Chapter 4
Surface Singularities, Seiberg–Witten Invariants of Their Links and Lattice Cohomology

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Contents

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Abstract The present note aims to focus on certain topological and analytical invariants of complex normal surface singularities and wishes to analyse their interferences. The first preliminary part introduces the needed notations, definitions and terminologies: e.g. resolutions, universal abelian coverings, natural line bundles on resolutions, links, spin^c structures on the links. Here we also recall certain vanishing theorems and statements connected with Serre's and Laufer's dualities. The next part presents two multivariable series, a topological one (associated with a dual resolution graph) and an analytic one (associated with the divisorial filtration), then we compare them. Then we introduce several topological invariants, as the Casson and Casson–Walker invariants, Turaev's torsion, the Seiberg–Witten invariant. By the 'Seiberg–Witten Invariant Conjecture' they are compared with the cohomology of the natural line bundles. In this discussion certain 'additivity formulae' will also be crucial. After a preparation (introduction of the weighted cubes) we continue with the presentation of the (topological) lattice cohomology and of the (topological) graded roots associated with rational homology sphere singularity links. They are exemplified by links of superisolated singularities, when the theory is also connected with the classification of irreducible rational cuspidal projective plane curves.

4.1 Introduction

Let (X, o) be a complex analytic normal surface singularity. The main motif of the present work is the following: what are the ties between analytic and topological invariants of (X, o)? Historically this program was started by Mumford, Artin and Laufer. Mumford realized the link as plumbed 3–manifold and proved that if the fundamental group of the link is trivial then the germ is (analytically) smooth [64]. Artin and Laufer characterized topologically the rational and minimally elliptic singularities (respectively), and computed several analytic invariants for them from the resolution graph [5, 6, 49, 50].

Let us exemplify a few pairs of analytic/topological objects, which play a central role in the text.

On the analytic side our fundamental objects are the dimensions of the sheaf cohomologies of line bundles on a resolution (including e.g. the geometric genus) and the multivariable Poincaré series of the divisorial filtration associated with a resolution. If the link of (X, o) is a rational homology sphere then we consider the universal abelian covering $(X_a, o) \to (X, o)$ too and the above listed analytic invariants associated with (X_a, o). These, reinterpreted at the level of (X, o) (and its resolutions) can be related with cohomological properties of the 'natural line bundles' on the resolution spaces $\widetilde{X}$ of (X, o).

On the topological side, the link, as an oriented 3-manifold, carries the Casson invariant whenever the link is an integral homology sphere. In the rational homology sphere case, it carries Casson–Walker invariant, the (refined) Turaev torsion, the Seiberg–Witten invariants, the lattice (co)homology and the graded roots.

Then, the Seiberg–Witten invariant (which agrees with the Euler characteristic of the lattice cohomology) will be compared with the ranks of cohomologies of line bundles (formulated by the Casson Invariant Conjecture of Neumann and Wahl whenever the link is an integral homology sphere, or by the Seiberg–Witten Invariant Conjecture of Nicolaescu and the author in the rational homology sphere case). Moreover, a topological multivariable Poincaré series (a 'zeta' function, associated with the dual graph) will be compared with its analytic counterpart provided by the divisorial filtration (as extensions of Campillo–Delgado–Gusein-Zade identity). The parallelism will be emphasized by several surgery and additivity formulae of a very similar shape present in both analytic and topological sides. (For more on such parallelisms see [77] as well.)

Regarding the topological invariants, the research of the author was greatly influenced by the work of Ozsváth and Szabó on Heegaard Floer theory of 3-manifolds. However, the techniques developed by the author to create a bridge between singularities and the low dimensional topology differ from those used in Heegaard Floer theory. The effort to create such a bridge had as a fruit and culminated in the lattice cohomology. It is defined combinatorially from the graph. Conjecturally it coincides with the Heegaard Floer cohomology. However, its definition and several of its properties resemble sheaf cohomology long exact sequences. Indeed, behind certain definitions and techniques in lattice cohomology theory one experiences certain generalizations of ideas of Laufer regarding computation sequences, used in sheaf cohomological arguments. In the new context these sequences appear as discrete 'homotopy deformation retracts'. Our presentation emphasises this continuity with Laufer's work.

The theory is exemplified by cyclic quotient, weighted homogeneous and superisolated singularities.

The presentation follows rather closely [66]. However, the present work concentrates mostly on the main statements and different connections and ideas behind the results, and basically we omit most of the proofs. The interested reader is invited to consult [66] for more information.

4.2 Resolution of Surface Singularities

4.2.1 Local Resolutions

Definition 4.2.1 Consider the germ (X, o) of a normal complex analytic surface singularity with singular points $o \in X$. Let $\phi : \widetilde{X} \to X$ be a proper analytic map, where X is a sufficiently small representative of (X, o). We also set $E := \phi^{-1}(o)$. We say that ϕ is a local *modification* of (X, o) if the restriction of ϕ induces an isomorphism $\widetilde{X} \setminus E \to X \setminus o$. Additionally, if $\widetilde{X}$ is smooth then we say that ϕ is a *resolution*.

Given two modifications $\phi_i : \widetilde{X}_i \to X_i$ $(i = 1, 2)$ of (X, o), we say that ϕ_1 *dominates* ϕ_2 if after replacing both representatives X_i of (X, o) by some smaller representative X, there exists an analytic map $\psi : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\phi_2 \circ \psi = \phi_1$.

A resolution is called *good* if all the irreducible components of E (with reduced structure) are smooth (in particular, they have no self-intersections), and intersect each other transversally.

A resolution is called *minimal* if it does not dominate (with ψ non-isomorphism) any other resolution. One defines similarly the *minimal good resolutions* as well.

Lemma 4.2.2 (Zariski's Main Theorem, see [120], [34, p. 280] for the Algebraic and [29, 30] for the analytic case) *Assume that (X, o) is a germ of a normal surface singularity and fix a resolution $\phi : \widetilde{X} \to X$, which is not an isomorphism. Then $E = \phi^{-1}(o)$ is connected, compact and one-dimensional.*

Definition 4.2.3 Let (X, o) be a normal surface singularity and ϕ a resolution.

(a) The analytic (reduced) curve E is called the *exceptional set (or curve)* of ϕ. We write $\{E_v\}_{v=1}^s$ (or, $\{E_v\}_{v\in\mathcal{V}}$) for the irreducible components of E and $g_v = g(E_v)$ denotes the geometric genus of (the normalization of) E_v.
(b) The intersection matrix I of ϕ consists of the intersection numbers $(E_v, E_u)_{v,u}$ in $\widetilde{X}$.
(c) Let $f : (X, o) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. Then the divisor $\mathrm{div}(f \circ \phi)$ on $\widetilde{X}$ decomposes into $\mathrm{div}_E(f \circ \phi) + S(f \circ \phi)$, abbreviated as $\mathrm{div}_E(f) + S(f)$, where $\mathrm{div}_E(f)$ is the part supported on E, while $S(f)$ is the *strict transform of the divisor of* f.

Example 4.2.4 Assume that (X, o) is smooth. Then by blowing up o we get a modification with an exceptional curve $E \simeq \mathbb{P}^1$ and $E^2 = -1$.

In general, if C is a curve on a smooth surface $\widetilde{X}$ with $C \simeq \mathbb{P}^1$ and $C^2 = -1$ then C is called *a* (-1)*-curve on* $\widetilde{X}$. By *Castelnuovo's Contractibility Criterion* any (-1)-curve appears as a blow up of a smooth point.

Assume that $\widetilde{X}$ is a smooth surface and C is an irreducible curve on it with $(C, C) < 0$, with genus $g(C)$, and the sum of the delta-invariants of its points is $\delta(C)$. Then by the adjunction formula $(K_{\widetilde{X}}, C) + (C, C) = -2 + 2g(C) + 2\delta(C) \geq -2$. Therefore, C is a (-1)-curve if and only if $(K_{\widetilde{X}}, C) < 0$.

The next statement guarantees the existence of a resolution, cf. [7, 35, 40, 43, 48, 57, 118, 119].

Theorem 4.2.5 *Let (X, o) be a normal surface singularity germ. Then the following facts hold.*

1. *A good resolution exists.*
2. *There is a unique minimal resolution and a unique minimal good resolution.*
3. *A resolution is minimal if and only if none of the curves E_v is a (-1)-curve.*
4. *A good resolution is minimal good if and only if any (-1)-curve intersects at least three other components.*

Remark 4.2.6 Since (X, o) is normal, $X \setminus \{o\}$ is smooth. Above, in the definition of the resolution, X was an open representative. However, (in topological discussions) we can assume additionally that X is contractible to $o \in X$ and it is closed with a compact and C^∞ boundary, cf. subsection 4.2.2. In particular, $\widetilde{X}$ has the homotopy type of E and it also has a C^∞ boundary $\partial\widetilde{X}$.

Proposition 4.2.7 (Du Val [16], see also [5, 48, 64]) *Let (X, o) be a normal surface singularity and ϕ a resolution. Then the intersection matrix $I := (E_v, E_u)_{v,u=1}^s$ is negative definite.*

Remark 4.2.8 The converse of Proposition 4.2.7 is also true. By a famous theorem of Grauert [28], any connected collection of (compact) curves on a smooth surface with negative definite intersection form can analytically be contracted to a normal singular point, hence it appears as the exceptional curve of a resolution of some normal surface singularity.

4.2.9 The Lattice Associated with a Resolution Let (X, o) be a complex normal surface singularity and let $\phi : \widetilde{X} \to X$ be a resolution. Here we take X sufficiently small and contractible (see 4.2.20).

Set $L := H_2(\widetilde{X}, \mathbb{Z})$. Since $\widetilde{X}$ has the homotopy type of E, L is freely generated by the classes of $\{E_v\}_v$ (still denoted by the same symbol E_v), and it becomes a lattice with the intersection form I. Define also $L' := H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$. It is dual to L. If for each $v \in \mathcal{V}$ one takes a transversal disc D_v to E_v (at a generic point of E_v), then their classes form a basis of L'. Furthermore, the homological map $L \to L'$ in the bases $\{E_v\}$ and $\{D_v\}$ is exactly the matrix I. Since I is non-degenerate, $L \to L'$ is injective. We write $H := L'/L$. Clearly, $|H| = |\mathrm{coker}(I)| = |\det(I)|$.

We extend the intersection form I of L to $L \otimes \mathbb{Q}$. By the perfect pairing between L and L', L' is identified with $\mathrm{Hom}(L, \mathbb{Z})$. On the other hand, $\mathrm{Hom}(L, \mathbb{Z})$ is also identified with those elements l' of $L \otimes \mathbb{Q}$ for which $(l', l) \in \mathbb{Z}$ for any $l \in L$. In the sequel we will think about L' in this way, as a sublattice of $L \otimes \mathbb{Q}$, and as an overlattice of L, endowed with the (rational) intersection form I.

Effective classes $l = \sum r_v E_v \in L'$ with all $r_v \in \mathbb{Q}_{\geq 0}$ are denoted by $L'_{\geq 0}$, and $L_{\geq 0} := L'_{\geq 0} \cap L$. There is a natural partial ordering in $L \otimes \mathbb{Q}$ associated with the bases $\{E_v\}_v$: we say that $l_1 \geq l_2$ if $l_1 - l_2$ is effective. We write $l_1 > l_2$ if $l_1 \geq l_2$ and $l_1 \neq l_2$. The cycle $\min\{l_1, l_2\}$ is the largest l with $l_1, l_2 \geq l$. If $l' = \sum_v r_v E_v$ is a rational cycle, its *support* $|l'|$ is $\cup_{v : r_v \neq 0} E_v$. Moreover, we set $\lfloor l' \rfloor := \sum_v \lfloor r_v \rfloor E_v$, and $\{l'\} := l' - \lfloor l' \rfloor$.

4.2.10 The Pontrjagin Dual of H We denote the Pontrjagin dual $\mathrm{Hom}(H, S^1)$ of H by $\widehat{H}$. Let $\theta : H \to \widehat{H}$ be the isomorphism $[l'] \mapsto e^{2\pi i (l', \cdot)}$ of H with $\widehat{H}$.

4.2.11 Lipman's Cones Associated with the Resolution [56] We prefer to replace the classes $[D_v] \in H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$, reinterpreted in L', by their 'opposites', denoted by E^*_v. That is, $E^*_v \in L' \subset L \otimes \mathbb{Q}$ satisfies $(E^*_v, E_w) = -1$ for $v = w$, and 0 otherwise. In particular, the vectors E^*_v, written in the base $\{E_v\}_v$, are exactly the columns of the matrix $-I^{-1}$, and $(I^{-1})_{vw} = (E^*_v, E^*_w)$.

Let $\mathcal{S}_{\mathbb{Q}} := \{l' \in L \otimes \mathbb{Q} \ : \ (l', E_v) \leq 0 \text{ for all } v \in \mathcal{V}\}$ be the anti-nef rational cone, $\mathcal{S}' := \mathcal{S}_{\mathbb{Q}} \cap L'$ and $\mathcal{S} := \mathcal{S}_{\mathbb{Q}} \cap L$. $\mathcal{S}'$ is generated over $\mathbb{Z}_{\geq 0}$ by the elements E^*_v.

The definition of the cone $\mathcal{S}$ is motivated by the following fact:

Lemma 4.2.12 *Let $f : (X, o) \to (\mathbb{C}, 0)$ be a holomorphic function, and ϕ a good resolution of (X, o). Then $\mathrm{div}_E(f) \in \mathcal{S} \setminus \{0\}$.*

The divisor $\mathrm{div}_E(f) = \sum_{w \in \mathcal{V}} m_w E_w$ satisfies $m_w > 0$ for all w. This is a general fact of all the elements of $\mathcal{S}'$ by the next corollary. In particular, $\mathcal{S}'$ is in the first quadrant. (This motivates the sign modification in the definition of E^*_v.)

Corollary 4.2.13

(a) Assume that $l = \sum_v r_v E_v$ with $r_v \in \mathbb{Q}$, $l \neq 0$, and $(l, E_v) \leq 0$ for all $v \in \mathcal{V}$. Then $r_v > 0$ for all $v \in \mathcal{V}$. In particular, all the entries of E^*_v are strictly positive.

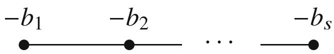
(b) For any fixed $l' \in L'$ the set $\{\tilde{l}' \in \mathcal{S}', \ \tilde{l}' \not\geq l'\}$ is finite.

4.2.14 The Resolution Graph Let (X, o) be a normal surface singularity and let $\phi : \widetilde{X} \to X$ be a *good resolution*. Denote by E the exceptional curve of ϕ with irreducible decomposition $\{E_v\}_{v\in\mathcal{V}}$. We construct a graph Γ as follows. Its *vertices* $\mathcal{V}$ correspond to the irreducible exceptional components. If two irreducible divisors corresponding to $v_1, v_2 \in \mathcal{V}$ have k intersection points then we connect v_1 and v_2 by k edges in Γ. The graph Γ is decorated as follows. Any vertex $v \in \mathcal{V}$ is decorated with the self-intersection $e_v := E_v^2$ and genus g_v of E_v (denoted as $[g_v]$). The valency (number of adjacent edges) of a vertex is denoted by κ_v.

Remark 4.2.15

(a) The graph Γ is connected by Lemma 4.2.2.
(b) The resolution is not unique, e.g. one can blow up a point of the exceptional divisor of a resolution. Accordingly, the graph Γ depends on the choice of ϕ. However, dual resolution graphs associated with different resolutions are connected by a sequence of blow ups and blow downs of vertices associated with (-1)-curves (well–defined modifications at the level of graphs).

Definition 4.2.16 A vertex of a graph with positive genus decoration, or adjacent to at least three edges, is called a *node*. A *string* is a ‘linear’ (sub)graph (with all genus-decorations zero) of type



Strings can be characterized by continued fractions.

Definition 4.2.17 To any two relative prime positive numbers n and q we associate the following (Hirzebruch, or negative) continued fraction:

$$\frac{n}{q} = [b_1, b_2, \ldots, b_s] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_s}}}, \quad b_1 \geq 1,\ b_2, \ldots, b_s \geq 2. \tag{4.1}$$

The entries $(b_1, \ldots, b_s)$ characterize a string graph with decorations $-b_1, \ldots, -b_s$.

For any pair n and q we also consider the *Dedekind sum*

$$\mathbf{s}(q, n) = \sum_{l=0}^{n-1} \left(\left(\frac{l}{n}\right)\right)\left(\left(\frac{ql}{n}\right)\right),$$

where $((x))$ is the *Dedekind symbol* (and $\{\cdot\}$ is the ‘fractional part’):

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Example 4.2.18 ([7, 35, 48, 105, 106]) For a normal surface singularity, the following conditions are equivalent. If (X, o) satisfies any of them, then it is called *Hirzebruch–Jung* or *cyclic quotient singularity*.

1. (X, o) is isomorphic with one of the 'model spaces' $\{X_{n,q}\}_{n,q}$, where $X_{n,q}$ is the normalization of $(\{xy^{n-q} = z^n\}, 0)$, where $0 < q < n,\ (n, q) = 1$.
2. There is an analytic covering $p : (X, o) \to (\mathbb{C}^2, 0)$ such that the reduced branch locus of p is $\{uv = 0\}$ in some local coordinates (u, v) of $(\mathbb{C}^2, 0)$.
3. The resolution graph Γ_X is a string (with $g_v = 0$ for any $v \in \mathcal{V}$).
4. (X, o) is the quotient singularity $(\mathbb{C}^2, 0)/\mathbb{Z}_n$ of the cyclic group $\mathbb{Z}_n = \{\xi \in \mathbb{C} : \xi^n = 1\}$ of order n, where the action is $\xi * (z_1, z_2) = (\xi z_1, \xi^q z_2)$ for some $0 < q < n$ with $(q, n) = 1$.

4.2.2 The Link

4.2.19 Let (X, o) be the germ of a normal complex analytic surface singularity and U a neighborhood of o. We fix a *real analytic* function $\rho : U \to [0, \infty)$ with $\rho^{-1}(0) = \{o\}$. In the sequel we write X_S for $\rho^{-1}(S)$ for different subsets S of $[0, \infty)$. The next theorem characterizes the local homeomorphism type of (X, o) showing its *conic structure*. For different levels of generality see [14, 18, 32, 54, 58, 59, 63].

Theorem 4.2.20 *There exists a sufficiently small* $\epsilon_0 > 0$ *such that for any* $0 < \epsilon \leq \epsilon_0$ *the inverse image* $X_{\{\epsilon\}} := \rho^{-1}(\epsilon)$ *is a* C^∞ *manifold of dimension three. Its* C^∞ *type is independent of the choice of* ϵ *and* ρ.

Moreover, the homeomorphism type of $(X_{[0,\epsilon]}, X_{\{\epsilon\}})$ *is independent of the choice of* ϵ *and* ρ*, and it is the same as the homeomorphism type of* (*real cone*$(X_{\{\epsilon\}})$, $X_{\{\epsilon\}}$)*, where the vertex corresponds to* o.

As $X_{[0,\epsilon]} \setminus \{o\}$ is a C^∞ manifold with a canonical orientation (induced by the complex structure), its boundary $X_{\{\epsilon\}}$ inherits a canonical orientation too.

Definition 4.2.21 The oriented diffeomorphism type of $X_{\{\epsilon\}}$ is called the *link of* X *at* o. It is denoted by $L(X, o)$.

Example 4.2.22

(a) Assume that X is a normal affine surface, which admits a good $\mathbb{C}^*$ action (cf. 4.2.3). Then $L(X, 0)$ is a Seifert 3-manifold.

(b) Consider the situation of Example 4.2.18(4). Set $S^3 = \{|z_1|^2 + |z_2|^2 = \epsilon\}$. Then the $\mathbb{Z}_n$-action preserves S^3, where it *acts freely*. Hence the link $L(X_{n,q}, o)$ is the lens space $L(n, q) = S^3/\mathbb{Z}_n$. Moreover, $L(n, q)$ and $L(m, p)$ are *orientation preserving diffeomorphic* if and only if $m = n$ and $p \in \{q, q'\}$, where $0 < q' < n$ and $qq' \equiv 1$ modulo n.

4.2.23 Links as Plumbed 3-Manifolds To any normal surface singularity (X, o) we associated its link $L(X, o)$ and its resolution graph Γ (well-defined up to blow up/down of (-1)-curves). The point is that they determine each other. Indeed, $L(X, o)$ is recovered from Γ via the *plumbing construction*, by considering Γ as a *plumbing graph*. For more details, see [37, 64, 87]. Note also that different plumbing graphs might produce diffeomorphic 3-manifold (via orientation preserving diffeomorphisms). However, if we restrict the plumbing construction to graphs which are *connected and have negative definite intersection matrix* then $M(\Gamma_1)$ and $M(\Gamma_2)$ are diffeomorphic if and only if the graphs are related by a sequence of (-1) blow ups and/or their inverses.

4.2.24 Homological Properties of the Link Let $\widetilde{X} = \phi^{-1}(\rho^{-1}([0, \epsilon]))$ as above with $0 < \epsilon \ll 1$. Since $i : L = H_2(\widetilde{X}, \mathbb{Z}) \to L' = H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$ is injective (see 4.2.9), the exact sequence of $(\widetilde{X}, \partial\widetilde{X})$ reads as

$$0 \to H_2(\widetilde{X}) \xrightarrow{i} H_2(\widetilde{X}, \partial\widetilde{X}) \to H_1(L_X) \to H_1(E) \to 0. \tag{4.2}$$

Set $g(\Gamma) := \sum_{v\in\mathcal{V}} g_v$ and let $c(\Gamma)$ be the number of independent cycles in Γ.

Proposition 4.2.25 ([37, 64, 107]) $L'/L = \mathrm{coker}(I) = \mathrm{Tors}(H_1(L_X, \mathbb{Z}))$, *and*

$$H_1(L_X, \mathbb{Z}) = \mathrm{coker}\,(I) \oplus H_1(E, \mathbb{Z}) = \mathrm{coker}\,(I) \oplus \mathbb{Z}^{2g(\Gamma)+c(\Gamma)}.$$

Hence, L_X is a rational homology sphere if and only if Γ is a tree with all $g_v = 0$, and L_X is an integral homology sphere when additionally $\det(-I) = 1$.

4.2.3 Example: Weighted Homogeneous Singularities

4.2.26 Definitions[99, 100] Fix some positive integers $(w_1, \dots, w_n)$. One defines the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ with weights $(w_1, \dots, w_n)$ by $t \cdot (x_1, \dots, x_n) = (t^{w_1}x_1, \dots, t^{w_n}x_n)$. A polynomial $f \in \mathbb{C}[x]$ is called weighted homogeneous of degree ℓ with respect to the weights $(w_1, \dots, w_n)$ if $f(t \cdot x) = t^\ell f(x)$, where $\ell \in \mathbb{Z}_{\geq 0}$.

Let us fix an affine algebraic variety $X \subset \mathbb{C}^n$. X is called weighted homogeneous with weights $\{w_i\}_i$ if it is stable with respect to the above action of $\mathbb{C}^*$. Since the weight are all positive the action on X is *good*, that is, the origin is contained in the closure of any orbit. If additionally we assume that $\gcd_i\{w_i\} = 1$ and $X \not\subseteq \cup_i\{x_i = 0\}$ then the action is *effective* too, that is, if $t \cdot x = x$ for all $x \in X$ then $t = 1$. If X is weighted homogeneous then its defining ideal is generated by weighted homogeneous polynomials. In particular, its affine coordinate ring is $\mathbb{Z}_{\geq 0}$-graded: $R = \oplus_{\ell\geq 0}R_\ell$. In fact, all finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebras correspond to affine varieties with good $\mathbb{C}^*$-action. However, note that the normality of $R = \oplus_{\ell\geq 0}R_\ell$ is not automatically guaranteed.

A normal analytic surface singularity (X_{an}, o) is called weighted homogeneous if there exists a normal affine surface X, which admits a good $\mathbb{C}^*$ action (with $w_i > 0$ and $\gcd_i\{w_i\} = 1$) and a singular point $o \in X$ such that (X_{an}, o) is analytically isomorphic with the (induced analytic germ) (X, o).

4.2.27 The Resolution [99] The dual graph of the minimal good resolution $\widetilde{X}$ of a weighted homogeneous germ is *star-shaped*.

A connected graph Γ is called *star-shaped* if it has a *central vertex* v_0, and $\Gamma \setminus v_0$ consists of $\nu \geq 0$ strings. Each string is connected to v_0 by an edge at one of the end-vertices of the string. In some cases, for a fixed Γ, the choice of the central vertex is not unique; e.g. if Γ itself is a string then any vertex can be central.

Next we recall some of the combinatorial properties of the star-shaped graphs.

We use the following notations: v_0 has self–intersection (Euler) number $-b_0$ and genus $g \geq 0$. The Euler numbers of the vertices v_{ji} of the jth string $(1 \leq j \leq \nu)$ are $-b_{j1}, \ldots, -b_{js_j}$, with $b_{ji} \geq 2$, determined by the continued fraction $\alpha_j/\omega_j = [b_{j1}, \ldots, b_{js_j}]$, where $\gcd(\alpha_j, \omega_j) = 1,\ 0 < \omega_j < \alpha_j$. For each j, v_0 is connected with v_{j1} by one edge. Set also $n_{j,i}/q_{j,i} := [b_{ji}, \ldots, b_{js_j}]$ with $\gcd\{n_{j,i}, q_{j,i}\} = 1$.

In such a case the plumbed 3-manifold $M(\Gamma)$ is a *Seifert fibered 3-manifold*, which means that $M(\Gamma)$ is foliated by circles such that any circle has a compact orientable saturated neighbourhood [38, 39, 87, 89, 108]. $M(\Gamma)$ and the foliation is characterized by the collection $(b_0, g; \{(\alpha_j, \omega_j)\}_j)$, called the *Seifert invariants*.

If either $g > 0$ or $\nu \geq 3$ then the choice of the central vertex is unique. In the sequel we assume this fact. The *virtual (or orbifold) Euler number* e and the *virtual Euler characteristic* χ are defined by

$$e := -b_0 + \sum_j \omega_j/\alpha_j, \quad \chi := 2 - 2g - \sum_j (\alpha_j - 1)/\alpha_j. \tag{4.3}$$

Note that for general star–shaped plumbing graphs $e < 0$ if and only if the intersection matrix $I = I(\Gamma)$ is negative definite.

Assume that $g = 0$ and let h_j denote the class $[E^*_{js_j}]$ $(j = 1, \ldots, \nu)$ and h_0 the class $[E^*_0]$ in $H = L'/L$. Then H is generated by $\{h_j\}_{j=0}^{\nu}$ with relations $b_0h_0 = \sum_{j=1}^{\nu} \omega_j h_j$ and $\alpha_j h_j = h_0$ $(j = 1, \ldots, \nu)$. Moreover, if $\mathfrak{o}$ be the order of h_0 in H and $\alpha := \mathrm{lcm}\{\alpha_1, \ldots, \alpha_\nu\}$ then (cf. [88]) $|H| = \alpha_1 \cdots \alpha_\nu |e|$ and $\mathfrak{o} = \alpha|e|$.

4.2.28 The Dolgachev–Pinkham–Demazure Formulae [103] Fix X normal, and let $R = \oplus_{\ell \geq 0} R_\ell$ be the graded algebra of X, and $P_X(t) = \sum_{\ell \geq 0} \dim R_\ell \cdot t^\ell$ the corresponding Poincaré series. Let $p_g = h^1(\mathcal{O}_{\widetilde{X}})$ be the geometric genus of (X, o) Assume next that L_X is a rational homology sphere, that is $g = 0$, and set

$$N(\ell) = \ell b_0 - \sum_j \lceil \ell\omega_j/\alpha_j \rceil. \tag{4.4}$$

Since $e < 0$ one has $\lim_{\ell\to\infty} N(\ell) = \infty$. Moreover, the following formulae hold:

$$P_X(t) = \sum_{\ell\geq 0} \max\{0, N(\ell)+1\}\, t^{\ell}, \quad \text{and} \quad p_g(X,o) = \sum_{\ell\geq 0} \max\{0, -N(\ell)-1)\}. \tag{4.5}$$

In particular, P_X and p_g are topological.

4.2.4 Example: Superisolated Singularities

4.2.29 Hypersurface superisolated singularities connect in a tautological way the theory of complex projective plane curves with normal surface singularities. They were introduced by I. Luengo [60]. For different applications see [3, 4, 60–62]. Before we start the definition of superisolated germs we review some basic facts and notations about plane curve singularities.

4.2.30 Invariants of Irreducible Plane Curve Singularities Let us fix first an irreducible plane curve singularity $(C, o) \subset (\mathbb{C}^2, 0)$. We write $\{(p_i, q_i)\}_i$ for its Newton pairs, $\Delta(t)$ for the characteristic polynomial (of the first homology of the Milnor fiber), $\mu = \deg\Delta(t)$ for the Milnor number. Furthermore, its delta-invariant $\delta(C)$ is the codimension of $n^*\mathcal{O}_{C,o} \subset \mathcal{O}_{\mathbb{C},o} = \mathbb{C}\{t\}$, where n is the normalization of (C, o). By Jung/Milnor's formula $\mu(C, o) = 2\delta(C)$ [41, 63].

The semigroup $\mathcal{S}_{C,o} \subset \mathbb{N}$ of (C, o) is the set of all the possible intersection multiplicities $(h, C)_o$, where $h \in \mathcal{O}_{\mathbb{C}^2,0}$. The delta-invariant $\delta(C)$ appears also as the cardinality of the finite set $\mathbb{N} \setminus \mathcal{S}_{C,o}$. The largest element of $\mathbb{N} \setminus \mathcal{S}_{C,o}$ is $\mu - 1$, and for $0 \leq k \leq \mu - 1$ one has the following 'gap-symmetry': $k \in \mathcal{S}_{C,o}$ if and only if $\mu - 1 - k \notin \mathcal{S}_{C,o}$. Moreover, by Campillo et al. [15]

$$\Delta(t)/(1-t) = \sum_{k\in\mathcal{S}} t^k. \tag{4.6}$$

Since $\Delta(1) = 1$ and $\Delta'(1) = \delta$, one gets $\Delta(t) = 1 + \delta(t-1) + (t-1)^2 \cdot Q(t)$ for some polynomial $Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i$ with integral coefficients. In fact, all the coefficients $\{\alpha_i\}_{i=0}^{\mu-2}$ are strict positive, and $\delta = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{\mu-2} = 1$. Indeed, by the above identity (4.6), one has $\delta + (t-1)Q(t) = \sum_{k\notin\mathcal{S}} t^k$, or $Q(t) = \sum_{k\notin\mathcal{S}}(t^{k-1} + \cdots + t + 1)$. This shows that

$$\alpha_i = \#\{k \notin \mathcal{S} \,:\, k > i\}. \tag{4.7}$$

4.2.31 Definition of Superisolated Singularities [60] A hypersurface singularity $(X, o) \subset (\mathbb{C}^3, 0)$ is called superisolated if the modification $\widetilde{X}$ of (X, o), induced by the blow up $0 \in \mathbb{C}^3$, is smooth. The definition guarantees that (X, o) is isolated. In fact, if X is not smooth, this $\widetilde{X}$ is exactly the *minimal* resolution of X.

Assume that (X, o) is the zero set of $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \cdots$, where f_j is homogeneous of degree j, $f_d \not\equiv 0$. Then (X, o) is superisolated if and only if the projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with (isolated) singularities $\{p_i\}_i$, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$. In this case the embedded topological type (and the equisingularity type) of f does not depend on the choice of f_j's for $j > d$, as long as f_{d+1} satisfies the above requirement. Therefore, those invariants of (X, o), which are stable with respect to equisingular deformations, depend only on C.

In the sequel we will assume that C is irreducible. In such a case the minimal resolution $\widetilde{X}$ has only one irreducible exceptional divisor, which is isomorphic to C, and C^2 in $\widetilde{X}$ is $-d$. Hence, the link of (X, o) is a rational homology sphere if and only if C is rational and all the plane curve singularities $(C, p_i) \subset (\mathbb{P}^2, p_i)$ are irreducible. (We use the terminology *cusp* for them.) Such a curve C is called *rational cuspidal plane curve*. We denote by μ_i and Δ_i (with the choice $\Delta_i(1) = 1$) the Milnor number and the characteristic polynomial of the local plane curve singularities $(C, p_i) \subset (\mathbb{P}^2, p_i)$. Then $\sum_i \mu_i = (d-1)(d-2)$.

The minimal good resolution is obtained from $\widetilde{X}$ by resolving the plane curve singularities $(C, p_i) \subset (\widetilde{X}, p_i)$. Note that the embedded topological types $(C, p_i) \subset (\widetilde{X}, p_i)$ and $(C, p_i) \subset (\mathbb{P}^2, p_i)$ agree. Hence, under the condition that C is irreducible and the link L_X is a rational homology sphere, the minimal good resolution graph Γ of (X, o) is the surgery graph described in 4.2.32. That is, the link of (X, o) is the oriented surgery 3-manifold $S^3_{-d}(\#_i K_i)$, where $(K_i \subset S^3)$ are the local knots of $(C, p_i) \subset (\mathbb{P}^2, p_i)$.

4.2.32 The Plumbing Graph of the Surgery Manifold $S^3_{-d}(\#_i K_i)$ with K_i Algebraic and d Arbitrary We fix an integer d and a collection of algebraic knots $\{K_i\}_{i=1}^{\nu}$ in S^3 (determined by irreducible plane curve singularities $(C_i, 0) \subset (\mathbb{C}^2, 0)$). Set the connected sum $K = K_1 \# \cdots \# K_\nu \subset S^3$ of the knots K_i. Then $S^3_{-d}(K)$ is a plumbed 3-manifold whose plumbing graph is constructed as follows. First, let Γ_i be the minimal good embedded resolution graph of $(C_i, 0) \subset (\mathbb{C}^2, 0)$ with a unique -1 vertex v_i which supports the strict transform. One also considers the cycle $Z_i = \mathrm{div}_{E(\Gamma_i)}(f_i) \in L(\Gamma_i)$ given by the local reduced equation f_i of $(C_i, 0)$; let m_i be the multiplicity in Z_i of the -1 curve of Γ_i. Then, in order to get the graph of $S^3_{-d}(K)$ from the disjoint union $\sqcup_i \Gamma_i$, one introduces a new vertex v_+, which is glued to each graph Γ_i via a new edge connecting v_+ and v_i, and one inserts the Euler decoration $-d - \sum_i m_i$ on v_+. The Euler decorations of $\{\Gamma_i\}_i$ stay unmodified. The resulting graph is negative definite if and only if $d > 0$. Furthermore, $|\det(I)| = |d|$.

4.2.33 A Restrictions Satisfied by the Combinatorial Type Consider a superisolated singularity. Let $\mathcal{S}_{C,p_i}$ be a semigroup of the local singularities (C, p_i). Fix an integer $0 \leq l < d$. In [24] is proved (via Bézout theorem) the following *Semigroup*

Distribution Inequality:

$$\min_{j_1+\cdots+j_\nu=ld+1}\ \sum_{i=1}^{\nu}\#\{\,\mathcal{S}_{C,p_i}\cap[0,\,j_i)\,\}\geq(l+1)(l+2)/2.$$

Moreover, in [24] the authors conjectured under the name *Semigroup Distribution Property*, that in the above inequality one has equality in any unicuspidal case. The general proof for any cusps was obtained by Borodzik and Livingston based on the d-invariant of Heegaard Floer theory [9]. That is, with the previous notations,

$$\min_{j_1+\cdots+j_\nu=ld+1}\ \sum_{i=1}^{\nu}\#\{\,\mathcal{S}_{C,p_i}\cap[0,\,j_i)\,\}=(l+1)(l+2)/2$$

for any rational cuspidal curve. In the unicuspidal case this reads as

$$\#\{\,\mathcal{S}_{C,p}\cap(\,(l-1)d,ld\,]\,\}=\min\{l+1,d\}\quad(l\geq 0).$$

4.2.5 Local Divisor Class Group

4.2.34 Sheaf Cohomological Properties of $\widetilde{X}$ Let us start this subsection with the following observations.

Let (X,o) be a complex normal surface singularity and let $\phi:\widetilde{X}\to X$ be a good resolution. In cohomological considerations, e.g. in the computation of $H^*(\widetilde{X},\mathbb{Z})$ or $H^*(\widetilde{X},\mathcal{F})$, we might take for $\widetilde{X}$ the space $\phi^{-1}(\rho^{-1}([0,\epsilon]))$, cf. 4.2.20. Therefore, for an analytic coherent sheaf and $q\geq 1$, $H^q(\widetilde{X},\mathcal{F})$ agrees with $(R^q\phi\mathcal{F})_o=\lim_{\to U}H^q(\phi^{-1}(U),\mathcal{F})$, where U runs over open sets $o\in U\subset X$.

By 'Theorem of formal functions', $(R^q\phi\mathcal{F})_o=\lim_{\leftarrow Z}H^q(Z,\mathcal{F}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}_Z)$, where Z runs over (larger and larger) effective cycles supported on E. In fact, for a line bundle $\mathcal{F}$ we have $H^{\geq 2}(\widetilde{X},\mathcal{F})=0$ and $H^1(\widetilde{X},\mathcal{F})=H^1(Z,\mathcal{F}\otimes\mathcal{O}_Z)$ for $Z\gg 0$, hence $\dim H^1(\widetilde{X},\mathcal{F})<\infty$. Furthermore, by Serre duality, for a locally free sheaf $\mathcal{F}$, $H^1_c(\widetilde{X},\mathcal{F})=H^1(\widetilde{X},\mathcal{F}^\vee\otimes\Omega^2_{\widetilde{X}})^*$. Note that for a divisor D supported on E and a locally free sheaf $\mathcal{F}$ on $\widetilde{X}$ we have $H^0(\widetilde{X}\setminus E,\mathcal{F}(D))=H^0(\widetilde{X}\setminus E,\mathcal{F})$ and $H^0(\widetilde{X}\setminus E,\mathcal{F})/H^0(\widetilde{X},\mathcal{F})$ is finite dimensional since it embeds into $H^1_c(\widetilde{X},\mathcal{F})$ [49].

4.2.35 The Picard Group Let $\mathrm{Pic}(\widetilde{X})=H^1(\widetilde{X},\mathcal{O}^*_{\widetilde{X}})$ denote the Picard group of $\widetilde{X}$, the group of isomorphism classes of analytic line bundles on $\widetilde{X}$. Recall also that the geometric genus of (X,o) is $p_g:=h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})$. (It is independent of the choice of the resolution.)

By duality, L' is isomorphic to $H^2(\widetilde{X},\mathbb{Z})$, hence it is the target of the first Chern class $c_1:\mathrm{Pic}(\widetilde{X})\to H^2(\widetilde{X},\mathbb{Z})$. This morphism is part of the following exact sequence induced by the exponential exact sequence of sheaves $0\to\mathbb{Z}_{\widetilde{X}}\to\mathcal{O}_{\widetilde{X}}\to$

$\mathcal{O}^*_{\widetilde{X}} \to 0$:

$$0 \to H^1(\widetilde{X}, \mathbb{Z}) \longrightarrow H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \xrightarrow{\varepsilon} \text{Pic}(\widetilde{X}) \xrightarrow{c_1} H^2(\widetilde{X}, \mathbb{Z}) \to 0. \tag{4.8}$$

Set

$$\text{Pic}^0(\widetilde{X}) := \ker(c_1) \simeq H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^1(\widetilde{X}, \mathbb{Z}) \simeq \mathbb{C}^{p_g}/H^1(E, \mathbb{Z}).$$

Since $H^1(\widetilde{X}, \mathbb{Z}) = \lim_{\to U} H^1(U, \mathbb{Z})$ and $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \lim_{\to U} H^1(U, \mathcal{O}_U)$, $E \subset U$, from (4.8) we also have $H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}}) = \lim_{\to U} H^1(U, \mathcal{O}^*_U)$. Furthermore, by Mumford [64], for any line bundle $\mathcal{L} \in H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})$ there exists $E \subset U \subset \widetilde{X}$ sufficiently small such that $\mathcal{L}|_U$ admits a meromorphic section over U. In particular, $\text{Pic}(\widetilde{X})$ can be identified with the group $\text{Cl}(\widetilde{X})$ of *local analytic divisors* near E modulo linear equivalence. More precisely, by a local analytic divisor we mean a sum $\sum_i n_i D_i$ of irreducible analytic divisors defined in a neighbourhood of E. Such a divisor is locally linear equivalent to zero if there exists a neighbourhood U of E, where all D_i are defined, and a meromorphic function on U such that $\text{div}(f) = \sum_i n_i(D_i \cap U)$.

The lattice L embeds into both $L' = H^2(\widetilde{X}, \mathbb{Z})$ and $\text{Pic}(\widetilde{X})$. For L' see 4.2.9, into $\text{Pic}(\widetilde{X})$ by $l \mapsto \mathcal{O}_{\widetilde{X}}(l)$. Similarly to the group $L'/L = \text{Tors}(H^2(X \setminus \{o\}, \mathbb{Z}))$ (cf. 4.2), $\text{Pic}(\widetilde{X})/L$ is also independent of the choice of the resolution $\widetilde{X}$. Indeed, the sequence

$$0 \to L \to \text{Pic}(\widetilde{X}) \xrightarrow{r} \text{Cl}(X, o) \to 0$$

is exact (cf. [64]), where $\text{Cl}(X, o)$ denotes the *local divisor class group* of (X, o). This is the class group of local Weil divisors of (X, o) modulo local Cartier divisors. If D is a local irreducible analytic divisor on $\widetilde{X}$, then its restriction to $\widetilde{X} \setminus E$ can be mapped to $X \setminus \{o\}$ by ϕ, and the class of its closure is $r(\mathcal{O}_{\widetilde{X}}(D))$. [This is exactly the definition of the natural map $\phi_* : \text{Cl}(\widetilde{X}) \to \text{Cl}(X, o)$, a reinterpretation of r.]

Hence we obtain the exact sequence

$$0 \to H^1(L_X, \mathbb{Z}) \to \mathbb{C}^{p_g} \to \text{Cl}(X, o) \xrightarrow{\bar{c}_1} \text{Tors}(H^2(L_X, \mathbb{Z})) \to 0. \tag{4.9}$$

The Chern class morphism $\bar{c}_1$—in the language of divisors and homology—has the form $\bar{c}'_1 : \text{Cl}(X, o) \to \text{Tors}(H_1(L_X, \mathbb{Z}))$, where $\bar{c}'_1$ assigns to a Weil divisor the homological class of its intersection with the link.

$\text{Cl}(X, o)$ coincides with the group of isomorphism classes of divisorial sheaves on (X, o). [If $\mathcal{F}$ is a divisorial sheaf, then $\mathcal{L} = (\phi^*(\mathcal{F}))^{\vee\vee}$ is locally free on $\widetilde{X}$, such that $\mathcal{L}|_{\widetilde{X}\setminus E} = \mathcal{F}|_{X\setminus\{o\}}$. By the above discussion $\mathcal{L}$ has the form $\mathcal{O}_{\widetilde{X}}(D)$, hence $\mathcal{F} = r(\mathcal{O}_{\widetilde{X}}(D))$, that is, $\mathcal{F}$ is associated with a Weil divisor $\phi_*(D)$.]

Example 4.2.36 If $j : X \setminus \{o\} \hookrightarrow X$ is the inclusion, then $\omega_X := j_*(\Omega^2(X \setminus \{o\}))$ is a divisorial sheaf. One can also write it in the form $\mathcal{O}_X(K_X)$ for a certain Weil divisor K_X. If $K_{\widetilde{X}}$ is a canonical divisor on $\widetilde{X}$, then K_X can be takes as $\phi_*(K_{\widetilde{X}})$ (or, $r(\Omega^2_{\widetilde{X}})$).

Definition 4.2.37 A Weil divisor of (X, o) (or its class) is called $\mathbb{Q}$-*Cartier*, if its class in $\mathrm{Cl}(X, o)$ has finite order. Its order is called its *index*.

4.2.6 Canonical Coverings

4.2.38 The germ of an analytic finite map $\pi : (Y, o) \to (X, o)$ (where (Y, o) and (X, o) are normal and $\pi^{-1}(o) = o$) is called o–ramified if the restriction $Y \setminus o \to X \setminus o$ is a regular (topological, unbranched) covering. An o–ramified covering is called G–covering if $Y \setminus o \to X \setminus o$ is Galois with deck transformations G. If $\pi : (Y, o) \to (X, o)$ is o–ramified, then there is a morphism $\widetilde{Y} \to \widetilde{X}$ at the level of (convenient) resolutions, and the pullback $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Y})$ induces a well–defined morphism $c^* : \mathrm{Cl}(X, o) \to \mathrm{Cl}(Y, o)$.

4.2.39 Let us recall a possibility how one can construct a cyclic o–ramified covering topologically. Let (X, o) be as above and let $\pi_1(L(X, o)) \to G$ be an epimorphism. Then, by Stein [110] it determines an o–ramified G–covering. E.g., if $L(X, o)$ is a $\mathbb{Q}HS^3$ link (that is, $H_1(L_X, \mathbb{Z}) = H = L'/L$) and we fix a character $\alpha \in \widehat{H}$, then it determines an epimorphism $\pi_1(L(X, o)) \to H \to \mathbb{Z}_N$ (for some N) and a Galois cyclic o–covering. In particular, if $L(X, o)$ is a $\mathbb{Q}HS^3$, and we start with a cycle $l' \in L'$, such that the order of $[l'] \in H$ is N, and we considered the character $\alpha := \theta([l']) \in \widehat{H}$, then we get a o–ramified $\mathbb{Z}_N$-covering $(X_\alpha, o) \to (X, o)$.

4.2.40 Next we associate a cyclic o–ramified covering $(X_D, o) \to (X, o)$ to any $\mathbb{Q}$-Cartier divisor D (in this case L_X is not necessarily a $\mathbb{Q}HS^3$).

Proposition 4.2.41 *Let D be a $\mathbb{Q}$-Cartier divisor of index N of (X, o). Then it determines a uniquely defined o-ramified Galois $\mathbb{Z}_N$-covering $c : (X_D, o) \to (X, o)$, where (X_D, o) is a normal surface singularity, and $c^*(D) = 0$ in $\mathrm{Cl}(X_D, o)$. The covering $c : (X_D, o) \to (X, o)$ depends only on the class of D in $\mathrm{Cl}(X, o)$.*

(In fact, the kernel of $c^* : \mathrm{Cl}(X, o) \to \mathrm{Cl}(X_D, o)$ is cyclic of order N and it is generated by the class of D.)

Indeed, adding a principal divisors to D we can assume that D is effective. Then $N \cdot D$ is an effective principal divisor of (X, o). Hence $N \cdot D = \mathrm{div}(f)$ for some holomorphic germ $f : (X, o) \to (\mathbb{C}, 0)$. Then define $X_{f,N}$ as the normalization of $\{(x, z) \in (X \times \mathbb{C}, (o, 0)),\ f(x) = z^N\}$. Then a local computation shows that the natural projection $c : (X_{f,N}, (o, 0)) \to (X, o)$ is o–ramified. The second statement claims that $\mathrm{div}(f \circ c)/N$ is an integral principal divisor of (X_D, o). But, indeed, this is exactly $\mathrm{div}(z)$.

Note also that the added principal divisors do not alter the isomorphism class of $X_{f,N}$. Indeed, (the normalized) $X_{fg^N,N}$ and $X_{f,N}$ are isomorphic.

4.2.42 The above facts can be used to define (in an analytic way) a covering associated with any $l' \in L'$. The construction depends on a choice, but it has no ambiguity whenever the link is a rational homology sphere. First, we associate to l' a $\mathbb{Q}$-Cartier divisor as follows. For parts *(a)–(b)* see [96, 112, 113].

Proposition 4.2.43

(a) *Fix a resolution $\phi : \widetilde{X} \to X$, $l' \in L'$, and let N be the order of its class in L'/L. Then there exists a divisor $D = D(l')$ on $\widetilde{X}$ such that one has a linear equivalence $N \cdot D \sim N \cdot l'$ and $c_1\mathcal{O}_{\widetilde{X}}(D) = l'$ (where Nl' is identified with an integral divisor supported on E). In particular, $\phi_*(D)$ has finite order N in* $\mathrm{Cl}(X, o)$.

(b) *If $H^1(\widetilde{X}, \mathbb{Z}) = 0$ then D is unique up to a linear equivalence. Hence, in this case, the correspondence $l' \mapsto \mathcal{O}_{\widetilde{X}}(D(l'))$ is a section of the exact sequence (4.8).*

(c) *If $H^1(\widetilde{X}, \mathbb{Z}) = 0$ then the covering associated with l' defined in 4.2.41 via $D(l')$ agrees with the covering associated with l' defined in 4.2.39 via the character* $\theta([l'])$.

Proof *(a)* Since c_1 is onto, there exists a divisor D_1 on $\widetilde{X}$ with $c_1\mathcal{O}_{\widetilde{X}}(D_1) = l'$. Hence $\mathcal{O}_{\widetilde{X}}(ND_1 - \mathrm{div}(Nl'))$ has the form $\epsilon(\mathcal{L})$ for some $\mathcal{L} \in \mathrm{Pic}^0(\widetilde{X}) = \mathbb{C}^{p_g}/H^1(\widetilde{X}, \mathbb{Z})$. Define D_2 so that $\mathcal{O}_{\widetilde{X}}(D_2) := \frac{1}{N}\mathcal{L} \in \mathrm{Pic}^0(\widetilde{X})$. Then $D := D_1 - D_2$ works. For *(b)* use the fact that $\mathrm{Pic}(\widetilde{X})$ is torsion free. For *(c)* use the definitions. □

Definition 4.2.44

(a) Write $\Omega^2_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ and assume that K_X is $\mathbb{Q}$-Cartier. Then the cyclic covering associated with K_X (as in 4.2.41) is called the *analytic canonical covering* of (X, o).

(b) Assume that the link of (X, o) is a rational homology sphere. The well-defined cyclic covering associated with $c_1(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}))$, constructed in 4.2.39 is called the *topological canonical covering* of (X, o).

If both assumptions are satisfied then the analytic and topological canonical coverings agree. However, if $H_1(\partial\widetilde{X}, \mathbb{Q}) = 0$, then the *topological canonical covering* is well-defined even if K_X is not $\mathbb{Q}$-Cartier.

4.2.7 Natural Line Bundles

4.2.45 Let $\phi : (\widetilde{X}, E) \to (X, o)$ be a good resolution and assume that $L(X, o)$ is a $\mathbb{Q}HS^3$. In the next discussion we identify the homology classes $l \in L$ and the integral divisors supported on E.

In the exact sequence (4.8) c_1 admits a natural group section s_L over the integral cycles $L \subset L'$. Indeed, for any $l \in L$ we can take $\mathcal{O}_{\widetilde{X}}(l) \in \mathrm{Pic}(\widetilde{X})$. Clearly $c_1(\mathcal{O}_{\widetilde{X}}(l)) = l$. In the sequel we extend s_L in a unique way to a natural group section $s : L' \to \mathrm{Pic}(\widetilde{X})$. Its existence is guaranteed by the facts that $H = L'/L$ is finite, while $\mathrm{Pic}^0(\widetilde{X}) \simeq \mathbb{C}^{p_g}$ is torsion free. In fact, we present several constructions of s, which emphasize its different geometrical aspects.

4.2.46 The Construction of s via $\mathrm{Cl}(X, o)$ [96]

For any $l' \in L'$ consider the divisor $D(l')$ provided by Lemma 4.2.43. Since $H^1(\widetilde{X}, \mathbb{Z}) = 0$, $D(l')$ is unique with the required properties of 4.2.43. Therefore one has a well-defined map $l' \mapsto s(l') = \mathcal{O}_{\widetilde{X}}(D(l'))$. By the uniqueness $D(l'_1 + l'_2) \sim D(l'_1) + D(l'_2)$, hence s is a homomorphism and a section of (4.8) as well.

Definition 4.2.47 The line bundles $s(l')$, indexed by $l' \in L'$, and denoted also by $\mathcal{O}_{\widetilde{X}}(l') := s(l')$, will be called *natural line bundles*.

Corollary 4.2.48

(a) A line bundle $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ is natural if and only if some power of it has the form $\mathcal{O}_{\widetilde{X}}(l)$ (in its usual classical sense) for an integral cycle $l \in L$. Equivalently, $\mathcal{L}$ is natural if and only if its projection by $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{X})/L = \mathrm{Cl}(X, o)$) has finite order (i.e., if it is $\mathbb{Q}$-Cartier).

(b) One has a natural isomorphism $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}^0(\widetilde{X}) \oplus L'$ given by $\mathcal{L} \mapsto (\mathcal{L} \otimes s(c_1\mathcal{L})^{-1}, c_1\mathcal{L})$. This induces a natural isomorphism $\mathrm{Cl}(X, o) \to \mathrm{Pic}^0(\widetilde{X}) \oplus H$.

In particular (since $\mathrm{Pic}^0(\widetilde{X})$ is torsion free), under this identification H is isomorphic with the group of $\mathbb{Q}$-Cartier divisor classes of (X, o).

4.2.49 The Universal Abelian Covering Let $c : (X_a, o) \to (X, o)$ be the universal abelian covering of (X, o). It is the Galois o–covering associated with $\pi_1(L_X) \to H_1(L_X, \mathbb{Z}) = L'/L$ (cf. [110]).

Let $\widetilde{c} : Z \to \widetilde{X}$ be the normalized pullback of c via ϕ. The (reduced) branch locus of $\widetilde{c}$ is included in E, and the Galois action of H extends to Z as well. Since E is a normal crossing divisor, the only singularities what Z might have are cyclic quotient singularities, cf. 4.2.18. Let $r : \widetilde{Z} \to Z$ be a resolution of these singular points such that $(\widetilde{c} \circ r)^{-1}(E)$ is a normal crossing divisor. Set $p := \widetilde{c} \circ r$.

$$\begin{array}{ccccc} \widetilde{Z} & \xrightarrow{r} & Z & \xrightarrow{\psi_a} & (X_a, o) \\ & & \downarrow{\scriptstyle \widetilde{c}} & & \downarrow{\scriptstyle c} \\ & & \widetilde{X} & \xrightarrow{\phi} & (X, o) \end{array} \tag{4.10}$$

4.2.50 The Construction of s via $p^* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Z})$ [71] One has the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & L & \to & L' & \to & H & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^H} & & \\ 0 & \to & L_a & \to & L'_a & \to & H_a & \to & 0 \end{array} \tag{4.11}$$

where the vertical arrows are pullbacks associated with $p = \widetilde{c} \circ r$ (e.g., p^* is the cohomology morphism $H^2(\widetilde{X}, \mathbb{Z}) \to H^2(\widetilde{Z}, \mathbb{Z})$ and the first arrow is the relative cohomology morphism), and the bottom line is the 'lattice exacts sequence' (4.2) associated with the resolution $\widetilde{Z} \to X_a$ of (X_a, o). We claim that:

$$p^H = 0. \tag{4.12}$$

In particular, $p^*(l') \in L_a$ for any $l' \in L'$, hence considering $p^*(l')$ as an integral divisor, the element $\mathcal{O}_{\widetilde{Z}}(p^*(l')) \in \mathrm{Pic}(\widetilde{Z})$ is well-defined.

Theorem 4.2.51 *The line bundle $\mathcal{O}_{\widetilde{Z}}(p^*(l'))$ is a pullback of a unique element $\mathcal{L}$ of* $\mathrm{Pic}(\widetilde{X})$. *This line bundle $\mathcal{L}$ will be denoted by $\mathcal{O}_{\widetilde{X}}(l')$. Moreover, $s : L' \to \mathrm{Pic}(\widetilde{X})$, defined by $l' \mapsto \mathcal{O}_{\widetilde{X}}(l')$, is a group section of c_1 in (4.8), which extends s_L.*

Furthermore, the definition of $\mathcal{O}_{\widetilde{X}}(l')$ is independent of the choice of the resolution $r : \widetilde{Z} \to Z$.

Proof Using the two exponential exact sequences one verifies that $p^* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Z})$ is injective and its image is the subgroup of invariants $(\mathrm{Pic}(\widetilde{Z}))^H$. On the other hand, $\mathcal{O}_{\widetilde{Z}}(p^*(l'))$ is H-invariant. □

4.2.52 The Construction of s via $c_*\mathcal{O}_{X_a,o}$ [42, 71, 96, 97]

Associated with the resolution $\phi : \widetilde{X} \to X$ we consider the 'unit closed-open cube' $Q := \{l' \in L' : \lfloor l' \rfloor = 0\}$. Obviously, for any $h \in H$ there is a unique element $r_h \in Q$, whose class is h. It is the minimal representative of h in the cone $L'_{\geq 0}$.

Theorem 4.2.53 ([71, 96, 97] (for the cyclic case see also [20–22])) *Assume, as above, that $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Consider the finite covering $\widetilde{c} : Z \to \widetilde{X}$. Then $\widetilde{c}_*\mathcal{O}_Z$ is a vector bundle and its H-eigensheaf decomposition has the form:*

$$\widetilde{c}_*\mathcal{O}_Z \simeq \oplus_{\alpha \in \widehat{H}} \mathcal{L}_\alpha, \tag{4.13}$$

where $\mathcal{L}_{\theta(h)} = \mathcal{O}_{\widetilde{X}}(-r_h)$ for any $h \in H$. In particular, $\widetilde{c}_*\mathcal{O}_Z \simeq \oplus_{l' \in Q} \mathcal{O}_{\widetilde{X}}(-l')$.

More generally, for any $l' \in L'$ one has

$$\widetilde{c}_*\mathcal{O}_Z(-\widetilde{c}^*(l')) \simeq \oplus_{h \in H} \mathcal{O}_{\widetilde{X}}(-r_h + \lfloor r_h - l' \rfloor). \tag{4.14}$$

Corollary 4.2.54 *The set of natural line bundles on $\widetilde{X}$ coincides with the set of line bundles of type $\mathcal{L} \otimes \mathcal{O}(l)$, where $\mathcal{L}$ is an eigensheaf of $\widetilde{c}_*\mathcal{O}_Z$ and $l \in L$. Or, via (4.14), the set of natural line bundles coincides with the set of eigensheaf of bundles of type $\widetilde{c}_*\mathcal{O}_Z(-\widetilde{c}^*(l'))$, $l' \in L'$.*

4.2.8 The Canonical Cycle

4.2.55 Fix any resolution $\widetilde{X}$. Let $K_{\widetilde{X}}$ be a *canonical divisor* (defined up to a linear equivalence), $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \Omega^2_{\widetilde{X}}$, and let $K = -Z_K$ be $c_1(\Omega^2_{\widetilde{X}}) \in L'$, the *canonical cycle of the resolution* ϕ. The cycle Z_K can be determined combinatorially from $(L', (\ ,\ ))$ via the adjunction formula, namely $(-Z_K + E_v, E_v) + 2 \cdot (1 - g(E_v) - \delta(E_v)) = 0$ for all $v \in \mathcal{V}$. (Here $\delta(E_v)$ is the sum of delta invariants of singularities of E_v.) In particular, $Z_K = 0$ if and only if $g(E_v) = \delta(E_v) = 0$ and $E_v^2 = -2$ for al v. In such a case (X, o) is an ADE singularity.

By Laufer [53], if the resolution is minimal, and $Z_K \neq 0$, then all the coefficients of Z_K are positive. Moreover, if $\widetilde{X}$ is a *minimal good resolution* and (X, o) is not of type ADE, then all the coefficients of Z_K are still positive.

Theorem 4.2.56 (Riemann–Roch Formula) *Fix a line bundle* $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ *and set* $c_1(\mathcal{L}) = l' \in L'$ *and* $k := -Z_K - 2l'$. *For any* $l \in L_{>0}$ *we consider the sheaf* $\mathcal{L} \otimes \mathcal{O}_l$ *on* l. *Then its analytic Euler characteristic satisfies*

$$\chi(\mathcal{L} \otimes \mathcal{O}_l) = -(l, l + k)/2. \tag{4.15}$$

We denote the combinatorial term from the right hand side of (4.15) by $\chi_k(l)$, or just by $\chi(l)$ if $k = -Z_K$. This expression motivates the following.

Definition 4.2.57 The set of characteristic elements are defined as

$$\mathrm{Char} = \mathrm{Char}(L) = \{k \in L' \ :\ (l, l + k) \in 2\mathbb{Z} \ \text{ for any } l \in L\}. \tag{4.16}$$

Note that $-Z_K$ is a characteristic element and $\mathrm{Char} = -Z_K + 2L'$.

The expression (4.15) can be extended to L', that is, for any $k \in \mathrm{Char}$ one defines $\chi_k : L' \to \mathbb{Q}$ by $\chi_k(l') := -(l', l' + k)/2$. If $k = -Z_K$ then we write $\chi := \chi_k$.

4.2.58 The expression $Z_K^2 + |\mathcal{V}|$ of the link behaves like a characteristic class in many index formulae. It is independent of the resolution. We have the following general formula for it.

Proposition 4.2.59 ([78]) $Z_K^2 + |\mathcal{V}|$ *in terms of the graph has the expression*

$$Z_K^2 + |\mathcal{V}| = 2 - 2b_1(L_X) + \sum_v (E_v^2 + 3) + \sum_{v,w} (2\chi(E_v) - \kappa_v)(2\chi(E_w) - \kappa_w)(E_v^*, E_w^*).$$

Example 4.2.60 ([36]) For the cyclic quotient singularity $X_{n,q}$ we have

$$Z_K^2 + |\mathcal{V}| = 2(n-1)/n - 12 \cdot \mathbf{s}(q, n).$$

Example 4.2.61 ([79]) For a star-shaped graph, with $\mathfrak{r} := \chi/e$, we have

$$Z_K^2 + |\mathcal{V}| = e\mathfrak{r}^2 + e + 5 - 12 \cdot \sum_{j=1}^{\nu} \mathbf{s}(\omega_j, \alpha_j).$$

Example 4.2.62 Assume that $L_X = S^3_{-d}(\#_i K_i)$ (cf. 4.2.32), with $\mu/2 = \delta = \sum_i \delta_i$ (the sum of delta-invariants of K_i) and arbitrary $d > 0$. Then $K^2 + |\mathcal{V}| = 1 - (d - 2 + \mu)^2/d$. If $\mu = (d-1)(d-2)$ (as in the superisolated case), then $K^2 + |\mathcal{V}| = 1 - d(d-2)^2$.

4.2.63 Splice Formula Assume that $L(X, o)$ is an integral homology sphere and let $\mathfrak{G}$ be the splice diagram associated with the plumbing graph Γ [19]. Assume that $\mathfrak{G}$ is obtained by splicing the diagrams $\mathfrak{G}_1$ and $\mathfrak{G}_2$ along the knots $K_1 \subset M(\mathfrak{G}_1)$, $K_2 \subset M(\mathfrak{G}_2)$. Let Γ_i be the plumbing graphs, which correspond to $\mathfrak{G}_i$. Recall also that $K_i \subset M(\mathfrak{G}_i)$ determines an open book decomposition, let μ_i be the first Betti number (Milnor number) of its fiber. Then one has the following.

Theorem 4.2.64 ([92])

$$\big(Z_K^2 + |\mathcal{V}|\big)(\Gamma) = \big(Z_K^2 + |\mathcal{V}|\big)(\Gamma_1) + \big(Z_K^2 + |\mathcal{V}|\big)(\Gamma_2) - 2 \cdot \mu_1 \cdot \mu_2.$$

Definition 4.2.65 The normal singularity (X, o) is called *Gorenstein* if $\Omega^2_{X\setminus\{o\}}$ is a holomorphically trivial line bundle, equivalently, if $Z_K \in L$ and one can choose for $K_{\widetilde{X}}$ the divisor $-Z_K$. Analogously, (X, o) is called *numerically Gorenstein* if $\Omega^2_{X\setminus\{o\}}$ is a topologically trivial complex line bundle.

Though Gorenstein (local) rings can be defined even without normality assumption, see e.g. [13], (e.g. complete intersections are Gorenstein even if they are not normal), here we discuss the Gorenstein property only for normal germs.

Lemma 4.2.66 ([17]) *(X, o) is numerically Gorenstein if and only if $Z_K \in L$.*

4.2.67 $\mathbb{Q}$-Gorenstein Singularities Let K_X be the canonical divisor of (X, o), cf. 4.2.36. Note that (X, o) is Gorenstein if and only if K_X is Cartier (invertible) at $o \in X$, that is, K_X is zero in $\mathrm{Cl}(X, o)$. Furthermore, if (X, o) is Gorenstein then any o-ramified covering (X', o) of (X, o) is Gorenstein. More generally, (X, o) is called $\mathbb{Q}$-Gorenstein, if there exists a positive integer r such that rK_X is a Cartier divisor at o (equivalently, if K_X has finite order in $\mathrm{Cl}(X, o)$). Again, if (X, o) is $\mathbb{Q}$–Gorenstein then any o-ramified covering (X', o) of (X, o) is $\mathbb{Q}$-Gorenstein. If $L(X, o)$ is $\mathbb{Q}HS^3$ then any numerically Gorenstein, $\mathbb{Q}$-Gorenstein singularity is Gorenstein.

4.2.68 Vanishing Theorems Fix a resolution and $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$. Then for $l_1, l_2 \in L_{>0}$ with $l_2 > l_1$ the morphisms $H^1(\widetilde{X}, \mathcal{L}) \to H^1(\mathcal{L} \otimes \mathcal{O}_{l_2})$ and $H^1(\mathcal{L} \otimes \mathcal{O}_{l_2}) \to H^1(\mathcal{L} \otimes \mathcal{O}_{l_1})$ are onto, and by the 'Theorem of formal functions' $H^1(\widetilde{X}, \mathcal{L}) = \varprojlim H^1(\mathcal{L} \otimes \mathcal{O}_l)$.

Theorem 4.2.69 ***Generalized Grauert–Riemenschneider Theorem*** *[31, 49, 104] Consider a line bundle* $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ *such that* $c_1(\mathcal{L}(-K_{\widetilde{X}})) \in \Delta - \mathcal{S}_{\mathbb{Q}}$ *for some* $\Delta \in L'$ *with* $\lfloor \Delta \rfloor = 0$*. Then for any* $l \in L_{>0}$ *one has the vanishing* $h^1(l, \mathcal{L}|_l) = 0$*. In particular,* $h^1(\widetilde{X}, \mathcal{L}) = 0$*.*

Corollary 4.2.70 *Write* $\lfloor Z_K \rfloor$ *as* $\lfloor Z_K \rfloor_+ - \lfloor Z_K \rfloor_-$ *with* $\lfloor Z_K \rfloor_+, \lfloor Z_K \rfloor_- \in L_{\geq 0}$ *and without common components. If* $\lfloor Z_K \rfloor_+ = 0$ *then* $p_g = 0$*. If* $\lfloor Z_K \rfloor_+ > 0$ *then for any* $Z \geq \lfloor Z_K \rfloor_+$*,* $Z \in L$*,* $p_g = h^1(\mathcal{O}_Z)$*.*

For certain cycles the Grauert-Riemenschneider Theorem 4.2.69 can be improved.

Proposition 4.2.71 (Lipman's Vanishing Theorem [56, Theorem 11.1]) *Take* $l \in L_{>0}$ *with* $h^1(\mathcal{O}_l) = 0$ *and* $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ *for which* $(c_1\mathcal{L}, E_v) \geq 0$ *for any* E_v *in the support of* l*. Then* $h^1(l, \mathcal{L}) = 0$*.*

4.2.9 The Role of the Monoids $\mathcal{S}$ and $\mathcal{S}'$

4.2.72 The monoids $\mathcal{S}$ and $\mathcal{S}'$ are combinatorially associated with a fixed resolution graph Γ, cf. 4.2.11.

Lemma 4.2.73 *For any fixed* $h \in H$ *set* $L'_h := \{l' \in L' \,:\, [l'] = h\}$*.*

(a) If $l'_1, l'_2 \in L'_h$ *then* $l' := \min\{l'_1, l'_2\} \in L'_h$ *too.*
(b) If $l'_1, l'_2 \in \mathcal{S}' \cap L'_h$ *then* $\min\{l'_1, l'_2\} \in \mathcal{S}' \cap L'_h$ *too.*

(For $l'_1, l'_2 \in L'$ *it can happen that* $\min\{l'_1, l'_2\}$*, defined in* $L \otimes \mathbb{Q}$*, is not in* L'*.)*

Proposition 4.2.74 *Let* $\widetilde{X} \to X$ *be a resolution of* (X, o) *as above.*

(a) For any $l' \in L'$ *there exists a unique minimal element* $e(l') \in L_{\geq 0}$ *with* $s(l') := l' + e(l') \in \mathcal{S}'$*.*
(b) $e(l')$ *can be found by the following (generalized Laufer's) algorithm. One constructs a 'computation sequence'* $z_0, z_1, \ldots, z_t \in L_{\geq 0}$ *with* $z_0 = 0$ *and* $z_{i+1} = z_i + E_{v(i)}$*, where the index* $v(i)$ *is determined by the following principle. Assume that* z_i *is already constructed. Then, if* $l' + z_i \in \mathcal{S}'$*, then one stops, and* $t = i$*. Otherwise, there exists at least one* $v \in \mathcal{V}$ *with* $(l' + z_i, E_v) > 0$*. Take for* $v(i)$ *one of these* v*'s. Then this algorithm stops after finitely many steps, and* $z_t = e(l')$*.*

Corollary 4.2.75 *For any* $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ *take* $c_1 := c_1(\mathcal{L})$ *and* $e := e(-c_1)$*. Then* $c_1(\mathcal{L}(-e)) = -s(-c_1) \in -\mathcal{S}'$ *and*

$$h^1(\mathcal{L}(-e)) - h^1(\mathcal{L}) = \chi(\mathcal{O}_e(c_1)) = \chi(e - c_1) - \chi(-c_1) \leq 0.$$

In particular, the computation of any $h^1(\mathcal{L})$ *can be reduced, modulo the combinatorics of* L*, to the computation of some* $h^1(\mathcal{L}')$ *with* $c_1(\mathcal{L}') \in -\mathcal{S}'$*.*

Example 4.2.76 If $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-l')$ for some $l' \in L'$ then 4.2.75 reads as

$$h^1(\mathcal{O}_{\widetilde{X}}(-s(l'))) - h^1(\mathcal{O}_{\widetilde{X}}(-l')) = \chi(\mathcal{O}_{e(l')}(-l')) = \chi(s(l')) - \chi(l') \leq 0.$$

The next consequence of Proposition 4.2.74 is the existence of the fundamental cycle.

Corollary 4.2.77

(a) *[5, 6] $\mathcal{S} \setminus \{0\}$ has a unique minimal element Z_{min}.*
(b) *[49] Z_{min} can be found by the following (Laufer's) algorithm. One constructs a computation sequence $z_1, \dots, z_t$ with $z_1 = E_w$ (arbitrarily chosen), and $z_{i+1} = z_i + E_{v(i)}$, where the index $v(i)$ is determined as follows. Assume that z_i is already constructed. Then, if $z_i \in \mathcal{S}$, then one stops, and $t = i$. Otherwise, there exists at least one $v \in \mathcal{V}$ with $(z_i, E_v) > 0$. Take for $v(i)$ one of these v's. Then this algorithm stops after finitely many steps, and $z_t = Z_{min}$ (independently of all the choices).*

The cycle $Z_{min} \in L_{>0}$ has several names in the literature: *minimal, fundamental, or Artin cycle*. The sequence from (b) is called the *Laufer's computation sequence for* Z_{min}.

4.2.78 The Representatives r_h and s_h Recall that for any $h \in H$, $r_h \in L'$ is the minimal representative of h in the cone $L'_{\geq 0}$. Replacing the cone $L'_{\geq 0}$ by $\mathcal{S}'$, by 4.2.73 we obtain the following.

Corollary 4.2.79 *For any $h \in H$ consider all the representatives $l' + L \subset L'$ of h. Then $(l' + L) \cap \mathcal{S}'$ has a unique minimal element s_h.*

Clearly $s_0 = 0$, and $s_h \geq r_h$. Strict inequality might appear (take e.g. the lens space $L(8, 5)$). $s_h = r_h$ if and only if $r_h \in \mathcal{S}'$, otherwise $s_h = s(r_h)$ in the sense of 4.2.74. Using 4.2.76 we obtain

$$\chi(s_h) \leq \chi(r_h). \tag{4.17}$$

Even at Euler-characteristic level, strict inequality can appear, see 4.2.89.

4.2.10 The Equivariant Geometric Genus and Laufer's Duality

4.2.80 The p_g–Formula of Laufer Let us discuss a different realizations of the geometric genus $p_g = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, where $\widetilde{X} \to X$ is any resolution.

By Serre duality $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})^*\simeq H^1_c(\widetilde{X},\Omega^2_{\widetilde{X}})$. In the exact sequence

$$H^0_c(\widetilde{X},\Omega^2_{\widetilde{X}})\to H^0(\widetilde{X},\Omega^2_{\widetilde{X}})\to H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})\to H^1_c(\widetilde{X},\Omega^2_{\widetilde{X}})\to H^1(\widetilde{X},\Omega^2_{\widetilde{X}})$$

$H^0_c(\widetilde{X},\Omega^2_{\widetilde{X}})=0$ while $H^1(\widetilde{X},\Omega^2_{\widetilde{X}})=0$ by 4.2.69. Hence,

Proposition 4.2.81 ([49])

$$H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})^*\simeq H^1_c(\widetilde{X},\Omega^2_{\widetilde{X}})=H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})/H^0(\widetilde{X},\Omega^2_{\widetilde{X}}), \tag{4.18}$$

where the last vector space is the space of global holomorphic 2-forms on $\widetilde{X}\setminus E$ up to those which can be extended holomorphically across $\widetilde{X}$.

Above, the set of poles can be bounded. Indeed, for any $Z\in L_{>0}$ consider the exact sequence of sheaves

$$0\to\Omega^2_{\widetilde{X}}\to\Omega^2_{\widetilde{X}}(Z)\to\mathcal{O}_Z(Z+K_{\widetilde{X}})\to 0.$$

Since $h^1(\Omega^2_{\widetilde{X}})=0$ (cf. 4.2.69) we get that

$$H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})=H^0(\mathcal{O}_Z(Z+K_{\widetilde{X}}))=H^1(\mathcal{O}_Z)^*. \tag{4.19}$$

Assume that $p_g\neq 0$. Then from 4.2.70*(a)* $h^1(\mathcal{O}_{\lfloor Z_K\rfloor_+})=p_g$, hence

$$p_g=\dim\,(H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(\lfloor Z_K\rfloor_+))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})). \tag{4.20}$$

This holds if $p_g=0$ too. Since $H^0(\widetilde{X},\Omega^2_{\widetilde{X}})\subset H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(\lfloor Z_K\rfloor_+))\subset H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})$, by (4.18) and (4.20) we get that $H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(\lfloor Z_K\rfloor_+))=H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})$. Hence, the poles of forms from $H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})$ are bounded by $\lfloor Z_K\rfloor_+$.

If (X,o) is numerically Gorenstein and $Z_K>0$ then $\chi(Z_K)=0$ and $h^0(\mathcal{O}_{Z_K})=h^1(\mathcal{O}_{Z_K})=p_g$. Hence, from the vanishing $h^1(\widetilde{X},\mathcal{O}(-Z_K))=0$ we obtain

$$p_g=\dim\,\big(H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-Z_K))\big). \tag{4.21}$$

If (X,o) is Gorenstein and $Z_K\geq 0$, via the isomorphism $\Omega^2_{\widetilde{X}}=\mathcal{O}_{\widetilde{X}}(-Z_K)$ the p_g formulae from (4.20) and (4.21) agree.

4.2.82 The Geometric Genus of the Universal Abelian Covering Assume that $L(X,o)$ is a $\mathbb{Q}HS^3$.

Let $(X_a,o)\to(X,o)$ be the universal abelian covering of (X,o), and consider the notations of the diagram (4.10). By definition, the geometric genus $p_g(X_a,o)$ of (X_a,o) is $h^1(\widetilde{Z},\mathcal{O}_{\widetilde{Z}})$. Recall that $r:\widetilde{Z}\to Z$ is the resolution of the cyclic quotient singularities of Z. Note that $r_*(\mathcal{O}_{\widetilde{Z}})=\mathcal{O}_Z$ (by the normality of Z), and $R^1r_*(\mathcal{O}_{\widetilde{Z}})=0$ since cyclic quotient singularities are rational (have geometric genus zero). Therefore, by Leray spectral sequence $p_g(X_a, o) = h^1(\mathcal{O}_Z)$. Since $\widetilde{c}$ is finite $h^1(\mathcal{O}_Z)$ equals $h^1(\widetilde{c}_*\mathcal{O}_Z)$, and it has an eigenspace decomposition $\oplus_{h\in H} H^1(\widetilde{c}_*\mathcal{O}_Z)_{\theta(h)}$. By Theorem 4.2.53 the dimension of the $\theta(h)$-eigenspace is

$$p_g(X_a, o)_{\theta(h)} := \dim\ H^1(\widetilde{c}_*\mathcal{O}_Z)_{\theta(h)} = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)).$$

By summation:

$$p_g(X_a, o) = \sum_{h\in H} h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)).$$

Clearly, for $h = 0$ we get $p_g(X_a, o)_{\theta(0)} = p_g(X, o)$.

Definition 4.2.83 If $H_1(L_X, \mathbb{Q}) = 0$ we define the equivariant geometric genus of (X, o) associated with $h \in H$ by $p_g(X_a, o)_{\theta(h)} = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h))$.

Via Proposition 4.2.75 it can also be expressed by s_h:

$$p_g(X_a, o)_{\theta(h)} = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-s_h)) + \chi(r_h) - \chi(s_h). \tag{4.22}$$

4.2.84 Laufer's formula (4.18) has the following generalization.

Proposition 4.2.85 *Assume that the link of (X, o) is a rational homology sphere and fix $h \in H$. Let l'_h be either r_h or s_h. Then*

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-l'_h))^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(l'_h))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)).$$

Remark 4.2.86 Since $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(r_h)) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(s_h))$, 4.2.85 gives

$$h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) - h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \dim\ H^0(\Omega^2_{\widetilde{X}}(s_h))/H^0(\Omega^2_{\widetilde{X}}(r_h)).$$

Write $s_h - r_h = \Delta$. Then from the proof of 4.2.85 one has $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(r_h)) = H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(s_h)) = H^1(\Omega^2_{\widetilde{X}}(s_h)|_\Delta) = 0$. Hence, the right hand side of the above identity is $\chi(\Omega^2_{\widetilde{X}}(s_h)|_\Delta) = \chi(r_h) - \chi(s_h)$, compatibly with (4.22).

4.2.87 In concrete computations it is always easier to find global sections than to determine higher cohomologies. This is one of the main advantages of the identity from 4.2.85. In several cases one can identify concrete basis for the vector space $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}(l'_h))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h))$, for $l'_h = r_h$ or s_h.

Example 4.2.88 $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h))$ **for weighted homogeneous singularities**, $g = 0$.

Assume that r_h in the dual basis is written as $r_h = a_0 E_0^* + \sum_{j,i} a_{ji} E_{ji}^*$. Define also $a_j := \sum_i n_{j,i+1} a_{ji}$ $(1 \le j \le \nu)$ and $N_{r_h}(\ell) = b_0\ell + a_0 - \sum_j \left\lceil \frac{\omega_j \ell - a_j}{\alpha_j} \right\rceil$. Then

$$h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) = \sum_{\ell \ge 0} \max\{0, -N_{r_h}(\ell) - 1\}. \tag{4.23}$$

Example 4.2.89 $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-s_h))$ **for weighted homogeneous singularities,** $g = 0$.

Set $s_h := \bar{a}_0 E_0^* + \sum_{j,i} \bar{a}_{ji} E_{ji}^*$ and $\bar{a}_j := \sum_i n_{j,i+1} \bar{a}_{ji}$ $(1 \le j \le \nu)$. Then

$$h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \sum_{\ell \ge 0} \max\{0, -N_{s_h}(\ell) - 1\}, \tag{4.24}$$

where $N_{s_h}(\ell) = b_0\ell + \bar{a}_0 - \sum_j \left\lceil \frac{\omega_j \ell - \bar{a}_j}{\alpha_j} \right\rceil$. Set $\Delta := s_h - r_h$ and let $\Delta_0 \in \mathbb{Z}_{\ge 0}$ be the E_0-coefficient of Δ. Then $N_{s_h}(\ell) = N_{r_h}(\ell + \Delta_0)$, hence

$$h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \sum_{\ell \ge \Delta_0} \max\{0, -N_{r_h}(\ell) - 1\}. \tag{4.25}$$

In particular,

$$h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) - h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \chi(r_h) - \chi(s_h) = \sum_{0 \le \ell < \Delta_0} \max\{0, -N_{r_h}(\ell) - 1\}.$$

This expression can be non-zero. Take e.g. the graph with $b_0 = 2$, and three legs all with invariants $(\alpha_j, \omega_j) = (3, 1)$. Then $s_h = \sum_{j=1}^3 E_{js_j}^*$, $r_h = s_h - E_0$, $\chi(s_h) = h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = 0$, and $\chi(r_h) = h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) = 1$.

Example 4.2.90 For a cyclic quotient germ $h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) = h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = 0$. (Use 4.2.53 and 4.2.71.)

4.2.11 Spin^c *Structures*

4.2.91 In the next discussion M is a link $L(X, o)$, which is a rational homology sphere.

M admits a spin^c structure. In fact, the set of spin^c structures $\mathrm{Spin}^c(M)$ is an $H^2(M, \mathbb{Z})$ torsor. Furthermore, the restriction $R : \mathrm{Spin}^c(\widetilde{X}) \to \mathrm{Spin}^c(M)$ is onto, where $\mathrm{Spin}^c(\widetilde{X})$ denotes the set of spin^c structures on $\widetilde{X}$. The natural cohomological morphism $H^2(\widetilde{X}, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is the factorization $L' \to L'/L$, $l' \mapsto [l']$. This projects Char onto Char$/L$. Then $c_1 : \mathrm{Spin}^c(\widetilde{X}) \twoheadrightarrow \mathrm{Char} \subset L'$ induces a map $c : \mathrm{Spin}^c(M) \twoheadrightarrow \mathrm{Char/L} \subset L'/L$ such that $c(R(\tilde{\sigma})) = [c_1(\tilde{\sigma})]$.

Moreover, $c([l'] * \sigma) = 2[l'] + c(\sigma)$ for any $[l'] \in L'/L$ and $\sigma \in \mathrm{Spin}^c(M)$.

While c_1 is injective, c in general is not. Its fibers are $H^1(M, \mathbb{Z}_2)$ torsors; $c^{-1}(0) \simeq \mathrm{Spin}(M)$. These facts will be explained next.

We consider the action of L on Char defined by $l*k := k+2l$. Let $\mathrm{Char}/2L$ be its orbit space. Then $\mathrm{Char}/2L$ is an L'/L torsor by the action induced by $l'*k = k+2l'$.

Moreover, the composition $R \circ c_1^{-1} : \mathrm{Char} \to \mathrm{Spin}^c(\widetilde{X}) \to \mathrm{Spin}^c(M)$ factorizes to $\mathrm{Char}/2L \to \mathrm{Spin}^c(M)$. This map is a bijection of L'/L torsors. In the sequel *we identify* $\mathrm{Spin}^c(M)$ *by this bijection*. Then $c : \mathrm{Spin}^c(M) \to \mathrm{Char}/L$ transforms into $c : \mathrm{Char}/2L \to \mathrm{Char}/L$. Its fibers are $\{l' \in L' : 2l' \in L\}/L \simeq H^1(M, \mathbb{Z}_2)$ torsors. The trivial element 0 of L'/L is in Char/L, and

$$c^{-1}(0) = (\mathrm{Char} \cap L)/2L \simeq \mathrm{Spin}(M),$$

where $\mathrm{Spin}(M)$ denotes the set of spin structures of M. (It is an $H^1(M, \mathbb{Z}_2)$ torsor.)

Definition 4.2.92 Let $M = L(X, o)$ be a singularity link. For any $k \in \mathrm{Char}$ we write $\widetilde{\sigma}(k)$ for that spin^c structure of $\widetilde{X}$ for which $c_1(\widetilde{\sigma}(k)) = k$. Similarly, let $\sigma[k] \in \mathrm{Spin}^c(M)$ be the restriction of $\widetilde{\sigma}(k)$ to M. The spin^c structure $\widetilde{\sigma}_{can}$ of $\widetilde{X}$ with $c_1(\widetilde{\sigma}) = K$ will be called *the canonical spin^c structure of* $\widetilde{X}$. Its restriction $\sigma_{can} \in \mathrm{Spin}^c(M)$ will be called the canonical spin^c structure of the link.

Lemma 4.2.93 *There is an involution* $\sigma \mapsto \overline{\sigma}$ *of* $\mathrm{Spin}^c(M)$ *which satisfies:* $c(\overline{\sigma}) = -c(\sigma)$, $\overline{[l'] * \sigma} = [-l'] * \overline{\sigma}$, *and* $\mathrm{Spin}(M) = \{\sigma \in \mathrm{Spin}^c(M) : \sigma = \overline{\sigma}\}$.

In algebraic geometry, by convention, the first Chern class of the 'canonical' line bundle is $K_{\widetilde{X}}$. Nevertheless, in simplectic geometry and differential topology, in the presence of an (almost) complex structure, the 'canonical' spin^c structure is usually defined via $-K_{\widetilde{X}}$. However, in this note we adopt the definition from Definition 4.2.92.

4.2.94 Definition of k_r Assume that the link is a rational homology sphere. Then $\mathrm{Spin}^c(\widetilde{X})$ is identified with the set of characteristic elements k on L', and if k and k' induces the same Spin^c structure on the link, then $k' = k + 2l$ for a certain $l \in L$. In this case $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$, hence the two functions χ_k and $\chi_{k'}$ can be easily compared, and they have identical qualitative properties. Therefore, for each class $[k] = k + 2L$ (that is, for each Spin^c structure $\sigma[k]$ of L_X), we choose a representive of $[k]$. Since the set of classes is indexed by H; we define the set of representatives by $k_r := K + 2s_h$, for each $h \in H$. Since $s_0 = 0$, for the trivial class $h = 0$ we get $\chi_{k_r} = \chi$.

Since for any $x \in L$ one has $\chi_{k_r}(x) = \chi(s_h + x) - \chi(s_h)$, the function χ_{k_r} defined on the integral lattice L (up to an additive constant $\chi(s_h)$) can be identified with χ acting on the (rationally) shifted lattice $s_h + L = \{l' \in L' : [l'] = h\}$.

4.3 Multivariable Series

4.3.1 The Divisorial Filtration

4.3.1 Let (X, o) be a normal surface singularity, and let $\phi : (\widetilde{X}, E) \to (X, o)$ be an arbitrary fixed resolution of (X, o). We will define an L–filtration of the local ring of (X, o) and a compatible H-equivariant L'–filtration of the local ring of (X_a, o) (where $H = L'/L$). In the whole discussion regarding the universal abelian covering (X_a, o) and the L'–filtration of its local ring we will assume that the link of (X, o) is a rational homology sphere. At the level of the L–filtration of the $\mathcal{O}_{X,o}$ this assumption is not needed.

4.3.2 The Module $\mathbb{Z}[[L']]$ Once a resolution is fixed, hence the natural basis $\{E_v\}_v$ of L is fixed too, $\mathbb{Z}[[L]]$ is identified with $\mathbb{Z}[\mathbf{t}^{\pm 1}] = \mathbb{Z}[[t_1^{\pm 1}, \dots, t_s^{\pm 1}]]$. It is contained in the larger module $\mathbb{Z}[[\mathbf{t}^{\pm 1/d}]] = \mathbb{Z}[[t_1^{\pm 1/d}, \dots, t_s^{\pm 1/d}]]$, the module of formal (Laurent) power series in variables $t_v^{\pm 1/d}$, where $d := |H|$. $\mathbb{Z}[[L']] \subset \mathbb{Z}[[\mathbf{t}^{\pm 1/d}]]$ consists of the $\mathbb{Z}$-linear combinations of monomials of type $\mathbf{t}^{l'} = t_1^{l'_1} \cdots t_s^{l'_s}$ where $l' = \sum_v l'_v E_v \in L'$. $\mathbb{Z}[[L']]$ also admits several $\mathbb{Z}$-submodules corresponding to different cones of L'; e.g. $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$, generated by monomials $\mathbf{t}^{l'}$ with $l' \in L'_{\geq 0}$, or $l' \in \mathcal{S}'$ respectively. Both $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$ have natural ring structure.

$\mathbb{Z}[[\mathcal{S}']]$ is a usual formal power series ring in variables $\{\mathbf{t}^{E_v^*}\}_v$: its elements are

$$\Phi(f)(\mathbf{t}) := f(\mathbf{t}^{E_1^*}, \dots, \mathbf{t}^{E_s^*}), \quad \text{where} \quad f(x_1, \dots, x_s) \in \mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, \dots, x_s]]. \tag{4.26}$$

Any series $S(\mathbf{t}) = \sum_{l'} a_{l'} \mathbf{t}^{l'} \in \mathbb{Z}[[L']]$ decomposes in a unique way as

$$S = \sum_{h \in H} S_h, \quad \text{where} \quad S_h = \sum_{[l']=h} a_{l'} \mathbf{t}^{l'}. \tag{4.27}$$

S_h is called the h-component of S. E.g., if $S(\mathbf{t}) := \Phi(f)(\mathbf{t})$ for some $f \in \mathbb{Z}[[\mathbf{x}]]$ as in (4.26) then

$$S_h(\mathbf{t}) = \frac{1}{|H|} \cdot \sum_{\rho \in \widehat{H}} \rho(h)^{-1} \cdot f(\rho([E_1^*])\mathbf{t}^{E_1^*}, \dots, \rho([E_s^*])\mathbf{t}^{E_s^*}). \tag{4.28}$$

4.3.2 The Analytic Series $H(\mathbf{t})$ and $P(\mathbf{t})$

Consider the diagram and the notations regarding the universal abelian covering from 4.2.49. Set $\phi_a = \psi_a \circ r$ and $p = \widetilde{c} \circ r$.

Recall that by (4.12) $p^*(l')$ is an *integral* cycle for any $l' \in L'$.

Definition 4.3.3 The L'–filtration on the local ring of holomorphic functions $\mathcal{O}_{X_a,o}$ is defined as follows. For any $l' \in L'$, we set

$$\mathcal{F}(l') := \{f \in \mathcal{O}_{X_a,o} \mid \mathrm{div}(f \circ \phi_a) \geq p^*(l')\}. \tag{4.29}$$

Notice that the natural action of H on (X_a, o) induces an action on $\mathcal{O}_{X_a,o}$, which keeps $\mathcal{F}(l')$ invariant. Therefore, H acts on $\mathcal{O}_{X_a,o}/\mathcal{F}(l')$ as well. For any $l' \in L'$, let $\mathfrak{h}(l')$ be the dimension of the $\theta([l'])$-eigenspace $(\mathcal{O}_{X_a,o}/\mathcal{F}(l'))_{\theta([l'])}$. Then one defines the Hilbert series $H(\mathbf{t})$ by

$$H(\mathbf{t}) := \sum_{l' \in L'} \mathfrak{h}(l') \cdot \mathbf{t}^{l'} \in \mathbb{Z}[[L']]. \tag{4.30}$$

Example 4.3.4 The 0-component of $H(\mathbf{t})$ is

$$H_0(\mathbf{t}) = \sum_{l \in L} \dim\ \big(\mathcal{O}_{X,o} \,/\, \{f \in \mathcal{O}_{X,o} : \mathrm{div}_E(f \circ \phi) \geq l\}\big) \cdot \mathbf{t}^l.$$

This is the Hilbert series of $\mathcal{O}_{X,o}$ associated with the divisorial filtration $L \ni l \mapsto \mathcal{F}_0(l) = \{f \in \mathcal{O}_{X,o} : \mathrm{div}_E(f \circ \phi) \geq l\}$ of all irreducible exceptional divisors of ϕ.

4.3.5 Next, we define the Poincaré series $P(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \mathfrak{p}(l')\mathbf{t}^{l'}$ associated with the filtration $\{\mathcal{F}(l')\}_{l'}$.

$$P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1}), \ \text{ or } \ \mathfrak{p}(l') = \sum_{I \subset \{1,\dots,s\}} (-1)^{|I|+1} \mathfrak{h}(l' + E_I), \quad (E_I = \sum_{v \in I} E_v). \tag{4.31}$$

It turns out that the series $P(\mathbf{t})$ is supported in $\mathcal{S}'$, and the following 'inversion identities' hold:

$$\mathfrak{h}(l') = \sum_{l \in L,\, l \ngeq 0} \mathfrak{p}(l' + l). \tag{4.32}$$

Proposition 4.3.6 *Let* $P_0(\mathbf{t}) = \sum_{l \in \mathcal{S}} \mathfrak{p}(l)\mathbf{t}^l$ *be the 0-component of* $P(\mathbf{t})$*. Then for* $l \in L$

$$h^1(\mathcal{O}_{\widetilde{X}}(-l)) = -\sum_{\tilde{l} \in L,\, \tilde{l} \ngeq l} \mathfrak{p}(\tilde{l}) + \chi(l) + p_g. \tag{4.33}$$

If $l \leq 0$*, then the sum on the right hand side is empty.*

If $l \in (-K_{\widetilde{X}} + \mathcal{S}') \cap L$ *then by the vanishing Theorem 4.2.69*

$$\sum_{\tilde{l} \in L,\ \tilde{l} \ngeq l} \mathfrak{p}(\tilde{l}) = \chi(l) + p_g. \tag{4.34}$$

That is, the counting function of the coefficients of $P_0(\mathbf{t})$*, associated with the special truncation* $\{\tilde{l} \in \mathcal{S},\ \tilde{l} \ngeq l\}$*, evaluated in the chamber* $-K + \mathcal{S}'$*, equals the quadratic polynomial* $\chi(l) + p_g$*.*

In particular, $P_0(\mathbf{t})$ *determines completely* p_g *and the functions* $l \mapsto \chi(l)$*,* $l \mapsto h^1(\mathcal{O}_{\widetilde{X}}(l))$ *(*$l \in L$*).*

4.3.7 The Equivariant Version of Proposition 4.3.6 Next, we assume that the link of (X, o) is a rational homology sphere. In particular, the universal abelian covering is well defined with its H-action. Recall that the geometric genus of (X_a, o) is the sum $\sum_h h^1(\mathcal{O}(-r_h))$ (of the equivariant genera of (X, o)) corresponding to the eigenspace decomposition of $H^1(\mathcal{O}_Z)$. Let l'_h be either r_h or s_h. Then for any fixed h the equivariant analogues of the formulae from Example 4.3.6 are the following.

For $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-l')$, where $l' \in L'$, $l' = l + l'_h$ with $l \in L$,

$$\begin{aligned} h^1(\mathcal{O}(-l')) &= -\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \ngeq l'} \mathfrak{p}(\tilde{l}') + \chi_{K+2l'_h}(l) + h^1(\mathcal{O}(-l'_h)) \\ &= -\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \ngeq l'} \mathfrak{p}(\tilde{l}') + \chi(l') + h^1(\mathcal{O}(-l'_h)) - \chi(l'_h). \end{aligned} \tag{4.35}$$

In particular, when $l' \in -K + \mathcal{S}'$, $l' = l + l'_h$ with $l \in L$,

$$\begin{aligned} \sum_{[\tilde{l}']=[l'],\ \tilde{l}' \ngeq l'} \mathfrak{p}(\tilde{l}') =& \chi_{K+2l'_h}(l) + h^1(\mathcal{O}(-l'_h)) \\ =& \chi(l') + h^1(\mathcal{O}(-l'_h)) - \chi(l'_h). \end{aligned} \tag{4.36}$$

Therefore, $P(\mathbf{t})$ determines completely each $h^1(\mathcal{O}_{\widetilde{X}}(l'))$ $(l' \in L')$.

Remark 4.3.8 The following comment is appropriate. In the above formulae (e.g. in 4.3.6 and 4.3.7) the term consisting of the sum of the coefficients of P can be replaced (via (4.32)) by the corresponding coefficient of the Hilbert series $H(\mathbf{t})$. E.g., (4.34), under the same assumption, reads as $\mathfrak{h}(l) = \chi(l) + p_g$. The corresponding versions in terms of the Hilbert series are simpler (and from the analytic point of view even more conceptual). The reason why we prefer above the summation expressions is the following. Later we will introduce the topological analogues of the above identities. The point is that $P(\mathbf{t})$ will have a topological analogue, namely $Z(\mathbf{t})$ (see subsection 4.3.3), however, the analogue of $H(\mathbf{t})$ will be defined ('merely') as the inversion of $Z(\mathbf{t})$, that is, by the summation of its coefficients. Hence, later we will hunt in the topological side for sum–expressions as above, where the coefficients of P will be replaced by those of Z.

4.3.3 The Topological Series $Z(\mathbf{t})$

4.3.9 We assume that L_X is a $\mathbb{Q}HS^3$ and we fix a good resolution as above.

Definition 4.3.10 We define the rational function $Z(\mathbf{t})$ in variables $x_v = \mathbf{t}^{E_v^*}$ by

$$Z(\mathbf{t}) := \Phi(z)(\mathbf{t}), \quad \text{where } \ z(\mathbf{x}) := \prod_{v\in\mathcal{V}} (1-x_v)^{\kappa_v - 2}. \tag{4.37}$$

Hence $Z(\mathbf{t}) = \prod_v (1-\mathbf{t}^{E_v^*})^{\kappa_v-2}$. By (4.28), its h-component for any $h \in H$ is

$$Z_h(\mathbf{t}) := \frac{1}{|H|} \cdot \sum_{\rho\in\widehat{H}} \rho(h)^{-1} \cdot \prod_{v\in\mathcal{V}} (1-\rho([E_v^*])\mathbf{t}^{E_v^*})^{\kappa_v-2}. \tag{4.38}$$

In the sequel we identify the rational function $Z(\mathbf{t})$ with its Taylor expansion at the origin, as an element of $\mathbb{Z}[[\mathcal{S}']]$ (cf. 4.26).

Example 4.3.11 (Splice Quotient Singularities) Splice quotient singularities were introduced by Neumann and Wahl in [91]. From any fixed graph Γ (for which $M(\Gamma)$ is a $\mathbb{Q}HS^3$ and Γ has some additional special arithmetical properties too, see below) one constructs a family of singularities with common equisingularity type, such that any member admits a distinguished resolution, whose dual graph is exactly Γ. The construction suggests that the analytic properties of the singularities constructed in this way are strongly linked with the fixed resolution and with its graph Γ. (Hence, the expectation is that certain analytic invariants might be computable from Γ.)

There are three different approaches how one can define the splice quotient singularities; they are based on different geometric properties: (a) the 'original' construction of Neumann–Wahl [91] (where Γ satisfies the additional *semigroup and the congruence conditions*), (b) the 'modified' version by Okuma [97] (where Γ satisfies the *monomial condition*), and (c) considering resolution of singularities satisfying the *end-curve condition* [93, 98]. It turns out that all these approaches provide the same family of singularities.

Rational singularities (where ϕ is an arbitrary resolution), minimally elliptic singularities, (where ϕ is a resolution in which the support of the minimal elliptic cycle is E), and weighted homogeneous singularities (where ϕ is the minimal good resolution) are splice quotient singularities.

Theorem 4.3.12 ([75]) *Assume that (X, o) admits a resolution ϕ, which satisfies the end curve condition, and $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Then $P(\mathbf{t}) = Z(\mathbf{t})$.*

Conversely, assume that the singularity (X, o) *satisfies* $H^1(\widetilde{X}, \mathbb{Z}) = 0$*, and we fix one of its good resolutions* ϕ*. If associated with* ϕ *one has* $P(\mathbf{t}) = Z(\mathbf{t})$*, then the 'end curve condition' for* ϕ *is also satisfied.*

Corollary 4.3.13 *Assume that* (X, o) *admits a resolution* ϕ*, which satisfies the end curve condition, and* $H^1(\widetilde{X}, \mathbb{Z}) = 0$*. Then* $h^1(\mathcal{O}_{\widetilde{X}}(l'))$ *is topological for any* $l' \in L'$*.*

Indeed, write $Z(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \mathfrak{z}(l')\mathbf{t}^{l'}$*. Then, after the identification* $P(\mathbf{t}) = Z(\mathbf{t})$*, the formulae from 4.3.7 read as follows:*

1. *For* $l' \in -K + \mathcal{S}'$

$$\sum_{[\tilde{l}'] = [l'],\ \tilde{l}' \ngeq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l' - r_h) + h^1(\mathcal{O}_{\widetilde{X}}(-r_h)); \tag{4.39}$$

2. *More generally, for* $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-l')$ *with arbitrary* $l' \in L'$*,*

$$h^1(\mathcal{O}_{\widetilde{X}}(-l')) = -\sum_{[\tilde{l}'] = [l'],\ \tilde{l}' \ngeq l'} \mathfrak{z}(\tilde{l}') + \chi_{K_{\widetilde{X}}+2r_h}(l' - r_h) + h^1(\mathcal{O}_{\widetilde{X}}(-r_h)). \tag{4.40}$$

4.3.4 Reductions of Variables in the Series $P(\mathbf{t})$ *and* $Z(\mathbf{t})$

For any fixed resolution ϕ, in the definition of the series $P(\mathbf{t})$ and $Z(\mathbf{t})$ one takes a variable t_v for each exceptional divisor E_v of ϕ. In most of the situations we strongly suspect that some of the variables are superfluous. E.g., if the resolution is not minimal, the non-essential exceptional components carry less information; the same is valid even for some of the exceptional curves of the minimal resolution, e.g. those with $\kappa_v = 2$. Moreover, certain exceptional divisors might have some intrinsic geometric meaning, and sometimes we wish to concentrate only on them.

4.3.14 We fix (X, o) as in 4.3.1 and the resolution ϕ. Let $\mathcal{I}$ be a non-empty subset of $\mathcal{V}$. Associated with it we consider formal series in variables $\{t_v\}_{v \in \mathcal{I}}$, denoted by $\mathbf{t}_{\mathcal{I}}$, and the projection $\pi_{\mathcal{I}} : L' \to L \otimes \mathbb{Q}$, $\pi_{\mathcal{I}}(\sum_{v \in \mathcal{V}} l'_v E_v) = \sum_{v \in \mathcal{I}} l'_v E_v$. We write

$$l'_{\mathcal{I}} := \pi_{\mathcal{I}}(l'), \ \text{ and } \ \mathbf{t}_{\mathcal{I}}^{l'} = \prod_{v \in \mathcal{I}} t_v^{l'_v} = \mathbf{t}^{l'}|_{t_v = 1 \text{ for all } v \notin \mathcal{I}}.$$

Here a word of warning is necessary. In the original case $\mathcal{I} = \mathcal{V}$, from a series $S(\mathbf{t}) = \sum_{l'} a_{l'}\mathbf{t}^{l'}$ we can recover its h-components S_h. Indeed, the monomial $a_{l'}\mathbf{t}^{l'}$ belongs to S_h if and only if $[l'] = h$. However, this property will be lost when we reduce the variable: from the information carried by $\pi_{\mathcal{I}}(l')$ one cannot recover $[l']$. Therefore, *the reduced h-components of a series* $S(\mathbf{t})$ *are defined as the* *reductions of the original h-components* $S_h(\mathbf{t})$ *(and they cannot be recovered from the reduced* S*).*

Definition 4.3.15 The **reduced series of** Z is defined as $Z_{\mathcal{I}}(\mathbf{t}_{\mathcal{I}}) := Z(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathcal{I}}$. Similarly, for any $h \in H$, $Z_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}}) := Z_h(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathcal{I}}$. Equivalently,

$$Z_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}}) := \frac{1}{|H|} \cdot \sum_{\rho\in\widehat{H}} \rho(h)^{-1} \cdot \prod_{v\in\mathcal{V}} \big(1-\rho([E_v^*])\mathbf{t}_{\mathcal{I}}^{E_v^*}\big)^{\kappa_v-2}. \tag{4.41}$$

The substitutions $\{t_v = 1\}_{v\notin\mathcal{I}}$ are well-defined since $Z(\mathbf{t})$ is supported on $\mathcal{S}'$, which has the special finiteness property 4.2.13.

4.3.16 Reducing Variables in Series $P(\mathbf{t})$ In the case of the analytic series $P(\mathbf{t})$ we can proceed, a priori, in two different ways. By the first one we reduce $P(\mathbf{t})$ 'blindly', as we did with $Z(\mathbf{t})$ in 4.3.15, via substitutions $t_v = 1$ for all $v \notin \mathcal{I}$. Again, this step is well-defined since P too is supported on $\mathcal{S}'$.

On the other hand, we can also repeat the original geometric definition of $P(\mathbf{t})$, as the multivariable Poincaré series associated with the divisorial filtration as in (4.31), however, at this time we will use the 'reduced set of divisors' indexed by $\mathcal{I}$. However, it turns out that the two approaches lead to the same object.

Corollary 4.3.17 *Assume that for a resolution* ϕ *and an element* $h \in H$ *the identity* $P_h(\mathbf{t}) = Z_h(\mathbf{t})$ *is valid. Then for the same* ϕ *and* h *and for any non-empty* $\mathcal{I} \subset \mathcal{V}$ *the 'reduced identity'* $Z_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}}) = P_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}})$ *(in* $\mathbb{Z}[[t_v^{1/\det(I)}, v \in \mathcal{I}]]$*) is valid too.*

In Sects. 4.3.5 and 4.3.6 we exemplify cases when $\mathcal{I}$ contains only one element. Our goal is to compare the analytic reduced series $P_{h,\mathcal{I}}$ with the topological series $Z_{h,\mathcal{I}}$.

4.3.5 Example: P and Z for Weighted Homogeneous Germs

Assume that (X, o) is weighted homogeneous and its minimal good resolution is star-shaped with $\nu \geq 3$. We set $\mathcal{I} = \{\text{central vertex } v_0\}$.

Our plan is to compare three filtrations and to show that they agree.

Firstly, the E_0-divisorial filtration coincides with the filtration given by the $\mathbb{C}^*$ action.

Assume next that $g = 0$, hence the universal abelian covering is well-defined, it is a Brieskorn isolated complete intersection singularity. Therefore, one has three equivariant $\mathbb{Z}$–filtrations of $\mathcal{O}_{X_a,o}$: the divisorial filtration $\mathcal{F}_{\mathcal{I}}$ associated with the central divisor E_0, the filtration/grading associated with the $\mathbb{C}^*$-action, and the monomial filtration $\mathcal{G}_{\mathcal{I}}$ associated with v_0.

The monomial filtration is determined by the following grading. If we denote the variables of the Brieskorn equations by $\{z_i\}_{i=1}^{\nu}$, then their degrees are $\deg(z_i) = \deg(E^*_{s_i}) = (\alpha_i|e|)^{-1}$ $(1 \leq i \leq \nu)$. The degree of the Brieskorn equations of the universal abelian covering are $|e|^{-1}$ (hence the Brieskorn exponent of z_i is α_i). This coincides exactly with the weights of the $\mathbb{C}^*$-action on (X_a, o). In particular, the monomial filtration and the filtration induced by the $\mathbb{C}^*$-action agree. Similarly as above, the filtration induced by the $\mathbb{C}^*$-action and the divisorial filtrations agree too.

The (common) Poincaré series of the above filtrations agree with the topological series $Z_{h,\mathcal{I}}(t)$ (the variable t corresponds to v_0). This fact can be seen in many different ways (see e.g. [79, 88, 103]). E.g.:

(i) The identity $P = Z$ was proved for any singularity which satisfies the end curve condition. Then the identity $P_{h,\mathcal{I}} = Z_{h,\mathcal{I}}$ follows from 4.3.17 (since the minimal good resolution of a weighted homogeneous germ satisfies the end curve condition).
(ii) If $h = 0$ then the Poincaré series of the graded $\mathcal{O}_{X,o}$ was computed analytically via the Dolgachev–Pinkham–Demazure technique, the output is identical with $Z_{h,\mathcal{I}}(t)$, cf. 4.2.28.

For any fixed $h \in H$, let $l'_h \in L'$ be one of its representatives. If $l'_h = a_0E^*_0 + \sum_{ik} a_{ik}E^*_{ik}$, then $l'_{\text{red}} := a_0E^*_0 + \sum_{ik} a_{ik}n^i_{k+1,s_i}E^*_{is_i}$ is still a representative, and

$$\mathfrak{a} := \pi_{\mathcal{I}}(l') = \pi_{\mathcal{I}}(l'_{\text{red}}) = -(E^*_0, l') = \frac{1}{|e|} \cdot \Big(a_0 + \sum_j \frac{a_j}{\alpha_j}\Big) \in \frac{1}{\mathfrak{o}}\mathbb{Z}.$$

The rational number $\mathfrak{a}$ modulo $\mathbb{Z}$ is independent of the choice of the representative l'_h, it depends only on h (and any integral shift can be realized by different choices). In particular, $\pi_{\mathcal{I}}(L + r_h) = \mathfrak{a} + \mathbb{Z}$.

The common Poincaré series is given by

$$P_{h,\mathcal{I}}(t) = \sum_{\ell\in\mathbb{Z},\ \ell\geq-\mathfrak{a}} \max\Big\{\, 0\,,\ 1 + a_0 + \ell b - \sum_j \Big\lceil \frac{\ell\omega_j - a_j}{\alpha_j} \Big\rceil \Big\} \cdot t^{\ell+\mathfrak{a}}.$$

With the choice $l'_h = r_h$ one has $\mathfrak{a} \in [0, 1)$.

This expression can also be compared with another expression obtained via a rather different construction, namely via the universal cycles $x(\ell)$ and their τ-function, cf. 4.7.22.

4.3.6 Example: P_0 and Z_0 for Superisolated Singularities

Next, we compute the one-variable $\{v_+\}$–reduced series P_0 and Z_0 for superisolated singularities associated with an irreducible curve C, and we formulate geometric properties and conjectures about their difference. Such properties might serve as combinatorial criteria for the existence of the rational cuspidal curve C with given topology.

4.3.18 Assume that (X, o) is a superisolated singularity with C irreducible and with a rational homology sphere link, cf. subsection 4.2.4. Let ϕ be its minimal good resolution described in 4.2.31 and 4.2.32. We set $\mathcal{I} = \{v_+\}$ (the vertex corresponding to the curve) and $h = 0$.

Set $\Delta(t) := \prod_i \Delta_i$. Then $\Delta(1) = 1$ and $d\Delta/dt(1) = \delta$, where $\delta = \sum_i \delta_i = (\sum_i \mu_i)/2 = (d-1)(d-2)/2$ is the sum of delta-invariants. Hence, Δ can be written as $\Delta(t) = 1+(t-1)\delta+(t-1)^2 Q(t)$ for an integral polynomial $Q(t) = \sum_{j=0}^{2\delta-2} \alpha_j t^j$ (see 4.2.30). For $\nu = 1$ one has $Q(t) = \sum_{s \notin \mathcal{S}_{C,p_1}} (1 + t + \cdots + t^{s-1})$, hence

$$\alpha_j = \#\{s \notin \mathcal{S}_{C,p_1} \ : \ s > j\} \quad (\text{if } \nu = 1). \tag{4.42}$$

Since $s \notin \mathcal{S}_{C,p_1}$ if and only if $2\delta - 1 - s \in \mathcal{S}_{C,p_1}$, we get

$$\alpha_{(d-3-j)d} = \#\{s \in \mathcal{S}_{C,p_1} \ : \ s \le jd\} \quad (\text{if } \nu = 1,\ 0 \le j \le d-3). \tag{4.43}$$

4.3.19 We wish to compare $P_{0,\mathcal{I}}(t)$ and $Z_{0,\mathcal{I}}(t)$. Firstly, $P_{0,\mathcal{I}}(t) = (1-t^d)/(1-t)^3$.

By the definition of $Z_{0,\mathcal{I}}$, and from A'Campo's formula (and using the fact that $H = \mathbb{Z}_d$ is generated by $[E_+]$), we obtain

$$Z_{0,\mathcal{I}}(t) = \frac{1}{d} \sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} .$$

Lemma 4.3.20 *The difference*

$$N(t) := Z_{0,\mathcal{I}}(t) - P_{0,\mathcal{I}}(t) = \frac{1}{d} \sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} - \frac{1-t^d}{(1-t)^3} \tag{4.44}$$

has the following properties:

(a) $N(0) = 0$, *and* $N(t)$ *is a symmetric polynomial:* $N(t) = t^{d-3} \cdot N(1/t)$.
(b)

$$N(t) = \sum_{j=0}^{d-3} \Big(\alpha_{(d-3-j)d} - \frac{(j+1)(j+2)}{2}\Big) t^{d-3-j}.$$

Assume that $\nu = 1$. Then 4.3.20*(b)* combined with (4.43) says that the Semigroup Distribution Property guarantees the vanishing of $N(t)$. However, for $\nu \ge 2$, $N(t) \ne 0$ might appear (see [24]). Several examples computed in [loc. cit.] supported the following (hasty) conjecture.

Conjecture 4.3.21 ([24]) All the coefficients of $N(t)$ are non-positive for any rational cuspidal curve.

If $\nu = 1$ then the conjecture is true since $N(t) \equiv 0$. If $\nu = 2$ then the Conjecture is true again, it follows from the Semigroup Distribution Property and certain lattice cohomology formulae of the link of superisolated singularities; the method even provides a conceptual meaning of the coefficients of $-N(t)$ in terms of ranks of certain first lattice cohomology groups. See subsection 4.9.2 for a detailed discussion.

However, the conjecture fails for certain curves with $\nu = 3$ [8].

A 'weaker' version of Conjecture 4.3.21 was formulated in [8], it is a numerical inequality (instead of a polynomial one); in fact, it is more in the spirit of the motivation of the original Conjecture 4.3.21, since it is a reformulation of an inequality between the geometric genus of a superisolated singularity and the normalized Seiberg–Witten invariant of the link (see again subsection 4.9.2 for the complete discussion).

Conjecture 4.3.22 ([8]) $N(1) \leq 0$ for any rational cuspidal curve.

Note that by Lemma 4.3.20*(b)* one has:

$$N(1) = \sum_{j=0}^{d-3} \alpha_{(d-3-j)d} - \frac{d(d-1)(d-2)}{6} = -p_g + \sum_{j=0}^{d-3} \alpha_{(d-3-j)d}. \tag{4.45}$$

Clearly, Conjecture 4.3.21 implies this second one, hence by the above discussion Conjecture 4.3.22 for $\nu \leq 2$ is also true. Moreover, in [8] a case-by-case verification provides its validity for all the 'known' curves (which, conjecturally, provide all the possible combinatorial types with $\nu \geq 3$).

4.3.7 The Periodic Constant of One-Variable Series

Definition 4.3.23 ([82, 3.9], [97]) Let $F(t) = \sum_{i\geq 0} a_i t^i$ be a formal power series. Suppose that there exist a positive integer p and a polynomial $\mathfrak{P}_p(t)$ such that $\sum_{0\leq i<pn} a_i = \mathfrak{P}_p(n)$ for every $n \in \mathbb{Z}_{>0}$. We call the constant term $\mathfrak{P}_p(0)$ the *periodic constant* of F and we denote it by $\mathrm{pc}(F)$. The integer p is called the 'period'. Furthermore, we extend the above definition to expressions of type $t^r \cdot F(t)$ via $\mathrm{pc}(t^r F(t)) := \mathrm{pc}(F(t))$, where F is a power series as above and $r \in \mathbb{Q} \cap [0, 1)$.

If the periodic constant exists then it is independent of the choice of the period p.

If F_1 and F_2 admit periodic constants, then the same is true for the series F_1+F_2, cF_1 (where $c \in \mathbb{C}$), $F_1(t^m)$ (where $m \in \mathbb{Z}_{>0}$). Moreover, $\mathrm{pc}(F_1 + F_2) = \mathrm{pc}(F_1) + \mathrm{pc}(F_2)$, $\mathrm{pc}(cF_1) = c \cdot \mathrm{pc}(F)$, $\mathrm{pc}(F_1(t^m)) = \mathrm{pc}(F_1(t))$.

If $F(t)$ is a finite sum (i.e. it is a polynomial), then $\mathrm{pc}(F)$ exists and equals $F(1)$.

For certain rational functions, one has the following equivalent description. (Here, we identify a rational function R with its Taylor expansion at the origin.) Clearly, any rational function can be written in a unique way as $R = R^+ + R^-$, where R^+ is a polynomial and R^- is a rational function of negative degree.

Lemma 4.3.24 *Let R be a rational function having poles only at infinity or at certain roots of unity. Then R admits a periodic constant and* $\mathrm{pc}(R) = R^+(1)$.

Example 4.3.25 Recall that for **cyclic quotients** (with $s > 1$) $Z(\mathbf{t}) = (1 - \mathbf{t}^{E_1^*})^{-1}(1 - \mathbf{t}^{E_s^*})^{-1}$, which equals also $P(\mathbf{t})$. We fix $\mathcal{I} = \{v_1\}$ and $h = e^{2\pi i a/n}$ $(0 \le a < n)$. Then $Z_{h,\mathcal{I}}$ equals $t^{a/n} \cdot \sum_{m\ge 0}(1 + \lfloor (a+nm)/q \rfloor)t^m$.

For the period it is convenient to take q, and one can check that $\mathrm{pc}(Z_{h,\mathcal{I}}) = 0$.

Example 4.3.26 Fix a **weighted homogeneous germ** with $g = 0$ and the representative r_h. Take $\mathcal{I}$ consisting of the central vertex E_0. Then, with the above notations (where $\mathfrak{a} \in [0, 1)$ stays for $-(r_h, E_0^*)$)

$$P_{h,\mathcal{I}}(t) = Z_{h,\mathcal{I}}(t) = \sum_{\ell\ge 0} \max\{0, 1 + N_{r_h}(\ell)\} t^{\ell+\mathfrak{a}}.$$

By a computation $Z^+_{h,\mathcal{I}}(t) = \sum_{\ell\ge 0} \max\{0, -1 - N_{r_h}(\ell)\} t^{\ell+\mathfrak{a}}$. Thus, by (4.23),

$$\mathrm{pc}(P_{h,\mathcal{I}}(t)) = \mathrm{pc}(Z_{h,\mathcal{I}}(t)) = \sum_{\ell\ge 0} \max\{0, -1 - N_{r_h}(\ell)\} = h^1(\mathcal{O}_{\widetilde{X}}(-r_h)).$$

4.3.8 Okuma's Additivity Formula

4.3.27 The Setup Consider a normal surface singularity (X, o) and fix one of its resolutions $\phi : \widetilde{X} \to X$. We fix a vertex $v \in \mathcal{V}$. Let $\cup_{j\in J}\Gamma_j$ be the connected components of the graph obtained from Γ by deleting v and its adjacent edges. Assume that v is connected to each Γ_j by exactly one edge. Let X' be the space obtained from $\widetilde{X}$ by contracting (via τ) all irreducible exceptional curves to normal points except E_v. It has $|J|$ normal singular points $\{o_j\}_j$, which are the images of the connected components of $E \setminus E_v$. Let X_j be a small Stein neighbourhood of o_j in X', and $\widetilde{X}_j = \tau^{-1}(X_j)$ its pre-image via the contraction $\tau : \widetilde{X} \to X'$. We denote the local singularities by (X_j, o_j). They are resolved by $\widetilde{X}_j$ with dual graphs Γ_j. Set $\tau(E) = E' \subset X'$. The resolution $\phi : \widetilde{X} \to X$ and the contraction $\tau : \widetilde{X} \to X'$ induce an analytic modification $\phi' : X' \to X$ with (irreducible) exceptional curve E'.

We say that the Assumption (C) is satisfied if

$$(C) \qquad nE' \subset X' \text{ is a Cartier divisor for a certain } n > 0.$$

Theorem 4.3.28 (Additivity for $\mathcal{O}_{\widetilde{X}}$ [97]) *If Assumption (C) is satisfied then $P_{0,\mathcal{I}}(t)$ admits a periodic constant and*

$$p_g(X, o) = \mathrm{pc}(P_{0,\mathcal{I}}(t)) + \sum_j p_g(X_j, o_j).$$

4.3.29 Additivity for Natural Line Bundles Assume that $H^1(\widetilde{X}, \mathbb{Z}) = 0$.

Theorem 4.3.30 *Set $\mathcal{I} = \{v\}$ and fix $h \in H$. Under the Assumption (C)*

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)) = \mathrm{pc}(P_{h,\{v\}}(t)) + \sum_j h^1(\widetilde{X}_j, \mathcal{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}).$$

4.4 The Seiberg–Witten Invariant Conjecture

4.4.1 The Casson Invariant

4.4.1 The Setup Let M be an oriented integral homology 3–sphere. The original definition of the Casson invariant $\lambda(M)$ given by Casson is based on a Heegaard splitting of M, and on the study of the space of conjugacy classes of SU_2-representations of different fundamental groups of the splitting [2, 26].

Here we will adopt a specific surgery formula of $\lambda(M)$ as starting definition, valid for any plumbed manifold $M(\Gamma)$. It was proved in the PhD thesis of A. Ratiu (Paris VII), and it follows also from the surgery formulae from Lescop's book [55].

Definition 4.4.2 Assume that M is the plumbed manifold of a connected negative definite graph Γ. Then

$$-24 \cdot \lambda(M) = \sum_{v \in \mathcal{V}} (E_v^2 + 3) + \sum_{v \in \mathcal{V}} (2 - \kappa_v)(E_v^*, E_v^*). \tag{4.46}$$

We extend the definition of λ by the same expression for non-connected graphs as well, (i.e., for connected sums of negative definite plumbed 3-manifolds). One verifies that the expression from the right hand side depends only on $M(\Gamma)$, i.e., it is stable to the plumbing calculus of negative definite plumbing graphs.

By a computation $\lambda(S^3) = 0$ and $\lambda(\Sigma(2,3,5)) = \lambda(\Sigma(2,3,7)) = -1$.

Example 4.4.3 If M is a Seifert 3-manifold, then

$$-24 \cdot \lambda(L_X) = \frac{1}{e}\Big(2 - \nu + \sum_{j=1}^{\nu} \frac{1}{\alpha_j^2}\Big) + e + 3 - 12 \cdot \sum_{j=1}^{\nu} \mathbf{s}(\omega_j, \alpha_j). \tag{4.47}$$

In this case (X, o) is a Brieskorn–Hamm complete intersection

$$\Big\{(z_1, \dots, z_\nu) \,:\, \sum_j a_{ij} z_j^{\alpha_j} = 0 \ \text{ for } \ 1 \le i \le \nu - 2\Big\}$$

with (a_{ij}) of full rank. Hence $L(X, o) = M = \Sigma(\alpha_1, \dots, \alpha_\nu)$. Furthermore, the integers $\{\alpha_k\}_k$ are pairwise relatively prime, and the integers ω_j's are determined from $\{\alpha_k\}_k$ by

$$\omega_j \cdot (\prod_k \alpha_k)/\alpha_j \equiv -1 \ (\text{mod}\ \alpha_j).$$

Hence

$$\mathbf{s}(\omega_j, \alpha_j) = -\mathbf{s}((\prod_k \alpha_k)/\alpha_j, \alpha_j).$$

In this case one also has $e^{-1} = -\prod_k \alpha_k$. Note also that

$$\lambda(\Sigma(\alpha_1, \ldots, \alpha_\nu)) = \lambda(\Sigma(\alpha_1, \ldots, \alpha_j, \alpha_{j+1} \cdots \alpha_\nu)) + \lambda(\Sigma(\alpha_1 \cdots \alpha_j, \alpha_{j+1}, \ldots, \alpha_\nu)). \tag{4.48}$$

In particular, the computation of $\lambda(\Sigma(\alpha_1, \ldots, \alpha_\nu))$ can be reduced to the case $\nu = 3$. On the other hand, if $M = \Sigma(\alpha_1, \alpha_2, \alpha_3)$, then one also has

$$\lambda(M) = -\tfrac{1}{2} \cdot \{\text{number of irreducible } SU_2\text{-representations of } \pi_1(M) \text{ up to conjugation}\}. \tag{4.49}$$

Additionally, in [11, 27] is proved that the Casson invariant is additive with respect to the splice decomposition. In particular, $\lambda(L(X, o))$ equals the sum of Casson invariants of the splice components of $L(X, o)$. Since all of them are of type $\Sigma(\alpha_1, \ldots, \alpha_\nu)$, we obtain that for any singularity link $\lambda(L(X, o)) \leq 0$, and $\lambda(L(X, o)) = 0$ if and only if $L(X, o) = S^3$.

4.4.2 The Casson Invariant Conjecture of Neumann–Wahl

Based on a result of Fintushel and Stern [26], valid for $\Sigma = \Sigma(\alpha_1, \alpha_2, \alpha_3)$, which identifies the irreducible SU_2-representations of $\pi_1(\Sigma)$ with Brieskorn formula for the signature of the Milnor fiber (cf. 4.49), Neumann and Wahl formulated the following conjecture.

Conjecture 4.4.4 (Casson Invariant Conjecture (CIC) [90]) Assume that (X, o) is an isolated complete intersection singularity of dimension two, whose link $L(X, o)$ is an integral homology sphere. Let $\sigma(F)$ be the signature of its Milnor fiber F. Then $\lambda(L(X, o)) = \sigma(F)/8$. *(Since the intersection form on the Milnor fiber is even, and the intersection form is unimodular, the signature is multiple of 8 by Serre [109, p. 53].)*

The conjecture would imply (via formulae of Durfee $\sigma(F) + 8p_g + Z_K^2 + |\mathcal{V}| = 0$ [17] and Laufer $\mu = 12p_g + Z_K^2 + |\mathcal{V}| - \text{rank}(H_1(L(X, o))))$ [51] that the Milnor number μ and the geometric genus p_g can also be computed from the abstract link.

Neumann and Wahl supported their conjecture by its verification for Brieskorn–Hamm complete intersection singularities and (hypersurface) suspensions [90]. More generally, the CIC was proved for any splice (complete intersection) singularity in [82].

4.4.3 *The Casson–Walker Invariant*

The Casson invariant defined for integral homology spheres has an extension to rational homology spheres given by Walker [116]. Similarly to the Casson invariant we adopt a working definition, valid for negative definite plumbed 3-manifolds, based on a surgery formula of [55].

Definition 4.4.5 Assume that $H = H_1(M(\Gamma), \mathbb{Z})$ is finite. We define

$$-\frac{24}{|H|} \cdot \lambda(M) = \sum_{v \in \mathcal{V}} (E_v^2 + 3) + \sum_{v \in \mathcal{V}} (2 - \kappa_v)(E_v^*, E_v^*). \tag{4.50}$$

Again, a direct verification shows that the right hand side depends only on M and it is independent of the choice of the negative definite graph Γ.

Example 4.4.6 If M is a Seifert 3-manifold with $\nu \geq 3$ then

$$-\frac{24}{|H|} \cdot \lambda(M) = \frac{1}{e}\Big(2 - \nu + \sum_{j=1}^{\nu} \frac{1}{\alpha_j^2}\Big) + e + 3 - 12 \cdot \sum_{j=1}^{\nu} \mathbf{s}(\omega_j, \alpha_j). \tag{4.51}$$

Example 4.4.7 For a lens space one has $\lambda(L(n, q)) = n \cdot \mathbf{s}(q, n)/2$.

Remark 4.4.8 The CIC identity $\lambda(L_X) = \sigma(F)/8$, expected in the case $H = 0$, does not extend in the same form to hypersurfaces with rational homology sphere links. For example, in the case of A_{n-1} germs, one has $\lambda(L(X, o)) = \lambda(L(n, n - 1)) = -(n-1)(n-2)/24$, while $\sigma/8 = -(n-1)/8$.

4.4.4 *Additivity Formulae for λ and $K^2 + |\mathcal{V}|$*

In the rational homology sphere case there is no natural splice decomposition, hence there is no analogues for the Casson–Walker invariant of the splice formula valid for integral homology spheres. However, we present another type of 'additivity formula', more in the spirit of Okuma's analytic additivity formulae 4.3.28. We start with some notations.

For $v, w \in \mathcal{V}$ we define $m_{vw} := -(E^*_v, E^*_w) = -(I^{-1})_{vw} \in \mathbb{Q}_{>0}$, and let κ_v be the valency of v in Γ as usual. Then for any fixed $v \in \mathcal{V}$ we set

$$\alpha_v := \sum_{w\in\mathcal{V}} (\kappa_w - 2)m_{vw}, \qquad \beta_v := \sum_{w\in\mathcal{V}} (\kappa_w - 2)m^2_{vw}. \tag{4.52}$$

4.4.9 For a fixed vertex v of Γ, we denote the connected components of $\Gamma \setminus v$ by $\{\Gamma_i\}_i$. We indicate by a subscript i when we consider an invariant in Γ_i, instead of Γ. We regard L_i as a sublattice of L and let $R_i : L' \to L'_i$ be the natural *cohomological restriction*, that is, $R_i(E^*_w) = E^*_{w,i}$ if $w \in \mathcal{V}_i$, and $R_i(E^*_w) = 0$ otherwise. By projection formula $(R_i(x), x_i)_{L'_i} = (x, x_i)_{L'}$ for any $x \in L'$ and $x_i \in L'_i$. Then R_i maps $\mathrm{Char}(\Gamma)$ into $\mathrm{Char}(\Gamma_i)$, and the canonical characteristic element K of $\mathrm{Char}(\Gamma)$ into the canonical characteristic element K_i of $\mathrm{Char}(\Gamma_i)$.

Theorem 4.4.10 *For any* $l' = \sum_w r_w E_w \in L'$

$$((K + 2l')^2 + |\mathcal{V}|) - \sum_i ((K_i + 2R_i(l'))^2 + |\mathcal{V}_i|) = 1 - \frac{(\alpha_v + 1 - 2r_v)^2}{m_{vv}}, \tag{4.53}$$

$$\frac{24}{|H|} \cdot \lambda - \sum_i \frac{24}{|H_i|} \cdot \lambda_i = -3 + \frac{1 - \beta_v}{m_{vv}}. \tag{4.54}$$

Example 4.4.11 Consider the **surgery 3-manifold** $M = S^3_{-d}(\#_i K_i)$ as in 4.2.32 with $d > 0$ and K_i algebraic with Alexander polynomial Δ_i. Let $\Delta(t) = \prod_i \Delta_i(t)$ and $\mu = \sum_i \mu_i = 2\delta$ as in 4.3.6. By a computation

$$24 \cdot \lambda = (d - 1)(d - 2) + 3\mu(\mu - 2) - 12 \cdot \Delta''(1).$$

If $\mu = (d - 1)(d - 2)$ then this transforms into $24\lambda = \mu(3\mu - 5) - 12 \cdot \Delta''(1)$.

4.4.5 The Reidemeister–Turaev Torsion: Generalities

For the general definition of the sign-refined torsion associated with spin^c–structures see the books of Turaev and work of Nicolaescu and Ranicki, see [94, 114, 115] and the references therein.

4.4.12 The Case of 3-Manifolds Assume that M is a closed connected 3-manifold without boundary with a fixed orientation. We assume that $H = H_1(M, \mathbb{Z})$ is finite.

Theorem 4.4.13 ([115]) *The 'universal abelian sign-refined torsion'*

$$\tau : \mathrm{Spin}^c(M) \to \mathbb{Q}[H]; \quad \sigma \mapsto \tau_\sigma = \sum_h \mathfrak{T}_\sigma(h)h \quad (\mathfrak{T}_\sigma(h) \in \mathbb{Q}) \tag{4.55}$$

has the following properties:

(a) ***Duality:*** *Consider the involution $\mathbb{Q}[H] \to \mathbb{Q}[H]$, given by $x = \sum_h a(h)h \mapsto \bar{x} := \sum_h a(h)h^{-1}$. Then $\tau_{\overline{\sigma}} = \overline{\tau_\sigma}$, or $\mathfrak{T}_{\overline{\sigma}}(h^{-1}) = \mathfrak{T}_\sigma(h)$.*

(b) ***H-equivariance:*** *$\tau_{h\sigma} = h\tau_\sigma$; that is, for any $g, h \in H$ one has $\mathfrak{T}_{g\sigma}(gh) = \mathfrak{T}_\sigma(h)$. In particular, for fixed $\sigma_0 \in \mathrm{Spin}^c(M)$ the coefficients $\{\mathfrak{T}_{\sigma_0}(h)\}_h$, or, for fixed $h_0 \in H$, the coefficients $\{\mathfrak{T}_\sigma(h_0)\}_\sigma$, determine the whole τ.*

(c) ***Augmentation:*** *Let $\mathfrak{aug} : \mathbb{Q}[H] \to \mathbb{Q}$ be the augmentation $\sum_h a(h)h \mapsto \sum_h a(h)$. Then, for any σ one has $\mathfrak{aug}(\tau_\sigma) = 0$. Equivalently,*

$$\sum_\sigma \mathfrak{T}_\sigma(h) = 0 \ \textit{for any } h.$$

4.4.14 The Fourier Transform We wish to have a dual description of the torsion in terms of Fourier transform. First we recall the definition of the Fourier transform.

Let H be a finite abelian group and let $\widehat{H} = \mathrm{Hom}(H, S^1)$ be its Pontryagin dual (the group of characters). If $\chi \in \widehat{H}$ then $\bar{\chi}$ denotes its conjugate: $\bar{\chi}(h) = \overline{\chi(h)}$.

The Fourier transform $\widehat{f} : \widehat{H} \to \mathbb{C}$ of a function $f : H \to \mathbb{C}$ satisfies

$$\widehat{f}(\chi) = \sum_{h \in H} f(h)\bar{\chi}(h), \quad f(h) = \frac{1}{|H|} \sum_{\chi \in \widehat{H}} \widehat{f}(\chi)\chi(h).$$

Example 4.4.15 For any σ set $f(h) := \mathfrak{T}_\sigma(h)$. Then $\widehat{f}(1) = \widehat{\mathfrak{T}_\sigma}(1) = \mathfrak{aug}(\tau_\sigma) = 0$.

Example 4.4.16 By 4.4.13*(a)–(b)* for any σ, χ, h one has

$$(a)\ \widehat{\mathfrak{T}_\sigma}(\chi) = \widehat{\mathfrak{T}_{\overline{\sigma}}}(\overline{\chi}), \quad (b)\ \widehat{\mathfrak{T}_\sigma}(\chi) = \chi(h) \cdot \widehat{\mathfrak{T}_{h\sigma}}(\chi). \tag{4.56}$$

4.4.6 The Reidemeister–Turaev Torsion of Graph 3-Manifolds

Let M be an oriented rational homology sphere 3-manifold associated with a connected negative definite plumbing graph Γ.

In 4.4.22 we provide a combinatorial expression in terms of Γ for the refined Reidemeister–Turaev torsion. The equivalence of this expression with the original definition of the refined torsion is proved in [78].

4.4.17 The Fourier Transform of $Z_{h,\mathcal{I}}(t)$ Assume that $\mathcal{I} = \{u\} \subset \mathcal{V}$ is a distinguished vertex, and for each $h \in H$ we consider the reduced series $Z_{h,\mathcal{I}}(t)$, where t is the variable corresponding to u. Set $m_{vu} := -(E^*_v, E^*_u) > 0$. From (4.38)

$$Z_{h,\{u\}}(t) = \frac{1}{|H|} \cdot \sum_{\chi \in \widehat{H}} \chi(h)^{-1} \cdot \prod_{v \in \mathcal{V}} \big(1 - \chi([E^*_v])t^{m_{vu}}\big)^{\kappa_v - 2}.$$

This shows that the Fourier transform of the map $h \mapsto Z_{h,\{u\}}(t)$ is

$$\widehat{Z_{\{u\}}(t)}(\bar{\chi}) = \prod_{v \in \mathcal{V}} \big(1 - \chi([E^*_v])t^{m_{vu}}\big)^{\kappa_v - 2}. \tag{4.57}$$

4.4.18 Character Values on Γ Since $\{[E^*_v]\}_v$ generate H, any character $\chi \in \widehat{H}$ is completely characterized by the values $\xi_v := \chi([E^*_v])$, $v \in \mathcal{V}$. These are roots of unity. When we wish to identify the character χ, we put its values $\{\xi_v\}_v$ as decorations on the vertices of the graph Γ. The collection $\{\chi([E^*_v])\}_{v,\chi}$ is a more subtle information than the abstract group $\widehat{H}$ itself: it shows the 'distribution along Γ' of the corresponding values of the characters as well. Since for any $v \in \mathcal{V}$ one has $e_v[E^*_v] + \sum_{(u,v) \text{ edge}}[E^*_u] = [-E_v] = 0$ in H (where $e_v = E^2_v$), for each χ one has

$$\xi_v^{e_v} \cdot \prod_{(u,v) \text{ edge}} \xi_u = 1. \tag{4.58}$$

Conversely, any collection of complex numbers $\{\xi_v\}_{v \in \mathcal{V}}$, $\xi_v \in S^1$, which satisfy (4.58) for any v, determines a character χ defined by $\chi([E^*_v]) = \xi_v$.

Furthermore, for any $\chi \in \widehat{H} \setminus \{1\}$, define the 'extended support' $\text{supp}^e(\chi)$ of χ as the set of those vertices $v \in \mathcal{V}$ for which either $\chi([E^*_v]) \neq 1$, or v has an adjacent vertex w such that $\chi([E^*_w]) \neq 1$.

Lemma 4.4.19 *Fix a character $\chi \in \widehat{H} \setminus \{1\}$.*

(a) For an arbitrary vertex u the limit $\lim_{t \to 1} \widehat{Z_{\{u\}}(t)}(\chi)$ exists and it is finite.
(b) This limit is independent of u whenever $u \in \text{supp}^e(\chi)$.

Remark 4.4.20 For $\chi = 1$, the Laurent expansion at 1 of the series $\widehat{Z_{\{u\}}(t)}(1)$ has a non-trivial principal part, hence $\lim_{t \to 1} \widehat{Z_{\{u\}}(t)}(1)$ is not finite.

4.4.21 In the sequel, the torsion $\sigma \in \text{Spin}^c(M) \mapsto \mathcal{T}_\sigma$, $\mathcal{T}_\sigma = \sum_h \mathcal{T}_\sigma(h)h \in \mathbb{Q}[H]$ is defined via the Fourier transform of $h \mapsto \mathcal{T}_\sigma(h)$ in the following way.

Definition 4.4.22

(a) For the trivial character $\widehat{\mathcal{T}_\sigma}(1) = 0$.
(b) If $\chi([E^*_v]) \neq 1$ for every v with $\kappa_v \neq 2$, then we set

$$\widehat{\mathcal{T}_\sigma}(\chi) = (\chi(h_\sigma))^{-1} \cdot \prod_{v \in \mathcal{V}} (1 - \chi([E^*_v]))^{\kappa_v - 2}, \quad \sigma = h_\sigma \sigma[K].$$

(c) If $\chi \neq 1$, but the assumption from (b) does not hold, then the formula from (b) is regularised as follows:

$$\widehat{\mathfrak{T}_\sigma}(\chi) = (\chi(h_\sigma))^{-1} \cdot \lim_{t\to 1} \prod_{v\in\mathcal{V}} (1-\chi([E_v^*]t^{m_{vu}}))^{\kappa_v-2} = (\chi(h_\sigma))^{-1} \cdot \lim_{t\to 1} \widehat{Z_{\{u\}}(t)}(\bar\chi),$$

for certain (any) $u = u_\chi \in \mathrm{supp}^e(\chi)$.

Theorem 4.4.23

(a) $\sigma \mapsto \mathfrak{T}_\sigma$ *defined in 4.4.22 and the refined Reidemeister–Turaev torsion 4.4.12 coincide.*

(b) $\mathfrak{T}$ *defined in 4.4.22 is independent of the choice of the resolution.*

Remark 4.4.24

(a) By Fourier inversion

$$\mathfrak{T}_\sigma(h) = \frac{1}{|H|} \cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} \chi(h)\cdot(\chi(h_\sigma))^{-1} \cdot \lim_{t_{u_\chi}\to 1} \widehat{Z_{\{u_\chi\}}(t_{u_\chi})}(\bar\chi).$$

One verifies that the Properties (4.56) are valid, hence $\{\mathfrak{T}_\sigma(h)\}_{\sigma,h}$ satisfy the duality and H-equivariance properties. Hence

$$\mathfrak{T}_\sigma(1) = \mathfrak{T}_{\overline{\sigma}}(1), \quad \text{and} \quad \mathfrak{T}_\sigma(1) = \mathfrak{T}_{h_\sigma\sigma[K]}(1) = \mathfrak{T}_{\sigma[K]}(-h_\sigma). \tag{4.59}$$

In particular, $\mathfrak{T}_{\sigma[K]}(h)h \in \mathbb{Q}[H]$ contains the same information as $\{\mathfrak{T}_\sigma(1)\}_\sigma$.

(b) From part (a),

$$\mathfrak{T}_\sigma(1) = \frac{1}{|H|} \cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} (\chi(h_\sigma))^{-1} \cdot \lim_{t_{u_\chi}\to 1} \widehat{Z_{\{u_\chi\}}(t_{u_\chi})}(\bar\chi).$$

Usually, for different characters χ one needs different regularization vertices u_χ. However, if $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi) \neq \emptyset$, then any $u \in \cap_{\chi\neq 1}\mathrm{supp}^e(\chi)$ might serve as a *common* regularization vertex (with a *common* variable $t = t_u$). In such a case, via $\widehat{Z_{\{u\}}(t)}(1) = Z_{\{u\}}(t)$,

$$\mathfrak{T}_\sigma(1) = \lim_{t\to 1}\Big(\frac{1}{|H|} \cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} (\chi(h_\sigma))^{-1} \cdot \widehat{Z_{\{u\}}(t)}(\bar\chi)\Big) = \lim_{t\to 1}\Big(Z_{h_\sigma,\{u\}}(t) - \frac{1}{|H|}\cdot Z_{\{u\}}(t)\Big).$$

We rewrite $\{Z_{h,\{u\}}(t)\}_h$ equivariantly as $Z_{H,\{u\}}(t) := \sum_{h\in H} Z_{h,\{u\}}(t)h \in \mathbb{Q}[[t]][H]$, and we set $N := \sum_h h \in \mathbb{Q}[H]$. Then, via $\mathfrak{T}_\sigma(1) = \mathfrak{T}_{\sigma[K]}(-h_\sigma)$,

$$\mathfrak{T}_{\sigma[-K]} = \overline{\mathfrak{T}_{\sigma[K]}} = \lim_{t\to 1}\Big(Z_{H,\{u\}}(t) - Z_{\{u\}}(t)\cdot\frac{N}{|H|}\Big) \in \mathbb{Q}[H]. \tag{4.60}$$

The identity (4.60) is not true in general, i.e. when $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)=\emptyset$.

The above formula already shows in this special case that the principal (pole) part of the Laurent series at $t=1$ of $Z_{h,\{u\}}(t)$ is independent of $h\in H$. This statement is true in general, even without the restriction $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)\neq\emptyset$.

(c) If Γ is star-shaped then the central vertex is an element of $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)$. Similarly, if H is cyclic, then again $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)\neq\emptyset$.

Example 4.4.25 (The Torsion of a Lens Space) We fix $\sigma=h_\sigma\sigma[K]\in \mathrm{Spin}^c(L_X)$. Then for $\chi\neq 1$

$$\widehat{\mathcal{T}_\sigma}(\chi)=\chi(h_\sigma)^{-1}\cdot(1-\chi([E_s^*]))^{-1}(1-\chi([E_1^*]))^{-1}.$$

Assume that $h_\sigma=a[E_s^*]$ for some $0\leq a<n$. Set $\xi:=\chi([E_s^*])$. Then,

$$\widehat{\mathcal{T}_\sigma}(\chi)=\frac{\xi^{-a}}{(1-\xi)(1-\xi^q)}\quad(\xi\neq 1),\ \text{and}\ \mathcal{T}_\sigma(1)=\frac{1}{n}\cdot\sum_{\xi^n=1\neq\xi}\frac{\xi^{-a}}{(1-\xi)(1-\xi^q)}. \tag{4.61}$$

4.4.7 Additivity Formula for the Torsion

We fix a graph Γ such that $M(\Gamma)$ is a rational homology sphere. For a vertex $v\in\mathcal{V}$ of Γ let $\{\Gamma_i\}_i$ be the connected components of $\Gamma\setminus v$. For any $\sigma\in\mathrm{Spin}^c(M(\Gamma))$ we define its restrictions $\sigma_i\in\mathrm{Spin}^c(M(\Gamma_i))$ as follows.

Choose $l'=\sum_w r_wE_w\in L'$ such that $r_v\in[0,1)$ so that $[l']=h_\sigma$ satisfies $\sigma=\sigma[2l'+K]=h_\sigma\sigma[K]\in\mathrm{Spin}^c(M(\Gamma))$. Then we set $\sigma_i=\sigma[R_i(2l'+K)]=[R_i(l')]\sigma[K_i]\in\mathrm{Spin}^c(M(\Gamma_i))$. (For R_i see paragraph 4.4.9.)

Theorem 4.4.26 ([12]) *Set $l'=\sum_w r_wE_w$, $r_v\in[0,1)$, $[l']=h_\sigma$ as above. Recall also the notations from (4.52)*

$$\alpha_v:=\sum_{w\in\mathcal{V}}(\kappa_w-2)m_{vw},\quad \beta_v:=\sum_{w\in\mathcal{V}}(\kappa_w-2)m_{vw}^2.$$

Then

$$\mathcal{T}_\sigma(1)(M(\Gamma))-\sum_i\mathcal{T}_{\sigma_i}(1)(M(\Gamma_i))=\mathrm{pc}(Z_{h_\sigma,\{v\}}(t^d))+\frac{1-\beta_v}{24m_{vv}}-\frac{(\alpha_v+1-2r_v)^2}{8m_{vv}}.$$

Corollary 4.4.27 *$\mathcal{T}_\sigma(1)(M(\Gamma))$ is a rational number.*

4.4.8 The Seiberg–Witten Invariant

In this section we fix a plumbed rational homology sphere 3-manifold M associated with a connected negative definite plumbing graph Γ. The Seiberg–Witten invariant of M, $\mathfrak{sw}$, associates to each spin^c structure $\sigma \in \mathrm{Spin}^c(M)$ of M a rational number $\mathfrak{sw}_\sigma$. Here, based on [95], we 'define' it as the refined Turaev torsion modified by the Casson–Walker invariant. Based on the formulae of the previous sections, this provides $\mathfrak{sw}$ combinatorially from Γ.

Definition 4.4.28 We define $\mathfrak{sw} : \mathrm{Spin}^c(M) \to \mathbb{Q}$, $\sigma \mapsto \mathfrak{sw}_\sigma$ by

$$\mathfrak{sw}_\sigma := \mathcal{T}_\sigma(1) - \lambda/|H|.$$

Example 4.4.29 If $H = 0$ then $\mathrm{Spin}^c(M)$ has only one element, and the corresponding Seiberg–Witten invariant is $-\lambda(M)$ (the negative of the Casson invariant).

4.4.30 Additivity Formula for the Seiberg–Witten Invariant The previous additivity formulae imply the following formula.

Theorem 4.4.31 ([12]) *Set $l' = \sum_w l'_w E_w$, $l'_v \in [0, 1)$, as in Theorem 4.4.26. Let $\sigma \in \mathrm{Spin}^c(M(\Gamma))$ be defined as $[l']\sigma[K] = \sigma[K+2l']$, and take also its restrictions $\sigma_i := [R_i(l')]\sigma[K_i] = \sigma[R_i(K+2l')]$ too. Set $h_\sigma = [l']$. Then one has the following identities:*

$$\mathfrak{sw}_\sigma(M(\Gamma)) - \sum_i \mathfrak{sw}_{\sigma_i}(M(\Gamma_i)) = \mathrm{pc}(Z_{h_\sigma,\{v\}}(t)) + \frac{1}{8} - \frac{(\alpha_v + 1 - 2r_v)^2}{8m_{vv}}.$$

and

$$\Big(\mathfrak{sw}_\sigma(M(\Gamma)) - \frac{(K+2l')^2 + |\mathcal{V}|}{8}\Big) - \sum_i \Big(\mathfrak{sw}_{\sigma_i}(M(\Gamma_i)) - \frac{(K_i + 2R_i(l'))^2 + |\mathcal{V}_i|}{8}\Big)$$
$$= \mathrm{pc}(Z_{h_\sigma,\{v\}}(t)).$$

Proof Combine Theorems 4.4.10 and 4.4.26 and use $\mathrm{pc}(S(t^d)) = \mathrm{pc}(S(t))$. □

This additivity formula should be compared with its 'analytic counterpart', namely with Okuma's additivity formula 4.3.30.

4.4.9 The Seiberg–Witten Invariant and the Series $Z(\mathbf{t})$

We prove two key formulae for the Seiberg–Witten invariant of a rational homology sphere link. One of them identifies it with a weighted Euler characteristic of (shifted) weighted cubes in a large rectangle of $L \otimes \mathbb{R}$, the other one with the constant term of the counting function of the coefficients of $Z(\mathbf{t})$. The proofs are based on additivity formulae of the compared invariants.

The similarities with the analytic counterpart (the series $P(\mathbf{t})$ and the equivariant genera) are emphasized.

4.4.32 In the next discussion we will use the weighted cubes, see also 4.6.3. Let us fix an element h of H and write $L'_h = \{l' \in L' : [l'] = h\}$. Recall that the set of 'combinatorial' q–cubes (associated with h) consists of pairs $(l', I) \in L'_h \times \mathcal{P}(\mathcal{V})$, $|I| = q$ $(q \in \mathbb{Z}_{\geq 0})$. (l', I) will be identified with the vertices $\{l' + \sum_{v\in I'} E_v\}_{I'\subset I}$ of an 'Euclidean' cube in $L \otimes \mathbb{R}$. One defines the weight function $w : L' \to \mathbb{Q}$, $w(l') := \chi(l')$, and also the a weight of the q–cubes

$$w((l', I)) = \max_{I'\subset I} \Big\{ w(l' + \sum_{v\in I'} E_v) \Big\}.$$

Assume that a set $A \subset L \otimes \mathbb{R}$ has the following property: if an Euclidean cube (as above) is in A then any face of any dimension of that cube is in A. For such a set A one defines the 'weighted Euler characteristic'

$$Eu_\chi(A) := \sum_{(l',I)\in A} (-1)^{|I|+1} w((l', I)).$$

Such a set A might appear as follows. For the fixed class $h \in L'/L$ one takes two representatives l'_1, $l'_2 \in L'_h$ with $l'_2 \leq l'_1$. Then $R_h = R_h(l'_2, l'_1)$ consists of the union of all combinatorial cubes (l', I), of any dimension, such that $[l'] = h$ and any vertex $l' + \sum_{v\in I'} E_v$ of (l', I) satisfies $l'_2 \leq l' + \sum_{v\in I'} E_v \leq l'_1$. Accordingly to the above identification, $R_h(l'_1, l'_2)$ will also denote the real rectangle $\{x \in L \otimes \mathbb{R} : l'_2 \leq x \leq l'_1\}$, or the union of all Euclidean cubes (with all vertices having class $[h]$) in this real rectangle.

Remark 4.4.33 For a fixed $h \in H$, we can consider two types of rectangles and weighted q–cubes, depending on the geometric situation. First, in the context of lattice cohomology (see e.g. 4.6.3, and in its preparation 4.5.2) we take integral lattice points and rectangles $R(l_2, l_1)$ and cubes with vertices in the lattice L, but we twist the weight function: we take χ_k (which generates w_k) with $k = K + 2l'_h$, for some representative l'_h of h.

Second, when we wish to relate the cubes with the coefficients of $Z(\mathbf{t})$ (as in the previous paragraph), we take shifted rectangles $R_h := R_h(l'_2, l'_1)$ $([l'_j] = h)$ with cubes (l', I) of type $[l'] = h$ in them, together with the usual untwisted Riemann–Roch-function $\chi = \chi_K$.

The two approaches can be compared easily (see also 4.6.3). Indeed, if $k = K + 2l'_h$, $[l'_h] = h$, then for $l \in L$ we have $\chi(l + l'_h) = \chi_k(l) + \chi(l'_h)$. In particular, with the notation $l'_j = l_j + l'_h$ $(l_j \in L)$, we have $R_h(l'_2, l'_1) = l'_h + R(l_2, l_1)$ as rectangles, and

$$Eu_\chi(R_h(l'_2, l'_1)) = Eu_{\chi_k}(R(l_2, l_1)) - \chi(l'_h).$$

4.4.34 Via the two incarnations of the weighted cubes (cf. 4.4.33) the next result is the 'pair' of Lemma 4.5.8.

Lemma 4.4.35 *Fix a class h and take a representative l'_0 of h in $-K+\mathcal{S}'$.*

(a) For any $l'\in L'$, $[l']=h$, $l'>l'_0$, there exists an E_v in the support of $l'-l'_0$ such that $w(l'-E_v)\leq w(l')$.
(b) There exists a computation sequence $\{\ell_i\}_{i\geq 0}$, $\ell_i\in L$, with $\ell_0=0$, and $\ell_{i+1}=\ell_i+E_{v(i)}$ for some $v(i)\in\mathcal{V}$ when $i\geq 0$, satisfying:

  (i) The coefficients of ℓ_i tend to infinity, that is $\lim_{i\to\infty}(\ell_i,-E^*_v)=\infty$ for all v.
  (ii) For any $i\geq 0$ one has $w(l'_0+\ell_i)\leq w(l'_0+\ell_{i+1})$.

(c) For any $l'<0$, with $[l']=h$, there exists $E_v\in|l'|$ such that $w(l'+E_v)\leq w(l')$.
(d) For any representatives l'_1, l'_2 of h, such that $l'_1\geq l'_0>0\geq l'_2$, $Eu_\chi(R_h(l'_2,l'_1))$ is independent of the choice of l'_1 and l'_2. In particular, with such choices, $h\mapsto Eu_\chi(R_h(l'_2,l'_1))$ is a numerical invariant of $h\in H=L'/L$.

Definition 4.4.36 The invariant provided by 4.4.35(d) will be denoted by $\bar{\mathfrak{s}}_h$.

4.4.37 Let $Z(\mathbf{t})=\sum_{l'\in L'}\mathfrak{z}(l')\mathbf{t}^{l'}$ be the combinatorial series defined in Sect. 4.3.3. Since Z is supported on $\mathcal{S}'$, the next sum in (4.62) is finite by 4.2.13.

Theorem 4.4.38 *Fix $h\in H$. For any $l'\in -K+\mathcal{S}'$ with $[l']=h$, the expression*

$$-\chi(l')+\sum_{l\in L,\, l\ngeq 0}\mathfrak{z}(l'+l) \tag{4.62}$$

depends only on the class h of l', and, in fact, it equals $\bar{\mathfrak{s}}_h$ defined in 4.4.36.

Theorem 4.4.39 ([73]) *For any* Γ *and* $[K+2l']\in \mathrm{Char}$ *one has* $\mathfrak{sw}_{\sigma[K+2l']}(M(\Gamma))=\bar{\mathfrak{s}}_{[l']}+(K^2+|\mathcal{V}|)/8$, *or,*

$$Eu_\chi(R_h(l'_2,l'_1))=\bar{\mathfrak{s}}_{[l']}=\mathfrak{sw}_{\sigma[K+2l']}(M(\Gamma))-(K^2+|\mathcal{V}|)/8. \tag{4.63}$$

The proof is based on the 'additivity formula' 4.4.31 and a similar formula valid for $\bar{\mathfrak{s}}_h$.

Therefore, Theorem 4.4.38 reads as follows.

Theorem 4.4.40 *Assume that $l'\in -K+\mathcal{S}'$ and Let $Z(\mathbf{t})=\sum_{l'\in L'}\mathfrak{z}(l')\mathbf{t}^{l'}$ be the combinatorial series defined in Sect. 4.3.3. Then*

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'}\mathfrak{z}(\tilde{l}')=\mathfrak{sw}_{\sigma[K+2l']}-\frac{(K+2l')^2+|\mathcal{V}|}{8}. \tag{4.64}$$

If we write $l' = r_h + l$ (where $h = [l']$ and $l \in L$), then (4.64) transforms into

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l) + \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2+|\mathcal{V}|}{8}. \tag{4.65}$$

In particular, in the chamber $l' = l + r_h \in -K + \mathcal{S}'$, the sum from the left hand side of the above identities is a multivariable quadratic function in l with constant term $\mathfrak{sw}_{\sigma[K+2r_h]} - ((K+2r_h)^2 + |\mathcal{V}|)/8$.

These formulae should be compared with those from (4.36) valid for the coefficients of the series P. The fact that in (4.36) (associated with the series P) the constant terms are the equivariant geometric genera, is rather natural. However, the fact that the constant terms in the above Theorem 4.4.40 (associated with Z, a rather 'simple' series) is the Seiberg–Witten invariant, is rather surprising. Nevertheless, the above identity provides a very natural, direct and conceptual explanation, how the Seiberg–Witten invariant might appear in the theory of singularity links.

Example 4.4.41 If Γ is numerically Gorenstein and $h = 0$ then (4.65) reads as

$$\sum_{l\in L,\ l\ngeq Z_K} \mathfrak{z}(l) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2+|\mathcal{V}|}{8}. \tag{4.66}$$

4.4.10 The Seiberg–Witten Invariant Conjecture/Coincidence

In this section we treat a set of potential identities connecting the analytic invariants with the topological ones, namely, the equivariant geometric genera with the Seiberg–Witten invariants of the link. Whenever these identities are valid they provide a topological description of the equivariant geometric genera. The identities are generalizations of the expectation of the Casson Invariant Conjecture to the case of singularities with rational homology sphere links.

Superisolated singularities in general do not satisfy SWIC, their case will be discussed in subsection 4.4.11.

4.4.42 Seiberg–Witten Invariant Conjecture/Coincidence (SWIC) [73, 75, 78] In this section we assume that the link of (X, o) is a rational homology sphere, and we fix a resolution $\widetilde{X} \to X$, and we keep all the notations associated with it. We say that (X, o) satisfies SWIC(r_h) for a certain $h \in H$ if the following identity holds

$$h^1(\widetilde{X}, \mathcal{O}(-r_h)) = \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2+|\mathcal{V}|}{8}. \tag{4.67}$$

We say that (X, o) satisfies the *equivariant* SWIC if (4.67) holds for every $h \in H$.

We say that (X, o) satisfies the SWIC if it satisfies SWIC(0), that is, if

$$p_g(X, o) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathcal{V}|}{8}. \tag{4.68}$$

The identity SWIC was formulated as a conjecture in [78] (while the equivariant case in [71]): the expectation was that it holds for any $\mathbb{Q}$-Gorenstein singularity. Although the conjecture can be verified for several subfamilies of singularities, since [61] we know that it is not true for the large class of $\mathbb{Q}$-Gorenstein singularities (see also 4.4.11 for the treatment of superisolated singularities, a family which produces several counterexamples). But even in the case of families when it fails, it still indicates interesting 'virtual' properties (e.g., in the superisolated case it has lead to the Semigroup Distribution Property). The limits of the validity of the SWIC are not clarified at this moment. Having in mind the existence of cases when the identities do not hold, one might say that its name as SWI 'Conjecture' is not totally justified, although this was its name in the literature. Hence, the reader might read the abbreviation SWIC as SWI 'Coincidence' too.

Example 4.4.43 Assume that (X, o) is Gorenstein and it admits a smoothing with smooth nearby (Milnor) fiber F. Then the signature satisfies $\sigma(F) + 8p_g + K^2 + |\mathcal{V}| = 0$, hence the SWIC (for $h = 0$) reads as

$$-\sigma(F)/8 = \mathfrak{sw}_{\sigma[K]}. \tag{4.69}$$

In this case, usually, $\sigma(F)/8$ is not an integer, see the germ A_n.

Example 4.4.44 Assume that (X, o) is a complete intersection with integral homology sphere link. Then $\mathcal{T}_{\sigma[K]}(1) = 0$, hence the SWIC reduces to the CIC (see 4.4.2):

$$\sigma(F)/8 = \lambda(L(X, o)).$$

Example 4.4.45 The identity $P(\mathbf{t}) = Z(\mathbf{t})$ (that is, the topological description via Z of the Poincaré series associated with the divisorial filtration) implies the equivariant SWIC. In particular, the identity $P_0(\mathbf{t}) = Z_0(\mathbf{t})$ implies SWIC. Indeed, for any $l' \in -K + \mathcal{S}'$ with $l' = l + r_h$ $(l \in L)$, from (4.36) one has

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathfrak{p}(\tilde{l}') = \chi_{K+2r_h}(l) + h^1(\mathcal{O}(-r_h)). \tag{4.70}$$

On the other hand, from (4.65),

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l) + \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K + 2r_h)^2 + |\mathcal{V}|}{8}. \tag{4.71}$$

For $l' \in -K + \mathcal{S}'$ and $l' = l + r_h$, we can regard the evaluation at $l = 0$ of the counting function $\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\not\geq l'} \mathrm{coeff}(\tilde{l}')$ as an operator. It associates with any multivariable series its 'multivariable periodic constant', cf. [45, 46]. In this sense, the above identities say that the periodic constant of P_h is $h^1(\mathcal{O}(-r_h))$, while of Z_h is $\mathfrak{sw}_{[K+2r_h]} - ((K+2r_h)^2 + |\mathcal{V}|)/8$.

Hence, if $P_h(\mathbf{t}) = Z_h(\mathbf{t})$ then the SWIC(r_h) automatically holds as well.

In fact, in order to have the SWIC(r_h) we need the validity of the above identities for a certain $l' \in -K + \mathcal{S}'$ ($[l'] = h$) only. Indeed, if a certain $l'_0 \in -K + \mathcal{S}'$, $[l'_0] = h$, has the property that $P_h(\mathbf{t}) - Z_h(\mathbf{t})$ is supported on $\{\tilde{l}' : \tilde{l}' \geq l'_0\}$, then by the above identities applied for this l'_0 we obtain SWIC(r_h). In such a case, again by the identities (4.70)–(4.71), even if $P_h(\mathbf{t}) \neq Z_h(\mathbf{t})$, their counting functions $l' \mapsto \sum_{[\tilde{l}']=[l'],\ \tilde{l}'\not\geq l'} \mathrm{coeff}(\tilde{l}')$ in the whole chamber $l' \in -K + \mathcal{S}'$ coincide (independently of the position of l'_0 in this chamber).

For a fixed h, the identity $P_h = Z_h$ is much stronger than the SWIC(r_h): examples when $P_h \neq Z_h$ but the SWIC(r_h) holds can be constructed.

4.4.46 Extension to the Other Natural Line Bundles

Recall that in 4.2.74 we proved that for any $l' \in L'$ there exists a unique minimal $s(l') \in \mathcal{S}'$ such that $s(l') - l' \in L_{\geq 0}$. We wish to compare $h^1(\mathcal{O}(-l'))$ and $h^1(\mathcal{O}(-s(l')))$ via the SWIC property.

We say that $l' \in L'$ satisfies the SWIC identity, denoted by SWIC(l'), if

$$\mathrm{SWIC}(l'): \qquad h^1(\widetilde{X}, \mathcal{O}(-l')) = \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2 + |\mathcal{V}|}{8}. \tag{4.72}$$

If this holds, then it obviously provides a topological description for $h^1(\widetilde{X}, \mathcal{O}(-l'))$.

By 4.2.76 one has

$$h^1(\widetilde{X}, \mathcal{O}(-s(l'))) - h^1(\widetilde{X}, \mathcal{O}(-l')) = \chi(s(l')) - \chi(l').$$

A computation shows that the right hand side of (4.72) behaves similarly. Hence

Proposition 4.4.47 *The SWIC(l') is valid if and only if SWIC($s(l')$) is valid. In particular, SWIC(r_h) is valid if and only if SWIC(s_h) holds.*

This shows that the validity of SWIC(r_h) implies the validity of SWIC(l') for all $l' \in L'_h$ with $s(l') = s_{[l']}$. This covers exactly those cycles $l' \in L'_h$ with $l' \leq s_{[l']}$ (including all cycles $l' = \sum_v l'_v E_v$ with $l'_v < 1$ for any v).

This topological characterization SWIC(l') of $h^1(\mathcal{O}(-l'))$ (modulo the validity of SWIC) in this 'negative' region $\{l' : l' \leq s_{[l']}\}$ can be compared with the vanishing $h^1(\mathcal{O}(-l')) = 0$ in the 'opposite positive' region $\{l' : l' \in -K + \mathcal{S}'\}$.

It is natural to ask the following question: what can one say in the case of an arbitrary l', which sits outside of these two regions.

Proposition 4.4.48 *If SWIC(r_h) holds then for any $l' \in L'_h$*

$$h^1(\mathcal{O}(-l')) = -\sum_{a\in L,\ a\not\geq 0} \mathfrak{p}(l'+a) + \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2+|\mathcal{V}|}{8}. \tag{4.73}$$

Additionally, if $P_h = Z_h$ (or, at least their counting functions coincide), then one has the following topological characterization of $h^1(\mathcal{O}(-l'))$:

$$h^1(\mathcal{O}(-l')) = -\sum_{a\in L,\ a\not\geq 0} \mathfrak{z}(l'+a) + \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2+|\mathcal{V}|}{8}. \tag{4.74}$$

Remark 4.4.49 Assume that the equivariant SWIC is true for (X, o). Then, taking the sum of the identities SWIC(r_h) from (4.67), and using $\sum_\sigma \mathfrak{T}_\sigma(1) = 0$, we get the following expression for the geometric genus of the universal abelian covering (X_a, o) in terms of the graph Γ:

$$p_g(X_a, o) = -\lambda(M(\Gamma)) - |H|\cdot\frac{K^2+|\mathcal{V}|}{8} + \sum_{h\in H}\chi(r_h).$$

Example 4.4.50 (SWIC is True for Cyclic Quotients) In this case the link is $L(n, q)$, $H = \mathbb{Z}_n$ and the spin^c structures are indexed by $\sigma = \sigma[K + 2aE_s^*]$, where $a \in \mathbb{Z}$ and $0 \leq a < n$. Set also $h = a[E_s^*] \in H$. Then

$$\mathfrak{T}_\sigma(1) = -\mathbf{s}(q,n) + \frac{n-1}{4n} - \frac{a}{2n} - \sum_{i=1}^{a}\left(\left(\frac{iq'}{n}\right)\right).$$

Since $\lambda/n = \mathbf{s}(q, n)/2$, cf. 4.4.7, we also have

$$\mathfrak{sw}_\sigma = -\frac{3}{2}\cdot\mathbf{s}(q,n) + \frac{n-1}{4n} - \frac{a}{2n} - \sum_{i=1}^{a}\left(\left(\frac{iq'}{n}\right)\right).$$

On the other hand, $(K + 2r_h)^2 + |\mathcal{V}|)/8 = (K^2 + |\mathcal{V}|)/8 - \chi(r_h)$ can also be computed explicitly. From 4.2.60 one has $(K^2+|\mathcal{V}|)/8 = (n-1)/4n - 3\mathbf{s}(q, n)/2$.

Furthermore, from 4.2.76 we have $h^1(\mathcal{O}(-s_h)) - h^1(\mathcal{O}(-r_h)) = \chi(s_h) - \chi(r_h)$. But $h^1(\mathcal{O}(-s_h)) = 0$ by the vanishing 4.2.71, while $h^1(\mathcal{O}(-r_h)) = p_g(X_a, o)_{\theta(h)} = 0$ (cf. 4.2.82) since the universal abelian covering (X_a, o) is smooth. Hence $\chi(r_h) = \chi(s_h)$, and its expression is

$$\chi(r_h) = \frac{a}{2n} + \sum_{i=1}^{a}\left(\left(\frac{iq'}{n}\right)\right).$$

In particular, the right hand side of SWIC(r_h) is zero, and the same is true for the left hand side because of the vanishing already mentioned.

Example 4.4.51 The equivariant SWIC is true for splice quotient singularities. In particular, it is true for rational, minimally elliptic and weighted homogeneous singularities (with $\mathbb{Q}HS^3$ link). The SWIC(0) is valid for all elliptic singularities and suspensions $\{z^n + f(x, y) = 0\}$, where f is irreducible (and with $\mathbb{Q}HS^3$ link).

4.4.11 SWIC and Superisolated Singularities

We assume that (X, o) is a superisolated singularity associated with the irreducible projective rational cuspidal curve C of degree d.

Though in many cases (e.g. for weighted homogeneous singularities) we discuss the SWIC together with equivariant SWIC, this is not the case for the superisolated germs. The main obstruction is that in the superisolated case (though $p_g(X, o)$ and $P_{0,\{v_+\}}(t)$ are extremely simple), usually we have very little information about the analytic properties of the universal abelian covering, e.g. about its geometric genus $p_g(X_a, o)$ (see e.g. [111]). Therefore, in this subsection we focus merely on the SWIC (for $h = 0$).

It turns out that for a superisolated singularity the SWIC is valid if and only if $N(1) = 0$, a property which is not always true, cf. subsection 4.3.6. Let us list first the involved invariants.

4.4.52 From Example 4.4.11 we have $K^2 + |\mathcal{V}| = -d(d-2)^2 + 1$ and $24\lambda = \mu(3\mu - 5) - 12 \cdot \Delta''(1)$ $(\mu = 2\delta)$. Moreover, the divisorial filtration associated with $\mathcal{I} = \{C\} = \{v_+\}$ agrees with the filtration associated with weights $(1, 1, 1)$, hence $P_{0,\mathcal{I}}(t) = (1 - t^d)/(1 - t)^3$. Since in the good resolution $\Gamma \setminus v_+$ supports only smooth germs, by 4.3.30 $p_g(X, o) = \mathrm{pc}(P_{0,\mathcal{I}}(t))$, which is $d(d-1)(d-2)/6$.

The definition of $Z_{\mathcal{I}}(t)$ compared with A'Campo formula [1] gives

$$Z_{0,\mathcal{I}}(t) = \frac{1}{d}\sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} \qquad \text{and} \quad Z_{\mathcal{I}}(t) = \frac{\Delta(t^{1/d})}{(1-t^{1/d})^2}.$$

Since H is generated by $[E_+^*]$, the vertex v_+ (corresponding to C) is a regularization vertex for any character. Therefore, from 4.4.24

$$\mathcal{T}_{\sigma[K]}(1) = \lim_{t\to 1}\Big(Z_{0,\mathcal{I}}(t) - \frac{1}{d}Z_{\mathcal{I}}(t)\Big) = \frac{1}{d}\sum_{\xi^d=1\neq\xi} \frac{\Delta(\xi)}{(1-\xi)^2}.$$

Following 4.3.6 we also consider

$$N(t) := Z_{0,\mathcal{I}}(t) - P_{0,\mathcal{I}}(t) = \frac{1}{d}\sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} - \frac{1-t^d}{(1-t)^3}.$$

Then

$$\lim_{t\to 1} N(t) = \mathfrak{T}_{\sigma[K]}(1) + \lim_{t\to 1}\Big(\frac{1}{d}\cdot\frac{\Delta(t^{1/d})}{(1-t^{1/d})^2} - \frac{1-t^d}{(1-t)^3}\Big).$$

If we write $\Delta(t) = 1 + \delta(t-1) + Q(t)(t-1)^2$ as in 4.3.6, then the limit can be computed in terms of d and $Q(1) = \Delta''(1)/2$. The computation provides

Proposition 4.4.53

$$N(1) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathcal{V}|}{8} - p_g.$$

This combined with (4.45) gives (with $Q(t) = \sum_{j=0}^{\mu-2}\alpha_j t^j$)

$$\mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathcal{V}|}{8} = \sum_{j=0}^{d-3}\alpha_{jd}.$$

Corollary 4.4.54

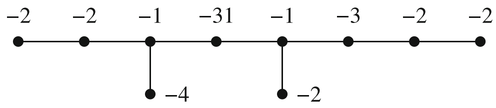
(a) SWIC for $h = 0$ is equivalent to $N(1) = 0$.
(b) The Conjecture 4.3.22 (which predicts that $N(1) \leq 0$ for any superisolated singularity) is equivalent to $\mathfrak{sw}_{\sigma[K]} - \frac{K^2+|\mathcal{V}|}{8} \leq p_g$.

Corollary 4.4.54 has the following consequences (for some of the arguments see the paragraphs after 4.3.21): via the 'Semigroup Distribution Property' 4.2.33, the SWIC (for $h = 0$) is valid whenever $\nu = 1$. In fact, in this case not only $N(1) = 0$, but even $N(t) \equiv 0$, i.e. $Z_{0,\mathcal{I}}(t) \equiv P_{0,\mathcal{I}}(t)$.

If $\nu = 2$ then the coefficients of $N(t)$ are non-positive, however, it can happen that $N(t) \neq 0$, see. e.g. several examples in [61]. Hence, if $\nu = 2$ and $N(t) \neq 0$ then the SWIC fails and $\mathfrak{sw}_{\sigma[K]} - \frac{K^2+|\mathcal{V}|}{8} < p_g$. (The difference will be interpreted in terms of lattice cohomology in 4.9.2.)

Remark 4.4.55 Though till now we tried to convince the reader that the SWIC, for certain analytic types, is a 'natural' reality, the superisolated case suggests the opposite. Indeed, for such germs, p_g depends only on d, but the topological side depends in a subtle way on the local singularity types of C (see above the formulae of λ and $\mathfrak{T}_{\sigma[K]}(1)$). Having in mind this subtle sensitivity to the local singularity data of C, the validity of SWIC (when it holds) is a true marvel.

Example 4.4.56 Let us analyse a particular case with more details. Assume $d = 5$, $\nu = 2$, and the two singularities have multiplicity sequence [3] and $[2_3]$. The graph Γ is presented below, and $N(t) = -2t$, hence SWIC fails: $p_g = 10$, while $-\lambda = 21/2$ and $\mathfrak{T}_{\sigma[K]}(1) = 2/5$, hence $\mathfrak{sw}_{\sigma[K]} - (K^2 + |\mathcal{V}|)/8 = 8$.



In fact, we can consider two analytic structures supported on this topological type (given by the graph). They are rather different, though both are very natural. The first is a superisolated hypersurface singularity, as analysed above. On the other hand, this topological type supports also a splice quotient singularity which satisfies SWIC, hence it has $p_g = 8$.

4.5 Weighted Cubes and the Spaces $S_{k,n}$

4.5.1 Weighted Cubes and Generalized Computation Sequences

To any good resolution graph Γ and characteristic element $k \in \mathrm{Char}$, we consider the weight function $\chi_k : L \to \mathbb{Z}$, and a natural cubical decomposition of $\mathbb{R}^s$ associated with the embedding $L \simeq \mathbb{Z}^s \hookrightarrow \mathbb{Z}^s \otimes \mathbb{R} = \mathbb{R}^s$, where $s = |\mathcal{V}|$ and the identification $L \simeq \mathbb{Z}^s$ is given by the base vectors $\{E_v\}_{v\in\mathcal{V}}$. Then, for each $n \geq \min_{l\in L}\{\chi_k(l)\}$, we define the topological space $S_{k,n}$, as the union of all cubes, which have all vertices of weight $\leq n$. We show that the homotopy type of the tower $\{S_{k,n}\}_n$ depends only on the 3-manifold $M(\Gamma)$ and on the spin^c structure associated with k. The tower $\{S_{k,n}\}_n$ carries an extremely deep information about $M(\Gamma)$; the final goal is to determine their homotopy types. Via the spaces $\{S_{k,n}\}_n$ this section prepares the theory of graded roots and lattice cohomology.

4.5.1 Cubes in $L \otimes \mathbb{R}$ and the Spaces $\{S_{k,n}\}_n$ [72] Fix a connected plumbing graph Γ with negative definite intersection form, and *we assume that the plumbed 3-manifold $M(\Gamma)$ is a rational homology sphere*.

We use the standard notations for the lattice L, which has the distinguished base elements $\{E_v\}_{v\in\mathcal{V}}$. Using this basis, one identifies L with $\mathbb{Z}^s$ with its fixed standard basis, still denoted by $\{E_v\}_{v\in\mathcal{V}}$.

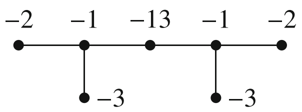
$\mathbb{Z}^s \otimes \mathbb{R} \simeq \mathbb{R}^s$ has a natural decomposition into cubes given by the inclusion $\mathbb{Z}^s \hookrightarrow \mathbb{R}^s$. The zero-dimensional cubes are exactly the lattice points $\mathbb{Z}^s$. Any $l \in \mathbb{Z}^s$ and subset $I \subset \mathcal{V}$ of cardinality q defines a q-dimensional cube $\square_q = (l, I)$, which has its vertices in the lattice points $(l + \sum_{v\in I'} E_v)_{I'}$, where I' runs over all subsets of I.

Next, we fix a characteristic element $k \in \mathrm{Char}$ and we consider the Riemann–Roch function $\chi_k : L \to \mathbb{Z}$, $\chi_k(l) = -(l, l + k)/2$. Here we regard χ_k as a weight function on the set of cubes: the weights of zero-dimensional cubes are defined by $w_0(l) = \chi_k(l)$, while, in general, $w_q((l, I)) := \max\{\chi_k(v) : v \text{ is a vertex of } (l, I)\}$.

Definition 4.5.2 For every $n \in \mathbb{Z}$, define $S_n \subset \mathbb{R}^s$ as the union of all the cubes $\square_q$, of any dimension, with $w(\square_q) \leq n$ (with induced topology of $\mathbb{R}^s$). Clearly, $S_n \neq \emptyset$ exactly when $n \geq m_k$, where $m_k := \min_{l \in \mathbb{Z}^s} \chi_k(l)$. If we wish to emphasize the k-dependence we write $S_{k,n}$.

One has the natural inclusions $S_{m_k} \subset \ldots \subset S_n \subset S_{n+1} \subset \ldots$. It turns out that the topology of the spaces $\{S_n\}_{n \geq m_k}$ might be rather interesting. The tower has a finiteness property: only finitely many S_n have nontrivial topology (are non-contractible), but an S_n with n 'small' might have rather complicated homology groups. In general it is rather hard to solve the corresponding Diophantine equations and to analyse the adjacent positions of the solutions (in order to get the cubes which build up the topological space S_n). However, this combinatorial/artihmetical structure can be extremely rich covering a big amount of deep information.

Example 4.5.3 ([72]) Consider the following graph:



A computation shows that $\chi \geq -1$. S_{-1} consists of two contractible connected components. The space S_0 has three connected components, two of them contractible, and the third has the homotopy type of the circle. The spaces S_n for $n \geq 1$ are contractible.

4.5.4 Assume that k and k' determine the same Spin^c–structure of $M(\Gamma)$, cf. 4.2.94, hence $k' = k + 2l$ for some $l \in L$. Then $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. This means that the transformation $x \mapsto x' := x - l$ realizes an identification of the 'S_n-spaces' associated with k and k': $S_{k,n} = S_{k',n-\chi_k(l)}$. Hence, fixing a representative k from each class $[k] \in \text{Spin}^c(M(\Gamma))$ we can speak about the tower of spaces $\{S_{k,n}\}_n$, indexed by $[k] \in \text{Spin}^c(M(\Gamma))$.

Proposition 4.5.5 ([72]) *The tower of spaces* $\{S_{k,n}\}_n$*, indexed by* $[k] \in \text{Spin}^c(M(\Gamma))$*, up to homotopy equivalence, depends only on* $M = M(\Gamma)$*, it is independent of the choice of the negative definite plumbing graph* Γ*, which provides* M*.*

Remark 4.5.6 A possible generalization of the set of weighted cubes and spaces S_n is provided via a set of compatible *weight functions*. Let $\mathcal{Q}_q$ denote the set of q–cubes. A set of functions $w_q : \mathcal{Q}_q \to \mathbb{Z}$ $(0 \leq q \leq |\mathcal{V}|)$ is called a *set of compatible weight functions* if the following hold:

(a) for any integer $n \in \mathbb{Z}$, the set $w_0^{-1}((-\infty, n])$ is finite;
(b) for any $\square_q \in \mathcal{Q}_q$ and for any of its faces $\square_{q-1} \in \mathcal{Q}_{q-1}$ one has $w_q(\square_q) \geq w_{q-1}(\square_{q-1})$.

Then one can define S_n as $\cup_q \{\square \in \mathcal{Q}_q : w_q(\square_q) \leq n\}$.

4.5.2 The Topology of the Spaces $\{S_{k,n}\}_n$

In order to analyse the topology of the space $S_n = S_{n,k}$ it is convenient to introduce the set of finite rectangles indexed by pairs $l_1, l_2 \in L$ with $l_1 \le l_2$.

Definition 4.5.7 For any such pair $l_1 \le l_2$ set $R(l_1, l_2) := \{x \in \mathbb{R}^s : l_1 \le x \le l_2\}$. Define also $R(l_1, \infty) := \{x \in \mathbb{R}^s : l_1 \le x\}$.

The point in the next lemma is that χ-monotone (non-increasing) computation sequences provide strong deformation retracts for the spaces $S_{k,n}$.

Lemma 4.5.8 *Fix $k \in$ Char and write $S_n = S_{k,n}$.*

(I) There exist $l_+ \in L$ and an increasing infinite sequence of cycles $\{l_i\}_{i\ge 0}$ ($l_i \in L$) with $l_0 = l_+$, such that

(a) for any $i \ge 0$ one has $l_{i+1} = l_i + E_{v(i)}$ for some $v(i) \in \mathcal{V}$,
(b) if $l_i = \sum_v m_{i,v} E_v$, then $\lim_{i\to\infty} m_{i,v} = \infty$ for all v,
(c) $\chi_k(l_{i+1}) \ge \chi_k(l_i)$.

Similarly, there exists $l_- \in L$ and an increasing infinite sequence of cycles $\{y_i\}_{i\ge 0}$, satisfying $y_0 = l_-$, the analogs of (a)–(b), and (c) $\chi_k(-y_{i+1}) \ge \chi_k(-y_i)$.

(II) Take l_- and l_+ as in (I). Without loss of generality we can assume that $-l_- \le l_+$. Then the inclusion $R(-l_-, \infty) \cap S_n \subset S_n$ and $R(-l_-, l_+) \cap S_n \subset S_n$ are homotopy equivalences for any $n \in \mathbb{Z}$.

Corollary 4.5.9 *For any $k \in$ Char the space S_n is contractible for any $n \gg 0$.*

Proof Fix $l_- \le l_+$ as in Lemma 4.5.8*(I)*. Let n be so large that $R(-l_-, l_+) \subset S_n$. Then, by Lemma 4.5.8*(II)* S_n has the same homotopy type as $R(-l_-, l_+)$. □

4.5.10 Distinguished Representatives and Their Spaces S_n As we already said in 4.5.4, if $k' = k + 2l$ for some $l \in L$ then $S_{k,n} = S_{k',n-\chi_k(l)}$. Hence, it is natural to choose one representative from each spin^c structure. For several results the choice is irrelevant, however, certain choices have certain advantages. Our preferred choice is the *distinguished representative*, or *distinguished characteristic element* $k_r := K + 2s_h$, cf. 4.2.94, where $s_h \in L'$ is the smallest representative of h in $\mathcal{S}'$, cf. 4.2.78.

A possible motivation for the choice of k_r is the following. Recall that the rationality criterion for graphs is $\chi(l) > 0$ for any $l \in L_{>0}$, hence it is decided in the 'first quadrant' $L_{\ge 0}$ of L. More generally, for arbitrary graphs, the essential properties of $\chi : L \to \mathbb{Z}$ are already coded in the restriction $\chi|L_{\ge 0}$. The choice $k_r = K + 2s_h$ guarantees that the essential properties of $\chi_{k_r} : L \to \mathbb{Z}$ are coded again in $L_{\ge 0}$ (or, equivalently, for a fixed h, the essential information of $\chi_{\mathbb{Q}}|\{l' \in L' : [l'] = h\}$ is coded in $\chi_{\mathbb{Q}}|s_h + L_{\ge 0}$.

Lemma 4.5.11 *Fix $h \in H$ and $k_r = K + 2s_h$ as above. Then the following facts hold.*

(a) In Lemma 4.5.8 one may take $l_- = 0$. This means that $R(0, \infty) \cap S_{k_r,n} \subset S_{k_r,n}$ is a homotopy equivalence for any n. In particular, by Lemma 4.5.8, there exists $l_+ \geq 0$ such that $R(0, l_+) \cap S_{k_r,n} \subset S_{k_r,n}$ is a homotopy equivalence for any n.

(b) Assume that $Z_K \geq 0$ (e.g., as in the minimal good resolution). Then one can take $l_+ = \lfloor Z_K \rfloor$. Hence, $S_{k_r,n}$ has the homotopy type of $R(0, \lfloor Z_K \rfloor) \cap S_{k_r,n}$.

(c) For any $x \geq 0$ one has $\chi_{k_r}(x) \geq \chi_K(x)$. Therefore, $\min \chi_{k_r} \geq \min \chi_K$.

(d) $S_{K,n}$ (i.e. when $h = 0$ and $s_h = 0$) is connected for all $n \geq 1$.

Example 4.5.12 (Characterization of Rational Graphs via the Spaces S_n [70]) Let Γ be a connected, negative definite plumbing graph whose plumbed 3-manifold is a rational homology sphere. Recall that Γ is rational if $\chi(l) > 0$ for any $l \in L_{>0}$. (In this case $p_g(X, o) = 0$ for any analytic type supported on the topological type determined by Γ.) Then the following facts are equivalent:

(a) Γ is rational;
(b) $S_{K,n}$ is contractible for every $n \geq \min \chi$;
(b') $S_{K,n}$ is connected for every $n \geq \min \chi$;
(c) $S_{k,n}$ is contractible for all $k \in \text{Char}$ and $n \geq \min \chi_k$.

Additionally, if Γ is rational and $k_r = K + 2s_h$, then $\min \chi_{k_r} = 0$.

Example 4.5.13 (Characterization of Elliptic Graphs via the Spaces $S_{K,n}$ [70]) Assume again that $M(\Gamma)$ is a $\mathbb{Q}HS^3$. Recall that Γ is elliptic if $\min \chi = 0$ and Γ is not rational. Then Γ is elliptic if and only if $S_{K,n}$ is empty for $n < 0$ and $S_{K,0}$ is not connected.

4.5.3 'Bad' Vertices, Almost Rational Graphs and Lattice Fibrations

We measure how far an arbitrary graph (tree) Γ is from being rational. Recall that decreasing all the self-intersection numbers of a tree, with all the vertices decorated by $g_v = 0$, we obtain a rational graph. The next definition aims to identify those vertices where such a decrease is really necessary. [Such a subset of $\mathcal{V}$ was already considered in different articles like [70, 72, 74, 102], mostly under the name 'bad vertices'. Since the definition of the 'badness' was not uniform here we use the notation SR for them, for several other families see [66].]

Definition 4.5.14 Let Γ be a negative definite connected tree with $M(\Gamma)$ a $\mathbb{Q}HS^3$.

A subset of vertices $\overline{\mathcal{V}} = \{v_1, \dots, v_\nu\} \subset \mathcal{V}$ is called *SR–set*, if by replacing the Euler numbers $e_v = E_v^2$ indexed by $v \in \overline{\mathcal{V}}$ by some more negative integers $e'_v \leq e_v$ we get a rational graph. A graph is called AR-graph ('almost rational graph') if it admits an SR–set of cardinality ≤ 1.

Example 4.5.15

(a) A possible SR–set can be chosen in many different ways, it is not determined uniquely even if it is minimal with this property.
(b) Usually we allow non-minimal SR–sets as well.
(c) Any rational graph is AR; for rational graphs the empty set is an SR–set. The class of AR graphs is closed while taking subgraphs or/and decreasing the Euler numbers.
(d) The set of nodes is an SR–set. Hence any star-shaped graph (with $g = 0$) is AR with $\overline{\mathcal{V}} = \{v_0\}$.
(e) Any elliptic graph with $H_1(L_X, \mathbb{Q}) = 0$ is AR.
(f) Consider the graph Γ of $S^3_{-d}(K)$ (for $d > 0$ and $K \subset S^3$ algebraic knot). Then Γ is AR: if we modify the -1 decoration of v_1 into -2, we get a sandwiched (hence rational) graph.
(g) Let $\{K_i\}_{i=1}^{\nu}$ be algebraic knots and set $K = \#_i K_i$. For $d > 0$ the negative definite graph Γ of $S^3_{-d}(K)$ is given in 4.2.32. Then the smallest SR–set consists of the set of (-1)-vertices (their number is ν).

4.5.16 Lattice Fibrations: Universal Cycles in the Fibers Let us give some intuition for the next construction.

If Γ is rational, then 0 is a χ_{k_r}–minimal lattice point, and $0 \hookrightarrow S_{k_r,n}$ ($n \geq 0$) admits a strong deformation retraction: there is a χ_{k_r}–non-increasing (combinatorial) flow contracting any $S_{k_r,n}$ (and $L \otimes \mathbb{R}$) to 0.

In general, let us start with the lattice L and a representative $k = K + 2l'_h$. Then (dictated by some 'badness properties' of some of the vertices, indexed by $\overline{\mathcal{V}}$) we will write the set of vertices $\mathcal{V}$ as a disjoint union $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$, such that any sublattice of type $\bar{l} + L(\mathcal{V}^*)$ (where $\bar{l} = \sum_{v \in \overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}})$) behaves as a rational lattice, that is, it can be contracted to one of its lattice points via a χ_k–non-increasing flow. (In other words, 'L, or the spaces S_n, project to $L(\overline{\mathcal{V}})$ with contractible fibers'.) On the other hand, the χ_k–minimal point of $\bar{l} + L(\mathcal{V}^*)$, where $\bar{l} + L(\mathcal{V}^*)$ contracts, depends essentially on $\bar{l}$; it is a crucial universal point $x_{l'_h}(\bar{l})$ of $\bar{l} + L(\mathcal{V}^*)$. The aim of different reduction theorems is to recover different invariants of the weighted lattice (L, χ_k) from $\{\chi_k(x_{l'_h}(\bar{l}))\}_{\bar{l} \in L(\overline{\mathcal{V}})}$.

In this subsection we define and analyse the points $x_{l'_h}(\bar{l})$. If $l'_h = s_h$ then some additional 'positivity' properties hold for them.

4.5.17 The Definition of the Lattice Points $x(\bar{l})$ Let us fix a resolution of a germ (whose link is not necessarily a rational homology sphere). Suppose we have a family of *distinguished* vertices $\overline{\mathcal{V}} := \{v_k\}_{k=1}^{\nu} \subseteq \mathcal{V}$ (usually chosen by a certain geometric property). Then we split the set of vertices $\mathcal{V}$ into the disjoint union $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$. Let $\{m_v(x)\}_v$ denote the coefficients of a cycle $x \in L \otimes \mathbb{Q}$, that is $x = \sum_{v \in \mathcal{V}} m_v(x) E_v$.

We use the notation $\bar{l} := \sum_{v \in \overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}})$, and we fix $h \in H$ and a representative $l'_h \in L'$ with $[l'_h] = h$. Then the cycles $x(\bar{l})$ are defined as follows.

Proposition 4.5.18 ([70, Lemma 7.6], [47]) *For any $\bar{l} \in L(\overline{\mathcal{V}})$ there exists a unique cycle $x(\bar{l}) \in L$ (depending on the choice of l'_h) satisfying the next properties:*

(a) $m_v(x(\bar{l})) = \ell_v$ for any distinguished vertex $v \in \overline{\mathcal{V}}$;
(b) $(x(\bar{l}) + l'_h, E_v) \le 0$ for every 'non-distinguished vertex' $v \in \mathcal{V}^*$;
(c) $x(\bar{l})$ is minimal with the two previous properties (with respect to $\le$).

Furthermore, the cycle $x(\bar{l})$ automatically satisfies

$$x(\bar{l}) + \bar{l}_1 \le x(\bar{l} + \bar{l}_1) \quad \text{for any } \bar{l}_1 \ge 0, \bar{l}_1 \in L(\overline{\mathcal{V}}). \tag{4.75}$$

If we wish to emphasize the dependence on l'_h we write $x_{l'_h}(\bar{l})$.

The cycles $x(\bar{l})$ satisfy the following universal property as well.

Lemma 4.5.19 *Assume that a certain $x \in L$ satisfies $m_v(x) = m_v(x(\bar{l}))$ for all $v \in \overline{\mathcal{V}}$, and $x \le x(\bar{l})$.*

*Then there is a **generalized Laufer's computation sequence** connecting x with $x(\bar{l})$. The sequence $\{z_i\}_{i=0}^t$ is constructed as follows. Set $z_0 = x$. Assume that z_i is already constructed. If for some $v \in \mathcal{V}^*$ one has $(z_i + s_h, E_v) > 0$ then take $z_{i+1} = z_i + E_{v(i)}$, where $v(i)$ is such an index. If z_i satisfies 4.5.18(b), then stop and set $t = i$. Then this procedure stops after finite steps and z_t is exactly $x(\bar{l})$.*

Along the computation sequence $\chi_k(z_{i+1}) \le \chi_k(z_i)$ for any $0 \le i < t$. Equality holds if $(z_i + l'_h, E_{v(i)}) = 1$.

In the case of an SR–set we have the following statement.

Proposition 4.5.20 *Let $\overline{\mathcal{V}}$ be an SR–set. Choose l'_h and set $k = K + 2l'_h$. Then $\bar{l} + L(\mathcal{V}^*) = \{x \in L : m_v(x) = m_v(x(\bar{l}))$ for all $v \in \overline{\mathcal{V}}\}$ contracts to $x(\bar{l})$ such that along the contraction χ_k is non-increasing. In particular, $\chi_k(x) \ge \chi_{k_r}(x(\bar{l}))$ for any $x \in \bar{l} + L(\mathcal{V}^*)$.*

4.5.4 Concatenated Computation Sequences of AR Graphs [70]

Assume that Γ is an AR resolution graph, let $\{v_0\}$ be an SR–set. In particular $M(\Gamma)$ is a rational homology sphere.

Theorem 4.5.21 *If Γ is AR, then for any $k \in$ Char and $n \ge m_k = \min \chi_k$ any connected component of $S_{k,n}$ is contractible.*

Note that the statement is independent of the choice of k in its class, cf. 4.5.10. In the sequel we will choose the distinguished representative k_r, and we write S_n for $S_{k_r,n}$. We also write $\mathcal{V} = \overline{\mathcal{V}} \sqcup \mathcal{V}^*$, where $\overline{\mathcal{V}} = \{v_0\}$. For each $\ell \in \mathbb{Z}$ we consider the cycles $\bar{l} := \ell E_{v_0} \in L(\overline{\mathcal{V}})$ and $x(\bar{l}) \in L$ from 4.5.16. We abridge $x(\ell E_{v_0})$ as $x(\ell)$.

In order to prove the theorem we construct an increasing path $\gamma = \{l_i\}_{i\geq 0}$ in L (with $\ell_{i+1} = \ell_i + E_{v(i)}$ for all i), which determines the 1-chain $C_\gamma := \cup_{i\geq 0}[l_i, l_{i+1}]$ of 1-cubes in $L \otimes \mathbb{R}$ (without any loop), such that $C_\gamma \cap S_n \hookrightarrow S_n$ is a homotopy equivalence. The construction and the statement of the theorem constitute the prototype of the more general Reduction Theorem 4.8.2 and also this was the original intuitive motivation and starting point in the definition of the graded roots, cf. 4.7 and 4.7.2.

The construction start as follows. By Lemma 4.5.11*(a)* the inclusion $R(0, \infty) \cap S_n \subset S_n$ admits a strong deformation retract. Hence we can restrict ourself to the intersection with the first quadrant. The path $\gamma = \{l_i\}_{i\geq 0}$ is defined as a series of concatenated computation sequences. It contains, as intermediate terms, all the universal cycles $\{x(\ell)\}_{\ell\geq 0}$ in an increasing order. The first term is $l_0 = x(0) = 0$. The part of the sequence starting from $x(\ell)$ and ending with $x(\ell+1)$ starts with $x(\ell)$ and the next term is $x(\ell) + E_{v_0}$. Then, the continuation is generalized Laufer-type computation sequence connecting $x(\ell)+E_{v_0}$ with $x(\ell+1)$. Indeed, the multiplicity of E_0 in both $x(\ell) + E_{v_0}$ and $x(\ell + 1)$ is $\ell + 1$, and by (4.75) $x(\ell + 1) \geq x(\ell) + E_{v_0}$. Hence Lemma 4.5.19 gives a computation sequence $\gamma^{(\ell+1)} = \{l_i^{(\ell+1)}\}_i$, which connects them. Then we proceed inductively.

Define $\tau(\ell) := \chi_{k_r}(x(\ell))$. Let o be the order of $E^*_{v_0}$ in L'/L and $p = m_{v_0}(oE^*_{v_0})$.

Lemma 4.5.22

(a) *The path* $\{l_i\}_i$ *is increasing:* $l_{i+1} = l_i + E_{v(i)}$.
(b) *For any* E_v*-coefficient one has* $\lim_{\ell\to\infty} m_v(x(\ell)) = \infty$ *(where* $v \in \mathcal{V}$*).*
(c) *(Quasiperiodicity)* $x(\ell + tp) = x(\ell) + toE^*_{v_0}$.
(d) χ_{k_r} *along each part (subsequence)* $\gamma^{(\ell)}$ *is constant.*
(e) $\tau(\ell + 1) = \tau(\ell) + 1 - (x(\ell) + s_h, E_{v_0})$.
(f) *There exists* ℓ_0 *such that* $\tau(\ell + 1) \geq \tau(\ell)$ *for* $\ell \geq \ell_0$.

Theorem 4.5.23 *Consider the 1-chain* $C_\gamma := \cup_{i\geq 0}[l_i, l_{i+1}]$ *in* $L \otimes \mathbb{R}$ *as above. Then for any* n *the inclusion* $C_\gamma \cap S_n \subset S_n$ *is a homotopy equivalence. In particular, since each connected component of* $C_\gamma \cap S_n$ *is contractible, Theorem 4.5.21 follows.*

Remark 4.5.24 In general, it is not easy to find the cycles $x(\ell)$. Fortunately, in several applications (see e.g. 4.7.3) one does not need all the coefficients of these cycles, only the values $\tau(\ell) = \chi_{k_r}(x(\ell))$. In most of the cases they are computed inductively using 4.5.22*(e)*, hence basically one needs only to know $(x(\ell), E_{v_0})$ for any ℓ.

Example 4.5.25 For the determination of the universal cycles $\{x(\ell)\}_\ell$ and the corresponding τ-function in the case of star-shaped graphs and surgery manifolds see 4.7.22, 4.7.4 and Sect. 4.9.

4.6 Lattice Cohomology

We provide two equivalent definitions for the lattice cohomology $\{\mathbb{H}^q\}_{q\geq 0}$ associated with a free $\mathbb{Z}$-module endowed with a fixed basis and with a set of 'compatible weight functions'. The first definition is based on the construction of a cochain complex. The second one involves the spaces $\{S_n\}_n$ introduced in 4.5.2. Once Γ is fixed, any characteristic element $k \in$ Char determines a set of weights (via the RR expression χ_k), hence the lattice cohomology $\mathbb{H}^*(\Gamma, k)$. It turns out that they depend only on $M(\Gamma)$ and $[k] \in \mathrm{Spin}^c(M(\Gamma))$. In 4.6.3 we show that the Euler characteristic of $\mathbb{H}^*(\Gamma, k)$ is the normalized Seiberg–Witten invariant of $M(\Gamma)$.

For more details see e.g. [71–73].

4.6.1 The Lattice Cohomology Associated with a System of Weights

We consider a free $\mathbb{Z}$-module, with a fixed basis $\{E_v\}_{v\in\mathcal{V}}$, denoted by $\mathbb{Z}^s$. It is also convenient to fix a total ordering of the index set $\mathcal{V}$, which in the sequel will be denoted by $\{1,\dots,s\}$. Our goal is to define a graded $\mathbb{Z}[U]$-module associated with the pair $(\mathbb{Z}^s, \{E_v\}_v)$ and a set of weights. First we set some notations regarding $\mathbb{Z}[U]$-modules.

4.6.1 $\mathbb{Z}[U]$-Modules Consider the graded $\mathbb{Z}[U]$-module $\mathcal{T} := \mathbb{Z}[U, U^{-1}]$, and (following [102]) denote by $\mathcal{T}_0^+$ its quotient by the submodule $U\cdot\mathbb{Z}[U]$. This has a grading in such a way that $\deg(U^{-d}) = 2d$ $(d \geq 0)$. Similarly, for any $n \geq 1$, the quotient of $U^{-(n-1)}\cdot\mathbb{Z}[U]$ by $U\cdot\mathbb{Z}[U]$ (with the same grading) defines the graded module $\mathcal{T}_0(n)$. Hence, $\mathcal{T}_0(n)$, as a $\mathbb{Z}$-module, is freely generated by $1, U^{-1}, \dots, U^{-(n-1)}$, and has finite $\mathbb{Z}$-rank n.

More generally, for any graded $\mathbb{Z}[U]$-module P with d-homogeneous elements P_d, and for any $r \in \mathbb{Q}$, we denote by $P[r]$ the same module graded (by $\mathbb{Q}$) in such a way that $P[r]_{d+r} = P_d$. Then set $\mathcal{T}_r^+ := \mathcal{T}_0^+[r]$ and $\mathcal{T}_r(n) := \mathcal{T}_0(n)[r]$. Hence, for $m \in \mathbb{Z}$, $\mathcal{T}_{2m}^+ = \mathbb{Z}\langle U^{-m}, U^{-m-1}, \dots\rangle$ as a $\mathbb{Z}$-module.

4.6.2 The Cochain Complex $\mathbb{Z}^s \otimes \mathbb{R}$ has a natural cellular decomposition into cubes (see also 4.5.1). The set of zero-dimensional cubes is provided by the lattice points $\mathbb{Z}^s$. Any $l \in \mathbb{Z}^s$ and subset $I \subset \mathcal{V}$ of cardinality q defines a q-dimensional cube, which has its vertices in the lattice points $(l + \sum_{v\in I'} E_v)_{I'}$, where I' runs over all subsets of I. On each such cube we fix an orientation. This can be determined, e.g., by the order $(E_{v_1}, \dots, E_{v_q})$, where $v_1 < \cdots < v_q$, of the involved base elements $\{E_v\}_{v\in I}$. The set of oriented q-dimensional cubes defined in this way is denoted by $\mathcal{Q}_q$ $(0 \leq q \leq s)$.

Let C_q be the free $\mathbb{Z}$-module generated by oriented cubes $\square_q \in \mathcal{Q}_q$. Clearly, for each $\square_q \in \mathcal{Q}_q$, the oriented boundary $\partial\square_q$ (of 'classical' cubical homology) has the form $\sum_k \varepsilon_k \square^k_{q-1}$ for some $\varepsilon_k \in \{-1, +1\}$. These are the *faces* of $\square_q$. It is clear that $\partial\circ\partial = 0$. But, obviously, the homology of the chain complex (C_*, ∂) (or, of the dual cochain complex $(\mathrm{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z}), \delta)$) is not very interesting: it is the (co)homology of $\mathbb{R}^s$. A more interesting (co)homology can be constructed as follows. For this, we consider a set of compatible *weight functions* $\{w_q\}_q$ as in 4.5.6. In the sequel sometimes we will omit the index q of w_q.

4.6.3 In the presence of any fixed set of compatible weight functions $\{w_q\}_q$ we define $\mathcal{F}^q$ as the set of morphisms $\mathrm{Hom}_{\mathbb{Z}}(C_q, \mathcal{T}^+_0)$ with finite support on $\mathcal{Q}_q$.

Notice that $\mathcal{F}^q$ is a $\mathbb{Z}[U]$-module by $(p * \phi)(\square_q) := p(\phi(\square_q))$ $(p \in \mathbb{Z}[U])$. Moreover, $\mathcal{F}^q$ has a $\mathbb{Z}$-grading: $\phi \in \mathcal{F}^q$ is homogeneous of degree $\deg(\phi) = d \in \mathbb{Z}$ if for each $\square_q \in \mathcal{Q}_q$ with $\phi(\square_q) \neq 0$, $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}^+_0$ of degree $d - 2\cdot w(\square_q)$. (In fact, the grading is $2\mathbb{Z}$-valued; hence, the reader interested only in the present construction may divide all the degrees by two. Nevertheless, we prefer to keep the present form in our presentation because of its resonance with the Heegaard Floer homology of the link.)

Next, we define $\delta_w : \mathcal{F}^q \to \mathcal{F}^{q+1}$. For this, fix $\phi \in \mathcal{F}^q$ and we show how $\delta_w\phi$ acts on a cube $\square_{q+1} \in \mathcal{Q}_{q+1}$. First write $\partial\square_{q+1} = \sum_k \varepsilon_k \square^k_q$, then set

$$(\delta_w\phi)(\square_{q+1}) := \sum_k \varepsilon_k\, U^{w(\square_{q+1}) - w(\square^k_q)}\, \phi(\square^k_q).$$

Lemma 4.6.4 $\delta_w \circ \delta_w = 0$, *i.e.* $(\mathcal{F}^*, \delta_w)$ *is a cochain complex.*

4.6.5 In fact, $(\mathcal{F}^*, \delta_w)$ has a natural **augmentation** too. Indeed, set $m_w := \min_{l\in\mathbb{Z}^s} w_0(l)$ and choose $l_w \in \mathbb{Z}^s$ such that $w_0(l_w) = m_w$. Then one defines the $\mathbb{Z}[U]$-linear map

$$\epsilon_w : \mathcal{T}^+_{2m_w} \longrightarrow \mathcal{F}^0$$

such that $\epsilon_w(U^{-m_w-s})(l)$ is the class of $U^{-m_w+w_0(l)-s}$ in $\mathcal{T}^+_0$ for any $l \in L$ and $s \geq 0$.

Lemma 4.6.6 ϵ_w *is injective, and* $\delta_w \circ \epsilon_w = 0$.

One verifies that both ϵ_w and δ_w are morphisms of $\mathbb{Z}[U]$-modules and are homogeneous of degree zero.

Definition 4.6.7 The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*(\mathbb{R}^s, w)$. The homology of the augmented cochain complex

$$0 \longrightarrow \mathcal{T}^+_{2m_w} \xrightarrow{\epsilon_w} \mathcal{F}^0 \xrightarrow{\delta_w} \mathcal{F}^1 \xrightarrow{\delta_w} \dots$$

is called the *reduced lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$.

If the pair $(\mathbb{R}^s, w)$ is clear from the context, we omit it from the notation.

For any $q \geq 0$ fixed, the $\mathbb{Z}$-grading of $\mathcal{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$; the homogeneous part of degree d is denoted by $\mathbb{H}^q_d$, or $\mathbb{H}^q_{red,d}$. Moreover, both $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$ admit an induced graded $\mathbb{Z}[U]$-module structure and $\mathbb{H}^q = \mathbb{H}^q_{red}$ for $q > 0$.

It is easy to see that $\mathbb{H}^*(\mathbb{R}^s, w)$ depends essentially on the choice of w.

Lemma 4.6.8 *One has a graded* $\mathbb{Z}[U]$*-module isomorphism* $\mathbb{H}^0 = \mathcal{T}^+_{2m_w} \oplus \mathbb{H}^0_{red}$.

4.6.9 Next, we present another realization of the modules $\mathbb{H}^*$. In 4.5.2 for each $n \in \mathbb{Z}$ we defined $S_n = S_n(w) \subset \mathbb{R}^s$ as the union of all the cubes $\square_q$ (of any dimension) with $w(\square_q) \leq n$. Clearly, $S_n = \emptyset$, whenever $n < m_w$. For any $q \geq 0$, set

$$\mathbb{S}^q(\mathbb{R}^s, w) := \oplus_{n \geq m_w} H^q(S_n, \mathbb{Z}).$$

Then $\mathbb{S}^q$ is $\mathbb{Z}$ (in fact, $2\mathbb{Z}$)-graded, the $d = 2n$-homogeneous elements $\mathbb{S}^q_d$ consist of $H^q(S_n, \mathbb{Z})$. Also, $\mathbb{S}^q$ is a $\mathbb{Z}[U]$-module; the U-action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. Moreover, for $q = 0$, the fixed base-point $l_w \in S_n$ provides an augmentation (splitting) $H^0(S_n, \mathbb{Z}) = \mathbb{Z} \oplus \tilde{H}^0(S_n, \mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$-modules

$$\mathbb{S}^0 = \mathcal{T}^+_{2m_w} \oplus \mathbb{S}^0_{red} = (\oplus_{n \geq m_w} \mathbb{Z}) \oplus (\oplus_{n \geq m_w} \tilde{H}^0(S_n, \mathbb{Z})).$$

Theorem 4.6.10 *There exists a graded* $\mathbb{Z}[U]$*-module isomorphism, compatible with the augmentations:*

$$\mathbb{H}^*(\mathbb{R}^s, w) = \mathbb{S}^*(\mathbb{R}^s, w).$$

4.6.11 Restrictions Assume that $T \subset \mathbb{R}^s$ is a subspace of $\mathbb{R}^s$ consisting of a union of some cubes (from $\mathcal{Q}_*$). Let $C_q(T)$ be the free $\mathbb{Z}$-module generated by q-cubes of T, $\mathcal{F}^q(T)$ be the restriction of $\mathcal{F}^q$ to $C_q(T)$. Then $(\mathcal{F}^*(T), \delta_w)$ is a complex, whose homology will be denoted by $\mathbb{H}^*(T, w)$. It has a natural graded $\mathbb{Z}[U]$-module structure. The restriction map induces a natural graded $\mathbb{Z}[U]$-module homogeneous homomorphism (of degree zero)

$$r^* : \mathbb{H}^*(\mathbb{R}^s, w) \to \mathbb{H}^*(T, w).$$

4.6.2 The Lattice Cohomology Associated with a Plumbing Graph

4.6.12 We consider a connected negative definite plumbing graph Γ and we assume that $M(\Gamma)$ is a $\mathbb{Q}HS^3$. We write $s := |\mathcal{V}|$. We also fix a characteristic element $k \in \mathrm{Char}$.

Note that Γ automatically and naturally provides a free $\mathbb{Z}$-module $L = \mathbb{Z}^s$ with a fixed bases $\{E_v\}_v$, cf. 4.2.9 and 4.5.1. Using Γ and k, we define a set of compatible weight functions w as in 4.5.1: $w_k(\square_q) = \max\{\chi_k(v) \ : \ v \text{ is a vertex of } \square_q\}$.

Definition 4.6.13 The $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\mathbb{R}^s, w)$ and $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$ obtained by these weight functions are called the *lattice cohomologies* associated with the pair (Γ, k) and are denoted by $\mathbb{H}^*(\Gamma, k)$, respectively $\mathbb{H}^*_{red}(\Gamma, k)$.

Proposition 4.6.14

(a) $\mathbb{H}^*_{red}(\Gamma, k)$ *is finitely generated over* $\mathbb{Z}$.
(b) $\mathbb{H}^0_{red,d}(\Gamma, K) = 0$ *for the canonical characteristic element* K *and* $d > 0$.

Remark 4.6.15 There is a symmetry present in the picture. Indeed, the involution $x \mapsto -x$ $(x \in L')$ induces identities $\chi_{-k}(-l) = \chi_k(l)$, hence isomorphisms

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, -k) \ \text{ and } \ \mathbb{H}^*_{red}(\Gamma, k) = \mathbb{H}^*_{red}(\Gamma, -k).$$

The involution $[k] \mapsto [-k]$ corresponds to the natural involution of $\mathrm{Spin}^c(M)$, cf. 4.2.93.

4.6.16 Assume that $[k] = [k']$, hence $k' = k+2l$ for some $l \in L$. Then $\chi_{k'}(x-l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. Therefore, the transformation $x \mapsto x' := x - l$ realizes the following identification:

Lemma 4.6.17 *If* $k' = k + 2l$ *for some* $l \in L$*, then:* $\mathbb{H}^*(\Gamma, k') = \mathbb{H}^*(\Gamma, k)[-2\chi_k(l)]$.

4.6.18 In fact, there is an easy way to choose one module from the multitude $\{\mathbb{H}^*(\Gamma, k)\}_{k\in[k]}$. Indeed, set $m_k = \min_{l\in L} \chi_k(l)$ as above. Since $(k + 2l)^2 = k^2 - 8\chi_k(l)$, we get

$$8m_k = k^2 - \max_{k'\in[k]} (k')^2 \leq 0. \tag{4.76}$$

Set $M_{[k]} := \{k \in [k] : m_k = 0\}$. Hence, if k_0 and $k_0 + 2l \in M_{[k]}$, then $-\chi_{k_0}(l) = 0$. In particular, for any fixed orbit $[k]$, any choice of $k_0 \in M_{[k]}$ provides the same module $\mathbb{H}^*(\Gamma, k_0)$, in the sequel denoted by $\mathbb{H}^*(\Gamma, [k])$. Hence, for any $k \in [k]$

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, [k])[2m_k]. \tag{4.77}$$

Proposition 4.6.19 *For each fixed* $[k] \in \mathrm{Spin}^c(M(\Gamma))$, $\mathbb{H}^*(\Gamma,[k])$ *depends only on* $M(\Gamma)$ *and is independent of the choice of the graph* Γ, *which provides* $M(\Gamma)$.

Next, consider the distinguished characteristic element k_r, cf. 4.5.10. The following statement follows from Lemma 4.5.11.

Proposition 4.6.20 *The restriction* $\mathbb{H}^*(\Gamma, k_r) \to \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$ *induced by the inclusion* $(\mathbb{R}_{\geq 0})^s \hookrightarrow \mathbb{R}^s$ *is an isomorphism of graded* $\mathbb{Z}[U]$ *modules.*

Remark 4.6.21 Assume that Γ is either rational or elliptic, in particular, $\min(\chi) = 0$. Then by 4.5.11 $\min(\chi_{k_r}) \geq 0$. Hence, by (4.76), in fact, $\min(\chi_{k_r}) = 0$.

Example 4.6.22 (Rational Graphs) Theorem 4.5.12 transforms into the following statement. The following facts are equivalent:

(a) Γ is rational;
(b) $\mathbb{H}^*_{red}(\Gamma, K) = 0$;
(b') $\mathbb{H}^0_{red}(\Gamma, K) = 0$;
(c) $\mathbb{H}^*_{red}(\Gamma, k) = 0$ for every $k \in \mathrm{Char}$.

Additionally, by Remark 4.6.21, if Γ is rational then $\mathbb{H}^0(\Gamma, k_r) = \mathcal{T}_0^+$ for any k_r.

Example 4.6.23 (Elliptic Graphs) Theorem 4.5.13 and Remark 4.6.21 transform into the following statement: Γ is elliptic if and only if $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+ \oplus \mathbb{H}^0_{red}(\Gamma, K)$ with $\mathbb{H}^0_{red}(\Gamma, K) \neq 0$. (In fact, if Γ is elliptic then $\mathbb{H}^0_{red}(\Gamma, K) = \mathcal{T}_0(1)^{\ell}$, where $\ell > 0$ is the length of the elliptic sequence in the sense of Laufer and Yau).

Example 4.6.24 (Almost Rational Graphs) By 4.5.21, if Γ is almost rational, $\mathbb{H}^q(\Gamma, k) = 0$ for any $q \geq 1$ and $k \in \mathrm{Char}$. (For $\mathbb{H}^0(\Gamma, k)$ see 4.7.3.)

Remark 4.6.25 The author knows no example when $\mathbb{H}^*(\Gamma, k)$ has a non-zero $\mathbb{Z}$-torsion element. *It is a challenge to prove that this cannot occur indeed.*

4.6.3 The Lattice Cohomology and the Seiberg–Witten Invariant

Fix Γ and k as above. Our goal is to identify the 'Euler characteristic' of the lattice cohomology $\mathbb{H}^*(\Gamma, k)$. Recall that by 4.6.14 $\mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^*_{red}(\Gamma, k)) < \infty$.

Definition 4.6.26 The Euler characteristic of $\mathbb{H}^*(\Gamma, k)$ is defined as

$$eu(\mathbb{H}^*(\Gamma, k)) := -m_k + \sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k)).$$

For motivation of the $-m_k$ term see 4.7.6 and the computations from below.

4.6.27 Fix l_- and l_+ and the rectangle $R = R(-l_-, l_+)$ as in Lemma 4.5.8. We define

$$Eu_{\chi_k}(R) := \sum_{\square_q \subset R} (-1)^{q+1} w_k(\square_q) \ \text{ and } \ Eu^{pol}_{\chi_k}(\mathfrak{q}) := \sum_{\square_q \subset R} (-1)^q \mathfrak{q}^{w_k(\square_q)} \in \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}].$$

In particular, if we write $Eu^{pol}_{w_k}(\mathfrak{q})/(1-\mathfrak{q})$ as $\sum_{n\geq m_k} a_n \mathfrak{q}^n$ then

$$a_n = \sum_{\square_q \subset R,\, w_k(\square_q)\leq n} (-1)^q = \chi_{top}(S_n \cap R),$$

where χ_{top} is the topological Euler characteristic. But, by 4.5.8, $S_n \cap R \hookrightarrow S_n$ is a homotopy equivalence, hence $a_n = \chi_{top}(S_n)$. This by 4.6.10 reads as

$$\frac{Eu^{pol}_{\chi_k}(\mathfrak{q}) - \mathfrak{q}^{m_k}}{1-\mathfrak{q}} = \sum_{n\geq m_k} (a_n - 1)\mathfrak{q}^n = \sum_{n\geq m_k} \Big(\sum_{q\geq 0} (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red,2n}(\Gamma, k)) \Big) \mathfrak{q}^n.$$

In particular, this expression is independent of the choice of R. Finally, by taking the limit $\lim_{\mathfrak{q}\to 1}$ we get

$$Eu_{\chi_k}(R) + m_k = \sum_{q\geq 0} (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k))),$$

or

$$Eu_{\chi_k}(R) = eu(\mathbb{H}^*(\Gamma, k)). \tag{4.78}$$

The above identity is a generalization to the level of weighted cubes of the classical fact that the Euler characteristic computed at the level of cubes equals the homological Euler characteristic.

4.6.28 Recall from 4.6.2 that if $k' = k + 2l$, $l \in L$, then $\mathbb{H}^*(\Gamma, k') = \mathbb{H}^*(\Gamma, k)[-2\chi_k(l)]$, hence the lattice cohomologies associated with different k's with the same class $[k]$ are equal up to a shift. This has no effect on $\sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k))$, however it has on m_k. This can be remedied either by choosing k from $M_{[k]}$ (cf. 4.6.18), or by taking k_r (cf. 4.6.16). Next we present another way to eliminate the above shift.

Let us replace the weight function $w_k(\square_q) := \{\chi_k(v) : v \text{ is a vertex of } \square_q\}$ by

$$\overline{w}_k(\square_q) := w_k(\square_q) + \mathfrak{d}_k, \ \text{ where } \ \mathfrak{d}_k := -\frac{k^2 + |\mathcal{V}|}{8} + \frac{K^2 + |\mathcal{V}|}{8} = \chi\left(\frac{k-K}{2}\right),$$

and denote the corresponding lattice cohomologies by $\overline{\mathbb{H}}^*(\Gamma, k)$. Then

Lemma 4.6.29 $\overline{\mathbb{H}}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k)[\mathfrak{d}_k]$ *is independent of the choice of k from* $[k]$.

Remark 4.6.30 In the spirit of 4.4.33, and with the notation $k = K+2l'_h$, $\overline{\mathbb{H}}^*(\Gamma, k)$ is the lattice cohomology of the cubes of l'_h+L, where the weight function is generated by the restriction of χ on this shifted lattice $l'_h + L$. (Indeed, for $l \in L$, $\chi(l + l'_h) = \chi_k(l) + \mathfrak{d}_k$.)

In particular, Theorem 4.4.39 combined with (4.78) give

Theorem 4.6.31 ([73])

$$eu(\mathbb{H}^*(\Gamma, k)) = \mathfrak{sw}_{\sigma[k]}(M(\Gamma)) - \frac{k^2 + |\mathcal{V}|}{8}.$$

4.6.32 The SWIC Revisited For any $h \in H$ assume that the representative l'_h is either r_h or s_h. Then via the extension 4.4.47 of the SWIC combined with 4.6.31 from above, the SWIC(h) is equivalent to

$$(\text{SWIC}(h)) \qquad h^1(\widetilde{X}, \mathcal{O}(-l'_h)) = eu(\mathbb{H}^*(\Gamma, K + 2l'_h)). \tag{4.79}$$

We wish to emphasize that to some extent this conjectured identity lead to the definition of graded roots and lattice cohomology (at least, of $\mathbb{H}^0$), see e.g. [70]. Indeed, for several singularities with AR graphs (e.g. for the weighted homogeneous germs) the left hand side was computed by a concatenated Laufer computations sequence, and its χ-fluctuation was reformulated as the key topological object at the right hand side too (cf. 4.5.4 and 4.7.3).

4.7 Graded Roots and Their Cohomologies

We introduce abstract graded roots (R, χ) and we define their cohomology $\mathbb{Z}[U]$-module $\mathbb{H}(R, \chi)$. We provide several constructions, which provide graded roots. One of them (cf. 4.7.2) associates a graded root $(R, \chi)_{\Gamma,k}$ with a plumbing graph Γ and a characteristic element k. It turns out that $\mathbb{H}^0(\Gamma, k) = \mathbb{H}((R, \chi)_{\Gamma,k})$. In particular, for any (Γ, k), the associated graded root is a geometrical/topological enhancement of $\mathbb{H}^0(\Gamma, k)$.

4.7.1 The Definition of Graded Roots and Their Cohomologies

In this subsection we follow [70, 71].

Definition 4.7.1 Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with end-vertices u and v. We say that R is a *graded root* with grading $\chi : \mathcal{V} \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$;
(b) $\chi(u) > \min\{\chi(v), \chi(w)\}$ for any $[u, v]$, $[u, w] \in \mathcal{E}$, $v \neq w$;
(c) χ is bounded below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$, and $|\chi^{-1}(k)| = 1$ if $k \gg 0$.

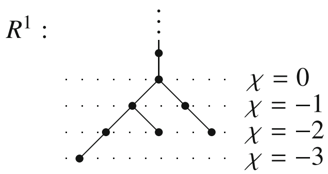
An isomorphism of graded roots is a graph isomorphism, which preserves the gradings.

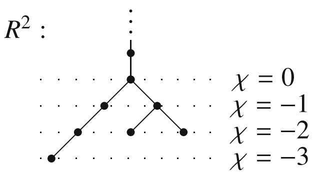
If (R, χ) is a graded root, and $r \in \mathbb{Z}$, then $(R, \chi)[r]$ denotes the same R with the new grading $\chi[r](v) := \chi(v) + r$.

Example 4.7.2

(1) For any integer $n \in \mathbb{Z}$, let $R_{(n)}$ be the tree with $\mathcal{V} = \{v^k\}_{k \geq n}$ and $\mathcal{E} = \{[v^k, v^{k+1}]\}_{k \geq n}$. The grading is $\chi(v^k) = k$.
(2) Let I be a finite index set. For each $i \in I$ fix an integer $n_i \in \mathbb{Z}$; and for each pair $i, j \in I$ fix $n_{ij} = n_{ji} \in \mathbb{Z}$ with the next properties: $n_{ii} = n_i, n_{ij} \geq \max\{n_i, n_j\}$, and $n_{jk} \leq \max\{n_{ij}, n_{ik}\}$ for any $i, j, k \in I$.
For any $i \in I$ consider $R_i := R_{(n_i)}$ with vertices $\{v_i^k\}$ and edges $\{[v_i^k, v_i^{k+1}]\}$, $(k \geq n_i)$. In the disjoint union $\sqcup_i R_i$, for any pair (i, j), identify v_i^k and v_j^k, resp. $[v_i^k, v_i^{k+1}]$ and $[v_j^k, v_j^{k+1}]$, whenever $k \geq n_{ij}$. Write $\bar{v}_i^k$ for the class of v_i^k. Then $\sqcup_i R_i/\sim$ is a graded root with $\chi(\bar{v}_i^k) = k$. It will be denoted by $R = R(\{n_i\}, \{n_{ij}\})$.
(3) Any map $\tau : \{0, 1, \ldots, T_0\} \to \mathbb{Z}$ produces a starting data for construction (2). Indeed, set $I = \{0, \ldots, T_0\}$, $n_i := \tau(i)$ $(i \in I)$, and $n_{ij} := \max\{n_k : i \leq k \leq j\}$ for $i \leq j$. Then $\sqcup_i R_i/\sim$ constructed in (2) using this data will be denoted by (R_τ, χ_τ).

For example, for $T_0 = 4$, take for the values of τ: $-3, -1, -2, 0$ and -2 (respectively $-3, 0, -2, -1$ and -2). Then the two graded roots are:





This construction can be extended to the case of a map $\tau : \mathbb{N} \to \mathbb{Z}$, whenever τ has the property that there exists some $k_0 \geq 0$ such that $\tau(k+1) \geq \tau(k)$ for any $k \geq k_0$. In this case one can take any $T_0 \geq k_0$ and construct the root associated with the restriction of τ to $\{0, \ldots, T_0\}$. It is independent of the choice of T_0. By definition, this is the root associated with τ.

Definition 4.7.3 (The (cohomology) $\mathbb{Z}[U]$-Modules Associated with a Graded Root) For any graded root (R, χ), let $\mathbb{H}(R, \chi)$ (briefly $\mathbb{H}(R)$) be the set of functions $\phi : \mathcal{V} \to \mathcal{T}_0^+$ with the following property: whenever $[v, w] \in \mathcal{E}$ with $\chi(v) < \chi(w)$, then $U \cdot \phi(v) = \phi(w)$. Clearly $\mathbb{H}(R)$ is a $\mathbb{Z}[U]$-module via $(U\phi)(v) = U \cdot \phi(v)$. Moreover, $\mathbb{H}(R)$ has a $\mathbb{Z}$-grading: the element $\phi \in \mathbb{H}(R)$ is homogeneous of degree $d \in \mathbb{Z}$ if for each $v \in \mathcal{V}$ with $\phi(v) \neq 0$, $\phi(v) \in \mathcal{T}_0^+$ is homogeneous of degree $d - 2\chi(v)$. Since $2\chi(v) + \deg \phi(v) = 2\chi(w) + \deg \phi(w)$, d is well-defined.

Note also that any ϕ as above is automatically finitely supported.

Remark 4.7.4 By the definitions $\mathbb{H}((R, \chi)[r]) = \mathbb{H}(R, \chi)[2r]$ for any $r \in \mathbb{Z}$.

Example 4.7.5

(a) $\mathbb{H}(R_n) = \mathcal{T}_{2n}^+$.
(b) The graded roots R^1 and R^2 constructed in 4.7.2(3) are not isomorphic but their $\mathbb{Z}[U]$-modules are isomorphic. Hence, in general, a graded root carries more information than its $\mathbb{Z}[U]$-module.

One has a natural graded $\mathbb{Z}[U]$ module isomorphism $\mathbb{H}(R, \chi) = \mathcal{T}_{2m}^+ \oplus \mathbb{H}_{red}(R, \chi)$, such that the $\mathbb{Z}[U]$-module $\mathbb{H}_{red}(R)$ has finite $\mathbb{Z}$-rank.

Proposition 4.7.6 *Let (R_τ, χ_τ) be a graded root associated with some function $\tau : \mathbb{N} \to \mathbb{Z}$, cf. 4.7.2(3). Then*

$$\operatorname{rank}_{\mathbb{Z}} \mathbb{H}_{red}(R_\tau, \chi_\tau) = -\tau(0) + \min_{i \geq 0} \tau(i) + \sum_{i \geq 0} \max\{\tau(i) - \tau(i+1), 0\}.$$

The summand $\mathcal{T}_{2m}^+$ of $\mathbb{H}(R_\tau, \chi_\tau)$ has index $m = \min_{i \geq 0} \tau(i) = \min_v \chi_\tau(v)$.

4.7.2 The Graded Root Associated with a Plumbing Graph

4.7.7 The Graded Root Associated with a System of Weigh Functions Fix a free $\mathbb{Z}$-module and a system of weights $\{w_q\}_q$. Consider the sequence of topological spaces (finite cubical complexes) $\{S_n\}_{n \geq m_w}$ with $S_n \subset S_{n+1}$. Let $\pi_0(S_n) = \{C_n^1, \ldots, C_n^{p_n}\}$ be the set of connected components of S_n.

Then we define the graded graph (R_w, χ_w) as follows. The vertex set $\mathcal{V}(R_w)$ is $\sqcup_{n \in \mathbb{Z}} \pi_0(S_n)$. The grading $\chi_w : \mathcal{V}(R_w) \to \mathbb{Z}$ is $\chi_w(C_n^j) = n$, that is, $\chi_w|_{\pi_0(S_n)} = n$.

Furthermore, if $C_n^i \subset C_{n+1}^j$ for some n, i and j, then we introduce an edge $[C_n^i, C_{n+1}^j]$. All the edges of R_w are obtained in this way.

Lemma 4.7.8 *(R_w, χ_w) satisfies all the required properties of the definition of a graded root, except maybe the last one: $|\chi_w^{-1}(n)| = 1$ whenever $n \gg 0$.*

4.7.9 The Graded Roots Associated with a Plumbing Graph Fix a graph and $k \in \mathrm{Char}$, their compatible weight functions and the graded cubes as in 4.6.12. The graded graph associated with this system of weight functions (cf. 4.7.7 and 4.7.8) is denoted by (R_k, χ_k).

For the system of weight functions induced by χ_k the sequence of spaces $\{S_n\}_n$ have a finiteness property: only finitely many S_n are not contractible, cf. 4.5.9.

Corollary 4.7.10

(a) (R_k, χ_k) *is a graded root.*
(b) $\mathbb{H}(R_k, \chi_k)$ *is a finitely generated* $\mathbb{Z}[U]$*-module, and* $\mathbb{H}_{red}(R_k, \chi_k)$ *is a finitely generated* $\mathbb{Z}$*-module.*

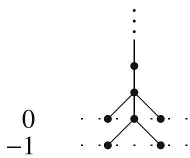
Remark 4.7.11 There are several natural **symmetries** in the picture.

(a) **The** Spin^c**-involution.** The involution $l' \mapsto -l'$ ($l' \in L'$) induces the identity $\chi_{-k}(-l) = \chi_k(l)$, hence an isomorphism of the graded roots $(R_k, \chi_k) \simeq (R_{-k}, \chi_{-k})$. ($[k] \mapsto [-k]$ is the natural involution of $\mathrm{Spin}^c(M(\Gamma))$, cf. 4.2.93.)
(b) **The Gorenstein symmetry.** If Γ is numerically Gorenstein then χ_K is stable with respect to the transformation $L \to L$, $x \mapsto Z_K - x$. This shows that (R_K, χ_K) has a $\mathbb{Z}_2$-symmetry.

More generally, if $k \in L$ (that is, k is spin) then $x \mapsto -k - x$ induces a $\mathbb{Z}_2$-symmetry of (R_k, χ_k).

Theorem 4.7.12 *Let* (R_k, χ_k) *be the graded root associated with* Γ *and* k*. Then* $\mathbb{H}(R_k, \chi_k) = \mathbb{H}^0(\Gamma, k)$.

Example 4.7.13 Consider the example from 4.5.3. Those computations show that the graded root (R_K, χ_K) is



Then $\mathbb{H}^0\Gamma, K) = \mathcal{T}^+_{-2} \oplus \mathcal{T}_{-2}(1) \oplus \mathcal{T}_0(1) \oplus \mathcal{T}_0(1)$, $\mathbb{H}^1(\Gamma, K) = \mathcal{T}_0(1)$ and $\mathbb{H}^q(\Gamma, K) = 0$ for $q \geq 2$.

4.7.14 Next, with the notations from 4.6.16, we have the analogues of 4.6.17, 4.77, 4.6.19:

Proposition 4.7.15

(a) *If* $k' = k + 2l$ *for some* $l \in L$*, then:* $(R_{k'}, \chi_{k'}) = (R_k, \chi_k)[-2\chi_k(l)]$.
(b) $(R_k, \chi_k) = (R_{[k]}, \chi_{[k]})[2m_k]$
(c) *The set* $(R_{[k]}, \chi_{[k]})$*, indexed by* $[k] \in \mathrm{Spin}^c(M(\Gamma))$*, depends only on* $M = M(\Gamma)$ *and is independent of the choice of the plumbing graph* Γ *which provides* M.

Example 4.7.16 (Rational Graphs) The following facts are equivalent:

(a) Γ is rational;
(b) $R_K = R_{(0)}$;

(c) $R_K = R_{(m)}$ for some $m \in \mathbb{Z}$;
(d) For all characteristic elements $k \in \text{Char}$, $R_k = R_{(m_k)}$ for some $m_k \in \mathbb{Z}$;

Recall from 4.6.21 that $\min \chi_{k_r} = 0$ for rational Γ. In particular, if Γ

Example 4.7.17 (Elliptic Graphs) Γ is elliptic; if and only if $(R_K, \chi_K) = R(\{n_i\}, \{n_{ij}\})$ for some index set I, $|I| = \ell + 1 \geq 2$, such that $n_i = 0$ for any $i \in I$ and $n_{ij} = 1$ for any pair $i \neq j$.

4.7.18 The following tasks appear very naturally.

Problem *Determine all the possible canonical* (R_K, χ_K) *(and non-canonical* (R_k, χ_k) *) graded roots.*

The possible resolution graphs are characterized by Grauert Theorem, namely they are connected and negative definite. For each negative definite graph (tree) we construct a canonical graded root in a direct combinatorial way. The problem is to find a combinatorial characterization of all of them.

Problem *Determine all the possible graded* $\mathbb{Z}[U]$*-modules, which might appear as* $\mathbb{H}^*(\Gamma, k)$ *for some* (Γ, k).

4.7.3 Graded Roots of Almost Rational Graphs

4.7.19 Assume that Γ is an AR graph, with SR-set $\{v_0\}$. We fix a distinguished characteristic element $k_r = K + 2s_h$ and we consider the universal cycles $\{x(\ell)\}_{\ell \geq 0}$ associated with (Γ, k_r), and their τ-function $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ defined as $\tau(\ell) := \chi_{k_r}(x(\ell))$, cf. 4.5.4. Associated with this τ-function we consider its graded root (R_τ, χ_τ) as well, cf. 4.7.2(3).

Theorem 4.7.20 *Assume that* Γ *is AR, and set* $k_r = K + 2s_h$ *for some* $h \in H$. *Then*

(a) $\mathbb{H}^q(\Gamma, k_r) = 0$ *for* $q \geq 1$*;*
(b) $\mathbb{H}^0(\Gamma, k_r) = \mathbb{H}(R_{k_r}, \chi_{k_r})$*;*
(c) $(R_{k_r}, \chi_{k_r}) = (R_\tau, \chi_\tau)$*;*
(d) $x(0) = 0$, $\tau(0) = 0$, $\tau(1) = 1 - (s_h, E_{v_0}) \geq 1$, $m_{k_r} = \min_{\ell \geq 0}\{\tau(\ell)\}$ *and*

$$eu(\mathbb{H}^*(\Gamma, k_r)) = -\min_{\ell}\{\tau(\ell)\} + \text{rank}_{\mathbb{Z}}(\mathbb{H}^0_{red}(\Gamma, k_r)) = \sum_{\ell \geq 0} \max\{\,\tau(\ell) - \tau(\ell+1), 0\,\}.$$

(e) $\tau(\ell) - \tau(\ell+1) = -1 + (x(\ell) + s_h, E_{v_0})$.

Remark 4.7.21

(a) The above theorem shows that for almost rational graphs, any graded tree (R_k, χ_k) is completely determined by the values of χ_k along a very natural (universal) infinite computation sequence (depending on k), which contains the elements $\{x(\ell)\}_{\ell\geq 0}$. (For the construction of the sequence see 4.5.4.) In particular, all the important vertices of R_k can be represented by some special cycles in L, which can be arranged in an increasing linear order (with respect to $\leq$).

(b) The set $\{x(\ell)\}_\ell$ usually is not very economical: only some of the $x(\ell)$'s carry substantial information, which will survive in (R_τ,χ_τ). The others are intermediate steps in some monotone paths. E.g., for rational singularities, $\chi(x(\ell+1))\geq\chi(x(\ell))$, hence only the information $\chi(x(0))=0$ is preserved in R_τ.

Example 4.7.22 (Star-Shaped Graphs) Assume that Γ is star-shaped with ν strings. In the sequel we will use the notations from 4.2.3. We also fix $l'_h=a_0E_0^*+\sum_{j=1}^{\nu}\sum_{t=1}^{s_j}a_{jt}E^*_{jt}$. The coefficients of l'_h also determine the integers $\tilde a_{jk}:=\sum_{t\geq k}n^j_{t+1,s_j}a_{jt}$ for $1\leq k\leq s_j$. We also write $a_j=\tilde a_{j1}$.

Γ is AR, where its SR-set consists of the central vertex, cf. 4.5.15(f). Hence, for any $\bar l=\ell E_0$ (and for the fixed l'_h and $k:=K+2l'_h$) we have a cycle $x(\bar l)$, which will be denoted simply by $x(\ell)$ ($\ell\in\mathbb{Z}$). The next expression describes the cycles $x(\ell)$ in terms of the Seifert invariants and the coefficients of l'_h.

Define the integers $\{v_{jk}\}$ $(1\leq j\leq\nu,\ 1\leq k\leq s_j)$ inductively by

$$v_{j1}:=\left\lceil\frac{\ell\omega_j-a_j}{\alpha_j}\right\rceil=\left\lceil\frac{\ell n^j_{2s_j}-\tilde a_{j1}}{n^j_{1s_j}}\right\rceil;\quad v_{jk}:=\left\lceil\frac{v_{j,k-1}n^j_{k+1,s_j}-\tilde a_{jk}}{n^j_{ks_j}}\right\rceil\ (1<k\leq s_j).$$

Then $x(\ell)=\ell E_0+\sum_{j,k}v_{jk}E_{jk}$.

Assume next that $g=0$ and $l'_h=s_h$, and set $\tau(\ell):=\chi_{k_r}(x(\ell))$ ($\ell\geq 0$). If $\ell=0$ then $x(0)=0$, hence $\tau(0)=0$ too. For $\ell\geq 0$ from 4.5.22 one gets

$$\tau(\ell+1)-\tau(\ell)=1-(x(\ell)+s_h,E_0)=1+a_0+\ell b_0-\sum_j\left\lceil\frac{\ell\omega_j-a_j}{\alpha_j}\right\rceil. \tag{4.80}$$

In particular,

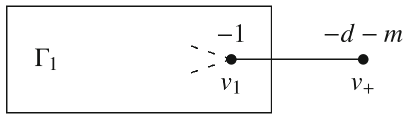
$$\tau(\ell)=\sum_{k=0}^{\ell-1}\Big(1+a_0+kb_0-\sum_j\left\lceil\frac{k\omega_j-a_j}{\alpha_j}\right\rceil\Big). \tag{4.81}$$

This can be compared with several similar expressions based on different other studies of weighted homogeneous germs or Seifert 3–manifolds.

4.7.4 Example: The Surgery Manifold $S^3_{-d}(K)$ [69, 71]

4.7.23 The Surgery Manifold $M(\Gamma)=S^3_{-d}(K)$ Fix $d\in\mathbb{Z}_{>0}$ and an *irreducible* plane curve singularity (C,o) with local algebraic knot $(K_1\subset S^3)$. Several invariants of (C, o) are listed in 4.2.30. For the shape and structure of the surgery 3-manifold $S^3_{-d}(K_1)$ see 4.2.32. If it appears as the link of a superisolated surface singularity associated with an irreducible rational unicuspidal curve (C, o) (cf. 4.2.31) then necessarily $(d-1)(d-2) = \mu(C, o)$. However, in the discussion below we will not assume this additional restriction (in particular, d can be any $d \in \mathbb{Z}_{>0}$). We use the following schematic diagram for Γ:

Γ :



The basis elements in $L = L(\Gamma)$ corresponding to v_1 and v_+ are denoted by E_1 and E_+. The lattice associated with Γ_1 is L_1, its dual is L'_1. The elements $\{E_v\}_{v \neq v_+}$ of L are identified with the basis elements of L_1.

Recall that Γ is an AR graph with $\overline{\mathcal{V}} = \{v_1\}$, cf. 4.5.15(f). In the sequel we follow [69, 71, 84].

Assume that (C, o) is determined by the function f; denote by Z that part of its divisor which is supported on compact curves. Set m for the E_1-multiplicity of Z. Then, $Z = E^*_1(\Gamma_1)$, hence $-(Z, Z)_{L_1} = m$. This combined with a determinant computation gives $\det(\Gamma) = d$. Since $\det(\Gamma_1) = 1$ the coefficient of E_+ in E^*_+ is $1/d$. Hence $[E^*_+]$ has order d in H, and $H = \mathbb{Z}_d$. We abridge $s_{a[E^*_+]}$ by s_a for $a = 0, 1, \ldots, d-1$.

Lemma 4.7.24 *$s_a = aE^*_+$ for any $a = 0, 1, \ldots, d-1$.*

4.7.25 Our goal is to determine $\{x_{k_r}(\ell)\}_{\ell \geq 0}$ for Γ and for any spin^c structure. If $k_r = K + 2aE^*_+$ for a certain a then we abridge $x_{k_r}(\ell)$ as $x_a(\ell)$, where $0 \leq a < d$.

Let us write $x_a(\ell)$ as $y_a(\ell) + n_a E_+$, where $n_a \in \mathbb{Z}_{\geq 0}$ and $y_a(\ell) \in L_1$. The inequality $(x_a(\ell) + aE^*_+, E_+) \leq 0$ reads as $n_a(m+d) \geq \ell - a$. Hence $n_a = \lceil (l-a)/(m+d) \rceil$.

On the other hand, for all other vertices $v \in \mathcal{V} \setminus \{v_+, v_1\}$ we have $(x_a(\ell) + aE^*_+, E_v) = (y_a(\ell), E_v)$, hence $y_a(\ell)$ is independent of a; let us denote it by $y(\ell)$. It satisfies the universal property *(a)-(b)-(c)* from 4.5.18 for the graph Γ_1, vertex v_1 and $l'_h = 0$. Namely, $y(\ell)$ is minimal with *(a)* $m_{v_1}(y(\ell)) = \ell$ and *(b)* $(y(\ell), E_v) \leq 0$ for any $v \neq v_1$. For example, $y(0) = 0$.

Proposition 4.7.26 *Let $Z = \mathrm{div}_{E(\Gamma_1)}(f) = E^*_1(\Gamma_1)$ be the cycle as above. Then*

(a) if $\ell = tm + \ell_0$ with $t \geq 0$ and $0 \leq \ell_0 < m$, then $y(\ell) = tZ + y(\ell_0)$;
(b) for any $\ell < m$ one has

$$(y(\ell), E_1) = \begin{cases} 1 & \textit{if} \quad \ell \notin \mathcal{S}_{C,o}; \\ 0 & \textit{if} \quad \ell \in \mathcal{S}_{C,o}. \end{cases}$$

Corollary 4.7.27 *Fix* $0 \leq a < d$ *and write* $\ell = tm + \ell_0$ *for some* $t \geq 0$ *and* $0 \leq \ell_0 < m$. *Then*

$$x_a(\ell) = t \cdot Z + y(\ell_0) + \left\lceil \frac{\ell - a}{m + d} \right\rceil E_+.$$

In particular,

$$(x_a(\ell), E_1) = -t + \left\lceil \frac{\ell - a}{m + d} \right\rceil + (y(\ell_0), E_1).$$

Furthermore, $\chi_{k_r}(x_a(0)) = 0$ *and for any* $\ell \geq 0$ *one has*

$$\chi_{k_r}(\,x_a(\ell+1)\,) - \chi_{k_r}(\,x_a(\ell)\,) = t + 1 - \left\lceil \frac{\ell - a}{m + d} \right\rceil - \begin{cases} 1 & \text{if } \ \ell_0 \notin \mathcal{S}_{C,o} \\ 0 & \text{if } \ \ell_0 \in \mathcal{S}_{C,o}. \end{cases} \tag{4.82}$$

4.7.28 The τ-Function τ_a According to 4.5.4 we set $\tau_a(\ell) := \chi_{k_r}(x_a(\ell))$. Then in (4.82) one has

$$\frac{\ell - a}{m + d} \leq t + 1,$$

hence $\tau_a(\ell + 1) - \tau_a(\ell) \geq -1$ for any ℓ, and $= -1$ only if

$$\frac{tm + \ell_0 - a}{m + d} > t \ \text{ and } \ \ \ell_0 \notin \mathcal{S}_{C,o}. \tag{4.83}$$

In order to analyze the cases when this holds, we will consider sequences $Seq(t) := \{tm + \ell_0 : 0 \leq \ell_0 < m\}$ for fixed $t \geq 0$. In such a sequence, notice that the very last element of $\mathbb{N} \setminus \mathcal{S}_{C,o}$, namely $\mu - 1 = 2\delta - 1$, is strictly smaller than $m - 1$, hence the complete set $\mathbb{N} \setminus \mathcal{S}_{C,o}$ sits in $\{0, \ldots, m - 1\}$. Therefore, in $Seq(t)$ there exists an ℓ_0 satisfying (4.83) if and only if

$$\frac{tm + 2\delta - 1 - a}{m + d} > t.$$

This is equivalent to $t \leq t_a$, for $t_a := \lfloor (2\delta - 2 - a)/d \rfloor$. In other words, if $\ell \geq T_0 := (t_a + 1)m$, then $\tau_a(\ell + 1) \geq \tau_a(\ell)$, hence those values of τ_a provide no contribution in the graded root. Moreover, for $t \in \{0, \ldots, t_a\}$, in $Seq(t)$ one has:

$$\Delta(\ell_0) := \tau_a(tm + \ell_0 + 1) - \tau_a(tm + \ell_0) = \begin{cases} 0 & \text{if } \ell_0 \leq td + a, \ \text{ and } \ell_0 \notin \mathcal{S}_{C,o}; \\ +1 & \text{if } \ell_0 \leq td + a, \ \text{ and } \ell_0 \in \mathcal{S}_{C,o}; \\ -1 & \text{if } \ell_0 > td + a, \ \text{ and } \ell_0 \notin \mathcal{S}_{C,o}; \\ 0 & \text{if } \ell_0 > td + a, \ \text{ and } \ell_0 \in \mathcal{S}_{C,o}. \end{cases}$$

In particular, $\Delta(\ell_0) \geq 0$ for any ℓ_0 with $0 \leq \ell_0 \leq td + a$, and $\Delta(\ell_0) \geq 0$ takes the value $+1$ exactly

$$A_t := \#\{s \in \mathcal{S}_{C,o} : s \leq td + a\}$$

times, otherwise it is zero. Furthermore, $\Delta(\ell_0) \leq 0$ for any $\ell_0 > td + a$ and it takes value -1 exactly

$$B_t := \#\{s \notin \mathcal{S}_{C,o} : s > td + a\}$$

times, otherwise it is zero. Recall that in 4.2.30 we rewrote $\Delta(t)$ as $1 + \delta(t-1) + (t-1)^2 Q(t)$, where $Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i$. The above B_t compared with (4.7) reads as $B_t = \alpha_{td+a}$.

Notice that both A_t and B_t are strictly positive (since $0 \in \mathcal{S}_{C,o}$, respectively $2\delta - 1 \notin \mathcal{S}_{C,o}$ and $2\delta - 1 > td + a$). This shows that

$$M_t := \max_{0 \leq \ell_0 < m} \tau_a(tm + \ell_0) = \tau_a(tm) + A_t = \tau_a((t+1)m) + B_t \tag{4.84}$$

and

$$M_t > \max\{\, \tau_a(tm), \tau_a(tm+m)\, \}.$$

Therefore, the graded root associated with the values $\{\tau_a(\ell)\}_{0 \leq \ell \leq (t_a+1)m}$ is the same as the graded root associated with the values

$$\tau_a(0), M_0, \tau_a(m), M_1, \tau_a(2m), M_2, \ldots, \tau_a(t_a m), M_{t_a}, \tau_a(t_a m + m).$$

Finally, since $\#\{s \notin \mathcal{S}_{C,o}\} = \delta$, one has $\delta - B_t = \#\{s \notin \mathcal{S}_{C,o} : s \leq td + a\}$, hence $\delta - B_t + A_t = td + a + 1$. Thus, by (4.84),

$$\tau_a((t+1)m) - \tau_a(tm) = td + a + 1 - \delta.$$

Since $\tau_a(0) = 0$, this gives $\tau_a(tm)$ inductively.

Clearly, the graded root associated with τ_a is the same as the graded root associated with $\tilde{\tau}_a : \{0, 1, 2, \ldots, 2t_a + 2\} \to \mathbb{Z}$, where $\tilde{\tau}_a(2t) := \tau_a(tm)$ and $\tilde{\tau}_a(2t+1) := M_t$.

The above discussion gives the following statement.

Theorem 4.7.29 *For each fixed $a = 0, 1, \dots, d-1$,—corresponding to the d different spin^c-structures of M—one defines the following objects :*

- $t_a := \left\lfloor \frac{2\delta-2-a}{d} \right\rfloor$, *($t_a \geq -1$ automatically) ;*
- *a function $\tau_a : \{0, 1, \dots, 2t_a + 2\} \to \mathbb{Z}$ by*

$$\begin{cases} \tau_a(2t) = d \cdot \frac{t(t-1)}{2} - t(\delta - 1 - a), \quad (t = 0, \dots, t_a + 1); \\ \\ \tau_a(2t+1) = \tau_a(2t+2) + \alpha_{td+a}, \quad (t = 0, \dots, t_a). \end{cases}$$

- *and the graded root $(R_{\tau_a}, \chi_{\tau_a})$ associated with τ_a.*

Then $(R_{\tau_a}, \chi_{\tau_a})$ is the graded root of M associated with (Γ, k_r).
Note also that $\min \tau_a = \tau_a(\,2\lceil t_a/2\rceil\,)$.

Remark 4.7.30

(a) Since for any $t \in \{0, \dots, t_a\}$, $\tau_a(2t+1) > \max\{\tau_a(2t), \tau_a(2t+2)\}$, the above representation of the graded root is the most 'economical': all the values are essential. This also shows that $(R_{\tau_a}, \chi_{\tau_a})$ has exactly $t_a + 2$ local minimum points, and they correspond to the values $\tau_a(2t)$, $t = 0, 1, \dots, t_a + 1$.
(b) The values $\tau_a(2t)$, $t = 0, 1, \dots, t_a + 1$ depend only on t, d and δ, that is, for these values no other information is needed from the semigroup $\mathcal{S}_{C,o}$.

Corollary 4.7.31

(a) $eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{t=0}^{t_a} \alpha_{td+a}$
(b) $\mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) = \sum_{t=0}^{t_a} \alpha_{td+a} + \frac{1}{8}(1 - \frac{(d+2\delta-2-2a)^2}{d})$.

Proof Use 4.7.6 for (a) and 4.6.31 and the identity $k_r^2 + |\mathcal{V}| = 1 - (d + 2\delta - 2 - 2a)^2/d$ for (b). □

Example 4.7.32 Assume $d = 1$. In this case M is an integral homology sphere; $a = 0$ and $t_0 = 2\delta - 2 = \mu - 2$. Moreover, $-(K^2 + |\mathcal{V}|)/4 = \delta(\delta - 1)$ and $\tau_0(2t) = t(t - 2\delta + 1)/2$. The reader is invited to draw the graded root and verify that

$$\mathbb{H}^0(\Gamma, K) = \big(\mathcal{T}_0^+ \oplus \mathcal{T}_0(\alpha_{\delta-1}) \oplus \bigoplus_{i=1}^{\delta-1} \mathcal{T}_{i(i+1)}(\,\alpha_{i-1+\delta}\,)^{\oplus 2}\big)[-\delta(\delta-1)].$$

4.7.5 Superisolated Singularities with One Cusp

4.7.33 In the sequel we will consider a superisolated singularity as in 4.2.31. For different invariants see 4.2.4, whose notations we will adopt. We will assume that C is a rational unicuspidal curve. We invite the reader to review the 'Semigroup

Distribution Inequality' from 4.2.33 and the 'Semigroup Distribution Property' from 4.2.33. The reinterpretations in terms of reduced Poincaré series can be found in 4.3.6, and the connection with the Seiberg–Witten Invariant Conjecture (as the basic motivation and source of the Semigroup Distribution Property) is presented in 4.4.11. Here we present further connections with the graded roots. We follow [25].

4.7.34 In this part we will compare the invariants of the link $M = S^3_{-d}(K)$ of the superisolated singularity with the corresponding invariants of the Seifert 3-manifold $\Sigma(d, d, d+1)$, the link of the hypersurface Brieskorn singularity $x^d+y^d+z^{d+1} = 0$. Before we state the next theorem, we recall that the plumbing graph of $S^3_{-d}(K)$ contains complete information about the embedded link $K \subset S^3$. Moreover, by the statements of 4.7.29, the graded root or lattice cohomology still preserves essential data about the Alexander polynomial. However, the Seifert 3-manifold $\Sigma(d, d, d+1)$ has information only about the degree μ of Δ via $(d-1)(d-2) = \mu$. The point is that the algebraic realizability of C (that is, the existence of an analytic superisolated singularity with link $S^3_{-d}(K)$) imposes the following very surprising necessary topological obstructions.

Theorem 4.7.35 ([25]) *The following facts are equivalent:*

(a) The Seiberg–Witten Invariant Conjecture is true for the superisolated germ.
(b) The Semigroup Distribution Property is true.
(c) The canonical graded roots of $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ are the same.
(d) The canonical lattice homologies of $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ are the same.
(e)

$$\Big(\mathfrak{sw}_{\sigma[K]}(M)-\frac{K^2+\#\mathcal{V}}{8}\Big)\Big|_{M=S^3_{-d}(K)} = \Big(\mathfrak{sw}_{\sigma[K]}(M)-\frac{K^2+\#\mathcal{V}}{8}\Big)\Big|_{M=\Sigma(d,d,d+1)}.$$

Recall that, in fact, the Semigroup Distribution Property *is true* by Borodzik and Livingston [9] (cf. 4.2.33), hence all the statements of 4.7.35 *are true* as well. However, we formulated above a weaker statement, only the *equivalence* of the above statements, whose proof is independent of the Heegaard Floer theory based proof of [9].

The proof of 4.7.35 is given in several steps. The starting point is that both 3-manifolds $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ are almost rational. In particular, in both cases, the canonical graded root can be determined via the τ-function, cf. 4.7.3. In the first case this is done explicitly in 4.7.29, while for the second case see 4.7.22.

Fact 1 Let us rewrite 4.7.29 for $S^3_{-d}(K)$ and for the canonical spin^c structure $a = 0$. Set $c_l := \alpha_{(d-3-l)d}$ and define $\tau : \{0, 1, \dots, 2d-4\} \to \mathbb{Z}$ by

$$\tau(2l) = \frac{l(l-1)}{2}d - l(\delta - 1), \quad \tau(2l+1) = \tau(2l+2) + c_{d-3-l}. \tag{4.85}$$

Then $(R_{can}, \chi_{can}) = (R_\tau, \chi_\tau)$.

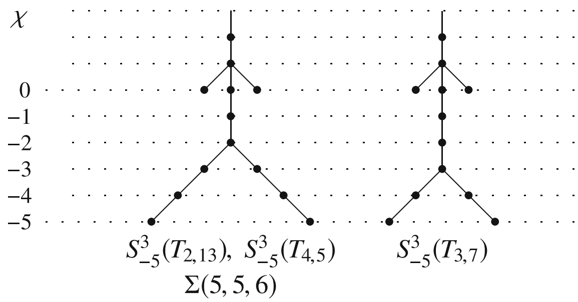
Fact 2 Consider next the Seifert manifold $\Sigma(d, d, d+1)$. Its canonical graded root is the following. For any $0 \le l \le d-3$ write $c^u_l := (l+1)(l+2)/2$, and $2\delta := (d-1)(d-2)$ and define $\tau^u : \{0, 1, \dots, 2d-4\} \to \mathbb{Z}$ by

$$\tau^u(2l) = \frac{l(l-1)}{2}d - l(\delta - 1), \quad \tau^u(2l+1) = \tau^u(2l+2) + c^u_{d-3-l}. \tag{4.86}$$

Then $(R_{can}, \chi_{can}) = (R_{\tau^u}, \chi_{\tau^u})$.

Next we compare 4.85 and 4.86: the graded roots associated with $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ coincide exactly when $c_l = c^u_l$ for every l. However, by the Semigroup Distribution Inequality (a consequence of the Bézout's Theorem, cf. 4.2.33) $c_l \ge c^u_l$ for every l. Hence $c_l = c^u_l$ for every l if and only if $\sum_l c_l = \sum_l c^u_l$. But this is exactly the vanishing of $N(1)$, cf. (4.3.20)*(b)*, hence 4.4.54 applies.

Example 4.7.36 Assume that $d = 5$ and C is unicuspidal and its singular point has only one Puiseux pair (a, b) with $a < b$. Then by the genus formula the possible values of (a, b) are (4, 5), (3, 7) and (2, 13). It turns out that the first and the third cases can be realized, while the second case not. This fact is compatible with the above Theorem 4.7.35. Indeed, the corresponding canonical graded roots (together with the root of $\Sigma(5, 5, 6)$) are shown in the next picture.



Remark 4.7.37 As we already mentioned in 4.2.33, the Semigroup Distribution Property (in the unicuspidal case) was partially verified in [24] and proved in [9]. The first approach is based on a case-by-case verification of the families of cuspidal rational projective curves which appear in the classification theorems. The second approach is based on the Heegaard Floer theory. The discussion from 4.7.39 traces a possible third approach, which would lead to a different proof, and would open a new chapter in the deformation theory of surface singularities.

Corollary 4.7.38 *The Seiberg–Witten Invariant Conjecture is true for superisolated germs associated with rational unicuspidal curves.*

4.7.39 Why $\Sigma(d, d, d+1)$? At the first glance the pairing of $S^3_{-d}(K)$ with $\Sigma(d, d, d+1)$ in Theorem 4.7.35 looks very unmotivated. In the next paragraphs we wish to convince the reader that this is not the case, and conjecturally a very deep structure might exist behind the scene.

Assume that the rational unicuspidal curve is given by $f_d(x, y, z) = 0$ in $\mathbb{P}^2$ (for notations see 4.2.31). We can fix the homogeneous coordinates in $\mathbb{P}^2$ in such a way that $z = 0$ intersects C generically. A possible choice for the superisolated singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is $f = f_d + z^{d+1}$. Write f_d as $\sum_{i=0}^{d} g_{d-i}(x, y)z^i$. Then g_d is a product of d linear factors corresponding to the points $C \cap \{z = 0\}$, hence the germ $g_d : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is equisingular with $(x, y) \mapsto x^d + y^d$.

Next, consider the following deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ of isolated hypersurface germs, given by $f_t(x, y, z) = f_d(x, y, tz) + z^{d+1} = \sum_i g_{d-i}(x, y)z^i t^i + z^{d+1}$. For $t \neq 0$ the deformation is μ-constant, the embedded topological type stays constant, and it is equivalent (up to such equivalences) to the type of f. However, for $t = 0$ it is equivalent (in similar sense) to the germ $x^d + y^d + z^{d+1}$.

Along this deformation not only does the embedded topological type jump (e.g. the Milnor number), but even the (non-embedded abstract) link as well: for $t \neq 0$ it is $S^3_{-d}(K)$, while for $t = 0$ it is $\Sigma(d, d, d + 1)$.

However, both graphs are AR and several key invariants stay stable. For example, in both cases $p_g = d(d - 1)(d - 2)/6$. On the other hand, if we compute the (resolution independent) invariant $K^2 + |\mathcal{V}|$ we realize that they are different. However, if we denote by K^2_{min} the self-intersection of K in the *minimal resolution*, then it turns out that in both cases it is $-d(d - 2)^2$. Hence we are dealing with a Gorenstein K^2_{min}-constant deformation. By a result of Laufer [52] such deformations admit a *very weak simultaneous resolution* (possible after a finite base change). This gives the possibility to compare the lattices associated with their minimal resolutions. Indeed, $S^3_{-d}(K)$ and $\Sigma(d, d, d + 1)$ admit certain non-minimal resolution graphs with lattices $L_{t\neq 0}$ and $L_{t=0}$ and a homological map $\iota : L_{t\neq 0} \to L_{t=0}$, which preserves the intersection matrices, the canonical classes, the χ-expression.

We formulate the next conjecture, whose positive answer would produce an extremely strong test for the existence of certain analytic deformations.

Conjecture 4.7.40 Along a K^2_{min}-constant deformation X_t of Gorenstein surface singularities, such that the links of $X_{t=0}$ and $X_{t\neq 0}$ are both rational homology spheres, the graded roots associated with the canonical spin^c structure of $X_{t=0}$ and of $X_{t\neq 0}$ are the same.

Note that along a deformation as in 4.7.40 we *cannot* expect the stability of the whole module $\{\mathbb{H}^q\}_q$. Indeed, for the deformation described in 4.7.39 valid for superisolated germs, for $t = 0$ we have an AR case with $\mathbb{H}^{\geq 1} = 0$. However, for $t \neq 0$, for certain superisolated germs with $\nu \geq 2$ we might have $\mathbb{H}^{\geq 1} \neq 0$. In fact, for any superisolated germ which produced a counterexample for the SWIC, along the above deformation the canonical Seiberg–Witten invariant is non-constant too.

4.8 The Reduction Theorem

4.8.1 Reduction Theorem for Lattice Cohomology

We consider a graph Γ as in 4.6.2. We also fix a distinguished class $k_r \in \mathrm{Char}$ and the corresponding lattice cohomology $\mathbb{H}^*(\Gamma, k_r)$. Recall that there is an isomorphism of graded $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\Gamma, k_r) \simeq \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$, where the second module is generated by weighted cubes in $(\mathbb{R}_{\geq 0})^s$, cf. 4.6.20. Here $s := |\mathcal{V}|$.

This $\mathbb{Z}[U]$-module was drastically simplified in the case of AR graphs, basically the cubes from $(\mathbb{R}_{\geq 0})^s$ were replaced by 0 and 1 dimensional cubes along an infinite increasing path (starting with $0 \in L$), cf. Theorem 4.7.20. Here the AR-assumption is really necessary: such a reduction to a 1-dimensional path (simplicial complex) cannot be done for any graph (e.g. when $\mathbb{H}^1 \neq 0$). In this subsection we discuss the analogue of this statement for an arbitrary graph.

Recall that the definition of an SR-set does not involve any $k \in \mathrm{Char}$, hence such a set can be uniformly used for any k_r. In this section we fix such an SR-set $\overline{\mathcal{V}} \subset \mathcal{V}$ as in 4.5.14, and any $k_r \in \mathrm{Char}$. Then, for each $\bar{l} = \sum_{v \in \overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}})$, with every $\ell_v \geq 0$, we define the universal cycle $x(\bar{l})$ associated with $\bar{l}$ and s_h (where $k_r = K + 2s_h$) as in 4.5.18. For several properties of the cycles $x(\bar{l})$ and of the values $\chi_{k_r}(x(\bar{l}))$ see 4.5.16. Let $\bar{s}$ be the cardinality of $\overline{\mathcal{V}}$. In the next paragraphs we follow [47].

4.8.1 Preparation for the Lattice Reduction Our goal is to replace the cubes of the lattice $\mathbb{R}^s$ (or from $(\mathbb{R}_{\geq 0})^s$) with cubes from $(\mathbb{R}_{\geq 0})^{\bar{s}}$. In order to run the theory we need to define the new weights. Define the function $\overline{w}_0 : (\mathbb{Z}_{\geq 0})^{\bar{s}} \to \mathbb{Z}$ by

$$\overline{w}_0(\bar{l}) := \chi_{k_r}(x(\bar{l})). \tag{4.87}$$

Then $\overline{w}_0$ defines a set $\{\overline{w}_q\}_{q=0}^{\bar{s}}$ of compatible weight functions by $\overline{w}_q(\square) = \max\{\overline{w}_0(v) \ : \ v \text{ is a vertex of } \square\}$, similarly as in 4.6.12. This system is denoted by $\overline{w}[k_r]$.

Here some comments are appropriate. We wish to emphasize that in the definition of the lattice cohomology the *lattice* (that is, the *linear*) *structure* in not used, it is not essential. The important structure consists of the weight-levels of the *lattice points* in some regions (e.g. quadrants, rectangles) and their neighboring properties. Note that in the new situation we do not use the linear structure of $\mathbb{Z}^{\bar{s}}$ either, and we do not even define the weights of the lattice points outside the first quadrant. Furthermore, $\bar{l} \mapsto \chi_{k_r}(x(\bar{l}))$ is a complicated arithmetical function (definitely not quadratic or polynomial).

Let us denote the associated lattice cohomology by $\mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r])$.

Theorem 4.8.2 (Reduction Theorem [47]) *There exists a graded $\mathbb{Z}[U]$-module isomorphism*

$$\mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r]). \tag{4.88}$$

Corollary 4.8.3 *Fix an arbitrary graph* Γ. *If it admits an SR-set of cardinality* $\bar{s}$ *then* $\mathbb{H}^q(\Gamma, k) = 0$ *for any* $q \geq \bar{s}$ *and* $k \in$ Char.

This vanishing can be proved by surgery exact sequences of lattice cohomology as well, see [74].

4.8.2 Reduction Theorem for $\mathbf{Z}(\mathbf{t})$

The Reduction Theorem has its effect on the relation of the lattice cohomology with the counting function of the coefficients of topological Poincaré series $Z(\mathbf{t})$ as well. Let us consider first the series $Z(\mathbf{t})$ written in terms of weighted cubes (cf. 4.4.33 and 4.4.40).

Theorem 4.8.4 *Fix* h, s_h *and* $k_r = K + 2s_h$ *as above. Let* $w = w[k_r]$ *be the system of weight associated with* k_r. *Then the following facts hold.*

(1)

$$Z_h(\mathbf{t}) = \sum_{l \in L} \Big(\sum_{I \subseteq \mathcal{V}} (-1)^{|I|+1} w((l, I)) \Big) \mathbf{t}^{l+s_h}.$$

(2) Fix some $l \in L$ *with* $l + s_h \in -K + \mathcal{S}'$. *Then*

$$\sum_{x \in L,\, x \not\geq l} \mathfrak{z}(x + s_h) = \chi_{k_r}(l) + eu(\mathbb{H}^*(\Gamma, k_r)).$$

4.8.5 The Reduced Series Let us return to the SR-set $\overline{\mathcal{V}}$, write $\mathcal{V}$ as $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$, and let $\pi : L' \to L(\overline{\mathcal{V}}) \otimes \mathbb{Q}$ be the projection to the $\overline{\mathcal{V}}$-coordinates. As usual, we also write $\mathbf{t}_{\overline{\mathcal{V}}} = \{t_v\}_{v \in \overline{\mathcal{V}}}$ for the variables of $L(\overline{\mathcal{V}})$, and $\mathbf{t}_{\overline{\mathcal{V}}}^{\bar{l}} = \prod_{v \in \overline{\mathcal{V}}} t_v^{\ell_v}$ for $\bar{l} = \sum_{v \in \overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}}) \otimes \mathbb{Q}$. For any $h \in H$ set $Z_{h,\overline{\mathcal{V}}}(\mathbf{t}_{\overline{\mathcal{V}}}) = Z_h(\mathbf{t})|_{t_v = 1 \text{ for all } v \in \mathcal{V}^*}$. It is supported on the projection of $\mathcal{S}' \cap (s_h + L)$. Write

$$Z_{h,\overline{\mathcal{V}}}(\mathbf{t}_{\overline{\mathcal{V}}}) = \sum_{\bar{l} \in L(\overline{\mathcal{V}})} \bar{\mathfrak{z}}_{\bar{l}+\pi(s_h)} \mathbf{t}_{\overline{\mathcal{V}}}^{\bar{l}+\pi(s_h)}.$$

Theorem 4.8.6 ([47]) *With the above notations (and* $\overline{w} = \overline{w}[k_r]$*)*

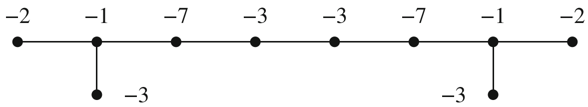
(1)

$$Z_{h,\overline{\mathcal{V}}}(\mathbf{t}_{\overline{\mathcal{V}}}) = \sum_{\bar{l} \in L(\overline{\mathcal{V}})} \Big(\sum_{\bar{I} \subseteq \overline{\mathcal{V}}} (-1)^{|\bar{I}|+1} \overline{w}((\bar{l}, \bar{I})) \Big) \mathbf{t}_{\overline{\mathcal{V}}}^{\bar{l}+\pi(s_h)}.$$

(2) There exists $\bar{l}_0 \in \pi(\mathcal{S})$ *such that for any* $\bar{l} \in \bar{l}_0 + \pi(\mathcal{S})$

$$\sum_{\bar{x} \not\geq \bar{l}} \bar{\mathfrak{z}}_{\bar{x}+\pi(s_h)} = \overline{w}(\bar{l}) + eu(\mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w})).$$

Example 4.8.7 Consider the following graph Γ



It is the minimal good resolution graph of the hypersurface singularity $x^{13} + y^{13} + x^2y^2 + z^3 = 0$. In particular, Z_K is integral.

In the sequel we will calculate the lattice cohomology of $M(\Gamma)$ associated with $k_r = K$. We choose the two nodes as an SR-set. Then Reduction Theorem 4.8.2 implies that $\mathbb{H}^*(\Gamma, K) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^2, \overline{w})$, where $\overline{w}(i, j) := \chi(x(i, j))$ for any $(i, j) \in (\mathbb{Z}_{\geq 0})^2$. It turns out that

$$\overline{w}(i+1, j) - \overline{w}(i, j) = 1 + i - \lceil (53i + j)/351 \rceil - \lceil i/2 \rceil - \lceil i/3 \rceil$$

$$\overline{w}(i, j+1) - \overline{w}(i, j) = 1 + j - \lceil (i + 53j)/351 \rceil - \lceil j/2 \rceil - \lceil j/3 \rceil.$$

Since $\pi(Z_K) = (14, 14)$, the projection of the rectangle $R(0, Z_K)$ is $\pi(R(0, Z_K)) = R((0, 0), (14, 14))$. Hence by Lemma 4.5.11*(b)* the rectangle $R((0, 0), (14, 14)) = \{(i, j) \in (\mathbb{R}_{\geq 0})^2 : (i, j) \leq (14, 14)\}$ contains all the needed information. The values $\overline{w}(i, j)$ are given in the next diagram. ((0, 0) is at the lower left corner.)

1	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	1	0	0	0	0	0	1	0	0	0	0	0	1	1

The large frames illustrate the generators of $H^0(S_{-1},\mathbb{Z})$, the small ones the generators of $H^0(S_0,\mathbb{Z})$ in degree 0 and the circle shows the generator of $H^1(S_0,\mathbb{Z})$. Hence,

$$\mathbb{H}^0(\Gamma,K)=\mathcal{T}^+_{-2}\oplus\mathcal{T}^3_{-2}(1)\oplus\mathcal{T}^2_0(1)\quad\text{and}\quad\mathbb{H}^1(\Gamma,K)=\mathcal{T}_0(1)\quad\text{and}\quad eu(\mathbb{H}^*(\Gamma,K))=5.$$

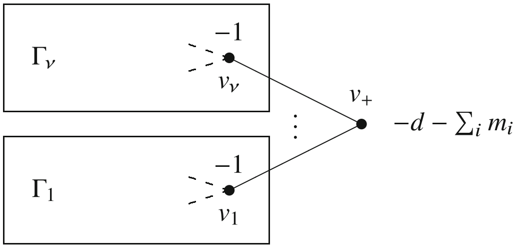
For several concrete formulae and other explicit examples when Γ has two nodes, and $\overline{\mathcal{V}}=\mathcal{N}$, see [44].

4.9 $\mathbb{H}^*$ of the Surgery Manifold $S^3_{-d}(\#_i K_i)$

4.9.1 Invariants of $M(\Gamma)=S^3_{-d}(\#_i K_i)$ for Any $d>0$ and for All Spin^c Structures [84]

4.9.1 Consider the notations of 4.2.32, or of 4.4.11 with $d>0$. Here we do not assume that $\mu=(d-1)(d-2)$ (as in the superisolated link case 4.2.4 or 4.3.6). In this subsection we follow [84]. By 4.2.32

$\Gamma=\Gamma(d)$:



The group H is $\mathbb{Z}_d$ and it is generated by the class of the dual of $E_+ := E_{v_+}$. Furthermore, as in Lemma 4.7.24 one has $s_{[aE^*_+]}=aE^*_+$ for any $a=0,1,\dots,d-1$. We will use the notations $h:=[aE^*_+]\in H$ and $k_r:=K+2aE^*_+\in$ Char. With $\mathcal{I}=\{v_+\}$ one has (cf. 4.4.11)

$$Z_{\mathcal{I}}(t)=\frac{\Delta(t^{1/d})}{(1-t^{1/d})^2}\quad\text{and}\quad Z_{h,\mathcal{I}}(t)=\frac{1}{d}\cdot\sum_{\xi^d=1}\xi^{-a}\frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2}.\tag{4.89}$$

Using $\Delta(t)=1+(t-1)\delta+(t-1)^2Q(t)$ and $Q(t)=\sum_{n=0}^{\mu-2}\alpha_n t^n$, by a computation

$$Z_{h,\mathcal{I}}(t)=\frac{t^{a/d}(a+1)+t^{1+a/d}(d-a-1)}{(t-1)^2}+\frac{\delta\cdot t^{a/d}}{t-1}+\sum_{n\equiv a\ (\mathrm{mod}\ d)}\alpha_n t^{n/d}.\tag{4.90}$$

Since the polynomial part $Z^+_{h,\mathcal{I}}(t)$ of this expression is $\sum_{n\equiv a \ (\mathrm{mod}\ d)} \alpha_n t^{n/d}$, we get

$$\mathrm{pc}(Z_{h,\mathcal{I}}(t)) = \mathrm{pc}(Z_{h,\mathcal{I}}(t^d)) = \sum_{n\equiv a \ (\mathrm{mod}\ d)} \alpha_n. \tag{4.91}$$

Next we apply the surgery formula from Theorem 4.4.31 for $v = v_+$ and $l' = aE^*_+$. Then $l'_{v_+} = a/d \in [0, 1)$. Furthermore, $R_i(aE^*_+) = 0$, hence all the contributions $\mathfrak{sw}_{\sigma[K_i]}(M(\Gamma_i)) - (K_i^2 + |\mathcal{V}_i|)/8$ vanish (since SWIC is valid for smooth germs). Therefore, from 4.4.31,

$$\mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) - \frac{k_r^2 + |\mathcal{V}|}{8} = \sum_{n\equiv a \ (\mathrm{mod}\ d)} \alpha_n. \tag{4.92}$$

This combined with Theorem 4.6.31 give

$$eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{n\equiv a \ (\mathrm{mod}\ d)} \alpha_n. \tag{4.93}$$

4.9.2 The Lattice Reduction In the next pages we follow closely [84].

The set $\overline{\mathcal{V}} := \{v_1, \dots, v_\nu\}$ of the (-1)-vertices form an SR-set, cf. 4.5.15(i). Set $E_1, \dots, E_\nu$ for the corresponding elements of L. Next we apply the Reduction Theorem from Sect. 4.8, whose notations we will adopt. Write $\bar{l} = \sum_{i=1}^{\nu} \ell_i E_i \in L(\overline{\mathcal{V}}) = \overline{L}$, and let $x_{k_r}(\bar{l})$ be the universal cycle associated with k_r and $\bar{l}$ as in 4.5.18 and Sect. 4.8. Set $\overline{w}(\bar{l}) := \chi_{k_r}(x(\bar{l}))$ as in (4.87). Then, by the Reduction Theorem 4.8.2 one has a graded $\mathbb{Z}[U]$-module isomorphism:

$$\mathbb{H}^*(\Gamma, k_r) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^\nu, \overline{w}). \tag{4.94}$$

For each $\ell_i \geq 0$ consider the cycle $y_i(\ell_i)$ determined in the graph Γ_i as in 4.7.25 and 4.7.26. Set $\Sigma m := \sum_i m_i$ and $\Sigma\ell := \sum_i \ell_i$ (and, in general, $\Sigma x := \sum_i x_i$ for $x \in \mathbb{R}^\nu$). Then the E_+-coefficient of $x_{k_r}(\bar{l})$ is $m_+(\bar{l}) = \lceil (\Sigma\ell - a)/(\Sigma m + d) \rceil$ and

$$x_{k_r}(\bar{l}) = \sum_i y_i(\ell_i) + \left\lceil \frac{\Sigma\ell - a}{\Sigma m + d} \right\rceil \cdot E_+. \tag{4.95}$$

Write $\ell_i = p_i m_i + \ell_{i,0}$ with $p_i \in \mathbb{Z}_{\geq 0}$ and $0 \leq \ell_{i,0} < m_i$. Let Z_i be the cycle $\mathrm{div}_{E(\Gamma_i)}(f_i) = E^*_i(\Gamma_i)$. Then $y_i(\ell_i) = p_i Z_i + y_i(\ell_{i,0})$ (cf. 4.7.26). Furthermore, if for any $i = 1, \dots, \nu$ we take $1_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at entry i) then $\overline{w}(0) = 0$, and

$$\overline{w}(\bar{l} + 1_i) - \overline{w}(\bar{l}) = p_i + 1 - \left\lceil \frac{\Sigma\ell - a}{\Sigma m + d} \right\rceil - \begin{cases} 1 & \text{if } \ \ell_{i,0} \notin \mathcal{S}_i \\ 0 & \text{if } \ \ell_{i,0} \in \mathcal{S}_i. \end{cases} \tag{4.96}$$

Here $\mathcal{S}_i$ is the abbreviation for the semigroup $\mathcal{S}_{C,p_i}$.

Next, we reduce $(\mathbb{R}_{\geq 0})^\nu$ to a finite multi-rectangle. We write $\mathbf{m}$ for the vector $(m_1, \dots, m_\nu)$, and $R(\bar{l}_1, \bar{l}_2)$ denotes the rectangle $\{x \in \mathbb{R}^\nu \,:\, \bar{l}_1 \leq x \leq \bar{l}_2\}$, as usual. Set also $R_p := R(p\mathbf{m}, (p+1)\mathbf{m})$.

Lemma 4.9.3

(a) Set $\tilde{p}_0 := \lceil (\mu - a - 1)/d \rceil$. *Then*

$$\mathbb{H}^*\big((\mathbb{R}_{\geq 0})^\nu, \overline{w}\big) \cong \mathbb{H}^*\big(R(0, \tilde{p}_0\,\mathbf{m}), \overline{w}\big) \cong \mathbb{H}^*\big(\cup_{0\leq p<\tilde{p}_0} R_p, \overline{w}\big).$$

(b) $\overline{w}(p\,\mathbf{m}) = p(1+a-\delta) + dp(p-1)/2$ *for any* $0 \leq p \leq \tilde{p}_0$.

(c) Fix $0 \leq p < \tilde{p}_0$. *Then, for any* $\bar{l} \in R_p \cap \overline{L}$, $\ell_i = pm_i + \ell_{i,0}$, *with* $\Sigma\ell \leq p(\Sigma m + d) + a + 1$ *one has:*

$$\overline{w}(\bar{l}) - \overline{w}(p\,\mathbf{m}) = \sum_i \#\{s \in \mathcal{S}_i \,:\, s \leq \ell_{i,0} - 1\}. \tag{4.97}$$

(d) Fix $0 \leq p < \tilde{p}_0$. *Then, for any* $\bar{l} \in R_p \cap \overline{L}$, $\ell_i = pm_i + \ell_{i,0}$, *with* $\Sigma\ell \geq p(\Sigma m + d) + a + 1$ *one has:*

$$\overline{w}(\bar{l}) - \overline{w}((p+1)\,\mathbf{m}) = \sum_i \#\{s \notin \mathcal{S}_i \,:\, s \geq \ell_{i,0}\}. \tag{4.98}$$

Consider the notation

$$T_p^- := \big\{\, x \in (\mathbb{R}_{\geq 0})^\nu \,:\, (\Sigma x - a - 1)/(\Sigma m + d) = p - 1 \big\}.$$

From the above facts we obtain the following.

Theorem 4.9.4 *Set* $\tilde{p}_0 := \lceil (\mu - a - 1)/d \rceil$ *as above and for any* $0 \leq p < \tilde{p}_0$ *consider*

$$\min T_{p+1}^- := \min\{\, \overline{w}(\bar{l}) \,:\, \bar{l} \in T_{p+1}^- \cap R_p \cap \overline{L}\}.$$

Then the following facts hold:

(a) $\overline{w}(p\,\mathbf{m}) \leq \min T_{p+1}^-$, $\overline{w}((p+1)\,\mathbf{m}) \leq \min T_{p+1}^-$.

(b) $m_{k_r} := \min \chi_{k_r} = \min_{0\leq p\leq\tilde{p}_0}\{\,\overline{w}(p\,\mathbf{m})\,\}$.

(c) Let $p_{\min}$ *be the smallest integer satisfying* $\overline{w}(p_{\min}\mathbf{m}) = m_{k_r}$. *Then*

$$\mathbb{H}^0_{red}(\Gamma, k_r) = \bigoplus_{0\leq p<p_{\min}} \mathcal{T}_{2\overline{w}(p\,\mathbf{m})}\big(\min T_{p+1}^- - \overline{w}(p\,\mathbf{m})\big)$$
$$\oplus \bigoplus_{p_{\min}\leq p<\tilde{p}_0} \mathcal{T}_{2\overline{w}((p+1)\,\mathbf{m})}\big(\min T_{p+1}^- - \overline{w}((p+1)\,\mathbf{m})\big).$$

(d) $\operatorname{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(\Gamma, k_r)$ *equals*

$$\sum_{0\le p<p_{\min}} \big(\min T^-_{p+1} - \overline{w}(p\,\mathbf{m})\big) + \sum_{p_{\min}\le p<\tilde p_0} \big(\min T^-_{p+1} - \overline{w}((p+1)\,\mathbf{m})\big),$$

or

$$-m_{k_r} + \operatorname{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(\Gamma, k_r) = \sum_{0\le p<\tilde p_0} \big(\min T^-_{p+1} - \overline{w}((p+1)\,\mathbf{m})\big).$$

(e) *For any* $q > 0$ *one has*

$$\mathbb{H}^q(\Gamma, k_r) = \bigoplus_{0\le p<\tilde p_0} \mathbb{H}^q(R_p, \overline{w}).$$

4.9.5 The Structure of $\mathbb{H}^{\ge 1}(R_p, \overline{w})$ The cohomology $\mathbb{H}^{\ge 1}(R_p, \overline{w})$ depends only on the $\overline{w}$-values at $p\,\mathbf{m}$, at $(p+1)\,\mathbf{m}$ and along T^-_{p+1}. Indeed, for any $n \in \mathbb{Z}$ consider S_n as in 4.5.2. Then for $n < \min T^-_{p+1}$ the space $S_n \cap R_p$ has the same homotopy type as the intersection of S_n with the two-element set $\{p\,\mathbf{m}, (p+1)\,\mathbf{m}\}$; while for $n \ge \min T^-_{p+1}$ it has the homotopy type of the suspension of $S_n \cap T^-_{p+1}$. In particular, all the nontrivial homogeneous elements of $\mathbb{H}^{\ge 1}(R_p, \overline{w})$ have degree $\ge \min T^-_{p+1}$, and one has the graded $\mathbb{Z}[U]$-module isomorphism

$$\mathbb{H}^q(R_p, \overline{w}) = \mathbb{H}^{q-1}_{red}(T^-_{p+1}, \overline{w}) \quad \text{for } q > 0. \tag{4.99}$$

4.9.6 The Structure of $\mathbb{H}^*(T^-_{p+1}, \overline{w})$**. The Modules** $\mathbb{H}^*(\mathbb{T}^-_n, \overline{W})$ In most of the notations above, we have omitted the symbol a codifying the characteristic element k_r. In fact, for any $p \ge 0$ and $a \in \{0, \dots, d-1\}$, T^-_{p+1} is

$$T^-_{p+1,a} := \{\bar l \,:\, \ell_i = pm_i + \ell_{i,0};\ \sum_i \ell_{i,0} = pd + a + 1\}.$$

Note that when p runs over $\mathbb{Z}_{\ge 0}$ and $a \in \{0, \dots, d-1\}$, the integer $n = pd + a$ runs over $\mathbb{Z}_{\ge 0}$. This motivates to consider for any $n \in \mathbb{Z}_{\ge 0}$

$$\mathbb{T}_n := \{(\ell_{1,0}, \dots, \ell_{\nu,0}) \in [0, m_1] \times \cdots \times [0, m_\nu] \,:\, \sum_i \ell_{i,0} = n + 1\}. \tag{4.100}$$

Then, for d and a fixed, $T^-_{p+1,a} = \mathbb{T}_{pd+a} + p\,\mathbf{m}$. If $p < \tilde p_0$ then $pd + a \le \mu - 2$, hence the relevant index set of the hyperplanes is $0 \le n \le \mu - 2$ (this can be compared with the index set $\{\alpha_n\}_{n=0}^{\mu-2}$ of the coefficients of $Q(t)$). The form $\mathbb{T}_{pd+a} + p\,\mathbf{m}$ shows also how they intersect the small rectangles: when we run a, an element of the set $\{\mathbb{T}_n + \lfloor n/d \rfloor \mathbf{m}\}_{0\le n\le \mu-2}$ intersects R_p if and only if $\lfloor n/d \rfloor = p$.

Up to the shift $\overline{w}(p\,\mathbf{m})$, which is constant on each $\mathbb{T}_n$, but otherwise depends on $p = \lfloor n/d \rfloor$, the weights on $\mathbb{T}_n \cap \mathbb{Z}^\nu$ are given by the right hand side of (4.97). Or, up to a shift $\overline{w}((p+1)\,\mathbf{m})$, the weights are given by (4.98). Following this second version we set the following weights for any $\mathbb{T}_n$:

$$\overline{W}((\ell_{1,0}, \ldots, \ell_{\nu,0})) = \sum_i \#\{s \notin \mathcal{S}_i \;:\; s \geq \ell_{i,0}\}. \tag{4.101}$$

That is, $\overline{W}|_{\mathbb{T}_n}(\bar{l} - p\,\mathbf{m}) = \overline{w}(\bar{l}) - \overline{w}((p+1)\,\mathbf{m})$, where $p = \lfloor n/d \rfloor$.

The weight function $\overline{W}$ restricted on all the level sets $\{\mathbb{T}_n\}_{n\geq 0}$ of $(\mathbb{Z}_{\geq 0})^\nu$ measures the very subtle distribution properties of the semigroups $\{\mathcal{S}_i\}_i$. Furthermore, up to a well-identified shift in degrees, the collection $(\mathbb{T}_n, \overline{W})$ provides all the lattice cohomologies $\mathbb{H}^*(\Gamma(d), k_r)$ for all the possible values d and a. Here, and in the next discussion, we denote the dependence of Γ on d by $\Gamma(d)$.

More precisely, for any d and $a \in \{0, \ldots, d-1\}$ and $q > 0$ one has:

$$\mathbb{H}^q(\Gamma(d), K + 2aE_+^*) = \bigoplus_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} \mathbb{H}^{q-1}_{red}(\mathbb{T}_n, \overline{W})[s_{n,d}], \tag{4.102}$$

where $s_{n,d}$ is the value of the shift $2\overline{w}((p+1)\,\mathbf{m}) = 2(p+1)(1+a-\delta)+d(p+1)p$ (with $p = \lfloor n/d \rfloor$). Moreover, the values $\{\min \overline{W}\,|\mathbb{T}_n\}_n$ and $s_{n,d}$ determine all the cohomology groups $\mathbb{H}^0(\Gamma(d), k_r)$ too. The second identity of (4.9.4)(d) together with (4.98) reads as:

$$-m_{k_r} + \mathrm{rank}\ \mathbb{H}^0_{red}(\Gamma(d), K + 2aE_+^*) = \sum_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} \min\{\overline{W}|_{\mathbb{T}_n}\}. \tag{4.103}$$

In particular, for any fixed $d > 0$ and $a \in \{0, \ldots, d-1\}$ one has:

$$\begin{aligned} eu(\mathbb{H}^0(\Gamma(d), K + 2aE_+^*)) &= \sum_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} \min\{\overline{W}|_{\mathbb{T}_n}\}, \\ eu(\mathbb{H}^*(\Gamma(d), K + 2aE_+^*)) &= \sum_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} -eu(\mathbb{H}^*(\mathbb{T}_n, \overline{W})). \end{aligned} \tag{4.104}$$

Example 4.9.7 For any $d > 0$ and $q > 0$ the summation of (4.102) over a gives

$$\mathbb{H}^q(\Gamma(d)) = \bigoplus_{a=0}^{d-1} \mathbb{H}^q(\Gamma(d), K + 2aE_+^*) = \bigoplus_{0\leq n\leq \mu-2} \mathbb{H}^{q-1}_{red}(\mathbb{T}_n, \overline{W})[s_{n,d}]. \tag{4.105}$$

On the right hand side of (4.105) the numbers $s_{n,d}$ depend on d, but the rank of the right hand side is independent of d. In particular, up to shifts of different direct sum blocks, $\oplus_{q>0}\mathbb{H}^q(\Gamma(d), k_r)$ is independent of the choice of the integer d. (This can also be deduced from the surgery exact sequences from [74].)

Example 4.9.8

(a) Assume that for a certain d and a one gets $\tilde{p}_0 = 0$. Then $\mathbb{H}^*_{red}(\Gamma, k_r) = 0$, and $\mathbb{H}^0(\Gamma, k_r) = \mathcal{T}^+_0$.
(b) Assume that for a certain d and a one gets $\tilde{p}_0 = 1$. Then $\mathbb{H}^*(\Gamma, k_r) = \mathbb{H}^*(R_0, \overline{w})$, hence everything is determined by $T^-_{1,a}$. Indeed,

$$\begin{aligned}\min T^-_{1,a} &= \min\Big\{\sum_i \#\{s \in \mathcal{S}_i\ :\ s \le \ell_i - 1\},\ \text{where}\ \sum_i \ell_i = a+1\Big\}\\ &= \min\Big\{\sum_i \#\{s \notin \mathcal{S}_i\ :\ s \ge \ell_i\},\ \text{where}\ \sum_i \ell_i = a+1\Big\} + 1 + a - \delta,\end{aligned}$$

$m_{k_r} = \min\{0, 1+a-\delta\}$, $\mathbb{H}^0_{red}(\Gamma, k_r)$ is generated by one element of degree $2\max\{0, 1+a-\delta\}$, rank $\mathbb{H}^0_{red}(\Gamma, k_r) = \min T^-_{1,a} - \max\{0, 1+a-\delta\}$, and finally for $q > 0$ one has $\mathbb{H}^q(\Gamma, k_r) = \mathbb{H}^{q-1}_{red}(T^-_{1,a}, \overline{w}) = \mathbb{H}^{q-1}_{red}(\mathbb{T}_a, \overline{W})[2(1+a-\delta)]$, $(T^-_{1,a} = \mathbb{T}_a + \mathbf{m})$.
(c) If $d \ge \mu - 1$ then $\tilde{p}_0 = 1$ for $a < \mu - 1$, and $\tilde{p}_0 = 0$ for $a \ge \mu - 1$.

Remark 4.9.9 Assume that we know all the cohomology groups $\{\mathbb{H}^*(\Gamma(d), k_r)\}_{k_r}$ for some specific d with $d \ge \mu - 1$. Then using them, and also the values $\overline{w}(p\mathbf{m}) = p(1+a-\delta) + dp(p-1)/2$ for all p, a and d, we can recover all the lattice cohomologies $\{\mathbb{H}^*(\Gamma(d), k_r)\}_{k_r}$ for any $d > 0$. [For this, use Example 4.9.8 and (4.102).]

Corollary 4.9.10 *For any $n \ge 0$ the coefficients of $Q(t) = \sum_n \alpha_n t^n$ satisfy*

$$\alpha_n = -eu\,(\mathbb{H}^*(\mathbb{T}_n, \overline{W})). \tag{4.106}$$

Proof Use the identities (4.93) and (4.104) for $d \gg 0$, cf. 4.9.9. □

Remark 4.9.11 Above we reduced several computations to the weight function $\overline{W}|_{\mathbb{T}_n}$. It was connected with the weight function provided by the reduction formula via $\overline{W}|_{\mathbb{T}_n}(\bar{l} - p\,\mathbf{m}) = \overline{w}(\bar{l}) - \overline{w}((p+1)\,\mathbf{m})$, where $p = \lfloor n/d \rfloor$. Since each $\overline{w}(p\,\mathbf{m})$ is computable from d, a, δ, cf. 4.9.3*(b)*, the lattice cohomology $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ is computable from d, a, δ and $\{\overline{W}|_{\mathbb{T}_n}\}_n$. On the other hand, by (4.101) $\overline{W}((\ell_{1,0}, \dots, \ell_{\nu,0}))$ equals $\sum_i \#\{s \notin \mathcal{S}_i : s \ge \ell_{i,0}\} = \sum_i \big(\delta_i - \#\{s_i \notin \mathcal{S}_i : s_i < \ell_{i,0}\}\big) = \sum_i(\delta_i - \ell_{i,0}) + \sum_i \#\{s_i \in \mathcal{S}_i : s_i < \ell_{i,0}\}$. Hence

$$\min\{\overline{W}|_{\mathbb{T}_n}\} = \delta - n - 1 + \min_{\sum_i \ell_{i,0} = n+1} \#\{s_i \in \mathcal{S}_i : s_i < \ell_{i,0}\}. \tag{4.107}$$

This motivates the replacement of the semigroup $\mathcal{S}_i$ with an equivalent object of it, with its 'counting function' $j \mapsto H_i(j)$,

$$H_i(j) := \#\{s \in \mathcal{S}_i \ : \ s < j\}. \tag{4.108}$$

From analytic point of view, $H_i(j)$ is the coefficient of t^j in the Hilbert function of the local singularity (C, p_i), associated with the filtration given by its normalization.

The above min-expression can be reformulated formally as follows. Consider any two functions H_1 and H_2 defined on integers and bounded from below. Then we define their *'minimum convolution'* (cf. [9, 5.3]), denoted by $H_1 \diamond H_2$ as $(H_1 \diamond H_2)(j) = \min_{j_1+j_2=j}\{H_1(j_1) + H_2(j_2)\}$.

Then from the counting functions $\{H_i\}_{i=1}^{\nu}$ associated with $\{\mathcal{S}_i\}_{i=1}^{\nu}$ we construct

$$H := H_1 \diamond H_2 \diamond \cdots \diamond H_\nu. \tag{4.109}$$

Since the operator $\diamond$ is associative and commutative, the function H is well-defined.

From the above discussion $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ is computable from d, a, δ and H.

Remark 4.9.12 In the above discussion (e.g. in 4.9.5–4.9.6), the space $\mathbb{T}_n$—intersection of a simplex with a rectangle—can be replaced by the supporting simplex. Indeed, set

$$\Sigma_n := \{(\ell_{1,0}, \ldots, \ell_{\nu,0}) \in (\mathbb{R}_{\geq 0})^\nu \ : \ \sum_i \ell_{i,0} = n+1\}. \tag{4.110}$$

A verification shows that $H^*_{red}(\mathbb{T}_n, \overline{W})$ is isomorphic with $H^*_{red}(\Sigma_n, \overline{W})$ for every $n \geq 0$. Furthermore, if $n > \mu - 2$ then $H^*_{red}(\mathbb{T}_n, \overline{W}) = 0$ automatically, hence in several formulae above (e.g. in the summations from (4.102) and (4.105)) the restrictions $n \leq \mu - 2$ can be safely neglected.

4.9.2 Superisolated Singularities with More Cusps

In this subsection we consider a superisolated singularity associated with an irreducible rational cuspidal curve. For different notations and statements regarding the analytic and topological type see Sects. 4.2.4, 4.3.6, 4.4.11, 4.7.4, and 4.9. In this subsection we follow [8].

Our goal is to discuss Conjectures 4.3.21 and 4.3.22 from the point of view of lattice cohomology. Let us recall the two statements. Set (cf. 4.3.20*(b)*)

$$N(t) = \sum_{l=0}^{d-3}\left(\alpha_{(d-3-l)d} - \frac{(l+1)(l+2)}{2}\right) t^{d-3-j}. \tag{4.111}$$

- Conjecture 4.3.21: all the coefficients of $N(t)$ are non-positive. We will refer to this as '**Conjecture C**' ('Conjecture regarding the coefficients of $N(t)$').
- Conjecture 4.3.22: $N(1)$ is non-positive. We will refer to this as the '**Conjecture I**' (we regard $N(1)$ as an 'index type invariant').

Clearly Conjecture C implies Conjecture I.

We will compare these statements with the Semigroup Distribution Property based on the properties of counting function H_i of the semigroups and also on a subtle connection with lattice cohomology.

We consider the counting functions H_i of the semigroups $\mathcal{S}_i$ (cf. (4.108)) and their minimum convolution H as in (4.109). Recall also (cf. 4.2.33) that the Semigroup Distribution Property (SDP) reads as $H(ld+1) = (l+1)(l+2)/2$ for any $l = 0, 1, \dots d-3$.

Example 4.9.13 (The case $\nu = 1$) In this case $\alpha_j = \#\{s \notin \mathcal{S}_1 : s > j\}$, cf. (4.42). From (4.43) $\alpha_{2\delta-2-j} = H_1(j+1)$ for $j = 0, \dots, 2\delta-2$. Hence, the α-coefficient needed in (4.111) is $\alpha_{(d-3-l)d} = \#\{s \in \mathcal{S}_1 : s \leq ld\} = H_1(ld+1)$. Recall that 4.2.33 (Bézout's Theorem) implies $\alpha_{(d-3-l)d} = H_1(ld+1) \geq (l+1)(l+2)/2$. This inequality and (4.111) show that for $\nu = 1$ Conjecture C is equivalent to $N(t) \equiv 0$. But, they are also equivalent to Conjecture I, since if $N(1) \leq 0$ then necessarily $N(t) \equiv 0$. Finally, the validity of all these statements follow from SDP.

However, for $\nu \geq 2$ the relationships are more subtle.

Theorem 4.9.14 ([8]) *With the above notations one has:*

1. *If $\nu = 2$, then $q_{2\delta-2-j} \leq H(j+1)$ for any $j = 0, 1, \dots 2\delta-2$. Therefore, for bicuspidal curves the SDP implies Conjecture C (hence Conjecture I too).*
2. *If $\nu \geq 3$, then the inequality $q_{2\delta-2-j} \leq H(j+1)$ does not hold in general, not even for $j = ld$ ($l = 0, 1, \dots, d-3$), needed for Conjectures C and I. Moreover, Conjecture C is not true in general, and Conjecture I behaves independently from SDP. (Conjecture I remains as a conjecture, though its validity is verified directly for all 'known' curves.)*

For a direct elementary proof of part *(1)* see [65].

4.9.15 Combinatorial Reformulations The next discussion aims to clarify the similarities and differences between the polynomial Q and the function H.

Let us start with ν semigroups $\{\mathcal{S}_i\}_{i=1}^{\nu}$ associated with local irreducible plane curve singularities. However, in the next discussion we will not require their realizability as singularities of a projective rational curve. [Regarding the realizability, we use the following terminology. If the sum δ of delta-invariants of the local singularity types is of form $2\delta = (d-1)(d-2)$ for some integer d, then we say that these ν local topological types are *combinatorial candidates* for the ν singularities of a rational cuspidal plane curve of degree d. If such a curve really exists then (SDP) is valid for the corresponding local data and d.], The semigroups determine their counting functions H_i by (4.108) and the minimal convolution H of the functions $\{H_i\}_i$ by (4.109). For convenience, define also the sequences $\{h_j^{(i)}\}_{j=0}^{\infty}$ by $h_j^{(i)} := H_i(j+1)$.

For any sequence $a = \{a_j\}_{j=0}^{\infty}$ denote by ∂a its *difference sequence*, i.e. $(\partial a)_j = a_j - a_{j-1}$ with the convention $a_{-1} = 0$. Similarly, we will denote by Σa the *sequence of partial sums*, i.e. $(\Sigma a)_j = a_0 + \cdots + a_j$. Of course, $\Sigma\partial a = a$ and $\partial\Sigma a = a$ for any sequence a.

By (4.108) and $\Delta_i(t) = (1-t)\cdot\sum_{s\in\mathcal{S}_i} t^s$ (cf. (4.6) the coefficient $c_j^{(i)}$ of t^j in $\Delta_i(t)$ can be written as $c_j^{(i)} = (\partial\partial h^{(i)})_j$. The coefficient sequence of a polynomial product is the *usual convolution* of coefficient sequences of the factors. Hence, the coefficient c_j of t^j in $\Delta(t) = \prod_i \Delta_i(t)$ is $c_j = \sum_{j_1+\cdots+j_\nu=j} c_{j_1}^{(1)}\cdots c_{j_\nu}^{(\nu)}$. Denoting the convolution of two sequences $a = \{a_j\}_{j=0}^{\infty}$ and $b = \{b_j\}_{j=0}^{\infty}$ by $a * b$, i.e. $(a*b)_j = \sum_{k=0}^{j} a_k b_{j-k}$, we get $c_j = (\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)})_j$. Let us define:

$$F(j) := (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j. \tag{4.112}$$

Before we identify F, let us recall some symmetry properties. From the symmetry of $\Delta = 1 + (t-1)\delta + (t-1)^2 Q(t)$ (and from $\delta = \sum_i \delta_i$)

$$\alpha_{2\delta-2-j} = \alpha_j + j + 1 - \delta \quad \text{for} \quad 0 \le j \le 2\delta - 2. \tag{4.113}$$

This (or the symmetry of each semigroup) implies also $H_i(j_i) = H_i(2\delta_i - j_i) + j_i - \delta_i$, from which one also obtains

$$H(2\delta - 2 - j + 1) = H(j+1) - j - 1 + \delta \quad \text{for every } j \in \mathbb{Z}. \tag{4.114}$$

Next, if $A(t) = \sum_j a_j t^j$ and $B(t) = \sum_j b_j t^j$ satisfy $A(t) = A(1) + (t-1)B(t)$, then $(\Sigma a)_j = A(1) - b_j$. This applied twice for Δ gives $(\Sigma\Sigma c)_j = j + 1 - \delta + \alpha_j$. Hence, then the definition of Q and (4.113) provide

$$\alpha_{2\delta-2-j} = (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j = F(j) \quad \text{for} \quad 0 \le j \le 2\delta - 2. \tag{4.115}$$

In other words, the H-values are obtained from $\{h^{(i)}\}_i$ by minimal convolution (shifted by one), while the F-coefficients (or α-coefficient in opposite order) are obtained by the composition of $\partial\partial$, the usual convolution, and the $\Sigma\Sigma$ operator.

Then one has the following reinterpretations in terms of F and H.

Let $C \subset \mathbb{C}P^2$ be a rational cuspidal curve of degree d with ν cusps of given topological types (in particular, $d(d-3) = 2\delta - 2$). Set $F(j) := (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j$, where $h_j^{(i)} = H_i(j+1)$, and H_i is the semigroup counting function of the i-th singularity. Set $H := H_1 \diamond \cdots \diamond H_\nu$. Then

$$\text{(Conjecture C)} \qquad F(ld) \le \frac{(l+1)(l+2)}{2} \quad \text{for all } l = 0, 1, \ldots, d-3. \tag{4.116}$$

$$\text{(Conjecture I)} \quad \sum_{l=0}^{d-3} F(ld) \leq \sum_{l=0}^{d-3} \frac{(l+1)(l+2)}{2} = \frac{d(d-1)(d-2)}{6}. \qquad (4.117)$$

$$\text{(SDP)} \qquad H(ld+1) = \frac{(l+1)(l+2)}{2} \quad \text{for all } l = 0, 1, \dots, d-3. \qquad (4.118)$$

Let us summarize the combinatorial situation. Starting from the semigroups of ν local singularities we define H and F.

If $\nu = 1$ (since $\Sigma\Sigma\partial\partial(h) = h$) then $F(j) = H(j+1)$ for each $j \in \mathbb{Z}_{\geq 0}$ (independently of realizability, hence not just for $j \in d \cdot \mathbb{Z}_{\geq 0}$).

On the other hand, for $\nu > 1$ the values $F(j)$ and $H(j+1)$ become different. Nevertheless, cf. Theorem 4.9.14*(1)* $F(j) \leq H(j+1)$ remains true for $\nu = 2$ and every integer $j \geq 0$, again by combinatorial (lattice cohomology) argument (independently of realizability and d).

With these facts in mind, it is tempting to conjecture that maybe *the inequality* $F(j) \leq H(j+1)$ *is always true—as a property of local singularity types—*, which would make Conjecture C a combinatorial corollary of SDP. But, for $\nu \geq 3$ there is no such relation between the local functions F and H.

4.9.16 Lattice Cohomological Reinterpretation Consider the combinatorial situation from 4.9.15. The semigroups $\mathcal{S}_i$ determine links $K_i \subset S^3$ of the corresponding (topological types) of plane curve singularities. Consider an arbitrary $d > 0$ and the surgery 3-manifold $S^3_{-d}(\#_i K_i)$ as in Sect. 4.9.

The next statements show a remarkable common feature of the functions F and H.

Theorem 4.9.17 *For any $d > 0$ and $0 \leq a < d$ the following facts hold:*

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} (H(j+1) + \delta - 1 - j),$$
$$= \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} H(2\delta - 2 - j + 1); \qquad (4.119)$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} (F(j) + \delta - 1 - j)$$
$$= \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} F(2\delta - 2 - j). \qquad (4.120)$$

Proof We will use the identities from (4.104). In the first one, note that by (4.101), (4.100), and (4.107) $\min(\overline{W}|_{\mathbb{T}_j})$ is $\delta - j - 1 + H(j+1)$ and (4.119) follows (for its second identity use (4.114)).

For the second identity, note that $-eu(\mathbb{H}^*(\mathbb{T}_j, \overline{W})$ equals α_j by (4.106), which is $F(2\delta - 2 - j)$ by (4.115). Then use again the symmetry (4.113). □

Remark 4.9.18 In fact, by Theorem 4.9.4, the integer d, the sum of delta-invariants δ and the function H completely determine the whole $\mathbb{H}^0$ as a graded $\mathbb{Z}[U]$-module (and not just its Euler characteristic).

Corollary 4.9.19 *Assume that $d(d-3) = 2\delta - 2$ (that is, d and $\{\mathcal{S}_i\}_i$ constitute a package of combinatorial candidates for algebraic realizability). Then*

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j\equiv -a (\text{mod } d)\\ 0\le j\le 2\delta-2}} H(j+1),$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j\equiv -a (\text{mod } d)\\ 0\le j\le 2\delta-2}} F(j).$$

This for $a = 0$ reads as

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) = \sum_{0\le l\le d-3} H(ld+1),$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) = \sum_{0\le l\le d-3} F(ld).$$

Since by 4.2.33 $H(ld+1) \geq (l+1)(l+2)/2$ for any $l = 0, \dots, d-3$, $\sum_{l=0}^{d-3} H(ld+1) = \sum_{l=0}^{d-3}(l+1)(l+2)/2$ is equivalent to SDP for every l (cf. (4.118)). In particular, in the presence of the algebraic realization, the valid SDP reads as:

$$\text{(SDP)} \quad eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) = d(d-1)(d-2)/6. \tag{4.121}$$

Furthermore, under the same realizability assumption, Conjecture I reads as:

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) \leq d(d-1)(d-2)/6. \tag{4.122}$$

They combined:

$$\text{(Conjecture I)} \quad eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) \leq eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right). \tag{4.123}$$

4.9.20 Proof of Conjecture I for $\nu = 2$ (via SDP)
First note that $\mathbb{H}^q(S^3_{-d}(\#_i K_i), k_r) = 0$ for any $q \geq \nu$ and any k_r (cf. 4.8.3). Then, for $\nu = 2$, one has $eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) = eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) - \mathrm{rank}_{\mathbb{Z}}\mathbb{H}^1(S^3_{-d}(\#_i K_i), K)$, hence (4.123) follows.

For $\nu \geq 3$ the similar argument does not work. From this point of view, it is even more surprising that in all the known cases Conjecture I still holds, cf. 4.9.14.

4.10 Lattice Cohomology and Heegaard Floer Homology

The Seiberg–Witten invariant is the (normalized) Euler-characteristic of the Seiberg–Witten monopole Floer homology of Kronheimer–Mrowka, or equivalently, of the Heegaard Floer homology of Ozsváth and Szabó. These theories had an extreme influence on the modern mathematics, solving (or disproving) a long list of old conjectures (e.g. Thom Conjecture, or conjectures regarding classification of 4-manifolds, or famous old problems in knot theory); see the long list of distinguished articles of Kronheimer–Mrowka or Ozsváth–Szabó. In [102] Ozsváth and Szabó provided a computation of the Heegaard Floer homology for some special plumbed 3-manifolds. This computation resonated incredibly with the theory of computation sequences used in Artin–Laufer program (see e.g. [50, 67, 68]). These two facts influenced considerably the definition of the lattice cohomology.

4.10.1 The Conjecture Connecting Lattice Cohomology and Heegaard Floer Theory

4.10.1 Short Review of Heegaard Floer Homology $HF^+(M)$ We assume that M is an oriented rational homology 3–sphere, and we restrict ourselves to the +–theory of Ozsváth and Szabó. The Heegaard Floer homology $HF^+(M)$ is a $\mathbb{Z}[U]$–module with a $\mathbb{Q}$–grading compatible with the $\mathbb{Z}[U]$–action, where $\deg(U) = -2$. Additionally, $HF^+(M)$ has another $\mathbb{Z}_2$–grading; $HF^+(M)_{even}$, respectively $HF^+(M)_{odd}$ denote the graded parts. Moreover, $HF^+(M)$ has a natural direct sum decomposition of $\mathbb{Z}[U]$–modules (compatible with all the gradings): $HF^+(M) = \oplus_\sigma HF^+(M, \sigma)$ indexed by the spin^c structures σ of M. For any $\sigma \in \mathrm{Spin}^c(M)$ one has

$$HF^+(M, \sigma) = \mathcal{T}^+_{d(M,\sigma)} \oplus HF^+_{red}(M, \sigma),$$

a graded $\mathbb{Z}[U]$–module isomorphism, and $HF^+_{red}(M,\sigma)$ has finite $\mathbb{Z}$–rank and an induced $\mathbb{Z}_2$–grading. One also considers

$$\chi(HF^+(M,\sigma)) := \text{rank}_{\mathbb{Z}}\, HF^+_{red,even}(M,\sigma) - \text{rank}_{\mathbb{Z}}\, HF^+_{red,odd}(M,\sigma).$$

Then the Seiberg–Witten invariant of (M,σ) equals $\chi(HF^+(M,\sigma)) - d(M,\sigma)/2$.

By changing the orientation we have $\chi(HF^+(M,\sigma)) = -\chi(HF^+(-M,\sigma))$ and $d(M,\sigma) = -d(-M,\sigma)$.

4.10.2 The Predicted Connection In [72] the author formulated the following

Conjecture 4.10.3 For any plumbed rational homology sphere associated with a connected negative definite graph Γ, and for any $k \in \text{Char}$, one has

$$d(M,[k]) = \max_{k'\in[k]} \frac{(k')^2 + |\mathcal{V}|}{4} = \frac{k^2 + |\mathcal{V}|}{4} - 2\cdot \min \chi_k.$$

Furthermore,

$$HF^+_{red,even}(-M,[k]) = \bigoplus_{p\ even} \mathbb{H}^p_{red}(\Gamma,[k])[-d],$$

and

$$HF^+_{red,odd}(-M,[k]) = \bigoplus_{p\ odd} \mathbb{H}^p_{red}(\Gamma,[k])[-d].$$

Both parts of the Conjecture were verified for almost rational graphs in [72], for two bad vertices in [101], see [72, 8.4] too. Otherwise, the Conjecture is still open.

Note that (conjecturally) $\mathbb{H}^*$ has a richer structure: its q–filtration $\mathbb{H}^* = \oplus_q \mathbb{H}^q$ collapses at the level of HF^+ to a $\mathbb{Z}_2$ odd/even filtration.

The fact that both theories have the same Euler characteristic support the above conjecture as well. Another supporting evidence is the following fact.

4.10.4 Coincidence of the Vanishing of the Reduced Theories By 4.6.22 the graph Γ is rational if and only if $\mathbb{H}^*_{red}(\Gamma) = 0$. On the other hand, following Ozsváth and Szabó, by definition, M is an L–space if and only if $HF^+_{red} = 0$. Their equivalence is predicted by Conjecture 4.10.3. This 'tip of the iceberg' statement was proved in [76]:

Theorem 4.10.5 *The following facts are equivalent for a connected negative definite graph* Γ*:*

(i) Γ *is a rational graph,*
(ii) $M = M(\Gamma)$ *is an* L*–space.*

$(i) \Rightarrow (ii)$ follows from lattice cohomology theory [70, 72], while $(ii) \Rightarrow (i)$ uses partly the following equivalence $(ii) \Leftrightarrow (iii)$, where (iii) means that $\pi_1(M)$ is not a left-orderable group. [A non trivial group G is said to be left-orderable if there exist a total order $<$ on G such that if $a < b$ then $ga < gb$ for every $g \in G$.] The equivalence $(ii) \Leftrightarrow (iii)$ was proved in [33] for any graph–manifold. For arbitrary 3–manifolds it was conjectured by Boyer, Gordon and Watson [10], for different developments and other references see [33, 76].

Problem 4.10.6 Characterize elliptic singularities (or other non-rational families of singularities) by a certain property of the fundamental group of the link.

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