Chapter 7
Rectangular Matrix-Variate Distributions

7.1. Introduction

Thus far, we have primarily been dealing with distributions involving real positive definite or Hermitian positive definite matrices. We have already considered rectangular matrices in the matrix-variate Gaussian case. In this chapter, we will examine rectangular matrix-variate gamma and beta distributions and also consider to some extent other types of distributions. We will begin with the rectangular matrix-variate real gamma distribution, a version of which was discussed in connection with the pathway model introduced in Mathai (2005). The notations will remain as previously specified. Lower-case letters such as x, y, z will denote real scalar variables, whether mathematical or random. Capital letters such as X, Y will be used for matrix-variate variables, whether square or rectangular. In the complex domain, a tilde will be placed above the corresponding scalar and matrix-variables; for instance, we will write $\tilde{x},\ \tilde{y},\ \tilde{X},\ \tilde{Y}$. Constant matrices will be denoted by upper-case letter such as A, B, C. A tilde will not be utilized for constant matrices except for stressing the point that the constant matrix is in the complex domain. When X is a $p\times p$ real positive definite matrix, then $A<X<B$ will imply that the constant matrices A and B are positive definite, that is, $A>O,\ B>O$, and further that $X>O,\ X-A>O,\ B-X>O$. Real positive definite matrices will be assumed to be symmetric. The corresponding notation for a $p\times p$ Hermitian positive definite matrix is $A<\tilde{X}<B$. The determinant of a square matrix A will be denoted by $|A|$ or $\det(A)$ whereas, in the complex case, the absolute value or modulus of the determinant of A will be denoted as $|\det(A)|$. When matrices are square, their order will be taken as being $p\times p$ unless specified otherwise. Whenever A is a real $p\times q,\ q\ge p$, rectangular matrix of full rank p, AA' is positive definite, a prime denoting the transpose. When A is in the complex domain, then AA^* is Hermitian positive definite where an A^* indicates the complex conjugate transpose of A. Note that all positive definite complex matrices are necessarily Hermitian. As well, $\mathrm{d}X$ will denote the wedge product of all differentials in the matrix X. If $X=(x_{ij})$ is a

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real $p \times q$ matrix, then $\mathrm{d}X = \wedge_{i=1}^{p} \wedge_{j=1}^{q} \mathrm{d}x_{ij}$. Whenever $X = (x_{ij})'$ is a $p \times p$ real symmetric matrix, $\mathrm{d}X = \wedge_{i \geq j} \mathrm{d}x_{ij} = \wedge_{i \leq j} \mathrm{d}x_{ij}$, that is, the wedge product of the $\frac{p(p+1)}{2}$ distinct differentials. As for the complex matrix $\tilde{X} = X_1 + iX_2,\ i = \sqrt{(-1)}$, where X_1 and X_2 are real, $\mathrm{d}\tilde{X} = \mathrm{d}X_1 \wedge \mathrm{d}X_2$.

7.2. Rectangular Matrix-Variate Gamma Density, Real Case

The most commonly utilized real gamma type distributions are the gamma, generalized gamma and Wishart in Statistics and the Maxwell-Boltzmann and Raleigh in Physics. The first author has previously introduced real and complex matrix-variate analogues of the gamma, Maxwell-Boltzmann, Raleigh and Wishart densities where the matrices are $p \times p$ real positive definite or Hermitian positive definite. For the generalized gamma density in the real scalar case, a matrix-variate analogue can be written down but the associated properties cannot be studied owing to the problem of making a transformation of the type $Y = X^{\delta}$ for $\delta \neq \pm 1$; additionally, when X is real positive definite or Hermitian positive definite, the Jacobians will produce awkward forms that cannot be easily handled, see Mathai (1997) for an illustration wherein $\delta = 2$ and the matrix X is real and symmetric. Thus, we will provide extensions of the gamma, Wishart, Maxwell-Boltzmann and Raleigh densities to the rectangular matrix-variate cases for $\delta = 1$, in both the real and complex domains.

The Maxwell-Boltzmann and Raleigh densities are associated with numerous problems occurring in Physics. A multivariate analogue as well as a rectangular matrix-variate analogue of these densities may become useful in extending the usual theories giving rise to these univariate densities, to multivariate and matrix-variate settings. It will be shown that, as was explained in Mathai (1999), this problem is also connected to the volumes of parallelotopes determined by p linearly independent random points in the Euclidean n-space, $n \geq p$. Structural decompositions of the resulting random determinants and pathway extensions to gamma, Wishart, Maxwell-Boltzmann and Raleigh densities will also be considered.

In the current nuclear reaction-rate theory, the basic distribution being assumed for the relative velocity of reacting particles is the Maxwell-Boltzmann. One of the forms of this density for the real scalar positive variable case is

$$f_1(x) = \frac{4}{\sqrt{\pi}} \beta^{\frac{3}{2}} x^2 \mathrm{e}^{-\beta x^2},\ 0 \leq x < \infty,\ \beta > 0, \tag{7.2.1}$$

and $f_1(x) = 0$ elsewhere. The Raleigh density is given by

$$f_2(x) = \frac{x}{\alpha^2} \mathrm{e}^{-\frac{x^2}{2\alpha^2}},\ 0 \leq x < \infty,\ \alpha > 0, \tag{7.2.2}$$

and $f_2 = 0$ elsewhere, and the three-parameter generalized gamma density has the form

$$f_3(x) = \frac{\delta\, b^{\frac{\alpha}{\delta}}}{\Gamma(\frac{\alpha}{\delta})} x^{\alpha-1} \mathrm{e}^{-bx^{\delta}},\ x \geq 0,\ b > 0,\ \alpha > 0,\ \delta > 0, \tag{7.2.3}$$

and $f_3 = 0$ elsewhere. Observe that (7.2.1) and (7.2.2) are special cases of (7.2.3). For derivations of a reaction-rate probability integral based on Maxwell-Boltzmann velocity density, the reader is referred to Mathai and Haubold (1988). Various basic results associated with the Maxwell-Boltzmann distribution are provided in Barnes et al. (1982), Critchfield (1972), Fowler (1984), and Pais (1986), among others. The Maxwell-Boltzmann and Raleigh densities have been extended to the real positive definite matrix-variate and the real rectangular matrix-variate cases in Mathai and Princy (2017). These results will be included in this section, along with extensions of the gamma and Wishart densities to the real and complex rectangular matrix-variate cases. Extensions of the gamma and Wishart densities to the real positive definite and complex Hermitian positive definite matrix-variate cases have already been discussed in Chap. 5. The Jacobians that are needed and will be frequently utilized in our discussion are already provided in Chaps. 1 and 4, further details being available from Mathai (1997). The previously defined real matrix-variate gamma $\Gamma_p(\alpha)$ and complex matrix-variate gamma $\tilde{\Gamma}_p(\alpha)$ functions will also be utilized in this chapter.

7.2.1. Extension of the gamma density to the real rectangular matrix-variate case

Consider a $p \times q,\ q \geq p$, real matrix X of full rank p, whose rows are thus linearly independent, and a real-valued scalar function $f(XX')$ whose integral over X is convergent, that is, $\int_X f(XX')\mathrm{d}X < \infty$. Letting $S = XX'$, S will be symmetric as well as real positive definite meaning that for every $p \times 1$ non-null vector Y, $Y'SY > 0$ for all $Y \neq O$ (a non-null vector). Then, $S = (s_{ij})$ will involve only $\frac{p(p+1)}{2}$ differential elements, that is, $\mathrm{d}S = \wedge_{i \geq j=1}^{p} \mathrm{d}s_{ij}$, whereas $\mathrm{d}X$ will contain pq differential elements $\mathrm{d}x_{ij}$'s. As has previously been explained in Chap. 4, the connection between $\mathrm{d}X$ and $\mathrm{d}S$ can be established via a sequence of two or three matrix transformations.

Let the $X = (x_{ij})$ be a $p \times q,\ q \geq p$, real matrix of rank p where the x_{ij}'s are distinct real scalar variables. Let A be a $p \times p$ real positive definite constant matrix and B be a $q \times q$ real positive definite constant matrix, $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ denoting the respective positive definite square roots of the positive definite matrices A and B. We will now determine the value of c that satisfies the following integral equation:

$$\frac{1}{c} = \int_X |AXBX'|^{\gamma} \mathrm{e}^{-\mathrm{tr}(AXBX')} \mathrm{d}X. \tag{i}$$

Note that $\mathrm{tr}(AXBX') = \mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}})$. Letting $Y = A^{\frac{1}{2}}XB^{\frac{1}{2}}$, it follows from Theorem 1.7.4 that $\mathrm{d}Y = |A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\mathrm{d}X$. Thus,

$$\frac{1}{c} = |A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\int_Y |YY'|^{\gamma}\mathrm{e}^{-\mathrm{tr}(YY')}\mathrm{d}Y. \qquad (ii)$$

Letting $S = YY'$, we note that S is a $p\times p$ real positive definite matrix, and on applying Theorem 4.2.3, we have $\mathrm{d}Y = \frac{\pi^{\frac{qp}{2}}}{\Gamma_p(\frac{q}{2})}|S|^{\frac{q}{2}-\frac{p+1}{2}}\mathrm{d}S$ where $\Gamma_p(\cdot)$ is the real matrix-variate gamma function. Thus,

$$\frac{1}{c} = |A|^{-\frac{q}{2}}|B|^{-\frac{p}{2}}\frac{\pi^{\frac{qp}{2}}}{\Gamma_p(\frac{q}{2})}\int_{S>O}|S|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(S)}\mathrm{d}S,\ A>O,\ B>O, \qquad (iii)$$

the integral being a real matrix-variate gamma integral given by $\Gamma_p(\gamma+\frac{q}{2})$ for $\Re(\gamma+\frac{q}{2}) > \frac{p-1}{2}$, where $\Re(\cdot)$ is the real part of $(\cdot)$, so that

$$c = \frac{|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}\Gamma_p(\gamma+\frac{q}{2})} \ \text{ for } \Re(\gamma+\frac{q}{2}) > \frac{p-1}{2},\ A>O,\ B>O. \qquad (7.2.4)$$

Let

$$f_4(X) = c\,|AXBX'|^{\gamma}\mathrm{e}^{-\mathrm{tr}(AXBX')} \qquad (7.2.5)$$

for $A>O,\ B>O,\ \Re(\gamma+\frac{q}{2}) > \frac{p-1}{2},\ X=(x_{ij}),\ -\infty < x_{ij} < \infty,\ i=1,\dots,p,\ j=1,\dots,q$, where c is as specified in (7.2.4). Then, $f_4(X)$ is a statistical density that will be referred to as the rectangular real matrix-variate gamma density with shape parameter γ and scale parameter matrices $A>O$ and $B>O$. Although the parameters are usually real in a statistical density, the above conditions apply to the general complex case.

For $p=1,\ q=1,\ \gamma=1,\ A=1$ and $B=\beta>0$, we have $|AXBX'| = \beta x^2$ and

$$\frac{|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\Gamma(\frac{q}{2})}{\pi^{\frac{qp}{2}}\Gamma_p(\gamma+\frac{q}{2})} = \frac{(\beta)^{\frac{1}{2}}\Gamma(\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{3}{2})} = \frac{2\sqrt{\beta}}{\sqrt{\pi}},$$

so that $c = \frac{2}{\sqrt{\pi}}\beta^{\frac{3}{2}}$ for $-\infty < x < \infty$. Note that when the support of $f(x)$ is restricted to the interval $0 \le x < \infty$, the normalizing constant will be multiplied by 2, $f(x)$ being a symmetric function. Then, for this particular case, $f_4(X)$ in (7.2.5) agrees with the Maxwell-Boltzmann density for the real scalar positive variable x whose density is given in (7.2.1). Accordingly, when $\gamma = 1$, (7.2.5) with c as specified in (7.2.4) will be referred to as the real rectangular matrix-variate Maxwell-Boltzmann density. Observe that for $\gamma = 0$, (7.2.5) is the real rectangular matrix-variate Gaussian density that was considered in Chap. 4. In the Raleigh case, letting $p = 1,\ q = 1,\ A = 1,\ B = \frac{1}{2\alpha^2}$ and $\gamma = \frac{1}{2}$,

$$|AXBX'|^{\gamma} = \Big(\frac{x^2}{2\alpha^2}\Big)^{\frac{1}{2}} = \frac{|x|}{\sqrt{2}|\alpha|} \text{ and } c = \frac{1}{\sqrt{2}\alpha}$$

which gives

$$f_5(x) = \frac{|x|}{2\alpha^2}\mathrm{e}^{-\frac{x^2}{2\alpha^2}},\ \ -\infty < x < \infty \ \text{ or } \ f_5(x) = \frac{x}{\alpha^2}\mathrm{e}^{-\frac{x^2}{2\alpha^2}},\ 0 \le x < \infty,$$

for $\alpha > 0$, and $f_5 = 0$ elsewhere where $|x|$ denotes the absolute value of x, which is the real positive scalar variable case of the Raleigh density given in (7.2.2). Accordingly, (7.2.5) with c as specified in (7.2.4) wherein $\gamma = \frac{1}{2}$ will be called the real rectangular matrix-variate Raleigh density.

From (7.2.5), which is the density for $X = (x_{ij}),\ p \times q,\ q \ge p$ of rank p, with $-\infty < x_{ij} < \infty,\ i = 1, \ldots, p,\ j = 1, \ldots, q$, we obtain the following density for $Y = A^{\frac{1}{2}}XB^{\frac{1}{2}}$:

$$f_6(Y)\mathrm{d}Y = \frac{\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}\Gamma_p(\gamma + \frac{q}{2})}|YY'|^{\gamma}\mathrm{e}^{-\mathrm{tr}(YY')}\mathrm{d}Y \tag{7.2.6}$$

for $\gamma + \frac{q}{2} > \frac{p-1}{2}$, and $f_6 = 0$ elsewhere. We will refer to (7.2.6) as the standard form of the real rectangular matrix-variate gamma density. The density of $S = YY'$ is then

$$f_7(S)\,\mathrm{d}S = \frac{1}{\Gamma_p(\gamma + \frac{q}{2})}|S|^{\gamma + \frac{q}{2} - \frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(S)}\mathrm{d}S \tag{7.2.7}$$

for $S > O,\ \gamma + \frac{q}{2} > \frac{p-1}{2}$, and $f_7 = 0$ elsewhere.

Example 7.2.1. Specify the distribution of $u = \mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}})$, the exponent of the density given in (7.2.5).

Solution 7.2.1. Let us determine the moment generating function (mgf) of u with parameter t. That is,

$$\begin{aligned} M_u(t) &= E[\mathrm{e}^{tu}] = E[\mathrm{e}^{t\,\mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}})}] \\ &= c\int_X |A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\gamma}\mathrm{e}^{-(1-t)\mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}})}\mathrm{d}X \end{aligned}$$

where c is given in (7.2.4). Let us make the following transformations: $Y = A^{\frac{1}{2}}XB^{\frac{1}{2}},\ S = YY'$. Then, all factors, except $\Gamma_p(\gamma + \frac{q}{2})$, are canceled and the mgf becomes

$$M_u(t) = \frac{1}{\Gamma_p(\gamma + \frac{q}{2})}\int_{S>O}|S|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}\mathrm{e}^{-(1-t)\mathrm{tr}(S)}\mathrm{d}S$$

for $1 - t > 0$. On making the transformation $(1 - t)S = S_1$ and then integrating out S_1, we obtain the following representation of the moment generating function:

$$M_u(t) = (1-t)^{-p(\gamma+\frac{q}{2})},\ 1-t>0,$$

which happens to be the mgf of a real scalar gamma random variables with the parameters $(\alpha = p(\gamma + \frac{q}{2}),\ \beta = 1)$, which owing to the uniqueness of the mgf, is the distribution of u.

Example 7.2.2. Let $U_1 = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}},\ U_2 = XBX',\ U_3 = B^{\frac{1}{2}}X'AXB^{\frac{1}{2}},\ U_4 = X'AX$. Determine the corresponding densities when they exist.

Solution 7.2.2. Let us examine the exponent in the density (7.2.5). By making use of the commutative property of trace, one can write

$$\mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}) = \mathrm{tr}[A(XBX')] = \mathrm{tr}(B^{\frac{1}{2}}X'AXB^{\frac{1}{2}}) = \mathrm{tr}[B(X'AX)].$$

Observe that the exponent depends on the matrix $A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$, which is symmetric and positive definite, and that the functional part of the density also involves its determinant. Thus, the structure is that of real matrix-variate gamma density; however, (7.2.5) gives the density of X. Hence, one has to reach U_1 from X and derive the density of U_1. Consider the transformation $Y = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$. This will bring X to Y. Now, let $S = YY' = U_1$ so that the matrix U_1 has the real matrix-variate gamma distribution specified in (7.2.7), that is, U_1 is a real matrix-variate gamma variable with shape parameter $\gamma + \frac{q}{2}$ and scale parameter matrix I. Next, consider U_2. Let us obtain the density of U_2 from the density (7.2.5) for X. Proceeding as above while ignoring A or taking $A = I$, (7.2.7) will become the following density, denoted by $f_{u_2}(U_2)$:

$$f_{u_2}(U_2)\mathrm{d}U_2 = \frac{|A|^{\gamma+\frac{q}{2}}}{\Gamma_p(\gamma+\frac{q}{2})}|U_2|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(AU_2)}\mathrm{d}U_2,$$

which shows that U_2 is a real matrix-variate gamma variable with shape parameter $\gamma + \frac{q}{2}$ and scale parameter matrix A. With respect to U_3 and U_4, when $q > p$, one has the positive semi-definite factor $X'BX$ whose determinant is zero; hence, in this singular case, the densities do not exist for U_3 and U_4. However, when $q = p$, U_3 has a real matrix-variate gamma distribution with shape parameter $\gamma + \frac{p}{2}$ and scale parameter matrix I and U_4 has a real matrix-variate gamma distribution with shape parameter $\gamma + \frac{p}{2}$ and scale parameter matrix B, observing that when $q = p$ both U_3 and U_4 are $q \times q$ and positive definite. This completes the solution.

The above findings are stated as a theorem:

Theorem 7.2.1. *Let $X = (x_{ij})$ be a real full rank $p \times q$ matrix, $q \geq p$, having the density specified in (7.2.5). Let U_1, U_2, U_3 and U_4 be as defined in Example 7.2.2. Then, U_1 is real matrix-variate gamma variable with scale parameter matrix I and shape parameter $\gamma + \frac{q}{2}$; U_2 is real matrix-variate gamma variable with shape parameter $\gamma + \frac{q}{2}$ and scale parameter matrix A; U_3 and U_4 are singular and do not have densities when $q > p$; however, and when $q = p$, U_3 is real matrix-variate gamma distributed with shape parameter $\gamma + \frac{p}{2}$ and scale parameter matrix I, and U_4 is real matrix-variate gamma distributed with shape parameter $\gamma + \frac{p}{2}$ and scale parameter matrix B. Further $|I_p - A^{\frac{1}{2}}XBXA^{\frac{1}{2}}| = |I_q - B^{\frac{1}{2}}X'AXB^{\frac{1}{2}}|$.*

***Proof*:** All the results, except the last one, were obtained in Solution 7.2.2. Hence, we shall only consider the last part of the theorem. Observe that when $q > p$, $|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}| > 0$, the matrix being positive definite, whereas $|B^{\frac{1}{2}}X'AXB^{\frac{1}{2}}| = 0$, the matrix being positive semi-definite. The equality is established by noting that in accordance with results previously stated in Sect. 1.3, the determinant of the following partitioned matrix has two representations:

$$\begin{vmatrix} I_p & A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ B^{\frac{1}{2}}X'A^{\frac{1}{2}} & I_q \end{vmatrix} = \begin{cases} |I_p|\,|I_q - (B^{\frac{1}{2}}X'A^{\frac{1}{2}})I_p^{-1}(A^{\frac{1}{2}}XB^{\frac{1}{2}})| = |I_q - B^{\frac{1}{2}}X'AXB^{\frac{1}{2}}| \\ |I_q|\,|I_p - (A^{\frac{1}{2}}XB^{\frac{1}{2}})I_q^{-1}(B^{\frac{1}{2}}X'A^{\frac{1}{2}})| = |I_p - A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}| \end{cases}.$$

7.2.2. Multivariate gamma and Maxwell-Boltzmann densities, real case

Multivariate usually means a collection of scalar variables, real or complex. Many real scalar variable cases corresponding to (7.2.1) or a multivariate analogue of thereof can be obtained from (7.2.5) by taking $p = 1$ and $A = b > 0$. Note that in this case, X is $1 \times q$, that is, $X = (x_1, \ldots, x_q)$, and XBX' is a positive definite quadratic form of the type

$$XBX' = (x_1, \ldots, x_q)B\begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}.$$

Thus, the density appearing in (7.2.5) becomes

$$f_8(X)\mathrm{d}X = \frac{b^{\gamma+\frac{q}{2}}|B|^{\frac{1}{2}}\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}\Gamma(\gamma+\frac{q}{2})}[XBX']^{\gamma}\mathrm{e}^{-b(XBX')}\mathrm{d}X \tag{7.2.8}$$

for $X=(x_1,\ldots,x_q),\ -\infty<x_j<\infty,\ j=1,\ldots,q,\ B=B'>O,\ b>0$, and $f_8=0$ elsewhere. Then, the density of $Y=B^{\frac{1}{2}}X'$ is given by

$$f_9(Y)\mathrm{d}Y = b^{\gamma+\frac{q}{2}}\frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}\Gamma(\gamma+\frac{q}{2})}(y_1^2+\cdots+y_q^2)^{\gamma}\mathrm{e}^{-b(y_1^2+\cdots+y_q^2)}\mathrm{d}Y \tag{7.2.9}$$

where $Y'=(y_1,\ldots,y_q),\ \ -\infty<y_j<\infty,\ j=1,\ldots,q,\ b>0,\ \gamma+\frac{q}{2}>0$, and $f_9=0$ elsewhere. We will take (7.2.8) as the multivariate gamma as well as multivariate Maxwell-Boltzmann density, and (7.2.9) as the standard multivariate gamma as well as standard multivariate Maxwell-Boltzmann density.

How can we show that (7.2.9) is a statistical density? One way consists of writing $f_9(Y)\mathrm{d}Y$ as $f_9(S)\mathrm{d}S$, applying Theorem 4.2.3 of Chap. 4 and writing $\mathrm{d}Y$ in terms of $\mathrm{d}S$ for $p=1$. This will yield the result. Another way is to integrate out variables $y_1,\ldots,y_q$ from $f_9(Y)\mathrm{d}Y$, which can be achieved via a general polar coordinate transformation such as the following: Consider the variables $y_1,\ldots,y_q,\ -\infty<y_j<\infty,\ j=1,\ldots,q$, and the transformation,

$$\begin{aligned}
y_1 &= r\ \sin\theta_1\\
y_j &= r\ \cos\theta_1\cos\theta_2\cdots\cos\theta_{j-1}\sin\theta_j,\ \ j=2,3,\ldots,q-1,\\
y_q &= r\ \cos\theta_1\cos\theta_2\cdots\cos\theta_{q-1},
\end{aligned}$$

for $-\frac{\pi}{2}<\theta_j\le\frac{\pi}{2},\ j=1,\ldots,q-2;\ \ -\pi<\theta_{q-1}\le\pi$, which was discussed in Mathai (1997). Its Jacobian is then given by

$$\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_q = r^{q-1}\left\{\prod_{j=1}^{q-1}|\cos\theta_j|^{q-j-1}\right\}\mathrm{d}r\wedge\mathrm{d}\theta_1\wedge\ldots\wedge\mathrm{d}\theta_{q-1}. \tag{7.2.10}$$

Under this transformation, $y_1^2+\cdots+y_q^2=r^2$. Hence, integrating over r, we have

$$\int_{r=0}^{\infty}(r^2)^{\gamma}r^{q-1}\mathrm{e}^{-br^2}\mathrm{d}r = \frac{1}{2}b^{-(\gamma+\frac{q}{2})}\Gamma(\gamma+\frac{q}{2}),\ \gamma+\frac{q}{2}>0. \tag{7.2.11}$$

Note that the θ_j's are present only in the Jacobian elements. There are formulae giving the integral over each differential element. We will integrate the θ_j's one by one. Integrating over θ_1 gives

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta_1)^{q-2}\mathrm{d}\theta_1 = 2\int_0^{\frac{\pi}{2}}(\cos\theta_1)^{q-2}\mathrm{d}\theta_1 = 2\int_0^1 z^{q-2}(1-z^2)^{-\frac{1}{2}}\mathrm{d}z$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{q-1}{2})}{\Gamma(\frac{q}{2})},\ q>1.$$

The integrals over $\theta_2, \theta_3, \ldots, \theta_{q-2}$ can be similarly evaluated as

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{q-2}{2})}{\Gamma(\frac{q-1}{2})},\ \frac{\Gamma(\frac{1}{2})\Gamma(\frac{q-3}{2})}{\Gamma(\frac{q-2}{2})},\ldots,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}$$

for $q > p-1$, the last integral $\int_{-\pi}^{\pi}\mathrm{d}\theta_{q-1}$ giving 2π. On taking the product, several gamma functions cancel out, leaving

$$\frac{2\pi[\Gamma(\frac{1}{2})]^{q-2}}{\Gamma(\frac{q}{2})} = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}. \tag{7.2.12}$$

It follows from (7.2.11) and (7.2.12) that (7.2.9) is indeed a density which will be referred to as the standard real multivariate gamma or standard real Maxwell-Boltzmann density.

Example 7.2.3. Write down the densities specified in (7.2.8) and (7.2.9) explicitly if

$$B = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ b=2 \text{ and } \gamma = 2.$$

Solution 7.2.3. Let us evaluate the normalizing constant in (7.2.8). Since in this case, $|B| = 2$,

$$c_8 = \frac{b^{\gamma+\frac{q}{2}}|B|^{\frac{1}{2}}\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}\Gamma(\gamma+\frac{q}{2})} = \frac{2^{2+\frac{3}{2}}2^{\frac{1}{2}}\Gamma(\frac{3}{2})}{\pi^{\frac{3}{2}}\Gamma(2+\frac{3}{2})} = \frac{2^6}{15\pi^{\frac{3}{2}}}. \tag{i}$$

The normalizing constant in (7.2.9) which will be denoted by c_9, is the same as c_8 excluding $|B|^{\frac{1}{2}} = 2^{\frac{1}{2}}$. Thus,

$$c_9 = \frac{2^{\frac{11}{2}}}{15\pi^{\frac{3}{2}}}. \tag{ii}$$

Note that for $X = [x_1, x_2, x_3]$, $XBX' = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ and $YY' = y_1^2 + y_2^2 + y_3^2$. Hence the densities $f_8(X)$ and $f_9(Y)$ are the following, where c_8 and c_9 are given in *(i)* and *(ii)*:

$$f_8(X) = c_8\Big[3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3\Big]^2 \mathrm{e}^{-2[3x_1^2+2x_2^2+x_3^2-2x_1x_2+2x_2x_3]}$$

for $-\infty < x_j < \infty,\ j = 1, 2, 3$, and

$$f_9(Y) = c_9 \left[y_1^2 + y_2^2 + y_3^2\right]^2 \mathrm{e}^{-2[y_1^2+y_2^2+y_3^2]},\ \text{ for } -\infty < y_j < \infty,\ j = 1, 2, 3.$$

This completes the computations.

7.2.3. Some properties of the rectangular matrix-variate gamma density

For the real rectangular matrix-variate gamma and Maxwell-Bolztmann distribution whose density is specified in (7.2.5), what might be the h-th moment of the determinant $|AXBX'|$ for an arbitrary h? This statistical quantity can be evaluated by looking at the normalizing constant c given in (7.2.4) since the integrand used to evaluate $E[|AXBX'|]^h$, where E denotes the expected value, is nothing but the density of X wherein γ is replaced by $\gamma + h$. Hence we have

$$E[|AXBX'|]^h] = \frac{\Gamma_p(\gamma + \frac{q}{2} + h)}{\Gamma_p(\gamma + \frac{q}{2})},\ \Re(h) > -\gamma - \frac{q}{2} + \frac{p-1}{2}. \tag{7.2.13}$$

In many calculations involving the Maxwell-Boltzmann density for the real scalar variable case x, one has to integrate a function of x, say $\nu(x)$, over the Maxwell-Boltzmann density, as can be seen for example in equations (4.1) and (4.2) of Mathai and Haubold (1988) in connection with a certain reaction-rate probability integral. Thus, the expression appearing in (7.2.13) corresponds to the integral of a power function over the Maxwell-Boltzmann density.

This arbitrary h-th moment expression also reveals an interesting point. By expanding the matrix-variate gamma functions, we have the following:

$$\frac{\Gamma_p(\gamma + \frac{q}{2} + h)}{\Gamma_p(\gamma + \frac{q}{2})} = \prod_{j=1}^{p} \frac{\Gamma(\gamma + \frac{q}{2} - \frac{j-1}{2} + h)}{\Gamma(\gamma + \frac{q}{2} - \frac{j-1}{2})} = \prod_{j=1}^{p} E(t_j)^h$$

where t_j is a real scalar gamma random variable with parameter $(\gamma + \frac{q}{2} - \frac{j-1}{2},\ 1),\ j = 1, \ldots, p$, whose density is

$$g_{(j)}(t_j) = \frac{1}{\Gamma(\gamma + \frac{q}{2} - \frac{(j-1)}{2})} t_j^{\gamma + \frac{q}{2} - \frac{(j-1)}{2} - 1} \mathrm{e}^{-t_j},\ t_j \geq 0,\ \gamma + \frac{q}{2} - \frac{(j-1)}{2} > 0, \tag{7.2.14}$$

and zero elsewhere. Thus structurally,

$$|AXBX'| = t_1 t_2 \cdots t_p \tag{7.2.15}$$

where $t_1, \ldots, t_p$ are independently distributed real scalar gamma random variables with t_j having the gamma density given in (7.2.14) for $j = 1, \ldots, p$.

7.2.4. Connection to the volume of a random parallelotope

First, observe that $|AXBX'| = |(A^{\frac{1}{2}}XB^{\frac{1}{2}})(A^{\frac{1}{2}}XB^{\frac{1}{2}})'| \equiv |UU'|$ where $U = A^{\frac{1}{2}}XB^{\frac{1}{2}}$. Then, note that U is $p\times q,\ q\ge p$, and of full rank p, and that the p linearly independent rows of U, taken in the order, will then create a convex hull and a parallelotope in the q-dimensional Euclidean space. The p rows of U represent p linearly independent vectors in the Euclidean q-space as well as p points in the same space. In light of (7.2.14), these random points are gamma distributed, that is, the joint density of the p vectors or the p random points is the real rectangular matrix-variate density given in (7.2.5), and the volume content of the parallelotope created by these p random points is $|AXBX'|^{\frac{1}{2}}$. Accordingly, (7.2.13) represents the $(2h)$-th moment of the random volume of the p-parallelotope generated by the p linearly independent rows of $A^{\frac{1}{2}}XB^{\frac{1}{2}}$. The geometrical probability problems considered in the literature usually pertain to random volumes generated by independently distributed isotropic random points, isotropic meaning that their associated density is invariant with respect to orthonormal transformations or rotations of the coordinate axes. For instance, the density given in (7.2.9) constitutes an example of isotropic form. The distributions of random geometrical configurations is further discussed in Chap. 4 of Mathai (1999).

7.2.5. Pathway to real matrix-variate gamma and Maxwell-Boltzmann densities

Consider a model of the following form for a $p\times q,\ q\ge p$, matrix X of full rank p:

$$f_{10}(X) = c_{10}|AXBX'|^{\gamma}|I - a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}},\ \alpha < 1, \tag{7.2.16}$$

for $A > O,\ B > O,\ a > 0,\ \eta > 0,\ I - a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} > O$ (positive definite), and $f_{10}(X) = 0$ elsewhere. It will be determined later that the parameter γ is subject to the condition $\gamma + \frac{q}{2} > \frac{p-1}{2}$. When $\alpha > 1$, we let $1-\alpha = -(\alpha-1),\ \alpha > 1$, so that the model specified in (7.2.16) shifts to the model

$$f_{11}(X) = c_{11}|AXBX'|^{\gamma}|I + a(\alpha-1)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{-\frac{\eta}{\alpha-1}},\ \alpha > 1 \tag{7.2.17}$$

for $\eta > 0,\ a > 0,\ A > O,\ B > O$, and $f_{11}(X) = 0$ elsewhere. Observe that $A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$ is symmetric as well as positive definite when X is of full rank p and $A > O,\ B > O$. For this model, the condition $\frac{\eta}{\alpha-1} - \gamma - \frac{q}{2} > \frac{p-1}{2}$ is required in addition to that applying to the parameter γ in (7.2.16). Note that when $f_{10}(X)$ and $f_{11}(X)$ are taken as statistical densities, c_{10} and c_{11} are the associated normalizing constants. Proceeding as in the evaluation of c in (7.2.4), we obtain the following representations for c_{10} and c_{11}:

$$c_{10}=|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}[a(1-\alpha)]^{p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}}\frac{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2})}{\Gamma_p(\gamma+\frac{q}{2})\Gamma_p(\frac{\eta}{1-\alpha}+\frac{p+1}{2})} \tag{7.2.18}$$

for $\eta>0,\ \alpha<1,\ a>0,\ A>O,\ B>O,\ \gamma+\frac{q}{2}>\frac{p-1}{2}$,

$$c_{11}=\frac{|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}[a(\alpha-1)]^{p(\gamma+\frac{q}{2})}\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}}\frac{\Gamma_p(\frac{\eta}{\alpha-1})}{\Gamma_p(\gamma+\frac{q}{2})\Gamma_p(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2})} \tag{7.2.19}$$

for $\alpha>1,\ \eta>0,\ a>0,\ A>O,\ B>O,\ \gamma+\frac{q}{2}>\frac{p-1}{2},\ \frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}>\frac{p-1}{2}$. When $\alpha\to 1_-$ in (7.2.18) and $\alpha\to 1_+$ in (7.2.19), the models (7.2.16) and (7.2.17) converge to the real rectangular matrix-variate gamma or Maxwell-Boltzmann density specified in (7.2.5). This can be established by applying the following lemmas.

Lemma 7.2.1.

$$\lim_{\alpha\to 1_-}|I-a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}}=\mathrm{e}^{-a\eta\,\mathrm{tr}(AXBX')}$$

and

$$\lim_{\alpha\to 1_+}|I+a(\alpha-1)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{-\frac{\eta}{\alpha-1}}=\mathrm{e}^{-a\eta\,\mathrm{tr}(AXBX')}. \tag{7.2.20}$$

***Proof*:** Letting $\lambda_1,\ldots,\lambda_p$ be the eigenvalues of the symmetric matrix $A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$, we have

$$|I-a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}}=\prod_{j=1}^{p}[1-a(1-\alpha)\lambda_j]^{\frac{\eta}{1-\alpha}}.$$

However, since

$$\lim_{\alpha\to 1_-}[1-a(1-\alpha)\lambda_j]^{\frac{\eta}{1-\alpha}}=\mathrm{e}^{-a\eta\lambda_j},$$

the product gives the sum of the eigenvalues, that is, $\mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}})$ in the exponent, hence the result. The same result can be similarly obtained for the case $\alpha>1$. We can also show that the normalizing constants c_{10} and c_{11} reduce to the normalizing constant in (7.2.4). This can be achieved by making use of an asymptotic expansion of gamma functions, namely,

$$\Gamma(z+\delta)\approx\sqrt{2\pi}\,z^{z+\delta-\frac{1}{2}}\,\mathrm{e}^{-z}\ \text{for } |z|\to\infty,\ \delta \text{ bounded}. \tag{7.2.21}$$

This first term approximation is also known as Stirling's formula.

Lemma 7.2.2.

$$\lim_{\alpha\to 1_-}[a(1-\alpha)]^{p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2})}{\Gamma_p(\frac{\eta}{1-\alpha}+\frac{p+1}{2})}=(a\eta)^{p(\gamma+\frac{q}{2})}$$

and

$$\lim_{\alpha\to 1_+}[a(\alpha-1)]^{p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\frac{\eta}{\alpha-1})}{\Gamma_p(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2})}=(a\eta)^{p(\gamma+\frac{q}{2})}. \tag{7.2.22}$$

Proof: On expanding $\Gamma_p(\cdot)$ using its definition, for $\alpha>1$, we have

$$\frac{[a(\alpha-1)]^{p(\gamma+\frac{q}{2})}\Gamma_p(\frac{\eta}{\alpha-1})}{\Gamma_p(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2})}=[a(\alpha-1)]^{p(\gamma+\frac{q}{2})}\prod_{j=1}^{p}\frac{\Gamma(\frac{\eta}{\alpha-1}-\frac{j-1}{2})}{\Gamma(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}-\frac{j-1}{2})}.$$

Now, on applying the Stirling's formula as given in (7.2.21) to each of the gamma functions by taking $z=\frac{\eta}{\alpha-1}\to\infty$ when $\alpha\to 1_+$, it is seen that the right-hand side of the above equality reduces to $(a\eta)^{p(\gamma+\frac{q}{2})}$. The result can be similarly established for the case $\alpha<1$.

This shows that c_{10} and c_{11} of (7.2.18) and (7.2.19) converge to the normalizing constant in (7.2.4). This means that the models specified in (7.2.16), (7.2.17), and (7.2.5) are all available from either (7.2.16) or (7.2.17) via the pathway parameter α. Accordingly, the combined model, either (7.2.16) or (7.2.17), is referred to as the pathway generalized real rectangular matrix-variate gamma density. The Maxwell-Boltzmann case corresponds to $\gamma=1$ and the Raleigh case, to $\gamma=\frac{1}{2}$. If either of the Maxwell-Boltzmann or Raleigh densities is the ideal or stable density in a physical system, then these stable densities as well as the unstable neighborhoods, described through the pathway parameter $\alpha<1$ and $\alpha>1$, and the transitional stages, are given by (7.2.16) or (7.2.17). The original pathway model was introduced in Mathai (2005).

For addressing other problems occurring in physical situations, one may have to integrate functions of X over the densities (7.2.16), (7.2.17) or (7.2.5). Consequently, we will evaluate an arbitrary h-th moment of $|AXBX'|$ in the models (7.2.16) and (7.2.17). For example, let us determine the h-th moment of $|AXBX'|$ with respect to the model specified in (7.2.16):

$$E[|AXBX'|^h]=c_{10}\int_X |AXBX'|^{\gamma+h}\,|I-a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}}\,\mathrm{d}X.$$

Note that the only change in the integrand, as compared to (7.2.16), is that γ is replaced by $\gamma+h$. Hence the result is available from the normalizing constant c_{10}, and the answer is the following:

$$E[|AXBX'|^h]=[a(1-\alpha)]^{-ph}\frac{\Gamma_p(\gamma+\frac{q}{2}+h)}{\Gamma_p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2})}{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2}+h)} \quad (7.2.23)$$

for $\Re(\gamma+\frac{q}{2}+h)>\frac{p-1}{2},\ a>0,\ \alpha<1$. Therefore

$$\begin{aligned}E[|a(1-\alpha)AXBX'|^h]&\\&=\frac{\Gamma_p(\gamma+\frac{q}{2}+h)}{\Gamma_p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2})}{\Gamma_p(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2}+h)}\\&=\prod_{j=1}^{p}\left\{\frac{\Gamma(\gamma+\frac{q}{2}-\frac{j-1}{2}+h)}{\Gamma(\gamma+\frac{q}{2}-\frac{j-1}{2})}\frac{\Gamma(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2}-\frac{j-1}{2})}{\Gamma(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+\frac{p+1}{2}-\frac{j-1}{2}+h)}\right\}\\&=\prod_{j=1}^{p}E\left(y_j^h\right)\end{aligned} \quad (7.2.24)$$

where y_j is a real scalar type-1 beta random variable with the parameters $(\gamma+\frac{q}{2}-\frac{j-1}{2},\ \frac{\eta}{1-\alpha}+\frac{p+1}{2}),\ j=1,\ldots,p$, the y_j's being mutually independently distributed. Hence, we have the structural relationship

$$|a(1-\alpha)AXBX'|=y_1\cdots y_p\,. \quad (7.2.25)$$

Proceeding the same way for the model (7.2.17), we have

$$E[|AXBX'|^h]=[a(\alpha-1)]^{-ph}\frac{\Gamma_p(\gamma+\frac{q}{2}+h)}{\Gamma_p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}-h)}{\Gamma_p(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2})} \quad (7.2.26)$$

for $\Re(\gamma+\frac{q}{2}+h)>\frac{p-1}{2},\ \Re(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}-h)>\frac{p-1}{2}$ or $-(\gamma+\frac{q}{2})+\frac{p-1}{2}<\Re(h)<\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}-\frac{p-1}{2}$. Thus,

$$\begin{aligned}E[|a(\alpha-1)AXBX'|^h]&=\prod_{j=1}^{p}\frac{\Gamma(\gamma+\frac{q}{2}-\frac{j-1}{2}+h)}{\Gamma(\gamma+\frac{q}{2}-\frac{j-1}{2})}\frac{\Gamma(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}-\frac{j-1}{2}-h)}{\Gamma(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2}-\frac{j-1}{2})}\\&=\prod_{j=1}^{p}E(z_j^h)\end{aligned} \quad (7.2.27)$$

where z_j is a real scalar type-2 beta random variable with the parameters $(\gamma + \frac{q}{2} - \frac{j-1}{2}, \frac{\eta}{\alpha-1} - \gamma - \frac{q}{2} - \frac{j-1}{2})$ for $j = 1, \ldots, p$, the z_j's being mutually independently distributed. Thus, for $\alpha > 1$, we have the structural representation

$$|a(\alpha - 1)AXBX'| = z_1 \cdots z_p\,. \tag{7.2.28}$$

As previously explained, one can consider the p linearly independent rows of $A^{\frac{1}{2}}XB^{\frac{1}{2}}$ as p vectors in the Euclidean q-space. Then, these p vectors are jointly distributed as rectangular matrix-variate type-2 beta, and $E[|AXBX'|^h] = E[|AXBX'|^{\frac{1}{2}}]^{2h}$ is the $(2h)$-th moment of the volume of the random parallelotope generated by these p q-vectors for $q > p$. In this case, the random points will be called type-2 beta distributed random points.

The real Maxwell-Boltzmann case will correspond $\gamma = 1$ and the Raleigh case, to $\gamma = \frac{1}{2}$, and all the above extensions and properties will apply to both of these distributions.

7.2.6. Multivariate gamma and Maxwell-Boltzmann densities, pathway model

Consider the density given in (7.2.16) for the case $p = 1$. In this instance, the $p \times p$ constant matrix A is 1×1 and we shall let $A = b > 0$, a positive real scalar quantity. Then for $\alpha < 1$, (7.2.16) reduces to the following where X is $1 \times q$ of the form $X = (x_1, \ldots, x_q),\ -\infty < x_j < \infty,\ j = 1, \ldots, q$:

$$\begin{aligned} f_{12}(X) = b^{\frac{q}{2}}|B|^{\frac{1}{2}}[a(1-\alpha)]^{(\gamma+\frac{q}{2})}\frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}}\frac{\Gamma(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+1)}{\Gamma(\gamma+\frac{q}{2})\Gamma(\frac{\eta}{1-\alpha}+1)}\\ \times [bXBX']^{\gamma}[1-a(1-\alpha)bXBX']^{\frac{\eta}{1-\alpha}} \end{aligned} \tag{7.2.29}$$

for $b > 0,\ B = B' > O,\ a > 0,\ \eta > 0,\ \gamma + \frac{q}{2} > 0,\ -\infty < x_j < \infty,\ j = 1, \ldots, q,\ 1 - a(1-\alpha)bXBX' > 0,\ \alpha < 1$, and $f_{12} = 0$ elsewhere. Note that

$$XBX' = (x_1, \ldots, x_q)B\begin{pmatrix} x_1\\ \vdots\\ x_q\end{pmatrix}$$

is a real quadratic form whose associated matrix B is positive definite. Letting the $1 \times q$ vector $Y = XB^{\frac{1}{2}}$, the density of Y when $\alpha < 1$ is given by

$$\begin{aligned} f_{13}(Y)\,\mathrm{d}Y = b^{\gamma+\frac{q}{2}}[a(1-\alpha)]^{(\gamma+\frac{q}{2})}\frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}}\frac{\Gamma(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+1)}{\Gamma(\gamma+\frac{q}{2})\Gamma(\frac{\eta}{1-\alpha}+1)}\\ \times [(y_1^2+\cdots+y_q^2)]^{\gamma}[1-a(1-\alpha)b(y_1^2+\cdots+y_q^2)]^{\frac{\eta}{1-\alpha}}\,\mathrm{d}Y, \end{aligned} \tag{7.2.30}$$

for $b > 0,\ \gamma + \frac{q}{2} > 0,\ \eta > 0,\ -\infty < y_j < \infty,\ j = 1, \ldots, q,\ 1 - a(1-\alpha)b(y_1^2 + \cdots + y_q^2) > 0$, and $f_{13} = 0$ elsewhere, which will be taken as the standard form of the real multivariate gamma density in its pathway generalized form, and for $\gamma = 1$, it will be the real pathway generalized form of the Maxwell-Boltzmann density in the standard multivariate case. For $\alpha > 1$, the corresponding standard form of the real multivariate gamma and Maxwell-Boltzmann densities is given by

$$f_{14}(Y)\mathrm{d}Y = b^{\gamma+\frac{q}{2}}[a(\alpha-1)]^{(\gamma+\frac{q}{2})}\frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}}\frac{\Gamma(\frac{\eta}{\alpha-1})}{\Gamma(\gamma+\frac{q}{2})\Gamma(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2})}$$
$$\times [(y_1^2+\cdots+y_q^2)]^{\gamma}[1+a(\alpha-1)b(y_1^2+\cdots+y_q^2)]^{-\frac{\eta}{\alpha-1}}\mathrm{d}Y. \qquad (7.2.31)$$

for $b > 0,\ \gamma + \frac{q}{2} > 0,\ a > 0,\ \eta > 0,\ \frac{\eta}{\alpha-1} - \gamma - \frac{q}{2} > 0,\ -\infty < y_j < \infty,\ j = 1, \ldots, q$, and $f_{14} = 0$ elsewhere. This will be taken as the pathway generalized real multivariate gamma density for $\alpha > 1$, and for $\gamma = 1$, it will be the standard form of the real pathway extended Maxwell-Boltzmann density for $\alpha > 1$. Note that when $\alpha \to 1_-$ in (7.2.30) and $\alpha \to 1_+$ in (7.2.31), we have

$$f_{15}(Y)\mathrm{d}Y = b^{\gamma+\frac{q}{2}}(a\eta)^{(\gamma+\frac{q}{2})}\frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}\Gamma(\gamma+\frac{q}{2})}$$
$$\times [(y_1^2+\cdots+y_q^2)]^{\gamma}\mathrm{e}^{-a\eta b(y_1^2+\cdots+y_q^2)}\mathrm{d}Y, \qquad (7.2.32)$$

for $b > 0,\ a > 0,\ \eta > 0,\ \gamma + \frac{q}{2} > 0$, and $f_{15} = 0$ elsewhere, which for $\gamma = 1$, is the real multivariate Maxwell-Bolzmann density in the standard form. From (7.2.30), (7.2.31), and thereby from (7.2.32), one can obtain the density of $u = y_1^2 + \cdots + y_q^2$, either by using the general polar coordinate transformation or the transformation of variables technique, that is, going from $\mathrm{d}Y$ to $\mathrm{d}S$ with $S = YY',\ Y$ being $1 \times p$. Then, the density of u for the case $\alpha < 1$ is

$$f_{16}(u) = b^{\gamma+\frac{q}{2}}[a(1-\alpha)]^{\gamma+\frac{q}{2}}\frac{\Gamma(\gamma+\frac{q}{2}+\frac{\eta}{1-\alpha}+1)}{\Gamma(\gamma+\frac{q}{2})\Gamma(\frac{\eta}{1-\alpha}+1)}u^{\gamma+\frac{q}{2}-1}[1-a(1-\alpha)bu]^{\frac{\eta}{1-\alpha}},\ \alpha < 1, \qquad (7.2.33)$$

for $b > 0,\ a > 0,\ \eta > 0,\ \alpha < 1,\ \gamma + \frac{q}{2} > 0,\ 1 - a(1-\alpha)bu > 0$, and $f_{16} = 0$ elsewhere, the density of u for $\alpha > 1$ being

$$f_{17}(u) = b^{\gamma+\frac{q}{2}}[a(\alpha-1)]^{\gamma+\frac{q}{2}}\frac{\Gamma(\frac{\eta}{\alpha-1})}{\Gamma(\gamma+\frac{q}{2})\Gamma(\frac{\eta}{\alpha-1}-\gamma-\frac{q}{2})}u^{\gamma+\frac{q}{2}-1}[1+a(\alpha-1)bu]^{-\frac{\eta}{\alpha-1}},\ \alpha > 1, \qquad (7.2.34)$$

for $b > 0,\ a > 0,\ \eta > 0,\ \gamma + \frac{q}{2} > 0,\ \frac{\eta}{\alpha - 1} - \gamma - \frac{q}{2} > 0,\ u \geq 0$, and $f_{17} = 0$ elsewhere. Observe that as $\alpha \to 1$, both (7.2.33) and (7.2.34) converge to the form

$$f_{18}(u) = \frac{(a\eta b)^{\gamma + \frac{q}{2}}}{\Gamma(\gamma + \frac{q}{2})} u^{\gamma + \frac{q}{2} - 1} \mathrm{e}^{-ab\eta u} \tag{7.2.35}$$

for $a > 0,\ b > 0,\ \eta > 0,\ u \geq 0$, and $f_{18} = 0$ elsewhere. For $\gamma = \frac{1}{2}$, we have the corresponding Raleigh cases.

Letting $\gamma = 1$ and $q = 1$ in (7.2.32), we have

$$\begin{aligned} f_{19}(y_1) &= \frac{b^{\frac{3}{2}}}{\Gamma(\gamma + \frac{1}{2})} (y_1^2)^{\gamma} \mathrm{e}^{-by_1^2} = \frac{2b^{\frac{3}{2}}}{\sqrt{\pi}} y_1^2 \mathrm{e}^{-by_1^2},\ -\infty < y_1 < \infty,\ b > 0 \\ &= \frac{4b^{\frac{3}{2}}}{\sqrt{\pi}} y_1^2 \mathrm{e}^{-by_1^2},\ 0 \leq y_1 < \infty,\ b > 0, \end{aligned} \tag{7.2.36}$$

and $f_{19} = 0$ elsewhere. This is the real Maxwell-Boltzmann case. For the Raleigh case, we let $\gamma = \frac{1}{2}$ and $p = 1,\ q = 1$ in (7.2.32), which results in the following density:

$$\begin{aligned} f_{20}(y_1) &= b(y_1^2)^{\frac{1}{2}} \mathrm{e}^{-by_1^2},\ -\infty < y_1 < \infty,\ b > 0 \\ &= 2b|y_1|\,\mathrm{e}^{-by_1^2},\ 0 \leq y_1 < \infty,\ b > 0, \end{aligned} \tag{7.2.37}$$

and $f_{20} = 0$ elsewhere.

7.2.7. Concluding remarks

There exist natural phenomena that are suspected to involve an underlying distribution which is not Maxwell-Boltzman but may be some deviation therefrom. In such instances, it is preferable to model the collected data by means of the pathway extended model previously specified for $p = 1,\ q = 1$ (real scalar case), $p = 1$ (real multivariate case) and the general matrix-variate case. The pathway parameter α will capture the Maxwell-Boltzmann case, the neighboring models described by the pathway model for $\alpha < 1$ and for $\alpha > 1$ and the transitional stages when moving from one family of functions to another, and thus, to all three different families of functions. Incidentally, for $\gamma = 0$, one has the rectangular matrix-variate Gaussian density given in (7.2.5) and its pathway extension in (7.2.16) and (7.2.17) or the general extensions in the standard forms in (7.2.30), (7.2.31), and (7.2.32) wherein $\gamma = 0$. The structures in (7.2.24), (7.2.27), and (7.2.28) suggest that the corresponding densities can also be written in terms of G- and H-functions. For the theory and applications of the G- and H-functions, the reader is referred to Mathai (1993) and Mathai et al. (2010), respectively. The complex analogues of some matrix-variate distributions, including the matrix-variate Gaussian, were introduced in Mathai and Provost (2006). Certain bivariate distributions are discussed in Balakrishnan and Lai (2009) and some general method of generating real multivariate distributions are presented in Marshall and Olkin (1967).

Example 7.2.4. Let $X = (x_{ij})$ be a real $p \times q,\ q \geq p$, matrix of rank p, where the x_{ij}'s are distinct real scalar variables. Let the constant matrices $A = b > 0$ be 1×1 and $B > O$ be $q \times q$. Consider the following generalized multivariate Maxwell-Boltzmann density

$$f(X) = c\,|AXBX'|^{\gamma}\,\mathrm{e}^{-[\mathrm{tr}(AXBX')]^{\delta}}$$

for $\delta > 0,\ A = b > 0,\ X = [x_1, \ldots, x_q]$. Evaluate c if $f(X)$ is a density.

Solution 7.2.4. Since X is $1 \times q$, $|AXBX'| = b[XBX']$ where XBX' is a real quadratic form. For $f(X)$ to be a density, we must have

$$1 = \int_X f(X)\mathrm{d}X = c\,b^{\gamma}\int_X [XBX']^{\gamma}\mathrm{e}^{-[(bXBX')]^{\delta}}\mathrm{d}X. \qquad (i)$$

Let us make the transformations $Y = XB^{\frac{1}{2}}$ and $s = YY'$. Then *(i)* reduces to the following:

$$1 = c\,b^{\gamma}\frac{\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}|B|^{-\frac{1}{2}}\int_0^{\infty} s^{\gamma+\frac{q}{2}-1}\mathrm{e}^{-(bs)^{\delta}}\mathrm{d}s. \qquad (ii)$$

Letting $t = b^{\delta}s^{\delta},\ b > 0,\ s > 0 \Rightarrow \mathrm{d}s = \frac{1}{\delta}\frac{t^{\frac{1}{\delta}-1}}{b}\mathrm{d}t$, *(ii)* becomes

$$\begin{aligned} 1 &= c\frac{\pi^{\frac{q}{2}}}{\delta\,b^{\frac{q}{2}}\Gamma(\frac{q}{2})|B|^{\frac{1}{2}}}\int_0^{\infty} t^{\frac{\gamma}{\delta}+\frac{q}{2\delta}-1}\mathrm{e}^{-t}\mathrm{d}t \\ &= c\frac{\pi^{\frac{q}{2}}\Gamma(\frac{\gamma}{\delta}+\frac{q}{2\delta})}{\delta\,b^{\frac{q}{2}}\Gamma(\frac{q}{2})|B|^{\frac{1}{2}}}. \end{aligned}$$

Hence,

$$c = \frac{\delta\,b^{\frac{q}{2}}\Gamma(\frac{q}{2})|B|^{\frac{1}{2}}}{\pi^{\frac{q}{2}}\Gamma(\frac{\gamma}{\delta}+\frac{q}{2\delta})}.$$

No additional conditions are required other than $\gamma > 0,\ \delta > 0,\ q > 0,\ B > O$. This completes the solution.

7.2a. Complex Matrix-Variate Gamma and Maxwell-Boltzmann Densities

The matrix-variate gamma density in the real positive definite matrix case was defined in equation (5.2.4) of Sect. 5.2. The corresponding matrix-variate gamma density in the complex domain was given in Eq. (5.2a.4). Those distributions will be extended to the rectangular matrix-variate cases in this section. A particular case of the rectangular matrix-variate gamma in the complex domain will be called the Maxwell-Boltzmann density in the complex matrix-variate case. Let $\tilde{X} = (\tilde{x}_{ij})$ be a $p \times q,\ q \ge p$, rectangular matrix of rank p in the complex domain whose elements $\tilde{x}_{ij}$ are distinct scalar complex variables. Let $|\mathrm{det}(\cdot)|$ denote the absolute value of the determinant of $(\cdot)$. Let A of order $p \times p$ and B of order $q \times q$ be real positive definite or Hermitian positive definite constant matrices. The conjugate transpose of $\tilde{X}$ will be denoted by $\tilde{X}^*$. Consider the function:

$$\tilde{f}(\tilde{X})\mathrm{d}\tilde{X} = \tilde{c}\,|\mathrm{det}(A\tilde{X}B\tilde{X}^*)|^{\gamma}\mathrm{e}^{-\mathrm{tr}(A\tilde{X}B\tilde{X}^*)}\mathrm{d}\tilde{X} \qquad (7.2a.1)$$

for $A > O,\ B > O,\ \Re(\gamma + q) > p - 1$ where $\tilde{c}$ is the normalizing constant so that $\tilde{f}(\tilde{X})$ is a statistical density. One can evaluate $\tilde{c}$ by proceeding as was done in the real case. Let

$$\tilde{Y} = A^{\frac{1}{2}}\tilde{X}B^{\frac{1}{2}} \Rightarrow \mathrm{d}\tilde{Y} = |\mathrm{det}(A)|^{q}|\mathrm{det}(B)|^{p}\mathrm{d}\tilde{X},$$

the Jacobian of this matrix transformation being provided in Chap. 1 or Mathai (1997). Then, $\tilde{f}(\tilde{X})$ becomes

$$\tilde{f}_1(\tilde{Y})\,\mathrm{d}\tilde{Y} = \tilde{c}\,|\mathrm{det}(A)|^{-q}|\mathrm{det}(B)|^{-p}|\mathrm{det}(\tilde{Y}\tilde{Y}^*)|^{\gamma}\mathrm{e}^{-\mathrm{tr}(\tilde{Y}\tilde{Y}^*)}\mathrm{d}\tilde{Y}. \qquad (7.2a.2)$$

Now, letting

$$\tilde{S} = \tilde{Y}\tilde{Y}^* \Rightarrow \mathrm{d}\tilde{Y} = \frac{\pi^{qp}}{\tilde{\Gamma}_p(q)}|\mathrm{det}(\tilde{S})|^{q-p}\mathrm{d}\tilde{S}$$

by applying Result 4.2*a*.3 where $\tilde{\Gamma}_p(q)$ is the complex matrix-variate gamma function, $\tilde{f}_1$ changes to

$$\tilde{f}_2(\tilde{S})\,\mathrm{d}\tilde{S} = \tilde{c}\,|\mathrm{det}(A)|^{-q}|\mathrm{det}(B)|^{-p}\frac{\pi^{qp}}{\tilde{\Gamma}_p(q)} \times |\mathrm{det}(\tilde{S})|^{\gamma+q-p}\mathrm{e}^{-\mathrm{tr}(\tilde{S})}\mathrm{d}\tilde{S}. \qquad (7.2a.3)$$

Finally, integrating out $\tilde{S}$ by making use of a complex matrix-variate gamma integral, we have $\tilde{\Gamma}_p(\gamma+q)$ for $\Re(\gamma+q) > p-1$. Hence, the normalizing constant $\tilde{c}$ is the following:

$$\tilde{c} = \frac{|\det(A)|^q |\det(B)|^p}{\pi^{qp}} \frac{\tilde{\Gamma}_p(q)}{\tilde{\Gamma}_p(\gamma+q)}, \ \Re(\gamma+q) > p-1, \ A > O, \ B > O. \quad (7.2a.4)$$

Example 7.2a.1. Evaluate the normalizing constant in the density in (7.2a.1) if $\gamma = 2, \ q = 3, \ p = 2,$

$$A = \begin{bmatrix} 3 & 1+i \\ 1-i & 2 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 0 & i \\ 0 & 2 & 1+i \\ -i & 1-i & 2 \end{bmatrix}.$$

Solution 7.2a.1. Note that A and B are both Hermitian matrices since $A = A^*$ and $B = B^*$. The leading minors of A are $|(3)| = 3 > 0$, $\begin{vmatrix} 3 & 1+i \\ 1-i & 2 \end{vmatrix} = (3)(2) - (1+i)(1-i) = 4 > 0$ and hence, $A > O$ (positive definite). The leading minors of B are $|(3)| = 3 > 0$, $\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6 > 0$, $|B| = 3\begin{vmatrix} 2 & 1+i \\ 1-i & 2 \end{vmatrix} + 0 + i\begin{vmatrix} 0 & 2 \\ -i & 1-i \end{vmatrix} = 3(4-2) + i(2i) = 4 > 0$. Hence $B > O$ and $|B| = 4$. The normalizing constant

$$\begin{aligned}\tilde{c} &= \frac{|\det(A)|^q |\det(B)|^p}{\pi^{pq}} \frac{\tilde{\Gamma}_p(q)}{\tilde{\Gamma}_p(\gamma+q)} \\ &= \frac{(4)^3(4)^2}{\pi^6} \frac{\tilde{\Gamma}_2(3)}{\tilde{\Gamma}_2(5)} = \frac{4^5}{\pi^6} \frac{\pi\Gamma(3)\Gamma(2)}{\pi\Gamma(5)\Gamma(4)} \\ &= \frac{2^7}{3^2\pi^6}.\end{aligned}$$

This completes the computations.

7.2a.1. Extension of the Matrix-Variate Gamma Density in the Complex Domain

Consider the density of $\tilde{X}$ is given in (7.2a.1) with $\tilde{c}$ given in (7.2a.4). The density of $\tilde{Y} = A^{\frac{1}{2}} \tilde{X} B^{\frac{1}{2}}$ is given by

$$f_1(\tilde{Y}) = \frac{\tilde{\Gamma}_p(q)}{\pi^{qp}\tilde{\Gamma}_p(\gamma+q)} |\det(\tilde{Y}\tilde{Y}^*)|^{\gamma} \mathrm{e}^{-\mathrm{tr}(\tilde{Y}\tilde{Y}^*)} \quad (7.2a.5)$$

for $\Re(\gamma+q) > p-1$, and $\tilde{f}_1 = 0$ elsewhere, and the density of $\tilde{S} = \tilde{Y}\tilde{Y}^*$ is

$$\tilde{f}_2(\tilde{S}) = \frac{1}{\tilde{\Gamma}_p(\gamma+q)}|\det(\tilde{S})|^{\gamma+q-p}\mathrm{e}^{-\mathrm{tr}(\tilde{S})},\ \Re(\gamma+q) > p-1, \tag{7.2a.6}$$

and $\tilde{f}_2 = 0$ elsewhere. Then, the density given in (7.2*a*.1), namely, $\tilde{f}(\tilde{X})$ for $\gamma = 1$ will be called the Maxwell-Boltzmann density for the complex rectangular matrix-variate case since for $p = 1$ and $q = 1$ in the real scalar case, the density corresponds to the case $\gamma = 1$, and (7.2*a*.1) for $\gamma = \frac{1}{2}$ will be called complex rectangular matrix-variate Raleigh density.

7.2a.2. The multivariate gamma density in the complex matrix-variate case

Consider the case $p = 1$ and $A = b > 0$ where b is a real positive scalar as the $p \times p$ matrix A is assumed to be Hermitian positive definite. Then, $\tilde{X}$ is $1 \times q$ and

$$A\tilde{X}B\tilde{X}^* = b\tilde{X}B\tilde{X}^* = b(\tilde{x}_1, \ldots, \tilde{x}_q)B\begin{pmatrix}\tilde{x}_1^*\\ \vdots\\ \tilde{x}_q^*\end{pmatrix}$$

is a positive definite Hermitian form, an asterisk denoting only the conjugate when the elements are scalar quantities. Thus, when $p = 1$ and $A = b > 0$, the density $\tilde{f}(\tilde{X})$ reduces to

$$\tilde{f}_3(\tilde{X}) = b^{\gamma+q}|\det(B)|\frac{\tilde{\Gamma}(q)}{\pi^q\tilde{\Gamma}(\gamma+q)}|\det(\tilde{X}B\tilde{X}^*)|^{\gamma}\mathrm{e}^{-b(\tilde{X}B\tilde{X}^*)} \tag{7.2a.7}$$

for $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_q)$, $B = B^* > O$, $b > 0$, $\Re(\gamma+q) > 0$, and $\tilde{f}_3 = 0$ elsewhere. Letting $\tilde{Y}^* = B^{\frac{1}{2}}\tilde{X}^*$, the density of $\tilde{Y}$ is the following:

$$\tilde{f}_4(\tilde{Y}) = \frac{b^{\gamma+q}\tilde{\Gamma}(q)}{\pi^q\tilde{\Gamma}(\gamma+q)}(|\tilde{y}_1|^2 + \cdots + |\tilde{y}_q|^2)^{\gamma}\mathrm{e}^{-b(|\tilde{y}_1|^2+\cdots+|\tilde{y}_q|^2)} \tag{7.2a.8}$$

for $b > 0, \Re(\gamma+q) > 0$, and $\tilde{f}_4 = 0$ elsewhere, where $|\tilde{y}_j|$ is the absolute value or modulus of the complex quantity $\tilde{y}_j$. We will take (7.2*a*.8) as the complex multivariate gamma density; when $\gamma = 1$, it will be referred to as the complex multivariate Maxwell-Boltzmann density, and when $\gamma = \frac{1}{2}$, it will be called complex multivariate Raleigh density. These densities are believed to be new.

Let us verify by integration that (7.2*a*.8) is indeed a density. First, consider the transformation $s = \tilde{Y}\tilde{Y}^*$. In view of Theorem 4.2a.3, the integral over the Stiefel manifold gives $\mathrm{d}\tilde{Y} = \frac{\pi^q}{\tilde{\Gamma}(q)}\tilde{s}^{q-1}\mathrm{d}\tilde{s}$, so that $\frac{\pi^q}{\tilde{\Gamma}(q)}$ is canceled. Then, the integral over $\tilde{s}$ yields

$b^{-(\gamma+q)}\tilde{\Gamma}(\gamma+q),\ \Re(\gamma+q)>0$, and hence it is verified that (7.2*a*.8) is a statistical density.

7.2a.3. Arbitrary moments, complex case

Let us determine the h-th moment of $u=|\det(A\tilde{X}B\tilde{X}^*)|$ for an arbitrary h, that is, the h-th moment of the absolute value of the determinant of the matrix $A\tilde{X}B\tilde{X}^*$ or its symmetric form $A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}}$, which is

$$E[|\det(A\tilde{X}B\tilde{X}^*)|^h=\tilde{c}\int_{\tilde{X}}|\det(A\tilde{X}B\tilde{X})|^{h+\gamma}\mathrm{e}^{-\mathrm{tr}(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})}\mathrm{d}\tilde{X}. \tag{7.2a.9}$$

Observe that the only change, as compared to the total integral, is that γ is replaced by $\gamma+h$, so that the h-th moment is available from the normalizing constant $\tilde{c}$. Accordingly,

$$E[u^h]=\frac{\tilde{\Gamma}_p(\gamma+q+h)}{\tilde{\Gamma}_p(\gamma+q)},\ \Re(\gamma+q+h)>p-1, \tag{7.2a.10}$$

$$=\prod_{j=1}^{p}\frac{\Gamma(\gamma+q+h-(j-1))}{\Gamma(\gamma+q-(j-1))}$$

$$=E(u_1^h)E(u_2^h)\cdots E(u_p^h) \tag{7.2a.11}$$

where the u_j's are independently distributed real scalar gamma random variables with parameters $(\gamma+q-(j-1),\ 1),\ j=1,\ldots,p$. Thus, structurally $u=|\det(A\tilde{X}B\tilde{X}^*)|$ is a product of independently distributed real scalar gamma random variables with parameters $(\gamma+q-(j-1),\ 1),\ \ j=1,\ldots,p$. The corresponding result in the real case is that $|AXBX'|$ is structurally a product of independently distributed real gamma random variables with parameters $(\gamma+\frac{q}{2}-\frac{j-1}{2},\ 1),\ \ j=1,\ldots,p$, which can be seen from (7.2.15).

7.2a.4. A pathway extension in the complex case

A pathway extension is also possible in the complex case. The results and properties are parallel to those obtained in the real case. Hence, we will only mention the pathway extended density. Consider the following density:

$$\tilde{f}_5(\tilde{X})=\tilde{c}_1|\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|^{\gamma}|\det(I-a(1-\alpha)A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|^{\frac{\eta}{1-\alpha}} \tag{7.2a.12}$$

for $a>0,\ \alpha<1,\ I-a(1-\alpha)A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}}>O$ (positive definite), $A>O,\ B>O,\ \eta>0,\ \Re(\gamma+q)>p-1$, and $\tilde{f}_5=0$ elsewhere. Observe that (7.2*a*.12) remains in the generalized type-1 beta family of functions for $\alpha<1$ (type-1 and type-2 beta densities in the complex rectangular matrix-variate cases will be considered in the next sections). If $\alpha > 1$, then on writing $1-\alpha = -(\alpha-1),\ \alpha > 1$, the model in (7.2*a*.12) shifts to the model

$$\tilde{f}_6(\tilde{X}) = \tilde{c}_2|\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|^{\gamma}|\det(I + a(\alpha-1)A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|^{-\frac{\eta}{\alpha-1}} \tag{7.2a.13}$$

for $a > 0,\ \alpha > 1,\ A > O,\ B > O,\ A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}} > O,\ \eta > 0,\ \Re(\frac{\eta}{\alpha-1} - \gamma - q) > p-1,\ \Re(\gamma+q) > p-1$, and $\tilde{f}_6 = 0$ elsewhere, where $\tilde{c}_2$ is the normalizing constant, different from $\tilde{c}_1$. When $\alpha \to 1$, both models (7.2*a*.12) and (7.2*a*.13) converge to the model $\tilde{f}_7$ where

$$\tilde{f}_7(\tilde{X}) = \tilde{c}_3|\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|^{\gamma}\mathrm{e}^{-a\,\eta\,\mathrm{tr}(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})} \tag{7.2a.14}$$

for $a > 0,\ \eta > 0,\ A > O,\ B > O,\ \Re(\gamma+q) > p-1,\ A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}} > O$, and $\tilde{f}_7 = 0$ elsewhere. The normalizing constants can be evaluated by following steps parallel to those used in the real case. They respectively are:

$$\tilde{c}_1 = |\det(A)|^q|\det(B)|^p[a(1-\alpha)]^{p(\gamma+q)}\frac{\tilde{\Gamma}_p(q)}{\pi^{qp}}\frac{\tilde{\Gamma}_p(\gamma+q+\frac{\eta}{1-\alpha}+p)}{\tilde{\Gamma}_p(\gamma+q)\tilde{\Gamma}_p(\frac{\eta}{1-\alpha}+p)} \tag{7.2a.15}$$

for $\eta > 0,\ a > 0,\ \alpha < 1,\ A > O,\ B > O,\ \Re(\gamma+q) > p-1$;

$$\tilde{c}_2 = |\det(A)|^q|\det(B)|^p[a(\alpha-1)]^{p(\gamma+q)}\frac{\tilde{\Gamma}_p(q)}{\pi^{qp}}\frac{\tilde{\Gamma}_p(\frac{\eta}{\alpha-1})}{\tilde{\Gamma}_p(\gamma+q)\tilde{\Gamma}_p(\frac{\eta}{\alpha-1}-\gamma-q)} \tag{7.2a.16}$$

for $a > 0,\ \alpha > 1,\ \eta > 0,\ A > O,\ B > O,\ \Re(\gamma+q) > p-1,\ \Re(\frac{\eta}{\alpha-1}-\gamma-q) > p-1$;

$$\tilde{c}_3 = (a\eta)^{p(\gamma+q)}|\det(A)|^q|\det(B)|^p\frac{\tilde{\Gamma}_p(q)}{\pi^{qp}\tilde{\Gamma}_p(\gamma+q)} \tag{7.2a.17}$$

for $a > 0,\ \eta > 0,\ A > O,\ B > O,\ \Re(\gamma+q) > p-1$.

7.2a.5. The Maxwell-Boltzmann and Raleigh cases in the complex domain

The complex counterparts of the Maxwell-Boltzmann and Raleigh cases may not be available in the literature. Their densities can be derived from (7.2*a*.8). Letting $p = 1$ and $q = 1$ in (7.2*a*.8), we have

$$\begin{aligned}\tilde{f}_8(\tilde{y}_1) &= \frac{b^{\gamma+1}}{\pi\tilde{\Gamma}(\gamma+1)}[|\tilde{y}_1|^2]^{\gamma}\mathrm{e}^{-b|\tilde{y}_1|^2},\ \tilde{y}_1 = y_{11}+iy_{12},\ i=\sqrt{(-1)},\\ &= \frac{b^{\gamma+1}}{\pi\tilde{\Gamma}(\gamma+1)}[y_{11}^2+y_{12}^2]^{\gamma}\mathrm{e}^{-b(y_{11}^2+y_{12}^2)}\end{aligned} \tag{7.2a.18}$$

for $b>0,\ \Re(\gamma+1)>0,\ -\infty<y_{1j}<\infty,\ j=1,2$, and $\tilde{f}_8=0$ elsewhere. We may take $\gamma=1$ as the Maxwell-Boltzmann case and $\gamma=\frac{1}{2}$ as the Raleigh case. Then, for $\gamma=1$, we have

$$\tilde{f}_9(\tilde{y}_1)=\frac{b^2}{\pi}|\tilde{y}_1|^2\mathrm{e}^{-b|\tilde{y}_1|^2},\ \tilde{y}_1=y_{11}+iy_{12}$$

for $b>0,\ -\infty<y_{1j}<\infty,\ j=1,2$, and $\tilde{f}_9=0$ elsewhere. Note that in the real case $y_{12}=0$ so that the functional part of $\tilde{f}_6$ becomes $y_{11}^2\mathrm{e}^{-by_{11}^2},\ -\infty<y_{11}<\infty$. However, the normalizing constants in the real and complex cases are evaluated in different domains. Observe that, corresponding to (7.2a.18), the normalizing constant in the real case is $b^{\gamma+\frac{1}{2}}/[\pi^{\frac{1}{2}}\Gamma(\gamma+\frac{1}{2})]$. Thus, the normalizing constant has to be evaluated separately. Consider the integral

$$\int_{-\infty}^{\infty}y_{11}^2\mathrm{e}^{-by_{11}^2}\mathrm{d}y_{11}=2\int_0^{\infty}y_{11}^2\mathrm{e}^{-by_{11}^2}\mathrm{d}y_{11}=\int_0^{\infty}u^{\frac{3}{2}-1}\mathrm{e}^{-bu}\mathrm{d}u=\frac{\sqrt{\pi}}{2b^{\frac{3}{2}}}.$$

Hence,

$$\begin{aligned}f_{10}(y_{11})&=\frac{2b^{\frac{3}{2}}}{\sqrt{\pi}}y_{11}^2\mathrm{e}^{-by_{11}^2},\ -\infty<y_{11}<\infty,\ b>0,\\&=\frac{4b^{\frac{3}{2}}}{\sqrt{\pi}}y_{11}^2\mathrm{e}^{-by_{11}^2},\ 0\le y_{11}<\infty,\ b>0,\end{aligned}\tag{7.2a.19}$$

and $f_{10}=0$ elsewhere. This is the real Maxwell-Boltzmann case. For the Raleigh case, letting $\gamma=\frac{1}{2}$ in (7.2a.18) yields

$$\begin{aligned}\tilde{f}_{11}(\tilde{y}_1)&=\frac{b^{\frac{3}{2}}}{\pi\Gamma(\frac{3}{2})}[|\tilde{y}_1|^2]^{\frac{1}{2}}\mathrm{e}^{-b(|\tilde{y}_1|^2)},\ b>0,\\&=\frac{2b^{\frac{3}{2}}}{\pi^{\frac{3}{2}}}[y_{11}^2+y_{12}^2]^{\frac{1}{2}}\mathrm{e}^{-b(y_{11}^2+y_{12}^2)},\ -\infty<y_{1j}<\infty,\ j=1,2,\ b>0,\end{aligned}\tag{7.2a.20}$$

and $\tilde{f}_{11}=0$ elsewhere. Then, for $y_{12}=0$, the functional part of $\tilde{f}_{11}$ is $|y_{11}|\,\mathrm{e}^{-by_{11}^2}$ with $-\infty<y_{11}<\infty$. The integral over y_{11} gives

$$\int_{-\infty}^{\infty}|y_{11}|\,\mathrm{e}^{-by_{11}^2}\mathrm{d}y_{11}=2\int_0^{\infty}y_{11}\,\mathrm{e}^{-by_{11}^2}\mathrm{d}y_{11}=b^{-1}.$$

Thus, in the Raleigh case,

$$f_{12}(y_{11}) = b\,|y_{11}|\,\mathrm{e}^{-by_{11}^2},\ \ -\infty < y_{11} < \infty,\ b > 0,$$
$$= 2\,b\,y_{11}\,\mathrm{e}^{-by_{11}^2},\ 0 \le y_{11} < \infty,\ b > 0, \tag{7.2a.21}$$

and $f_{12} = 0$ elsewhere. The normalizing constant in (7.2a.18) can be verified by making use of the polar coordinate transformation: $y_{11} = r\cos\theta,\ \ y_{12} = r\sin\theta$, so that $\mathrm{d}y_{11} \wedge \mathrm{d}y_{12} = r\,\mathrm{d}r \wedge \mathrm{d}\theta,\ 0 < \theta \le 2\pi,\ 0 \le r < \infty$. Then,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[y_{11}^2 + y_{12}^2]^{\gamma}\mathrm{e}^{-(y_{11}^2+y_{12}^2)}\mathrm{d}y_{11}\wedge\mathrm{d}y_{12} = 2\pi\int_0^{\infty}(r^2)^{\gamma}\mathrm{e}^{-br^2}r\mathrm{d}r$$
$$= \pi\,b^{-(\gamma+1)}\,\Gamma(\gamma+1)$$

for $b > 0,\ \Re(\gamma+1) > 0$.

Exercises 7.2

7.2.1. Supply a proof to (7.2.9) by using Theorem 4.2.3.

7.2.2. Derive the exact density of the determinant in (7.2.15) for $p = 2$.

7.2.3. Verify the results in (7.2.18) and (7.2.19).

7.2.4. Derive the normalizing constants $\tilde{c}_1$ in $(7.2a.12)$ and $\tilde{c}_2$ in $(7.2a.13)$.

7.2.5. Derive $\tilde{c}_3$ in $(7.2a.14)$ by integrating out over $\tilde{X}$.

7.2.6. Approximate $\tilde{c}_1$ and $\tilde{c}_2$ of Exercise 7.2.4 by making use of Stirling's approximation, and then show that the result agrees with that in Exercise 7.2.5.

7.2.7. Derive (state and prove) for the complex case the lemmas corresponding to Lemmas 7.2.1 and 7.2.2.

7.3. Real Rectangular Matrix-Variate Type-1 and Type-2 Beta Densities

Let us begin with the real case. Let $A > O$ be $p\times p$ and $B > O$ be $q\times q$ where A and B are real constant matrices. Let $X = (x_{ij})$ be a $p\times q,\ q \ge p$, matrix of distinct real scalar variables x_{ij}'s as its elements, X being of full rank p. Then, $A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} > O$ is real positive definite where $A^{\frac{1}{2}}$ is the positive definite square root of the positive definite matrix A. Let $|(\cdot)|$ represent the determinant of $(\cdot)$ when $(\cdot)$ is real or complex, and $|\det(\cdot)|$

be the absolute value of the determinant of $(\cdot)$ when $(\cdot)$ is in the complex domain. Consider the following density:

$$g_1(X) = C_1|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\gamma}|I - A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\beta-\frac{p+1}{2}} \tag{7.3.1}$$

for $A > O,\ B > O,\ I - A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} > O,\ \Re(\beta) > \frac{p-1}{2},\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}$, and $g_1 = 0$ elsewhere, where C_1 is the normalizing constant. Accordingly, $U = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} > O$ and $I-U > O$ or U and $I-U$ are both positive definite. We now make the transformations $Y = A^{\frac{1}{2}}XB^{\frac{1}{2}}$ and $S = YY'$. Then, proceeding as in the case of the rectangular matrix-variate gamma density discussed in Sect. 7.2, and evaluating the final part involving S with the help of a real positive definite matrix-variate type-1 beta integral, we obtain the following normalizing constant:

$$C_1 = |A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\frac{\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}}\frac{\Gamma_p(\gamma + \frac{q}{2} + \beta)}{\Gamma_p(\beta)\Gamma_p(\gamma + \frac{q}{2})} \tag{7.3.2}$$

for $A > O,\ B > O,\ \Re(\beta) > \frac{p-1}{2},\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}$. Usually the parameters associated with a statistical density are real, which is the case for γ and β. Nonetheless, the conditions will be stated for general complex parameters. When the density of X is as given in g_1, the density of $Y = A^{\frac{1}{2}}XB^{\frac{1}{2}}$ is given by

$$g_2(Y) = \frac{\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}}\frac{\Gamma_p(\gamma + \frac{q}{2} + \beta)}{\Gamma_p(\beta)\Gamma_p(\gamma + \frac{q}{2})}|YY'|^{\gamma}|I - YY'|^{\beta-\frac{p+1}{2}} \tag{7.3.3}$$

for $\Re(\gamma + \frac{q}{2}) > \frac{p-1}{2},\ YY' > O,\ I - YY' > O$, and $g_2 = 0$ elsewhere. When X has the density specified in (7.3.1), the density of $S = YY'$ is given by

$$g_3(S) = \frac{\Gamma_p(\gamma + \frac{q}{2} + \beta)}{\Gamma_p(\beta)\Gamma_p(\gamma + \frac{q}{2})}|S|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}|I - S|^{\beta-\frac{p+1}{2}} \tag{7.3.4}$$

for $\Re(\beta) > \frac{p-1}{2},\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2},\ S > O,\ I - S > O$, and $g_3 = 0$ elsewhere, which is the usual real matrix-variate type-1 beta density. Observe that the density $g_1(X)$ is also available from the pathway form of the real matrix-variate gamma case introduced in Sect. 7.2.

Example 7.3.1. Let $U_1 = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}},\ U_2 = XBX',\ U_3 = B^{\frac{1}{2}}X'AXB^{\frac{1}{2}}$ and $U_4 = X'AX$. If X has the rectangular matrix-variate type-1 beta density given in (7.3.1), evaluate the densities of $U_1,\ U_2,\ U_3$ and U_4 whenever possible.

Solution 7.3.1. The matrix U_1 is already present in the density of X, namely (7.3.1). Now, we have to convert the density of X into the density of U_1. Consider the transformations $Y = A^{\frac{1}{2}}XB^{\frac{1}{2}},\ S = YY' = U_1$ and the density of S is given in (7.3.4). Thus, U_1 has a real matrix-variate type-1 beta density with the parameters $\gamma + \frac{q}{2}$ and β. Now, on applying the same transformations as above with $A = I$, the density appearing in (7.3.4), which is the density of U_2, becomes

$$g_3(U_2)\mathrm{d}U_2 = \frac{\Gamma_p(\gamma+\frac{q}{2}+\beta)}{\Gamma_p(\gamma+\frac{q}{2})\Gamma(\beta)}|A|^{\gamma+\frac{q}{2}}|U_2|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}|I-AU_2|^{\beta-\frac{p+1}{2}}\mathrm{d}U_2 \qquad (i)$$

for $\Re(\beta) > \frac{p-1}{2},\ \Re(\gamma+\frac{q}{2}) > \frac{p-1}{2},\ A > O,\ U_2 > O,\ I - A^{\frac{1}{2}}U_2A^{\frac{1}{2}} > O$, and zero elsewhere, so that U_2 has a scaled real matrix-variate type-1 beta distribution with parameters $(\gamma+\frac{q}{2},\ \beta)$ and scaling matrix $A > O$. For $q > p$, both $X'AX$ and $B^{\frac{1}{2}}X'AXB^{\frac{1}{2}}$ are positive semi-definite matrices whose determinants are thus equal to zero. Accordingly, the densities do not exist whenever $q > p$. When $q = p$, U_3 has a $q \times q$ real matrix-variate type-1 beta distribution with parameters $(\gamma+\frac{p}{2},\ \beta)$ and U_4 is a scaled version of a type-1 beta matrix variable whose density is of the form given in *(i)* wherein B is the scaling matrix and p and q are interchanged. This completes the solution.

7.3.1. Arbitrary moments

The h-th moment of the determinant of $U = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$ with h being arbitrary, is available from the normalizing constant given in (7.3.2) on observing that when the h-th moment is taken, the only change is that γ turns into $\gamma + h$. Thus,

$$\begin{aligned} E[|U|^h] &= E[|YY'|^h] = E[|S|^h] \\ &= \frac{\Gamma_p(\gamma+\frac{q}{2}+h)}{\Gamma_p(\gamma+\frac{q}{2})}\frac{\Gamma_p(\gamma+\frac{q}{2}+\beta)}{\Gamma_p(\gamma+\frac{q}{2}+\beta+h)} &(7.3.5)\\ &= \prod_{j=1}^{p}\frac{\Gamma(\gamma+\frac{q}{2}-\frac{j-1}{2}+h)}{\Gamma(\gamma+\frac{q}{2}-\frac{j-1}{2})}\frac{\Gamma(\gamma+\frac{q}{2}+\beta-\frac{j-1}{2})}{\Gamma(\gamma+\frac{q}{2}+\beta-\frac{j-1}{2}+h)} &(7.3.6)\\ &= E[u_1^h]E[u_2^h]\cdots E[u_p^h] &(7.3.7)\end{aligned}$$

where $u_1,\ldots,u_p$ are mutually independently distributed real scalar type-1 beta random variables with the parameters $(\gamma+\frac{q}{2}-\frac{j-1}{2},\ \beta),\ j = 1,\ldots,p$, provided $\Re(\beta) > \frac{p-1}{2}$ and $\Re(\gamma+\frac{q}{2}) > \frac{p-1}{2}$.

7.3.2. Special case: $p = 1$

For the case $p = 1$, let the positive definite 1×1 matrix A be the scalar $b > 0$ and X which is $1 \times q$, be equal to $(x_1, \ldots, x_q)$. Then,

$$AXBX' = b(x_1, \ldots, x_q)B \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}$$

is a real quadratic form, the matrix of the quadratic form being $B > O$. Letting $Y = XB^{\frac{1}{2}}$, $\mathrm{d}Y = |B|^{\frac{1}{2}}\mathrm{d}X$, and the density of Y, denoted by $g_4(Y)$, is then given by

$$\begin{aligned} g_4(Y) &= b^{\gamma+\frac{q}{2}} \frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}} \frac{\Gamma(\gamma + \frac{q}{2} + \beta)}{\Gamma(\gamma + \frac{q}{2})\Gamma(\beta)} |YY'|^{\gamma} \\ &\quad \times |I - bYY'|^{\beta-1},\ YY' > O,\ I - bYY' > O,\ \Re(\gamma + \frac{q}{2}) > 0,\ \Re(\beta) > 0 \\ &= b^{\gamma+\frac{q}{2}} \frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}} \frac{\Gamma(\gamma + \frac{q}{2} + \beta)}{\Gamma(\gamma + \frac{q}{2})\Gamma(\beta)} [y_1^2 + \cdots + y_q^2]^{\gamma} \\ &\quad \times [1 - b(y_1^2 + \cdots + y_q^2)]^{\beta-1},\ Y = (y_1, \ldots, y_q), \end{aligned} \tag{7.3.8}$$

for $b > 0$, $\Re(\gamma + 1) > 0$, $\Re(\beta) > 0$, $1 - b(y_1^2 + \ldots + y_q^2) > 0$, and $g_4 = 0$ elsewhere. The form of the density in (7.3.8) presents some interest as it appears in various areas of research. In reliability studies, a popular model for the lifetime of components corresponds to (7.3.8) wherein $\gamma = 0$ in both the scalar and multivariate cases. When independently distributed isotropic random points are considered in connection with certain geometrical probability problems, a popular model for the distribution of the random points is the type-1 beta form or (7.3.8) for $\gamma = 0$. Earlier results obtained assuming that $\gamma = 0$ and the new case where $\gamma \neq 0$ in geometrical probability problems are discussed in Chapter 4 of Mathai (1999). We will take (7.3.8) as the standard form of the real rectangular matrix-variate type-1 beta density for the case $p = 1$ in a $p \times q$, $q \geq p$, real matrix X of rank p. For verifying the normalizing constant in (7.3.8), one can apply Theorem 4.2.3. Letting $S = YY'$, $\mathrm{d}Y = \frac{\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}|S|^{\frac{q}{2}-1}\mathrm{d}S$, which once substituted to $\mathrm{d}Y$ in (7.3.8) yields a total integral equal to one upon integrating out S with the help of a real matrix-variate type-1 beta integral (in this case a real scalar type-1 beta integral); accordingly, the constant part in (7.3.8) is indeed the normalizing constant. In this case, the density of $S = YY'$ is given by

$$g_5(S) = b^{\gamma+\frac{q}{2}} \frac{\Gamma(\gamma + \frac{q}{2} + \beta)}{\Gamma(\gamma + \frac{q}{2})\Gamma(\beta)} |S|^{\gamma+\frac{q}{2}-1} |I - bS|^{\beta-1} \tag{7.3.9}$$

for $S > O,\ b > 0,\ \Re(\gamma + \frac{q}{2}) > 0,\ \Re(\beta) > 0$, and $g_5 = 0$ elsewhere. Observe that this S is actually a real scalar variable.

As obtained from (7.3.8), the type-1 beta form of the density in the real scalar case, that is, for $p = 1$ and $q = 1$, is

$$g_6(y_1) = b^{\gamma+\frac{1}{2}} \frac{\Gamma(\gamma + \frac{1}{2} + \beta)}{\Gamma(\gamma + \frac{1}{2})\Gamma(\beta)} [y_1^2]^{\gamma} [1 - b y_1^2]^{\beta-1}, \tag{7.3.10}$$

for $b > 0,\ \beta > 0,\ \gamma + \frac{1}{2} > 0,\ -\frac{1}{\sqrt{b}} < y_1 < \frac{1}{\sqrt{b}}$, and $g_6 = 0$ elsewhere. When the support is $0 < y_1 < \frac{1}{\sqrt{b}}$, the above density which is symmetric, is multiplied by two.

7.3a. Rectangular Matrix-Variate Type-1 Beta Density, Complex Case

Consider the following function:

$$\tilde{g}_1(\tilde{X}) = \tilde{C}_1 |\det(A^{\frac{1}{2}} \tilde{X} B \tilde{X}^* A^{\frac{1}{2}})|^{\gamma} |\det(I - A^{\frac{1}{2}} \tilde{X} B \tilde{X}^* A^{\frac{1}{2}})|^{\beta-p} \tag{7.3a.1}$$

for $A > O,\ B > O,\ \Re(\beta) > p - 1,\ \Re(\gamma + q) > p - 1,\ I - A^{\frac{1}{2}} \tilde{X} B \tilde{X}^* A^{\frac{1}{2}} > O$, and $\tilde{g}_1 = 0$ elsewhere. The normalizing constant can be evaluated by proceeding as in the real rectangular matrix-variate case. Let $\tilde{Y} = A^{\frac{1}{2}} \tilde{X} B^{\frac{1}{2}}$ so that $\tilde{S} = \tilde{Y}\tilde{Y}^*$, and then integrate out $\tilde{S}$ by using a complex matrix-variate type-1 beta integral, which yields the following normalizing constant:

$$\tilde{C}_1 = |\det(A)|^q |\det(B)|^p \frac{\tilde{\Gamma}_p(q)}{\pi^{qp}} \frac{\tilde{\Gamma}_p(\gamma + q + \beta)}{\tilde{\Gamma}_p(\gamma + q)\tilde{\Gamma}_p(\beta)} \tag{7.3a.2}$$

for $\Re(\gamma + q) > p - 1,\ \Re(\beta) > p - 1,\ A > O,\ B > O$.

7.3a.1. Different versions of the type-1 beta density, the complex case

The densities that follow can be obtained from that specified in (7.3a.1) and certain related transformations. The density of $\tilde{Y} = A^{\frac{1}{2}} \tilde{X} B^{\frac{1}{2}}$ is given by

$$\tilde{g}_2(\tilde{Y}) = \frac{\tilde{\Gamma}_p(q)}{\pi^{qp}} \frac{\tilde{\Gamma}_p(\gamma + q + \beta)}{\tilde{\Gamma}_p(\gamma + q)\tilde{\Gamma}_p(\beta)} |\det(\tilde{Y}\tilde{Y}^*)|^{\gamma} |\det(I - \tilde{Y}\tilde{Y}^*)|^{\beta-p} \tag{7.3a.3}$$

for $\Re(\beta) > p - 1,\ \Re(\gamma + q) > p - 1,\ I - \tilde{Y}\tilde{Y}^* > O$, and $\tilde{g}_2 = 0$ elsewhere. The density of $\tilde{S} = \tilde{Y}\tilde{Y}^*$ is the following:

$$\tilde{g}_3(\tilde{S}) = \frac{\tilde{\Gamma}_p(\gamma + q + \beta)}{\tilde{\Gamma}_p(\gamma + q)\tilde{\Gamma}_p(\beta)} |\det(\tilde{S})|^{\gamma+q-p} |\det(I - \tilde{S})|^{\beta-p} \tag{7.3a.4}$$

for $\Re(\beta) > p - 1,\ \Re(\gamma + q) > p - 1,\ \tilde{S} > O,\ I - \tilde{S} > O$, and $\tilde{g}_3 = 0$ elsewhere.

7.3a.2. Multivariate type-1 beta density, the complex case

When $p = 1$, $\tilde{X}$ is the $1 \times q$ vector $(\tilde{x}_1, \ldots, \tilde{x}_q)$ and the 1×1 matrix A will be denoted by $b > 0$. The resulting density will then have the same structure as that given in (7.3a.1) with p replaced by 1 and A replaced by $b > 0$:

$$\tilde{g}_4(\tilde{X}) = \tilde{C}_2 |\det(\tilde{X} B \tilde{X}^*)|^{\gamma} |\det(I - b\tilde{X} B \tilde{X}^*)|^{\beta-1} \tag{7.3a.5}$$

for $B > O,\ b > 0,\ \Re(\beta) > p - 1,\ \Re(\gamma + q) > p - 1,\ I - b\tilde{X} B \tilde{X}^* > O$, and $\tilde{g}_4 = 0$ elsewhere, where the normalizing constant $\tilde{C}_2$ is

$$\tilde{C}_2 = b^{\gamma+q} |\det(B)| \frac{\tilde{\Gamma}(q)}{\pi^q} \frac{\tilde{\Gamma}(\gamma + q + \beta)}{\tilde{\Gamma}(\gamma + q)\tilde{\Gamma}(\beta)} \tag{7.3a.6}$$

for $b > 0,\ B > O,\ \Re(\beta) > 0,\ \Re(\gamma + q) > 0$. Letting $\tilde{Y} = \tilde{X} B^{\frac{1}{2}}$ so that $\mathrm{d}\tilde{Y} = |\det(B)|\mathrm{d}\tilde{X}$, the density of $\tilde{Y}$ reduces to

$$\tilde{g}_5(\tilde{Y}) = b^{\gamma+q} \frac{\tilde{\Gamma}(q)}{\pi^q} \frac{\tilde{\Gamma}(\gamma + q + \beta)}{\tilde{\Gamma}(\gamma + q)\tilde{\Gamma}(\beta)} |\det(\tilde{Y}\tilde{Y}^*)|^{\gamma} |\det(I - b\tilde{Y}\tilde{Y}^*)|^{\beta-1} \tag{7.3a.7}$$

$$= b^{\gamma+q} \frac{\tilde{\Gamma}(q)}{\pi^q} \frac{\tilde{\Gamma}(\gamma + q + \beta)}{\tilde{\Gamma}(\gamma + q)\tilde{\Gamma}(\beta)} [|\tilde{y}_1|^2 + \cdots + |\tilde{y}_q|^2]^{\gamma} [1 - b(|\tilde{y}_1|^2 + \cdots + |\tilde{y}_q|^2)]^{\beta-1} \tag{7.3a.8}$$

for $\Re(\beta) > 0,\ \Re(\gamma + q) > 0,\ 1 - b(|\tilde{y}_1|^2 + \cdots + |\tilde{y}_q|^2) > 0$, and $\tilde{g}_5 = 0$ elsewhere. The form appearing in (7.3a.8) is applicable to several problems occurring in various areas, as was the case for (7.3.8) in the real domain. However, geometrical probability problems do not appear to have yet been formulated in the complex domain. Let $\tilde{S} = \tilde{Y}\tilde{Y}^*$, $\tilde{S}$ being in this case a real scalar denoted by s whose density is

$$\tilde{g}_6(s) = b^{\gamma+q} \frac{\tilde{\Gamma}(\gamma + q + \beta)}{\tilde{\Gamma}(\gamma + q)\tilde{\Gamma}(\beta)} s^{\gamma+q-1}(1 - bs)^{\beta-1},\ b > 0 \tag{7.3a.9}$$

for $\Re(\beta) > 0,\ \Re(\gamma + q) > 0,\ s > 0,\ 1 - bs > 0$, and $\tilde{g}_6 = 0$ elsewhere. Thus, s is real scalar type-1 beta random variable with parameters $(\gamma + q, \beta)$ and scaling factor $b > 0$. Note that in the real case, the distribution was also a scalar type-1 beta random variable, but having a different first parameter, namely, $\gamma + \frac{q}{2}$, its second parameter and scaling factor remaining β and b.

Example 7.3a.1. Express the density (7.3a.5) explicitly for $b=5,\ p=1,\ q=2,\ \beta=3,\ \gamma=4,\ \tilde{X}=[\tilde{x}_1,\ \tilde{x}_2]=[x_1+iy_1, x_2+iy_2]$, where $x_j,\ y_j,\ j=1,2$, are real variables, $i=\sqrt{(-1)}$, and

$$B=\begin{bmatrix}3 & 1+i\\ 1-i & 1\end{bmatrix}.$$

Solution 7.3a.1. Observe that since $B=B^*$, B is Hermitian. Its leading minors being $|(3)|=3>0$ and $\begin{vmatrix}3 & 1+i\\ 1-i & 1\end{vmatrix}=(3)(1)-(1+i)(1-i)=3-2=1>0$, B is also positive definite. Letting $Q=\tilde{X}B\tilde{X}^*$,

$$\begin{aligned}Q&=3(x_1+iy_1)(x_1-iy_1)+(1)(x_2+iy_2)(x_2-iy_2)+(1+i)(x_1+iy_1)(x_2-iy_2)\\&\quad+(1-i)(x_2+iy_2)(x_1-iy_1)\\&=3(x_1^2+y_1^2)+(x_2^2+y_2^2)+(1+i)[x_1x_2+y_1y_2-i(x_1y_2-x_2y_1)]\\&\quad+(1-i)[x_1x_2+y_1y_2-i(x_2y_1-x_1y_2)]\\&=3(x_1^2+y_1^2)+(x_2^2+y_2^2)+2(x_1x_2+y_1y_2)+2(x_1y_2-x_2y_1).\end{aligned}\tag{i}$$

The normalizing constant being

$$\begin{aligned}b^{\gamma+q}|\det(B)|\frac{\tilde{\Gamma}(q)}{\pi^q}\frac{\tilde{\Gamma}(\gamma+q+\beta)}{\tilde{\Gamma}(\gamma+q)\tilde{\Gamma}(\beta)}&=5^6(1)\frac{\Gamma(2)}{\pi^2}\frac{\Gamma(9)}{\Gamma(6)\Gamma(3)}\\&=\frac{5^6}{\pi^2}\frac{(1!)(8!)}{(5!)(2!)}=\frac{5^6(168)}{\pi^2}.\end{aligned}\tag{ii}$$

The explicit form of the density (7.3a.5) is thus the following:

$$\tilde{g}_4(\tilde{X})=\frac{5^6(168)}{\pi^2}Q^3[1-5Q]^2,\ 1-3Q>0,\ Q>0,$$

and zero elsewhere, where Q is given in *(i)*. It is a multivariate generalized type-1 complex-variate beta density whose scaling factor is 5. Observe that even though $\tilde{X}$ is complex, $\tilde{g}_4(\tilde{X})$ is real-valued.

7.3a.3. Arbitrary moments in the complex case

Consider again the density of the complex $p\times q,\ q\ge p$, matrix-variate random variable $\tilde{X}$ of full rank p having the density specified in (7.3a.1). Let $\tilde{U}=A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}}$. The h-th moment of the absolute value of the determinant of $\tilde{U}$, that is, $E[|\det(\tilde{U})|^h]$, will now be determined for arbitrary h. As before, note that when the expected value is taken, the only change is that the parameter γ is replaced by $\gamma+h$, so that the moment is available from the normalizing constant present in (7.3a.2). Thus,

$$E[|\det(\tilde{U})|^h]=\frac{\tilde{\Gamma}_p(\gamma+q+h)}{\tilde{\Gamma}_p(\gamma+q)}\frac{\tilde{\Gamma}_p(\gamma+q+\beta)}{\tilde{\Gamma}_p(\gamma+q+\beta+h)} \tag{7.3a.10}$$

$$=\prod_{j=1}^{p}\frac{\Gamma(\gamma+q+h-(j-1))}{\Gamma(\gamma+q-(j-1))}\frac{\Gamma(\gamma+q+\beta-(j-1))}{\Gamma(\gamma+q+\beta-(j-1)+h)} \tag{7.3a.11}$$

$$=E[u_1]^hE[u_2]^h\cdots E[u_p]^h \tag{7.3a.12}$$

where $u_1,\ldots,u_p$ are independently distributed real scalar type-1 beta random variables with the parameters $(\gamma+q-(j-1),\ \beta)$ for $j=1,\ldots,p$. The results are the same as those obtained in the real case except that the parameters are slightly different, the parameters being $(\gamma+\frac{q}{2}-\frac{j-1}{2},\ \beta),\ j=1,\ldots,p$, in the real domain. Accordingly, the absolute value of the determinant of $\tilde{U}$ in the complex case has the following structural representation:

$$|\det(\tilde{U})|=|\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|=|\det(\tilde{Y}\tilde{Y}^*)|=|\det(\tilde{S})|=u_1\cdots u_p \tag{7.3a.13}$$

where $u_1,\ldots,u_p$ are mutually independently distributed real scalar type-1 beta random variables with the parameters $(\gamma+q-(j-1),\ \beta),\ j=1,\ldots,p$.

We now consider the scalar type-1 beta density in the complex case. Thus, letting $p=1$ and $q=1$ in (7.3a.8), we have

$$\tilde{g}_7(\tilde{y}_1)=b^{\gamma+1}\frac{1}{\pi}\frac{\tilde{\Gamma}(\gamma+1+\beta)}{\tilde{\Gamma}(\gamma+1)\tilde{\Gamma}(\beta)}[|\tilde{y}_1|^2]^{\gamma}[1-b|\tilde{y}_1|^2]^{\beta-1},\ \tilde{y}_1=y_{11}+iy_{12}$$

$$=b^{\gamma+1}\frac{1}{\pi}\frac{\Gamma(\gamma+1+\beta)}{\Gamma(\gamma+1)\Gamma(\beta)}[y_{11}^2+y_{12}^2]^{\gamma}[1-b(y_{11}^2+y_{12}^2)]^{\beta-1} \tag{7.3a.14}$$

for $b>0,\ \Re(\beta)>0,\ \Re(\gamma)>-1,\ -\infty<y_{1j}<\infty,\ j=1,2,\ 1-b(y_{11}^2+y_{12}^2)>0$, and $\tilde{g}_7=0$ elsewhere. The normalizing constant in (7.3a.14) can be verified by making the polar coordinate transformation $y_{11}=r\cos\theta,\ y_{12}=r\sin\theta$, as was done earlier.

Exercises 7.3

7.3.1. Derive the normalizing constant C_1 in (7.3.2) and verify the normalizing constants in (7.3.3) and (7.3.4).

7.3.2. From $E[|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|]^h$ or otherwise, derive the h-th moment of $|XBX'|$. What is then the structural representation corresponding to (7.3.7)?

7.3.3. From (7.3.7) or otherwise, derive the exact density of $|U|$ for the cases (1): $p = 2$; (2): $p = 3$.

7.3.4. Write down the conditions on the parameters γ and β in (7.3.6) so that the exact density of $|U|$ can easily be evaluated for some $p \ge 4$.

7.3.5. Evaluate the normalizing constant in (7.3.8) by making use of the general polar coordinate transformation.

7.3.6. Evaluate the normalizing constant in (7.3a.2).

7.3.7. Derive the exact density of $|\mathrm{det}(\tilde{U})|$ in (7.3a.13) for (1): $p = 2$; (2): $p = 3$.

7.4. The Real Rectangular Matrix-Variate Type-2 Beta Density

Let us consider a $p \times q,\ q \ge p$, matrix X of full rank p and the following associated density:

$$g_8(X) = C_3|AXBX'|^{\gamma}|I + AXBX'|^{-(\gamma+\frac{q}{2}+\beta)} \tag{7.4.1}$$

for $A > O,\ B > O,\ \Re(\beta) > \frac{p-1}{2},\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}$, and $g_8 = 0$ elsewhere. The normalizing constant can be seen to be the following:

$$C_3 = |A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\frac{\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}}\frac{\Gamma_p(\gamma + \frac{q}{2} + \beta)}{\Gamma_p(\gamma + \frac{q}{2})\Gamma_p(\beta)} \tag{7.4.2}$$

for $A > O,\ B > O,\ \Re(\beta) > \frac{p-1}{2},\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}$. Letting $Y = A^{\frac{1}{2}}XB^{\frac{1}{2}}$, its density denoted by $g_9(Y)$, is

$$g_9(Y) = \frac{\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}}\frac{\Gamma_p(\gamma + \frac{q}{2} + \beta)}{\Gamma_p(\gamma + \frac{q}{2})\Gamma_p(\beta)}|YY'|^{\gamma}|I + YY'|^{-(\gamma+\frac{q}{2}+\beta)} \tag{7.4.3}$$

for $\Re(\beta) > \frac{p-1}{2},\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}$, and $g_9 = 0$ elsewhere. The density of $S = YY'$ then reduces to

$$g_{10}(S) = \frac{\Gamma_p(\gamma + \frac{q}{2} + \beta)}{\Gamma_p(\gamma + \frac{q}{2})\Gamma_p(\beta)}|S|^{\gamma+\frac{q}{2}-\frac{p+1}{2}}|I + S|^{-(\gamma+\frac{q}{2}+\beta)}, \tag{7.4.4}$$

for $\Re(\beta) > \frac{p-1}{2},\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}$, and $g_{10} = 0$ elsewhere.

7.4.1. The real type-2 beta density in the multivariate case

Consider the case $p = 1$ and $A = b > 0$ in (7.4.1). The resulting density has the same structure, with A replaced by b and X being the $1 \times q$ vector $(x_1, \ldots, x_q)$. Letting $Y = XB^{\frac{1}{2}}$, the following density of $Y = (y_1, \ldots, y_q)$, denoted by $g_{11}(Y)$, is obtained:

$$g_{11} = b^{\gamma+\frac{q}{2}} \frac{\Gamma(\frac{q}{2})}{\pi^{\frac{q}{2}}} \frac{\Gamma(\gamma + \frac{q}{2} + \beta)}{\Gamma(\gamma + \frac{q}{2})\Gamma(\beta)} [y_1^2 + \cdots + y_q^2]^{\gamma} \times [1 + b(y_1^2 + \cdots + y_q^2)]^{-(\gamma+\frac{q}{2}+\beta)} \quad (7.4.5)$$

for $b > 0$, $\Re(\gamma + \frac{q}{2}) > 0$, $\Re(\beta) > 0$, and $g_{11} = 0$ elsewhere. The density appearing in (7.4.5) will be referred to as the standard form of the real rectangular matrix-variate type-2 beta density. In this case, A is taken as $A = b > 0$ and $Y = XB^{\frac{1}{2}}$. What might be the standard form of the real type-2 beta density in the real scalar case, that is, when it is assumed that $p = 1,\ q = 1,\ A = b > 0$ and $B = 1$ in (7.4.1)? In this case, it is seen from (7.4.5) that

$$g_{12}(y_1) = b^{\gamma+\frac{1}{2}} \frac{\Gamma(\gamma + \frac{1}{2} + \beta)}{\Gamma(\gamma + \frac{1}{2})\Gamma(\beta)} [y_1^2]^{\gamma} [1 + by_1^2]^{-(\gamma+\frac{1}{2}+\beta)},\quad -\infty < y_1 < \infty, \quad (7.4.6)$$

for $\Re(\beta) > 0,\ \Re(\gamma + \frac{1}{2}) > 0,\ b > 0$, and $g_{12} = 0$ elsewhere.

7.4.2. Moments in the real rectangular matrix-variate type-2 beta density

Letting $U = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$, what would be the h-th moment of the determinant of U, that is, $E[|U|^h]$ for arbitrary h? Upon determining $E[|U|^h]$, the parameter γ is replaced by $\gamma + h$ while the other parameters remain unchanged. The h-th moment which is thus available from the normalizing constant, is given by

$$E[|U|^h] = \frac{\Gamma_p(\gamma + \frac{q}{2} + h)}{\Gamma_p(\gamma + \frac{q}{2})} \frac{\Gamma_p(\beta - h)}{\Gamma_p(\beta)} \quad (7.4.7)$$

$$= \prod_{j=1}^{p} \frac{\Gamma(\gamma + \frac{q}{2} - \frac{j-1}{2} + h)}{\Gamma(\gamma + \frac{q}{2} - \frac{j-1}{2})} \frac{\Gamma(\beta - \frac{j-1}{2} - h)}{\Gamma(\beta - \frac{j-1}{2})} \quad (7.4.8)$$

$$= E[u_1^h]E[u_2^h] \cdots E[u_p^h] \quad (7.4.9)$$

where $u_1, \ldots, u_p$ are mutually independently distributed real scalar type-2 beta random variables with the parameters $(\gamma + \frac{q}{2} - \frac{j-1}{2},\ \beta - \frac{j-1}{2}),\ j = 1, \ldots, p$. That is, (7.4.9) or (7.4.10) gives a structural representation to the determinant of U as

$$|U| = |A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}| = |YY'| = |S| = u_1 \cdots u_p \quad (7.4.10)$$

where the u_j's are mutually independently distributed real scalar type-2 beta random variables as specified above.

Example 7.4.1. Evaluate the density of $u = |A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|$ for $p = 2$ and the general parameters $\gamma,\ q,\ \beta$ where X has a real rectangular matrix-variate type-2 beta density with the parameter matrices $A > O$ and $B > O$ where A is $p \times p$, B is $q \times q$ and X is a $p \times q,\ q \geq p$, rank p matrix.

Solution 7.4.1. The general h-th moment of u can be determined from (7.4.8). Letting $p = 2$, we have

$$E[u^h] = \frac{\Gamma(\gamma + \frac{q}{2} - \frac{1}{2} + h)\Gamma(\gamma + \frac{q}{2} + h)}{\Gamma(\gamma + \frac{q}{2} - \frac{1}{2})\Gamma(\gamma + \frac{q}{2})} \frac{\Gamma(\beta - \frac{1}{2} - h)\Gamma(\beta - h)}{\Gamma(\beta - \frac{1}{2})\Gamma(\beta)} \qquad (i)$$

for $-\gamma - \frac{q}{2} + \frac{1}{2} < \Re(h) < \beta - \frac{1}{2},\ \beta > \frac{1}{2},\ \gamma + \frac{q}{2} > \frac{1}{2}$. Since four pairs of gamma functions differ by $\frac{1}{2}$, we can combine them by applying the duplication formula for gamma functions, namely,

$$\Gamma(z)\Gamma(z + 1/2) = \pi^{\frac{1}{2}}\, 2^{1-2z}\Gamma(2z). \qquad (ii)$$

Take $z = \gamma + \frac{q}{2} - \frac{1}{2} + h$ and $z = \gamma + \frac{q}{2} - \frac{1}{2}$ in the first set of gamma ratios in *(i)* and $z = \beta - \frac{1}{2} - h$ and $z = \beta - \frac{1}{2}$ in the second set of gamma ratios in *(i)*. Then, we have the following:

$$\begin{aligned}\frac{\Gamma(\gamma + \frac{q}{2} - \frac{1}{2} + h)\Gamma(\gamma + \frac{q}{2} + h)}{\Gamma(\gamma + \frac{q}{2} - \frac{1}{2})\Gamma(\gamma + \frac{q}{2})} &= \frac{\pi^{\frac{1}{2}}\, 2^{1-2\gamma-q+1-2h}\Gamma(2\gamma + q - 1 + 2h)}{\pi^{\frac{1}{2}}\, 2^{1-2\gamma-q+1}\Gamma(2\gamma + q - 1)}\\ &= 2^{-2h}\frac{\Gamma(2\gamma + q - 1 + 2h)}{\Gamma(2\gamma + q - 1)} \qquad (iii)\\ \frac{\Gamma(\beta - \frac{1}{2} - h)\Gamma(\beta - h)}{\Gamma(\beta - \frac{1}{2})\Gamma(\beta)} &= \frac{\pi^{\frac{1}{2}}\, 2^{1-2\beta+1+2h}\Gamma(2\beta - 1 - 2h)}{\pi^{\frac{1}{2}}\, 2^{1-2\beta+1}\Gamma(2\beta - 1)}\\ &= 2^{2h}\frac{\Gamma(2\beta - 1 - 2h)}{\Gamma(2\beta - 1)}, \qquad (iv)\end{aligned}$$

the product of *(iii)* and *(iv)* yielding the simplified representation of the h-th moment of u that follows:

$$E[u^h] = \frac{\Gamma(2\gamma + q - 1 + 2h)}{\Gamma(2\gamma + q - 1)}\frac{\Gamma(2\beta - 1 - 2h)}{\Gamma(2\beta - 1)}.$$

Now, since $E[u^h] = E[u^{\frac{1}{2}}]^{2h} \equiv E[y^t]$ with $y = u^{\frac{1}{2}}$ and $t = 2h$, we have

$$E[y^t] = \frac{\Gamma(2\gamma + q - 1 + t)}{\Gamma(2\gamma + q - 1)}\frac{\Gamma(2\beta - 1 - t)}{\Gamma(2\beta - 1)}. \qquad (v)$$

As t is arbitrary in *(v)*, the moment expression will uniquely determine the density of y. Accordingly, y has a real scalar type-2 beta distribution with the parameters $(2\gamma+q-1,\ 2\beta-1)$, and so, its density denoted by $f(y)$, is

$$
\begin{aligned}
f(y)\mathrm{d}y &= \frac{\Gamma(2\gamma+q+2\beta-2)}{\Gamma(2\gamma+q-1)\Gamma(2\beta-1)}y^{2\gamma+q-2}(1+y)^{-(2\gamma+q+2\beta-2)}\mathrm{d}y\\
&= \frac{\Gamma(2\gamma+q+2\beta-2)}{\Gamma(2\gamma+q-1)\Gamma(2\beta-1)}\frac{1}{2}u^{-\frac{1}{2}}u^{\gamma+\frac{q}{2}-1}(1+u^{\frac{1}{2}})^{-(2\gamma+q+2\beta-2)}\mathrm{d}u.
\end{aligned}
$$

Thus, the density of u, denoted by $g(u)$, is the following:

$$
g(u) = \frac{1}{2}\frac{\Gamma(2\gamma+q+2\beta-2)}{\Gamma(2\gamma+q-1)\Gamma(2\beta-1)}u^{\gamma+\frac{q-1}{2}-1}(1+u^{\frac{1}{2}})^{-(2\gamma+q+2\beta-2)}
$$

for $0 \le u < \infty$, and zero elsewhere, where the original conditions on the parameters remain the same. It can be readily verified that $g(u)$ is a density.

7.4.3. A pathway extension in the real case

Let us relabel $f_{10}(X)$ as specified in (7.2.16), as $g_{13}(X)$ in this section:

$$
g_{13}(X) = C_4|AXBX'|^{\gamma}|I-a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}} \tag{7.4.11}
$$

for $A > O,\ B > O,\ \eta > 0,\ a > 0,\ \alpha < 1,\ 1-a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} > O$, and $g_{13} = 0$ elsewhere, where C_4 is the normalizing constant. Observe that for $\alpha < 1,\ a(1-\alpha) > 0$ and hence the model in (7.4.11) is a generalization of the real rectangular matrix-variate type-1 beta density considered in (7.3.1). When $\alpha < 1$, the normalizing constant C_4 is of the form given in (7.2.18). For $\alpha > 1$, we may write $1-\alpha = -(\alpha-1)$, so that $-a(1-\alpha) = a(\alpha-1) > 0,\ \alpha > 1$ in (7.4.11) and the exponent $\frac{\eta}{1-\alpha}$ changes to $-\frac{\eta}{\alpha-1}$; thus, the model appearing in (7.4.11) becomes the following generalization of the rectangular matrix-variate type-2 beta density given in (7.4.1):

$$
g_{14}(X) = C_5|AXBX'|^{\gamma}|I+a(\alpha-1)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{-\frac{\eta}{\alpha-1}} \tag{7.4.12}
$$

for $A > O,\ B > O,\ \eta > 0,\ a > 0,\ \alpha > 1$ and $g_{14} = 0$ elsewhere. The normalizing constant C_5 will then be different from that associated with the type-1 case. Actually, in the type-2 case, the normalizing constant is available from (7.2.19). The model appearing in (7.4.12) is a generalization of the real rectangular matrix-variate type-2 beta model considered in (7.4.1). When $\alpha \to 1$, the model in (7.4.11) converges to a generalized form of the real rectangular matrix-variate gamma model in (7.2.5), namely,

$$
g_{15}(X) = C_6|AXBX'|^{\gamma}\mathrm{e}^{-a\,\eta\,\mathrm{tr}(AXBX')} \tag{7.4.13}
$$

where

$$C_6 = (a\eta)^{p(\gamma+\frac{q}{2})}\frac{|A|^{\frac{q}{2}}|B|^{\frac{p}{2}}\Gamma_p(\frac{q}{2})}{\pi^{\frac{qp}{2}}\Gamma_p(\gamma+\frac{q}{2})} \tag{7.4.14}$$

for $a > 0,\ \eta > 0,\ A > O,\ B > O,\ \Re(\gamma + \frac{q}{2}) > \frac{p-1}{2}$, and $g_{15} = 0$ elsewhere. More properties of the model given in (7.4.11) have already been provided in Sect. 4.2. The real rectangular matrix-variate pathway model was introduced in Mathai (2005).

7.4a. Complex Rectangular Matrix-Variate Type-2 Beta Density

Let us consider a full rank $p \times q,\ q \geq p$, matrix $\tilde{X}$ in the complex domain and the following associated density:

$$\tilde{g}_8(\tilde{X}) = \tilde{C}_3\,|\det(A\tilde{X}B\tilde{X}^*)|^{\gamma}|\det(I + A\tilde{X}B\tilde{X}^*)|^{-(\beta+\gamma+q)} \tag{7.4a.1}$$

for $A > O,\ B > O,\ \Re(\beta) > p - 1,\ \Re(\gamma + q) > p - 1$, and $\tilde{g}_8 = 0$ elsewhere, where $\tilde{C}_3$ is the normalizing constant. Let

$$\tilde{Y} = A^{\frac{1}{2}}\tilde{X}B^{\frac{1}{2}} \Rightarrow \mathrm{d}\tilde{Y} = |\det(A)|^{q}|\det(B)|^{p}\mathrm{d}\tilde{X},$$

and make the transformation

$$\tilde{S} = \tilde{Y}\tilde{Y}^* \Rightarrow \mathrm{d}\tilde{Y} = \frac{\pi^{qp}}{\tilde{\Gamma}_p(q)}|\det(\tilde{S})|^{q-p}\mathrm{d}\tilde{S}.$$

Then, the integral over $\tilde{S}$ can be evaluated by means of a complex matrix-variate type-2 beta integral. That is,

$$\int_{\tilde{S}}|\det(\tilde{S})|^{\gamma+q-p}|\det(I + \tilde{S})|^{-(\beta+\gamma+q)}\mathrm{d}\tilde{S} = \frac{\tilde{\Gamma}_p(\gamma+q)\tilde{\Gamma}_p(\beta)}{\tilde{\Gamma}_p(\gamma+q+\beta)} \tag{7.4a.2}$$

for $\Re(\beta) > p-1,\ \Re(\gamma+q) > p-1$. The normalizing constant $\tilde{C}_3$ as well as the densities of $\tilde{Y}$ and $\tilde{S}$ can be determined from the previous steps. The normalizing constant is

$$\tilde{C}_3 = |\det(A)|^{q}|\det(B)|^{p}\frac{\tilde{\Gamma}_p(q)}{\pi^{qp}}\frac{\tilde{\Gamma}_p(\gamma+q+\beta)}{\tilde{\Gamma}_p(\gamma+q)\tilde{\Gamma}_p(\beta)} \tag{7.4a.3}$$

for $\Re(\beta) > p - 1,\ \Re(\gamma + q) > p - 1$. The density of $\tilde{Y}$, denoted by $\tilde{g}_9(\tilde{Y})$, is given by

$$\tilde{g}_9(\tilde{Y}) = \frac{\tilde{\Gamma}_p(q)}{\pi^{qp}}\frac{\tilde{\Gamma}_p(\gamma+q+\beta)}{\tilde{\Gamma}_p(\gamma+q)\tilde{\Gamma}_p(\beta)}|\det(\tilde{Y}\tilde{Y}^*)|^{\gamma}|\det(I + \tilde{Y}\tilde{Y}^*)|^{-(\gamma+q+\beta)} \tag{7.4a.4}$$

for $\Re(\beta) > p-1,\ \Re(\gamma+q) > p-1$, and $\tilde{g}_9 = 0$ elsewhere. The density of $\tilde{S}$, denoted by $\tilde{g}_{10}(\tilde{S})$, is the following:

$$\tilde{g}_{10}(\tilde{S}) = \frac{\tilde{\Gamma}_p(\gamma+q+\beta)}{\tilde{\Gamma}_p(\gamma+q)\tilde{\Gamma}_p(\beta)}|\det(\tilde{S})|^{\gamma+q-p}|\det(I+\tilde{S})|^{-(\beta+\gamma+q)} \qquad (7.4a.5)$$

for $\Re(\beta) > p-1,\ \Re(\gamma+q) > p-1$, and $\tilde{g}_{10} = 0$ elsewhere.

7.4a.1. Multivariate complex type-2 beta density

As in Sect. 7.3a, let us consider the special case $p=1$ in (7.4a.1). So, let the 1×1 matrix A be denoted by $b>0$ and the $1\times q$ vector $\tilde{X} = (\tilde{x}_1,\ldots,\tilde{x}_q)$. Then,

$$A\tilde{X}B\tilde{X}^* = b\tilde{X}B\tilde{X}^* = b(\tilde{x}_1,\ldots,\tilde{x}_q)B\begin{pmatrix}\tilde{x}_1^*\\ \vdots\\ \tilde{x}_q^*\end{pmatrix} \equiv b\,\tilde{U} \qquad (a)$$

where, in the case of a scalar, an asterisk only designates a complex conjugate. Note that when $p=1$, $\tilde{U} = \tilde{X}B\tilde{X}^*$ is a positive definite Hermitian form whose density, denoted by $\tilde{g}_{11}(\tilde{U})$, is obtained as:

$$\tilde{g}_{11}(\tilde{U}) = b^{\gamma+q}|\det(B)|\frac{\tilde{\Gamma}(q)}{\pi^q}\frac{\tilde{\Gamma}(\gamma+q+\beta)}{\tilde{\Gamma}(\gamma+q)\tilde{\Gamma}(\beta)}|\det(\tilde{U})|^{\gamma}|\det(I+b\tilde{U})|^{-(\gamma+q+\beta)} \quad (7.4a.6)$$

for $\Re(\beta) > 0,\ \Re(\gamma+q) > 0$ and $b>0$, and $\tilde{g}_{11} = 0$ elsewhere. Now, letting $\tilde{X}B^{\frac{1}{2}} = \tilde{Y} = (\tilde{y}_1,\ldots,\tilde{y}_q)$, the density of $\tilde{Y}$, denoted by $\tilde{g}_{12}(\tilde{Y})$, is obtained as

$$\tilde{g}_{12}(\tilde{Y}) = b^{\gamma+q}\frac{\tilde{\Gamma}(q)}{\pi^q}\frac{\tilde{\Gamma}(\gamma+q+\beta)}{\tilde{\Gamma}(\gamma+q)\tilde{\Gamma}(\beta)}[|\tilde{y}_1|^2+\cdots+|\tilde{y}_q|^2]^{\gamma}[1+b(|\tilde{y}_1|^2+\cdots+|\tilde{y}_q|^2)]^{-(\gamma+q+\beta)} \qquad (7.4a.7)$$

for $\Re(\beta) > 0,\ \Re(\gamma+q) > 0,\ b>0$, and $\tilde{g}_{12} = 0$ elsewhere. The constant in (7.4a.7) can be verified to be a normalizing constant, either by making use of Theorem 4.2a.3 or a $(2n)$-variate real polar coordinate transformation, which is left as an exercise to the reader.

Example 7.4a.1. Provide an explicit representation of the complex multivariate density in (7.4a.7) for $p=2,\ \gamma=2,\ q=3,\ b=3$ and $\beta=2$.

Solution 7.4a.1. The normalizing constant, denoted by $\tilde{c}$, is the following:

$$\begin{aligned}\tilde{c} &= b^{\gamma+q}\frac{\tilde{\Gamma}(q)}{\pi^q}\frac{\tilde{\Gamma}(\gamma+q+\beta)}{\tilde{\Gamma}(\gamma+q)\tilde{\Gamma}(\beta)}\\ &= 3^5\frac{\Gamma(3)}{\pi^3}\frac{\Gamma(7)}{\Gamma(5)\Gamma(2)} = 3^5\frac{(2!)}{\pi^3}\frac{(6!)}{(4!)(1!)} = \frac{(60)3^5}{\pi^3}.\end{aligned} \qquad (i)$$

Letting $\tilde{y}_1 = y_{11} + iy_{12},\ \tilde{y}_2 = y_{21} + iy_{22},\ \tilde{y}_3 = y_{31} + iy_{32},\ y_{1j},\ y_{2j},\ y_{3j},\ j = 1, 2,$ being real and $i = \sqrt{(-1)}$,

$$Q = |\tilde{y}_1|^2 + |\tilde{y}_2|^2 + |\tilde{y}_3|^2 = (y_{11}^2 + y_{12}^2) + (y_{21}^2 + y_{22}^2) + (y_{31}^2 + y_{32}^2). \qquad (ii)$$

Thus, the required density, denoted by $\tilde{g}_{12}(\tilde{Y})$ as in (7.4a.7), is given by

$$\tilde{g}_{12}(\tilde{Y}) = \tilde{c}Q^2(1 + 3Q)^{-7},\ -\infty < y_{ij} < \infty,\ i = 1, 2, 3,\ j = 1, 2.$$

where $\tilde{c}$ is specified in *(i)* and Q, in *(ii)*. This completes the solution.

The density in (7.4a.6) is called a complex multivariate type-2 beta density in the general form and (7.4a.7) is referred to as a complex multivariate type-2 beta density in its standard form. Observe that these constitute only one form of the multivariate case of a type-2 beta density. When extending a univariate function to a multivariate one, there is no such thing as a unique multivariate analogue. There exist a multitude of multivariate functions corresponding to specified marginal functions, or marginal densities in statistical problems. In the latter case for instance, there are countless possible copulas associated with some specified marginal distributions. Copulas actually encapsulate the various dependence relationships existing between random variables. We have already seen that one set of generalizations to the multivariate case for univariate type-1 and type-2 beta densities are the type-1 and type-2 Dirichlet densities and their extensions. The densities appearing in (7.4a.6) and (7.4a.7) are yet another version of a multivariate type-2 beta density in the complex case.

What will be the resulting distribution when $q = 1$ in (7.4a.7)? The standard form of this density then becomes the following, denoted by $\tilde{g}_{13}(\tilde{y}_1)$:

$$\tilde{g}_{13}(\tilde{y}_1) = b^{\gamma+1}\frac{1}{\pi}\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma+1)\Gamma(\beta)}[|\tilde{y}_1|^2]^{\gamma}[1 + b|\tilde{y}_1|^2]^{-(\gamma+1+\beta)} \qquad (7.4a.8)$$

for $\Re(\beta) > 0,\ \Re(\gamma + 1) > 0,\ b > 0$, and $\tilde{g}_{13} = 0$ elsewhere. We now verify that this is indeed a density function. Let $\tilde{y}_1 = y_{11} + iy_{12},\ y_{11}$ and y_{12} being real scalar quantities and $i = \sqrt{(-1)}$. When $\tilde{y}_1$ is in the complex plane, $-\infty < y_{1j} < \infty,\ j = 1, 2$. Let us make a polar coordinate transformation. Letting $y_{11} = r\cos\theta$ and $y_{12} = r\sin\theta$, $\mathrm{d}y_{11} \wedge \mathrm{d}y_{12} = r\,\mathrm{d}r \wedge \mathrm{d}\theta,\ 0 \le r < \infty,\ 0 < \theta \le 2\pi$. The integral over the functional part of (7.4a.8) yields

$$\int_{\tilde{y}_1} |\tilde{y}_1|^{2\gamma}[1+|\tilde{y}_1|^2]^{-(\gamma+1+\beta)}\mathrm{d}\tilde{y}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[y_{11}^2+y_{12}^2]^{\gamma}$$
$$\times [1+b(y_{11}^2+y_{12}^2)]^{-(\gamma+1+\beta)}\mathrm{d}y_{11}\wedge \mathrm{d}y_{12}$$
$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}[r^2]^{\gamma}[1+br^2]^{-(\gamma+1+\beta)}r\ \mathrm{d}\theta\wedge \mathrm{d}r$$
$$= (2\pi)\Big(\frac{1}{2}\Big)\int_{t=0}^{\infty}t^{\gamma}(1+bt)^{-(\gamma+1+\beta)}\mathrm{d}t$$

which is equal to

$$\pi\ b^{-(\gamma+1)}\frac{\Gamma(\gamma+1)\Gamma(\beta)}{\Gamma(\gamma+1+\beta)}.$$

This establishes that the function specified by (7.4*a*.8) is a density.

7.4a.2. Arbitrary moments in the complex type-2 beta density

Let us consider the h-th moment of $|\det(\tilde{U})| = |\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|$ in (7.4*a*.1). Since the only change upon integration is that γ is replaced by $\gamma+h$, the h-th moment is available from the normalizing constant in (7.4*a*.2):

$$E[|\det(\tilde{U})|^h] = \frac{\Gamma_p(\gamma+q+h)}{\Gamma_p(\gamma+q)}\frac{\Gamma_p(\beta-h)}{\Gamma_p(\beta)} \tag{7.4a.9}$$
$$= \prod_{j=1}^{p}\frac{\Gamma(\gamma+q-(j-1)+h)}{\Gamma(\gamma+q-(j-1))}\frac{\Gamma(\beta-(j-1)-h)}{\Gamma(\beta-(j-1))} \tag{7.4a.10}$$
$$= E[u_1^h]\cdots E[u_p^h] \tag{7.4a.11}$$

where $u_1,\ldots,u_p$ mutually independently distributed real scalar type-2 beta random variables with the parameters $(\gamma+q-(j-1),\ \beta-(j-1)),\ \ j=1,\ldots,p$. Thus, $|\det(\tilde{U})|$ has the structural representation

$$|\det(\tilde{U})| = |\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})| = |\det(\tilde{Y}\tilde{Y}^*)| = |\det(\tilde{S})| = u_1\cdots u_p \tag{7.4a.12}$$

where the $u_1,\ldots,u_p$ are as previously defined. The density for a complex scalar type-2 beta random variable is provided in (7.4*a*.8).

7.4a.3. A pathway version of the complex rectangular matrix-variate type-1 beta density

Consider the model specified in (7.2*a*.12), that is,

$$\tilde{g}_{14}(\tilde{X}) = \tilde{C}_4|\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|^{\gamma}|\det(I-a(1-\alpha)A^{\frac{1}{2}}\tilde{X}B\tilde{X}^*A^{\frac{1}{2}})|^{\frac{\eta}{1-\alpha}} \tag{7.4a.13}$$

for $a>0,\ \alpha<1,\ A>O,\ B>O,\ \eta>0,\ I-a(1-\alpha)A^{\frac{1}{2}}\tilde{X}B\tilde{X}^{*}A^{\frac{1}{2}}>O$, and $\tilde{g}_{14}=0$ elsewhere, where $\tilde{C}_4$ is the normalizing constant given in (7.2a.15). When $\alpha<1$, the model appearing in (7.4a.13) is a generalization of the complex rectangular matrix-variate type-1 beta model considered in (7.3a.1). When $\alpha>1$, we write $1-\alpha=-(\alpha-1),\ \alpha>1$ and re-express $\tilde{g}_{14}$ as

$$\tilde{g}_{15}(\tilde{X})=\tilde{C}_5|\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^{*}A^{\frac{1}{2}})|^{\gamma}|\det(I+a(\alpha-1)A^{\frac{1}{2}}\tilde{X}B\tilde{X}^{*}A^{\frac{1}{2}})|^{-\frac{\eta}{\alpha-1}}\qquad(7.4a.14)$$

for $a>0,\ \alpha>1,\ \eta>0,\ A>O,\ B>O$, and $\tilde{g}_{15}=0$ elsewhere, where the normalizing constant $\tilde{C}_5$ is the same as the one in (7.2a.16). Observe that the model in (7.4a.14) is a generalization of the complex rectangular matrix-variate type-2 beta model in (7.4.1). When $q\to 1$, the models in (7.4a.13) and (7.4a.14) both converge to the following model:

$$\tilde{g}_{16}(\tilde{X})=\tilde{C}_6|\det(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^{*}A^{\frac{1}{2}})|^{\gamma}\mathrm{e}^{-a\,\eta\,\mathrm{tr}(A^{\frac{1}{2}}\tilde{X}B\tilde{X}^{*}A^{\frac{1}{2}})}\qquad(7.4a.15)$$

for $a>0,\ \eta>0,\ A>O,\ B>O$, and $\tilde{g}_{16}=0$ elsewhere, where the normalizing constant $\tilde{C}_6$ is the same as that in (7.2a.17). The model specified in (7.4a.15) is a generalization of the complex rectangular matrix-variate gamma model considered in (7.2a.1). Thus, model in (7.4a.13) contains all the three models (7.4a.13), (7.4a.14), and (7.4a.15), which are generalizations of the models given in (7.3a.1), (7.4a.1), and (7.2a.1), respectively. The pathway model in the complex domain, namely (7.4a.13), was introduced in Mathai and Provost (2006). Additional properties of the pathway model have already been discussed in Sect. 7.2a.

Exercises 7.4

7.4.1. Following the instructions or otherwise, derive the normalizing constant $\tilde{C}_3$ in (7.4a.3).

7.4.2. By integrating over $\tilde{Y}$, show that (7.4a.4) is a density.

7.4.3. Evaluate the normalizing constant in (7.4a.7) by using (1): Theorem 4.2a.3; (2): a $(2n)$-variate real polar coordinate transformation.

7.4.4. Given the standard real matrix-variate type-2 beta model in (7.4.5), evaluate the marginal joint density of $y_1,\ldots,y_r,\ r<p$.

7.4.5. Evaluate the density in (7.4a.4) explicitly for $p=1$ and $q=2$.

7.4.6. Given the standard complex matrix-variate type-2 beta model in (7.4a.7), evaluate the joint marginal density of $\tilde{y}_1, \ldots, \tilde{y}_r,\ r < p$.

7.4.7. Derive the density of $|\det(\tilde{U})|$ in (7.4a.12) for the cases (1): $p = 1$; (2): $p = 2$.

7.4.8. Derive the density of $|U|$ in (7.4.10) for (1): $p = 2$; (2): $p = 3$.

7.5,7.5a. Ratios Involving Rectangular Matrix-Variate Random Variables

Since scalar variables such as type-1 beta, type-2 beta, F, Student-t and Cauchy variables are all associated with ratios of independently distributed random variables, we will explore ratios involving rectangular matrix-variate random variables. Such ratios will yield the rectangular matrix-variate versions of the aforementioned ratios of scalar variables. Let the $p \times n_1,\ p \le n_1$, full rank matrix X_1 and the $p \times n_2,\ p \le n_2$, full rank matrix X_2 be independently distributed real matrix-variate random variables having the rectangular matrix-variate gamma densities specified in (7.2.5), that is,

$$f_j(X_j) = \frac{|A_j|^{\frac{n_j}{2}}|B_j|^{\frac{p}{2}}\Gamma_p(\frac{n_j}{2})}{\pi^{\frac{n_j p}{2}}\Gamma_p(\gamma_j + \frac{n_j}{2})}|A_j X_j B_j X_j'|^{\gamma_j}\mathrm{e}^{-\mathrm{tr}(A_j X_j B_j X_j')},\ j = 1, 2, \qquad (7.5.1)$$

for $A_j > O,\ B_j > O,\ \Re(\gamma_j + \frac{n_j}{2}) > \frac{p-1}{2}$, and $n_j \ge p$ where A_j is $p \times p$ and B_j is $n_j \times n_j$. Then, owing to the statistical independence of the variables, the joint density of X_1 and X_2 is $f(X_1, X_2) = f_1(X_1)f_2(X_2)$. Consider the ratios

$$U_1 = \Big(\sum_{j=1}^{2}\Big(A_j^{\frac{1}{2}}X_j B_j X_j' A_j^{\frac{1}{2}}\Big)\Big)^{-\frac{1}{2}}\Big(A_1^{\frac{1}{2}}X_1 B_1 X_1' A_1^{\frac{1}{2}}\Big)\Big(\sum_{j=1}^{2}\Big(A_j^{\frac{1}{2}}X_j B_j X_j' A_j^{\frac{1}{2}}\Big)\Big)^{-\frac{1}{2}} \qquad (i)$$

and

$$U_2 = \Big(A_2^{\frac{1}{2}}X_2 B_2 X_2' A_2^{\frac{1}{2}}\Big)^{-\frac{1}{2}}\Big(A_1^{\frac{1}{2}}X_1 B_1 X_1' A_1^{\frac{1}{2}}\Big)\Big(A_2^{\frac{1}{2}}X_2 B_2 X_2' A_2^{\frac{1}{2}}\Big)^{-\frac{1}{2}}. \qquad (ii)$$

Let us derive the densities of U_1 and U_2. Letting $V_j = A_j^{\frac{1}{2}}X_j B_j^{\frac{1}{2}}$, we have $\mathrm{d}X_j = |A_j|^{-\frac{n_j}{2}}|B_j|^{-\frac{p}{2}}\mathrm{d}V_j$. Denoting the joint density of V_1 and V_2 by $g(V_1, V_2)$, it follows that $f(X_1, X_2)\mathrm{d}X_1 \wedge \mathrm{d}X_2 = g(V_1, V_2)\mathrm{d}V_1 \wedge \mathrm{d}V_2$ and so,

$$g(V_1, V_2)\mathrm{d}V_1\wedge\mathrm{d}V_2 = \Big\{\prod_{j=1}^{2}\frac{\Gamma_p(\frac{n_j}{2})}{\pi^{\frac{n_j p}{2}}\Gamma_p(\gamma_j + \frac{n_j}{2})}\Big\}|V_1V_1'|^{\gamma_1}|V_2V_2'|^{\gamma_2}\mathrm{e}^{-\mathrm{tr}(V_1V_1'+V_2V_2')}\mathrm{d}V_1\wedge\mathrm{d}V_2.$$

Letting $W_j = V_j V_j'$, $\mathrm{d}V_j = \frac{\pi^{\frac{n_j p}{2}}}{\Gamma_p(\frac{n_j}{2})}|W_j|^{\frac{n_j}{2}-\frac{p+1}{2}}\mathrm{d}W_j$, and the joint density of W_1 and W_2, denoted by $h(W_1, W_2)$, is the following:

$$h(W_1, W_2) = \left\{\prod_{j=1}^{2} \frac{1}{\Gamma_p(\gamma_j + \frac{n_j}{2})}\right\} |W_1|^{\gamma_1+\frac{n_1}{2}-\frac{p+1}{2}} |W_2|^{\gamma_2+\frac{n_2}{2}-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(W_1+W_2)}. \quad (7.5.2)$$

Note that $U_1 = (W_1 + W_2)^{-\frac{1}{2}} W_1 (W_1 + W_2)^{-\frac{1}{2}}$ and $U_2 = W_2^{-\frac{1}{2}} W_1 W_2^{-\frac{1}{2}}$. Then, given the relationship between independently distributed matrix-variate gamma variables and a type-1 matrix-variate beta variable and a type-2 matrix-variate beta variable, U_1 and U_2 are distributed as real matrix-variate type-1 beta and type-2 beta random variables, respectively, both with the parameters $(\gamma_1 + \frac{n_1}{2}, \gamma_2 + \frac{n_2}{2})$, that is,

$$U_1 \sim \text{ type-1 beta}\Big(\gamma_1 + \frac{n_1}{2}, \gamma_2 + \frac{n_2}{2}\Big) \text{ and } U_2 \sim \text{ type-2 beta}\Big(\gamma_1 + \frac{n_1}{2}, \gamma_2 + \frac{n_2}{2}\Big).$$

Thus, we have the following result:

Theorem 7.5.1. *Let X_1 of dimension $p \times n_1$, $p \le n_1$, and X_2 of dimension $p \times n_2$, $p \le n_2$, be rank p matrices that are independently distributed rectangular real matrix-variate gamma random variables whose densities are specified in (7.5.1). Then, as defined in (i) and (ii), U_1 and U_2 are respectively real matrix-variate type-1 beta and type-2 beta distributed with the same parameters $(\gamma_1 + \frac{n_1}{2}, \gamma_2 + \frac{n_2}{2})$. Thus they have the following densities, denoted by $g_j(U_j)$, $j = 1, 2$:*

$$g_1(U_1)\mathrm{d}U_1 = c\,|U_1|^{\gamma_1+\frac{n_1}{2}-\frac{p+1}{2}}|I - U_1|^{\gamma_2+\frac{n_2}{2}-\frac{p+1}{2}}\mathrm{d}U_1,\ O < U_1 < I, \quad (7.5.3)$$

and zero elsewhere, and

$$g_2(U_2)\mathrm{d}U_2 = c\,|U_2|^{\gamma_1+\frac{n_1}{2}-\frac{p+1}{2}}|I + U_2|^{-(\gamma_1+\gamma_2+\frac{n_1}{2}+\frac{n_2}{2})}\mathrm{d}U_2,\ U_2 > O, \quad (7.5.4)$$

where

$$c = \frac{\Gamma_p(\gamma_1 + \gamma_2 + \frac{n_1}{2} + \frac{n_2}{2})}{\Gamma_p(\gamma_1 + \frac{n_1}{2})\Gamma_p(\gamma_2 + \frac{n_2}{2})},\ \Re\Big(\gamma_j + \frac{n_j}{2}\Big) > \frac{p-1}{2},\ j = 1, 2.$$

Analogous derivations will yield the densities of $\tilde{U}_1$ and $\tilde{U}_2$, the corresponding matrix-variate random variables in the complex domain:

Theorem 7.5a.1. *Let $\tilde{X}_1$ of dimension $p\times n_1,\ p\le n_1$, and $\tilde{X}_2$ of dimension $p\times n_2,\ p\le n_2$, be full rank rectangular matrix-variate complex gamma random variables that are independently distributed whose densities are*

$$\tilde{f}_j(\tilde{X}_j)\mathrm{d}\tilde{X}_j=\frac{|A_j|^{n_j}|B_j|^p\tilde{\Gamma}_p(n_j)}{\pi^{n_jp}\tilde{\Gamma}_p(\gamma_j+n_j)}\,|\mathrm{det}(A_j\tilde{X}_jB_j\tilde{X}_j^*)|^{\gamma_j}\mathrm{e}^{-\mathrm{tr}(A_j\tilde{X}_jB_j\tilde{X}_j^*)}\mathrm{d}\tilde{X}_j \qquad (7.5a.1)$$

where $A_j=A_j^*>O$ and $B_j=B_j^*>O$ with A_j being $p\times p$ and B_j, $n_j\times n_j$, $p\le n_j,\ j=1,2$. Letting

$$\tilde{U}_1=\Big(\sum_{j=1}^{2}A_j^{\frac{1}{2}}\tilde{X}_jB_j\tilde{X}_j^*A_j^{\frac{1}{2}}\Big)^{-\frac{1}{2}}(A_1^{\frac{1}{2}}\tilde{X}_1B_1\tilde{X}_1^*A_1^{\frac{1}{2}})\Big(\sum_{j=1}^{2}A_j^{\frac{1}{2}}\tilde{X}_jB_j\tilde{X}_j^*A_j^{\frac{1}{2}}\Big)^{-\frac{1}{2}}$$

and

$$\tilde{U}_2=(A_2^{\frac{1}{2}}\tilde{X}_2B_2\tilde{X}_2^*A_2^{\frac{1}{2}})^{-\frac{1}{2}}(A_1^{\frac{1}{2}}\tilde{X}_1B_1\tilde{X}_1^*A_1^{\frac{1}{2}})(A_2^{\frac{1}{2}}\tilde{X}_2B_2\tilde{X}_2^*A_2^{\frac{1}{2}})^{-\frac{1}{2}}, \qquad (7.5a.2)$$

the densities of $\tilde{U}_1$ and $\tilde{U}_2$, denoted by $\tilde{g}_j(\tilde{U}_j),\ j=1,2$, are respectively given by

$$\tilde{g}_1(\tilde{U}_1)\mathrm{d}\tilde{U}_1=\tilde{c}\,|\mathrm{det}(\tilde{U}_1)|^{\gamma_1+n_1-p}|\mathrm{det}(I-\tilde{U}_1)|^{\gamma_2+n_2-p}\mathrm{d}\tilde{U}_1,\ O<\tilde{U}_1<I, \qquad (7.5a.3)$$

and

$$\tilde{g}_2(\tilde{U}_2)\mathrm{d}\tilde{U}_2=\tilde{c}\,|\mathrm{det}(\tilde{U}_2)|^{\gamma_1+n_1-p}|\mathrm{det}(I+\tilde{U}_2)|^{-(\gamma_1+\gamma_2+n_1+n_2)}\mathrm{d}\tilde{U}_2,\ \tilde{U}_2>O, \qquad (7.5a.4)$$

where

$$\tilde{c}=\frac{\tilde{\Gamma}_p(\gamma_1+\gamma_2+n_1+n_2)}{\tilde{\Gamma}_p(\gamma_1+n_1)\tilde{\Gamma}_p(\gamma_2+n_2)},\ \Re(\gamma_j+n_j)>p-1,\ j=1,2.$$

The densities specified in (7.5.4) and (7.5a.4) happen to be quite useful in real-life applications. Connections of the type-2 beta distribution to the F-distribution, the Student-t^2 distribution and the distribution of the sample correlation coefficient when the population is Gaussian, have already been pointed out in the course of our previous discussions with respect to the scalar, vector variable and matrix-variate cases. Some further relationships are next pointed out. Let $\{Y_1,\ldots,Y_n\}$ constitutes a simple random sample where $Y_j\overset{iid}{\sim}N_p(\mu,\Sigma),\ \Sigma>O,\ j=1,\ldots,n$, and the sample matrix be denoted by $\mathbf{Y}=[Y_1,Y_2,\ldots,Y_n]$; letting $\bar{Y}=\frac{1}{n}[Y_1+\cdots+Y_n]$ and the matrix of sample means be $\bar{\mathbf{Y}}=[\bar{Y},\ldots,\bar{Y}]$, the sample sum of products (corrected) matrix is $S=(\mathbf{Y}-\bar{\mathbf{Y}})(\mathbf{Y}-\bar{\mathbf{Y}})'$, which is unaffected by μ. We have determined that S follows a real Wishart distribution having $m=n-1$ degrees of freedom, and that when μ is known to be a null vector, $\mathbf{YY}'$ is real Wishart matrix with n degrees of freedom. Now, consider $A^{\frac{1}{2}}Y_j \overset{iid}{\sim} N_p(A^{\frac{1}{2}}\mu, A^{\frac{1}{2}}\Sigma A^{\frac{1}{2}})$; when $\mu = O$, the sample sum of products matrix is $A^{\frac{1}{2}}YY'A^{\frac{1}{2}}$, which can be expressed in the form of the product of matrices appearing in (7.5.1) with $B = I$. Hence, we can regard $A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$ (or equivalently $AXBX'$ in the determinant in (7.5.1)) as a weighted sample sum of products matrix with sample sizes n_1 and n_2. Then, the type-2 beta density in (7.5.4) with U_2 replaced by $\frac{n_1}{n_2}U_2$ corresponds to a generalized *real rectangular matrix-variate F-density having* n_1 *and* n_2 *degrees of freedom* where U_2 is as defined in *(ii)*. Moreover, for $\gamma_1 = 0 = \gamma_2$, this density will correspond to a *rectangular matrix-variate Student-t density*. The material included in Sect. 7.5,7.5a may not be available in the literature.

7.5.1. Multivariate F, Student-*t* and Cauchy densities

The densities appearing in (7.5.4) and (7.5*a*.4) for the $p\times p$ positive definite matrices U_2 and $\tilde{U}_2$ have $\mathrm{d}U_2$ and $\mathrm{d}\tilde{U}_2$ as differential elements. A positive definite matrix such as U_2, can be expressed as $U_2 = TT'$ where T of dimension $p\times n_1,\ p\le n_1$, has rank p, and we can write $\mathrm{d}U_2$ in terms of $\mathrm{d}T$. We can also consider the format $U_2 = TCT'$ where $C > O$ is an $n_1\times n_1$ positive definite constant matrix. In other words, we can arrive at the format in (7.4.1) from (7.5.4), and correspondingly obtain (7.4*a*.1) from (7.5*a*.4). Let us re-examine the expressions given in (7.4.1) and (7.4*a*.1), which could be referred to as *rectangular matrix-variate F and Student-t densities in the real and complex cases* for specific values of the parameters β and γ. Now, let $p = 1$ and $A = a > 0$ in (7.4.1) wherein a location parameter vector μ is inserted. The resulting density is

$$h(X)\mathrm{d}X = \frac{a^{\gamma+\frac{q}{2}}|B|^{\frac{1}{2}}\Gamma(\frac{q}{2})\Gamma(\gamma+\frac{q}{2}+\beta)}{\pi^{\frac{q}{2}}\Gamma(\gamma+\frac{q}{2})\Gamma(\beta)}[(X-\mu)B(X-\mu)']^{\gamma} \times [1+a(X-\mu)B(X-\mu)']^{-(\gamma+\frac{q}{2}+\beta)}\mathrm{d}X \tag{7.5.5}$$

where X and μ are $1\times q$ row vectors, the corresponding density in the complex domain, denoted by $\tilde{h}(\tilde{X})$, being the following:

$$\tilde{h}(\tilde{X})\mathrm{d}\tilde{X} = \frac{|a|^{\gamma+q}|\mathrm{det}(B)|\Gamma(q)\Gamma(\gamma+q+\beta)}{\pi^{q}\,\Gamma(\gamma+q)\Gamma(\beta)}[(\tilde{X}-\tilde{\mu})B(\tilde{X}-\tilde{\mu})^*]^{\gamma} \times [1+a(\tilde{X}-\tilde{\mu})B(\tilde{X}-\tilde{\mu})^*]^{-(\gamma+q+\beta)}\mathrm{d}\tilde{X}. \tag{7.5a.5}$$

For specific values of the parameters, the densities appearing in (7.5.5) and (7.5a.5) can be respectively called the *multivariate F and Student-t densities* in the real and complex domains. With a view to model certain types of signal processes, (Kondo et al., 2020) made use of a special form of the complex multivariate Student-t wherein $\gamma = 0,\ a = \frac{2}{\nu}$ and $\beta = \frac{\nu}{2}$, which is given next.

7.5a.1. A complex multivariate Student-t having ν degrees of freedom

$$\tilde{h}_1(\tilde{X})\mathrm{d}\tilde{X} = \frac{2^q\Gamma(\frac{\nu}{2}+q)}{(\nu\pi)^q\Gamma(\frac{\nu}{2})|\det(\Sigma)|}\Big[1+\frac{2}{\nu}(\tilde{X}-\tilde{\mu})\Sigma^{-1}(\tilde{X}-\tilde{\mu})^*\Big]^{-(\frac{\nu}{2}+q)}\mathrm{d}\tilde{X}. \quad (7.5a.6)$$

A complex multivariate Cauchy density that, as well, is mentioned in Kondo et al. (2020), can be obtained by letting $\nu = 1$ in (7.5a.6). We conclude this section with its representation, denoted by $\tilde{h}_2(\tilde{X})$:

7.5a.2. A complex multivariate Cauchy density

$$\tilde{h}_2(\tilde{X})\mathrm{d}\tilde{X} = \frac{2^q\,\Gamma(\frac{1}{2}+q)}{\pi^q\,\Gamma(\frac{1}{2})|\det(\Sigma)|}[1+2(\tilde{X}-\tilde{\mu})\Sigma^{-1}(\tilde{X}-\tilde{\mu})^*]^{-(q+\frac{1}{2})}\mathrm{d}\tilde{X}. \quad (7.5a.7)$$

Exercises 7.5

7.5.1. Derive the complex densities in (7.5a.3) and (7.5a.4).

7.5.2. Derive the normalizing constant in (7.5a.6) by integrating out the functional portion of this density.

7.5.3. Derive the normalizing constant in (7.5a.7) by integrating out the functional portion of this density.

7.5.4. Derive the density in (7.5a.6) from complex q-variate Gaussian densities.

7.5.5. Derive the density in (7.5a.7) from complex q-variate Gaussian densities.

7.6. Rectangular Matrix-Variate Dirichlet Density, Real Case

For the real matrix-variate type-1 and type-2 Dirichlet models involving sets of real positive definite matrices, the reader is referred to Sects. 5.8.6 and 5.8.7. The corresponding rectangular matrix-variate cases will be considered in this section. Let $A_j > O,\ j = 1,\ldots,k$, be $p\times p$ real positive definite constant matrices, and $B_j,\ j = 1,\ldots,k$, be $q_j\times q_j$ real positive definite constant matrices. Let $X_j,\ j = 1,\ldots,k$, be $p\times q_j,\ q_j \ge p$, rank p real matrices whose elements are distinct real scalar variables. Then, consider the real-valued scalar function of $X_1, \ldots, X_k$,

$$f_1(X_1, \ldots, X_k) = C_k |A_1 X_1 B_1 X_1'|^{\gamma_1} \cdots |A_k X_k B_k X_k'|^{\gamma_k} \\ \times |I - A_1^{\frac{1}{2}} X_1 B_1 X_1' A_1^{\frac{1}{2}} - \cdots - A_k^{\frac{1}{2}} X_k B_k X_k' A_k^{\frac{1}{2}}|^{\gamma_{k+1} - \frac{p+1}{2}} \tag{7.6.1}$$

for $A_j > O,\ B_j > O,\ A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}} > O,\ j = 1, \ldots, k,\ I - \sum_{j=1}^{k} A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}} > O,\ \Re(\gamma_j + \frac{q_j}{2}) > \frac{p-1}{2},\ j = 1, \ldots, k$, and $f_1 = 0$ elsewhere, where C_k is the normalizing constant. This normalizing constant can be evaluated as follows: Letting

$$Y_j = A_j^{\frac{1}{2}} X_j B_j^{\frac{1}{2}} \Rightarrow \mathrm{d}Y_j = |A_j|^{\frac{q_j}{2}} |B_j|^{\frac{p}{2}} \mathrm{d}X_j,\ j = 1, \ldots, k, \tag{i}$$

the joint density of $Y_1, \ldots, Y_k$, denoted by $f_2(Y_1, \ldots, Y_k)$, is given by

$$f_2(Y_1, \ldots, Y_k) = \Big\{ \prod_{j=1}^{k} |A_j|^{-\frac{q_j}{2}} |B_j|^{-\frac{p}{2}} \Big\} C_k |Y_1 Y_1'|^{\gamma_1} \ldots |Y_k Y_k'|^{\gamma_k} \\ \times |I - \sum_{j=1}^{k} Y_j Y_j'|^{\gamma_{k+1} - \frac{p+1}{2}}. \tag{7.6.2}$$

Now, let

$$S_j = Y_j Y_j' \Rightarrow \mathrm{d}Y_j = \frac{\pi^{\frac{q_j p}{2}}}{\Gamma_p(\frac{q_j}{2})} |S_j|^{\frac{q_j}{2} - \frac{p+1}{2}} \mathrm{d}S_j,\ j = 1, \ldots, k. \tag{ii}$$

Then, the joint density of $S_1, \ldots, S_k$, which follows, is a real matrix-variate type-1 Dirichlet density:

$$f_3(S_1, \ldots, S_k) = C_k \Big\{ \prod_{j=1}^{k} |A_j|^{-\frac{q_j}{2}} |B_j|^{-\frac{p}{2}} \frac{\pi^{\frac{q_j p}{2}}}{\Gamma_p(\frac{q_j}{2})} \Big\} \\ \times \Big\{ \prod_{j=1}^{k} |S_j|^{\gamma_j + \frac{q_j}{2} - \frac{p+1}{2}} \Big\} |I - S_1 - \cdots - S_k|^{\gamma_{k+1} - \frac{p+1}{2}} \tag{7.6.3}$$

for $S_j > O,\ \Re(\gamma_j + \frac{q_j}{2}) > \frac{p-1}{2},\ j = 1, \ldots, k$. Next, on integrating out $S_1, \ldots, S_k$, by making use of a type-1 real matrix-variate Dirichlet integral that was defined in Sect. 5.8.6, we have

$$\frac{\{\prod_{j=1}^{k} \Gamma_p(\gamma_j + \frac{q_j}{2})\} \Gamma_p(\gamma_{k+1})}{\Gamma_p(\sum_{j=1}^{k+1} \gamma_j + \sum_{j=1}^{k} \frac{q_j}{2})},\ \Re(\gamma_j + \frac{q_j}{2}) > \frac{p-1}{2},\ j = 1, \ldots, k, \tag{iii}$$

and $\Re(\gamma_{k+1}) > \frac{p-1}{2}$. Then, as obtained from (*i*), (*ii*) and (*iii*), the normalizing constant is

$$C_k = \Big\{\prod_{j=1}^{k}|A_j|^{\frac{q_j}{2}}|B_j|^{\frac{p}{2}}\frac{\Gamma_p(\frac{q_j}{2})}{\pi^{\frac{q_jp}{2}}}\frac{1}{\Gamma_p(\gamma_j+\frac{q_j}{2})}\Big\}$$
$$\times\frac{\Gamma_p(\gamma_1+\cdots+\gamma_{k+1}+\frac{q_1}{2}+\cdots+\frac{q_k}{2})}{\{\prod_{j=1}^{k}\Gamma_p(\gamma_j+\frac{q_j}{2})\}\Gamma_p(\gamma_{k+1})} \tag{7.6.4}$$

for $A_j > O,\ B_j > O,\ q_j \ge p,\ \Re(\gamma_j + \frac{q_j}{2}) > \frac{p-1}{2},\ j = 1,\ldots,k$, and $\Re(\gamma_{k+1}) > \frac{p-1}{2}$.

7.6.1. Certain properties, real rectangular matrix-variate type-1 Dirichlet density

Letting $U = \sum_{j=1}^{k} A_j^{\frac{1}{2}}X_jB_jX_j'A_j^{\frac{1}{2}}$, what might be the distributions of U and $I-U$? In the real scalar case, one could have easily evaluated the moments $E[(1-u)^h]$ for arbitrary h, which would have automatically determined the distribution of $1-u$, and therefrom that of u. In the matrix-variate case as well, one can readily determine the h-th moment of the determinant of $I-U$, $E[|I-U|^h]$, and the unique resulting distribution. However, the distribution of a determinant being unique does not imply that the distribution of the corresponding matrix is unique. Thus, we have to resort to other approaches for obtaining the distributions of U and $I-U$. Consider the following transformation:

$$V_j = A_j^{\frac{1}{2}}X_jB_jX_j'A_j^{\frac{1}{2}},\ j=1,\ldots,k-1,\ V_k=\sum_{j=1}^{k}A_j^{\frac{1}{2}}X_jB_jX_j'A_j^{\frac{1}{2}} = U.$$

Then $A_k^{\frac{1}{2}}X_kB_kX_k'A_k^{\frac{1}{2}} = U - V_1 - \cdots - V_{k-1}$ and $I - \sum_{j=1}^{k}A_j^{\frac{1}{2}}X_jB_jX_j'A_j^{\frac{1}{2}} = I - V_k = I - U$. Noting that

$$\mathrm{d}X_1\wedge\ldots\wedge\mathrm{d}X_k = \Big\{\prod_{j=1}^{k}|A_j|^{-\frac{q_j}{2}}|B_j|^{-\frac{p}{2}}\frac{\pi^{\frac{q_jp}{2}}}{\Gamma_p(\frac{q_j}{2})}\Big\}\mathrm{d}V_1\wedge\ldots\wedge\mathrm{d}V_{k-1}\wedge\mathrm{d}U, \tag{7.6.5}$$

the joint density of $V_1,\ldots,V_{k-1},\ U$, denoted by $f_3(V_1,\ldots,V_{k-1},U)$, is seen to be

$$f_3(V_1,\ldots,V_{k-1},U) = \frac{\Gamma_p(\sum_{j=1}^{k}(\gamma_j+\frac{q_j}{2})+\gamma_{k+1})}{\{\prod_{j=1}^{k}\Gamma_p(\gamma_j+\frac{q_j}{2})\}\Gamma_p(\gamma_{k+1})}\Big\{\prod_{j=1}^{k-1}|V_j|^{\gamma_j+\frac{q_j}{2}-\frac{p+1}{2}}\Big\}$$
$$\times|U-V_1-\cdots-V_{k-1}|^{\gamma_k+\frac{q_k}{2}-\frac{p+1}{2}}|I-U|^{\gamma_{k+1}-\frac{p+1}{2}}, \tag{7.6.6}$$

where

$$|U-V_1-\cdots-V_{k-1}| = |U|\,|I-U^{-\frac{1}{2}}V_1U^{-\frac{1}{2}}-\cdots-U^{-\frac{1}{2}}V_{k-1}U^{-\frac{1}{2}}|.$$

Letting $W_j = U^{-\frac{1}{2}} V_j U^{-\frac{1}{2}},\ j = 1, \ldots, k-1$, for fixed U we have

$$\mathrm{d}V_1 \wedge \ldots \wedge \mathrm{d}V_{k-1} = |U|^{(k-1)(\frac{p+1}{2})} \mathrm{d}W_1 \wedge \ldots \wedge \mathrm{d}W_{k-1}.$$

Now, the joint density of $W_1, \ldots, W_{k-1}$ and U, denoted by $f_4(W_1, \ldots, W_{k-1}, U)$, is the following:

$$f_4(W_1, \ldots, W_{k-1}, U) = C_k' \, |U|^{\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}) - \frac{p+1}{2}} |I - U|^{\gamma_{k+1} - \frac{p+1}{2}} \Big\{ \prod_{j=1}^{k-1} |W_j|^{\gamma_j + \frac{q_j}{2} - \frac{p+1}{2}} \Big\}$$
$$\times\, |I - W_1 - \cdots - W_{k-1}|^{\gamma_k + \frac{q_k}{2} - \frac{p+1}{2}} \tag{7.6.7}$$

where C_k' is the normalizing constant. We then integrate out $W_1, \ldots, W_{k-1}$ by using a $(k-1)$-variate type-1 Dirichlet integral, this yielding the result:

$$\Big\{ \prod_{j=1}^{k} \Gamma_p(\gamma_j + \frac{q_j}{2}) \Big\} \Big/ \Gamma_p\Big(\sum_{j=1}^{k} (\gamma_j + \frac{q_j}{2} \Big) \text{ for } \Re(\gamma_j + \frac{q_j}{2}) > \frac{p-1}{2},\ j = 1, \ldots, k.$$

Accordingly, the marginal density of U is the following:

$$f_5(U) = \frac{\Gamma_p(\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}) + \gamma_{k+1})}{\Gamma_p(\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}))\Gamma_p(\gamma_{k+1})} |U|^{\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}) - \frac{p+1}{2}} |I - U|^{\gamma_{k+1} - \frac{p+1}{2}} \tag{7.6.8}$$

for $O < U < I$, $\Re(\gamma_j + \frac{q_j}{2}) > \frac{p-1}{2},\ j = 1, \ldots, k,\ \Re(\gamma_{k+1}) > \frac{p-1}{2}$, and $f_5 = 0$ elsewhere. Thus, U is a real matrix-variate type-1 beta with the parameters $(\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}),\ \gamma_{k+1})$ and therefore that $I - U$ is a real matrix-variate type-1 beta with the parameters $(\gamma_{k+1},\ \sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}))$. These results are now stated as a theorem.

Theorem 7.6.1. *Consider the density given in (7.6.1). Let* $U = \sum_{j=1}^{k} A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}}$. *Then,* U *has a real matrix-variate type-1 beta distribution whose parameters are* $(\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}),\ \gamma_{k+1})$ *and* $I - U$ *is distributed as a real matrix-variate type-1 beta with the parameters* $(\gamma_{k+1},\ \sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}))$.

The h-th moment of the determinant of the matrix $I - U$ can be evaluated either from Theorem 7.6.1 or from Eq. (7.6.1). This h-th moment of the determinant, which can be worked out from the normalizing constant appearing in (7.6.8), is

$$E[|I - U|^h] = \frac{\Gamma_p(\gamma_{k+1} + h)}{\Gamma_p(\gamma_{k+1})} \frac{\Gamma_p(\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}) + \gamma_{k+1})}{\Gamma_p(\sum_{j=1}^{k}(\gamma_j + \frac{q_j}{2}) + \gamma_{k+1} + h)} \tag{7.6.9}$$

for $\Re(\gamma_j+\frac{q_j}{2})>\frac{p-1}{2}$, $\Re(\gamma_{k+1})>\frac{p-1}{2}$. Observe that a representation of the h-th moment of the determinant of U cannot be derived from (7.6.9). However, $E[|U|^h]$ can be readily evaluated from Theorem 7.6.1:

$$E[|U|^h]=\frac{\Gamma_p(\sum_{j=1}^k(\gamma_j+\frac{q_j}{2})+h)}{\Gamma_p(\sum_{j=1}^k(\gamma_j+\frac{q_j}{2}))}\frac{\Gamma_p(\sum_{j=1}^k(\gamma_j+\frac{q_j}{2})+\gamma_{k+1})}{\Gamma_p(\sum_{j=1}^k(\gamma_j+\frac{q_j}{2})+\gamma_{k+1}+h)} \tag{7.6.10}$$

for $\Re(\gamma_j+\frac{q_j}{2})>\frac{p-1}{2},\ j=1,\ldots,k,\ \Re(\gamma_{k+1})>\frac{p-1}{2}$. Upon expanding the $\Gamma_p(\cdot)$'s in terms of $\Gamma(\cdot)$'s, the following structural representations are obtained:

$$|I-U|=u_1\cdots u_p\,, \tag{7.6.11}$$

$$|U|=v_1\cdots v_p\,, \tag{7.6.12}$$

where $u_1,\ldots,u_p$ are independently distributed real scalar type-1 beta random variables with the parameters $(\gamma_{k+1}-\frac{j-1}{2},\ \sum_{j=1}^k(\gamma_j+\frac{q_j}{2})),\ j=1,\ldots,k$, and $v_1,\ldots,v_k$ are independently distributed real scalar type-1 beta random variables with the parameters $(\sum_{j=1}^k(\gamma_j+\frac{q_j}{2})-\frac{j-1}{2},\ \gamma_{k+1}),\ j=1,\ldots,k$.

7.6.2. A multivariate version of the real matrix-variate type-1 Dirichlet density

For $p=1$, consider the joint density of $Y_1,\ldots,Y_k$ in $f_2(Y_1,\ldots,Y_k)$, which shall be denoted by $f_6(Y_1,\ldots,Y_k)$. Then,

$$f_6(Y_1,\ldots,Y_k)=\Big\{\prod_{j=1}^k\frac{\Gamma(\frac{q_j}{2})}{\pi^{\frac{q_j}{2}}}\Big\}\frac{\Gamma(\sum_{j=1}^k(\gamma_j+\frac{q_j}{2})+\gamma_{k+1})}{\Big\{\prod_{j=1}^k\Gamma(\gamma_j+\frac{q_j}{2})\Big\}\Gamma(\gamma_{k+1})}\Big\{\prod_{j=1}^k|Y_jY_j'|^{\gamma_j}\Big\}$$
$$\times|I-Y_1Y_1'-\cdots-Y_kY_k'|^{\gamma_{k+1}-\frac{p+1}{2}}, \tag{7.6.13}$$

the conditions on the parameters remaining as previously stated. Note that Y_j is of the form $Y_j=(y_{j1},\ldots,y_{jq_j})$, so that $Y_jY_j'=y_{j1}^2+\cdots+y_{jq_j}^2$. Thus, in light of its structure, the density appearing in (7.6.13) has interesting properties. For instance, it can be observed that all the subsets of $Y_1,\ldots,Y_k$ also have densities belonging to the same family. Accordingly, the marginal density of Y_1 is the following:

$$f_7(Y_1) = \frac{\Gamma(\frac{q_1}{2})}{\pi^{\frac{q_1}{2}}} \frac{\Gamma(\gamma_1 + \frac{q_1}{2} + \gamma_2)}{\Gamma(\gamma_1 + \frac{q_1}{2})\Gamma(\gamma_2)} [y_{11}^2 + \cdots + y_{1q_1}^2]^{\gamma_1}$$
$$\times [1 - y_{11}^2 - \cdots - y_{1q_1}^2]^{\gamma_2 + \gamma_1 + \frac{q_1}{2} - \frac{p+1}{2}} \quad (7.6.14)$$

for $-\infty < y_{1r} < \infty,\ r = 1, \ldots, q_1,\ 0 < y_{11}^2 + \cdots + y_{1q_1}^2 < 1,\ \Re(\gamma_1 + \frac{q_1}{2}) > 0,\ \Re(\gamma_2) > 0$, and $f_7 = 0$ elsewhere. As has already been mentioned, the structure in (7.6.14) is related to geometrical probability problems involving type-1 beta distributed isotropic random points. Thus, (7.6.13) suggests the possibility of generalizing such geometrical probability problems in connection with a type-1 Dirichlet density as the underlying density for the random points. This does not appear to have yet been discussed in the literature on geometrical probability.

The complex case of the type-1 rectangular matrix-variate Dirichlet density, the real and complex cases of the rectangular matrix-variate type-2 Dirichlet density and their generalized forms can be similarly handled; hence, they will not be further discussed. Certain of these cases are brought up in this section's exercises.

Note 7.6.1. One could also consider a pathway version of the model appearing in Eq. (7.6.1). Let us replace the second line in (7.6.1) by

$$|I - a(1-\alpha)(A_1^{\frac{1}{2}} X_1 B_1 X_1' A_1^{\frac{1}{2}} + \cdots + A_k^{\frac{1}{2}} X_k B_k X_k' A_k^{\frac{1}{2}})|^{\frac{\eta}{1-\alpha} - \frac{p+1}{2}}$$

where $a > 0,\ \alpha < 1,\ \eta > 0$ are real scalar and γ_{k+1} by $\frac{\eta}{1-\alpha}$, and denote the resulting moded by f_8 whose corresponding equation number will be referred to as (7.6.15). Observe that (7.6.15) belongs to a generalized type-1 Dirichlet family of models and that the new normalizing constant will be denoted C_{k1}. For $\alpha > 1$, write $-a(1-\alpha) = a(\alpha-1) > 0$, and then $\frac{\eta}{1-\alpha} = -\frac{\eta}{\alpha-1}$. Number the resulting model of (7.6.1) as f_9, with (7.6.16) as the associated equation number. Note that (7.6.16) is actually a generalized type-2 Dirichlet model whose normalizing constant, denoted C_{k2}, will be different. Taking the limits as $\alpha \to 1_-$ in (7.6.15) and $\alpha \to 1_+$ in (7.6.16), both the models f_8 in (7.6.15) and f_9 in (7.6.16) will converge to a model f_{10} whose associated equation number will be (7.6.17), wherein the second line corresponding to the second line in (7.6.1) will be

$$\mathrm{e}^{-a\,\eta\,\mathrm{tr}(A_1^{\frac{1}{2}} X_1 B_1 X_1' A_1^{\frac{1}{2}} + \cdots + A_k^{\frac{1}{2}} X_k B_k X_k' A_k^{\frac{1}{2}})},$$

this limiting model having its own normalizing constant denoted by C_{k3}. As well, it can be established that, under the above limiting process, both C_{k1} and C_{k2} will converge to C_{k3}. Now, observe that the matrices $X_1, \ldots, X_k$ in model f_{10} are mutually independently distributed real rectangular matrix-variate gamma random variables. This turns out to be an unforeseen result as, in this case, the pathway parameter α is also seen to control the dependence to independence transitional stages. Results analogous to those obtained in Sects. 7.6, 7.6.1, and 7.6.2 could similarly be derived within the complex domain.

Exercises 7.6

7.6.1. Construct, in the complex domain, the rectangular matrix-variate type-1 Dirichlet density corresponding to the density specified in (7.6.1) and determine the associated normalizing constant.

7.6.2. Establish, in the complex domain, a theorem corresponding to Theorem 7.6.1.

7.6.3. Establish, for the complex case, the structural representations corresponding to (7.6.11) and (7.6.12).

7.6.4. Construct a real rectangular matrix-variate type-2 Dirichlet density corresponding to the density in (7.6.1).

7.6.5. Construct a complex rectangular matrix-variate type-2 Dirichlet density corresponding to the density in (7.6.1).

7.6.6. When the $p \times q_j$ matrices X_j's jointly have a real type-2 Dirichlet density with the parameter matrices $A_j > O,\ B_j > O$ as in (7.6.1) where A_j is $p \times p$, B_j is $q_j \times q_j$ and X_j is $p \times q_j,\ q_j \ge p,\ j = 1, \ldots, k$, of full rank p, establish that $U = [I + \sum_{j=1}^{k}(A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}})]^{-1}$ has a real matrix-variate type-1 beta distribution and specify its parameters. What about the density of $\sum_{j=1}^{k}(A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}})$ in this case?

7.6.7. Answer the questions in Exercise 7.6.6 for the corresponding type-2 Dirichlet density in the complex domain, replacing X_j by $\tilde{X}_j$ and X_j' by $\tilde{X}_j^*$.

7.6.8. For the real type-2 Dirichlet density in Exercise 7.6.4, determine $E[|U|^h]$ for U as specified in Exercise 7.6.6.

7.6.9. Extend all the results obtained in Sect. 7.6 to the complex domain.

7.6.10. Derive, in the complex domain, results that are analogous to those obtained for the real case in Note 7.6.1, while keeping $a,\ \eta$ and α real.

7.7. Generalizations of the Real Rectangular Dirichlet Models

The first author and his collaborators have considered several types of generalizations to the type-1 and type-2 Dirichlet models for real positive definite matrices and Hermitian positive definite matrices. We will propose certain extensions of those results to rectangular matrix-variate cases, both in the real and complex domains. Again, let X_j be a

$p\times q_j,\ q_j\ge p$, matrix of full rank p having distinct real scalar variables as its elements, for $j=1,\ldots,k$. Let the constant real positive definite matrices $A_j>O$ and $B_j>O$, where A_j is $p\times p$ and B_j is $q_j\times q_j$, $j=1,\ldots,k$, be as defined in Sect. 7.6. Consider the real model

$$\begin{aligned}f_{11}(X_1,\ldots,X_k)&=D_k|A_1^{\frac{1}{2}}X_1B_1X_1'A_1^{\frac{1}{2}}|^{\gamma_1}|I-A_1^{\frac{1}{2}}X_1B_1X_1'A_1^{\frac{1}{2}}|^{\beta_1}\\&\quad\times|A_2^{\frac{1}{2}}X_2B_2X_2'A_2^{\frac{1}{2}}|^{\gamma_2}|I-\sum_{j=1}^{2}A_j^{\frac{1}{2}}X_jB_jX_j'A_j^{\frac{1}{2}}|^{\beta_2}\ldots\\&\quad\times|A_k^{\frac{1}{2}}X_kB_kX_k'A_k^{\frac{1}{2}}|^{\gamma_k}|I-\sum_{j=1}^{k}A_j^{\frac{1}{2}}X_jB_jX_j'A_j^{\frac{1}{2}}|^{\gamma_{k+1}+\beta_k-\frac{p+1}{2}}\end{aligned}\tag{7.7.1}$$

for $\Re(\gamma_j+\frac{q_j}{2})>\frac{p-1}{2},\ j=1,\ldots,k,\ \Re(\alpha_{k+1})>\frac{p-1}{2}$, and other conditions to be specified later, where D_k is the normalizing constant. For evaluating the normalizing constant, consider the following transformations:

$$Z_j=A_j^{\frac{1}{2}}X_jB_j^{\frac{1}{2}}\Rightarrow \mathrm{d}X_j=|A_j|^{-\frac{q_j}{2}}|B_j|^{-\frac{p}{2}}\mathrm{d}Z_j,\ \ j=1,\ldots,k,\tag{i}$$

so that the model f_{11} changes to f_{12} where

$$\begin{aligned}f_{12}(Z_1,\ldots,Z_k)&=D_k\Big\{\prod_{j=1}^{k}|A_j|^{-\frac{q_j}{2}}|B_j|^{-\frac{p}{2}}\Big\}|Z_1Z_1'|^{\gamma_1}\\&\quad\times|I-Z_1Z_1'|^{\beta_1}|Z_2Z_2|^{\gamma_2}\\&\quad\times|I-Z_1Z_1'-Z_2Z_2'|^{\beta_2}\cdots|Z_kZ_k'|^{\gamma_k}|I-\sum_{j=1}^{k}Z_jZ_j'|^{\gamma_k+\beta_k-\frac{p+1}{2}}.\end{aligned}\tag{7.7.2}$$

Now, letting

$$Z_jZ_j'=S_j\Rightarrow \mathrm{d}Z_j=\frac{\pi^{\frac{q_jp}{2}}}{\Gamma_p(\frac{q_j}{2})}|S_j|^{\frac{q_j}{2}-\frac{p+1}{2}}\mathrm{d}S_j,\ \ j=1,\ldots,k,\tag{ii}$$

the model becomes

$$f_{13}(S_1,\ldots,S_k)=\Big\{\prod_{j=1}^{k}|A_j|^{-\frac{q_j}{2}}|B_j|^{-\frac{p}{2}}\frac{\pi^{\frac{q_jp}{2}}}{\Gamma_p(\frac{q_j}{2})}\Big\}|S_1|^{\gamma_1+\frac{q_1}{2}-\frac{p+1}{2}}$$
$$\times|I-S_1|^{\beta_1}|S_2|^{\gamma_2+\frac{q_2}{2}-\frac{p+1}{2}}|I-S_1-S_2|^{\beta_2}\cdots$$
$$\times|S_k|^{\gamma_k+\frac{q_k}{2}-\frac{p+1}{2}}|I-\sum_{j=1}^{k}S_j|^{\gamma_k+\beta_k-\frac{p+1}{2}}. \quad (7.7.3)$$

Now, consider the transformation (5.8.20), namely,

$$S_1=Y_1$$
$$S_2=(I-Y_1)^{\frac{1}{2}}Y_2(I-Y_1)^{\frac{1}{2}}$$
$$S_j=(I-Y_1)^{\frac{1}{2}}\cdots(I-Y_{j-1})^{\frac{1}{2}}Y_j(I-Y_{j-1})^{\frac{1}{2}}\cdots(I-Y_1)^{\frac{1}{2}} \quad (7.7.4)$$

for $j=2,\ldots,k$. Then $Y_1,\ldots,Y_k$ will be independently distributed real matrix-variate type-1 beta random variables with the parameters $(\alpha_j=\gamma_j+\frac{q_j}{2},\ \delta_j),\ j=1,\ldots,k$, where

$$\delta_j=\gamma_{j+1}+\frac{q_{j+1}}{2}+\cdots+\gamma_{k+1}+\frac{q_{k+1}}{2}+\beta_j+\cdots+\beta_k,\ j=1,\ldots,k,\ \text{and}\ q_{k+1}=0. \quad (iii)$$

The normalizing constant D_k is thus the following:

$$D_k=\left\{\prod_{j=1}^{k}|A_j|^{\frac{q_j}{2}}|B_j|^{\frac{p}{2}}\frac{\Gamma_p(\frac{q_j}{2})}{\pi^{\frac{q_jp}{2}}}\right\}\frac{\Gamma_p(\sum_{j=1}^{k}(\delta_j+\alpha_j))}{\prod_{j=1}^{k}[\Gamma_p(\alpha_j)\Gamma_p(\delta_j)]} \quad (7.7.5)$$

for $\alpha_j>\frac{p-1}{2},\ \delta_j>\frac{p-1}{2},\ j=1,\ldots,k$, where the α_j's and δ_j's are as previously given. Properties parallel to those pointed out in Sects. 7.1–7.6 can also be studied for the model specified in (7.7.1). The marginal distributions of subsets of the matrices $X_1,X_2,\ldots,X_k$, taken in the order, will belong to the same family of densities. There exist other generalizations of the type-1 and type-2 Dirichlet models. For all such generalizations, one can extend the results to the rectangular matrix-variate cases in both the real and complex domains.

Exercises 7.7

7.7.1. Develop the transformation corresponding to (7.7.4) for the real type-2 Dirichlet case. Specify the Jacobians of the transformation (7.7.4) and the corresponding transformation for the type-2 case.

7.7.2. Verify the result for δ_j in *(iii)* following (7.7.4) and develop the expression corresponding to δ_j for the type-2 Dirichlet case.

7.7.3. Derive the joint marginal density of $X_1, \ldots, X_r,\ r < k$, by integrating out the matrices starting with X_k in (7.7.1).

7.7.4. Develop, in the complex domain, the model corresponding to (7.7.1) and derive its associated normalizing constant.

7.7.5. If possible, derive the density of $U = \sum_{j=1}^{k} A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}}$ where the X_j's, $j = 1, \ldots, k$, jointly have the density given in (7.7.1).

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