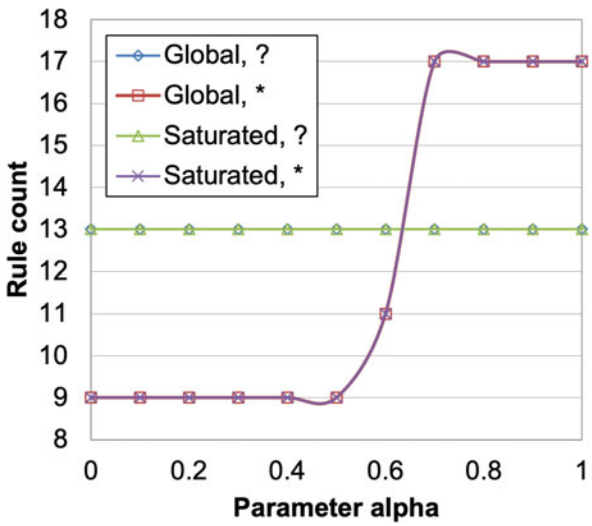


**Fig. 19.1** The number of rules for the *bankruptcy* data set

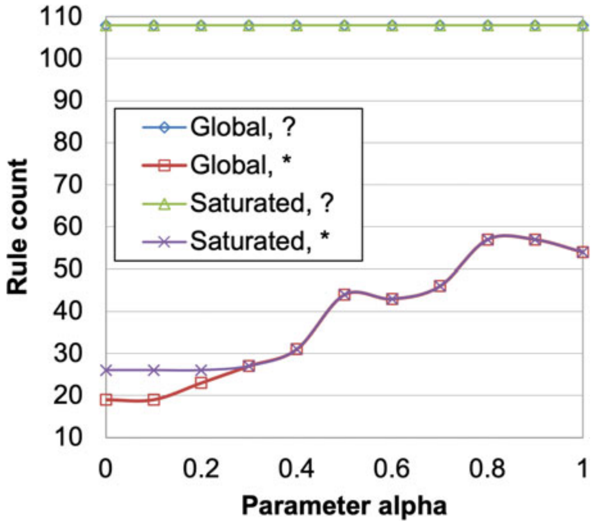


**Fig. 19.2** The number of rules for the *breast cancer* data set

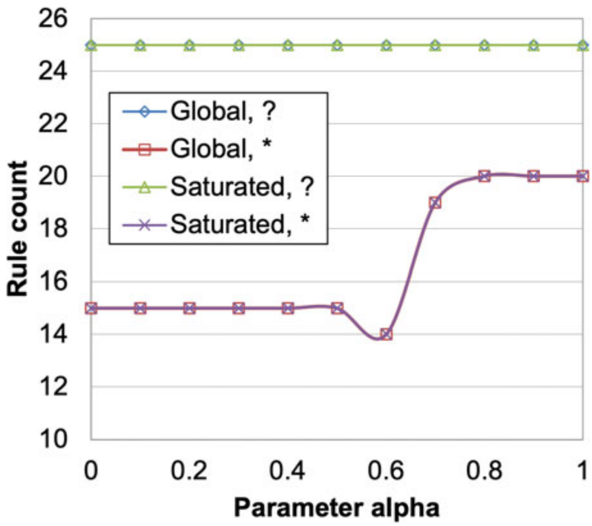


**Fig. 19.3** Th number of rules for the *echocardiogram* data set

These four methods were compared by applying the distribution-free Friedman rank sum test and then by the post hoc test (distribution-free multiple comparisons based on the Friedman rank sums), with a 5% level of significance.

For all data sets, except for the *iris* data set with lost values, there is no significant difference between rule sets induced using global and saturated probabilistic approximations. For the *iris* data set with lost values, the size of rule sets induced using global probabilistic approximations is smaller than the size of rule sets induced from probabilistic approximations.

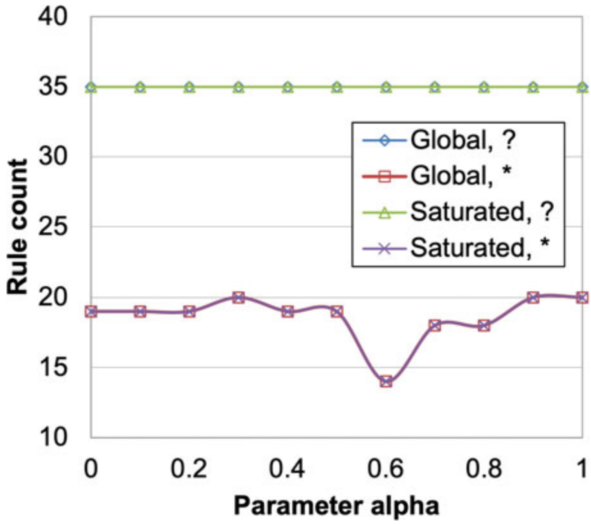


**Fig. 19.4** The number of rules for the *hepatitis* data set

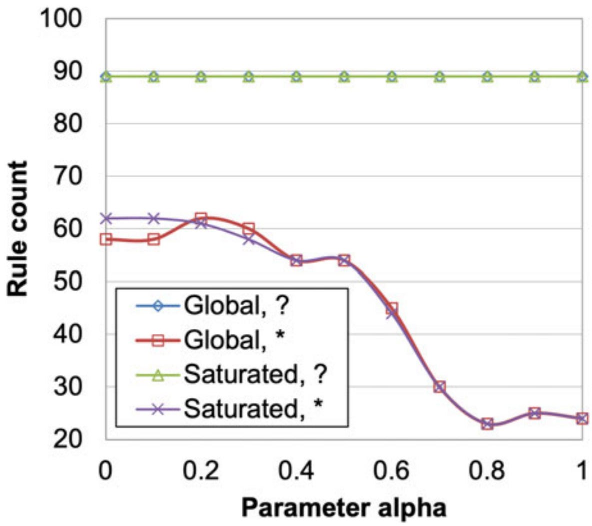


**Fig. 19.5** The number of rules for the *image segmentation* data set

For two data sets, *bankruptcy* and *iris*, there is no significant difference for the size of rule sets induced from data sets with lost values and "do not care" conditions, but there are significant differences when it comes to the total number of conditions: for the *bankruptcy* data set with "do not care" conditions, the total number of cases is significantly smaller than that for lost values, while for the *iris* data set, for the total number of conditions, it is the other way around. For four data sets, *breast cancer*, *hepatitis*, *image segmentation*, and *wine recognition*, the size of the induced rule sets and the total number of conditions are smaller for data sets with "do not care" conditions than that for data sets with lost values. For two data sets,

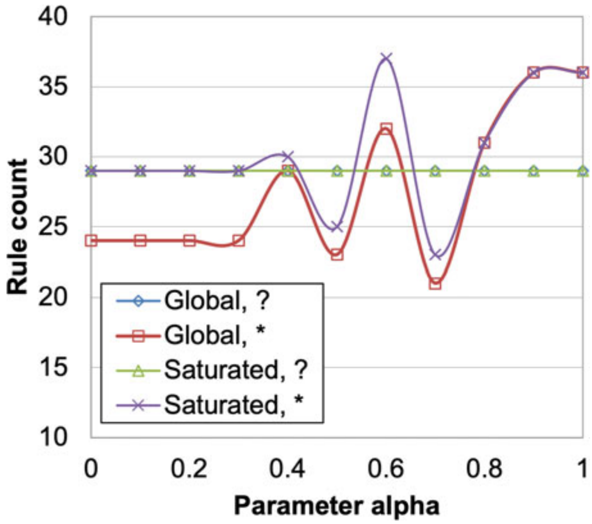


**Fig. 19.6** The number of rules for the *iris* data set

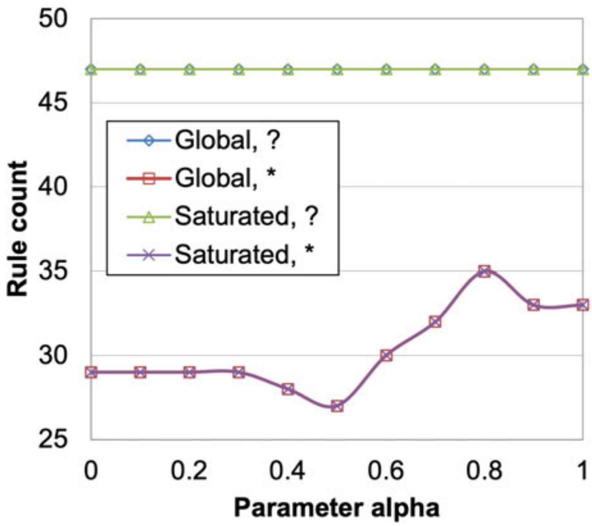


**Fig. 19.7** The number of rules for the *lymphography* data set

*echocardiogram* and *lymphography*, the size of induced rule sets is smaller for lost values. It is obvious that such rule sets have a small number of rules, but rules are more complicated, with the large total number of conditions.

We may conclude that, in general, the difference between using both kinds of probabilistic approximations, global and saturated, is insignificant. On the other hand, for reducing the number of rules or the total number of conditions induced from data sets with "do not care" conditions and lost values, we should try both possibilities.

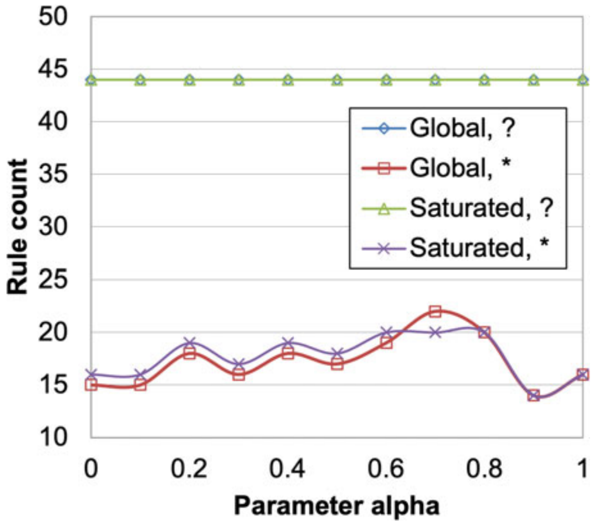


**Fig. 19.8** The number of rules for the *wine recognition* data set

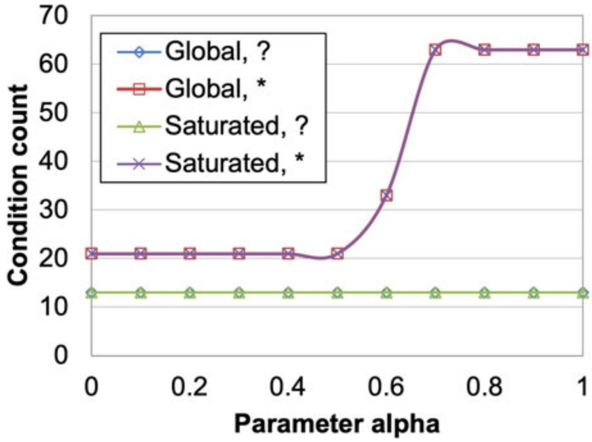


**Fig. 19.9** The total number of conditions for the *bankruptcy* data set

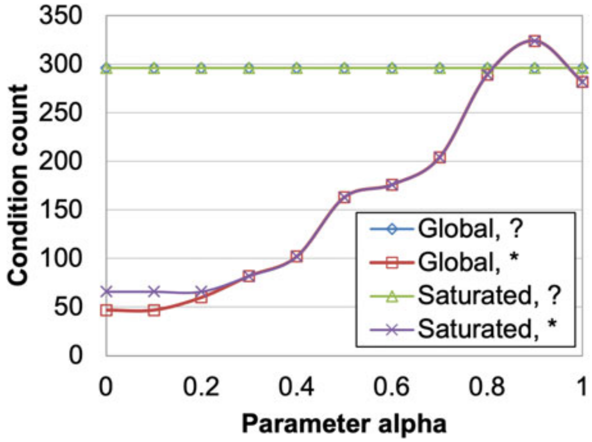


**Fig. 19.10** The total number of conditions for the *breast cancer* data set

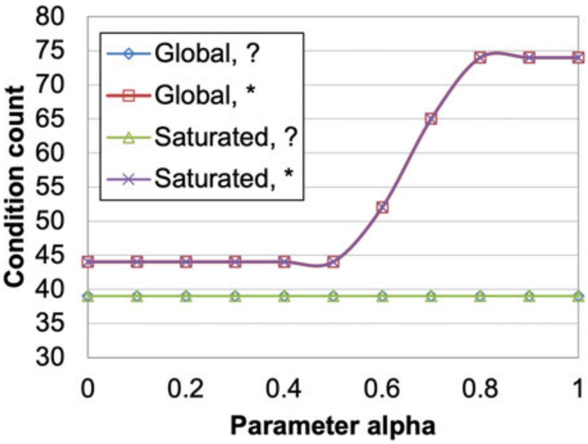


**Fig. 19.11** The total number of conditions for the *echocardiogram* data set

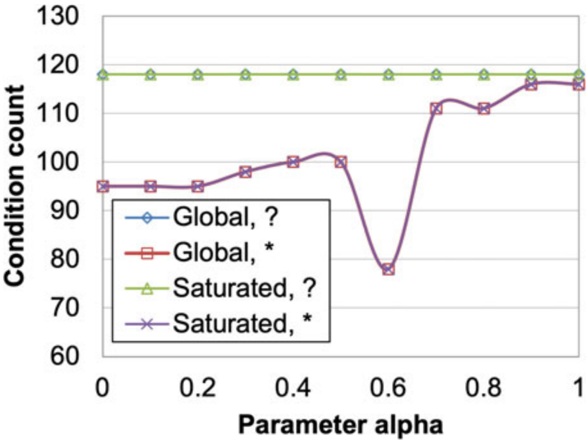


**Fig. 19.12** The total number of conditions for the *hepatitis* data set

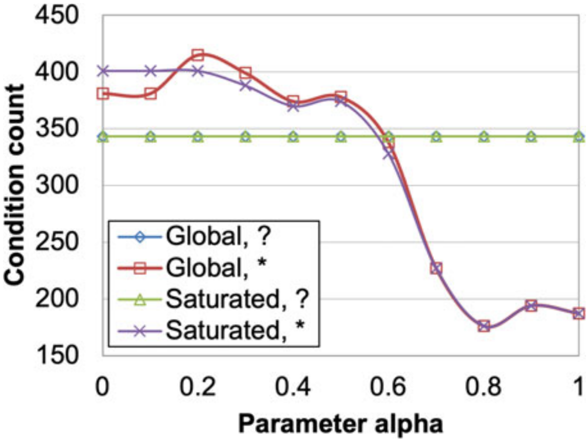


**Fig. 19.13** The total number of conditions for the *image segmentation* data set

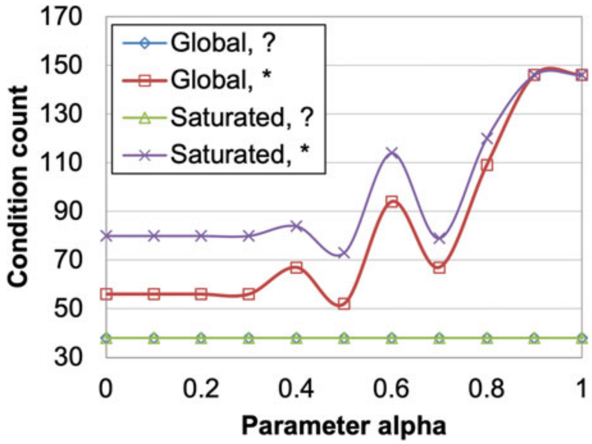


**Fig. 19.14** The total number of conditions for the *iris* data set

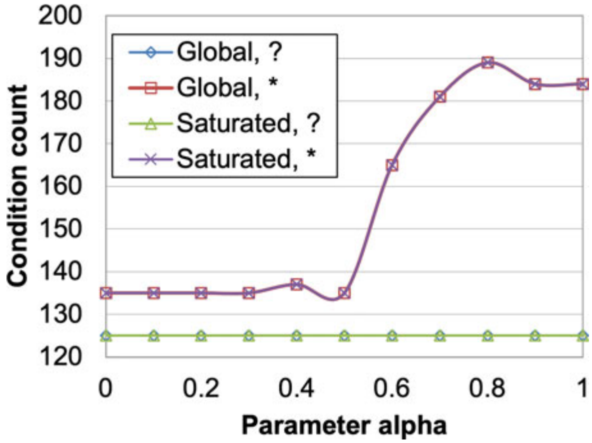


**Fig. 19.15** The total number of conditions for the *lymphography* data set

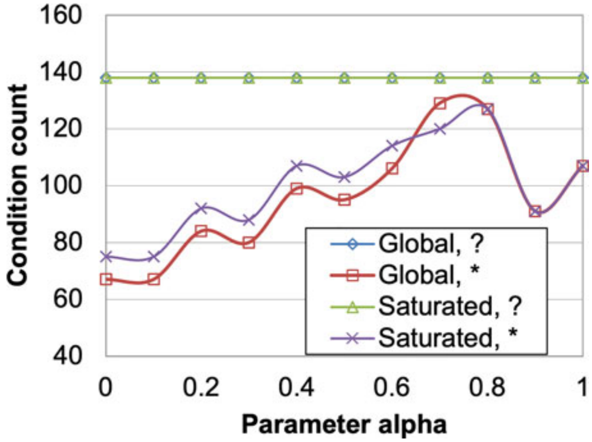


**Fig. 19.16** The total number of conditions for the *wine recognition* data set

## 19.5 Conclusions

We compared four methods for mining incomplete data sets, combining two interpretations of missing attribute values with two types of probabilistic approximations. Our criterion of quality was complexity of induced rule sets. As follows from our experiments, there were significant differences between the four methods. However, in general, the difference between used probabilistic approximations is not significant. The only significant difference is between the two interpretations of missing attribute values. The main conclusion is that both interpretations should be tested.

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# Chapter 20
# Rule Confirmation Measures: Properties, Visual Analysis and Applications

**Izabela Szczech, Robert Susmaga, Dariusz Brzezinski, and Jerzy Stefanowski**

**Abstract** According to Bayesian confirmation theory, for a $E \rightarrow H$ rule, evidence $E$ confirms hypothesis $H$ when $E$ and $H$ are positively probabilistically correlated. Surprisingly, this leads to a plethora of non-equivalent quantitative measures that attempt to measure the degree in which $E$ confirms $H$. This observation has triggered research on the differentiating characteristics of confirmation measures—their analytical properties, tools for visual inspection of those properties, and the applications of confirmation measures in rule-based systems. This chapter constitutes an extensive overview that covers the analysis and development of rule confirmation measures and their properties. It moves from research on desirable properties of confirmation measures, through visualization methods that support this process, to current applications of rule confirmation measures and lines of future research in the field.

## 20.1 Introduction

Rules are one of the oldest and intensively investigated knowledge representations in artificial intelligence, intelligent data analysis, and logic-based methodology of science and linguistics. They are typically represented as *if* <premise> *then* <conclusion> statements [52], but their form and ways of creating them depend on the research context and applications. Here, we will mainly refer to rules used in artificial intelligence, machine learning, and data mining. In artificial intelligence, rule-based systems were studied since the 1960s [8], in particular as production rules for expert systems, but also in other logic-based approaches, multi-agent systems, fuzzy set-based systems, or natural language processing. These rules have richer syntax than those considered in machine learning; in particular, rule consequences may lead to some actions on data [34]. Moreover, they are usually acquired through

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interviews with domain experts and are exploited in complex systems with the use of inference engines, which leads to a chain of reasoning with many rules.

On the other hand, in machine learning or data mining rules are considered in either the *descriptive perspective*, which aims at discovery of hidden patterns in data, or in the *predictive perspective* [18, 51]. In the latter case, one automatically learns a set of rules from the training data, where rules sufficiently cover the set of labelled examples and they are used to make a prediction for possible instances. Predictive rule learning leads mainly to *classification rules* with either single class labels or more complex targets, such as multiple labels, structured output, ordered classes, or continuous target variables [17]. Descriptive rule learning, on the other hand, includes unsupervised approaches, such as association rules or contrast patterns, and supervised approaches such as subgroup discovery; see their presentation in [17]. Furthermore, the most popular propositional (attribute-value representation) rule learning has also been extended to relational learning [32].

This rich research area resulted in a great number of algorithms and many applications in various domains; for a comprehensive survey, see [17]. Although machine learning is currently dominated by deep neural networks, statistical approaches, and classifier ensembles, rules are still appreciated because of the compactness of the representation of the discovered knowledge, their interpretability, usefulness to explain the decisions of the systems, insight into its internal operations, and possible using in other systems. Many researchers, even ones known for developing other paradigms, e.g., Ross Quinlan, claim that the naturalness of rules' symbolic form corresponds to a high level of comprehensibility of humans. Moreover, recently rules are receiving renewed interest due to their interpretability [33], which plays an important factor in explainable AI and the newly proposed regulatory right to explanation [47].

Recall that Roman Slowinski has actively carried out research on various types of rules and their usage in many fields. Since the 1980s he has been studying rules in rough set theory, mainly in the descriptive perspective, where rules are used to simplify and generalize large and redundant data tables with inconsistent descriptions, see, e.g., his pioneering works in their application to medical diagnosis, pharmacy of drugs, technical diagnostics, or financial analysis ([49] includes many chapters on these topics). Then in the 1990s, he proposed many important generalizations of rough sets, e.g., to deal with incomplete or imprecise data. Nevertheless, the most important and influential contribution of Roman Slowinski, along with Salvatore Greco and Benedetto Matarazzo, to Multicriteria Decision Aiding is the *methodology of decision rule preference modeling* [25, 26]. In this methodology, the decision-maker expresses preferential information in terms of examples of decisions and looks for simple rules justifying these decisions, which is a different paradigm than utility functions or outranking relations [26]. An important aspect of this approach is the possibility of handling inconsistencies in the preferential information resulting from hesitations of the decision-maker. It is mathematically based on the newly introduced dominance-based rough set approach. The decision rules constituting the preference model are induced from this preferential information by using rule induction algorithms adapted to handle the dominance principle and ordered properties of criteria and decision classes [27]. Roman Slowinski in his research has also paid attention to the usefulness of various measures evaluating rough set decision rules [21].

It is important to note that the usefulness of induced rules for their interpretation or usage to support explanations of the intelligent systems is associated with such characteristics as their compactness. The number of rules, non-redundancy, and quality should be acceptable for humans in order to be interpreted by them. In the case of classification rules, it is strongly related to pruning rules and guiding the search for good quality candidates for rules. However, it is even more visible in the descriptive perspective. In particular association rules become hard to interpret and difficult to extract if the rule mining process is not guided by the interestingness of the extracted associations. Thus, there is a need to evaluate the usefulness of rules to allow users to focus only on the most relevant patterns found in the data. Such evaluation is typically carried out using *attractiveness measures* (also called interestingness measures) [36].

The above-mentioned need for rule evaluation in different domains has resulted in a plethora of attractiveness measures, each with its own set of characteristics. The plurality of these measures makes the choice of the measure for the task at hand hard and nontrivial. In this context, the analysis of *properties of attractiveness measures* is a valid and important research topic [21, 56]. Using measures satisfying desirable properties helps discovering useful rules since the measures reflect rational expectations towards their behavior. Among various properties, an important role is played by the property of *Bayesian confirmation* [21, 35, 46]. Measures characterized by this property quantify the degree to which the evidence in the rule's premise provides support *for* or *against* the hypothesized piece of evidence in the rule's conclusion [15]. Therefore, by definition confirmation measures make it possible to distinguish meaningful rules for which the premise confirms the conclusion.

The formal analysis of confirmation measures with respect to their properties is a challenging and laborious task that often requires advanced mathematical methods. That is why several visualization methods [10, 53, 54] have been developed to help understand the characteristic properties of various confirmation measures. Such knowledge can be applied in practice while selecting measures to guide rule list pruning or improve rule interpretability.

In this chapter, we provide an overview of research concerning rule confirmation measures and their properties. After providing the definition of Bayesian confirmation, other confirmation perspectives and rule confirmation measures in Sect. 20.2, works on valuable groups of properties of confirmation measures (monotonicity, symmetry, properties inspired by extreme values of confirmation) are discussed in Sect. 20.3. Section 20.4 presents visualization methods proposed to inspect confirmation measures. In Sect. 20.5 we discuss how confirmation measures and visualization tools have been applied in different fields. Finally, in Sect. 20.5.5 we discuss lines of future research on confirmations measures.

## 20.2 Bayesian Confirmation and Rule Confirmation Measures

Rules are usually induced from a dataset being a set of objects, *learning examples*, characterized by a set of attributes. Rules are consequence relations between a condition $E$ and a conclusion $H$, typically denoted as $E \rightarrow H$ ("*if* $E$ *then* $H$"). The condition formula $E$ (*premise*, *evidence*) is a conjunction of elementary conditions created on the basis of values of the attributes describing the examples, and the conclusion formula $H$ (*decision*, *hypothesis*) indicates the target class of the example satisfying the condition part.

Recall that rule induction can be considered in two contexts: predictive and descriptive [18, 51]. The common problem of both of them, however, is that the number of generated rules can often be overwhelmingly high. This can hinder the expert from analyzing or deploying the rules. Thus, there arises the need to evaluate the relevance and usefulness of the induced rules, which is commonly done using quantitative measures, known as attractiveness (or interestingness) measures [19, 36].

Generally, attractiveness measures can be divided into subjective and objective ones [52]. The subjective measures incorporate domain knowledge and the beliefs of an expert, whereas the objective ones are independent of the application domain and the user and are calculated on the basis of the dataset being analyzed.

### *20.2.1 Bayesian Confirmation*

Within objective attractiveness measures, a particular group called (*Bayesian*) *confirmation measures* can be distinguished. Their common feature is that they satisfy *the property of Bayesian confirmation* [15, 21, 35, 46] (or simply: confirmation), which can be regarded as an expectation that a measure obtains positive values when the rule's premise increases the knowledge about the conclusion, zero when the premise does not influence the conclusion at all, and finally, negative values when the premise has a negative impact on the conclusion.

Formally, for a given rule $E \rightarrow H$, an attractiveness measure $c(H, E)$ satisfies the property of Bayesian confirmation, when:

- $c(H, E) > 0$ if and only if $P(H|E) > P(H)$
- $c(H, E) = 0$ if and only if $P(H|E) = P(H)$
- $c(H, E) < 0$ if and only if $P(H|E) < P(H)$

where $P(H|E)$ is the conditional probability of the conclusion given the premise, and $P(H)$ is the probability of the conclusion.

Interestingly, there are also other, alternative formulations of the property of confirmation [15, 35]. In fact, one can define the property of confirmation in four different ways (referred to as *perspectives*) [22, 24], each based on different probabilities. Apart from the Bayesian confirmation perspective using $P(H|E)$ and $P(H)$, there are: strong Bayesian ($P(H|E)$, $P(H|\neg E)$), likelihoodist ($P(E|H)$, $P(E)$), and strong likelihoodist ($P(E|H)$, $P(E|\neg H)$) perspectives. Because of the different probabilities used, each of the perspectives naturally emphasizes different aspects of the concept of confirmation and also obtains undefined values in different situations (e.g., when $P(E) = 0$, then $P(H|E)$ becomes undefined in the Bayesian and strong Bayesian perspectives). However, despite those differences, all four perspectives are logically equivalent, which means that they "switch" between positive values, zero, and negative values in the same situations, provided these are not undefined [20, 22]. It can also be proved that the conditions forming each of the four perspectives can boil down to one formulation called *the general definition of confirmation* [24].

### *20.2.2 Rule Confirmation Measures*

Thorough discussion about introducing confirmation measures as a tool valuable for assessing the quality of rules induced from data within the rough set approach and, more generally, within data mining, machine learning, and knowledge discovery was presented in [21]. Greco, Pawlak, and Slowinski pointed out there that using the confirmation property and quantitative confirmation theory for data analysis allows to benefit from the results of such prominent researchers as Carnap [9], Hempel [29], and Popper [46].

The usefulness of confirmation measures in general and their supremacy in terms of interpretability over measures not satisfying the property of confirmation can be illustrated with an example proposed by Popper [21, 46] of rolling a dice, where the possible results are in the set $\{1, 2, 3, 4, 5, 6\}$. For an exemplary premise $E =$*"the result is divisible by 2"*, consider two alternative conclusions: $H_1 =$*"the result is six"* and $H_2 =$*"the result is not six"*. Let us first evaluate the rules $E \rightarrow H_1$ and $E \rightarrow H_2$ with confidence, which is a very popular attractiveness measure that does not possess the property of confirmation. Confidence is defined as the ratio of the number of objects satisfying the rule's premise and conclusion to the number of objects satisfying the premise: $confidence(E \rightarrow H) = P(E \wedge H)/P(E) = P(H|E)$. For the first rule $confidence(E \rightarrow H_1) = 1/3$, and for the second rule $confidence(E \rightarrow H_2) = 2/3$. The second rule is thus unambiguously and highly preferred over the first one, provided only the confidence measure is used. Furthermore, the high value of confidence for $E \rightarrow H_2$ would most likely encourage the user to take actions described by the rule.

The evaluation of the two rules with any confirmation measure leads, however, to completely different results. In particular, rule $E \rightarrow H_1$ is characterized by a positive value of confirmation measures because $P(H_1|E) = 1/3 > P(H_1) = 1/6$. On the other hand, rule $E \rightarrow H_2$ is characterized by a negative value of confirmation measures as $P(H_2|E) = 2/3 < P(H_2) = 5/6$. As a result, rule $E \rightarrow H_2$ is in fact misleading since its premise decreases the probability of obtaining the conclusion. Thus, following the high value of confidence of the second rule would not be beneficial for the user. This illustrates the applicability of confirmation measures, which owing to their scales, clearly indicate the misleading (disconfirming, $c(H, E) < 0$) rules, as well as the limitations of the popular confidence, quite commonly used in various applications as the default measure.

The list of confirmation measures proposed in the literature is long [15, 35] as the conditions in the property of confirmation itself define only when the measure should obtain positive values, zero, or negative values, giving a wide space for expressing alternative measures of confirmation, differing however with respect to various other properties.

Originally, the definitions of confirmation measures were often expressed using probabilities, see, e.g., measure: $D(H, E) = P(H|E) - P(H)$ [13]. However, in the context of a particular dataset, specific information on whether a given piece of evidence $E$ or hypothesis $H$ holds or not is often estimated with four discrete, non-negative values:

- $a$: the number of objects in the dataset for which both $E$ and $H$ hold
- $b$: the number of objects in the dataset for which the premise $E$ does not hold and the conclusion $H$ holds
- $c$: the number of objects in the dataset for which the premise $E$ holds, but the conclusion $H$ does not
- $d$: the number of objects in the dataset for which neither $E$ nor $H$ holds

These values may collectively be stored in a $2 \times 2$ table, referred to as the contingency table (Table 20.1).

The list of exemplary confirmation measures expressed also using such a frequentionistic approach is presented in Table 20.2. Measures $c_1(H, E)$ and $c_2(H, E)$ are defined using parameters $\alpha$ and $\beta$, where $\alpha + \beta = 1$ and $\alpha > 0$, $\beta > 0$. Observe that parameters $\alpha$ and $\beta$ can be used to closen the new measure to $Z(H, E)$ or $A(H, E)$, i.e., to Bayesian or likelihoodist inspirations.

The choice of an attractiveness measure for the task at hand, though limited by the desirable property of confirmation itself, requires non-trivial and careful consideration of many alternative measures since the property of confirmation is not the only factor that guarantees the right choice. Different confirmation measures were defined to reflect different characteristic behavior of the measures, formalized as measure properties (surveyed in [9, 14, 19, 21, 56]). Identification of measure properties makes it possible to group measures according to their similar behavior, providing much-needed insight into the process of measure selection.

**Table 20.1** An exemplary contingency table of the rule's premise and conclusion

| | $H$ | $\neg H$ | $\Sigma$ |
|---|---|---|---|
| $E$ | $a$ | $c$ | $a + c$ |
| $\neg E$ | $b$ | $d$ | $b + d$ |
| $\Sigma$ | $a + b$ | $c + d$ | $n$ |

**Table 20.2** Popular confirmation measures

| | |
|---|---|
| $D(H,E) = P(H\|E) - P(H) = \frac{a}{a+c} - \frac{a+b}{n} = \frac{ad-bc}{n(a+c)}$ | [13] |
| $M(H,E) = P(E\|H) - P(E) = \frac{a}{a+b} - \frac{a+c}{n} = \frac{ad-bc}{n(a+b)}$ | [37] |
| $S(H,E) = P(H\|E) - P(H\|\neg E) = \frac{a}{a+c} - \frac{b}{b+d} = \frac{ad-bc}{(a+c)(b+d)}$ | [11] |
| $N(H,E) = P(E\|H) - P(E\|\neg H) = \frac{a}{a+b} - \frac{c}{c+d} = \frac{ad-bc}{(a+b)(c+d)}$ | [43] |
| $C(H,E) = P(E \wedge H) - P(E)P(H) = \frac{a}{n} - \frac{(a+c)(a+b)}{n^2} = \frac{ad-bc}{n^2}$ | [9] |
| $F(H,E) = \frac{P(E\|H) - P(E\|\neg H)}{P(E\|H) + P(E\|\neg H)} = \frac{\frac{a}{a+b} - \frac{c}{c+d}}{\frac{a}{a+b} + \frac{c}{c+d}} = \frac{ad-bc}{ad+bc+2ac}$ | [31] |
| $Z(H,E) = \begin{cases} 1 - \frac{P(\neg H\|E)}{P(\neg H)} = \frac{ad-bc}{(a+c)(c+d)} & \text{in case of confirmation} \\ \frac{P(H\|E)}{P(H)} - 1 = \frac{ad-bc}{(a+c)(a+b)} & \text{in case of disconfirmation} \end{cases}$ | [12] |
| $A(H,E) = \begin{cases} \frac{P(E\|H) - P(E)}{1 - P(E)} = \frac{ad-bc}{(a+b)(b+d)} & \text{in case of confirmation} \\ \frac{P(H) - P(H\|\neg E)}{1 - P(H)} = \frac{ad-bc}{(b+d)(c+d)} & \text{in case of disconfirmation} \end{cases}$ | [22] |
| $c_1(H,E) = \begin{cases} \alpha + \beta A(H,E) & \text{in case of confirmation when } c = 0 \\ \alpha Z(H,E) & \text{in case of confirmation when } c > 0 \\ \alpha Z(H,E) & \text{in case of disconfirmation when } a > 0 \\ -\alpha + \beta A(H,E) & \text{in case of disconfirmation when } a = 0 \end{cases}$ | [22] |
| $c_2(H,E) = \begin{cases} \alpha + \beta Z(H,E) & \text{in case of confirmation when } b = 0 \\ \alpha A(H,E) & \text{in case of confirmation when } b > 0 \\ \alpha A(H,E) & \text{in case of disconfirmation when } d > 0 \\ -\alpha + \beta Z(H,E) & \text{in case of disconfirmation when } d = 0 \end{cases}$ | [22] |
| $c_3(H,E) = \begin{cases} A(H,E)Z(H,E) & \text{in case of confirmation} \\ -A(H,E)Z(H,E) & \text{in case of disconfirmation} \end{cases}$ | [22] |
| $c_4(H,E) = \begin{cases} min(A(H,E), Z(H,E)) & \text{in case of confirmation} \\ max(A(H,E), Z(H,E)) & \text{in case of disconfirmation} \end{cases}$ | [22] |

## 20.3 Properties of Confirmation Measures

### *20.3.1 Property of Monotonicity M*

Among properties regarded as desirable within confirmation measures, an important place is given to the property of monotonicity $M$ [21], ensuring monotonic dependency of the measure on the number of objects satisfying (supporting) or not the premise and/or the conclusion of the rule. The property $M$ emerges from the argument that for a decision rule $E \rightarrow H$, an attractiveness measure should give the credibility of the proposition: $H$ is satisfied more frequently when $E$ is satisfied rather than when $E$ is not satisfied. As proved by Greco, Pawlak, and Slowinski in [21], this requirement is satisfied, when the confirmation measure $c(H, E)$ is a function:

- non-decreasing with respect to $a$ and $d$
- non-increasing with respect to $b$ and $c$

Confirmation measures satisfying these monotonic requirements with respect to $a$, $b$, $c$, and $d$ are referred to as measures possessing the property $M$.

The property $M$ with respect to $a$ means that any evidence in which the premise and the conclusion hold together increases (or at least does not decrease) the confirmation of the rule $E \rightarrow H$. Analogous interpretation holds for $d$. On the other hand, the property $M$ with respect to $b$ means that any evidence in which $E$ does not hold and $H$ holds decreases (or at least does not increase) the confirmation of the rule $E \rightarrow H$. Analogously with respect to $c$. Property of monotonicity $M$ can be, thus, interpreted as a formal indication of conditions permitting passing from a situation of non-confirmation (or disconfirmation) to a situation of confirmation (i.e., when $H$ is satisfied more frequently when $E$ is satisfied rather than when $E$ is not satisfied). Such passage is permitted by an increase of $a$ or $d$, or by a decrease of $b$ or $c$.

Among measures satisfying the property $M$, one can find, e.g., confirmation measures $S(H, E)$, $N(H, E)$, $F(H, E)$, $Z(H, E)$ [21, 22]. However, there are also confirmation measures that do not possess property $M$ (e.g., measure $D(H, E)$), which only indicates that the quest for a valuable measure for a task at hand usually cannot be successfully completed with a single property.

## *20.3.2 Symmetry Properties*

Quite a lot of attention has been given in the literature to a large set of properties, commonly referred to as *symmetry properties*, characterizing how the value of a confirmation measure relates to its value obtained after the rule's premise and conclusion are interchanged and/or negated [12, 14, 15, 22]. The studies considering symmetry properties differ in the set of considered symmetries and in the way the symmetries are assessed as either desirable or undesirable.

Eells et al. [14] have analyzed a set of popular confirmation measures from the viewpoint of four properties of symmetry, earlier also discussed by Carnap [9]:

- evidence symmetry (ES): $c(H, E) = -c(H, \neg E)$
- inversion (or commutativity) symmetry (IS): $c(H, E) = c(E, H)$
- hypothesis symmetry (HS): $c(H, E) = -(\neg H, E)$
- evidence-hypothesis (or total) symmetry (EHS): $c(H, E) = c(\neg H, \neg E)$

The evidence symmetry, for example, considers how the value of a confirmation measure $c(H, E)$ relates to its value obtained for the situation when the rule's premise is negated, whereas the inversion symmetry considers switching positions of the rule's premise and conclusion. Eells et al. [14] used a series of examples of drawing cards from a deck to assess the symmetries as desirable or not. They argued that only hypothesis symmetry ($HS$) is a desirable property, leaving $ES$, $IS$, and $EHS$ as properties that should not be satisfied by valuable measures.

More recently, Crupi et al. [12] have argued for an extended and systematic treatment of the issue of symmetry properties. They propose to analyze a confirmation measure $c(H, E)$ with respect to all combinations obtained by applying the negation operator to the premise, hypothesis, or both, and/or by inverting $E$ and $H$. Additionally, they advocate that such a set of seven symmetries should be analyzed separately for the case of confirmation (i.e., when $P(H|E) > P(H)$) and for the case of disconfirmation (i.e., when $P(H|E) < P(H)$), forming the final set of 14 analyzed symmetries. Crupi et al. [12] followed the steps of Eells et al. for the preferential assessment of symmetries. Their results concur with respect to $ES$, $IS$, $HS$, and $EHS$, however only in the case of confirmation.

Later, the symmetry properties have been revised by Greco, Slowinski, and Szczech [23] in the light of the statement that a confirmation measure should give an account of the credibility that $H$ is satisfied more frequently when $E$ is satisfied rather than when $E$ is not satisfied. The analysis conducted in [23] revealed that it is enough to consider the symmetry properties only in case of confirmation, which further implied that the set of desirable properties contains $ES$, $HS$, and their composition, i.e., $EHS$.

The above-shown differences in conclusions stemming from the works published by different authors on symmetry properties inspired Susmaga et al. [55] to look at symmetry properties and their preferential assessment from the viewpoint of group theory [30]. Their considerations revealed a kind of incompleteness in the previous works, understood as a situation in which a composition of two symmetries from the proposed set results in a symmetry that does not belong to that set. The desirable completeness, however, is guaranteed if the symmetries and their compositions form an algebraic group (symmetries as the elements of the group and the symmetry composition as the operation in the group). In that context, the necessary extensions of the sets of symmetries discussed in the literature can be easily made, forming groups and consequently eliminating the incompleteness phenomenon [55].

Interestingly, group-theoretic aspects can also be further applied to the discussion about assessing the symmetries as desirable and undesirable. Their application can actually be regarded as a way to make the preferential assessments of symmetries objective, i.e., independent of the argumentation proposed earlier by different authors. Using group-theoretic aspects for such assessment, naturally, raises a need for necessary assumptions that provide preferential assessments about the compositions of symmetries, starting with the assumption expressing whether the group's neutral element is to be considered as desirable or undesirable. In [55] it has been argued that the neutral element should be considered as desirable. Following that, other assumptions about the assessments of symmetry compositions have been made. The analysis of the acknowledged preferential assessments of symmetries has shown, however, that the assumptions are violated by some authors, causing inconsistencies in their proposed approaches. In fact, as argued in [55], there exists only one consistent division of the set of symmetries into the desirable and undesirable ones that has also proper interpretations in the context of rule evaluation. That division places in the set of desirable properties only: $ES$, $HS$, and $EHS$, which makes it concordant with the preferential assessment proposed by Greco, Slowinski, and Szczech [23].

Analysis of selected confirmation measures with respect to the desirable symmetry properties revealed that measures $S(H, E)$, $N(H, E)$, $c_3(H, E)$, and $c_4(H, E)$ are particularly valuable [24].

### *20.3.3 Properties Inspired by Extreme Values of Confirmation*

To handle the plurality of alternative confirmation measures, some authors [12, 20, 22] have reached to considering inductive logic as an extrapolation from classical deductive logic, giving rise to new properties:

- Logicality $L$, and its generalization called *weak* $L$, indicating the conditions under which the confirmation measures should obtain their maximal or minimal values [12, 15, 22]
- $Ex_1$, and its generalization called *weak* $Ex_1$, assuring that any conclusively confirmatory rule is assigned a higher value of interestingness measure than any rule that is not conclusively confirmatory, and any conclusively disconfirmatory rule is assigned a lower value than any rule that is not conclusively disconfirmatory [12, 22]
- $Maximality/minimality$ assuring that measures obtain maximal values if and only if $b = c = 0$, and minimal values if and only if $a = d = 0$ [20]

A summary analyzing which confirmation measures possess which of the above properties can be found in [24]. As its result, measures $S(H, E)$, $N(H, E)$, $c_3(H, E)$, and $c_4(H, E)$ were recommended for finding meaningful rules.

It is worth noting that the list of measure properties proposed in the literature is quite long, and the final choice of properties should always be made with respect to the task at hand. It is reasonable, though, that such choice should precede the choice of the measure for rule evaluation. Formal analysis of measures with respect to their properties is undoubtedly a challenging and laborious task that often requires potentially advanced mathematical methods. Interestingly though, it can be effectively supported by visualizing the measure.

## 20.4 Visual Analysis of Measure Properties

The process of analyzing the properties of confirmation measures can be aided by visualization methods. The improvement of that process is profitable not only in the context of time but also in that it supports methods of new measure development (both automatic and semi-automatic methods). Tackling the above problems, [53, 54] introduced and applied a barycentric visualization method for confirmation measures, which allows fast and easy analysis of measures in their entire domains. Such a comprehensive insight into all values that the analyzed measure can possibly obtain allows to infer about the measure's behavior in any situation (represented by particular domain values) and has thus the advantage over visualizations in partial contexts based on particular datasets and rule sets.

The barycentric visualization of confirmation measures displays the measures as colored tetrahedra. The method is based on the fact that confirmation measures share a four-dimensional domain, represented in the form of a $2 \times 2$ non-negative matrix with a positive sum, the values of which fully determine $E$ and $H$ (see Table 20.1), so they may represent numerically the evidence and the hypothesis (the formal parameters of any confirmation measure). This matrix simultaneously represents an element of a $3D$ simplex, which can be depicted as a $3D$ tetrahedron in a $4D$ barycentric coordinate system.

More precisely, given a constant $n > 0$ and $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, with $a + b + c + d = n$, matrix $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ represents a point of the tetrahedron. Position of this point is fully determined by values of $a$, $b$, $c$, and $d$ in a manner characteristic to barycentric coordinate systems. Consider all possible values of $a$. When $a = 1$ (the first extreme), the point is situated in vertex $A$ of the tetrahedron, so its position is fully determined. When $a = 0$ (the second extreme), the point is situated in face $BCD$ of the tetrahedron, with its actual position within this face being determined by the values of $b$, $c$, and $d$ (which then satisfy $b + c + d = 1 - a = 1 - 0 = 1$). In intermediate stages, i.e., for $a \in (0, 1)$, the point is situated in a triangular cross-section that is parallel to face $BCD$ and cuts the $BCD$–$A$ height of the tetrahedron at fraction $a$ from $BCD$. Again, the actual position of the point within this cross-section is determined by the values of $b$, $c$, and $d$ (which then satisfy $b + c + d = 1 - a < 1$). Similar for the remaining variables.

Interestingly, as soon as a face or an above-mentioned cross-section emerges, it constitutes a $2D$ equilateral triangle (and thus a $2D$ simplex), to be naturally interpreted in a $3D$ barycentric coordinate system. With $a$ constant (as above), which reduces the number of considered dimensions by one, positions of points in this system are fully determined by values of $b$, $c$, and $d$. When, e.g., $b = 1$ (the first extreme), then, similarly, the point is situated in vertex $B$ of the triangle. When $b = 0$ (the second extreme), the point is situated in edge $CD$ of the triangle, with its actual position being determined by the values of $c$ and $d$ (which then satisfy $c + d = 1 - a - b = 1 - 0 - 0 = 1$), and so on. Notice that this dimensionality reduction principle gives the barycentric visualization a characteristic recursive flavor.

After having constructed the tetrahedron, coloring each of its points according to a given color map makes it possible to visualize various functions $f(a, b, c, d) \equiv f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right])$, including the confirmation measures. An exemplary color map is shown in Fig. 20.1. Interestingly, if the visualized function $f(a, b, c, d)$ admits non-numeric values ($+\infty$, $NaN$ and $-\infty$), they may also be visualized, however, with a color map that is disjoint with the previous one. In the visualizations presented below only $NaN$ values occur and (if present) are rendered in magenta.

An exemplary 2-view (two sides of the tetrahedron) visualization of function $f(a, b, c, d) = c_3(H, E)$ is shown in Fig. 20.2. An exemplary parallelogram (the net of the tetrahedron) visualization of function $f(a, b, c, d) = D(H, E)$ is shown in Fig. 20.3. Notice that the visualizations of the functions in Figs 20.2 and 20.3, although probably most comprehensible, are in a way incomplete, as they show only the exterior of the shape, which correspond to extreme values of the functions: face $BCD$ to $a = 0$ ($b + c + d = 1$), edge $CD$ to $a = b = 0$ ($c + d = 1$), vertex $D$ to $a = b = c = 0$ ($d = 1$), and so on. These views must thus be sometimes used together with views showing the interior of the shape (i.e., for $a, b, c, d \in (0, 1)$). In many cases, however, especially when the visualized functions are continuous and monotonic, views of the exterior of the tetrahedron are sufficient to fully conceptualize these functions.

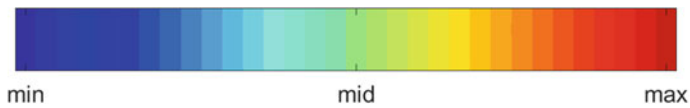


**Fig. 20.1** The color maps for the values of the visualized function

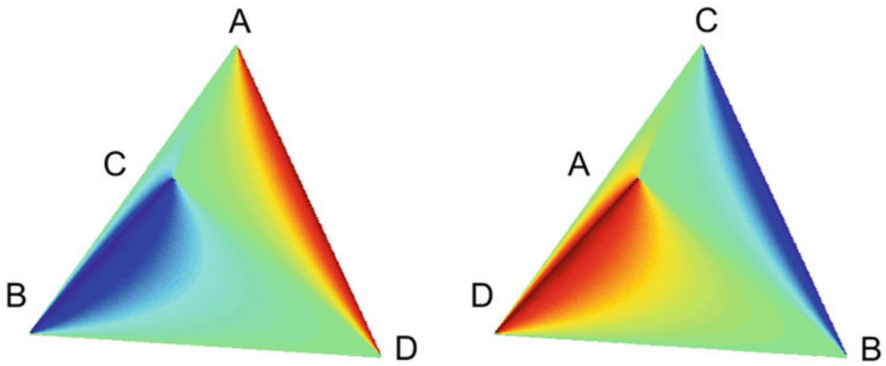


**Fig. 20.2** Tetrahedron visualization of $f(a, b, c, d) = c_3(H, E)$

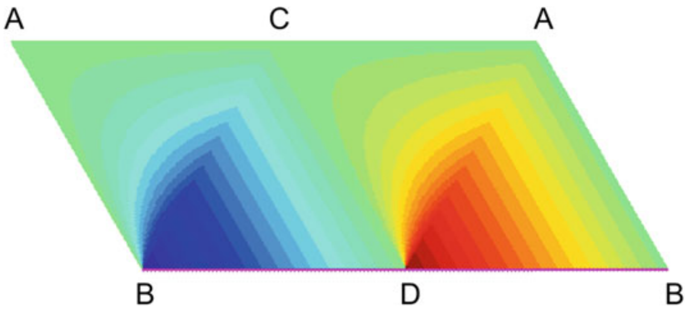


**Fig. 20.3** A 2D "parallelogram" visualization of $D(H, E)$

### 20.4.1 Characteristic Regions of Confirmation Measures

The barycentric visualization can be used to support basic analyses of confirmation measures, e.g., for fast detection of the location of the measure's extremes or zero values (in the measure's domain, i.e., in particular points of the tetrahedron), or for detection of areas in which the measure's values increase/decrease. Such analyses can facilitate, e.g., checking if two measures are oridinally equivalent, which is an important aspect not only of measures already acknowledged in the literature but also of new propositions. Notice how visualizations instantly reveal basic characteristics of the visualized functions, e.g., $c_3(H, E)$ (see Fig. 20.2):

- is fully defined (no undefined values in its domain)
- attains its maximum in the whole of edge $AD$
- attains its minimum in the whole of edge $BC$

while $D(H, E)$ (see Fig. 20.3):

- has undefined values in the whole edge $BD$
- attains its maximum at face $ABD$ towards vertex $D$
- attains its minimum at face $BCD$ towards vertex $B$

An important characteristic region common for all confirmation measures is their neutrality zone (the region characterized by zero values). Figure 20.4 depicts this region (rendered using a gray-scale color map). The saddle-like shape of this region divides the tetrahedron into two subregions of positive, i.e., confirmatory values (around edge $AD$), and negative, i.e., disconfirmatory values (around edge $BC$). Notice that as opposed to the zero region, the location of extreme and non-numeric regions is not common to all confirmation measures.

Moreover, the barycentric visualization method can also be applied to analyses of groups of measures, by visualizing the differences between measures or variances for groups of measures. This, in turn, swiftly identifies the regions of the tetrahedron where the measures of the group vary the least or the most. The practitioners could then decide on using only one representative of a low-variant group (as such

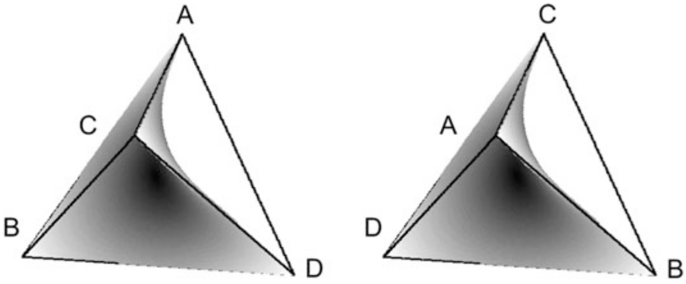


**Fig. 20.4** A 2-view 3D visualization of the neutral region (common for all confirmation measures)

measures tend to produce fairly consistent evaluations), and thus avoid potential redundancy and its undesirable impact on the efficiency of the rule evaluation process.

Interestingly, visualization of confirmation measures can also support the analyses of confirmation measures with respect to their properties. Visual measure analysis regarding such properties as monotonicity $M$, symmetry properties, logicality $L$, $Ex_1$, or maximality/minimality is discussed in detail in the following sections.

### *20.4.2 Monotonicity-Related Properties: M*

In terms of $a$, $b$, $c$, and $d$ property $M$ requires that the confirmation measure is a function $f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right])$ that is non-decreasing with respect to $a$ and $d$, and non-increasing with respect to $b$ and $c$. To confirm $M$, the four types of triangular cross-sections that are parallel to the four faces of the tetrahedron must be examined. The colors of appropriate points of cross-sections corresponding to increasing values of $a$ and $d$ must reveal non-decreasing values of the measure, while colors of appropriate points of cross-sections corresponding to increasing values of $b$ and $c$ must reveal non-increasing values of the measure. Clearly, the interior views of the tetrahedron are required to fully verify this property. However, by the presented dimensionality reduction principle of the barycentric visualization, monotonicity $M$ may also be observed in all considered cross-sections, including the faces, of the tetrahedron. For brevity, only the faces, which do not require any interior views, will be discussed here. Consider face $BCD$, which implies $a = 0$. The colors of appropriate points of $BCD$ corresponding to increasing values of $d$ must reveal non-decreasing values of the measure, while the colors of appropriate points of $BCD$ corresponding to increasing values of both $b$ and $c$ must reveal non-increasing values of the measure. Similarly for other faces.

Clearly, measure $D(H, E)$ in Fig. 20.3 violates those expectations by revealing the increasing values of the measure when $c$ increases, allowing to draw conclusions that $D(H, E)$ does not possess property $M$.

### 20.4.3 Symmetry-Related Properties: $ES$, $HS$, and $EHS$

In terms of $a$, $b$, $c$, and $d$ property of evidence symmetry, $ES$, requires that the confirmation measure is a function that satisfies $f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]) = -f(\left[\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}\right])$. To confirm $ES$, the parallelogram rotated by 180° about its middle (this leads to the exchange of rhombi $ABDC$ and $CDBA$, with their orientation changed to upside-down) and rendered with reversed color map must be verified to be the same.

In terms of $a$, $b$, $c$, and $d$ property of hypothesis symmetry, $HS$, requires that the confirmation measure is a function that satisfies $f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]) = -f(\left[\begin{smallmatrix} c & a \\ d & b \end{smallmatrix}\right])$. To confirm $HS$, the parallelogram transformed by shifting the left-hand side rhombus to the right and the right-hand side rhombus to the left (this leads to the exchange of rhombi $ABDC$ and $CDBA$, with their orientation unchanged) and rendered with reversed color map must be verified to be the same.

In terms of $a$, $b$, $c$, and $d$ property of evidence-hypothesis symmetry, $EHS$, requires that the confirmation measure is a function that satisfies $f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]) = f(\left[\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right])$. To confirm $EHS$, in the parallelogram: rhombus $ABDC$ rotated by 180 ° must be verified to be the same, and rhombus $CDBA$ rotated by 180° must be verified to be the same.

Notice that also the interior views of the tetrahedron are required to fully verify any of these three properties. However, often enough, the analysis of only the external views of a tetrahedron can showcase counterexamples, as is for measure $D(H, E)$ in Fig. 20.3 in case of $ES$ and $EHS$.

### 20.4.4 Extrema-Related Properties: L, $Ex_1$, and $Maximality/Minimality$

In terms of $a$, $b$, $c$, and $d$ property $L$ requires that if $c = 0$ ($a = 0$) then the confirmation measure $f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right])$ obtains its maximal (minimal) value. To confirm $L$, the corresponding faces of the tetrahedron ($ABD$ for $c = 0$ and $BCD$ for $a = 0$) must be verified to have the right color ("maximal" and "minimal," respectively). Notice that only exterior view of the tetrahedron is required to fully verify this property.

Property $Ex_1$ reverses the implications of $L$ and in terms of $a$, $b$, $c$, and $d$ it requires that if the confirmation measure $f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right])$ obtains its maximal (minimal) value, then $c = 0$ ($a = 0$). To confirm $Ex_1$, the regions of the right color ("maximal" and "minimal," respectively) must be verified to be located in the corresponding faces of the tetrahedron ($ABD$ for $c = 0$ and $BCD$ for $a = 0$, respectively). Notice that also the interior views of the tetrahedron are required to fully verify this property.

In terms of $a$, $b$, $c$, and $d$ property $maximality/minimality$ requires that the confirmation measure $f(\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right])$ obtains its maximal (minimal) value if and only if $b = c = 0$ ($a = d = 0$). To confirm $maximality/minimality$, the corresponding edges of the tetrahedron ($AD$ for $b = c = 0$ and $BC$ for $a = d = 0$) must be verified to be the only regions of the tetrahedron of the right color ("maximal" and "minimal," respectively). Notice that also the interior views of the tetrahedron are required to fully verify this property.

The analysis of the external views of measure $D(H, E)$ in Fig. 20.3 can showcase counterexamples to property $L$, $Ex_1$ as well as $maximality/minimality$.

## 20.5 Current and Future Applications to Data Mining and Machine Learning

### *20.5.1 Multi-Criteria Selection of Non-dominated Rules*

Evaluation of rules can be performed with respect to more than one attractiveness measure in order to provide a multi-context description of the quality of the rule, when a single measure is an insufficient indicator. In case of such a multi-criteria rule evaluation, the best rules are the non-dominated ones, i.e., those for which there does not exist any other rule that is better on at least one evaluation criterion and not worst on any other. The set of all non-dominated rules, with respect to particular attractiveness measures, is referred to as the Pareto-optimal set or the Pareto-optimal border. Confirmation measures have been a component of various multi-criteria evaluation spaces and proved to be a useful tool in data mining and machine learning applications.

In [1] Bayardo and Agrawal introduced a support[1]-confidence evaluation space in order to limit the set of association rules. In such space, for a set of rules with fixed conclusion, the Pareto-optimal border includes optimal rules according to several different interestingness measures, such as, e.g., gain, Laplace, lift [1]. This practically useful result has been revised by Brzezinska, Greco, and Slowinski [2] by substituting confidence with a confirmation measure $F(H, E)$ [31]. Interestingly, all the profits of using support-confidence evaluation space are preserved in the new space due to a particular monotone link between confidence and $F(H, E)$. As a result, working in a support-$F(H, E)$ evaluation space allows to identify the most interesting rules according to various attractiveness measures. Additionally, as presented in Fig. 20.5, the semantics of the scale of confirmation measures allows

[1] $support(H, E) = a$, i.e., it is the number of examples in the dataset satisfying both the rule's premise and conclusion.

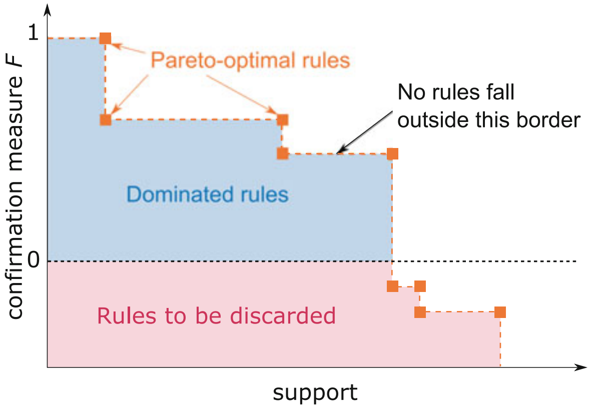


**Fig. 20.5** Support-$F(H, E)$ evaluation space with Pareto-optimal border and area of rules with negative confirmation indicated

to immediately discard all meaningless (i.e., characterized by $F(H, E) < 0$) rules, which is not so straightforward if confidence is used [2]. Particularly in association rule mining, elimination of disconfirmatory rules can significantly limit the set of induced rules, allowing to focus on the practically useful ones [50, 56].

Selection of non-dominated rules has also been studied with respect to support and anti-support[2] measures [2]. Though these two measures are just basic frequencies, and neither of them is a confirmation measure, putting them together allows to mine all rules maximizing any measure enjoying the property of monotonicity $M$ (including confirmation measures). The support–anti-support Pareto-optimal set includes all the rules from the support-$F(H, E)$ Pareto-optimal border. Its practical usefulness could be further increased by limiting the set only to rules with positive confirmation [2].

### 20.5.2 *Post-processing of Rules Induced from Imbalanced Data*

Classification of imbalanced data is yet another task that illustrates the usage of confirmation measures for pruning rules. In *imbalanced data* one class (further called the *minority class*) is underrepresented compared to the majority class, which constitutes difficulties for learning classifiers.

Rule classifiers are quite sensitive to class imbalanced data and fail to accurately recognize instances from the minority class. The BRACID rule induction algorithm is one of the most recent proposals to address this problem in a comprehensive manner [39]. One of the key characteristics of BRACID is its use of so called *types of difficulty* of learning examples, estimated by the analysis of the neighborhood of

[2] $anti\text{-}support(H, E) = c$, i.e., it is the number of counterexamples to the rule in the dataset.

these examples [41]. The assignment of difficulty type to each example influences the rule generalization, as for the unsafe minority example it is possible to generate additional rules covering it. As a result, the number of minority class rules, as well as their support, is increased and they are more likely to win with the stronger majority rules while classifying new instances. The experimental evaluation of the classification performance of BRACID over 22 popular imbalanced datasets showed that it significantly outperformed many standard rule classifiers as well as other rule approaches specialized for class imbalance [39].

Although BRACID proved to be an accurate classifier for binary imbalanced tasks, its potentially high number of rules may restrict some applications and may lower the possibility of analyzing the rules by humans. Therefore the authors of BRACID proposed a special post-pruning strategy [40]. This strategy was extended to use two confirmation measures $S(H, E)$ and $N(H, E)$ in a *weighted rule covering* algorithm [42], where each rule is evaluated with a product of a *rule support* and $S(H, E)$ or $N(H, E)$ *confirmation measure*. The usefulness of this approach was experimentally studied over a dozen of benchmark imbalanced datasets [42]. The general observations from these experiments showed that each variant of pruning improved the considered attractiveness measures. Besides increasing average values of $S(H, E)$ and $N(H, E)$ measures, the average rule supports for both minority and majority classes were higher. For instance, for *cmc* data the average rule supports increased from 6.59 to 18.57 examples in the minority class, and from 7.3 to 21.0 examples in the majority class. Classification performance of such pruned rules did not decrease much compared to the non-pruned BRACID rules.

### *20.5.3 Classification by Association and Biomedical Data*

The potential of applying confirmation measures in rule classification was implemented in the CM-CAR algorithm [3]. CM-CAR starts from mining frequent item sets, which are then used to form classification rules. Next, it uses two user-defined lists of attractiveness measures to sort and filter the rules, making it possible to divide the responsibility for the predictive and descriptive aspects of its classification model. Computational experiments conducted on 20 datasets from the UCI repository compared 12 confirmation measures. The use of each confirmation measure was evaluated using several classifier performance measures, which indicated that measures $c_1(H, E)$, $F(H, E)$, and $Z(H, E)$ are really good for general-purpose rule pruning and sorting, while in the context of balanced/imbalanced problems also measures $N(H, E)$, $S(H, E)$, and $c_3(H, E)$ stand out. The results of the experiments in [3] demonstrated that confirmation measures can be successfully applied to evaluate rules induced from data of various characteristics, when looking for a reduced set of rules with good descriptive characteristics while maintaining satisfactory predictive performance. A more recent approach incorporating selected confirmation measures in classification is GuideR [48], a user-guided rule induction algorithm, which has the possibility to introduce user preferences or domain knowledge to the rule learning process.

Confirmation measures in predictive systems were practically applied to biomedical data. Pieszko et al. have analyzed hematological inflammation markers in the prediction of short-term acute coronary syndrome outcomes through a dominance-based rough set classifier [45]. The proposed approach identified diabetes, systolic and diastolic blood pressure, and prothrombin time as having the highest value of confirmation measure $S(H, E)$ in the detection of in-hospital mortality. A similar approach was used to analyze the antimicrobial activity of bis-quaternary imidazolium chlorides, chemical compounds with good anti-electrostatic properties used in cosmetic, textile, and pharmaceutical industries [44]. Confirmation measures were also part of the medical data analyses by Nahar et al., who used Tertius [16] to detect factors that contribute to heart disease in males and females [38].

### *20.5.4 Tetrahedron Visualizations for Classifier Evaluation Measures*

The idea of using the barycentric coordinate system to visualize measures was recently adapted to classifier evaluation measures [6]. For a binary classification problem, the entries *TP*, *TN*, *FN*, *FP* from a confusion matrix follow a constrained sum rule $n = TP + FP + FN + TN$, which is similar to that observed in confirmation measures. By adapting the tetrahedron visualization technique, the paper puts forward ten properties, which should be taken into account when examining classification performance measures, particularly in the context of imbalanced data. The proposed properties included the analysis of measure minima, maxima, monotonicity, symmetry, and undefined values. Importantly, all properties were designed to be verifiable by means of visual inspection. The efficiency of the proposed visual analysis technique was demonstrated by comparing 22 popular measures (e.g., accuracy, balanced accuracy, F1-score, G-mean, precision, recall, specificity) using the discussed set of properties. An interactive web application allowing users to perform such analyses has been made publicly available.[3] The application aids the analysis of complete ranges of performance measures based on any $2 \times 2$ contingency matrix. The tool operates in a barycentric coordinate system using a 3D tetrahedron, which can be rotated, zoomed, cut, parameterized, and animated. The application is capable of visualizing 86 predefined measures, as well as helping prototype new measures by visualizing user-defined formulas [5].

This concept of analyzing classifier evaluation measures was further extended in [7], where measure gradients were studied. Since every possible confusion matrix and its corresponding measure value can be visualized as a point in the barycentric space, one can also calculate the gradient of the analyzed measure and depict it

[3] https://dabrze.shinyapps.io/Tetrahedron/.

as an arrow. The gradient shows the direction of the greatest rate of increase of the measure's value and its magnitude is the rate itself. In the case of evaluation measures, this can be translated to the direction of changes in the confusion matrix that causes the greatest increase in the measure's value. Using this visualization technique, one can observe the extent to which it is possible to obtain a higher measure value simply by changing class proportions in the dataset. This can be used to show which measures, and to what amount, are susceptible to such changes [7].

### *20.5.5 Future Directions*

Previous studies on the use of confirmation measures focused primarily on the selection or filtering of the set of already generated rules. An independent interesting problem is to investigate the usefulness of these measures in the rule induction process itself. Perhaps confirmation measures could be beneficial in controlling the search of possible candidates for conditions within a rule and emerge as an extension to the currently used heuristics [17]. It would also be worth examining whether some of the considered confirmation measures could contribute to constructing compact sets of rules having both good interpretative properties and leading to a high prediction on new instances. Initial, however limited, attempts were started in [57], but this topic requires further and more extensive research.

Yet another direction concerns extending interpretation of discovered rules by establishing the most important elementary conditions in rules. Slowinski and his co-authors initiated a similar approach in [28], which was based on using conjoint measures, but it did not use confirmation measures so far.

It would also be interesting to further investigate the notion of Pareto-optimal borders in spaces constructed using confirmation measures (e.g., support-$F(H, E)$). Up till now, only the very Pareto-optimal sets were considered, whereas it would be captivating to include also subsequent Pareto-borders, i.e., Pareto-sets emerging after hiding the previous non-dominated set. Adding these deeper borders could be a way of increasing the number of covered instances while keeping the benefits of discovering objectively best (non-dominated) rules.

Finally, the visualization approach initially proposed for confirmation measures and later extended to classifier evaluation measures could also further inspire and aid the analysis of other measures, in particular ensemble diversity measures. Since most diversity measures are based on proportions of correct/incorrect answers of two component classifiers [4], they use a $2 \times 2$ non-negative matrix similar to those used to construct confirmation and binary classification measures. Such visual analyses of diversity measures could help explain the way diversity affects the performance of classifier ensembles.

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# Chapter 21
# An Approach to Combining Adherence-to-Therapy and Patient Preference Models for Evaluation of Therapies in Patient-Centered Care

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**Abstract** Adherence to therapy is one of the major determinants of therapy success, while non-adherence leads to worsening of patient condition and increased healthcare costs. The aim of our work is to evaluate therapies recommended by a clinical practice guideline in order to select a therapy that is most suited for a patient's adherence profile and accounts for patient's preferences. We define three broad categories of adherence—good, moderate, and poor. Each category is associated with a single adherence profile that defines patient characteristics and thus describes a typical patient population for that category. Moreover, each category is also associated with an adherence model that defines therapeutic characteristics linked to adherence (e.g., complexity of therapy). We assume that each patient has a preference model that defines preferences for specific therapies (e.g., an attitude toward invasiveness of therapy). Adherence and patient preference models are constructed from preferential information elicited using multiple-criteria decision analysis methods, and they are represented as value functions. Once a patient has been associated with an adherence profile, both models are used to

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evaluate therapies generated from a guideline. We present an illustrative clinical scenario describing a patient with atrial fibrillation to demonstrate our proposed approach.

## 21.1 Introduction

One of the most significant barriers to effective medical therapies is patients' non-adherence, understood as a failure to follow advice and recommendations from their health care provider. A Cochrane review by Haynes et al. [1] concluded that interventions for improving therapy adherence may have a far greater impact on clinical outcomes than therapy advancements. The consequences of non-adherence include worsening health condition and increased prevalence of comorbid diseases that further increases health care utilization and costs [2]. In 2014, the Institute for Healthcare Informatics estimated that between 100 and 300 billion of avoidable health care costs have been attributed to patients' non-adherence in the USA annually, representing 3–10% of the total US health care expenditures [3]. In England, between one-third and a half of all medications prescribed for long-term conditions are not taken as recommended with an estimated cost of £300 million annually to the National Health Service [4, 5]. According to the World Health Organization, in developed countries, adherence to long-term therapies in the general population is around 50% and is even lower in developing countries [6].

The last number of years has seen a rise in the advocacy for patient-centered care that supports the active involvement of patients in decision-making about preferred therapy. The Institute of Medicine defines patient-centered care as:

> Providing care that is respectful of and responsive to individual patient preferences, needs, and values, and ensuring that patient values guide all clinical decisions [7].

According to the same report [7], patient-centered care can also influence the effectiveness of therapy. The studies have shown that patients whose therapy is deemed patient-centered are more likely to adhere to therapy recommendations [1].

This chapter operationalizes patient-centered care and builds on our earlier work in modeling therapy generation from clinical practice guidelines (CPG) [8]. Here we describe how we created adherence profiles by identifying high-level adherence-related criteria characterizing patients (e.g., cognitive function). Adherence profile is associated with a specific adherence category—*good*, *moderate*, or *poor*, and each category is associated with an adherence model. All adherence models use the same set of therapy-related criteria (e.g., complexity of therapy); however, the values of the criteria are evaluated differently for each adherence model (e.g., low complexity of a therapy may be evaluated as more important for a poor adherence model than for a good adherence one). Each adherence model is subsequently combined with a patient preference model that uses patient-specific preferential criteria for a therapy, such as the cost or the perceived trade-off between the risks of potential adverse events and the overall benefits of a therapy.

In order to develop each model, multiple-criteria decision-making analysis is used to elicit preferential information. Preferences regarding therapies are elicited from clinicians who assess examples of alternative therapies defined in terms of the therapy-related criteria. Patient preferences are elicited from patients who assess alternative therapies defined in terms of patient-related criteria. Specifically, patients are presented with a number of possible therapies (a subset of all applicable therapies) which they assess according to their preferences. Laboratory of Intelligent Decision Support Systems at the Institute of Computing Science, Poznan University of Technology led by Professor Roman Slowinski is a world-renowned research center working on novel methods of elicitation and processing of preferential information. Under Professor Slowinski's leadership, this group has developed the GRIP (Generalized Regression with Intensities of Preference) method [9] that accepts preferential information in the form of pairwise comparisons of reference alternatives.

We used this method for processing of clinicians' and patients' preferential information. Specifically, GRIP computes scoring functions that are converted into scores that are combined to evaluate proposed therapies generated from a CPG. If there is a list of alternative and clinically equivalent therapies, the combined adherence to therapy and patient preference scoring functions are used to rank these therapies according to a patient's adherence category and their preferential information for a therapy. Scores or ranked lists can then be used during a patient–physician encounter when deciding on the most appropriate therapy for the patient. We demonstrate an application of our approach using a case study for non-valvular atrial fibrillation (AF). The contribution of our approach is to building adherence models that capture patient characteristics associated with engagement with therapy in general. These models are then combined with patient preference models for an evaluation of therapies from a CPG in a holistic and patient-centered manner.

## 21.2 Background

A large body of literature has investigated the causes of therapy non-adherence and categorized non-adherence factors as patient-, disease-, therapy-, health care system-, and socio-economic-related [10–18]. In this chapter, we focus only on patient- and therapy-related factors. A systematic review [11] identified patient-related factors as demographics, smoking or alcohol intake, cognitive function, and history of adherence and therapy-related factors as route of administration, complexity, duration, medication side effects, and the degree of behavioral change required.

In the UK, the National Institute of Clinical Excellence's (NICE) guideline on involving patients in decisions about therapy [19] recommends a patient-centered approach that encourages informed adherence as part of a frank and open discussion between a patient and a clinician. The guideline recognizes that non-adherence may be the norm (or is at least very common) and encourages patients to discuss non-adherence and any doubts or concerns they have about therapy. A systematic review of interventions to improve adherence to self-administered medications for chronic diseases in the USA examined the effectiveness of interventions such as case management, decision aids, face-to-face education with pharmacists, educational and behavioral support via telephone, mail and/or video, access to medical records, reminders, risk communications, and shared decision-making [14]. The review concluded that from a patient's perspective, case management and patient education with behavioral support improved medication adherence for more than one condition. Other reviews have shown the positive impact of case management and shared decision-making on adherence [20, 21]. In particular, research has shown that patient-centered interactions such as patient involvement in care and the individualization of patient care promote adherence and lead to improved health outcomes [21]. Researchers have also investigated randomized controlled experimental designs in which some participants are randomly allocated to treatments (a "randomization arm") and others receive their preferred treatment ("a preference arm"), and they demonstrated that allowing patients to choose their treatments improves treatment adherence [22].

In line with recommendations such as that issued by NICE [19] and the findings of research that advocate a patient-centered approach to improve adherence, the aim of our work is to combine non-adherence factors and patient's preferences for specific therapies in order to support patients and physicians in choosing a therapy that a patient prefers and therefore is more likely to adhere to. Much of the literature on therapy adherence is focused on examining the effectiveness of various interventions including technologies such as automated reminders [23]. Others have modeled adherence using simulations of patient cohorts, for example using Markov models to evaluate potential clinical and economic implications of non-adherence [24]. However, few have focused on adherence in a broader context and on using patient characteristics to build models for adherence that reflect how patients engage with therapy and incorporating these into computer-based tools as we propose here.

A large body of work exists on modeling patient preferences by eliciting preferences from patients, often in the form of utilities, and then either using these preferences as a basis for discussion with physicians about possible therapy options or employing preference information as parameters in automated decision-making aids. For example, multi-criteria analysis tools including the analytic hierarchy process (AHP), discrete choice, or conjoint analysis have been widely used to compare sets of alternatives to derive patient preferences for therapies. Dolan et al. [25] conducted an AHP analysis of the choice among five screening regimens for colon cancer using patient preference data including sensitivity of the screening regime, possible side effects, estimated costs, and convenience of the procedure. Van Til et al. [26] investigated the effect of a priori information on preferences (described with eight decision criteria) for stroke treatment in a discrete choice experiment. Rosenfeld et al. [27] conducted a study of patient preferences for HIV treatment utilizing a forced-choice paired-comparison method of all possible pairs of eight different treatment options for the disease. Preferences were analyzed using binary multidimensional scaling to determine the utility of paired-comparison models. Our proposal overlaps with the research described in [25–27], as we also elicit patient preferences to represent them formally for use in computer-based decision aids. However, we combine preferences with adherence-to-therapy-related criteria to gain a complete evaluation of possible therapies.

Other methods of incorporating patient preferences include using formal tools such as Markov decision analysis and decision tree models to simulate possible progression of disease and using the results of such simulations to help patients make decisions about therapy. Thomson et al. [28] used Markov decision analysis to model decision-making about warfarin treatment in patients with AF. The model was run for combinations of age, sex, blood pressure, and risk factors and assessed results in terms of quality-adjusted life years and costs. Thomson et al. [29] also created a decision tree model that incorporates the Framingham risk score to assess a patient's risk of coronary artery disease, together with a detailed assessment of the patient's current lifestyle and their willingness to change behavior. The model assesses individual's preferences for different treatment outcomes, before providing patients with guidance on what might be the best treatment option for them. The MobiGuide system [30] uses a hybrid decision tree and Markov model to elicit patients' preferences related to the progression of disease together with prevention and management of the risk of thromboembolism in AF as part of shared patient–physician decision support system.

Incorporating patient preferences into therapy selection has been shown to have a positive impact on adherence, thus supporting the benefit of patient participation in the therapy development process. Bhosle et al. [31] showed a positive correlation with patient preferences, therapy satisfaction, and adherence to medications compared with other therapies in a longitudinal cohort study of psoriasis patients. Wilder et al. [32] compared patients with medications prescribed according to their preferences and patients with medications prescribed solely according to the CPGs and observed a 10% increase in adherence to medications among patients who were prescribed medications that they preferred. Chrystyn et al. [33] demonstrated the impact of patients' preferences for inhalers measured in terms of durability, ergonomics, and ease of use on therapy compliance and health status in chronic obstructive pulmonary disease. A study assessing several patient-centered preferences for HIV HAART therapy found adherence was best among those patients with high levels of behavioral intention [34].

Given the demonstrated links between patient preferences and improved adherence to therapy, we propose to model and combine factors associated with adherence to therapy and patient preferences. This allows for the patient-centered evaluation of therapies taking into account patient preferences and their adherence profile. The proposed approach supports patients and physicians in choosing a therapy that is best suited to their specific needs. In the next section, we give an overview of AF including a description of few possible therapies. In Sect. 21.4, we outline adherence profiles and present how they are linked with adherence and patient preference models. We also discuss our approach for combining them for evaluating therapies generated from a CPG. We conclude the paper with a discussion and plans for future work.

## 21.3 Atrial Fibrillation

AF is one of the most prevalent types of cardiac arrhythmias and accounts for about 30% of hospitalizations for arrhythmias. There are different types of AF and different therapy options that impact how AF is managed. In this chapter, we focus on pharmacological therapies of non-valvular AF to prevent stroke. These therapies are relatively complex, and a number of new ones have recently been approved for the condition.

An important aspect in AF management for stroke prevention is using oral anticoagulant medications (OACs). Common OACs include vitamin K antagonists (VKAs) such as warfarin, but they are associated with a number of problems, including drug–drug and drug–food interactions, a narrow therapeutic window, and the need for routine coagulation control to test how long it takes for blood to clot (the test is called the International Normalized Ratio or INR). As such VKA therapy is often poorly adhered to, and only half of the patients maintain INR at the target range [35]. New direct oral anticoagulant medications (DOACs) offer faster, more predictable, and sustainable anticoagulation with a fixed-dosage administration characterized by fewer interactions and no need for laboratory INR level control.

In clinical trials, DOACs were found to be at least as effective as VKAs in stroke prevention and associated with lower bleeding and risk of death from cardiovascular causes [36]. However, DOAC-based therapy comes with challenges that include a lack of antidote for bleeding, shorter half-lives that require more diligence on the part of the patient to ensure the drug is taken strictly according to the dosing schedule, increased cost of medication, and limited opportunities to enforce monitoring the level of anticoagulation [37]. In addition, the choice of DOAC is influenced by individual patient characteristics, including risk of stroke, risk of bleeding, and comorbidity (e.g., renal dysfunction) [38].

## 21.4 Materials and Methods

Figure 21.1 outlines our proposed approach. Currently, we assume there are 3 possible adherence categories—poor, moderate, or good (this may change in the future as discussed in the last section). Each category is associated with an adherence profile representing a typical patient population for that category, and each profile is in turn linked to an adherence model. Adherence profile is characterized with profile-related criteria, while adherence model is characterized with therapy-related criteria. The process of evaluating therapies starts with a new patient matched with an adherence profile (1) for the identification of the most likely adherence category. This allows a patient to be assigned to the adherence model linked with their profile (2). The patient has a unique preference model characterized with patient-related criteria defining their therapy preferences. Both models (adherence and patient

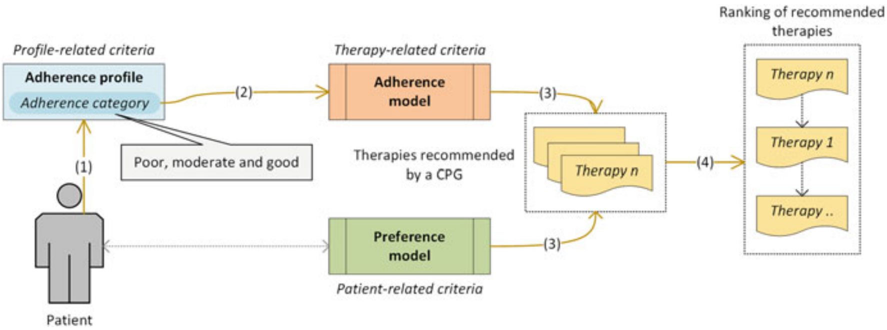


Fig. 21.1 High-level view of our proposed approach

preference) are used to evaluate therapies from a CPG (3) in order to rank them (4). These steps are described in a greater detail in the next subsections.

### 21.4.1 Adherence Profile—Identifying Criteria

The adherence profile is composed of patient characteristics and as such is used to assign a patient to one of the three adherence categories. We use the systematic review by Jin et al. [11] as a basis for defining patient criteria having the greatest effects on adherence. We recognize that there are other criteria associated with adherence, for example, subjective aspects such as patient's beliefs and attitude toward therapy and similarly some demographic criteria such as gender, ethnicity, and educational level. However, as noted by Jin et al. [11], the effects of many of these criteria are inter-related, and their impact on adherence is not well understood.

Therefore, the criteria we chose to use in developing adherence profile are as follows:

- *Age*: A majority of the studies reviewed by Jin et al. [11] showed that age was associated with adherence and the effect of age could be divided into 3 major groups: the elderly group (over 55 years old), the middle-age group (40–55 years old), and the young group (under 40 years old). In general, elderly patients without cognitive impairments and other physical difficulties are relatively more compliant. Middle-aged and young patients are likely to have other priorities (e.g., work commitments meaning they are unable to follow therapy or spend a long time waiting for clinic appointments) in their daily life impacting adherence. Low adherence also occurs in adolescents often marked by rebellious behavior or desire to live a normal life and children who may need help from their parents or guardians to implement therapies.
- *Cognitive function*: Impaired cognitive function and forgetfulness are widely reported factors that cause non-adherence with therapy.

- *History of adherence*: Patients with a prior history of good adherence are likely to adhere to new therapies.
- *Tobacco smoking or alcohol intake*: Several studies found that patients who smoked or drank alcohol were more likely to be non-adherent.

### *21.4.2 Adherence Profile—Selecting an Adherence Category*

Table 21.1 shows criteria and values for each adherence profile and how they are used to delineate adherence categories.

Characterizing patients using the same set of criteria (age, cognitive function, history of adherence, and tobacco smoking or alcohol intake) allows to match a new patient against the three adherence profiles in order to select an adherence category for the patient. Specifically, we use the Euclidean distance to find the closest profile for a given patient and assign the patient to the adherence category indicated by the profile.

Suppose the following hypothetical patient:

> Jack Burns is a 71 years old retiree diagnosed with non-valvular AF. He is an erratic visitor to the clinic, often failing to show up to his appointments. He is a diagnosed type 2 diabetic, being treated with metformin and sitagliptin to control blood sugar levels, but his glycated hemoglobin (Hb1AC) level is critically high (9.2%). When questioned regarding his medication, he is vague in his answers regarding when and how often he takes them. During his visits, his wife has expressed concern regarding his memory. Recent Mini–Mental State Examination scores for Jack were at 17 indicating moderate cognitive impairment. He freely admits to spending most of his time in the pub with friends.

Based on the collected data, the distance between Mr. Burns' profile and each adherence profile is computed as shown in Table 21.2. Since the *poor* adherence profile is closest to the patient description, Mr. Burns is assigned to the poor adherence category.

**Table 21.1** Adherence profiles and corresponding categories

| Profile-related criteria | Values | | |
|---|---|---|---|
| Age | >55 | 40–55 | <40 |
| Cognitive function | Normal | Mildly impaired | Moderate–severe impaired |
| History of compliance | Good | None or moderate | Poor |
| Tobacco or alcohol intake | None | Moderate | Heavy |
| Adherence category | Good | Moderate | Poor |

**Table 21.2** Matching Mr. Burns against adherence profiles

| | Good | Moderate | Poor |
|---|---|---|---|
| Distance | 3.00 | 2.23 | 1.73 |

### *21.4.3 Adherence Model—Identifying Criteria*

Each adherence category is linked with an adherence model. These models define therapeutic criteria with the greatest effects on adherence as identified in the systematic review by Jin et al. [11]. Criteria we use are as follows (in brackets, we give their shorter names used in subsequent text and tables):

- *Route of administration* (*administration*): Medications with a convenient way of administration (e.g., oral medication) are likely to make patients adhere. Likewise, difficulty in administration (e.g., injections or multi-route administrations) contributes to non-adherence.
- *Therapy complexity* (*complexity*): The rate of adherence decreases as the number of daily doses of medications and their timing in relation to other medications or meals varies.
- *Therapy side effects* (*side effects*): Side or adverse effects threaten a patient's adherence.
- *Degree of behavioral change required* (*behavioral change*): Larger degrees of required behavioral change associated with following a therapy are linked with poor adherence.

### *21.4.4 Adherence Model—Capturing Preferential Information Regarding Therapies*

An adherence model allows identifying therapies that should be preferred or are easier to accept for patients in a given adherence category. For example, assuming that the complexity of a therapy can be defined by the dosing schedule; a therapy that involves taking one dosage of medication per day has a low complexity and is preferred for a patient with poor adherence in order to encourage them to adhere, while a therapy that involves multiple doses has higher complexity and may be more appropriate for a patient with moderate or good adherence. Therefore, each adherence model uses the same set of criteria; however, marginal scores associated with specific values of these criteria are different for each category.

Adherence models are created from information elicited from physicians using the GRIP method [39]. Physicians are first asked to compare alternative therapies for each of the adherence category, and then GRIP constructs an additive value function that combines marginal value functions for each criterion—route of administration, therapy complexity, therapy side effects, and the degree of behavioral change. For example, Table 21.3 shows the adherence model for a poor adherence category. It includes four marginal value functions (for each of the considered therapy-related criteria) each represented as a set of marginal scores for all possible criterion values. Higher marginal scores indicate values that are preferred.

**Table 21.3** Adherence model for a poor adherence category (score indicates a marginal score for a given value)

| Administration | | Complexity | | Side effects | | Behavioral change | |
|---|---|---|---|---|---|---|---|
| Value | Score | Value | Score | Value | Score | Value | Score |
| Convenient | 0.22 | Low | 0.26 | Low | 0.26 | Low | 0.26 |
| Somewhat inconvenient | 0.10 | Medium | 0.13 | Medium | 0.10 | Medium | 0.06 |
| Inconvenient | 0.00 | High | 0.00 | High | 0.00 | High | 0.00 |

### *21.4.5 Patient Preference Models*

Patients may have specific preferences for therapies that, for example, are characterized with such criteria as risk of an adverse event or the cost. In the same manner that physicians compared alternative therapies in terms of therapy-related criteria (see previous subsection), patients compare alternative therapies in terms of patient-related criteria. These pairwise comparisons are then used by GRIP to construct value functions constituting patients' preference models.

Consider again the hypothetical patient Jack Burns who according to his adherence profile has been assigned to a poor adherence category. He is now asked to provide preferences for therapies so his preference model can be constructed. Specifically, in this scenario we use two criteria as an example, and Mr. Burns is asked to compare alternative therapies in terms of the availability of a reversal agent in the event of a bleed (potential adverse event) and the cost of therapy. It is important to note that patient preferences and criteria can be disease-specific and different sets can be used for different diseases (e.g., patient preferences related to hypertension therapies might be different than those related to AF therapies). Value functions reflecting the preferences for Mr. Burns are elicited and shown in Table 21.4. Mr. Burns prefers therapies where a reversal agent is available, since he is very concerned about the potential non-mitigated risk of bleeding, and he is not concerned with the cost of therapy as all medications are available to him via the UK National Health Service.

**Table 21.4** Patient preferences for a therapy (score indicates a marginal score for a given value)

| Reversal agent | | Cost | |
|---|---|---|---|
| Value | Score | Value | Score |
| Available | 1.00 | Inexpensive | 0.00 |
| None | 0.00 | Expensive | 0.00 |

## 21.5 Calculations

Each therapy recommended by a CPG is characterized with values of criteria considered by the adherence and patient preference models. This representation allows evaluating each therapy according to the adherence and patient preferences. We apply both models to each therapy and obtain two scores for each—one relating to adherence and the other relating to patient preferences.

Returning to our AF example, Table 21.5 shows descriptive characteristics of possible therapies from the AF guideline [40] that include VKA and three different DOACS—dabigatran (DAB), rivaroxaban (RIV), and apixaban (APIX). Table 21.6

**Table 21.5** Descriptive characteristics of AF therapies in terms of therapy- and patient-related criteria

| | VKA | DAB | RIV | APIX |
|---|---|---|---|---|
| *Therapy-related criteria* | | | | |
| Administration | Pill injection | Pill | Pill | Pill |
| Complexity | Once daily INR monitoring 1–2 times per week | Twice daily no INR monitoring | Once daily no INR monitoring | Twice daily no INR monitoring |
| Side effects | A wide range of medication interaction, food and drink interactions, bleed-ing/bruising (more) | Bleeding/bruising (less), nausea, vomiting, dyspepsia (more) | Bleeding/bruising (less), nausea, vomiting, dyspepsia (less) | Bleeding/bruising (less), nausea, vomiting, dyspepsia (less) |
| Behavioral change | High | Low | Low | Low |
| *Patient-related criteria* | | | | |
| Reversal agent | Available | Available | None | None |
| Cost | £10–50 (excluding monitoring costs) | £600 | £600 | £640 |

**Table 21.6** AF therapies characterized with regard to criteria and values used in the adherence and patient preference models

| | VKA | DAB | RIV | APIX |
|---|---|---|---|---|
| Administration | Convenient | Convenient | Convenient | Convenient |
| Complexity | High | Medium | Low | Medium |
| Side effects | Medium | Low | Low | Low |
| Behavioral change | High | Low | Low | Low |
| Reversal agent | Available | Available | None | None |
| Cost | Inexpensive | Expensive | Expensive | Expensive |

**Table 21.7** Applying adherence and patient preference models to evaluate AF therapies

| | VKA | DAB | RIV | APIX |
|---|---|---|---|---|
| Administration | 0.22 | 0.22 | 0.22 | 0.22 |
| Complexity | 0.00 | 0.13 | 0.26 | 0.13 |
| Side effects | 0.10 | 0.26 | 0.26 | 0.26 |
| Behavioral change | 0.00 | 0.26 | 0.26 | 0.26 |
| Score for therapy-related criteria | 0.32 | 0.87 | **1.00** | 0.87 |
| Reversal agent | 1.00 | 1.00 | 0.00 | 0.00 |
| Cost | 0.00 | 0.00 | 0.00 | 0.00 |
| Score for patient-related criteria | **1.00** | **1.00** | 0.00 | 0.00 |

The numbers in bold indicate the largest (best) scores

shows these therapies characterized with criteria and values used in the adherence and patient preference models.

The adherence and patient preference models can now be applied to evaluate therapies from the AF CPG as shown in Table 21.7. As a result of applying the adherence model for Mr. Burns, rivaroxaban (RIV) is the preferred therapy as indicated by the *score for therapy-related criteria*. Applying the patient preference model results in the highest score (*score for patient-related criteria*) for VKA or DAB, reflecting the patient's preference for the availability of a reversal agent. The scores as they are shown in Table 21.7 can be used during a conversation between the patient and the physician in deciding on the most appropriate therapy.

Alternatively, the final scores may be combined and used to rank the therapies. Given the two partially ordered sets of scores, a lexicographical ordering may be applied to combine the scores into one set. For example, such an ordering may first consider scores for therapy-related criteria from Table 21.7 to create an initial ranking:

1. RIV
2. DAB, APIX
3. VKA

Next, the patient criteria scores from Table 21.7 are considered. As there is a tie between the therapy-related scores for DAB and APIX, the patient criteria score is used to resolve it. As DAB was preferred to APIX by the patient, it becomes the second ranked therapy followed by APIX and VKA resulting in a final ordering of:

1. RIV
2. DAB
3. APIX
4. VKA

We note that the presentation of preferences as have shown in the example in Table 21.7 is in itself a multi-criteria decision problem as it outlines a scenario where the best option needs to be selected given that there are two different points of view.

We present a conservative approach with lexicographical ordering where expert knowledge (captured by adherence models constructed with the help of clinical experts) has precedence over patient preferences; however, more sophisticated approaches could be considered. For example, the scores could be combined using weights dependent on the adherence profile (such weights would become part of the profiles). In such an approach, the preference of patients assigned to the good adherence category would have a stronger impact on therapy selection than preferences of patients assigned to moderate or poor adherence categories.

## 21.6 Results and Discussion

We have presented an approach to support therapy adherence by modeling patient and therapy-related factors that impact adherence and combined these with patient preferences for therapy evaluation, ranking, and selection.

Previous work in the field on modeling patient preferences has focused on finding the best methods for preference elicitation and combination [25–27] with a particular focus on determining which methods are most amenable to patients. Research on therapy adherence typically takes a general view, for example, focusing on cost and effectiveness of non-adherence using simulated patient cohorts [24], or a very specific view on the effectiveness of interventions (e.g., reminders such as text messages or manual follow-up [23]). The approach proposed in this chapter bridges these gaps by formalizing models for adherence and patient preferences using the GRIP method and combining them to evaluate therapies in a holistic manner. Instead of taking a general or specific view, we create a formal model for adherence that considers patient characteristics that are strongly associated with adherence (or non-adherence). With regard to patient preferences, our preference model can be customized for a given disease. This allows for a flexible approach that confronts the impact of poor adherence, while at the same time allowing for a patient-centered view of care that involves patients and their preferences in medical decision-making.

## 21.7 Conclusions

A substantial number of patients struggle to adhere to therapy, and the consequences of poor adherence on health and subsequently a health care system are a topical issue. Furthermore, there is a need for the incorporation of patient-centered methods into medical decision-making, in particular consideration of patient preferences for therapy. We presented an approach for eliciting preferences regarding therapies by asking clinicians to assess examples of alternative therapies, while patient preferences are elicited from patients who assess alternative therapies defined in terms of patient-related criteria. Both sets of preferences are processed using the

GRIP method that computes scoring functions that are converted into scores that are combined to evaluate and rank therapies from a CPG. These scores or ranked lists of therapies can be used as part of a patient–physician conversation when deciding on the most appropriate therapy for a patient. We demonstrated our approach using an illustrative case study for non-valvular AF.

We intend to extend our approach in a number of ways. We plan to use data mining techniques (e.g., clustering) on patient data to define finer-grained adherence categories beyond the 3 groups (poor, moderate, and good). This will allow associating several adherence profiles with a single adherence category. Furthermore, when associating a patient with an adherence category, the approach should support contextualized profile-related criteria. For example, some diseases affect mostly elderly, or some patients may not have been on any medication and thus have no prior history of adherence. This will require using an advanced multiple-criteria decision-making analysis method such as ELECTRE TRI-nC [41], which can assess multiple profiles within a single adherence category. We intend to use large patient cohorts from online sources to gather representative data (e.g., from patient support groups such as StopAFib.org). Such data, in addition to clinical characteristics, includes information that cannot be found in medical records. For example, it includes detailed information on preferences as well information about past adherence that can be used for validating the performance of our models. We also intend to elicit preferences from patients for therapy-related criteria defined by our adherence model to supplement those already supplied by physicians. Such preferences will be modeled in the same manner as physician preferences for therapies and used as an additional piece of information when evaluating therapies from a CPG. Finally, we intend to investigate more advanced schemes for combining scores provided by the models and for rank ordering therapies. We also plan to incorporate adherence and patient preference models into our existing CPG mitigation framework to improve the framework's ability to personalize suggested therapies for multi-morbid patients [8].

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