José Luis Cisneros-Molina
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José Seade *Editors*

Handbook of Geometry and Topology of Singularities III

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Preface

This is the third volume of the *Handbook of the Geometry and Topology of Singularities*, a subject which is ubiquitous in mathematics, appearing naturally in a wide range of different areas of knowledge. The scope of singularity theory is vast, its purpose is multifold. This is a meeting point where many areas of mathematics, and science in general, come together.

Let us reminisce Bernard Teissier's words in his foreword to Volume I of the Handbook:

> I claim that Singularity Theory sits inside Mathematics much as Mathematics sits inside the general scientific culture. The general mathematical culture knows about the existence of Morse theory, parametrizations of curves, Bézout's theorem for plane projective curves, zeroes of vector fields and the Poincaré-Hopf theorem, catastrophe theory, sometimes a version of resolution of singularities, the existence of an entire world of commutative algebra, etc. But again, for the singularist, these and many others are lineaments of a single landscape and she or he is aware of its connectedness. Moreover, just as Mathematics does with science in general, singularity theory interacts energetically with the rest of Mathematics, if only because the closures of non singular varieties in some ambient space or their projections to smaller dimensional spaces tend to present singularities, smooth functions on a compact manifold must have critical points, etc. But singularity theory is also, again in a role played by Mathematics in general science, a crucible where different types of mathematical problems interact and surprising connections are born.

The Handbook has the intention of covering a wide scope of singularity theory, presenting articles on various aspects of the theory and its interactions with other areas of mathematics in a reader-friendly way. The authors are world experts; the various articles deal with both classical material and modern developments.

The first Volume I of this collection gathered ten articles concerning foundational aspects of the theory. This includes:

– The combinatorics and topology of plane curves and surface singularities
– An introduction to four of the classical methods for studying the topology and geometry of singular spaces, namely resolution of singularities, deformation theory, Stratifications, and slicing the spaces *à la* Lefschetz
– Milnor fibrations and their monodromy

– Morse theory for stratified spaces and constructible sheaves
– Simple Lie algebras and simple singularities

Volume II also consists of ten articles. These cover foundational aspects of the theory as well as some important relations with other areas of mathematics. They include:

– The analytic classification of plane curve singularities and the existence of complex and real algebraic curves in the plane with prescribed singularities
– An introduction on the limits of tangents to a complex analytic surface, a subject that originates in Whitney's work
– Introductions to Zariski's equisingularity and intersection homology, which are two of the main current viewpoints for studying singularities
– An overview of Milnor's fibration theorem for real and complex singularities, as well as an introduction to Massey's theory of Lê cycles
– A discussion of mixed singularities, which are real analytic singularities with a rich structure that allows their study via complex geometry
– The study of intersections of concentric ellipsoids in $\mathbb{R}^n$ and its relation with several areas of mathematics, from holomorphic vector fields to singularity theory, toric varieties, and moment-angle manifolds
– A review of the topology of quasi-projective varieties and generalizations about the complements of plane curves and hypersurfaces in projective space

This Volume III also consists of ten chapters. Some of these complement topics explored previously in Volumes I and II, while other chapters bring in important new subjects. Let us say a few words about the content of this volume, though each chapter has its own abstract, introduction and a large bibliography for further reading. There is also a global index of terms at the end.

Chapters 1 and 2 have as common thread the much celebrated Thom-Mather theory. In 1944, Whitney studied mappings $\mathbb{R}^n \to \mathbb{R}^{2n-1}$, the first pair of dimensions not covered by his immersion theorem, showing that in these setting singularities cannot be avoided in general. He then introduced the concept of stable mappings and characterized the stable mappings from $\mathbb{R}^n$ to $\mathbb{R}^p$ with $p \geq 2n - 1$, and also those from the plane into itself, showing that in all these cases the stable mappings form a dense set in the space of smooth proper mappings. Whitney conjectured that the density of stable mappings would hold for any pair (n, p). However, René Thom showed that this is not the case by giving a counterexample. Thom then conjectured that the topologically stable maps are always dense and gave an outline of the proof. The complete proof was given by John Mather, who, from 1965 to 1975, solved almost completely the program drawn by Thom for the stability problem. This is known as Thom-Mather theory.

Chapter 3 is about Zariski's equisingularity, previously envisaged in Parusiński's chapter in Volume II. Among the various notions of equivalence of singularities, topological equisingularity is one of the oldest and easiest to define, but it is far from being well understood. Several challenging questions remain open. In this chapter, the author surveys developments in topological equisingularity, some of its relations with other equisingularity notions, and hints on new possible approaches to old questions based in algebro-geometric methods, Floer theory, and Lipschitz geometry. Topological equisingularity questions were crucial motivation sources for the development of the Computer Algebra program SINGULAR; this is explained in an appendix by G.-M. Greuel and G. Pfister.

Chapter 4 somehow fits within the classical interplay between normal singularities in complex surfaces and 3-manifold theory, which has been studied for decades and was discussed from a topological perspective in F. Michel's chapter in Volume I. Now the author looks at the subject from another perspective, bringing in subtle structures. Given a complex analytic normal surface singularity $(X, 0)$ we know that its topology is fully determined by its link L_X, a 3-manifold which is the intersection of X with a sufficiently small sphere in the ambient space, centered at 0. The main motif of this chapter is studying the ties between analytic and topological invariants of $(X, 0)$. Historically, this program was started by Artin and Laufer, which characterized topologically the rational and minimally elliptic singularities (respectively), and computed several analytic invariants from the resolution graph. This question brings us into the theory of the Casson and Casson-Walker invariant, the (refined) Turaev torsion, Seiberg-Witten invariants, lattice (co)homology, Heegaard-Floer theory, and other important invariants of 3-manifolds. This chapter starts from well-known elementary facts about surface singularities and brings us to the depths of this rich and interesting theory.

Chapters 5–7 discuss different aspects of the theory of Chern classes for singular varieties. For complex manifolds, their Chern classes are by definition those of its tangent bundle. These are important invariants that encode deep geometric and topological information. When we consider singular varieties, there is not a unique way of extending this concept. This somehow depends on which properties of Chern classes we are interested in, or how we extend the notion of the tangent bundle over the singular set. In these chapters, the authors introduce in elementary ways the various notions of Chern classes for singular varieties and their relations with other invariants of singular varieties. Chapter 5 gives a thorough account of the subject, from the birth of the theory of Chern classes up to the modern theories of motivic, bivariant, and Hirzebruch characteristic classes. Chapter 6 has Segre classes as its core. These classes are an important ingredient in Fulton-MacPherson intersection theory and provide a powerful mean for studying Chern classes of vector bundles in the algebraic setting. Several important invariants of algebraic varieties may be expressed in terms of Segre classes. The goal of that chapter is to survey several invariants specifically arising in singularity theory which may be defined or recast in terms of Segre classes. Chapter 7 looks at the subject from a topological viewpoint, focusing on the relations between local and global invariants, particularly indices of vector fields, the Milnor number, and Lê cycles. It includes for completeness an introduction to the Hirzebruch-Riemann-Roch theorem and its generalizations to singular varieties that give rise to several of the recent developments in the subject.

Chapter 8 studies the residues in complex analytic varieties that arise from the localization of characteristic classes via Alexander duality. A paradigm for this theory is the classical theorem of Poincaré-Hopf that can be understood as providing a localization of the top Chern class of a complex manifold at the singularities of a vector field. This was beautifully extended by Baum and Bott for singular holomorphic foliations on complex manifolds, providing expressions for certain Chern numbers in terms of residues localized at the singular set of the foliation. The theory that the author presents in this chapter starts with the study of residues of singular holomorphic foliations, later transferred to the index theory of holomorphic self-maps. The philosophy behind is rather simple. Namely, once we have some kind of vanishing theorem on the non-singular part of a geometric object such as a foliation, certain characteristic classes are localized at the set of singular points, and the localization gives rise to residues via the Alexander duality. The author explains how the relative Čech-de Rham theorem allows us to deal with the problem from both the topological and differential geometric viewpoints, and the comparison of the two yields various interesting expressions of the residues and applications.

Chapter 9 surveys applications of mixed Hodge theory to the study of isolated singularities. Hodge theory deals with the cohomology of smooth complex projective varieties, or more generally, compact Kähler manifolds. A choice of a Riemannian metric enables one to define the Laplace operator Δ on differential forms, and each de Rham cohomology class contains exactly one closed form ω with $\Delta\omega = 0$, the harmonic representative. One has the Hodge decomposition of cohomology classes via their harmonic representatives:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the subspace of $H^k(X, \mathbb{C})$ consisting of classes of forms containing harmonic forms of type (p, q). Using Leray's theory of sheaves and resolution of singularities, Grothendieck defined the de Rham cohomology of complex algebraic varieties in purely algebraic terms. A generalization of Hodge theory to arbitrary complex algebraic varieties was then developed by Deligne. He showed that the cohomology of a complex algebraic variety carries a slightly more general structure, which presents $H^k(X, \mathbb{C})$ as a successive extension of Hodge structures of decreasing weights. This generalization is called a mixed Hodge structure.

We close this volume with Chap. 10, a detailed introduction of the theory of constructible sheaf complexes in the complex algebraic and analytic settings. All concepts are illustrated by many interesting examples and relevant applications, while some important results are presented with complete proofs. This chapter is intended as a broadly accessible user's guide to those topics, providing the readers not only with a presentation of the subject but also with concrete examples and applications that motivate the general theory. The authors introduce the main results of stratified Morse theory in the framework of constructible sheaves, a subject discussed also in Goresky's chapter in Volume I of this Handbook. Constructible sheaf complexes and especially perverse sheaves have become indispensable tools for studying complex algebraic and analytic varieties. They have seen spectacular applications in geometry and topology, and several of these are discussed in this chapter.

This handbook is addressed to graduate students and newcomers to the theory, as well as to specialists who can use it as a guidebook. It provides an accessible account of the state of the art in several aspects of the subject, its frontiers, and its interactions with other areas of research. This will continue with a Volume IV, which will cover other aspects of singularity theory, and a Volume V, which will focus on holomorphic foliations, a remarkably important subject on its own that has close connections with singularity theory.

We thank Bernard Teissier for allowing us to use his words above and for valuable and inspiring comments.

Cuernavaca, Mexico — José Luis Cisneros Molina
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Chapter 1
Old and New Results on Density of Stable Mappings

Maria Aparecida Soares Ruas

The analysis of the conditions for a map-germ to be finitely determined and of the degree of determinacy involves the most important of the local aspects of singularity theory.

C. T. C. Wall [108]

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Abstract Density of stable maps is the common thread of this paper. We review Whitney's contribution to singularities of differentiable mappings and Thom-Mather theories on C^∞ and C^0-stability. Infinitesimal and algebraic methods are presented in order to prove Theorems A and B on density of proper stable and topologically stable mappings $f : N^n \to P^p$. Theorem A states that the set of proper stable maps is dense in the set of all proper maps from N to P, if and only if the pair (n, p) is in *nice dimensions,* while Theorem B shows that density of topologically stable maps holds for any pair (n, p). A short review of results by du Plessis and Wall on the range in which proper smooth mappings are C^1-stable is given. A Thom-Mather map is a topologically stable map $f : N \to P$ whose associated k-jet map $j^k f : N \to P$ is transverse to the Thom-Mather stratification in $J^k(N, P)$. We give a detailed description of Thom-Mather maps for pairs (n, p) in the boundary of the nice dimensions. The main open question on density of stable mappings is to determine the pairs (n, p) for which Lipschitz stable mappings are dense. We discuss recent results by Nguyen, Ruas and Trivedi on this subject, formulating conjectures for the density of Lipschitz stable mappings in the boundary of the nice dimensions. At the final section, Damon's results relating $\mathcal{A}$-classification of map-germs and $\mathcal{K}_V$ classification of sections of the discriminant $V = \Delta(F)$ of a stable unfolding of f are reviewed and open problems are discussed.

1.1 Introduction

Although Riemann, Klein, Poincaré and other great mathematicians of the nineteenth century already used deep topological concepts in their work, the birth of algebraic and differential topology as formal sub-areas of Mathematics occurred in the first half of the twentieth century.

After previous works of Whitehead, Veblen and others, the American mathematician Hassler Whitney introduced fundamental concepts and proved strong results in differential topology such as the well known *strong Whitney embedding theorem* and *weak Whitney embedding theorem*. The first one states that any smooth real m-dimensional manifold can be smoothly embedded in $\mathbb{R}^{2m}$, while the latter says that any continuous mapping of an n-dimensional manifold to an m-dimensional manifold may be approximated by a smooth embedding provided that $m > 2n$. Furthermore, replacing embedding by immersion in this last statement the result holds for all $m \geq 2n$. His survey paper *Topological properties of differentiable* *manifolds* published in 1937 [111] contains many contributions he made in those early years of differential topology.

In 1944, Whitney [113] studied the first pair of dimensions not covered by his immersion theorem. For mappings f from $\mathbb{R}^n$ to $\mathbb{R}^{2n-1}$ Whitney proved that singularities cannot be avoided in general. He introduced the *semi regular mappings* as proper mappings $f : \mathbb{R}^n \to \mathbb{R}^{2n-1}$ whose only singularities are the generalized cross-caps (Whitney umbrellas) points. Away from singular points, f is an immersion with transverse double points, and when $n = 2$, a finite number of triple points may also appear in the image of f. These are the only stable singularities in these dimensions. However, only later, Whitney introduced the notion of stable mappings.

Abstract spaces and their topological properties were known by then, so that the notion of *stability* of systems and mappings appeared naturally. It appeared first in dynamical systems, introduced by A. Andronov and L. Pontryagin [1] for a class of autonomous differential systems on the plane, under the name of "systèmes grossiers". The term "structural stability" appears in the english language edition of the book by Andronov and Chaikin, edited under the direction of Solomon Lefschetz in 1949 [2] (see also [91]). It also appears in other pioneering papers on the subject, among them the paper *On structural stability* by Mauricio Peixoto [78], published in 1959.

The notion of stable mappings was formulated by Whitney in [115] around the middle of last century. He characterized stable mappings from $\mathbb{R}^n$ to $\mathbb{R}^p$ with $p \geq 2n - 1$ in [112] and stable mappings from the plane into the plane in [114], showing in these cases that stable mappings form a dense set in the space of smooth proper mappings.

The article Whitney [114] published in 1955 is a landmark, considered by many to be the cornerstone of the theory of singularities. The stable singularities of mappings of the plane into the plane are folds and cusps and any proper smooth mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be approximated by a stable mapping. Whitney conjectured that density of stable mappings would hold for any pair (n, p). However René Thom showed, in his 1959 lecture at Bonn, that this is not the case by given an example of a map $f : \mathbb{R}^9 \to \mathbb{R}^9$ that appears generically in a 1-parameter family of maps.

Thom conjectured that the topologically stable maps are always dense and gave an outline of the proof. The complete proof was given by John Mather, who from 1965 to 1975, solved almost completely the program drawn by René Thom for the problem of stability.

Mather found several characterizations of stability and proved that the set $\mathcal{S}^\infty(N, P)$ of stable mappings is dense in the set $C^\infty_{pr}(N, P)$ of smooth proper mappings, from the n-dimensional manifold N to the p-dimensional manifold P, if and only if (n, p) is in the *nice dimensions,* which he completely characterized in [63]. Based on Thom's ideas, he also proved in [65, 66] that the set of topologically stable mappings $\mathcal{S}^0(N, P)$ in $C^\infty_{pr}(N, P)$ is residual for all pairs (n, p).

The 1970s was blooming period for singularity theory. Along with Mather's work, René Thom's book on catastrophe theory [94] and Arnold's seminal classification of simple singularities of functions [3] also had a great impact. These works paved the intense development of the theory of the following decades. The deep understanding of stable mappings, versal unfoldings and finite determinacy transformed singularity theory into an organizing center for several areas of mathematics and sciences.

The common thread of these notes is the question of density of stable mappings in $C^\infty_{pr}(N, P)$. We outline the solutions of the various formulations of this problem: C^∞, C^0 and C^l, $1 \leq l < \infty$ stability. The remaining open problem in this setting is density of Lipschitz stable mappings. Recent progress in the solution of this problem appear in [75, 88].

We give an account of tools for the proofs of the main theorems including the notion of infinitesimal stability, the generalized Malgrange's theorem, Thom's transversality theorem, mappings of finite singularity type and finite determinacy of Mather's groups. Whitney and Thom's results on stratified sets and maps are fundamental pieces of the theory. For an account of these topics we refer to David Trotman's article in Volume 1 of this Handbook.

In these notes we concentrate on the discussion of real singularities. The infinitesimal methods discussed here also hold true for holomorphic mappings. For an account on Mather's theory of $\mathcal{A}$-equivalence and the description of the topology of stable perturbations of $\mathcal{A}$-finitely determined holomorphic germs the reader may consult the notes by David Mond and Juan José Nuño-Ballesteros in this Handbook [70].

Related topics to those discussed in these notes, as well as new developments of the theory, are given in the subsections *Notes* at the end of each section. The final section includes a discussion of open problems in the theory of singularities of smooth mappings.

1.2 Setting the Problem

Let $C^\infty(N, P) = \{f : N \to P, f \in C^\infty\}$ be the set of smooth mappings from N to P, where N and P are smooth manifolds of dimension n and p respectively. The topology on $C^\infty(N, P)$ is the C^∞-Whitney topology.

We review here the contributions of singularity theory to solve the following problem.

Problem 1.2.1 Find an open and dense set $\mathcal{S}$ in $C^\infty(N, P)$ and describe all singularities of mappings $f \in \mathcal{S}$.

The relevant equivalence is $\mathcal{A}$-equivalence.

Definition 1.2.2 Two smooth maps $f, g : N \to P$ are $\mathcal{A}$-*equivalent* if there exist C^∞ diffeomorphisms $h : N \to N$ and $k : P \to P$ such that the following diagram commutes

$$\begin{array}{ccc} N & \xrightarrow{f} & P \\ h \big\downarrow & \circlearrowleft & \big\downarrow k \\ N & \xrightarrow[g]{} & P \end{array}$$

Definition 1.2.3 The map $f : N \to P$ is *stable* ($\mathcal{A}$-*stable*) if there exists a neighborhood W of f in $C^\infty(N, P)$, such that $g \underset{\mathcal{A}}{\sim} f$ for every $g \in W$.

Replacing C^∞-diffeomorphisms by homeomorphisms, C^l-diffeomorphisms, $l > 0$ or bi-Lipschitz homeomorphisms in Definitions 1.2.2 and 1.2.3 we get respectively the definitions of C^0-$\mathcal{A}$, C^l-$\mathcal{A}$ ($l > 0$), *bi-Lipschitz-*$\mathcal{A}$ *equivalences* and of *topological stability*, C^l*-stability, or Lipschitz stability of maps* in $C^\infty(N, P)$.

Before starting the discussion of Problem 1.2.1, we review some notation and definitions.

The Whitney $C^\infty-$ topology in $C^\infty(N, P)$ was defined by John Mather in [57]. We review it here (more details can be found in the book of Golubitsky and Guillemin [40]).

For $x \in N$, $y \in P$ and for a non-negative integer k, we denote by $J^k(N, P)_{x,y}$ the set of k-jets of map-germs $(N, x) \to (P, y)$. When $N = \mathbb{R}^n$, $P = \mathbb{R}^p$, we denote $J^k(n, p)$ the set of polynomial mappings $f : \mathbb{R}^n \to \mathbb{R}^p$ of degree $\leq k$, such that $f(0) = 0$.

The set $J^k(N, P) = \bigcup_{x \in N, y \in P} J^k(N, P)_{x,y}$ is the k-jet space of mappings from N to P. The set $J^k(N, P)$ is a smooth manifold (theorem 2.7 in [40]). Moreover, it has the structure of a fibre bundle with basis $N \times P$.

Let U be an open set in $J^k(N, P)$ and

$$M(U) = \{f \in C^\infty(N, P) |\ j^k f(N) \subset U\}.$$

The family of sets $\{M(U)\}$ where U is an open set of $J^k(N, P)$ is a basis for a topology in $C^\infty(N, P)$ (note that $M(U) \cap M(V) = M(U \cap V)$). This topology is called the Whitney C^k-topology.

Denote by W_k the set of open subsets of $C^\infty(N, P)$ in the Whitney C^k-topology. The Whitney C^∞-topology is the topology whose basis is $W = \cup_{k=0}^\infty W_k$.

Given a metric d on $J^k(N, P)$, compatible with its topology and a nonnegative continuous function $\delta : N \to \mathbb{R}$ we can define a basic neighborhood of $f \in C^\infty(N, P)$ as follows

$$B_\delta(f) = \{g \in C^\infty(N, P) |\ d(j^k f(x), j^k g(x)) < \delta(x), \forall x \in N\}.$$

When N is compact, f_n converges to f in the Whitney C^k-topology if and only if $j^k f_n$ converges uniformly to $j^k f$. On noncompact manifolds f_n converges to f in the Whitney C^k-topology if and only there exists a compact $K \subset N$, such $j^k f_n$ converges to $j^k f$ uniformly in K, and there exists n_0 such that $f_n \equiv f$ in $N \setminus K$ for any $n \geq n_0$ (for details see the book by Golubitsky and Guillemin [40]).

Thus we can see that there is a great difference in the Whitney topology depending on whether or not the domain N is a compact manifold.

When N is not compact, the Whitney C^k-topology is a very fine topology, with many open sets. As a consequence, dense sets in $C^\infty(N, P)$ are very large sets, and theorems characterizing these sets in $C^\infty(N, P)$ are strong results.

1.2.1 The work of Hassler Whitney: from 1944 to 1958

The foundations of the theory were Whitney's work, in which he formulated the problem of classifying singularities that can not be eliminated by small perturbations, and completely succeeded in solving it for maps from $\mathbb{R}^n$ to $\mathbb{R}^p$ with $p \geq 2n - 1$ in Whitney [112] and from $\mathbb{R}^2$ to $\mathbb{R}^2$ in Whitney [114].

The article [114] published in 1955 is a magnificent work dedicated to maps from the plane into the plane. In the introduction to the article, Whitney presents a complete review of the existing results and future perspectives of the theory. We reproduce it here: *"Let f_0 be a mapping of an open set R in n-space E^n into m-space E^m. Let us consider, along with f_0, all the mappings f which are sufficiently good approximations to f_0. By the Weierstrass Approximation Theorem, there are such mappings f which are analytic; in fact, (see [110, Lemma1]) we may make f approximate to f_s throughout R arbitrarily well, and if f_0 is r-smooth, (i.e., has continuous partial derivatives of order $\leq r$), we may make corresponding derivatives of f approximate those of f_0.*

Supposing f is smooth, (i.e., 1-smooth), the Jacobian matrix $J(f)$ of f is defined (using fixed coordinate systems); we say the point $p \in R$ is a regular point *or* singular point *of f, according as $J(f)$ is of maximal rank (i.e., of rank* $\min(n, m)$*) or lesser rank. In general we cannot expect f to be free of singular points. A fundamental problem is to determine what sort of singularities any good approximations f to f_0 must have; what sort of sets they occupy, what f is like near such points, what topological properties hold with references to them, etc.*

Some special cases of this problem have been studied as follows:

(a) For $m = 1$, we have a real valued function in R. It was shown by M. Morse in Theorem 1.6 of [73], that f may be chosen so that the singular points (called critical points here) are isolated, the "Hessian" being non-zero at each." "Moreover, each critical point may be assigned a "type number"; topological relations among these were given by Morse [72].

(b) If $m \geq 2n$, we may find an f with no singular points; see (a) and (b) of Theorem 2 in [110].

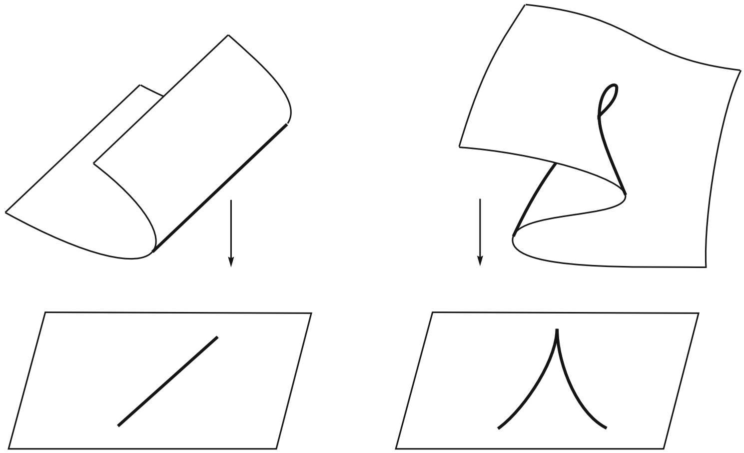


Fig. 1.1 Folds and cusps

(c) *If* $m = 2n - 1$, *we may obtain an* f *with singular points: see [112]. For each such point* $p \in R$, *coordinate systems* $(x_1, x_2, \ldots, x_n)$ *in* E^n *and* $(u_1, u_2, \ldots, u_m)$ *in* E^m *may be chosen, in which* f, *near* p, *has the form*

$$u_1 = x_1^2, \quad u_i = x_i, \quad u_{n+i-1} = x_1 x_i, \quad (i = 2, \ldots, n).$$

The singularities are studied from a topological point of view in [113].

(d) *Some beginnings have been made for the other pairs of values* $(n.m)$ *by N. Wolfsohn, [120], but no complete classification of the singularities exist in these cases. Thus the smallest pair of values for which the problem is open is the pair* $(2, 2)$, *i.e for mappings of the plane into the plane; it is this case that we treat here. In this case, there can be "folds" lying along curves and isolated "cusps" on the folds "* (Fig. 1.1).

We review Whitney's results in this section.

Let $f : U \to \mathbb{R}^2$ be a smooth mapping defined on the open set $U \subset \mathbb{R}^2$. With coordinates systems (x, y) in U and (u, v) in the target, the Jacobian of f is given by

$$J(f) = u_x v_y - u_y v_x.$$

A point $p \in U$ is *regular* or *singular* according as $J(f)(p) \neq 0$ or $J(f)(p) = 0$. A singular point (x_0, y_0) is *good* if the derivatives $\frac{\partial J(f)}{\partial x}(x_0, y_0)$ and $\frac{\partial J(f)}{\partial y}(x_0, y_0)$ do not vanish simultaneously. We say that f is *good* if every singular point of f is good. This condition implies that the set $S(f)$ of singular points of f is a regular curve. If f is good and p is a singular point, let $\phi : (-\epsilon, \epsilon) \to \mathbb{R}^2$ be a parametrization of the singular set $\mathcal{S}(f)$ in a neighborhood of $p \in \mathcal{S}(f)$ such that $\phi(0) = p$. Then, we define

(i) If $(f \circ \phi)'(0) \neq 0$, we say p is *fold point* of f.
(ii) $(f \circ \phi)'(0) = 0$ and $(f \circ \phi)''(0) \neq 0$, we say p is a *cusp point* of f.

These definitions are independent of the parametrization chosen for $S(f)$ in a neighborhood of p.

One can easily see that at a fold point, the restriction of f to its singular set is non singular, while a cusp point is a singular point of this restriction.

It follows from the definition that cusp points are isolated.

Definition 1.2.4 (Whitney [114], p. 379) Let f be a good map. We say that p is an *excellent point* of f if it is a regular, fold or cusp point of f. If each point $p \in U$ is excellent we say f is *excellent.*

Any smooth map can be approximated in the C^r-Whitney topology, $r \geq 3$, by an excellent map.

Theorem 1.2.5 (Whitney [114], Theorem 13A) *Let f_0 be a mapping from $U \subset \mathbb{R}^2$ to $\mathbb{R}^2$, where U is an open set in $\mathbb{R}^2$. Then arbitrarily near f_0 there is an excellent mapping f. If f_0 is r-smooth and ϵ is a positive continuous function in U, we make f an (r, ϵ)-approximation of f.*

Prior to Thom's transversality theorem [92], Whitney introduced the method of characterizing in the jet space the set of jets with degenerate singularities, the so called "bad set".

In addition, methods of producing generic C^r-perturbations of any given map were also introduced by him. The goal was to find sufficiently close perturbations that would avoid the bad set.

For polynomial maps from the plane into plane, the bad set are the polynomial maps admitting singularities more degenerate than folds and cusps.

Folds and cusps have simple normal forms.

Theorem 1.2.6 (Whitney [114], Theorems 15A and 15B)

1. *Let p be a fold point of the r-smooth mapping f of $\mathbb{R}^2$ into $\mathbb{R}^2$, with $r \geq 3$. Then $(r-3)$-smooth coordinate systems (x, y), (u, v) may be introduced about p and $f(p)$ respectively, in terms of which f takes the form*

$$u = x^2, v = y \tag{1.1}$$

2. *Let p be a cusp point of the r-smooth mapping f of $\mathbb{R}^2$ into $\mathbb{R}^2$, with $r \geq 12$. Then $(\frac{r}{2} - 5)$-smooth coordinate systems (x, y), (u, v) may be introduced about p and $f(p)$ respectively, in terms of which f takes the form*

$$u = xy - x^3, v = y \tag{1.2}$$

While the proof of (1) is not hard, Whitney's proof of the normal form in a neighborhood of a cusp point p follows by an ingenious sequence of changes of coordinates in the source and target. The tool is essentially the implicit function theorem.

Today, there are simpler proofs of this result, based on current tools of singularity theory: see for instance, Theorem 2.4, Chapter VI in Golubitsky and Guillemin's book [40] or Example 3.6 in Mond and Ballesteros [69].

The notion of stable mappings is due to Whitney. In order to characterize them, in addition to the local behavior of stable singularities, it is necessary to explain the behavior of multiple points. For maps from the plane into the plane the following holds.

Theorem 1.2.7 *Let $f : N^2 \to P^2$ be a smooth map, N and P 2-dimensional manifolds, N compact. Then f is C^∞- stable if and only if the following conditions hold.*

1. *f is excellent and hence $\mathcal{S}(f)$ is a regular curve, with at most a finite number of cusp points.*
2. *If p_1 and p_2 are singular points of f, $f(p_1) = f(p_2)$, then p_1 and p_2 are not cusp points. Moreover the fold lines intersect transversaly at $f(p_1) = f(p_2)$.*
3. *The restriction of f to $\mathcal{S}(f)$ has no triple points.*

Whitney formulated in [115] a general approach to defining a stratification in jet space and to define locally generic mappings as those whose r-jets were transversal to the strata of the stratification, for every $r \in \mathbb{N}^*$. The article contains an explicit description of generic singularities for pairs (n, p) such that $n, p \leq 5$.

He asked the question whether for any pair of dimensions (n, p), the stable maps could be characterized by transversality to a finite collection of submanifolds in jet space, so that one could apply Thom's transversality theorem to prove that a smooth map could be always approximated by stable maps.

However, in a course taught at the University of Bonn in 1959, René Thom showed with an example that it is not always possible to approximate a given map by C^∞ stable mappings (See Sect. 1.6, on Thom's example). In fact, in the notes *Singularity of differentiable mappings I,* written by Harold Levine [96], Thom sketched the proof that C^2-stable mappings do not form an open set in $C^\infty(N, P)$, when $n = p = 9$ and he formulated conjectures that promoted a great development in the theory in the following decades. In particular, Thom conjectured the density of topologically stable mappings, proved by John Mather in 1971. We discuss René Thom and John Mather's contributions in the next section.

1.2.2 René Thom and John Mather: from 1958 to 1970

We start by reviewing the subjects covered by R. Thom in his course at the University of Bonn. H. Levine's notes are divided into three chapters.

Chapter I, named "Jets" introduces the notion of jet spaces, the action of the group $\mathcal{A}$ in jet space and $\mathcal{A}^r$-invariant manifolds, denominated, in the notes, *critical varieties* in $J^r(n, p)$. The set $\mathcal{S}_k$ of 1−jets of corank k and its topological closure $\overline{\mathcal{S}}_k$ in $J^1(n, p)$ were defined.

In Chapter II, entitled "Singularities of mappings", Thom's transversality theorem was stated and proved. We remark however that the topology in the space of mappings in Thom's proof was the weakest topology making the mapping

$$j^r : C^\infty(N, P) \to C^\infty(N, J^r(N, P))$$
$$f \to j^r f$$

continuous. The topology in the second space was the compact open topology. The transversality theorem in [96] was stated as follows: For $s > r \geq 0$, let W be a codimension q, C^{s-r} submanifold of $J^r(N, P)$, $s - r > \dim N - q$. Then the set of mappings $f \in C^\infty(N, P)$, such that $j^r f \pitchfork W$ is dense in $C^\infty(N, P)$. The notion of second order singularities $S_{h,k}$ in $J^2(n, p)$ was introduced. These sets are connected to the singular points $S_h \subset J^1(n, p)$ by the relation: if $j^1 f \pitchfork S_k$, then $(j^2 f)^{-1}(S_{k,h}) = S_h(S_k(f))$. The general definition of the singular varieties $S_{k_1,\ldots,k_r} \subset J^r(N, P)$, introduced in [96] was better formulated by J.M. Boardman, in 1967, in [11]. Mather's account in [64] is the clearest.

Remark 1.2.8 In the following sections the sets S_k and $S_{k,h}$ will be denoted by Σ^k and $\Sigma^{k,h}$, respectively.

In Chapter III, "Equivalence and stability", Thom formulated the problem of characterizing singularities determined by their jet of some order. The name *finitely determined germs,* was later given by John Mather [58], who also gave necessary and sufficient conditions for finite determinacy. The notion of C^s-stable mappings and the example illustrating that C^2 stable mappings are not dense when $n = p = 9$ were discussed in that chapter.

The notion of homotopic stability was also introduced. A mapping $f : N \to P$ is *homotopically stable* if for every homotopy $F : N \times I \to P$ of f, there exist t_0 and homotopies of diffeomorphisms $\phi_t : N \to N$, $0 \leq t \leq t_0$, $\psi_t : P \to P$, of 1_N and 1_P such that $F_t = \psi_t \circ f \circ \phi_t$, $t < t_0$.

The program for the theory of stable mappings originated from the contributions of Whitney and Thom consisted of finding pairs of dimensions (n, p), for which there exists a set of mappings $\mathcal{S} \subset C^\infty(N^n, P^p)$, with the following properties:

1. $\mathcal{S}$ is a residual set in $C^\infty(N^n, P^p)$,
2. The maps $f \in \mathcal{S}$ are C^∞-stable,
3. There exists a finite number of polynomial normal forms such that every singular point of $f \in \mathcal{S}$ is equivalent to a normal form in this list.

In a memorable series of six articles from 1968 to 1971, John Mather found several characterizations of stability and provided theorems answering almost completely the question of density of stable maps.

The main results on density of stable mappings are stated below. The proofs are based on ideas of René Thom developed by Mather in the sequence of papers, on Stability of C^∞-mappings, I to VI, [56–58, 60–63, 65, 66]. In these notes we review the main steps leading to the proofs of Theorems A and B.

Let $C^\infty_{pr}(N, P)$ be the set of proper smooth mappings $f : N \to P$.

Theorem A (Density of Stable Mappings in the Nice Dimensions, Mather [61, 63]) *The set $\mathcal{S}^\infty(N, P)$ of proper stable mappings $f : N \to P$ is dense in $C^\infty_{pr}(N, P)$ if and only if (n, p) is in the nice dimensions.*

See Sect. 1.5 for the definition of the nice dimensions.

Theorem B (Density of Topologically Stable Mappings, Mather [65, 66]) *The set $\mathcal{S}^0(N, P)$ of proper topologically stable mappings is dense in $C^\infty_{pr}(N, P)$.*

The main tools in the proofs of Theorems A and B are the notion of infinitesimal stability, Thom's transversality theorem, the generalized Malgrange theorem, the notions of mappings of finite singularity type and contact equivalence, finite determinacy and unfoldings of Mather's groups, properties of Whitney stratified sets and Thom's isotopy theorems. Such notions and results form the framework of the theory of singularities of differentiable mappings.

We organize the contents of the next sections as follows.

In Sect. 1.3 we introduce infinitesimally stable and transverse stable mappings. The main goal of the section is to discuss Theorem 1.3.11 which establishes the equivalence between these notions and stable mappings.

Section 1.4 gives a short presentation of the infinitesimal machinery of singularity theory. We introduce the contact group $\mathcal{K}$ defined by Mather as a tool to classify stable singularities. For Mather's groups $\mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}$ and $\mathcal{K}$ we define $\mathcal{G}$-finitely determined germs and prove the Infinitesimal Criterion for $\mathcal{G}$-determinacy. We finish the section with a discussion of maps of finite singularity type (FST), a global version of $\mathcal{K}$-finitely determined germs, which plays a central role in the proof of Theorem B.

In Sect. 1.5 we define the nice dimensions and give an outline of the proof of Theorem A.

Section 1.6 gives a detailed presentation of Thom's example, illustrating that the set of stable maps in $C^\infty_{pr}(\mathbb{R}^9, \mathbb{R}^9)$ is not dense.

Section 1.7 is dedicated to the proof of density of topologically stable mappings $f : N \to P$, when N is compact manifold. The general lines of the proof are discussed, although the details are omitted.

Section 1.8 gives a systematic presentation of the topologically stable singularities in the boundary of the nice dimensions. Much of the section is well known to experts, however the organized presentation of the Thom-Mather stratification in jet space and the discussion of properties of topologically stable mappings in these dimensions do not appear in the literature.

The question of the density of Lipschitz stable mappings is still open. We report in Sect. 1.9 some recent results of Ruas and Trivedi [88] and Nguyen, Ruas and Trivedi [75] on this subject.

In Sect. 1.10, Damon's results relating $\mathcal{A}$-classification of map-germs and $\mathcal{K}_V$ classification of sections of the discriminant $V = \Delta(F)$ of a stable unfolding of f are reviewed and open problems are discussed.

1.3 Equivalent Notions of Stability

Mather defined infinitesimally stable mappings in [57], in order to introduce infinitesimal deformations of a map as a tool to study stability. The main goal in this section is to review Mather's result that, for proper mappings, stability and infinitesimal stability are equivalent notions.

First, we introduce some notation. Let $C^\infty(N) = \{\lambda : N \to \mathbb{R}\}$ be the ring of smooth functions defined on the smooth manifold N.

We denote by Θ_f the $C^\infty(N)$-module of vector fields along f, defined as follows

$$\Theta_f = \{\sigma : N \to TP| \ \pi_2 \circ \sigma = f\}$$

where $\pi_2 : TP \to P$ is the projection of the tangent bundle TP into P.

Let $f^*(TP)$ denote the pull-back bundle over N via f. Then the module Θ_f is the set of sections of this bundle.

Similarly,

$$\Theta_N = \{\xi : N \to TN| \ \pi_1 \circ \xi = I_N\}$$

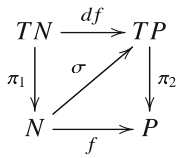
is the set of sections of the tangent bundle of N, and

$$\Theta_P = \{\eta : P \to TP| \ \pi_2 \circ \eta = I_P\},$$

the set of sections of the tangent bundle of P, where I_N and I_P are the identities.,

The set Θ_N is a $C^\infty(N)$-module, while Θ_P is a module over the ring $C^\infty(P)$.

We have the following diagram and homomorphisms



$$tf : \Theta_N \to \Theta_f$$
$$\xi \mapsto tf(\xi)$$

where $tf(\xi)(x) = df_x(\xi(x))$,

$$\begin{aligned} \omega f : \Theta_P &\to \Theta_f \\ \eta &\mapsto \omega f(\eta) = \eta \circ f \end{aligned}$$

The map tf is a homomorphism of $C^\infty(N)$-modules. The map $f : N \to P$ induces a ring homomorphism

$$\begin{aligned} f^* : C^\infty(P) &\to C^\infty(N) \\ \phi &\mapsto f^*(\phi) = \phi \circ f. \end{aligned}$$

We say that the map ωf is a homomorphism over $f^*(C^\infty(P))$ (or alternatively a $C^\infty(P)$-module homomorphism via f).

Notice that $\omega f(\eta_1 + \eta_2) = (\eta_1 + \eta_2) \circ f = \omega f(\eta_1) + \omega f(\eta_2)$ and $\omega f(\alpha\eta) = (\alpha \circ f)(\eta \circ f) = (\alpha \circ f)\omega f(\eta)$, for any $\alpha \in C^\infty(P)$ and any $\eta_1, \eta_2 \in \Theta_p$.

Definition 1.3.1 The map $f : N \to P$ is *infinitesimally stable* if for any $\sigma \in \Theta_f$, there are sections $\xi \in \Theta_N$ and $\eta \in \Theta_P$ such that $\sigma = tf(\xi) + \eta \circ f$. Equivalently, we can say that $\Theta_f = tf(\Theta_N) + \omega f(\Theta_P)$.

Example 1.3.2 If N is compact, $1 - 1$ immersions and submersions $f : N \to P$ are infinitesimally stable.

Infinitesimal stability has a local counterpart that we define now. Recall that two maps $f, g : N^n \to P^p$ define the same *germ* at $x = a$ if they agree in some neighborhood of a. The point $x = a$ is the *source* of the germ and $b = f(a)$ is its *target*. The analogues of the above notations for a germ $f : (N, a) \to (P, b)$ can be obtained replacing N by (N, a) and P by (P, b) in the previous notation. However to simplify notation, we take local coordinates such that $a = 0 \in \mathbb{R}^n$ and $f(a) = 0 \in \mathbb{R}^p$, denoting the germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. In this case, we use the usual notation:

$\mathcal{E}_n = \{\lambda : (\mathbb{R}^n, 0) \to \mathbb{R}\}$ is the local ring of C^∞ function germs at the origin. Its unique maximal ideal is $\mathcal{M}_n = \{\lambda \in \mathcal{E}_n |\ \lambda(0) = 0\}$.

$\mathcal{E}_n^p = \{f : (\mathbb{R}^n, 0) \to \mathbb{R}^p\}$ is a free $\mathcal{E}_n$-module of rank p, also denoted by $\mathcal{E}_{n,p}$.

The local version of the previous diagram is

$$\begin{array}{ccc} (T\mathbb{R}^n, 0) & \xrightarrow{df} & (T\mathbb{R}^p, 0) \\ \downarrow \pi_1 & \nearrow \sigma & \downarrow \pi_2 \\ (\mathbb{R}^n, 0) & \xrightarrow[f]{} & (\mathbb{R}^p, 0) \end{array}$$

The set

$$\Theta_f = \{\sigma : (\mathbb{R}^n, 0) \to (T\mathbb{R}^p, 0) |\ \pi_2 \circ \sigma = f\}$$

is the $\mathcal{E}_n$-module of rank p consisting of germs of vector fields along f. When f is the identity in $\mathbb{R}^n$, respectively in $\mathbb{R}^p$, we obtain

$$\Theta_n = \{\xi : (\mathbb{R}^n, 0) \to (T\mathbb{R}^n, 0) |\ \pi_1 \circ \xi = \mathrm{id}_{\mathbb{R}^n}\}$$

and

$$\Theta_p = \{\eta : (\mathbb{R}^p, 0) \to (T\mathbb{R}^p, 0) |\ \pi_2 \circ \eta = \mathrm{id}_{\mathbb{R}^p}\}$$

We now define the groups acting on $\mathcal{E}_n^p$.

Definition 1.3.3 Let

$$\mathcal{R} = \{h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0),\ \text{germs of } C^\infty - \text{diffeomorphisms in } (\mathbb{R}^n, 0)\},$$
$$\mathcal{L} = \{k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0),\ \text{germs of } C^\infty - \text{diffeomorphisms in } (\mathbb{R}^p, 0)\},$$

and $\mathcal{A} = \mathcal{R} \times \mathcal{L}$.

The actions of the groups $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{A}$ are as follows

$$\begin{array}{ccc} \mathcal{R} \times \mathcal{E}_n^p \to \mathcal{E}_n^p & \mathcal{L} \times \mathcal{E}_n^p \to \mathcal{E}_n^p & \mathcal{A} \times \mathcal{E}_n^p \to \mathcal{E}_n^p \\ (h, f) \mapsto f \circ h^{-1}, & (k, f) \mapsto k \circ f, & ((h, k), f) \mapsto k \circ f \circ h^{-1}. \end{array}$$

These notions extend to multigerms. Let $S = \{x_1, x_2, \dots x_s\}$ be a finite subset of $\mathbb{R}^n$.

Definition 1.3.4 A *multigerm* at $\mathcal{S} = \{x_1, \dots, x_s\}$ is the germ of a smooth map

$$f = \{f_1, f_2, \dots f_s\} : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y),\ \ f_i(x_i) = y,\ i = 1, \dots, s.$$

By a local change of coordinates at each $x_i \in \mathcal{S}$, we can take $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ and we let $\mathcal{M}_S \mathcal{E}_{n,S}^p$ be the vector space of these map-germs, and call f_i, $i = 1, \dots, s$ a *branch* of f.

The previous notations for monogerms extend naturally to multigerms. As before Θ_f and $\Theta_{n,S}$ are $\mathcal{E}_{n,S}$-modules. The map $tf : \Theta_{n,S} \to \Theta_f$ is an $\mathcal{E}_{n,S}$-module homomorphism defined by $tf(\xi)(x) = df_x(\xi(x))$.

The map-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ induces the ring homomorphism

$$\begin{aligned} f^* : \mathcal{E}_p &\to \mathcal{E}_{n,S} \\ \gamma &\mapsto f^*(\gamma) = \gamma \circ f, \end{aligned}$$

and we say that the map

$$\begin{aligned} \omega f : \Theta_p &\to \Theta_f. \\ \eta &\mapsto \omega f(\eta) = \eta \circ f \end{aligned}$$

is a *homomorphism over* $f^*(\mathcal{E}_p)$ (or alternatively, an $\mathcal{E}_p$-module homomorphism via f).

Definition 1.3.5 Two germs $f, g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ are $\mathcal{A}$-equivalent ($f \underset{\mathcal{A}}{\sim} g$) if there exist $h : (\mathbb{R}^n, S) \to (\mathbb{R}^n, S)$ and $k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that $g = k \circ f \circ h^{-1}$.

Definition 1.3.6 The germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ is *infinitesimally stable* if

$$tf(\Theta_{n,S}) + \omega f(\Theta_p) = \Theta_f$$

Remark 1.3.7 When we refer to an infinitesimally stable multigerm $f : (N, S) \to (P, y)$, we use the notation

$$tf(\Theta_{(N,S)}) + \omega f(\Theta_{(P,y)}) = \Theta_f.$$

Definition 1.3.8 For the groups $\mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A}$, and any multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$, we define the *tangent space* $T\mathcal{G}_f$ and the *extended tangent space* $T\mathcal{G}_e f$ as follows:

$$\begin{aligned} T\mathcal{R}f &= tf(\mathcal{M}_n\Theta_{n,\mathcal{S}}) & T\mathcal{R}_e f &= tf(\Theta_{n,\mathcal{S}}) \\ T\mathcal{L}f &= \omega f(\mathcal{M}_p\Theta_p) & T\mathcal{L}_e f &= \omega f(\Theta_p) \\ T\mathcal{A}f &= tf(\mathcal{M}_S\Theta_{n,\mathcal{S}}) + \omega f(\mathcal{M}_p\Theta_p) & T\mathcal{A}_e f &= tf(\Theta_{n,\mathcal{S}}) + \omega f(\Theta_p) \end{aligned}$$

One can give a heuristic justification for the definition of the tangent space for the groups $\mathcal{G}$ in the above definition. They can be seen as the set of "tangent vectors" at the origin, to "paths" f_t, such that $f_0 = f$, and f_t is contained in the $\mathcal{G}$−orbit of f. A careful calculation in the case $\mathcal{G} = \mathcal{A}$, beginning with $f_t = \psi_t \circ f_t \circ \phi_t$ and differentiating with respect to t, is done on pages 60–61 of the book of Mond and Nuño-Ballesteros [69].

For any group $\mathcal{G}$ acting on $\mathcal{E}_{n,S}$ the $\mathcal{G}$-codimension and the $\mathcal{G}_e$-codimension to the $\mathcal{G}$-orbit of f, are given by

$$\mathcal{G}\text{-cod } f = \dim_{\mathbb{R}} \frac{\mathcal{M}_S\Theta_f}{T\mathcal{G}f} \text{ and } \mathcal{G}_e - \text{cod } f = \dim_{\mathbb{R}} \frac{\Theta_f}{T\mathcal{G}_e f}.$$

Note that a map-germ $f \in \mathcal{E}_{n,S}$ is infinitesimally stable if and only if $\mathcal{A}_e$-cod $f = 0$.

Definition 1.3.9 A mapping $f : N \to P$ is *locally infinitesimally stable at* $\mathcal{S} = \{x_1, \ldots, x_s\} \subset N$ if the germ of f at $\mathcal{S}$ is infinitesimally stable.

The next theorem shows that for proper mappings infinitesimal stability is locally a condition of finite order. That is, if the equations can be solved locally to order $p = \dim P$, then they can be solved globally.

Theorem 1.3.10 (Theorem 1.6, Chapter 5, [40]) *Let $f : N \to P$ be a smooth and proper C^∞ mapping. Then f is infinitesimally stable if and only if for every $y \in P$ and every finite set $\mathcal{S} \subset f^{-1}(y)$, with no more than $(p+1)$ points, we have*

$$\Theta_f = tf(\Theta_{(N,\mathcal{S})}) + \omega f(\Theta_{(P,y)}) + \mathcal{M}_{\mathcal{S}}^{p+1}\Theta_f.$$

The proof of the necessity in Theorem 1.3.10 is obvious. To prove the sufficiency, the main tool is the generalized Malgrange Preparation Theorem proved by Mather in [57]. See Proposition 1.4.21 and Corollary 1.4.23. A complete proof of this theorem is given in Chapter 5, section 1 of [40].

Our main goal in this section is to discuss the following theorem.

Theorem 1.3.11 (Mather [62], Theorem 4.1) *The following conditions are equivalent in $C^\infty_{pr}(N, P)$ for a proper mapping $f : N \to P$.*

1. *f is stable,*
2. *f is infinitesimally stable.*
3. *f is transverse stable.*

We present the main steps of the proof of Theorem 1.3.11. Initially we discuss the notion of transverse stability.

1.3.1 Transverse Stability and the Proof of 2. ⇔ 3.

The idea of transverse stability consists in defining a stratification in jet space, such that the strata of this stratification are invariant by the action of the group $\mathcal{A}$ in jet space. A map is *transverse stable* if its k-jet is transversal to this stratification. To make this notion more precise, we introduce the r-fold k-jet bundle, following Mather [62].

Let N and P be manifolds. Let $N^{(r)} = \{(x_1, x_2, \ldots, x_r) \in N^r |\ x_i \neq x_j \text{ if } i \neq j\}$. Let $\pi_N : J^k(N, P) \to N$ denote the projection where $J^k(N, P)$ is the bundle of k-jets. We define ${}_rJ^k(N, P) = (\pi_N^r)^{-1}(N^{(r)})$ where $\pi_N^r : J^k(N, P)^r \to N^r$ is the projection.

It follows that

$${}_rJ^k(N, P) = \{(z_1, \ldots, z_r) \in J^k(N, P)^r, \text{ such that } \pi_N(z_i) \neq \pi_N(z_j),\ \text{ if } i \neq j\}.$$

The set ${}_rJ^k(N, P)$ is a fibre bundle over $N^{(r)} \times P^r$, and we call it the *r fold k-jet bundle* of mappings of N into P.

If $f : N \to P$ is a C^∞ mapping, we define

$${}_rj^k f : N^{(r)} \to {}_rJ^k(N, P)$$

by

$${}_rj^k f(x_1, \ldots, x_r) = (j^k f(x_1), \ldots, j^k f(x_r))$$

The action of the group $\mathcal{A}$ in ${}_rJ^k(N, P)$ is defined as follows. If $(h, h') \in \mathcal{A}$, $z = (z_1, \ldots, z_r) \in {}_rJ^k(N, P)$, $x_i = \pi_N z_i$, and $j^k f_i(x_i) = z_i$, then $(h, h')z = (z'_1, \ldots, z'_r)$ where $z'_i = j^k(h' \circ f_i \circ h^{-1})h(x_i)$. We denote by $\mathcal{A}^k$ the group of k-jets of elements in $\mathcal{A}$.

Proposition 1.3.12 (Mather [62], Proposition 1.4) *An $\mathcal{A}^k$ orbit W in ${}_rJ^k(N, P)$ is a submanifold.*

Definition 1.3.13 $f : N \to P$ is *transverse* stable if ${}_rj^k f : N^{(r)} \to {}_rJ^k(N, P)$ is transverse to every $\mathcal{A}^k$ orbit W in ${}_rJ^k(N, P)$.

An important remark is that in order to understand the local structure of the orbits in ${}_rJ^k(N, P)$ it is sufficient to understand the structure of the orbits in $\pi_P^r(\Delta_r)$, where $\Delta_r \subset P^r$ is the diagonal (see Mather [62] for details). In other words, it suffices to take jets with sources $\mathcal{S} = \{x_1, \ldots, x_r\}$ for which $f(x_1) = \cdots = f(x_r)$.

The next proposition gives a characterization of transversality of ${}_rj^k f$ to W; it is an important step in the proof of Theorem 1.3.11.

Proposition 1.3.14 (Mather [62], Proposition 2.6) *${}_rj^k f$ is transverse to W at x if and only if,*

$$tf(\Theta_{(N,\mathcal{S})}) + \omega f(\Theta_{(P,y)}) + \mathcal{M}_{\mathcal{S}}^{k+1}\Theta_f = \Theta_f,$$

where $y = f(x)$, $\mathcal{S} = f^{-1}(y) = \{x_1, \ldots, x_r\}$.

From Proposition 1.3.14 and Theorem 1.3.10 we obtain the proof of 2. $\iff$ 3. in Theorem 1.3.11.

That 1. implies 3. in Theorem 1.3.11 follows from a general fact, and it is not hard to show.

In fact, let $f : N \to P$ be a stable mapping. It follows from the transversality theorem that f can be well approximated by a mapping $g : N \to P$, such that g is transverse stable and $g \underset{\mathcal{A}}{\sim} f$. That is, there is $(h, k) \in \mathcal{A}$ such that $g = k \circ f \circ h^{-1}$. Now, transversality is preserved by $\mathcal{A}$-equivalence, hence f is transverse stable as well, as we wanted to show.

We have proved 1. $\Rightarrow$ 2. $\Leftrightarrow$ 3..

Mather proved in [60], Theorem 1 that if f is proper and infinitesimally stable then it is stable, that is 2. $\Rightarrow$ 1..

His proof follows from the following result.

Theorem 1.3.15 (Mather [60], Theorem 2) *If f is proper and infinitesimally stable, then there exists a neighborhood U of f in $C^\infty(N, P)$ and continuous mappings $H_1 : U \to Diff^\infty(N)$ and $H_2 : U \to Diff^\infty(P)$ such that $H_1(f) = 1_N$, $H_2(f) = 1_P$ and $g = H_2(g) \circ f \circ H_1(g)$, for $g \in U$.*

Du Plessis and Wall [82] introduced the notion of *W-strongly* stable mappings as stable mappings $F : N \to P$ admitting a neighborhood U in $C^\infty(N, P)$ satisfying the conditions stated in Theorem 1.3.15.

The main difficult to prove that stable mappings are W-strongly stable is that in the Whitney C^∞ topology, the composition of mappings is not continuous. However continuity holds when one restricts to proper mappings. The strong stability of non proper functions was recently discussed by Kenta Hayano in [42].

It follows that the result 2. $\Rightarrow$ 1. is an easy consequence of Theorem 1.3.15.

The hypothesis that f is proper cannot be omitted, as we see in the following example.

Example 1.3.16 ([60], pp. 267) Let $N = (-1, 1) \cup (1, 2)$, $P = (-1, 1)$, and

$$f|_{(-1,1)} : (-1, 1) \to (-1, 1) \qquad f|_{(1,2)} : (1, 2) \to (-1, 1)$$
$$x \mapsto x^2 \qquad\qquad x \mapsto 2 - x$$

We can verify that f is infinitesimally stable, as the restrictions to $(-1, 1)$ and $(1, 2)$ are.

However, f is not stable since it has the following non-stable property: for any $a \in P$, $f^{-1}(a)$ contains either 0, 1 or 3 points.

The reader can find in [62] the discussion of which implications in Theorem 1.3.11 depend on the hypothesis that f is proper.

In the next example we illustrate the role of the Whitney C^∞-topology in the characterization of stable mappings.

Example 1.3.17 The cusp map

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto F(x, y) = (x, y^3 + xy)$$

is a stable mapping when the topology in $C^\infty_{pr}(\mathbb{R}^2, \mathbb{R}^2)$ is the Whitney topology. This follows from Whitney's theorem as we discussed in Sect. 1.2.1. We can also apply Mather's result: the map F is proper and infinitesimally stable, hence it is stable

Let $F_n(x, y) = (x, y^3 + xy + \frac{x^2}{n} y)$. The singular set of F_n is the set Σ_n defined by $3y^2 + x + \frac{x^2}{n} = 0$. For each n, F_n has two cusp points: $(0, 0)$ and $(-n, 0)$.

We can easily see that $F_n \to F$ in $C^\infty_{pr}(\mathbb{R}^2, \mathbb{R}^2)$ with the topology of uniform convergence on compact sets. Hence F is not stable when one considers this topology in $C^\infty_{pr}(\mathbb{R}^2, \mathbb{R}^2)$.

1.3.2 Notes

The definitions and properties of infinitesimally stable mappings also hold for real and complex analytic germs. However, care is necessary to characterize stable maps $f : N \to P$, when f is a holomorphic map between complex manifolds N and P. In fact,Thom's transversality theorem does not hold in general in this case.

See discussion by F. Forstnerič, [34] and examples given by S. Kaliman and M. Zaĭdenberg in [46]. In a recent paper, S. Trivedi [99] proves that the set of maps between Stein manifolds and Oka manifolds, transverse to a countable collection of submanifolds in the target is dense in the space of holomorphic maps with the weak topology. The results hold, in particular, for holomorphic maps $f : \mathbb{C}^n \to \mathbb{C}^p$, as the complex spaces satisfy the hypothesis of the theorem.

A related problem is the characterization of topologically stable polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^p$. M. Farnick, Z. Jeloneck and M.A. S. Ruas [32], characterize topologically stable polynomial mappings $F : \mathbb{C}^2 \to \mathbb{C}^2$ in the space $\Omega_{\mathbb{C}^2}(d_1, d_2)$ of polynomial mappings of degree bounded by (d_1, d_2). Locally stable singularities are folds and cusps, but the behavior of generic polynomial mappings at infinity imposes new restrictions. The number of cusps of a topological stable $F \in \Omega_{\mathbb{C}^2}(d_1, d_2)$ is given by $c(F) = d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7$. In particular, when $d_1 = 1$ and $d_2 = 3$, $c(F) = 2$.

1.4 Finite Determinacy of Mather's Groups

Mather's groups are the groups $\mathcal{G} = \mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{C}$.

The contact group $\mathcal{K}$, defined by Mather in [58] plays a fundamental role in the classification of stable singularities. In Sects. 1.4.1 and 1.4.3 we define the group $\mathcal{K}$, discuss properties of $\mathcal{K}$-equivalence and their role in the study of stable mappings.

The problem of classification of stable singularities motivated the introduction of the notion of $\mathcal{G}$-finitely determined germs [58]. For the groups $\mathcal{G} = \mathcal{R}$ or $\mathcal{K}$, finite determinacy was studied by J. Tougeron in [97] and chapter II of [98]. When $\mathcal{G} = \mathcal{A}$ or $\mathcal{L}$, the first results are due to Mather's in [58]. Infinitesimal criteria of finite determinacy for $\mathcal{G} = \mathcal{A}$ and $\mathcal{L}$ depend on the Preparation Theorem. We discuss the infinitesimal criterion for Mather's group in Sect. 1.4.2. In Sect. 1.4.4 we introduce the basic properties of maps of finite singularity type.

1.4.1 The Contact Group

Definition 1.4.1 The *contact group* $\mathcal{K}$ is the set of pairs of germs of diffeomorphisms (h, H), where $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ such that $\pi_1 \circ H = h$, $(\pi_2 \circ H)(x, 0) = 0$ where π_1 and π_2 are the projections into $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively.

Notice that $H(x, y) = (h(x), H_2(x, y))$, $H_2(x, 0) = 0$.

The set of pairs $(h, H) \in \mathcal{K}$, such that h is the identity $I_{\mathbb{R}^n}$ form a subgroup of $\mathcal{K}$, usually denoted by $\mathcal{C}$.

Definition 1.4.2 Let $f, g \in \mathcal{E}_n^p$. We say that f and g are contact equivalent, $f \underset{\mathcal{K}}{\sim} g$, if there is a pair $(h, H) \in \mathcal{K}$ such that $H(x, f(x)) = (h(x), g(h(x))$.

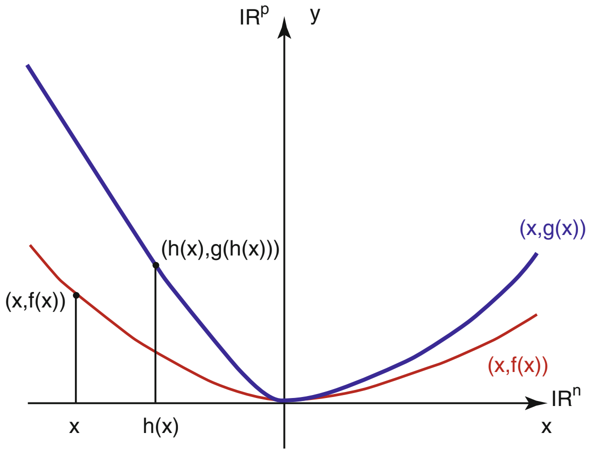


Fig. 1.2 Contact equivalence

Remark 1.4.3 Notice that if $f \underset{\mathcal{K}}{\sim} g$, then the diffeomorphism $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ sends graph(f) into graph(g), leaving $\mathbb{R}^n \times \{0\}$ invariant (see Fig. 1.2). This geometric viewpoint of contact equivalence was extended by Montaldi [71] as follows: two pairs of germs of submanifolds of $\mathbb{R}^m$ have the same contact type if there is a germ of diffeomorphism of $\mathbb{R}^m$ taking one pair to the other. Moreover, he proved in [71], that the contact type of a pair of germs of manifolds is completely characterized by the $\mathcal{K}$-equivalence class of a convenient map. This result is one the fundamental pieces of the applications of singularity theory to differential geometry (see Bruce and Giblin [13] and Izumiya, Romero-Fuster, Ruas and Tari, [45]).

The *tangent space* and the *extended tangent space* of $\mathcal{K}$-equivalence are, respectively

$$T\mathcal{K}f = tf(\mathcal{M}_n\Theta_n) + f^*(\mathcal{M}_p)\Theta_f$$
$$T\mathcal{K}_e f = tf(\Theta_n) + f^*(\mathcal{M}_p)\Theta_f$$

We also define $\mathcal{K}$-cod $f = \dim_{\mathbb{R}} \frac{\mathcal{M}_n\Theta_f}{T\mathcal{K}f}$ and $\mathcal{K}_e$-cod $f = \dim_{\mathbb{R}} \frac{\Theta_f}{T\mathcal{K}_e f}$.

The following result was first proved by Mather in [61].

Proposition 1.4.4 (Gibson [38], Proposition 2.2, Mond and Nuño-Ballesteros [69], Section 4.4)

The following statements are equivalent.

(1) Two map-germs $f, g \in \mathcal{E}_n^p$ are $\mathcal{K}$-equivalent.
(2) There exists a germ of diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that

$$h^* f(\mathcal{M}_p)\mathcal{E}_n = g^*(\mathcal{M}_p)\mathcal{E}_n.$$

The local algebra we introduce now is an useful invariant of $\mathcal{K}$-equivalence. For a given map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ we define the *local algebra* of f as

$$Q(f) = \frac{\mathcal{E}_n}{f^*(\mathcal{M}_p)\mathcal{E}_n}.$$

It follows from the previous proposition that the isomorphism class of $Q(f)$ is a $\mathcal{K}$- invariant. Furthermore, it is a complete invariant of $\mathcal{K}$-equivalence for germs f with finite $\mathcal{K}$-codimension. More precisely, we have

Theorem 1.4.5 *If f and g are map-germs with finite $\mathcal{K}$-codimension it follows that*

$$f \underset{\mathcal{K}}{\sim} g \textit{ if and only if the local algebras } Q(f) \textit{ and } Q(g) \textit{ are isomorphic.}$$

Remark 1.4.6 For complex analytic germs the hypothesis of $\mathcal{K}-$determinacy in Theorem 1.4.5 is not needed.

Example 1.4.7 Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a germ of rank r. Then, up to $\mathcal{A}$-equivalence, we can take F in the normal form $F(x, y) = (x, f(x, y))$, $x \in \mathbb{R}^r$, $y \in \mathbb{R}^{n-r}$, with $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{p-r}, 0)$ and $j^1 f(0, 0) \equiv 0$. Let $f_0 : (\mathbb{R}^{n-r}, 0) \to (\mathbb{R}^{p-r}, 0)$ be the rank zero germ $f_0(y) = f(0, y)$. Then $Q(F) = Q(f_0)$.

If $\mathcal{K}$-cod $f_0 < \infty$ and $Q(F) \cong Q(f_0)$ it follows that F is $\mathcal{K}$-equivalent to the suspension $F_0(x, y) = (x, f_0(y))$ of f_0.

As we shall see in the next section, germs $f \in \mathcal{E}_n^p$ of finite $\mathcal{K}$-codimension are finitely $\mathcal{K}$-determined, and in this case $\mathcal{K}(f) = \mathcal{K}(z)$, where $z = j^k f(0)$ for some k.

Now, for each positive integer k, we set

$$Q_k(f) = \frac{\mathcal{E}_n}{f^*(\mathcal{M}_p)\mathcal{E}_n + \mathcal{M}_n^{k+1}}.$$

$Q_k(f)$ is the local algebra of $z = j^k f(0)$. We can also write $Q_k(f) = Q(z)$.

It is not hard to show that $z \underset{\mathcal{K}^k}{\sim} z'$ if and only if $Q_k(z)$ and $Q_k(z')$ are isomorphic.

This definition can be extended to k-jets of a multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ $S = \{x_1, x_2, \ldots, x_s\}$. By a *contact class* in $J^k(N, P)$ we mean an equivalence class of ${}_sJ^k(N, P)$ under the relation of $\mathcal{K}^k$-equivalence.

1.4.2 *Finitely Determined Germs*

Let $\mathcal{G}$ be a group acting in the space of germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. We say that f is finitely $\mathcal{G}$*-determined* if there exists a positive integer k such that for all $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $j^k g(0) = j^k f(0)$, it follows that $f \underset{\mathcal{G}}{\sim} g$. We say that f

is *$\mathcal{G}$-finitely determined* if f is k-determined for some k. The denomination *$\mathcal{G}$-finite* germs is also widely used.

Finite determinacy has been an important subject in singularity theory for many decades and the bibliography in this topic is extensive.

With regard to results on necessary and sufficient conditions of finite determinacy and estimates of the order of determinacy we refer to Mather [58], Gaffney [36, 37], du Plessis [79], Damon [24] and Du Plessis, Bruce and Wall [14]. The survey article by Terry Wall [108] is a complete account of the theory of finite determinacy for Mather's groups $\mathcal{G} = \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{K}$ and $\mathcal{C}$ until 1981. See also the clear presentation (with examples) in Chapter 6 of the book of Mond and Nuño-Ballesteros [69].

An important advance appeared in [24] in which J. Damon defined the *geometric subgroups of $\mathcal{K}$*, a large class of subgroups for which the theory of finite determinacy can be formulated as for Mather's group.

The following theorem, known as *infinitesimal criterion* gives necessary and sufficient conditions for finite determinacy. The original result is due to Mather [58]. We give here an improved version due to Gaffney [37] and du Plessis [79]. The statement and proof of Theorem 1.4.8 are slight modifications of T. Wall [108, Theorem 1.2]. The reader can also compare with Theorem 2.2.12 of the article of Mond and Nuño-Ballesteros in this Handbook [70].

Theorem 1.4.8 *For each $f \in \mathcal{E}_n^p$, $\mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}, \mathcal{K}$ the following conditions are equivalent*

(1) f *is finitely $\mathcal{G}$-determined,*
(2) *for some r,* $T\mathcal{G}f \supset \mathcal{M}_n^r \Theta_f$,
(3) $\mathcal{G}$-cod $f < \infty$,
(4) $\mathcal{G}_e$-cod $f < \infty$.

More precisely, if we set $\epsilon = 1$ for $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$ and $\epsilon = 2$ for $\mathcal{G} = \mathcal{L}, \mathcal{A}$,

(i) *If f is k-$\mathcal{G}$-determined then $T\mathcal{G}f \supset \mathcal{M}_n^{k+1}\Theta_f$,*
(ii) *If $T\mathcal{G}f \supset \mathcal{M}_n^{k+1}\Theta_f$, then f is $(\epsilon k + 1)$-$\mathcal{G}$-determined.*
(iii) *If $T\mathcal{G}f + \mathcal{M}_n^{\epsilon k+2}\Theta_f \supset \mathcal{M}_n^{k+1}\Theta_f$, then $T\mathcal{G}f \supset \mathcal{M}_n^{k+1}\Theta_f$.*

This section is mainly devoted to describe this result. Although the theory applies to multigerms, for simplicity we restrict our discussion to monogerms $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$.

The successful approach to finite determinacy was inspired by the action of a Lie group on finite dimensional manifolds. The following lemma is due to Mather.

Lemma 1.4.9 (Mather [61], Lemma 3.1) *Let G be a Lie group, M a C^∞ manifold and $\alpha : G \times M \to M$ a C^∞ action. Let V be a connected C^∞-submanifold of M. Then V is contained in an orbit of α if and only if*

(a) *For all $v \in V$, $T_vG \cdot v \supseteq T_vV$, and*
(b) $\dim T_v(G \cdot v)$ *is the same for all $v \in V$.*

Our groups are not Lie groups, and our function spaces are not Banach manifolds. But, the solution to the problem of finding sufficient conditions for a germ $f \in \mathcal{E}_n^p$ to be finitely determined, consists in reducing our infinitesimal approach to jet spaces.

Suppose f is k-G-determined. Then, given $g \in \mathcal{E}_n^p$, $j^k g(0) = j^k f(0)$, the one-parameter family

$$\bar{f} : (\mathbb{R}^n \times \mathbb{R}, 0 \times \mathbb{R}) \to (\mathbb{R}^p \times \mathbb{R}, 0)$$
$$(x,t) \mapsto \bar{f}(x,t) = (1-t)f(x) + tg(x)$$

has a constant k-jet $j^k \bar{f}_t(0) = j^k f(0) + tj^k(g-f)(0) = j^k f(0)$.

We will identify $\bar{f}$ with a "line" L_t in $\mathcal{E}_n^p$. Our problem is to show that L_t is contained in a unique orbit.

A sufficient condition is to find a 1-parameter family h_t of elements in $\mathcal{G}$ such that $h_0 = 1 \in \mathcal{G}$, $h_t(0) = 0$, $h_t \cdot f_t = f$, for any $t \in \mathbb{R}$. These conditions say that the family $\bar{f}$ is *$\mathcal{G}$-trivial*. As in the case of stable singularities, the next step is to search for an infinitesimal condition, giving an equivalent characterization of triviality in terms of vector fields.

This step, in principle, is not hard: the equation $h_t \cdot f_t = f$ implies that $\frac{\partial}{\partial t}(h_t \cdot f_t) = 0$ leading to the desired infinitesimal condition. The converse follows from integration of vector fields.

For any group $\mathcal{G}$ acting on $\mathcal{E}_n^p$, we call this result "the Thom-Levine lemma." We now specialize to $\mathcal{G} = \mathcal{A}$, as this case includes all difficulties of the proof of the infinitesimal criterion.

Definition 1.4.10 A 1-parameter family $\bar{f} : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p, 0)$, $\bar{f}(x,0) = f(x)$ is $\mathcal{A}$-trivial if there is a pair (h,k) of 1-parameter families of germs of diffeomorphisms

$$h : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0) \qquad k : (\mathbb{R}^p \times \mathbb{R}, 0) \to (\mathbb{R}^p, 0)$$
$$(x,t) \mapsto h(x,t) \qquad (y,t) \mapsto k(y,t)$$

such that $h(x,0) = x, k(y,0) = y,\ h_t(0) = 0, k_t(0) = 0$ and

$$k_t \circ f_t \circ h_t = f.$$

Remark 1.4.11 We also use the notation $F(x,t) = (\bar{f}(x,t),t)$, $H(x,t) = (h(x,t),t)$ and $K(y,t) = (k(y,t),t)$ for the corresponding 1-parameter unfoldings. In this notation F is $\mathcal{A}$-trivial if $K \circ F \circ H = f \times \mathrm{Id}_{\mathbb{R}}$. We denote by $\partial \cdot F$ the vector field in $(\mathbb{R}^n \times \mathbb{R}, 0)$ with zero component in the $\frac{\partial}{\partial t}$ direction, that is $dF(\frac{\partial}{\partial t}) = (\partial \cdot F, 1)$.

The next result is known as the Thom-Levine lemma (see [58, 69, 79]).

Proposition 1.4.12 *Let* $f \in \mathcal{E}_n^p$ *and* $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0)$, $F(x,t) = (\bar{f}(x,t),t)$, $\bar{f}(0,t) = 0$, $\bar{f}(x,0) = f(x)$, *the germ at* 0 *of a* 1*-parameter* *unfolding of F. Then F is $\mathcal{A}$-trivial if and only there exist vector fields* $V : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ *with* $V(x,t) = v(x,t) + \frac{\partial}{\partial t},\ v(x,t) = \sum_{i=1}^n v_i(x,t)\frac{\partial}{\partial x_i}$, $v_i(0,t) = 0$ *for* $i = 1, \ldots, n$ *and* $W : (\mathbb{R}^p \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0)$ *with* $W(y,t) = w(y,t) + \frac{\partial}{\partial t},\ w(y,t) = \sum_{j=1}^p w_j(y,t)\frac{\partial}{\partial y_j},\ w_j(0,t) = 0$ *for* $j = 1, \ldots, p$. *such that*

$$\partial \cdot F(x,t) = \sum_{i=1}^n \frac{\partial \bar{f}}{\partial x_i}(x,t) \cdot v_i(x,t) + w \circ F(x,t). \tag{1.3}$$

Proof We give here an idea of the proof. The reader may consult, for instance, Mather [58, p. 144], du Plessis [79, p. 174], or Mond and Nuño-Ballesteros [69, p. 37] for a complete proof.

If F is a trivial unfolding of f, $K \circ F \circ H = f \times 1_{\mathbb{R}}$ and then $\partial \cdot (K \circ F \circ H) = 0$ and we apply the chain rule to get (1.3).

Conversely, if condition (1.3) holds, we consider the systems of differential equations in $(\mathbb{R}^n \times \mathbb{R}, 0)$ and $(\mathbb{R}^p \times \mathbb{R}, 0)$, respectively:

$$\begin{cases} \dot{x} = v(x,t) \\ v(0,t) = 0 \end{cases} \qquad \begin{cases} \dot{y} = w(y,t) \\ w(0,t) = 0 \end{cases} \tag{1.4}$$

We can integrate these vector fields to obtain 1−parameter families h_t and k_t of diffeomorphisms of $(\mathbb{R}^n \times \mathbb{R}, 0)$ and $(\mathbb{R}^p \times \mathbb{R}, 0)$, respectively, such that $h_0(x) = x,\ h_t(0) = 0;\ k_0(y) = y,\ k_t(0) = 0$ and $k_t \circ \bar{f}_t \circ h_t = f$.

□

Condition (1.3) in Proposition 1.4.12 admits an useful algebraic formulation. First, we introduce some notation.

Given the 1-parameter unfolding $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0),\ F(x,t) = (\bar{f}(x,t), t)$ with $\bar{f}(x,0) = f(x)$, as before, Θ_F denotes the $\mathcal{E}_{n+1}$ module of vector fields along F. However, here it will be more convenient to consider the submodule of Θ_F defined as:

$$\Psi_F = \{\sigma \in \Theta_F \,|\, \text{the } \mathbb{R}\text{-component of } \sigma \text{ is zero}\}.$$

Similarly, Ψ_{n+1} and Ψ_{p+1} denote vector fields in $(\mathbb{R}^n \times \mathbb{R}, 0)$ and $(\mathbb{R}^p \times \mathbb{R}, 0)$ respectively, with zero $\mathbb{R}$-components.

The restrictions of the homomorphisms tF and ωF give respectively the $\mathcal{E}_{n+1}$-homomorphism $tF : \Psi_{n+1} \to \Psi_F$ and the $\mathcal{E}_{p+1}$-homomorphism via F^*, $\omega F : \Psi_{p+1} \to \Psi_F$.

With this notation, we can see that (1.3) holds if and only if

$$\partial \cdot F \in tF(\mathcal{M}_n \Psi_{n+1}) + \omega F(\mathcal{M}_p \Psi_{p+1}) \tag{1.5}$$

holds.

We call $T\mathcal{A}_{un}(F) = tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1})$, the *$\mathcal{A}$-tangent space* of the unfolding F. Similarly $T\mathcal{K}_{un}(F) = tF(\mathcal{M}_n\Psi_{n+1}) + F^*(\mathcal{M}_{p+1})\Psi_{p+1}$ is the *$\mathcal{K}-$tangent space* of F.

We now turn to the algebraic tools we need in the proof of Theorem 1.4.8.

In the cases $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$ the proof of the infinitesimal criterion of $\mathcal{G}$-determinacy will follow from the following elementary result.

Lemma 1.4.13 (Nakayama's Lemma) *Let R be a commutative ring, M an ideal such that for $x \in M$, $(1 + x)$ is invertible. Let C be a finitely generated R-module, A a submodule, then*

(i) if $A + M \cdot C = C$, then $A = C$,
(ii) if R is a k-algebra, and $\dim_k(\frac{C}{A+M^{d+1}C}) \leq d$ then $M^d \cdot C \subseteq A$.

An equivalent formulation of condition (i) in Lemma 1.4.13 is the following

(i') If $MC = C$, then $C = 0$.

When $\mathcal{G} = \mathcal{L}$ or $\mathcal{A}$, we need a fairly deep result, the generalized Malgrange preparation theorem (see Golubitsky and Guillemin [40], Martinet [54, 55], Wall [108]).

Theorem 1.4.14 (Preparation Theorem) *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a C^∞ map-germ, E a finitely generated $\mathcal{E}_n$-module. If $\dim_{\mathbb{R}}(\frac{E}{f^*(\mathcal{M}_p)\cdot E}) < \infty$, then E is finitely generated as $\mathcal{E}_p$-module (via f).*

The next proposition is a consequence of the Preparation theorem. It is an useful tool to study $\mathcal{A}$-finite determinacy.

Proposition 1.4.15 (Bruce, du Plessis and Wall [14], Lemma 2.6) *Let C be a finitely generated $\mathcal{E}_n$-module, $B \subset C$ a finitely generated $\mathcal{E}_n$-submodule, $A \subset f^*(\mathcal{M}_p)C$ a finitely generated $\mathcal{E}_p$-submodule (via f), and M a proper, finitely generated ideal in $\mathcal{E}_n$. If*

$$MC \subset A + B + M(f^*(\mathcal{M}_p) + M)C$$

then $MC \subset A + B$.

We are now ready to prove Theorem 1.4.8.

Proof of Theorem 1.4.8 First we notice that (i) and (ii) give respectively the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). The implication (2) $\Rightarrow$ (3) is trivial since $\mathcal{M}_n^k\Theta_f$ has finite codimension.

It is easy to prove the equivalence between (3) and (4). The implication (3) $\Rightarrow$ (2) will follow from (iii), as we now explain.

For any $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}, \mathcal{L}, \mathcal{A}$ let

$$c_k = \dim_{\mathbb{K}} \frac{\mathcal{M}_n\Theta_f}{T\mathcal{G}f + \mathcal{M}_n^k\Theta_f}, \ k \geq 1.$$

Since $\mathcal{G}$-cod $f < \infty$, the sequence

$$0 = c_1 \leq c_2 \leq \cdots \leq \mathcal{G}\text{-cod } f$$

is finite.

Then, there exists s such that $c_k = c_s$ for all $k \geq s+1$. It follows that $T\mathcal{G}f + \mathcal{M}_n^s\Theta_f = T\mathcal{G}f + \mathcal{M}_n^k\Theta_f$ for all $k \geq s+1$. In particular $\mathcal{M}_n^s\Theta_f \subseteq T\mathcal{G}f + \mathcal{M}_n^k\Theta_f$ for all $k \geq s+1$. Taking $k = s+1$, when $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}$ and $k = 2s$, when $\mathcal{G} = \mathcal{A}, \mathcal{L}$, we obtain the statement in (iii) from which the result follows.

It suffices to prove (i), (ii) and (iii). For a clearer presentation, we first prove (iii).

If $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}$, the result follows easily by Nakayama's Lemma. If $\mathcal{G} = \mathcal{A}$ (the argument for $\mathcal{G} = \mathcal{L}$ is similar) we apply Proposition 1.4.15 taking $C = \Theta_f$, $M = \mathcal{M}_n^{k+1}$, $B = tf(\mathcal{M}_n\Theta_n)$ and $A = \omega f(\mathcal{M}_p\Theta_p)$.

We leave the details as an exercise to the reader.

(i) Necessary condition for finite determinacy.

This is not hard. A map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is k-$\mathcal{G}$-determined if $\mathcal{G}f$ contains all germs $g \in \mathcal{E}_n^p$, such that $j^k g(0) = j^k f(0)$. Let us denote this set by $\mathtt{W}$.

Let

$$\pi^l : \mathcal{E}_n^p \to J^l(n, p)$$
$$g \to j^l g(0).$$

As $\mathcal{G}f \supset \mathtt{W}$, then $\pi^l(\mathcal{G}f) \supset \pi^l(\mathtt{W})$. Thus we also get that

$$\text{the tangent space of } \pi^l(\mathcal{G}f) \supset \text{the tangent space of } \pi^l(\mathtt{W}). \tag{1.6}$$

Notice that for all $l > k$, the set $\pi^l(\mathtt{W})$ is the affine subspace of $J^l(n, p)$ consisting of all l-jets whose k-jet is $j^k f(0)$. Hence we can rewrite (1.6) as

$$T\mathcal{G}f + \mathcal{M}_n^{l+1}\Theta_f \supset \mathcal{M}_n^{k+1}\Theta_f,\ l > k.$$

The result now follows from (iii) taking $l = k+1$ for $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$ and $l = 2k+1$ when $\mathcal{G} = \mathcal{A}$ or $\mathcal{L}$.

(ii) Sufficient condition for finite determinacy.

Let $f, g \in \mathcal{E}_n^p$, $j^{\epsilon k+1} f(0) = j^{\epsilon k+1} g(0)$, $\epsilon = 1$ or 2, $F(x,t) = (\bar{f}(x,t), t)$, where $\bar{f}(x,t) = (1-t)f(x) + tg(x)$, $t \in [0, 1]$.

(I) $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$.

In these cases the hypothesis

$$T\mathcal{G}f \supset \mathcal{M}_n^{k+1}\Theta_f \tag{1.7}$$

implies

$$T\mathcal{G}_{un}(F) + \mathcal{M}_{n+1}^{k+2}\Psi_F \supseteq \mathcal{M}_n^{k+1}\Psi_F. \tag{1.8}$$

The proof that (1.7) implies (1.8) is not hard, but we omit it (the reader may consult Wall [108] or du Plessis [79]).

The tangent spaces $T\mathcal{G}_{un}(F)$, $\mathcal{G} = \mathcal{R}, C$ or $\mathcal{K}$, are finitely generated $\mathcal{E}_{n+1}$-modules, so we can apply Nakayama's lemma to (1.8) with $C = T\mathcal{G}_{un}(F) + \mathcal{M}_n^{k+1}\Psi_F$, $A = T\mathcal{G}_{un}(F)$ and $M = \mathcal{M}_{n+1}$ to get $T\mathcal{G}_{un}(F) \supseteq \mathcal{M}_n^{k+1}\Psi_F$.

Now, $\partial \cdot F = g - f \in \mathcal{M}_n^{k+2}\Psi_F$, and we can apply the Thom-Levine lemma to prove that F is $\mathcal{G}$-trivial in some neighborhood of $t = 0$. For a proof of the Thom-Levine lemma for $\mathcal{G} = \mathcal{K}$ see du Plessis et al. [39]. Notice that $j^{k+1}\bar{f}_t(0) = j^{k+1}f(0)$, and the hypothesis (ii) holds for $\bar{f}_a$, for any $a \in [0,1]$, so that the arguments of the proof also hold to prove that F is $\mathcal{G}$-trivial in a small neighborhood of $t = a$ for any $a \in [0,1]$. Hence f is $(k+1)$-$\mathcal{G}$-determined, $\mathcal{G} = \mathcal{R}$, C or $\mathcal{K}$.

(II) $\mathcal{G} = \mathcal{L}$ or $\mathcal{A}$.

In these cases, $T\mathcal{G}_{un}(F)$ is not an $\mathcal{E}_{n+1}$-module in general. Let $\mathcal{G} = \mathcal{A}$ (the case $\mathcal{G} = \mathcal{L}$ follows as a particular case).

$$TA_{un}(F) = tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1}),$$

$$F(x,t) = (\bar{f}(x,t), t),\ \ \bar{f}(x,t) = (1-t)f(x) + tg(x),$$

and $j^{2k+1}f(0) = j^{2k+1}g(0)$

First notice that if $F_0(x,t) = (f(x), t)$ is the suspension of f, the hypothesis $\mathcal{M}_n^{k+1}\Theta_f \subseteq tf(\mathcal{M}_n\Theta_n) + \omega f(\mathcal{M}_p\Theta_p)$ implies that

$$\mathcal{M}_n^{k+1}\Theta_{F_0} \subseteq tF_0(\mathcal{M}_n\Psi_{n+1}) + \omega F_0(\mathcal{M}_p\Psi_{p+1}) + (t\mathcal{M}_n^{k+1} + \mathcal{M}_n^{2k+2})\Psi_{F_0}.$$

Notice that $\mathcal{M}_n^{k+1}\Psi_{F_0} \subset \mathcal{M}_n^{k+1}\Theta_f + t\mathcal{M}_n^{k+1}\Psi_{F_0}$.

The next step is to verify that similar inclusion holds replacing F_0 by F, $j^{2k+1}\bar{f}_t(0) = j^{2k+1}f(0)$, that is

$$\mathcal{M}_n^{k+1}\Psi_F \subset tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1}) + (t\mathcal{M}_n^{k+1} + \mathcal{M}_n^{2k+2})\Psi_F \tag{1.9}$$

(see sublemma 2.2 in du Plessis [79]).

If we can show that the term $(t\mathcal{M}_n^{k+1} + \mathcal{M}_n^{2k+2})\Psi_F$ can be eliminated in (1.9) then the Thom-Levine lemma can be applied to prove that F is $\mathcal{A}$-trivial.

To achieve this goal Malgrange's preparation theorem will be the fundamental tool.

Multiplying (1.9) by $\mathcal{M}_n^{k+1}$ and since $\mathcal{M}_n^{k+1}\omega F(\mathcal{M}_p\Psi_{p+1}) \subset F^*(\mathcal{M}_p)\mathcal{M}_n^{k+1}\Psi_F$, we get

$$\mathcal{M}_n^{2k+2}\Psi_F \subset tF(\mathcal{M}_n^{k+2}\Psi_{n+1}) + F^*(\mathcal{M}_p)\mathcal{M}_n^{k+1}\Psi_F + (t + \mathcal{M}_n^{k+1})\mathcal{M}_n^{2k+2}\Psi_F. \tag{1.10}$$

The $\mathcal{E}_{n+1}$-module

$$E = \frac{tF(\mathcal{M}_n^{k+2}\Psi_{n+1}) + F^*(\mathcal{M}_p)\mathcal{M}_n^{k+1}\Psi_F + \mathcal{M}_n^{2k+2}\Psi_F.}{tF(\mathcal{M}_n^{k+2}\Psi_{n+1}) + F^*(\mathcal{M}_p)\mathcal{M}_n^{k+1}\Psi_F}$$

is finitely generated, since it is a quotient of finitely generated modules. Moreover, from (1.10) we get that $E = (t + \mathcal{M}_n^{k+1})E$, and by Nakayama's lemma it follows that $E = 0$. Then, we get

$$\mathcal{M}_n^{2k+2}\Psi_F \subset tF(\mathcal{M}_n^{k+2}\Psi_{n+1}) + F^*(\mathcal{M}_p)\mathcal{M}_n^{k+1}\Psi_F. \tag{1.11}$$

Using (1.11) to replace part of the remainder term in (1.9), we get

$$\mathcal{M}_n^{k+1}\Psi_F \subset tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1}) + (t + F^*(\mathcal{M}_p))\mathcal{M}_n^{k+1}\Psi_F. \tag{1.12}$$

Let E' be the $F^*(\mathcal{E}_{p+1})$-module

$$E' = \frac{tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1}) + \mathcal{M}_n^{k+1}\Psi_F}{tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1})}.$$

Using (1.12), it follows that $E' = (t + F^*(\mathcal{M}_p))E'$. Notice that the ideal $\langle t\rangle + F^*(\mathcal{M}_p)$ is contained in $F^*(\mathcal{M}_{p+1})$, so it follows that $E' = F^*(\mathcal{M}_{p+1})E'$.

To apply Nakayama's lemma, one has to show that E' is a $F^*(\mathcal{E}_{p+1})$-module finitely generated. For this, let the finitely generated $\mathcal{E}_{n+1}$-module

$$E'' = \frac{tF(\mathcal{M}_n\Psi_{n+1}) + \mathcal{M}_n^{k+1}\Psi_F}{tF(\mathcal{M}_n\Psi_{n+1})}.$$

Notice that the inclusion

$$tF(\mathcal{M}_n\Psi_{n+1}) + \mathcal{M}_n^{k+1}\Psi_F \subset tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1}) + \mathcal{M}_n^{k+1}\Psi_F$$

induces an epimorphism of $F^*(\mathcal{E}_{n+1})$-modules $E'' \to E'$ so that if E'' is a finitely generated $F^*(\mathcal{E}_{p+1})$-module, then E' also is.

From Malgrange preparation theorem, E'' is a finitely generated $F^*(\mathcal{E}_{p+1})$-module if and only if

$$\dim_{\mathbb{R}} \frac{E''}{F^*(\mathcal{M}_{p+1})E''} < \infty. \tag{1.13}$$

Now

$$\frac{E''}{F^*(\mathcal{M}_{p+1})E''} \simeq \frac{tF(\mathcal{M}_n\Psi_{n+1}) + \mathcal{M}_n^{k+1}\Psi_F}{tF(\mathcal{M}_n\Psi_{n+1}) + F^*(\mathcal{M}_{p+1})\mathcal{M}_n^{k+1}\Psi_F}$$

It follows from (1.11) that

$$tF(\mathcal{M}_n^{k+2}\Psi_{n+1}) + F^*(\mathcal{M}_{p+1})\mathcal{M}_n^{k+1}\Psi_F \supset \mathcal{M}_n^{2k+2}\Psi_F.$$

As $t \in F^*(\mathcal{M}_{p+1})$, we also get that

$$tF(\mathcal{M}_n^{k+2}\Psi_{n+1}) + F^*(\mathcal{M}_{p+1})\mathcal{M}_n^{k+1}\Psi_F \supset \mathcal{M}_{n+1}^{k+1}\mathcal{M}_n^{k+1}\Psi_F,$$

so that,

$$\dim_{\mathbb{R}} \frac{E''}{F^*(\mathcal{M}_{p+1})E''} \leq \dim_{\mathbb{R}} \frac{\mathcal{M}_n^{k+1}\Psi_F}{\mathcal{M}_{n+1}^{k+1}\mathcal{M}_n^{k+1}\Psi_F} < \infty$$

Then we can apply Nakayama's lemma to (1.12) to get that $E' = 0$, so that $\mathcal{M}_n^{k+1}\Psi_F \subset tF(\mathcal{M}_n\Psi_{n+1}) + \omega F(\mathcal{M}_p\Psi_{p+1})$.

To conclude we proceed as in part (I). □

The following result follows from Theorem 1.4.8 and Mather's lemma.

Proposition 1.4.16 *Let $f \in \mathcal{E}_n^p$, $\epsilon = 1$ when $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$ and $\epsilon = 2$ when $\mathcal{G} = \mathcal{L}, \mathcal{A}$. Then f is k-$\mathcal{G}$-determined if and only if $\mathcal{M}_n^{k+1}\Theta_g \subset T\mathcal{G}g + \mathcal{M}_n^{\epsilon(k+1)}\Theta_g$ for all $g \in \mathcal{E}_n^p$ such that $j^k g(0) = j^k f(0)$.*

We see in the next example that the converse of condition (i) in Theorem 1.4.8 does not hold, that is, the condition $T\mathcal{G}f \supseteq \mathcal{M}_n^{k+1}\Theta_f$ does not imply in general that f is k-$\mathcal{G}$-determined.

Example 1.4.17 Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $f(x, y) = x^3 + y^3$, and $\mathcal{G} = \mathcal{R}$. Then

$$T\mathcal{R}f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \mathcal{M}_2 = \mathcal{M}_2^3$$

but f is not 2-$\mathcal{R}$-determined as $j^2 f(0) \equiv 0$.

A successful approach to a necessary and sufficient condition for finite determinacy appears in [14] where J. Bruce, A. du Plessis and C.T.C. Wall prove this condition for *unipotent subgroups* of $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{K}$, $\mathcal{L}$ or $\mathcal{A}$.

Let $\mathcal{G}_s = \{h \in \mathcal{G} |\ j^s h(0) = j^s 1_{\mathcal{G}}\}$ where $1_{\mathcal{G}}$ is the identity of $\mathcal{G}$, and $J^s\mathcal{G}$ the Lie group of s-jets of elements of $\mathcal{G}$. The sets $\mathcal{G}_s,\ s \geq 1$ are unipotent subgroups of $\mathcal{G}$. A special case of the main result in [14] is the following:

Theorem 1.4.18 (Bruce, du Plessis, Wall [14]) *A C^∞ map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is r-$\mathcal{G}_s$-determined $(s \geq 1)$ if and only if $\mathcal{M}_n^{r+1}\Theta_f \subset T\mathcal{G}_s(f)$.*

1.4.3 Classification of Stable Singularities

We consider here the problem of classification of stable germs with respect to $\mathcal{A}$-equivalence. The main result is the following

Theorem 1.4.19 (Mather [61]) *If f, g are stable germs then $f \underset{\mathcal{A}}{\sim} g$ if and only if the algebras $Q(f)$ and $Q(g)$ are isomorphic.*

The proof of this theorem follows from the following property holding for infinitesimally stable germs: $\mathcal{A}^{p+1}z = \mathcal{K}^{p+1}z \cap St^{p+1}$, where $z = j^{p+1}f(0)$, and St^{p+1} is the set of all stable jets in $J^{p+1}(n, p)$. We omit the complete proof, however the main steps leading to the proof are given.

Example 1.4.20 The hypothesis that f and g are stable is essential. For instance, let $f(x, y) = (x, y^3 + xy)$ and $g(x, y) = (x, y^3)$. Both algebras $Q(f)$ and $Q(g)$ are isomorphic to $\frac{\mathcal{E}_1}{(y^3)}$, but f and g are not $\mathcal{A}$-equivalent. In fact, f is stable and g is not.

The condition that $f \in \mathcal{E}_n^p$ is infinitesimally stable is determined by its $p+1$-jet. In fact the following holds:

Proposition 1.4.21 (Mather [61], Proposition I.I) *The map-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ is stable if and only if*

$$tf(\Theta_{(n,S)}) + \omega f(\Theta_p) + (f^*(\mathcal{M}_p) + \mathcal{M}_S^{p+1})\Theta_f = \Theta_f. \tag{1.14}$$

Proof We need to show that (1.14) implies

$$tf(\Theta_{(n,S)}) + \omega f(\Theta_p) = \Theta_f.$$

The proof is similar to the proof of Proposition 1.4.15 but simpler.
Let $D = tf(\Theta_{(n,S)}) + f^*(\mathcal{M}_p)\Theta_f$. Note that

$$\omega f(\mathcal{M}_p\Theta_p) \subset f^*(\mathcal{M}_p)\Theta_f \subset D.$$

Then

$$\dim_{\mathbb{R}} \frac{\Theta_f}{\mathcal{M}_S^{p+1}\Theta_f + D} \leq \dim_{\mathbb{R}} \frac{\omega f(\Theta_p)}{\omega f(\mathcal{M}_p\Theta_p)} \leq p.$$

The result then follows by Lemma 1.4.13 (ii). □

Remark 1.4.22 Mather gives in [61], Proposition (I.6), a simple geometric characterization of a stable multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$, $S = \{x_1, x_2, \ldots, x_r\}$. Recall that if V is a vector space and $H_1, \ldots, H_r$ are subspaces of V, then $H_1, \ldots, H_r$ are in *general position* if for every sequence of integers $i_1, \ldots, i_l$ with $1 \leq i_1 \leq \cdots \leq i_l \leq r$, we have $\text{cod}(H_{i_1} \cap \cdots \cap H_{i_l}) = \text{cod}(H_{i_1}) + \cdots + \text{cod}(H_{i_l})$.

Let $f_i : U_i \to \mathbb{R}^p,\ i = 1, \ldots, r$ be a representative of the germ $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}^p, 0)$. Denote by $X_i = \{x \in U_i \,|\, (f_i, x) \underset{\mathcal{A}}{\sim} (f_i, x_i)\}$ where (f_i, x) denotes the germ $f_i : (\mathbb{R}^n, x) \to (\mathbb{R}^p, 0),\ i = 1, \ldots, r$. Since f is infinitesimally stable, the sets X_i are submanifolds. Mather's result states that the multigerm f is stable if and only if each branch $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}^p, 0)$, $i = 1 \ldots r$ is infinitesimally stable and the images $f_i(X_i), i = 1, \ldots, r$ are in general position.

Corollary 1.4.23 *An infinitesimally stable germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is $(p+1)$-$\mathcal{A}$-determined.*

Proof Notice that Proposition 1.4.21 implies that if $j^{p+1}g(0) = j^{p+1}f(0)$, then g is also infinitesimally stable.

It is also clear that every such g is $\mathcal{A}$-finitely determined, say l-$\mathcal{A}$-determined. Then, we can apply Proposition 1.4.16 to get the result. □

As the local algebra is a complete invariant for the classification of stable germs, we can ask:

– Can we provide a normal form of a stable germ whose local algebra is a given algebra Q?

The answer was given by Mather [61] and we review it here (see also section 1.2.5 of the Mond and Nuño-Ballesteros in this Handbook [70]).

We start with a rank zero $\mathcal{K}$-finitely determined $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $f = (f_1, f_2, \ldots, f_p)$. Let

$$Q(f) = \frac{\mathcal{E}_n}{f^*(\mathcal{M}_p)\mathcal{E}_n} = \frac{\mathcal{E}_n}{\langle f_1, \ldots, f_p\rangle \mathcal{E}_n}.$$

Since f is $\mathcal{K}$-finitely determined, the quotient

$$Nf = \frac{\Theta_f}{tf(\Theta_n) + f^*(\mathcal{M}_p)\Theta_f + \omega f(\Theta_p)} \tag{1.15}$$

is a finite dimensional $\mathbb{R}$-vector space of dimension r and we can choose $\sigma_i \in \mathcal{E}_n^p, i = 1, \ldots, r$ such that

$$Nf = \mathbb{R}\{\sigma_1, \ldots, \sigma_r\}, \tag{1.16}$$

For practical purposes, note that the vector space Nf admits the following simpler characterization:

$$Nf \simeq \frac{\mathcal{M}_n\Theta_f}{tf(\Theta_n) + f^*\mathcal{M}_p\Theta_f}$$

Let $F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0)$ be the linear r-parameter unfolding of f defined by

$$F(x, u) = (f(x) + \sum_{i=1}^{n} u_i \sigma_i(x), u). \tag{1.17}$$

Then F is infinitesimally stable. In fact, from (1.16) we get

$$\Theta_f = tf(\Theta_n) + \omega f(\Theta_p) + f^*(\mathcal{M}_p)\Theta_f + \mathbb{R}\{\sigma_1, \ldots, \sigma_r\},$$

which implies that

$$\Psi_F = tF(\Psi_{n+r}) + \omega F(\Psi_{p+r}) + F^*(\mathcal{M}_{p+r})\Psi_F + \mathcal{E}_r\{\sigma_1, \ldots, \sigma_r\}, \tag{1.18}$$

where $\mathcal{E}_r\{\sigma_1, \ldots, \sigma_r\}$ denotes the $\mathcal{E}_r$-module generated by $\{\sigma_1, \ldots, \sigma_r\}$. Notice that $F^*(\mathcal{M}_p)\mathcal{E}_{n+r} \supset \langle u_1, \ldots, u_r\rangle\mathcal{E}_{n+r}$. Then, it follows from that

$$\Theta_F = tF(\Theta_{n+r}) + \omega F(\Theta_{p+r}) + F^*(\mathcal{M}_{p+r})\Theta_F,$$

and it follows from Proposition 1.4.21 that F is infinitesimally stable.

Example 1.4.24

(a) $\mathcal{A}_k$ singularities

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $f(x) = x^{k+1}$. Then $Nf = \mathbb{R}\{1, x, \ldots, x^{k-1}\}$. From the above construction, we obtain that

$$F : \mathbb{R} \times \mathbb{R}^{k-1} \to \mathbb{R} \times \mathbb{R}^{k-1}$$

$$(x, u) \mapsto F(x, u) = (x^{k+1} + \sum_{i=1}^{k-1} u_i x^i, u),$$

is infinitesimally stable.

(b) $\Sigma^{2,0}$ singularities $B^{\pm}_{2,2} = (x^2 \pm y^2, xy)$

(We use here du Plessis and Wall notation [82]. They are denoted $\mathrm{I}_{2,2} = (x^2 + y^2, xy)$ and $\mathrm{II}_{2,2} = (x^2 - y^2, xy)$ by Mather [61].)

Normal forms for infinitesimally stable singularities whose local algebra are $B^{\pm}_{2,2}$ are

$$F : (\mathbb{R}^2 \times \mathbb{R}^2, 0) \to (\mathbb{R}^2 \times \mathbb{R}^2, 0)$$

$$(x, y, u, v) \mapsto F(x, y, u, v) = (x^2 \pm y^2 + ux + vy, xy, u, v).$$

As a consequence of the results of this section we can state the following addendum to Theorem 1.3.11.

Theorem 1.4.25 (Mather [62], Addendum to Theorem 4.1) *Let $r \leq p+1$ and $k \geq p$. Let $f : N \to P$ be a proper C^∞ mapping. Then the following conditions are equivalent*

(a) *f is stable.*
(b) *${}_r j^k f$ is transversal to every contact class in ${}_r j^k(N, P)$.*
(c) *For every subset S of N having r or fewer points, such that $f(S)$ is a single point $y \in P$, we have*

$$tf(\Theta_{(N,S)}) + \omega f(\Theta_{(P,y)}) + \mathcal{M}_S^{k+1}\Theta_f = \Theta_f$$

1.4.4 Maps of Finite Singularity Type

Another fundamental notion introduced by Mather in [65] was the notion of mappings of finite singularity type, denoted by FST. Properties of such mappings are also discussed in [39].

A mapping $f : N \to P$ will be said of *finite singularity type* if $E = \frac{\Theta_f}{tf(\Theta_N)}$ is a finite module over $C^\infty(P)$ via f.

We can also define similarly the notion of FST for multigerms $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$.

Local properties of mappings of finite singularity type follow from our previous discussion. The critical set of f is the set $\Sigma(f)$ of non-submersive points of f.

Let $F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0)$ with $F(x, u) = (\bar{f}(x, u), u)$ and $\bar{f}(x, 0) = f(x)$. If F is a stable germ, we say that F is a *parametrized stable unfolding* of f.

Theorem 1.4.26 *Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$. The following are equivalent.*

(1) *f is of FST.*
(2) *f is $\mathcal{K}$-finitely determined.*
(3) *f admits a stable parametrized unfolding.*

Moreover, these conditions imply

(4) *for every sufficiently small representative $f : U \to V$, $f|_{\Sigma(f)} : \Sigma(f) \to V$ is proper and has finite fibers.*

Remark 1.4.27 We say that $f : X \to Y$ has finite fibers (or, is finite-to-one) if for every $y \in Y$, $f^{-1}(y)$ has a finite number of points.

Proof The equivalence (1) $\Leftrightarrow$ (2) follows from the Preparation Theorem. In fact $E = \frac{\Theta_f}{tf(\Theta_{(n,S)})}$ is a finitely generated $f^*(\mathcal{E}_p)$-module if and only if $\mathcal{K}_e$-cod $f = \dim_{\mathbb{R}} \frac{\Theta_f}{tf(\Theta_{(n,S)})+f^*(\mathcal{M}_p)\Theta_f} < \infty$.

We saw in Sect. 1.4.3 that a $\mathcal{K}$-finitely determined germ has a stable unfolding; so that (2) $\Rightarrow$ (3). We saw in Example 1.4.7 that $Q(f) = Q(f_0)$, so that (3) $\Rightarrow$ (2).

It is sufficient to prove (4) for infinitesimally stable germs. In this case, the general position condition implies that for any $y \in V$, $f^{-1}(y) \cap \Sigma(f)$ has at most p points (see Remark 1.4.22). □

We shall need some extra conditions to formulate the theory of FST mappings $f : N \to P$. The condition that f has a parametrized stable unfolding is fairly easily computable, but it does not always have a global version (see Mather [65] for counter examples).

Definition 1.4.28 Let $f : N \to P$ be smooth. We say that $\{F, N', P', i, j\}$ is an *unfolding* of f if we have a commutative diagram

$$\begin{array}{ccc} N' & \xrightarrow{F} & P' \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ N & \xrightarrow{f} & P \end{array}$$

where N', P' are smooth manifolds, F is a smooth mapping, i, j are closed smooth embeddings, $i(N) = F^{-1}(j(P))$ and F is transverse to j.

Theorem 1.4.29 (Mather [66], Proposition 7.2)
Let $f : N \to P$ be smooth and N compact. Then f is of finite singularity type if and only if there exists an unfolding $\{F, N', P', i, j\}$ of f such that F is proper and infinitesimally stable.

1.4.5 Notes

All the results in this section remain true if we replace smooth germs by real analytic or complex analytic germs. In particular, the notion of $\mathcal{G}$-finite determinacy for $\mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}$ and $\mathcal{K}$ is independent of whether we consider f as a real analytic, C^∞ or complex analytic map-germ. The Infinitesimal Criterion of $\mathcal{G}$-finite determinacy holds with essentially the same proof replacing Malgrange Preparation Theorem by Weirstrass Preparation Theorem. We use the same notation $\mathcal{O}_n$ for the local rings of real analytic or complex analytic map-germs at the origin. The maximal ideal in both cases is also denoted by $\mathcal{M}_n$. The set $\mathcal{O}_n^p$ denotes the $\mathcal{O}_n$-module of real or complex analytic map-germs from $(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The following result explains the relation among finite determined germs in these different modules.

Proposition 1.4.30 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map-germ. The following are equivalent*

(i) f is k-$\mathcal{G}$-determined in the space of real analytic map-germs $\mathcal{O}_n^p$.
(ii) f is k-$\mathcal{G}$-determined in $\mathcal{E}_n^p$.
(iii) The complexification of f, $f_{\mathbb{C}} : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, is k-$\mathcal{G}$-determined in the space $\mathcal{O}_n^p$ of holomorphic map-germs.

In the complex case there are useful geometric characterization of $\mathcal{G}$-finite determinacy. The main result characterizes $\mathcal{G}$-finite determined germs as map-germs with isolated instability. The case $\mathcal{G} = \mathcal{A}$ was stated by Mather and proved by Gaffney. For a complete account we refer to Wall [108] or Mond and Nuño-Ballesteros [69]. See also Mond and Nuño-Ballesteros article in this Handbook [70]

Theorem 1.4.31 (Geometric Criterion of Finite Determinacy) *A holomorphic map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, is $\mathcal{A}$-finite if and only if there is a neighborhood U of 0 in $\mathbb{C}^n$ such that for every finite subset $S \subset U \setminus \{0\}$, the multigerm of f at S is $\mathcal{A}$-stable.*

The geometric condition of this theorem (isolated instability) holds for any real $\mathcal{A}$-finite map-germ. However, the converse statement does not hold. For a simple example, let $f(x, y) = (x^2 + y^2)^2$. As $\Sigma(f) = \{0\}$, the origin is an isolated instability, but f is not $\mathcal{A}$-finitely determined.

1.5 The Nice Dimensions

We discuss in this section the main steps in the proof of theorem A. Mather proved in [61] that for a pair of positive integers (n, p), there exists a smallest Zariski closed $\mathcal{K}^k$-invariant set $\Pi^k(n, p)$ in the set $J^k(n, p)$ such that $J^k(n, p) \setminus \Pi^k(n, p)$ is the union of finitely many $\mathcal{K}^k$-orbits. The set $\Pi^k(n, p)$ is the "bad set." It is in fact the set of k-jets in $J^k(n, p)$ of "modality" ($\mathcal{K}$-modality) greater than or equal to 1 (see Sect. 1.8.1 for the definition of modality).

We review Mather's construction of $\Pi^k(n, p)$. For each $r, k \in \mathbb{N}$ we define $W_r^k(n, p)$ as the set of $z \in J^k(n, p)$ such that $\mathcal{K}^k$-cod $z \geq r$. This set is a closed algebraic subset of $J^k(n, p)$. Let $W_r^k(n, p)^*$ denote the union of all irreducible components of $W_r^k(n, p)$ whose codimension is less than r. We let $\Pi^k(n, p) = \cup_{r \geq 0} W_r^k(n, p)^*$. The following properties hold:

- $\Pi^k(n, p)$ is a closed algebraic subset of $J^k(n, p)$.
- Let $\pi_k : J^{k+1}(n, p) \to J^k(n, p)$ be the projection. It follows that $\pi_k^{-1}(\Pi^k(n, p)) \subset \Pi^{k+1}(n, p)$, hence $\operatorname{cod} \Pi^{k+1}(n, p) \leq \operatorname{cod} \Pi^k(n, p)$.
- There exists a k big enough for which the codimension of $\Pi^k(n, p)$ attains its minimum. For this k, $\operatorname{cod} \Pi^k(n, p)$ is denoted $\sigma(n, p)$.

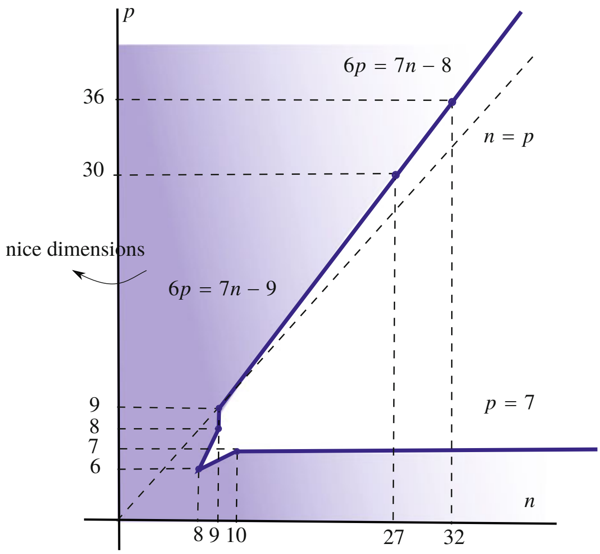


Fig. 1.3 Boundary of nice dimensions

Mather calculated $\sigma(n, p)$ in [63] and the result is as follows (Fig. 1.3):
Case 1: $n \leq p$

$$\sigma(n, p) = \begin{cases} 6(p-n)+8 & \text{if } p-n \geq 4 \text{ and } n \geq 4 \\ 6(p-n)+9 & \text{if } 3 \geq p-n \geq 0 \text{ and } n \geq 4 \text{ or if } n = 3 \\ 7(p-n)+10 & \text{if } n = 2 \\ \infty & \text{if } n = 1 \end{cases}$$

Case 2: $n > p$

$$\sigma(n, p) = \begin{cases} 9 & \text{if } n = p+1 \\ 8 & \text{if } n = p+2 \\ n-p+7 & \text{if } n \geq p+3 \end{cases}$$

Definition 1.5.1 A pair (n, p) is in the *nice dimensions* if $n < \sigma(n, p)$.

Suppose k has the property that $\operatorname{cod} \Pi^k(n, p) = \sigma(n, p)$. If (n, p) is in the nice dimensions, then there exists an analytically trivial stratification $\mathcal{S}^k(n, p)$ of $J^k(n, p) \setminus \Pi^k(n, p)$ such that the strata are a finite number of $\mathcal{K}$-orbits. To get a stratification of the whole jet space $J^k(N, P)$, we add to $S^k(n, p)$ a Whitney regular stratification of $\Pi^k(n, p)$ (it exists since $\Pi^k(n, p)$ is an algebraic closed set of $J^k(n, p)$).

Table 1.1 $\mathcal{K}$-orbits of stable germs $n = p \leq 8$

Type	Name	Normal form	Conditions	$\mathcal{K}$-cod $\leq n$
Σ^1	A_j	(x^{j+1})	$1 \leq j \leq n$	j
$\Sigma^{2,0}$	$B^{\pm}_{p,q}$	$(xy, x^p \pm y^q)$	$2 \leq p, q \leq n-2$	$p+q$
$\Sigma^{2,0}$	$B^{*}_{p,p}$	$(x^2 + y^2, x^p)$	$3 \leq p \leq 4$	$2p$
$\Sigma^{2,1}$	C_{2k-1}	$(x^2 + y^3, y^3)$		7
$\Sigma^{2,1}$	C_{2k}	$(x^2 + y^3, xy^2)$		8

This stratification of $J^k(n, p)$ induces a partition of $J^k(N, P)$ by $\mathcal{K}$-orbit bundles whose restriction to $J^k(N, P) \setminus \Pi^k(N, P)$ is denoted by $S^k(N, P)$.

As we saw in Theorem 1.4.25, stable mappings can be characterized by transversality of the k-jet extension $j^k f : N \to J^k(N, P)$ to the $\mathcal{K}^k$-orbits.

When $\sigma(n, p) > n$, transversality to the strata of the stratification $J^k(N, P)$, implies that $j^k f(N) \cap \Pi^k(N, P) = \emptyset$. Hence Theorem A follows from Thom's transversality theorem.

Example 1.5.2 (Stable Singularities When $n = p \leq 8$) We refer to [69] for the list of stable singularities in the nice dimensions.

When $n = p$, $\sigma(n, p) = 9$, then (n, n) is a nice pair of dimensions if and only if $n \leq 8$. The set $\Pi^k(n, n) \subset J^k(n, n)$, $k \geq n + 1$, $n \leq 8$ is the closure of all $\mathcal{K}^k$-orbits of $\mathcal{K}^k$-codimension greater than or equal to $n + 1$. In particular, $\Sigma^3(n, n) \subset \Pi^k(n, n)$, where $n \leq 8$ since cod $\Sigma^3 = 9$. The strata of the stratification $\mathcal{S}^k(n, n)$, $k \geq n + 1$, $n \leq 8$ are presented in Table 1.1:

Remark 1.5.3 Classification of stable singularities in the nice dimensions. Mather classified the stable germs in the nice dimensions as an application of results and arguments in [63]. He gave complete proofs of the classification of the local algebras of singularities of type Σ^1 and $\Sigma^{2,0}$ and outlined the classification of $\Sigma^{2,1}$ and Σ^{n-p+1} singularities. Further classification of simple and unimodular algebras were performed by Arnold [4], Wall [109], Dimca and Gibson [27–29] and Damon [18–20].

A remarkable property of stable map-germs in the nice dimensions is that, with respect to suitable coordinates, all singularities are weighted homogeneous. For many years, this property was considered to be true but there was no reference of a written proof.

This result was recently proved by Mond and Nuño-Ballesteros [69, theorem 7.6]. Their proof is based on Mather's classification of local algebras of stable germs in the nice dimensions and on the direct construction of the normal forms of their minimal stable unfoldings. This property of the nice dimensions plays a crucial role in the proof of Damon and Mond [26] that the $\mathcal{A}_e$-codimension is less than or equal to the rank of the vanishing homology of the discriminant (the discriminant Milnor number) for map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \geq p$ and (n, p) nice dimensions.

1.5.1 Notes

Non Proper Stable Mappings C^∞ non-proper stable mappings were discussed by du Plessis and Vosegaard [81] and more recently by Kenta Hayano [42].

For proper maps $f : N \to P$, Mather proves that stability, strong stability, infinitesimal stability an local infinitesimal stability are equivalent notions. In [81], du Plessis and Vosegaard prove that these notions are equivalent when f is a quasi-proper map with closed discriminant.

The purpose of Hayano's paper, [42], is to give a sufficient condition for strong stability of non-proper smooth functions $f : N \to \mathbb{R}$. He introduces the notion of *end-triviality* of smooth mappings, which controls the behavior of f around the ends of the source manifold N. He shows that a Morse function is stable if it is end-trivial at any point in its discriminant.

The extra-nice dimensions. When the pair (n, p) is in the nice dimensions and the source N is compact, an important problem in the applications of singularity theory to topology of manifolds is the characterization of generic singularities of 1-parameter paths between two stable maps; they are also known as *pseudo-isotopies.* A 1-parameter family $F : N \times [0, 1] \to P$ connecting two non equivalent stable maps always intersects the set of non stable maps at a finite number of values of the parameter, the bifurcation points. The classification of singularities of bifurcation points in generic families of maps is an important step in results on elimination of singularities (see for instance [7, 50]) and on results about the topology of the space of smooth maps such as [16, 44, 104].

We say that a family $F : N \times [0, 1] \to P$ is a *locally stable family* if $F_t : N \to P$ is stable for all $t \in [0, 1]$ except possibly a finite number of values $\{t_1, \dots, t_k\}$ and the non stable singularities of F_t are a finite number of points x_j at which $\mathcal{A}_e$-cod$(F_{t_i}) = 1$.

In [6] Oset Sinha, Ruas and Wik Atique obtain a result parallel to Mather's characterization of the nice dimensions. They define the *extra-nice dimensions* and (see Fig. 1.4) prove that the subset of stable 1-parameter families in $C^\infty(N \times [0, 1], P)$ is dense if and only (n, p) is in the extra-nice dimensions.

In Sect. 1.10 we relate the condition that (n, p) is in the extra-nice dimensions to the geometry of sections of the discriminant of stable maps in dimensions $(n + 1, p + 1)$.

1.6 Thom's Example

If a pair of dimensions (n, p) is not in the nice range of dimensions, then there exists an open non void subset U of $C^\infty(N, P)$, such that U is the union of an uncountable number of $\mathcal{A}_e$-orbits. This property was first proved by René Thom when $n = p = 9$. We review Thom's example [96] here. The pair $n = p = 9$ is in the *boundary of the nice dimensions*, which consists of pairs (n, p) such that $\sigma(n, p) = n$.

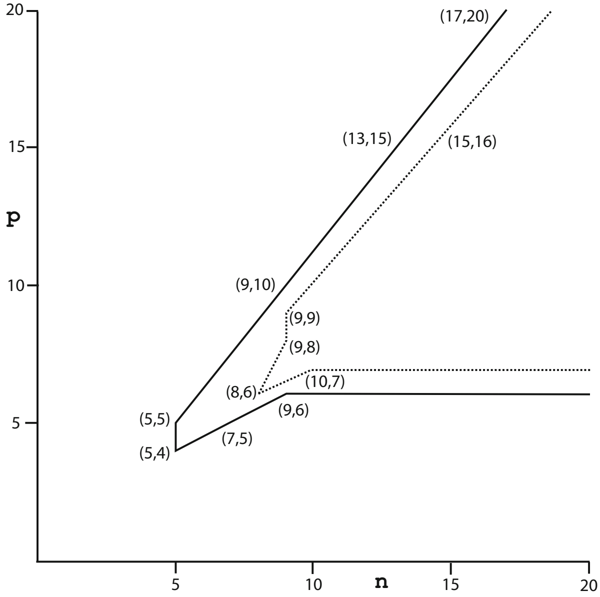


Fig. 1.4 Extra-nice dimensions

The construction of Thom's example was based on the following

1. The set of mappings $F : N \to P$, $\dim N = \dim P = n$, such that $j^k F \pitchfork \Sigma^r(N, P)$, where $\Sigma^r(N, P) = \{\sigma \in J^k(N, P) | \operatorname{corank} \sigma = r\}$, $0 \le r \le n$ is a residual set of $C^\infty(N, P)$.
2. $\operatorname{cod}_{J^k(N,P)} \Sigma^r(N, P) = r^2$.
3. When $r = 3$, $n = 9$, there exists a 1-parameter family of non $\mathcal{K}$-equivalent mappings $F_\lambda : \mathbb{R}^9 \to \mathbb{R}^9$, such that $j_1^k F : \mathbb{R} \times \mathbb{R}^9 \to J^k(\mathbb{R}^9, \mathbb{R}^9)$ is transversal to $\Sigma^3(\mathbb{R}^9, \mathbb{R}^9)$, where $j_1^k F$ denotes the $k-$jet with respect do the variable x.

The sets Σ^r are the first order Boardman symbols and it is an easy exercise to prove that they are codimension r^2 submanifolds of $J^k(N, P)$ when $\dim(N) = \dim(P)$. Hence (1) follows from Thom's transversality theorem.

It is sufficient to verify (3) for map-germs $F : (\mathbb{R}^9, 0) \to (\mathbb{R}^9, 0)$, such that corank $F(0) = 3$. By changing coordinates in source and target, it follows that F can be written in the form $F(x, u) = (f(x, u), u)$, $x = (x_1, x_2, x_3)$, $u = (u_1, \ldots, u_6)$, $f_0(x) = f(x, 0)$, where $f_0 : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ has zero rank.

The local algebras $Q(F)$ and $Q(f_0)$ are isomorphic. As we saw in Example 1.4.7, F is $\mathcal{K}$-equivalent to a suspension of f_0. The 2-jet $j^2 f_0$ is a quadratic polynomial mapping $q : \mathbb{R}^3 \to \mathbb{R}^3$, which determines a net of real quadrics. Non degenerate nets of quadrics over the complex numbers were classified by C. T. C. Wall in [107]. Over the reals, the classification was given by Wall and Edwards in [30]. The complete classification of real nets of quadrics can be found in [82, chapter 8, table 8.21].

For our purpose here, it suffices to remark that the set $\Sigma^{3,3}$ has a Zariski open set, denoted by W_2, defined by the union of the $J^2\mathcal{K}$-orbits of the unimodular family:

$$\begin{aligned}(f_0)_\lambda :&(\mathbb{R}^3,0)\to(\mathbb{R}^3,0)\\ &(x_1,x_2,x_3)\to(x_1^2+\lambda x_2x_3,\, x_2^2+\lambda x_1x_3,\, x_3^2+\lambda x_2x_3)\end{aligned} \tag{1.19}$$

with $\lambda(\lambda^3+8)(\lambda^3-1)\neq 0$.

For each λ, $(f_0)_\lambda$ is a homogeneous polynomial map of degree 2, hence the $J^2\mathcal{K}$-action in W_2 coincides with the action of the linear group $\mathcal{G}=GL(3)\times GL(3)$ in W_2. Notice that the dimension of the linear group $\mathcal{G}$ is 18, as well as the dimension of W_2.

However $\mathcal{G}$ contains a one dimensional subgroup which acts trivially on W_2, namely $\{(cI_{\mathbb{R}^3},\frac{1}{c^2}I_{\mathbb{R}^3})\}$, c a non zero number. Hence the orbits have codimension at least 1 in W_2.

We can prove that the family (1.19) is 2-determined with respect to $\mathcal{K}$-equivalence. It follows that W_2 determines the $\mathcal{K}$-invariant sets $W_2^k=(\pi_2^k)^{-1}(W_2)$, where $\pi_2^k : J^k(9,9)\to J^2(9,9)$. Moreover, $\operatorname{cod}_{J^k(9,9)} W_2 = 9$, and $\mathcal{K}\text{-cod}(f_0)_\lambda =$ 10.

In other words, $\sigma(9,9)=\operatorname{cod} W_2 = 9$, so that the unimodular stratum W_2 cannot be avoided by a generic set of proper mappings $F:\mathbb{R}^9\to\mathbb{R}^9$. As a consequence, stable mappings are not dense when $n=p=9$.

For each $\lambda\notin\{0,-2,1\}$, $(f_0)_\lambda$ admits the topologically stable unfolding

$$\begin{aligned}F_\lambda :&(\mathbb{R}^9,0)\to(\mathbb{R}^9,0)\\ &(x,u)\mapsto(f_\lambda(x,u),u)\end{aligned} \tag{1.20}$$

where $f_\lambda(x,u)=(x_1^2+\lambda x_2x_3+u_1x_2+u_2x_3,\, x_2^2+\lambda x_1x_3+u_3x_1+u_4x_3,\, x_3^2+\lambda x_1x_2+u_5x_1+u_6x_2)$.

We will discuss the topological stability of F_λ in Sect. 1.8.

1.7 Density of Topologically Stable Mappings

From the previous example, it becomes clear that outside the nice dimensions, one has to loosen the formulation of Problem 1.2.1 to obtain a solution. Mather considered in [64] two possible ways.

One might hope that the space of mappings f whose germ f_x at each point $x\in N$ is $\mathcal{A}$-finitely determined is an open and dense subset in $C^\infty_{pr}(N,P)$. However, Mather gave in [59] an example which shows that this set is not always dense. In [80] du Plessis defined the *semi-nice dimensions* as the pairs (n,p) for which finite determinacy holds in general (see Definition 1.7.6). The complement of the semi-nice dimensions is essentially made of pairs (n, p) where singularities of $\mathcal{K}$-modality greater than or equal to 2 occur generically (see [80], [109]).

The second way to try to solve the problem is based on ideas due to Thom, and led to Theorem B on density of C^0stable mappings in $C^\infty_{pr}(N, P)$.

In his article *Local topological properties of differentiable mappings* [93], Thom describes the topological structure of differentiable mappings, outlining the proof of the topological stability theorem.

Theorem 1.7.1 (Theorem 4, [93]) *Let z be any jet in $J^r(n, p)$. Then, there exists a positive integer s depending only on r, n and p, and a proper algebraic variety Σ in $\pi_s^{-1}(z) \subset J^{r+s}(n, p)$ such that any jet in $\pi_s^{-1}(z)$ outside Σ is C^0-$\mathcal{A}$-finitely determined. Moreover, any two mappings realizing such jet are locally weakly stratified and isotopic.*

A complete proof of this theorem follows from the proof of the *Main Theorem* in A. Varchenko's article with the same title, *Local topological properties of differentiable mappings* [103] (see also [101, 102]). He also proves in [103] a *stratification theorem*, although he states in the paper he does not know whether Mather's density theorem follows from his stratification theorem, or whether the stratification theorem can be proved by Mather's methods.

Mather gave in 1970, an outline of a complete proof of Theorem B. His proof was published in the Proceedings of the Symposium of Dynamical Systems, held in Salvador, Bahia [64]. As remarked by him, he expected to publish a book in which the details of the proof would appear. In the Spring 1970, he gave a series of lectures and the notes appeared as a booklet published in the same year by the Harvard Printing Office. The notes also discuss the Thom-Whitney theory of stratified sets and stratified mappings. They were recently republished in the Bulletin of the American Mathematical Society [56].

Complete proofs of Theorem B were given in 1976, independently, by Gibson, Wirthmüller, du Plessis and Looijenga in [39] and by Mather in [66]. Both proofs are based on Thom's ideas and Mather's outline [64]. In what follows we refer to Theorem B as the Thom-Mather theorem.

The book [39] comprises the notes of a seminar on Topological Stability of Smooth Mappings held at the Department of Pure Mathematics in the University of Liverpool, during the academic year 1974–75. The main objective was to organize a complete proof of the Topological Stability Theorem, for which no published complete account existed. The book has become a fundamental reference on the subject.

The proof in [39] and [66] are similar and they rely on the following ingredients:

(1) Properties of Whitney regular stratifications
(2) Łojasiewicz theorem, giving the existence of Whitney regular stratification of semialgebraic sets.
(3) Properties of stable mappings and mappings of finite singularity type (FST). A fundamental property of mappings of FST is the existence of a stable unfolding.

(4) Thom's second isotopy theorem, applied to show that families of mappings transverse to the Thom-Mather stratification are topologically trivial.

For a review of stratification theory and Thom's isotopy theorems in the differentiable category, we also refer to the paper by David Trotman, in Volume I of this Handbook. We only make a brief presentation of basic concepts and results.

Let V be a subset of a smooth manifold N of class C^k. A C^k-*stratification* of V is a filtration by closed subsets

$$V = V_d \supset V_{d-1} \supseteq \cdots \supseteq V_1 \supseteq V_0$$

such that each difference $V_i \setminus V_{i-1}$ is C^k-manifold of dimension i, or is empty. Each connected component of $V_i \setminus V_{i-1}$ is a *stratum* of dimension i. It follows that V is disjoint union of strata $\{X_\alpha\}_{\alpha \in A}$, and we say that V is a *stratified set.*

For the purposes of these notes we assume that the stratified sets $V = \cup_{\alpha \in A} X_\alpha$ are *locally finite* and satisfy the *frontier condition* (see Gibson et al. book [39] or Trotman [100] for the definition).

Let V be a subset of $\mathbb{R}^n$ and $\{X_\alpha\}_{\alpha \in A}$ a stratification of V. Whitney defined regularity conditions (a) and (b), seeking for stratifications topologically trivial along strata.

Definition 1.7.2 (Whitney's Conditions (a) and (b)) Let X and Y be strata of $\{X_\alpha\}_{\alpha \in A}$, such that $Y \subset \overline{X} \setminus X$.

(a) The pair (X, Y) satisfies Whitney's condition (a) at $y \in Y$ if: for all sequences $(x_m) \in X$ with $x_m \to y$, such that $T_{x_m} X$ converges to a subspace $T \subset \mathbb{R}^n$ (in Grassmannian of dim X- planes in $\mathbb{R}^n$), then $T \supset T_y Y$.

(b) The pair (X, Y) satisfies Whitney's condition (b) at $y \in Y$ if: for all sequences $(x_m) \in X$ and $(y_m) \in Y$, with $x_m \to y$, $y_m \to y$, such that $\{T_{x_m} X\}$ converges to T and the lines $\overline{x_m y_m}$ converges to a line ℓ one has $\ell \in T$.

It was pointed out by Mather in his notes on topological stability that Whitney (b) implies Whitney (a). The reader may verify this as an exercise. We say that the stratification is *Whitney regular* if every pair of strata (X_α, X_β) satisfies (b) (hence also satisfies (a)) at every point in X_β.

These regularity conditions are local and can be easily extended to stratified sets of a manifold N.

Whitney [116, 117] proved in 1965 that any analytic variety in $\mathbb{R}^n$ or $\mathbb{C}^n$ admits a regular stratification whose strata are analytic. This result was extended to semi-analytic sets by Łojasiewicz [49], also in 1965. For the purposes of this section, the relevant result is the existence theorem for semialgebraic sets. We refer to Thom [95] and Wall [105] for accessible proofs.

Definition 1.7.3 Let $f : N \to P$ be a smooth mapping and $A \subseteq N$, $B \subseteq P$ sets with $f(A) \subset B$. A *stratification* of $f : A \to B$ is a pair $(\mathcal{X}, \mathcal{X}')$, such that $\mathcal{X}$ is a

Whitney stratification of A, $\mathcal{X}'$ is a Whitney stratification of B, and the following conditions hold

- f maps strata to strata.
- If $X \in \mathcal{X},\ X' \in \mathcal{X}'$, $f(X) \subset X'$ then $f : X \to X'$ is a submersion.

Definition 1.7.4 Let $f : N \to P$ and $\mathcal{X}$ and $\mathcal{X}'$ as in Definition 1.7.3. Given X_α, X_β strata of $\mathcal{X}$, $x \in X_\beta$ we say that X_α is *Thom regular* over X_β at $x \in X_\beta$ relative to f when the following holds: for every sequence $(x_i) \in X_\alpha,\ x_i \to x$ such that ker $(d_{x_i}(f|_{X_\alpha}))$ converges to T in the appropriate Grasmannian, then ker $d_x(f|_{X_\beta}) \subseteq T$. We say that X_α *is Thom regular over* X_β *relative to* f when this condition hold for all $x \in X_\beta$.The pair $(\mathcal{X}, \mathcal{X}')$ is a *Thom stratification* for f when Thom's regularity condition holds for all pair of strata (X_α, X_β) with $X_\beta \subset \overline{X}_\alpha$. The triple $(f, \mathcal{X}, \mathcal{X}')$ with f a smooth mapping and $(\mathcal{X}, \mathcal{X}')$ a Thom stratification for f is called a *Thom stratified mapping*.

1.7.1 How to Stratify Mappings and Jet Spaces

We first discuss the Thom-Mather stratification in jet space and how to stratify stable mappings and mappings of finite singularity type. Then, we discuss why mappings transverse to the Thom-Mather stratification are topologically stable.

The idea of the proof is to construct a stratification $\mathcal{A}^l(N, P)$, of a big open subset of $J^l(N, P)$, with the following property: if l is sufficiently large, then for any mapping $f : N \to P$ which is multitransverse to $\mathcal{A}^l(N, P)$, then the locally finite manifold partition $\mathcal{B} = ((j^l f)^{-1}\mathcal{A}^l(N, P))$ is a Whitney stratification which extends to a Thom stratification $(\mathcal{B}, \mathcal{B}')$ of f.

Let $z \in J^l(n, p)$ and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ such that $j^l f(0) = z$.

Following Gibson et al. [39], we let

$$\chi_z = \dim_{\mathbb{R}} \frac{\Theta_f}{tf(\Theta_n) + (f^*(\mathcal{M}_p) + \mathcal{M}_n^l)\Theta_f}$$

We define $W^l(n, p) = \{z \in J^l(n, p) \mid \chi_z \geq l\}$. $W^l(n, p)$ is the *bad set*, and the following hold

(a) If $z \in J^l(n, p) \setminus W^l(n, p)$, then any $f \in \mathcal{E}_n^p$ such that $j^l f(0) = z$ is l-$\mathcal{K}$-determined.
(b) $W^l(n, p)$ is $\mathcal{K}$-invariant.
(c) $W^l(n, p)$ is a real algebraic variety in $J^l(n, p)$.

To verify (a) notice that, if $\chi_z \leq l - 1$, then

$$tf(\Theta_n) + (f^*(\mathcal{M}_p) + \mathcal{M}_n^l)\Theta_f \supset \mathcal{M}_n^{l-1}\Theta_f. \tag{1.21}$$

Then we can multiply (1.21) by $\mathcal{M}_n$ and the result follows from Theorem 1.4.8.

It follows from (a) that map-germs $f \in \mathcal{E}_n^p$ such that $z = j^l f(0)$ satisfy $\chi_z \leq l - 1$ are of finite singularity type. In the following proposition we prove that the property of FST holds in general.

Proposition 1.7.5 (Gibson et al. [39], Theorem 7.2) *The following conditions hold:*

(i) $\operatorname{cod} W^{l+1}(n, p) \geq \operatorname{cod} W^l(n, p)$.
(ii) $\lim_{l \to \infty} \operatorname{cod} W^l(n, p) = \infty$.
(iii) *There is a subbundle $W^l(N, P) \subset J^l(N, P)$ naturally associated to $W^l(n, p)$. Moreover, when N is compact, mappings $f : N \to P$ such that $j^l f(N) \cap W^l(N, P) = \emptyset$ are of finite singularity type.*

Definition 1.7.6 We say that a property $\mathcal{P}$ of map-germs *holds in general* if the sets $W^l_{\mathcal{P}}(n, p) = \{z \in J^l(n, p) | \ z$ does not satisfy $\mathcal{P}\}$, satisfy (i) and (ii) (see [108]).

While condition (i) in Proposition 1.7.5 can be easily verified, we can prove (ii) as follows.

Given $z \in W^l(n, p)$, find $z' \in W^{l+q}(n, p)$, $\pi_l(z') = z$, where $\pi_l : W^{l+q}(n, p) \to W^l(n, p)$ is the projection, such that $z' \notin W^{l+q}(n, p)$ (see Bruce, Ruas and Saia [15], for a simpler proof of this result).

As $W^l(n, p)$ is a real algebraic variety, it follows from Łojasiewicz's result [49] that it has a Whitney stratification with semialgebraic strata. Condition (iii) is immediate. Notice that conditions (i) and (ii) imply that we can choose sufficiently high l for which $\operatorname{cod} W^l(n, p) > n$. Then, the mappings $f : N \to P$ which are multitransverse to $\mathcal{A}^l(N, P)$ satisfy the condition $j^l f(N) \cap W^l(N, P) = \emptyset$.

Our problem now is to construct a stratification $\mathcal{A}^l(n, p)$ of $J^l(n, p) \setminus W^l(n, p)$ whose members are $\mathcal{K}$-invariant sets $S_j = \{z \in J^l(n, p) \setminus W^l(n, p) | \ \mathbf{cod}\, z = j\}$, for $j = 0, 1, 2, \ldots$. The definition of $\mathbf{cod}\, z$ will be given in the sequel.

We shall see that $\mathcal{K}^l$-equivalent jets z and z' have the same codimension, i.e., $\mathbf{cod}\, z = \mathbf{cod}\, z'$. This number does not coincide with the $\mathcal{K}^l$-codimension.

Although we know that contact classes are smooth submanifolds of the jet spaces, it is not clear at this point that the collection S_j defines a stratification of $J^l(n, p) \setminus W^l(n, p)$. To define $\mathbf{cod}\, z$ and to understand the structure of the strata S_j in $\mathcal{A}^l(n, p)$, we first discuss shortly how to stratify infinitesimally stable mappings and mappings of FST. Recall that for any smooth map $f : N \to P$, the *critical set* of f is $\Sigma(f) = \{x \in N | \ df_x : T_x N \to T_{f(x)}$ is not surjective$\}$ and the *discriminant* of f is $\Delta(f) = f(\Delta(f))$.

We saw in Sect. 1.4 that if $f : N \to P$ is infinitesimally stable, the restriction $f|_{\Sigma(f)} : \Sigma(f) \to P$ is proper and uniformly finite-to-one. In fact for any $y \in P$, $\#(f^{-1}(y) \cap \Sigma(f)) \leq p$. Moreover, if $f^{-1}(y) \cap \Sigma(f) = \{x_1, x_2, \ldots, x_s\}$ the multigerm $f : (N, S) \to (P, y)$ has a representative equivalent to a polynomial mapping $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^p$, where U and V are open sets in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. In other words f is a semialgebraic map defined on semialgebraic subsets. Then we can apply the basic theorems of Whitney and Lojasiewicz to construct Whitney stratifications $\mathcal{S}$ of N and $\mathcal{S}'$ of P with the following properties

1. For each stratum X of $\mathcal{S}$, there is a stratum Y of $\mathcal{S}'$ such that $f(X) \subset Y$.
2. For each stratum Y of $\mathcal{S}'$, it follows that $f^{-1}(Y) \setminus \Sigma(f)$ is a stratum of $\mathcal{S}$.
3. For each stratum X of $\mathcal{S}$, such that $X \subset \Sigma(f)$, we have that $\dim X = \dim Y$ and $f : X \to Y$ is an immersion, where Y is the stratum of $\mathcal{S}'$ which contains $f(X)$.

Notice that from 2. it follows that $N \setminus \Sigma(f)$ is a union of strata. Hence, $\Sigma(f)$ is also a union of strata.

Now, if $f : (N, x_0) \to (P, y_0)$ is a stable germ, for any small representative that we also denote by f, the stratum $X \in \mathcal{S}$ which contains x_0 is connected and its codimension is strictly greater than the codimension of any other stratum of $\mathcal{S}$. This number depends only of f. We call it *the codimension* of f, and we write $\mathbf{cod}\, f$. A germ f has codimension zero if and only if it is of maximal rank.

This notion generalizes to map-germs of finite singularity type.

Definition 1.7.7 Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a map of finite singularity type. We define $\mathbf{cod}\, f$ *at* $x = 0$ as the codimension of a stable unfolding of f.

Notice that this number is well defined. In fact, if $F : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^p \times \mathbb{R}^s, 0)$ and $F' : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0)$ are stable unfoldings of f and if, say, $r = s+k$, then it follows that $F \times Id$ is equivalent to F', where Id is the identity map in $\mathbb{R}^k$. Then $\mathbf{cod}\,(F \times Id) = \mathbf{cod}\, F'$, and it easy to see that $\mathbf{cod}\, F = \mathbf{cod}\,(F \times Id)$. Now the following result follows easily.

Proposition 1.7.8 *If* $f \underset{\mathcal{K}}{\sim} f'$ *then* $\mathbf{cod}\, f = \mathbf{cod}\, f'$.

The properties of the stratification $\mathcal{A}^l(N, P)$ can be summarized in the following results.

Proposition 1.7.9 *Let* $f : (N, x_0) \to (P, y_0)$ *be a smooth map-germ with an unfolding* $F : (N', x'_0) \to (P', y'_0)$*, as in the diagram*

$$\begin{array}{ccc} (N', x'_0) & \xrightarrow{F} & (P', y'_0) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ (N, x_0) & \xrightarrow[f]{} & (P, y_0). \end{array}$$

Then the following conditions are equivalent

(i) $j^l f \notin W^l(N, P)$ *and* $j^l f$ *is transverse to* $\mathcal{A}^l(N, P)$.
(ii) $j^l F \notin W^l(N', P')$ *and* $j^l F$ *is transverse to* $\mathcal{A}^l(N', P')$*, and in addition if* $X \in (j^l F)^{-1}\mathcal{A}^l(N', P')$ *contains* x'_0*, then* i *is transverse to* N'.

Proposition 1.7.10 (Gibson et al., [39], Proposition 3.3, Chapter 4) *Let* $f : N \to P$ *be a proper smooth mapping multi-transverse to* $\mathcal{A}^l(N, P)$ *and such that*

$j^l f(N) \cap W^l(N, P) = \emptyset$. Let $\mathcal{S} = (j^l f)^{-1}\mathcal{A}^l(N, P)$ and $\mathcal{S}' = \{f(X) \mid X \in \mathcal{S}\} \cup \{P \setminus f(N)\}$. Then $(\mathcal{S}, \mathcal{S}')$ is a Thom stratification of f.

Remark 1.7.11 The pair $(\mathcal{S}, \mathcal{S}')$ in Proposition 1.7.10 has a minimality property which uniquely characterizes it among all possible pairs. We refer to Gibson et al., [39] or Mather [66] for details.

1.7.2 Proof that Topologically Stable Mappings are Dense (Mather, [66], §8)

Initially, we state the Thom-Mather topological stability theorem, whose proof we outline in this section. Theorem B will follow from this result and Thom's transversality theorem.

Theorem 1.7.12 *If $f : N \to P$ is proper and for some (and hence for all) $k \geq p + 1$, $j^k f$ is multitransverse to the Thom-Mather stratification of $J^k(N, P)$, then f is strongly C^∞-stable.*

Given $f : N \to P$, we will show that we can approximate it by a topologically stable mapping. First, we approximate f by a mapping $f_1 : N \to P$ of finite singularity type (Proposition 1.7.5). Then, we can choose an unfolding (F, N', P', i, j) of f_1 such that F is proper and infinitesimally stable. Let $\mathcal{S}'_{N'}$ and $\mathcal{S}'_{P'}$ be stratifications of N' and P', respectively satisfying conditions (1)–(3) in Sect. 1.7.1.

By Thom's transversality theorem, we can approximate j by $j_2 : P \to P'$ such that j_2 is transverse to the strata of $\mathcal{S}'_{P'}$. Moreover we may suppose $j_2 = j$ outside a compact neighborhood of $f(N)$.

Since F is transverse to j, it follows that F is transverse to j_2 for j_2 sufficiently close to j.

The set $N_2 = F^{-1}(j_2(P))$ is a smooth manifold. One can show that there is a diffeomorphism $i_2 : N \to N_2$ close to $i : N \to N'$.

We let $f_2 : j_2^{-1} \circ F \circ i_2 : N \to P$. It follows from construction that f_2 is close to f in the C^∞ topology. We claim that f_2 is topologically stable.

The proof is based on the following facts from the construction we have made:

(i) (F, N', P', i_2, j_2) is an unfolding of f_2;
(ii) F is proper and infinitesimally stable;
(iii) j_2 is transverse to the stratification $\mathcal{S}'_{P'}$ of P'.

Let g be a small perturbation of f_2, so that we can suppose f_2 and g are connected by a small arc g_t in $C^\infty(N, P), t \in [0, 1]$, $g_0 = f_2, g_1 = g$. We can lift g_t to an arc G_t in $C^\infty(N', P')$ such that $G_0 = F$ and (G_t, N', P', i, j) is an unfolding of g_t. Moreover, we may suppose that $G_t = F$ outside of a sufficiently small compact neighborhood of $i(N)$.

From Theorem 1.3.15, it follows that there exist one parameter families of diffeomorphisms $(H_t, K_t) \in \mathcal{A}$, $H_0 = Id_{N'}$, $K_0 = Id_{P'}$, such that $F = K_t \circ G_t \circ H_t^{-1}$, for all $t \in [0, 1]$.

Now consider the commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{i} & N' & \xrightarrow{H_t} & N' \\ {\scriptstyle g_t}\downarrow & & {\scriptstyle G_t}\downarrow & & {\scriptstyle F}\downarrow \\ P & \xrightarrow{j} & P' & \xrightarrow{K_t} & P' \end{array}$$

Since (G_t, N', P', i, j) is an unfolding of g_t, it follows that $(F, N', P', H_t \circ i, K_t \circ j)$ is also an unfolding of g_t. Let $G(x, t) = (g_t(x), t)$, $\tilde{H}(x, t) = H_t(x)$ and $\tilde{K}(y, t) = K_t(y)$. Then we have the following commutative diagram

$$\begin{array}{ccc} N \times I & \xrightarrow{\tilde{H}} & N' \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle F} \\ P \times I & \xrightarrow{\tilde{K}} & P' \end{array}$$

So,we have that the triple $(F, \mathcal{S}'_{N'}, \mathcal{S}'_{P'})$ is a Thom stratified map, and i and j are transverse respectively to $\mathcal{S}'_{N'}$ and $\mathcal{S}'_{P'}$. Then taking g sufficiently close to f_2, $H_t \circ i$ and $K_t \circ j$ are also transverse to $\mathcal{S}'_{N'}$ and $\mathcal{S}'_{P'}$, respectively.

It follows that these stratifications pull back to the Whitney's stratifications $\tilde{H}^*(\mathcal{S}_{N'})$ and $\tilde{K}^*(\mathcal{S}'_{P'})$ in $N \times I$ and $P \times I$, respectively.

Moreover, each $N \times \{t\}$, $P \times \{t\}$ is transverse to $\tilde{H}^*(\mathcal{S}_{N'})$ and $\tilde{K}^*(\mathcal{S}'_{P'})$, and conditions (1)–(3) are satisfied.

Then, we may apply the Thom's second isotopy lemma (Gibson et al., [39, theorem 5.8, Chapter II]) and conclude that $f_2 = g_0$ is topologically equivalent to $g = g_1$.

1.7.3 The Geometry of Topological Stability

Whether C^0-stability and C^∞-stability are equivalent notions in the nice dimensions is a question not answered by the Thom-Mather theory. The first steps towards such result appear in Robert May's thesis [67, 68]. Mays's results were followed by a series of papers by Damon [19–21], who proved in [21] that C^∞-stability is equivalent to a stronger notion of C^0-stability.

Some of the ideas introduced in these papers form part of the basis for Andrew du Plessis and Terry Wall's book on topological stability. The book, *The geometry of topological stability,* [82] published in 1995, is a deep contribution to the subject of topological stability of smooth mappings. They are motivated by the problems left unanswered in the Thom-Mather theory. One such problem is that it is very difficult to determine explicitly the Thom-Mather stratification $\mathcal{A}^k(n, p)$ in the complement of the nice dimensions and its boundary. Another problem is that the transversality to the Thom-Mather stratification is not a *necessary* condition for topological stability. In fact, this follows from a combination of results of Looijenga [51] and Bruce [12] as we see in Examples 1.7.15 and 1.7.16 below. Du Plessis and Wall give partial answers to the following two conjectures:

Conjecture (i) (Conjecture 1.3 in [82]) The smooth map $f : N \to P$ is W-strongly C^0-stable if and only if it is quasi-proper and locally C^0-stable.
Following [82], we say that a map f is *quasi-proper* if there is a neighborhood V of the discriminant $\Delta(f)$ in P such that the restriction of f to $f^{-1}(V)$, $f : f^{-1}(V) \to V$, is a proper map.

Conjecture (ii) If N is compact, $f : N \to P$ is C^0-stable if and only if it is locally C^0-stable.

Conjecture (iii) (Conjecture 1.4 in [82]) There exist a $\mathcal{K}$-invariant semi-algebraic stratification $\mathcal{B}^k(n, p)$ of $J^k(n, p) \setminus W^k(n, p)$ such that a smooth map $f : N \to P$ is locally C^0-stable if and only if, for k such that $\operatorname{cod} W^k(n, p) > n$, $j^k f$ avoids $W^k(n, p)$ and is multitransverse to $\mathcal{B}^k(n, p)$.

We summarize now the main results of [82].

Theorem 1.7.13 (Theorem 1.5, [82])

(i) If $f : N \to P$ is W-strongly C^0-stable, then it is quasi-proper and locally C^0-stable.
(ii) If $f : N \to P$ is quasi-proper, of a finite singularity type over a neighborhood of its discriminant, and locally tamely P-C^0-stable, then it is W-strongly C^0-stable.

The local P-C^0-stability is a very strong form of local C^0-stability. We refer to [82, p. 113], for the definition of *tame P-C^0-stability*.

Theorem 1.7.14 (Theorem 1.6, [82]) *There exist $\mathcal{K}$-invariant algebraic subsets $Y^k(n, k)$ in $J^k(n, k)$ with $W^k(n, k) \subseteq Y^k(n, k)$, and a $\mathcal{K}$-invariant stratification $\mathcal{B}^k(n, p)$ of $J^k(n, k) \setminus Y^k(n, k)$ with the following properties:*

(a) If $f : N \to P$ is locally C^0-stable, or if N is compact and f is C^0-stable, then $j^k f$ is multitransverse to $\mathcal{B}^k(N, P)$; moreover, if codim $Y^k(n, p) \geq n$, *then $j^k f$ avoids $Y^k(N, P)$.*
(b) If $f : N \to P$ is such that $j^k f$ avoids $Y^k(N, P)$ and is multitransverse to $\mathcal{B}^k(N, P)$, then f is locally tamely C^0-stable.

In the range of dimensions $n < \operatorname{codim} Y^k(n, p)$, the results imply that Conjectures (i) and (ii), with W^k replaced by Y^k, hold (see [82, pg. 8]).

We finish this section with two examples illustrating two rather delicate questions in the theory of C^0-stability.

Example 1.7.15 (The Simple Elliptic Singularity $\tilde{E}_8$) The simple elliptic singularities $\tilde{E}_8$ in $\mathbb{K}^3$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is the $\mathcal{K}$-unimodular family of hypersurfaces with isolated singularities defined by

$$\tilde{E}_8: \quad f_\lambda(x_0, x_1, x_2) = x_0^2 + x_1^3 + x_2^6 + \lambda x_0 x_1 x_2.$$

The family f_λ is weighted homogeneous of type $(3, 2, 1; 6)$, then the Milnor number $\mu(f_\lambda)$ is constant and equal to 10. When $\mathbb{K} = \mathbb{C}$, it was shown by Looijenga [51] that the stable unfolding of f_λ is topologically trivial along the moduli parameter λ.

From Sect. 1.4.3, (1.17), it follows that the stable unfolding of f_λ can be given as

$$\begin{aligned} F : (\mathbb{C}^3 \times \mathbb{C}^8 \times \mathbb{C}, 0) &\to (\mathbb{C} \times \mathbb{C}^8 \times \mathbb{C}, 0) \\ (x, u, \lambda) &\mapsto (\tilde{f}(x, u, \lambda), u, \lambda) \end{aligned}$$

with $x = (x_0, x_1, x_2)$, $u = (u_1, \dots, u_8)$, $\tilde{f}_\lambda(x, u) = \tilde{f}(x, u, \lambda)$, $\tilde{f}_\lambda(x, 0) = f_\lambda(x)$, and

$$\begin{aligned} \tilde{f}(x, u, \lambda) = x_0^2 + x_1^3 + x_2^6 + \lambda x_0 x_1 x_2 + u_1 x_1 + u_2 x_2 + u_3 x_1 x_2 \\ + u_4 x_2^2 + u_5 x_1 x_2^2 + u_6 x_2^3 + u_7 x_1 x_2^3 + u_8 x_2^4. \end{aligned}$$

For all λ sufficiently small, including $\lambda = 0$, $F_\lambda : (\mathbb{C}^{11}, 0) \to (\mathbb{C}^9, 0)$ is topologically stable. See Looijenga [51] and Bruce [12].

On the other hand, the construction of the Thom-Mather stratification $\mathcal{A}^k(n, p)$ in $J^k(n, p) \setminus W^k(n, p)$ as discussed in Sects. 1.7.2 and 1.7.3 reduces to the problem of finding a minimal Whitney stratification of jets of finite singularity type. However, Bruce proved that at $\lambda = 0$ the Whitney condition (b) fails (see [12], Proposition 2 and Example 3(a)). The failure of condition (b) can be geometrically detected as follows: the number of cusps ($\mathcal{A}_2$-singularities) of the intersection of the discriminant $\Delta(F)$ with a family of 2-planes transversal to $\Delta(F)$ jumps from 12 to 13 at $\lambda = 0$. This number is an invariant of the Thom-Mather stratification [12, Proposition 2].

If follows that the germ $F_0 : (\mathbb{C}^{11}, 0) \to (\mathbb{C}^9, 0)$ is topologically stable, but $j^k F_0$ is not transverse to the Thom-Mather stratification.

Example 1.7.16 (May [67] and du Plessis and Wall [82], Section 4.1)

Lef $f : \mathbb{R} \to \mathbb{R}$ be the proper map whose graph is illustrated in Fig. 1.5. Its singular set $\Sigma(f)$ is $\mathbb{Z} \subset \mathbb{R}$, and the critical values are $F(0) = 0$, $f(n) = n + 1$, for $n > 0$ and odd, and $f(n) = n - 1$ for $n > 0$ and even; while $f(-x) = -f(x)$.

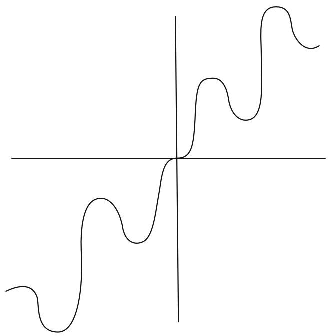


Fig. 1.5 C^0-stable non transversal map

For example, we may define, as in du Plessis and Wall [82],

$$f(x) = \begin{cases} x^3 & x \in [\frac{-1}{4}, \frac{1}{4}], \\ n + 1 - (x - n)^2 & x \in [n - \frac{1}{4}, n + \frac{1}{4}], n \in \mathbb{N}, n \text{ odd} \\ n - 1 + (x - n)^2 & x \in [n - \frac{1}{4}, n + \frac{1}{4}], n \in \mathbb{N}, n \text{ even} \end{cases}$$

with f defined on the remaining intervals so that it is monotone (with $f' \neq 0$) on each interval and C^∞ everywhere.

One can see that f is C^0-stable. However it is not transverse to the Boardman manifold Σ^1 at the origin. In fact, f cannot be transverse to any invariant stratification of jet space. Thus C^0-stability of proper maps $f : \mathbb{R} \to \mathbb{R}$ cannot be characterized by multitransversality to any stratification.

Notice that f is not locally C^0-stable, then it follows from Theorem 1.7.13(i) that f is not strongly stable.

1.7.4 Notes

In the recent paper *On the smooth Whitney fibering conjecture* [74] Murolo, du Plessis and Trotman give a remarkable improvement of the first Thom-Mather isotopy theorem for Whitney stratified sets. The result follows from their proof, in the same paper, of the smooth version of the Whitney fibering conjecture for Bekka (c)-regular stratifications. The original conjecture made by Whitney in [116] in the real and complex, local analytic and global algebraic cases, was proved by Parusinski and Paunescu [77] in 2014.

As an application of the results, in Sect. 1.9 of the paper, the authors give a sufficient condition for a smooth map between two smooth manifolds to be strongly topologically stable [74, Theorem 13].

This result in turn, implies the long-awaited improvements of Mather's topological stability theorem, which we state below.

Corollary 1.7.17 (Corollary 11, [74]) *Lef $f : N \to P$ be a quasi-proper smooth map of finite singularity type whose l-jet avoids $W^l(N, P)$ and is multi-transverse to $\mathcal{A}^l(N, P)$. Then f is strongly topologically stable.*

Corolllary 1.7.17 has the following immediate consequence.

Corollary 1.7.18 (Corollary 12, [74]) *The space of strong topologically stable maps is dense in the space of quasi-proper maps between two smooth manifolds.*

1.8 The Boundary of the Nice Dimensions

In this section we give a systematic presentation of the Thom-Mather singularities in the boundary of the nice dimensions (BND). Much of the material presented here is well known to experts. However, it seems that the organized presentation of the construction of the Thom-Mather stratification of $J^k(n, p)$ when (n, p) is a pair in BND combined with the discussion of the properties of topologically stable mappings in these dimensions do not appear in the literature. The results come from Mather [61, 63], Damon [22, 23], du Plessis and Wall [82] and Ruas [90] and recent results by Ruas and Trivedi [88].

We only give an outline of most of the proofs but we present the full details in the case $n = p = 9$.

We also review du Plessis and Wall main result in [83] that C^1-stable mappings are dense if and only if (n, p) is in the nice dimensions.

1.8.1 A Candidate for the Thom-Mather Stratification in BND

The main reference for this section is Ruas and Trivedi [88]. We saw that a pair (n, p) is in *the boundary of nice dimensions* if $\sigma(n, p) = n$, where $\sigma(n, p) = \operatorname{cod} \pi^k(n, p)$, $k \geq p + 1$, and $\pi^k(n, p)$ is the smallest Zariski closed $\mathcal{K}^k$-invariant set in $J^k(n, p)$ such that its complement in $J^k(n, p)$ is the union of finitely many $\mathcal{K}^k$-orbits.

In the nice dimensions $\sigma(n, p) > n$, so it follows that the strata of the stratification of $J^k(n, p) \setminus \pi^k(n, p)$ are the simple $\mathcal{K}^k$-orbits of $\mathcal{K}$-codimension $\leq n$. However, at the BND, there are strata of codimension n in $\pi^k(n, p)$; these strata cannot be avoided by transversal maps.We shall see that for all pairs (n, p) in BND with the exception of the pair $(10, 7)$ these strata are unimodular strata consisting of the union of a one-parameter family of $\mathcal{K}$-orbits. When $(n, p) = (10, 7)$, surprisingly, the Thom-Mather stratification also has a bimodal strata which is the union of a two parameter family of $\mathcal{K}$-orbits. We call the pair $(10, 7)$ *the exceptional pair in BND.*

We recall here the notion of modality (or modularity). This notion can be defined for any geometric subgroup of $\mathcal{K}$, but here we refer to modularity for group $\mathcal{K}$.

Let $z \in J^k(n, p)$ and denote by $K^*(z)$ the union of all $\mathcal{K}^k$-orbits of codimension equal to the codimension of $\mathcal{K}^k(z)$ in $J^k(n, p)$. Suppose $K_*(z)$ is the connected component of $K^*(z)$ in which z lies. Then we say that $z \in J^k(n, p)$ is r-modular if

$$\text{cod } K_*(z) = \text{cod } K^k \cdot z - r.$$

We say that 1-modular jets are *unimodular*, 2-modular jets are *bimodular* and so on. Also, if the union of unimodular jets is a submanifold of $J^k(n, p)$, as it happens in our case, we call this union a *unimodular stratum.*

The bad set $\tilde{\Pi}^k(n, p)$ in this case is a proper Zariski closed subset of $\Pi^k(n, p)$ such that cod $\tilde{\Pi}^k(n, p) \geq n+1$ and $\Pi^k(n, p)\backslash\tilde{\Pi}^k(n, p)$ is the union of the connected components of a unique unimodular family, while for the pair (10, 7) this set is the union of the unimodular and the bimodular families.

We stratify $J^k(n, p) \setminus \tilde{\Pi}^k(n, p)$ by taking as strata the $\mathcal{K}$-orbits of the stable maps and the modular strata. We call this stratification $\Sigma^k_{bnd}(n, p)$ (see [88]).

In the global setting we have the following situation. Let N, P and $J^k(N, P)$ as before. Denote by $\tilde{\Pi}(N, P)$ the subbundle of $J^k(N, P)$ with fibers $\tilde{\Pi}(n, p)$. Then the codimension of $J^k(N, P) \setminus \tilde{\Pi}(N, P)$ is equal to the codimension of $\tilde{\Pi}(n, p)$ in $J^k(n.p)$. Moreover, the stratification $\Sigma^k_{bnd}(n, p)$ induces a stratification on $J^k(N, P) \setminus \tilde{\Pi}(N, P)$, denoted by $\Sigma^k_{bnd}(N, P)$.

The following result appears in [88].

Theorem 1.8.1 (Ruas and Trivedi, [88], Theorem 3.1) *The set of maps $f : N \to P$ such that $j^k f(N) \cap \tilde{\Pi}(N, P) = \emptyset$ and $j^k f$ is transverse to the strata of $\Sigma^k_{bnd}(N, P)$ is open in $C^\infty(N, P)$ with the Whitney topology.*

The (a) regularity of $\Sigma^k_{bnd}(N, P)$ follows from the above result and the Main Theorem in Trotman [100].

Corollary 1.8.2 *The stratification $\Sigma^k_{bnd}(n, p)$ is (a)-regular.*

We prove in Theorem 1.8.4 that maps transverse to $\Sigma^k_{bnd}(N, P)$ are Thom-Mather maps for any pair (n, p) in BND.

1.8.2 The Unimodular Strata in BND

The results in this section are local and hold for map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $f \in \mathcal{E}^p_n$ or $f \in \mathcal{O}^p_n$. From Mather's calculations in [63], it follows that the following pairs lie in the boundary of the nice dimensions:

(i) $\mathbf{n} \leq \mathbf{p}$:

(1) The case $\sigma(n, p) = 6(p - n) + 9$ for $3 \geq p - n \geq 0$ and $n \geq 4$ or $n = 3$, gives $(n, p) \in \{(9, 9), (15, 16), (21, 23), (27, 30)\}$.

(2) The case $\sigma(n, p) = 6(p - n) + 8$ for $p - n \geq 4$ and $n \geq 4$, gives $(n, p) \in \{(6t + 2, 7t + 1); t \geq 5\}$.

(ii) $\mathbf{n > p}$:

(1) The case $\sigma(n, p) = 9$ for $n = p + 1$, gives $(n, p) = (9, 8)$.
(2) The case $\sigma(n, p) = 8$ for $n = p + 2$, gives $(n, p) = (8, 6)$.
(3) The case $\sigma(n, p) = n - p + 7$ for $n \geq p + 3$ gives $(n, p) \in \{(10 + k, 7) : k \geq 0\}$.

The strategy to find the strata of $\Sigma_{bnd}(n, p)$ has the following steps:

(1) inspecting the classification of the local algebras $Q(z)$, $z \in J^k(n, p)$, such that $\mathcal{K}$-cod$(z) \leq n$. By Mather's results these algebras are simple and for each such algebra $Q(z)$ there exists a stable germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, such that $Q(f) \simeq Q(z)$;
(2) listing the unimodular algebras of $\mathcal{K}$-codimension $n + 1$, whose union makes the unimodular strata of the stratification;
(3) Excluding the existence of bimodular strata of codimension n for pairs (n, p) in BND except $(10, 7)$. For $(n, p) = (10, 7)$ we include the classification of the bimodular strata.

A detailed discussion of simple and unimodular algebras appears in Chapter 8 of the book of du Plessis and Wall [82]. For the convenience of the reader we give the precise references of the classifications. First a word about the notation. We use mainly Thom's notation, and the relevant here are the first and second order Thom-Boardman symbols Σ^r and $\Sigma^{r,s}$, respectively, $r = 1, 2, 3, 4$. Mather's adaptation $\Sigma^{r(s)}$ also appears, as it is useful for 2-jet classification. A germ f in Σ^r may be regarded as an unfolding of a germ f_0 with rank zero and source dimension r. When we look at the second degree terms, the notation s in $\Sigma^{r(s)}$ indicates how many independent components the 2-jet of f_0 has.

We first describe the unimodular strata in the boundary of the nice dimensions, based on the presentation in Ruas and Trivedi [88],

1.8.2.1 Case 1: $n \leq p$

(1) $\mathbf{(n, p) = (9, 9)}$

The first unimodular family in this case is the one parameter family of type $\Sigma^{3,0}$ ($\Sigma^{3(3)}$ in Mather's notation) introduced in Sect. 1.6:

$$f_\lambda : (\mathbb{K}^3, 0) \to (\mathbb{K}^3, 0) \qquad (1.22)$$
$$(x, y, z) \mapsto (x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy)$$

with $\lambda \neq 0, -2, 1$.

Calculating the $\mathcal{K}$-tangent space of f_λ we find that $\mathcal{K}$-cod$(f_\lambda) = 10$, for $\lambda \neq 0, -2, 1$. The sets $(-\infty, 0)$, $(0, -2)$, $(-2, 1)$, $(1, \infty)$ parametrize orbits in the connected components of the unimodular strata of codimension 9.

(2) $(\mathbf{n}, \mathbf{p}) = (\mathbf{15}, \mathbf{16})$

The unimodular stratum in these dimensions is related to the moduli stratum in dimensions (9, 9) in the following way. From a result of Serre and Berger (see Eisenbud [31, Proposition 2]) it follows that for analytic map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ the class of the Jacobian $J(f)$ is a non-zero element in the local algebra $Q(f)$. Moreover, the ideal generated by $J(f)$ in this algebra is the unique minimal non-zero ideal in $Q(f)$. It also follows that the residue class of $J^2(f)$ in $Q(f)$ is zero.

The unimodular family here is

$$f_{1\lambda} : (\mathbb{K}^3, 0) \to (\mathbb{K}^4, 0),\ f_{1\lambda}(x, y, z) = (f_\lambda(x, y, z), J(f_\lambda)(x, y, z)), \tag{1.23}$$

where f_λ is the map given in (1.22) and $J(f_\lambda)(x, y, z) = xyz$. The following holds

$$\mathcal{K}\text{- cod}(f_{1\lambda}) = \mathcal{K}\text{- cod}(f_\lambda) + (\delta(f_\lambda) - 2) = 16$$

where $\delta(f_\lambda) = \dim_{\mathbb{K}} Q(f_\lambda) = 8$. The unimodular stratum in $J^k(15, 16),\ k \geq 3$ is the union of all corank 3 k-jets $z \in J^k(15, 16)$, $\mathcal{K}$-equivalent to a suspension of $f_{1\lambda}$.

(3) $(\mathbf{n}, \mathbf{p}) = (\mathbf{21}, \mathbf{23})$

In this case the unimodular family is

$$f_{2\lambda} : (\mathbb{K}^3, 0) \to (\mathbb{K}^5, 0),\ f_{2\lambda}(x, y, z) = (f_{1\lambda}(x, y, z), 0). \tag{1.24}$$

(4) $(\mathbf{n}, \mathbf{p}) = (\mathbf{27}, \mathbf{30})$

The unimodular family here is

$$f_{3\lambda} : (\mathbb{K}^3, 0) \to (\mathbb{K}^6, 0),\ f_{3\lambda}(x, y, z) = (f_{2\lambda}(x, y, z), 0). \tag{1.25}$$

Remark 1.8.3 The following formula holds (du Plessis and Wall [82], Chapter 8)

$$\mathcal{K}\text{- cod}(f_{i\lambda}) = \mathcal{K}\text{- cod}(f_\lambda) + (p - n)(\dim_{\mathbb{R}} Q(f_\lambda) - 2),$$

for $i = 1, 2, 3$, $p = n + i$.

(5) $(\mathbf{n}, \mathbf{p}) = (\mathbf{6t + 2}, \mathbf{7t + 1})$ **for** $\mathbf{t \geq 5}$

When $t = 5$ the unimodular stratum is defined by

$$f_\lambda : (\mathbb{K}^4, 0) \to (\mathbb{K}^8, 0),\ f_\lambda(x, y, z, w) = (u_1, u_2, \ldots, u_8)$$

where

$$u_1 = x^2 + y^2 + z^2 \qquad u_2 = y^2 + \lambda z^2 + w^2 \qquad u_3 = xy \qquad u_4 = xz$$
$$u_5 = xw \qquad u_6 = yz \qquad u_7 = yw \qquad u_8 = zw$$

1.8.2.2 Case 2: $n > p$

(6) $(\mathbf{n}, \mathbf{p}) = (\mathbf{8}, \mathbf{6})$

The smallest pair (n, p) with $n > p$ in the boundary of the nice dimensions is $(8, 6)$. The unimodular stratum is given by the following one-parameter family of maps

$$f_\lambda : (\mathbb{K}^4, 0) \to (\mathbb{K}^2, 0),$$
$$f_\lambda(x, y, z, w) = (x^2 + y^2 + z^2, y^2 + \lambda z^2 + w^2), \ \lambda \neq 0, 1.$$

(7) $(\mathbf{n}, \mathbf{p}) = (\mathbf{9}, \mathbf{8})$

The unimodular family here is

$$f_\lambda : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0),$$
$$f_\lambda(x, y) = x^4 + y^4 + \lambda x^2 y^2, \lambda \neq \pm 2.$$

(8) $(\mathbf{n}, \mathbf{p}) = (\mathbf{10} + \mathbf{k}, \mathbf{7})$ for $\mathbf{k} \geq \mathbf{0}$

In this case, the unimodular family is

$$f_\lambda : (\mathbb{K}^{4+k}, 0) \to (\mathbb{K}, 0),$$
$$f_\lambda(x, y, z, w_0, \ldots, w_k) = x^3 + y^3 + z^3 + \lambda xyz + \sum_{i=0}^{k} \delta_i w_i^2,$$

for $\delta = \pm 1, \ i = 0, \ldots, k, \ \lambda^3 \neq -1$.

The pair $(n, p) = (10, 7)$ is the exceptional pair in BND. It follows from Wall [109] that the following two parameter moduli family of Σ^5 singularities has codimension $n = 10$, providing for this pair of dimensions a new relevant strata.

$$f_\lambda : (\mathbb{K}^5, 0) \to (\mathbb{K}^2, 0),$$
$$f_\lambda(x) = (\sum_{i=1}^{5} a_i x_i^2, \sum_{i=1}^{5} b_i x_i^2), \ \ a_i b_j - a_j b_i \neq 0, \ i \neq j. \tag{1.26}$$

Theorem 1.8.4 *For each pair (n, p) in the boundary of the nice dimensions the following hold:*

*(a) If $(n, p) \neq (10, 7)$ the strata of $\Sigma^k_{bnd}(n, p)$ are the $\mathcal{K}^k$-orbits of the stable germs of $\mathcal{K}$-codimension $\leq n$ and the unimodular strata of codimension n defined by the connected components of the unimodular families described in Sects. 1.8.2.1 and 1.8.2.2. If $(n, p) = (10, 7)$, besides the unimodular strata defined in 1.8.2.2**(8)**, there is an exceptional bimodular strata as defined in* (1.26)*.*

(b) Maps $f : N \to P$ such that $j^k f$ is transverse to the strata of $\Sigma^k_{bnd}(n, p)$ are Thom-Mather maps for any pair (n, p) in BND.

Proof The proof consists on a careful inspection of the tables of simple and unimodular singularities in order to list the relevant strata and to verify that the codimension of the set $\tilde{\Pi}^k(n, p)$, $k \geq p + 1$ is greater than or equal to $n + 1$. We give an outline of the proof.

I. $\mathbf{n \leq p}$

For $(n, p) \in \{(9, 9), (15, 16), (21, 23), (27, 30)\}$ the relevant Boardman types are Σ^1, $\Sigma^{2,0}$, $\Sigma^{2,1}$ and Σ^3. We first analyze the pair $(9, 9)$.

Case (1) $\mathbf{(n, p) = (9, 9)}$

All singularities of type Σ^1 and $\Sigma^{2,0}$ are simple. A complete list of strata of type $\Sigma^{2,1}$ has been given by Dimca and Gibson [28]. See also Table 8.4 in du Plessis and Wall [82].

The first unimodular family of type $\Sigma^{2,1}$ is

$$I_{2,3} : (x^2 - \eta y^4, xy^3 + cy^5),\ c^2 \neq 0, \eta. \tag{1.27}$$

It follows that the $\mathcal{K}$-codimension of each orbit is 12, the unimodular stratum has codimension 11, so that this family does not appear generically when $n = p = 9$. As a consequence, the relevant $\Sigma^{2,1}$ strata in this case are simple $\mathcal{K}$-orbits. Notice that cod $\Sigma^{2,2}(9, 9) \geq 10$ and then the $\Sigma^{2,2}$ singularities do not appear generically in $J^k(9, 9)$.

The next Boardman symbol is Σ^3, and as we saw in Sect. 1.8.2.1, the relevant strata are the connected components of the unimodular family (1).

We list all the strata in Table 1.2.

The set $\tilde{\Pi}^k(9, 9)$ is the finite union of the following Zariski closed sets of codimension ≥ 10 in $J^k(9, 9)$, $k \geq 10$:

$$\tilde{\Pi}^k(9, 9) = \tilde{\Pi}^k_1 \cup \tilde{\Pi}^k_2 \cup \tilde{\Pi}^k_{j \geq 3}$$

Table 1.2 $(n, p) = (9, 9)$

Type	Name	Normal form	Conditions	$\mathcal{K}$-cod ≤ 9
Σ^1	A_j	(x^{j+1})	$1 \leq j \leq 9$	$j \leq 9$
$\Sigma^{2,0}$	$B^{\pm}_{p,q}$	$(xy, x^p \pm y^q)$	$2 \leq p \leq q \leq 8$	$4 \leq p + q \leq 9$
$\Sigma^{2,0}$	$B^{*}_{p,p}$	$(x^2 + y^2, x^p)$	$p = 3, 4$	$5 \leq 2p \leq 9$
$\Sigma^{2,1}$	C_{2k-1}	$(x^2 + y^3, y^{k+2})$	$k = 1, 2$	$2k + 5 \leq 9$
$\Sigma^{2,1}$	C_{2k}	$(x^2 + y^3, xy^{k+1})$	$k = 1$	$2k + 6 \leq 9$
$\Sigma^{3,0}$		$(x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy)$	$\lambda \neq -2, 0, 1$	10

where

$$\tilde{\Pi}^k_1 = \{\sigma \in J^k(9, 9), \sigma \in \Sigma^1, \mathcal{K}^k\text{-cod}(\sigma) \geq 10\}$$

$$\tilde{\Pi}^k_2 = \{\sigma \in J^k(9, 9), \sigma \in \Sigma^2, \mathcal{K}^k\text{-cod}(\sigma) \geq 10\}$$

$$\tilde{\Pi}^k_{j\geq 3} = \{\sigma \in J^k(9, 9), \ \sigma \in \Sigma^j, \ j \geq 3, \mathcal{K}^k\text{-cod}(\sigma) \geq 11\}$$

□

Cases (2) (15, 16); (3) (21, 23); (4) (27, 30)
The singularities of type Σ^1 and $\Sigma^{2,0}$ are simple. The classification of the singularities of type $\Sigma^{2,1}$ and their invariants in these cases can be found in Tables 8.7, 8.8 and 8.9 of [82]. The first unimodular family of type $\Sigma^{2,1}$, when $n < p$, is $\overline{D}_{3,5}$ (also denoted by $\overline{J}_{2,3,5,5}$ in [82]).

The normal forms are

$$f_{1\lambda}(x, y) = (x^2 \pm y^4, xy^3 + cy^5, y^6)$$

$$f_{2\lambda}(x, y) = (x^2 \pm y^4, xy^3 + cy^5, y^6, 0)$$

$$f_{3\lambda}(x, y) = (x^2 \pm y^4, xy^3 + cy^5, y^6, 0, 0)$$

From (1.27), we get

$$\mathcal{K}\text{-cod}(f_{i\lambda}) = \mathcal{K}\text{-cod}(f_\lambda) + +i(\dim_{\mathbb{R}} Q(f_\lambda) - 2),$$

for $i = 1, 2, 3$ where

$$f_\lambda(x, y) = (x^2 \pm y^4, xy^3 + cy^5). \tag{1.28}$$

Then $\mathcal{K}$-cod$(f_{i\lambda}) = 12 + i(10 - 2), i = 1, 2, 3$ and these singularities do not appear generically in BND. As in Case (1), for $n = 9 + 6i$, $i = 1, 2, 3$ with the help of Tables 8.7, 8.9, 8.9 and 8.11 in [82] we can verify that the strata of type Σ^1, $\Sigma^{2,0}$, $\Sigma^{2,1}$ and $\Sigma^{2,2}$ are $\mathcal{K}$-orbits of $\mathcal{K}$-codimension $\leq 9 + 6i$, $i =$

1, 2, 3 and the unimodular strata defined in (1.23), (1.24), and (1.25). Moreover, cod $\tilde{\Pi}^k(n, p) \geq n + 1$.

Cases (5) $(\mathbf{6t+2, 7t+1}), \mathbf{t \geq 5}$
The relevant Boardman types here are Σ^1, $\Sigma^{2,0}$, $\Sigma^{2,1}$, $\Sigma^{2,2}$, Σ^3 and Σ^4. As before Σ^1, $\Sigma^{2,0}$ are simple, and the moduli strata of type $\Sigma^{2,1}$ has normal form $f_{3\lambda} : (\mathbb{K}^2, 0) \to (\mathbb{K}^{t+1}, 0), t \geq 5$,

$$f_{3\lambda}(x, y) = (x^2 \pm y^4, xy^3 + cy^5, y^6, \underbrace{0, \ldots, 0}_{\text{t-1}}),$$

where $f_\lambda(x, y) = (x^2 \pm y^4, xy^3 + cy^5)$. Since $\mathcal{K}$-cod$(f_\lambda) = 12$, then $\mathcal{K}$-cod$(f_{3\lambda}) \geq 12 + (t-1)(10-2) = 4 + 8t > 6t + 2$, and it follows that this family is not generic when $(n, p) = (6t + 2, 7t + 1), t \geq 5$.

The $\Sigma^{2,2}$ germs of order 3 appear in du Plessis and Wall [82, Section 8.5, Tables 8.10 and 8.11]. The type $\Sigma^{2,2}$ is subdivided (see [82]) into types $\Sigma^{2,2(j)}$, where j is the rank of the kernel of the third intrinsic derivative. It follows that codim $\Sigma^{2,2(j)} = 6e + 10 + j(e + j - 2)$, where $e = p - n$. With a simple calculation we get that the relevant are $j = 0, 1$. Based on Table 8.10 of [82] we can verify that $\tilde{\Pi}(6t+2, 7t+1)$ contains the closure of the $\mathcal{K}$-orbit $(x^3 \pm xy^2, x^2y, y^3, 0, 0, 0)$ (type $E\text{-}Q_4^I$).

Germs of type Σ^n, $n = 3, 4$ are classified in [82], Section 8.6.

For $n = 3$, the more delicate analysis is that of singularities of type $\Sigma^{3(2)}$. Based on Tables 8.15, 8.17 and 8.20 in [82], it follows that the moduli does not occur in strata of codimension $\leq 6t - 2$, $t \geq 5$. It follows then that $\tilde{\Pi}(6t + 2, 7t + 1) \cap \overline{\Sigma^{3(2)}}$ is the closure of $\mathcal{K}$-orbits of codimensions $> 6t + 2$.

For the singularities of type $\Sigma^{3(3)}$, the best algebra of this type is the unimodular family whose normal form is $f_{4\lambda} = (f_{3\lambda}, 0)$, where $f_{3\lambda}$ is as in Sect. 1.8.2.1 **(4)**.

We know that $\mathcal{K}$-cod$(f_{3\lambda}) = 28$ and $\delta(f_{3\lambda}) = 7$, so that $\mathcal{K}$-cod$(f_{4\lambda}) = 28+6 = 34 > 32$. As the family is 1-modal it follows that the codimension of the stratum is 33, then this singularity does not occur generically in (32, 26). It is easy to extend this argument to all pairs $(6t + 2, 7t + 1)$, $t > 5$.

The first singularity of type Σ^4 in (32, 36) is the unimodular family Sect. 1.8.2.1 **(5)**. The $\mathcal{K}$-cod (f_λ) is 33 and the codimension of the stratum is 32.

It follows from our description that cod $\tilde{\Pi}(6t + 2, 7t + 1) > 6t + 2$.

Cases (6) (8, 6); (7) $(\mathbf{10+k, 7})$ $\mathbf{k > 0}$
These cases are simpler, since the deformations of the algebras have to be a simple function singularity, i.e., a singularity from Arnold's list of simple singularities of functions [3]. We can obtain the complete list from the adjacencies of simple and unimodular singularities from Arnold's [5].

The exceptional pair (10, 7) has two modular strata

(i) The unimodular family $f_\lambda(x, y, z, w) = x^3 + y^3 + z^3 + \lambda xyz + w^2$ with $\mathcal{K}$-cod$(f_\lambda) = 11$ and codimension of the stratum equal to 10.

(ii) The bimodular family $f_\lambda(x) = (\sum_{i=1}^{5} a_i x_i^2, \sum_{j=1}^{5} b_j x_j^2)$, $a_i b_j - a_j b_i \neq 0$, $1 \leq i, j \leq 5$, $i \neq j$.

1.8.3 Topological Triviality of Unimodular Families

Results on C^0-$\mathcal{A}$-triviality of the unimodular families of mappings appeared few years after Mather's theorem, due mainly to Eduard Looijenga [51, 52] and Jim Damon [22, 23].

In the 1977 paper Looijenga obtained explicit examples of topologically stable map-germs which are not analytically stable. He studied the simple elliptic singularities:

$$\tilde{E}_6 : \quad f(z_0, \ldots, z_n) = z_1(z_1 - z_0)(z_1 - \lambda z_0) + z_0 z_2^2 + Q(z_3, \ldots, z_n), \ (n \geq 2);$$

$$\tilde{E}_7 : \quad f(z_0, \ldots, z_n) = z_1 z_0(z_1 - z_0)(z_1 - \lambda z_0) + Q(z_2, \ldots, z_n), \ (n \geq 1);$$

$$\tilde{E}_8 : \quad f(z_0, \ldots, z_n) = z_1(z_1 - z_0^2)(z_1 - \lambda z_0^2) + Q(z_2, \ldots, z_n), \ (n \geq 1).$$

where Q is any nondegenerate quadratic form. He proved that two simple-elliptic singularities in the same family have topologically equivalent semi-universal deformations. As a consequence he obtained the C^0-$\mathcal{A}$-triviality of the stable unfolding of these singularities along the moduli parameter.

Remark 1.8.5 The family $\tilde{E}_6$ is analytically equivalent to the family 1.8.2.2 **(8)** and $\tilde{E}_7$ is analytically equivalent to the family 1.8.2.2 **(7)**. The family $\tilde{E}_8$ does not occur generically in BND.

Looijenga's approach to this problem is based on the weighted homogeneity of the germs together with algebraic calculations to solve a localized form of equation for infinitesimal C^∞ or analytic triviality.

Wirthmüller [119] extended Looijenga's results proving the topological triviality of the versal unfolding of non-simple hypersurfaces germs along the Hessian deformation parameter. These results were further extended by J. Damon [22, 23] for unfoldings F of "non-negative weight" of a weighted homogeneous polynomial germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. His main result applies to a large class of unimodular families, which includes all unimodular families in the boundary of the nice dimensions.

Theorem 1.8.6 (Damon, [22]) *If f is a weighted homogeneous $\mathcal{A}$-finitely determined germ, then any polynomial unfolding of f of non-negative weight is topologically trivial*

Damon's result apply to weighted homogeneous $\mathcal{A}$-finitely determined germs f of type $(w_1, \ldots, w_n; d_1, d_2, \ldots, d_p)$ and their unfoldings of weighted degree equal to or higher than the weighted degree of f.

The unimodular families in the boundary of the nice dimensions satisfy an even stronger condition: up to the addition of a quadratic form, the $\mathcal{K}$-orbits $\mathcal{K}(f_\lambda)$ in Sects. 1.8.2.1 and 1.8.2.2 have a homogeneous normal form; in other words we can take weights $w_1 = w_2 = \cdots = w_n = 1$, and if we write $f_\lambda : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$, $f_\lambda = (f_{1\lambda}, f_{2\lambda}, \ldots, f_{t\lambda})$, then $f_{i\lambda}$ is homogeneous of degree $d_i,\ i = 1, \ldots, t$. As in Sect. 1.4.3 let

$$N(f_\lambda) \simeq \frac{\Theta(f_\lambda)}{TK_e(f_\lambda) + \omega f_\lambda(\Theta_t)}.$$

Notice that since f_λ has rank 0, it follows that $N(f_\lambda) \simeq \frac{\mathcal{M}_s\Theta(f_\lambda)}{TK_e(f_\lambda)}$.

Let $J(f_\lambda)$ be the ideal generated by the $t \times t$ minors of f_λ and let $I(f_\lambda) = J(f_\lambda) + f_\lambda^*(\mathcal{M}_p)$. Notice that when $s < t$, $I(f_\lambda) = f_\lambda^*(\mathcal{M}_p)$.

Lemma 1.8.7

(a) If

$$I_\lambda^1 = \langle x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy, xyz\rangle, \quad \lambda \neq -2, 0, 1$$

and

$$I_\lambda^2 = \langle x^2 + y^2 + z^2, y^2 + \lambda z^2 + w^2, xy, xz, xw, yz, yw\rangle, \quad \lambda \neq 0, 1$$

then $I_\lambda^i \supseteq \mathcal{M}^3,\ i = 1, 2.$

(b) For each normal form (1) *to* (5) *in Sect. 1.8.2.1 and* (6) *in Sect. 1.8.2.2,* $TK_e(f_\lambda) \supseteq \mathcal{M}^3\Theta(f_\lambda)$.

(c) For the normal form (8) *in 1.8.2.2,* $J(f_\lambda) \supseteq \mathcal{M}^4$.

(d) For the normal form (7) *in 1.8.2.2,* $J(f_\lambda) \supseteq \mathcal{M}^5$.

Proof $(a), (c)$ and (d) follows from easy calculations, using the corresponding normal forms.

To prove (b) notice that if $I(f_\lambda) = J(f_\lambda) + f_\lambda^*(\mathcal{M}_t)$, it follows that $I(f_\lambda)\Theta(f_\lambda) \subset TK_e(f_\lambda)$, and the result follows from (a). □

With the help of the above Lemma it is an easy task to find, for each normal form, (1) to (5) in Sect. 1.8.2.1 and (6) to (8) in Sect. 1.8.2.2, a monomial basis for the normal space $N(f_\lambda)$, so that we can write

$$N(f_\lambda) \cong \mathbb{K}\{\sigma_1, \sigma_2, \ldots \sigma_r, \sigma_m\}$$

where the r generators $\sigma_j = (\sigma_{1j}, \sigma_{2j}, \ldots, \sigma_{tj}) \in \theta(f_\lambda),\ j = 1, \ldots, r$ have the following property: each coordinate $\sigma_{ij},\ i = 1, \ldots, t$ of σ_j satisfies the following condition

$$\text{degree}\, \sigma_{ij} < \text{degree}\, f_{i\lambda} \quad i = 1, \ldots, t, \quad j = 1, \ldots, r.$$

The generator $\sigma_m = (\sigma_{1m}, \sigma_{2m}, \ldots, \sigma_{tm})$ is the direction of the modulus and the degree $\sigma_{im} = \text{degree} f_{i\lambda}$ for $i = 1, \ldots, t$.

For each $\lambda = \lambda_0$, the stable unfolding of f_{λ_0} is the map-germ

$$F : (\mathbb{K}^s \times \mathbb{K}^r \times \mathbb{K}, 0) \to (\mathbb{K}^t \times \mathbb{K}^r \times \mathbb{K}, 0) \tag{1.29}$$
$$(x, u, \lambda) \mapsto (\tilde{f}(x, u, \lambda), u, \lambda),$$

$x = (x_1, \ldots, x_s),, u = (u_1, \ldots, u_r)$, and

$$\tilde{f}(x, u, \lambda) = f(x, \lambda_0) + \sum_{j=1}^{r} u_j \sigma_j(x) + \lambda \sigma_m(x).$$

For each λ_0, with the exception of a finite number of exceptional values, we obtain the normal form of the unimodular topologically stable singularity:

$$F_{\lambda_0} : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0),$$

with

$$F_{\lambda_0}(x, u) = (\tilde{f}_{\lambda_0}(x, u), u), \tag{1.30}$$

where

$$\tilde{f}_{\lambda_0}(x, u) = f(x, \lambda_0) + \sum_{j=1}^{r} u_j \sigma_j(x). \tag{1.31}$$

and $n = s + r, \ p = t + r$.

Remark 1.8.8 Notice that F_{λ_0} is unfolding of $f_{\lambda_0}(x)$ by terms σ_j of smaller degree. Damon's in [22] refers to F_{λ_0} as unfolding of negative weight of f_{λ_0} (see section 2 in Damon [23]).

A similar construction can be made for the exceptional pair $(n, p) = (10, 7)$. The bimodal family $f_\lambda = (\mathbb{K}^5, 0) \to (\mathbb{K}^2, 0), \ \lambda = \lambda_1, \lambda_2$ has a normal space

$$N(f_\lambda) \simeq \mathcal{R}\{\sigma_1, \ldots, \sigma_r, \sigma_m^1, \sigma_m^2\},$$

where $\{\sigma_m^1, \sigma_m^2\}$ generates the bimodal plane and degree $\sigma_m^i = \text{degree} f_\lambda = 2, \ i = 1, 2$. The normal form of the topologically stable singularity is given by (1.30).

We display these normal forms in Tables 1.3, 1.4, 1.5, and 1.6. To simplify notation we denote the canonical basis in $(\mathbb{R}^t, 0)$ by $\{e_i, i = 1, \ldots, t\}$, so that an element $g \in \mathcal{E}_s^t$ can be written as $g(x) = \sum_{i=1}^r g_i(x) e_i$.

We remark that, with convenient choices of weights for the variables $u_1, \ldots, u_r$, each normal form F_{λ_0} is a weighted homogeneous germ. To apply Damon's result

Table 1.3 $6(p-n)+9=n,\ 3 \leq p-n \leq 0$

(n,p)	$f=(f_1,\dots,f_t)$	Unfolding monomials $< m$	r	σ_m
$(9,9)$	$f_\lambda =$ $(x^2+\lambda yz,\ y^2+\lambda xz,\ z^2+\lambda xy)$ $\lambda \neq -2, 0, 1$	$\{y,z\}e_1,\ \{x,z\}e_2,$ $\{x,y\}e_3$	6	$yze_1+xze_2+xye_3$
$(15,16)$	$f_{1\lambda}=(f_\lambda, Jf_\lambda),\ Jf_\lambda = xyz$	$\{y,z\}e_1,\ \{x,z\}e_2,$ $\{x,y\}e_3,\ \{x,y,z\}e_4,$ $\{yz,xz,xy\}e_4,$	12	$yze_1+xze_2+xye_3$
$(21,23)$	$f_{2\lambda}=(f_{1\lambda},0)$	$\{y,z\}e_1,\ \{x,z\}e_2,$ $\{x,y\}e_3,\ \{x,y,z\}e_4$ $\{yz,xz,xy\}e_4$ $\{x,y,z\}e_5,\ \{yz,xz,xy\}e_5$	18	$yze_1+xze_2+xye_3$
$(27,30)$	$f_{3\lambda}=(f_{2\lambda},0)$	$\{y,z\}e_1,\ \{x,z\}e_2,$ $\{x,y\}e_3,\ \{x,y,z\}e_4$ $\{yz,xz,xy\}e_4$ $\{x,y,z\}e_5,\ \{yz,xz,xy\}e_5$ $\{x,y,z\}e_6,\ \{yz,xz,xy\}e_6$	24	$yze_1+xze_2+xye_3$

Table 1.4 $6(p-n)+8,\ p-n \geq 4,\ n \geq 4$

(n,p)	$f=(f_1,\dots,f_t)$	Unfolding monomials $< m$	r	σ_m
$(6s+2, 7s+1)$ $s \geq 5$	$f_\lambda := (x^2+y^2+z^2,\ y^2+\lambda z^2+w^2,$ $xy, xz, xw, yz, yw, zw, \underbrace{0,\dots,0}_{s-5})$ $t=s+3,\ s\geq 5$	$\{x,y\}e_1,\ \{z,x\}e_2$ $\{x,y,z,w\}e_{3+i}$ $0 \leq i \leq s,\ s \geq 5$	$6s-2$ $s \geq 5$	z^2e_2

Table 1.5 $n > p$

(n,p)	$f=(f_1,\dots,f_t)$	Unfolding monomials $< m$	r	σ_m
$(8,6)$	$(x^2+y^2+z^2,\ y^2+\lambda z^2+$ $w^2),\ \lambda \neq 1$	$\{y,w\}e_1, \{x,z\}e_2$	4	z^2e_2
$(10+k,7),\ k \geq 0$	$x^3+y^3+z^3+\lambda xyz+$ $\sum_{i=1}^k \delta_i w_i^2,$ $\delta_i = \pm 1,\ \lambda^3 \neq -1$	$\{x,y,z,yz,xz,xy\}e_1$	6	$xyze_1$
$(9,8)$	$x^4+y^4+\lambda x^2y^2,\ \lambda \neq \pm 2$	$\{x,y,x^2,xy,y^2,x^2y,xy^2\}e_1$	7	$x^2y^2e_1$

Table 1.6 Bimodular strata

Exceptional pair	Complex normal form	Unfolding monomials $< m$, $m=2$	r	σ_m^1, σ_m^2
$(10,7)$	$f_{\lambda_1\lambda_2} = (p(x), q(x))$ $p(x) = \sum_{i=1}^4 x_i^2$ $q(x) =$ $x_2^2+\lambda_1 x_3^2+\lambda_2 x_4^2+x_5^2$ $\lambda_i \neq 0, 1\ i=1,2$	$\{x_2,x_3,x_4,x_5\}e_1$ $\{x_1\}e_2$	5	$\{x_3^2, x_4^2\}e_2$

(Theorem 1.8.6) we need to show that F_{λ_0} is $\mathcal{A}$-finitely determined. The relevant property of F_{λ_0} is that the $\mathcal{A}$-orbit is open in the $\mathcal{K}$-orbit, as we now explain.

Definition 1.8.9 Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $\mathcal{A}$-finitely determined map-germ. The *$\mathcal{A}$-orbit of f is open in the $\mathcal{K}$-orbit of f* if $T\mathcal{A}(f) = T\mathcal{K}(f)$.

Given a pair (n, p) and a $\mathcal{K}^k$-orbit in $J^k(n, p)$, if this $\mathcal{K}^k$-orbit does not contain an infinitesimally stable map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $j^k f(0) \in \mathcal{K}^k$, we can ask whether there exist f such that $\mathcal{A}^k(f)$ is open in $\mathcal{K}^k(f)$. This was introduced by Ruas [90] as an approach to the $\mathcal{A}$-classification problem. The non existence of f with such property implies that all map-germs $f \in \mathcal{K}^k$ are non-simple. The following necessary and sufficient condition for the existence of an open orbit in $\mathcal{K}(f)$ was given in [90] (see also Rieger and Ruas [85]).

Proposition 1.8.10 (Ruas, [90], Theorem 5.1, Rieger and Ruas, [85], Prop.4.6) *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a $\mathcal{K}$-finitely determined germ and denote by $\{v_1, v_2, \ldots, v_r\}$ a basis for $N = \frac{\theta_f}{T\mathcal{A}_e f + f^*\mathcal{M}_p\theta_f}$. The $\mathcal{A}$-orbit of f is open in the $\mathcal{K}$-orbit of f if $f_i v_j \in T\mathcal{A}f$, mod$(f^*\mathcal{M}_p^2\theta_f)$ for $i = 1, \ldots, p$, $j = 1, \ldots, r$.*

To apply proposition 1.8.10 to the unimodular singularities at BND we introduce the following notation, where F_λ is as in Eq. (1.30).

Let

$$T_{F_\lambda} = F_\lambda^*(\mathcal{M}_p)\{\sigma_1, \sigma_2, \ldots, \sigma_r\} + tF_\lambda(\mathcal{M}_{s+n}\Psi_{s+r}) + \omega F_\lambda(\mathcal{M}_{t+r}\Psi_{t+r}).$$

This is a $F_\lambda^*(\mathcal{E}_{t+r})$-submodule of Ψ_{F_λ} consisting of elements of $T\mathcal{A}(F_\lambda)$ with zero components in the $\mathbb{R}^r$ direction (see Sect. 1.4.2).

Corollary 1.8.11 *Let F_λ as in (1.30). Then $\mathcal{A}(F_\lambda)$ is open in $\mathcal{K}(F_\lambda)$ is and only if*

(i) $(\tilde{f}_\lambda)_i \cdot \sigma_m \in T_{F_\lambda} + F^*(\mathcal{M}_p^2)\Psi_{F_\lambda}$, $i = 1, \ldots, t$.
(ii) $u_j \cdot \sigma_m \in T_{F_\lambda} + F^*(\mathcal{M}_p^2)\Psi_{F_\lambda}$, $i = 1, \ldots, r$.

Remark 1.8.12 Taking the quotient $\frac{T_{F_\lambda}}{\mathcal{M}_u T_{F_\lambda}}$ in condition (i) of Corollary 1.8.11, we get

$$(i_0)\ \ (f_\lambda)_i \cdot \sigma_m \in \frac{T_{F_\lambda}}{\mathcal{M}_u T_{F_\lambda}} \simeq f^*(m_t)\{\sigma_1, \ldots \sigma_r\} + tf_\lambda(m_s\Theta_s) + \omega f_\lambda(M_t\Theta_t. \tag{1.32}$$

The $f^*(\theta_t)$-module $\frac{T_{F_\lambda}}{\mathcal{M}_u T_{F_\lambda}}$ is $im(z_0)$ in Damon's notation (see definition of z_0 in section 1 of Damon [23]).

Condition (i_0) is a necessary condition for the property $T\mathcal{A}(F_\lambda) = T\mathcal{K}(F_\lambda)$ to hold.

We collect in the following proposition the relevant properties of F_{λ_0}.

Theorem 1.8.13 *Let (n, p) be a pair in BND and $F_{\lambda_0} : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ the unimodular map-germ as in (1.30). Then for all $\lambda_0 \in \mathbb{K}$, except a finite number of exceptional values the following hold:*

(a) F_{λ_0} *is $\mathcal{A}$-finitely determined.*
(b) $\mathcal{A}_e\text{-cod}\, F_{\lambda_0} = 1$.
(c) *The $\mathcal{A}$-orbit of F_{λ_0} is open in $\mathcal{K}(F_{\lambda_0})$.*

Proof First notice that $(c) \Leftrightarrow (b) \Rightarrow (a)$. In fact if (c) holds, $T\mathcal{A}(F_{\lambda_0}) = T\mathcal{K}(F_{\lambda_0})$. We saw that $\mathcal{K}\text{-cod}(F_{\lambda_0}) = n + 1$. Now, for any $\mathcal{A}$-finitely determined $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$, $S = \{x_1, \ldots, x_s\}$, the following formula due to L. Wilson [118] holds (see Rieger [84] for a proof):

$$\mathcal{A}_e\text{-cod}(f) = \mathcal{A}\text{-cod}(f) + s(p - n) - p.$$

Applying this formula with $s = 1$, it follows that $\mathcal{A}_e\text{-cod}(F_{\lambda_0}) = 1 \Leftrightarrow \mathcal{A}\text{-cod}(f) = n + 1$ and the equivalence $(c) \Leftrightarrow (b)$ follows from this. It is also clear that $(b) \Rightarrow (a)$.

We now want to verify (c) (or equivalently (b)). for each normal form $F_\lambda : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, with $F_\lambda(x, u) = (\tilde{f}_\lambda(x, u), u)$, $\tilde{f}_\lambda(x, u) = f_\lambda(x) + \sum_{j=1}^{r} u_j \cdot \sigma_j(x)$, $\text{degree}(\sigma_j) < \text{degree}(f_\lambda)$, $j = 1, \ldots, r$.

To verify (c), we verify condition (i) and (ii) in Corollary 1.8.11 to F_λ. We do it case by case, collecting calculations that appeared previously in the literature.

(1) Cases $(n, p) = \{(9, 9), (15, 16), (21, 23), (27, 30)\}$.

These were solved by Damon in Example 2 and Proposition 8.2, §8 in [23]. Notice that Damon uses Wall's normal form for the $\Sigma^{3,0}$ unimodular family

$$f_\lambda = (2xz + y^2, 2yz, x^2 + 3gy^2 - cz^2), c \neq 0, \quad c + 9g^2 \neq 0.$$

Here c is fixed and g is the modulus. □

(2) Cases $(n, p) = (8, 6)$ and $(n, p) = (32, 36)$.

We first consider $(n, p) = (8, 6)$.
$F_\lambda : (\mathbb{K}^8, 0) \to (\mathbb{K}^6, 0)$, $F_\lambda = (\tilde{f}_\lambda, u)$, where

$$\tilde{f}_\lambda(x, y, z, w, u) = (x^2 + y^2 + z^2 + u_1 y + u_2 w, y^2 + \lambda z^2 + w^2 + u_3 x + u_4 z).$$

It follows from Lemma 1.8.7 that F_λ is 2-determined with respect to the group $\mathcal{K}$, if $\lambda \neq 0, 1$. The following follow from simple calculations

(i) $J(f_\lambda) + f_\lambda^*(\mathcal{M}_2)$ contains the mixed monomials xy, xz, xw, yz, yw, zw.
(ii) If $\alpha = x^4, y^4, z^4, w^4$, then $\alpha e_i \in T\mathcal{A} f_\lambda$ $i = 1, 2 (\text{mod}\, J(f_\lambda)\Theta(f_\lambda))$.

Using (i) and (ii) it follows that the conditions of Corollary 1.8.11 hold, and $\mathcal{A}(F_\lambda)$ is open in $\mathcal{K}(F_\lambda)$.

We leave the calculations of the pair $(n, p) = (32, 36)$ as an exercise for the reader.

(3) Cases $(n, p) = (9, 8)$ and $(n, p) = (10 + k, 7),\ k \geq 0$.

These cases follows from Looijenga [52], Lemma 2.2.

Remark 1.8.14 A similar result holds for the bimodular strata in the pair $(10, 7)$ replacing $\mathcal{A}_e\text{-cod}(F_\lambda) = 1$ by $\mathcal{A}_e\text{-cod}(F_\lambda) = 2$.

We summarize the discussion of this section stating the following results.

Corollary 1.8.15 *Let (n, p) be a pair in BND and $F_{\lambda_0} : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ the unimodular map-germ as in (1.30). Then for all $\lambda_0 \in \mathbb{K}$, except for a finite number of exceptional values, the one parameter unfolding $F : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ of F_{λ_0}, as in (1.29), is $\mathcal{A}$-topologically trivial.*

Proof The proof follows from Theorem 1.8.13 and Damon's result (Theorem 1.8.6). □

Corollary 1.8.16 *Let (n, p) be a pair in BND. Then a Thom-Mather map $f : N^n \to P^p$ has at most a finite set of points $S = \{x_1, \ldots, x_r\}$ such that for all $x_i \in S$, $j^k f(x_i) \in \mathcal{A}_M$, $j^k f \pitchfork \mathcal{A}_M$, where $\mathcal{A}_M$ is any of the modal stratum of $\mathcal{A}^k(N, P)$. Moreover, if $f(x_i) = y_i$, $i = 1, \ldots r$ then $f^{-1}(y_i) \cap \Sigma(f) = \{x_i\}$, $i = 1, \ldots, r$. The restriction of f to $N \setminus S$ is an infinitesimally stable map.*

1.8.4 Notes

Density of C^1 Stable Mappings In [83], du Plessis and Wall determine the precise range of dimensions where C^1-stable maps are dense. This property holds if and only if the pair (n, p) is in the nice dimensions.

A parallel result is also obtained when C^1-stability is replaced by ∞-C^1-determinacy. We say that a map-germ $f \in \mathcal{E}_n^p$ is ∞-determined with respect to C^1-$\mathcal{A}$-equivalence if the C^1-$\mathcal{A}$-orbit of f contains all $g \in \mathcal{E}_n^p$ such that $j^\infty g(0) = j^\infty f(0)$. We can also denote the group C^1-$\mathcal{A}$ by $\mathcal{A}^{(1)}$.

The paper [83] appeared in 1989. In contrast with the C^0 and C^∞ cases much less was known in the C^1 case. Wall [106] sketched in 1980 the proof that C^1-stable maps are not dense when $n = 8$ and $p = 6$ and Mather [59] proved that finite $\mathcal{A}^{(1)}$determinacy does not hold in general for map-germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^{n+1}, 0)$, with $n \geq 15$.

The main result of [83] is the following theorem: (A) if (n, p) is in complement of the nice dimensions, then for any smooth manifolds N, P there is a nonempty open subset $U \subset C^\infty(N, P)$ containing no C^1-stable mapping. (B) If (n, p) is in the complement of semi-nice dimensions (see [80, 109] for details) with the exception of the pairs (14.14), (15, 15), (16, 16), (12, 11), (14, 12) and (15, 13), then for any pair of smooth manifolds N, P there is a nonempty open subset $U \subset C^\infty$ containing no map all of whose point-germs are ∞-$\mathcal{A}^1$-determined.

The proof of this theorem follows the line of the proof of the corresponding C^∞ result. It is shown that C^1stability implies transversality and ∞-$\mathcal{A}^{(1)}$-determinacy implies transversality off the base-point to the fibres of a $\mathcal{K}$-invariant fibred submanifold of $J^r(n, p)$ in the complement of the set $W^r(n, p)$ of r-jets with $\mathcal{K}^r$-modality ≥ 1. This follows from the property that stability an determinacy conditions imply a weak form of transversality (the preimage is a C^1-submanifold). To strengthen this to actual transversality the use of unfolding theory and a perturbation lemma of R.D. May [67] were the important tools.

Several notions of C^1-invariance of submanifolds of jet space are discussed in [82]. In particular, the C^1-invariance of the Thom-Boardman varieties and, in some cases, of $\mathcal{K}^r$-orbits within them are obtained.

1.9 Density of Lipschitz Stable Mappings

We discuss here the problem of density of Lipschitz stable mappings, which is still widely open.

In [76] Nguyen, Ruas and Trivedi introduced the *Lipschitz nice dimensions (LND)* as the pairs (n, p) for which the set $\mathcal{S}^{Lip}(N, P)$ of Lipschitz stable mappings is *dense* in $C^\infty_{pr}(N^n, P^p)$.

When N is compact, it is clear that the LND contains Mather's nice dimensions, since every C^∞ stable mapping is Lipschitz stable. The main purpose in Nguyen, Ruas and Trivedi [76] is to give an answer for the following conjectures.

Conjecture 1.9.1 The Lipschitz nice dimensions contains Mather's nice dimensions and its boundary.

Conjecture 1.9.2 The result in Conjecture 1.9.1 is sharp, that is, if (n, p) is in the complement of the nice dimensions or its boundary then $\mathrm{S}^{Lip}(N, P)$ is not a dense set in $C^\infty(N, P)$.

The following result is proved by Ruas and Trivedi [88].

Theorem 1.9.3 (Section 6, [88]) *The unimodular strata in the boundary of the nice dimensions are bi-Lipschitz $\mathcal{K}$-trivial.*

Remark 1.9.4 The exceptional unimodular strata when $(n, p) = (10, 7)$ also safisfies bi-Lipschitz $\mathcal{K}$-triviality condition.

We first review the notions of $\mathcal{K}$-equivalence and $\mathcal{K}$-triviality of r-parameter deformations.

Definition 1.9.5 A bi-Lipschitz $\mathcal{K}$-equivalence of r-parameter deformations is a pair (H, K) of bi-Lipschitz germs $H : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^r \times \mathbb{R}^n, 0)$ and $K : (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^p, 0)$ with H an r-parameter unfolding at 0 of the germ of the identity map of $\mathbb{R}^n$, and K an r-parameter unfolding at 0 of the germ of the identity in $\mathbb{R}^n \times \mathbb{R}^p$ such that the following diagram commutes

$$\begin{array}{ccccc}
(\mathbb{R}^r \times \mathbb{R}^n, 0) & \xrightarrow{i} & (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi} & (\mathbb{R}^r \times \mathbb{R}^n, 0) \\
\uparrow H & & \uparrow K & & \uparrow H \\
(\mathbb{R}^r \times \mathbb{R}^n, 0) & \xrightarrow{j} & (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi} & (\mathbb{R}^r \times \mathbb{R}^n, 0)
\end{array}$$

Here i is the canonical inclusion and π is the canonical projection. Two r-parameter deformations Φ and Ψ of f are bi-Lipschitz $\mathcal{K}$-equivalent if there exist a bi-Lipschitz $\mathcal{K}$-equivalence (H, K) as above such that

$$K \circ (id, \Phi) = (id, \Psi) \circ H.$$

If (H, K) has the special property that H is the germ of the identity on $\mathbb{R}^n$, then (H, K) is said to be a $\mathcal{C}$-equivalence and Φ and Ψ are said to be $\mathcal{C}$-equivalent deformations.

Definition 1.9.6 An r-parameter deformation Φ of a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is *bi-Lipschitz $\mathcal{K}$-trivial* (resp. *bi-Lipschitz $\mathcal{C}$-trivial*) if it is bi-Lipschitz $\mathcal{K}$-equivalent (resp. bi-Lipschitz $\mathcal{C}$-equivalent) to the deformation $\Psi : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, given by $\Psi(u, x) = f(x)$.

A sufficient condition for bi-Lipschitz $\mathcal{K}$-triviality is the following Thom-Levine type lemma.

Lemma 1.9.7 *Let* $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p, 0)$ *be a one parameter deformation of* $f : (\mathbb{R}^r, 0) \to (\mathbb{R}^p, 0)$. *If there exist a* $p \times p$ *matrix* (a_{ij}) *(not necessarily invertible) with entries germs of Lipschitz functions* $(\mathbb{R}^n \times \mathbb{R}, 0)$ *and a germ of a Lipschitz vector field* X *of the form*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x, t) \frac{\partial}{\partial x_i}$$

with $X_i(0, t) = 0$ *such that*

$$X \cdot \begin{bmatrix} F_1 \\ \vdots \\ F_p \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \dots & \vdots \\ a_{p1} & \dots & a_{pp} \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_p \end{bmatrix} \tag{1.33}$$

Then, F *is a bi-Lipschitz $\mathcal{K}$-trivial deformation.*

The proof follows from the fact the integration of a Lipschitz vector field gives a bi-Lipschitz flow. In fact, the bi-Lipschitz trivialization in source is given by integrating the vector field X and that in the product is given by integration of the vector field W, where

$$W(x, y, t) = \frac{\partial}{\partial t} + \sum_{i=1}^{p} W_i(x, y, t)\frac{\partial}{\partial y_i}$$

where $W_i(x, y, t) = \sum_{j=1}^{p} a_{ij} y_j$

The converse of the above lemma is not known and so we say that a one parameter deformation is *strongly bi-Lipschitz $\mathcal{K}$-trivial* if the conditions of the above lemma hold.

If $X_i(x, t) \equiv 0,\ i = 1, \ldots, n$, condition (1.33) implies that F is $\mathcal{C}$-trivial.

A case by case proof of the bi-Lipschitz $\mathcal{K}$-triviality of the unimodular strata Sect. 1.8.2.1 and 1.8.2.2 is given in Ruas and Trivedi [88]. The cases $n \leq p$ and $n > p$ are treated separately.

When $n \leq p$, the modal families are families of finite maps. For them, $\mathcal{K}$-determinacy holds if and only if $\mathcal{C}$-determinacy holds (see Wall [108], Prop. 2.4). In this case, we can apply the Lipschitz Thom-Levine lemma to prove the bi-Lipschitz $\mathcal{C}$-triviality of these families.

We discuss here the case $n = p = 9$.

Lemma 1.9.8 (Ruas and Trivedi, [88], Lemma 6.1) *The unimodular family Sect. 1.8.2.1 (1)*

$$F(x, y, z, \lambda) = (x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy),$$

$\lambda \neq -2, 0, 1$, *is strongly bi-Lipschitz $\mathcal{C}$-trivial.*

Proof Let $\mathcal{I}$ be the $\mathcal{E}_4$-ideal generated by the components of F, i.e.,

$$\mathcal{I} = \langle x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy\rangle.$$

We can prove that $\mathcal{I} \supset \mathcal{M}_3^4\mathcal{E}_4$, where $\mathcal{M}_3$ is the ideal generated by x, y, z. More precisely

$$\mathcal{I} \cdot \mathcal{M}_3^2\mathcal{E}_4 = \mathcal{M}_3^4\mathcal{E}_4 \tag{1.34}$$

Consider the following control function $\rho(x, y, z, \lambda) = \sqrt{F_1^2 + F_2^2 + F_3^2}$. Since F_λ is $\mathcal{C}$-finitely determined and homogeneous of degree 2 for all $\lambda \neq -2, 0, 1$, there exist constants c and c', (see Ruas [86]), such that

$$c'||(x, y, z)||^2 \leq \rho(x, y, z, \lambda) \leq c||(x, y, z)||^2$$

From (1.34) it follows that there exists a 3×3 matrix (a_{ij}) with entries in $\mathcal{M}_3^4\mathcal{E}_4$ such that

$$\rho^2(x, y, z, \lambda) \begin{bmatrix} \frac{\partial F_1}{\partial \lambda} \\ \frac{\partial F_2}{\partial \lambda} \\ \frac{\partial F_3}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

□

Now consider the germ of the vector field V on $(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, 0)$ defined by

$$V = \frac{\partial}{\partial \lambda} + \frac{1}{\rho^2} \left\{ \sum_{j=1}^{3} a_{1j} Y_j \frac{\partial}{\partial Y_1} + \sum_{j=1}^{3} a_{2j} Y_j \frac{\partial}{\partial Y_2} + \sum_{j=1}^{3} a_{3j} Y_j \frac{\partial}{\partial Y_3} \right\}$$

where $(Y_1, Y_2, Y_3) = Y$ are the target coordinates. Notice that $\frac{a_{ij} Y_j}{\rho^2}$ are continuous in a neighborhood of the origin in $(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, 0)$, but the derivative with respect to x, y, z are not bounded, so that V is not Lipschitz. However we can modify V to get a Lipschitz vector field $V' = pV$ where $p : (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ is defined as follows.

Let $D_1 = \{||Y|| \le c_1 ||(x, y, z, \lambda)||\}$ and $D_2 = \{||Y|| \ge c_2 ||(x, y, z, \lambda)||\}$ be cones in $(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})$ with $c_1 < c_2$ and let p be defined by

$$p(x, y, z, \lambda, Y) = \begin{cases} 1 & \text{if } (x, y, z, \lambda, Y) \in D_1 \\ 0 & \text{if } (x, y, z, \lambda, Y) \in D_2 \end{cases}$$

and $0 < p(x, y, z, \lambda, Y) < 1$ if $c_1 ||(x, y, z, \lambda)|| < |Y| < c_2 ||(x, y, z, \lambda)||$, such that the derivative of $p(x, y, z, \lambda, Y)$ with respect to any coordinate is bounded by a real number K (see Ruas [86] for details).

The integration of V' will give a bi-Lipschitz C-trivialization of F by the Thom-Levine criterion. This completes the proof.

Remark 1.9.9 For any fixed $\lambda = \lambda_0 \neq -2, 0, 1$, the deformation $F(x, y, z, \lambda)$ in Lemma 1.9.8 is semialgebraic and satisfies the condition $\frac{|F_\lambda(x,y,z)|}{|F_{\lambda_0}(x,y,z)|}$ is bounded for any (x, y, z, λ) in $(\mathbb{R}^3 \times \mathbb{R}, 0)$. Then we can also apply Theorem 3.1 of Ruas and Valette [89] to prove that F_λ is semialgebraically bi-Lipschitz $\mathcal{K}$-trivial. Notice however that the conclusion in Lemma 1.9.8 is stronger, as we prove that the family F_λ is strongly bi-Lipschitz $\mathcal{K}$-trivial.

The bi-Lipschitz $\mathcal{K}$-triviality of the Thom-Mather stratification along the unimodular strata in the boundary of the nice dimensions suggest that mappings transverse to this stratification are bi-Lipschitz stable.

A natural approach to prove Conjecture 1.9.1 is to follow the proof of Theorem 1.8.6, taking into account that the pair (n, p) is in the boundary of the nice dimensions.

We saw in Corollary 1.8.16 that a Thom-Mather map $f : N^n \to P^p$, (n, p) in the boundary of the nice dimensions has at most a finite set of points $S = \{x_1, \dots, x_\ell\}$ such that for all $x_i \in S$, $j^k f(x_i) \in \mathcal{A}_M$, $j^k f \pitchfork \mathcal{A}_M$, where $\mathcal{A}_M$ is the modal stratum. Moreover by multi-transversality, if $f(x_i) = y_i$, $i = 1, \dots \ell$ then

$$f^{-1}(y_i) \cap \Sigma(f) = \{x_i\},\ i = 1, \dots, \ell.$$

Clearly, f is an infinitesimally stable mapping in the complement of S.

To prove that f is Lipschitz stable it would be sufficient to prove that each unimodular family F_λ (see Sect. 1.8.3), and also the bimodular family when $(n, p) = (10, 7)$, is bi-Lipschitz $\mathcal{A}$-trivial.

Let

$$F(x, u, \lambda) = (\bar{f}(x, u, \lambda), u, \lambda)$$

be the (weighted homogeneous) normal form of a unimodular family in BND as in (1.29), where $x = (x_1, \dots, x_s)$, $u = (u_1, \dots, u_r)$, $s+r = n$ and $\bar{f} = (\bar{f}_1, \dots, \bar{f}_t)$.

Following the proof of Theorem 1.8.6, we can find weighted homogeneous vector fields V and W in source and target respectively, given by:

$$V(x, u, \lambda) = \sum_{j=1}^{s} v_j(x, u, \lambda)\frac{\partial}{\partial x_j} + \sum_{i=1}^{r} \overline{v}_i(\tilde{f}, u, \lambda)\frac{\partial}{\partial u_i} + \frac{\partial}{\partial \lambda}$$

where $x = (x_1, \dots, x_s)$, $u = (u_1, u_2, \dots, u_r)$ and $\tilde{v}_i(0, 0, \lambda) = v_j(0, 0, \lambda) = 0$,

$$W(X, U, \lambda) = \sum_{j=1}^{t} w_j(X, U, \lambda)\frac{\partial}{\partial X_j} + \sum_{i=1}^{r} \overline{w}_i(X, U, \lambda)\frac{\partial}{\partial U_i} + \frac{\partial}{\partial \lambda}$$

where $X = (X_1, \dots, X_t)$, $U = (U_1, \dots, U_r)$, and $\overline{w}_i(0, 0, \lambda) = w_j(0, 0, \lambda) = 0$, (capital letters denote the coordinates in the target), and a weighted homogeneous control function $\rho(X, U, \lambda)$ such that

$$(\rho \circ F)(x, u, \lambda)\frac{\partial \tilde{f}}{\partial \lambda} = -\sum_{j=1}^{3} \frac{\partial \tilde{f}}{\partial x_j} v_j(x, u, \lambda) - \sum_{i=1}^{6} \frac{\partial \tilde{f}}{\partial u_i}\overline{v}_i(\tilde{f}, u, \lambda) + \tilde{W}(\tilde{f}, u, \lambda) \tag{1.35}$$

where $\tilde{W} = (w_1, \dots, w_t)$.

It follows from (1.35) that the vector fields $\mathcal{X}(x, u, \lambda) = \frac{1}{(\rho \circ F)(x,u,\lambda)} V(x, u, \lambda)$ and $\mathcal{Y}(X, U, \lambda) = \frac{1}{\rho(X,U,\lambda)} W(X, U, \lambda)$ satisfy the equation $DF(\mathcal{X}) = \mathcal{Y} \circ F$. Moreover, they are continuous and can be integrated to give the topological $\mathcal{A}$-triviality of F along the moduli space.

If we can prove that $\mathcal{X}$ and $\mathcal{Y}$ are Lipschitz vector fields in the source and target, respectively, the bi-Lipschitz $\mathcal{A}$-triviality of F would follow from the Lipschitz version of the Thom-Levine lemma.

1.10 Sections of Discriminant of Stable Germs: Open Problems

An important consequence of Theorems **A** and **B** is that we can approximate any map $f : U \subset \mathbb{K}^n \to \mathbb{K}^p$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, by a stable mapping if (n, p) is in the nice dimensions or else by a topologically stable map if (n, p) is not in the nice dimensions.

For a map-germ of finite singularity type $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, a stable perturbation can be realized as the generic member of a 1-parameter unfolding $\bar{f}(x, t) = (f_t(x), t)$ of f. More precisely, $\bar{f}$ is a *stabilization* of f if there exists a representative $\bar{f} : U \to V \times T$ such that $f_t : U \cap (\mathbb{K}^n \times \{t\}) \to V$ is stable for all $t \neq 0$.

When $\mathbb{K} = \mathbb{C}$, the stable perturbation of f is uniquely determined up to $\mathcal{A}$-equivalence when (n, p) is in the nice dimensions and up to C^0-$\mathcal{A}$-equivalence otherwise. When $\mathbb{K} = \mathbb{R}$, there may exists a finite number of nonequivalent stabilizations of f. On the reals, in general $t > 0$ and $t < 0$ give non-equivalent perturbations of f (see Mond and Nuño-Ballesteros in this Handbook or [69] for details).

The geometry of the stable perturbations $\bar{f}$ are associated to invariants of the germ f.

We discuss here this important tool in singularity theory.

Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be a germ of a finite singularity type and F its stable unfolding:

$$\begin{array}{ccc} (\mathbb{K}^{n'}, 0) & \xrightarrow{F} & (\mathbb{K}^{p'}, 0) \\ \uparrow & & \uparrow g \\ (\mathbb{K}^{n}, 0) & \xrightarrow[f]{} & (\mathbb{K}^{p}, 0) \end{array} \tag{1.36}$$

where g is the germ of an immersion transverse to F.

Let $V = \Delta(F)$ be the discriminant of F (recall that when $n < p$ the discriminant is the image $F(\mathbb{K}^n)$.) Damon in [25] described a relation between $\mathcal{A}$-equivalence and properties of the discriminant V. This relation is valid for all pairs (n, p) and directly relates $\mathcal{A}_e$-codimension of f with a codimension of the germ at 0 of $g(\mathbb{K}^p)$ as a section of the discriminant. The idea of using sections of the discriminant to determine $\mathcal{A}$-determinacy properties of f appears in [53, 54] (see also du Plessis

[80]). However, the precise relation between an equivalence relation for germs of immersions g and the $\mathcal{A}$-equivalence of f was derived in [25].

Given the germ of a variety $(V, 0) \subset (\mathbb{K}^{p'}, 0)$ Damon defined the group $\mathcal{K}_V$ of *contact equivalences preserving* V which acts on the set of germs $g : (\mathbb{K}^p, 0) \to (\mathbb{K}^{p'}, 0)$ (the map-germs g are in $\mathcal{E}_p^{p'}$ when $\mathbb{K} = \mathbb{R}$ or in $\mathcal{O}_p^{p'}$ when $\mathbb{K} = \mathbb{C}$.)

The *contact group* $\mathcal{K}_V$ is defined as follows:

$$\mathcal{K}_V = \{(h, H) \in \mathcal{K} | \, H(\mathbb{K}^p \times V) \subseteq \mathbb{K}^p \times V\}$$

(see Definition 1.4.1).

The action of $\mathcal{K}_V$ on $\mathcal{E}_p^{p'}$ or $\mathcal{O}_p^{p'}$ is defined as in Definition 1.4.1. We can also define the similar notions for unfoldings.The group $\mathcal{K}_V$ is a geometric subgroup of the contact group, so that the machinery of singularity theory applies to $\mathcal{K}_V$-equivalence. In particular the infinitesimal and the geometric criteria for $\mathcal{K}_V$-determinacy.

We can define

$$T\mathcal{K}_V \cdot g = tg(\mathcal{M}_p\Theta_p) + \epsilon_p\{\eta_i \circ g, \; i = 1, \dots, m\}$$

$$T\mathcal{K}_{V_e} \cdot g = tg(\Theta_p) + \epsilon_p\{\eta_i \circ g, \; i = 1, \dots, m\}$$

where η_i, $i = 1, \dots, m$ are the generators of Θ_V, the $\epsilon_{p'}$-module of vector fields in $\mathbb{K}^{p'}$ tangent to the variety V at its smooth points. Equivalently, Θ_V is the $\epsilon_{p'}$ module of derivations of $\Theta_{p'}$ which preserve the ideal defining V. The notation Der(-log V) proposed by Saito for the module of these vector fields as well the notation g^*(Der(-log V)) for the ϵ_p-module $\epsilon_p\{\eta_i \circ g, \; i = 1, \dots, m\}$ are also widely used. See section 2.9 of the article of Nuño-Ballesteros and David Mond in this Handbook [70].

$T\mathcal{K}_V g$ and $T\mathcal{K}_{V_e} g$ are $\mathcal{O}_p$-modules when $\mathbb{K} = \mathbb{C}$ and $\mathcal{E}_p$-modules when $\mathbb{K} = \mathbb{R}$.

With the notations as in (1.36) we can state the main results in [25] as follows.

1. g has finite $\mathcal{K}_V$-codimension if and only if f has finite $\mathcal{A}$-codimension.
2. If $N\mathcal{A}_e f$ and $N\mathcal{K}_{V_e} g$ denote the normal spaces to $\mathcal{A}_e f$ and $\mathcal{K}_{V_e} g$, respectively, then

$$N\mathcal{A}_e f \cong N\mathcal{K}_{V_e} g.$$

3. $\mathcal{A}_e$-codimension$(f) = \mathcal{K}_{V_e}$-codimension(g).
4. Conditions 1. to 3. hold for multigerms $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$.

The geometric characterization of $\mathcal{K}_V$-equivalence holds only for holomorphic map-germs $f \in \mathcal{O}_n^p$, namely: $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is $\mathcal{A}$- finitely determined if and only if g is transverse to the strata of V away from the origin. For real germs, the geometric condition is a necessary condition for $\mathcal{K}_V$ finite determinacy, but the converse does not hold.

Damon's theory builds a solid bridge between singularity theory of mappings and topology of singular varieties. This connection has been used successfully for the past three decades. We follow this approach to formulate some open problems in singularity theory, related to the subject discussed in this paper.

1.10.1 Geometry of Sections of Discriminant of Stable Mappings in the Nice Dimensions

Let $(n+1, p+1)$ be a nice pair of dimensions and $F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$ a minimal stable map-germ. Minimal here means that $\{0\} \in \mathbb{K}^{n+1}$ is a stratum of the stratification of F by stable types. A hyperplane section $H = g(\mathbb{K}^p)$ transversal to the discriminant $V = \Delta(F) \subset \mathbb{K}^{p+1}$ away from the origin pulls back by F to an $\mathcal{A}$-finite map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$.

From Damon's result 3. above, it follows that if (n, p) is in the semi-nice dimensions (see Sect. 1.7.1) there exists an open and dense set $\mathcal{I}$ of immersions $g : (\mathbb{K}^p, 0) \to (\mathbb{K}^{p+1}, 0)$ such that the pull back of g by F is an $\mathcal{A}$-finite map germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ whose $\mathcal{A}_e$-codimension is minimal, that is,

$$\mathcal{A}_e\text{-cod}\, f \leq \mathcal{A}_e\text{-cod}\, f', \quad \text{for all } f' \underset{\mathcal{K}}{\sim} f.$$

As F is a minimal stable unfolding of f we may ask: is there a map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $Q(f) \cong Q(F)$ such that $\mathcal{A}_e\text{-cod}\, f = 1$, which in this case implies that $\mathcal{A}$-orbit of f is open in its $\mathcal{K}$-orbit?

It follows form Proposition 1.8.10 that this condition holds if and only if it holds for a general linear hyperplane section (see [41] for the case $(n, n+1)$). Notice however that sections of $\Delta(F)$ minimizing $\mathcal{A}_e$-codimension are not necessarily linear (see section 3.1 in [6]). The complete answer to the question above appears in [6].

Theorem 1.10.1 ([6], Theorem 4.6) *If the pair (n, p) is in the extra-nice dimensions, then every stable germ $F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$ admits a section of $\mathcal{A}_e$-codimension 1 $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. The converse is true if $(n+1, p+1)$ is in the nice dimensions.*

Corollary 1.10.2 *If $\mathbb{K} = \mathbb{C}$ and (n, p) is in the extra-nice dimensions any two generic hyperplane sections g and g' of the discriminant $\Delta(F)$ of a stable germ $F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$ pull back by F to $\mathcal{A}$-equivalent germs $f, f' : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. Moreover $\mathcal{A}_e\text{-cod}\, f = \mathcal{A}_e\text{-cod}\, f'$.*

Remark 1.10.3 When $\mathbb{K} = \mathbb{C}$, $p \leq n+1$ and (n, p) is in the nice dimensions, the topology of the stabilization of holomorphic $\mathcal{A}_e$-codimension 1, corank 1 germs and multigerms is well understood. See [17] where T. Cooper, D. Mond and Wik-Atique classify these singularities and study the topology of their stabilizations.

Problem 1. To study the geometry of generic hyperplane sections of the discriminant of stable mappings in $(n+1, p+1)$ when (n, p) is in extra-nice dimensions and its boundary.

Problem 2. To study equisingularity of families of generic hyperplane sections $g_t(\mathbb{C}^p)$ of the discriminant $\Delta(F)$ of stable map-germs $F : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{p+1}, 0)$ where $g_t : (\mathbb{C}^p, 0) \to (\mathbb{C}^{p+1}, 0)$ are germs of immersions, when (n, p) is in the boundary of extra-nice dimensions. These pair of extra-nice dimensions have been calculated in [6].

(i) $n \leq p, \ \ 4p = 5n - 5, \ \ p \geq 5.$
(ii) $n > p, \ \ (n, p) = \{(5, 4), (7, 5), (9 + k, 6), \ k \geq 0\}.$

Observe that these families are always topologically trivial. However the Whitney equisingularity and the bi-Lipschitz triviality of these families are open questions.

Conjecture 1.10.4 At the boundary of the extra-nice dimensions any two generic immersions $g, g' : (\mathbb{C}^p, 0) \to (\mathbb{C}^{p+1}, 0)$ are bi-Lipschitz $\mathcal{K}_V$-equivalent and they define bi-Lipschitz $\mathcal{A}$-equivalent germs $f, \ f' : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

Problem 3. Apply the geometric approach discussed in this section to study the bi-Lipschitz $\mathcal{G}$-classification of analytic map-germs $f \in \mathcal{O}_n^p$ where $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}, \mathcal{L}, \mathcal{A}$ or more generally, any geometric subgroup of $\mathcal{K}$. The Lipschitz theory of singularity is an almost completely open problem. See [87] for an account on bi-Lipschitz $\mathcal{G}$-classification of function germs $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}$ and references therein [8–10, 33, 35, 43, 47, 48, 75, 89].

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Chapter 2
Singularities of Mappings

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Abstract We summarise some of the basic theory of $\mathscr{A}$-equivalence (right-left equivalence) of germs of maps due to John Mather and others, and then go on to explain how to calculate some of the key invariants: the $\mathscr{A}_e$-codimension, the determinacy degree, and a minimal versal unfolding, introducing a new technique which lead to easily implementable computer algorithms. We then describe the topology of stable perturbations of $\mathscr{A}$-finite germs, and in particular the image and discriminant Milnor number for germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n + 1 \geq p$. We give a new formula for the discriminant Milnor number which, once again, can be implemented in a computer algorithm. The survey continues with a brief introduction to multiple point spaces and the image computing spectral sequence, and to the study of the Fitting ideals. We end with a number of open problems.

2.1 Introduction

The part of the theory of singularities of mappings we are concerned with is the local theory: the study and classification of (multi- or mono-) germs of smooth or analytic maps $(\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, under the action of the group $\mathscr{A}$ of right-left equivalence—i.e. pairs of germs of diffeomorphisms[1] φ in the source and ψ in the target, acting by $(\varphi, \psi) \cdot f = \psi \circ f \circ \varphi^{-1}$. The article by Maria Aparecida Ruas in this volume surveys the global theory of singularities, with an emphasis on the density of the set of stable mappings.

In this brief survey we look at its key ideas and techniques, with an emphasis on calculation, much of it through the study of a couple of examples, and with only occasional proofs. We will consistently refer to our recently published monograph [48] for proofs and a more detailed account. However, we do introduce and prove here some extensions of earlier results, which serve to make some of the basic notions more easily calculable—see Theorems 2.2.23 and 2.3.9.

In the first part of the article, Sect. 2.2, we deal simultaneously with the germs of real C^∞ maps and complex analytic maps. The theorems and calculations for the C^∞ and analytic cases are the same,[2] even though the rings of germs of C^∞ functions at $0 \in \mathbb{R}^n$, $\mathscr{E}_n$, and of germs of analytic functions at 0 in $\mathbb{C}^n$, $\mathcal{O}_n$, are very different. We refer to both $\mathbb{R}$ and $\mathbb{C}$ as $\mathbb{F}$, to both real C^∞ and complex analytic functions as "smooth", and to both $\mathscr{E}_n$ and $\mathcal{O}_n$ as $\mathcal{O}_n$.

[1] We use this term for biholomorphisms in the complex analytic case as well as for invertible smooth germs in the C^∞ case.

[2] See e.g. [48, Proposition 3.8].

The second part of the article, Sects. 2.3–2.5, is concerned with the notion of a stable perturbation of an unstable map-germ, which plays a rôle in the theory rather analogous to the rôle of the Milnor fibre in the theory of isolated complete intersection singularities (ICIS). In particular we are concerned with the relation between the rank of the vanishing homology in the discriminant or image of a stable perturbation, and the $\mathscr{A}_e$-codimension of the germ, when $p \leq n+1$. This relation bears a striking resemblance to the $\mu \geq \tau$ relation familiar from the case of an ICIS. The vanishing homology of the image can be calculated by Morse theory when $p \leq n+1$, and described with the help of the so-called image-computing spectral sequence when $p \geq n+1$. The two cases are considered in Sects. 2.3 and 2.4 respectively.

The article by Maria Aparecida Ruas in this volume explores the global theory in the real C^∞ case.

2.2 Thom-Mather Theory

Finding pairs of germs of diffeomorphisms directly and explicitly is impractical; they are generally constructed by integrating pairs of vector fields. Thus, we are naturally led to infinitesimal methods, in which we begin to understand a germ by calculating its "tangent space" for $\mathscr{A}$-equivalence, $T\mathscr{A}f$, which we will shortly define. In fact the codimension of $T\mathscr{A}f$, and of the closely associated space $T\mathscr{A}_e f$, in the space of all infinitesimal deformations, $\theta(f)$, give the most evident $\mathscr{A}$-invariant of a map-germ, and provide a natural hierarchy of complexity with which to structure their classification.

From $T\mathscr{A}f$ and the "extended" tangent space $T\mathscr{A}_e f$, in Sect. 2.2.2 we derive information about the possible deformations of f, via the notion of *versal unfolding* defined below, and also about the extent to which f is determined, up to $\mathscr{A}$-equivalence, by a finite segment of its power series expansion.

In Sect. 2.2.3 we discuss the notion of *finite determinacy*: a germ f is k-determined (for $\mathscr{A}$-equivalence) if any other germ with the same Taylor series up to degree k is $\mathscr{A}$-equivalent to it, and finitely determined if this holds for some finite k. It was proved by Mather in [44] that finite determinacy is equivalent to having finite $\mathscr{A}$-codimension.

In most of this section we consider only the case of "monogerms", when $|S| = 1$. Sect. 2.2.4 extends some of the results and calculations to the case of multi-germs. One should not ignore these: they form an integral and essential part of the theory (see e.g. [66], and the examples of codimension 1 singularities in Sect. 2.3 below).

Most progress in relating the geometric behaviour of a germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ to the algebraic properties of $T\mathscr{A}_e f$ has been made in the two cases where the set $D(f)$ of critical values is a hypersurface, namely where $n \geq p$, and where $p = n+1$, where $D(f)$ is simply the image of f. If f is $\mathscr{A}$-finite, and f_t is a stable perturbation of f, then because $D(f)$ is a hypersurface, and f is unstable only at 0, $D(f_t)$ has the homotopy type of a wedge of spheres of dimension $p-1$. The number of spheres in the wedge provides the most important discrete geometrical invariant, which we study in some detail in Sect. 2.3.

2.2.1 The $\mathscr{A}_e$ and $\mathscr{A}$ Tangent Space

Throughout this article we work with germs. Most of the notions have global versions, which can be found in the article of Maria Aparecida Ruas in this volume. Let $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ be a smooth germ. First, $\theta(f)$ is the space of infinitesimal deformations,

$$\left\{ \left.\frac{df_t}{dt}\right|_{t=0} : f_t \text{ is a 1-parameter deformation of } f \right\};$$

its members determine sections $x \mapsto (df_t/dt)_{t=0} \in T_{f(x)}\mathbb{F}^p$ of $f^*(T\mathbb{F}^p)$. It is a free $\mathcal{O}_n$-module; coordinates $Y_1, \ldots, Y_p$ on $\mathbb{F}^p$ provide it with the basis $\partial/\partial Y_1, \ldots, \partial/\partial Y_p$.

Then $T\mathscr{A} f$ is its subspace

$$\left\{ \left.\frac{d(\psi_t \circ f \circ \varphi_t)}{dt}\right|_{t=0} : \psi_t \text{ and } \varphi_t \text{ are 1-parameter deformations of } \mathrm{id}_{\mathbb{F}^p} \text{ and } \mathrm{id}_{\mathbb{F}^n}, \text{ both fixing } 0 \right\},$$

consisting of $\mathscr{A}$-trivial infinitesimal deformations. The slightly larger space $T\mathscr{A}_e f$ omits the requirement that ψ_t and φ_t should fix 0. A calculation shows that

$$\left.\frac{d(\psi_t \circ f \circ \varphi_t)}{dt}\right|_{t=0} = df \circ \left(\left.\frac{d\varphi_t}{dt}\right|_{t=0} \right) + \left(\left.\frac{d\psi_t}{dt}\right|_{t=0} \right) \circ f. \tag{2.1}$$

As indicated above, $\left.\frac{d\varphi_t}{dt}\right|_{t=0}$ and $\left.\frac{d\psi_t}{dt}\right|_{t=0}$ belong to $\theta_n := \theta(\mathrm{id}_{\mathbb{F}^n})$ and $\theta_p := \theta(\mathrm{id}_{\mathbb{F}^p})$—which means they are germs of vector fields at 0 in $\mathbb{F}^n$ and $\mathbb{F}^p$ respectively. Both θ_n and θ_p are free modules, over $\mathcal{O}_n$ and $\mathcal{O}_p$ respectively, with bases $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and $\partial/\partial Y_1, \ldots, \partial/\partial Y_p$ once coordinates are chosen. The requirement that $\varphi_t(0) = 0$ and $\psi_t(0) = 0$ means that $\left.\frac{d\varphi_t}{dt}\right|_{t=0}$ and $\left.\frac{d\psi_t}{dt}\right|_{t=0}$ vanish at 0, and so lie in $\mathfrak{m}_n\theta_n$ and $\mathfrak{m}_p\theta_p$, where $\mathfrak{m}_k$ is the maximal ideal of $\mathcal{O}_k$. Every vector field integrates to a flow, so by (2.1),

$$T\mathscr{A} f = \{df \circ \xi + \eta \circ f : \xi \in \mathfrak{m}_n\theta_n, \eta \in \mathfrak{m}_p\theta_p\}$$

and

$$T\mathscr{A}_e f = \{df \circ \xi + \eta \circ f : \xi \in \theta_n, \eta \in \theta_p\}.$$

It is customary to write $df \circ \xi$ as $tf(\xi)$ and $\eta \circ f$ as $\omega f(\eta)$, emphasising the rôle of the homomorphism $tf : \theta_n \to \theta(f)$ sending ξ to $df \circ \xi$ and the homomorphism $\omega f : \theta_p \to \theta(f)$ sending η to $\eta \circ f$. So

$$T\mathscr{A} f = tf(\mathfrak{m}_n\theta_n) + \omega f(\mathfrak{m}_p\theta_p)$$

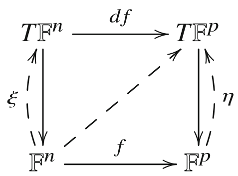
and

$$T\mathscr{A}_e f = tf(\theta_n) + \omega f(\theta_p).$$

Note that tf is $\mathcal{O}_n$-linear, while ωf is $\mathcal{O}_p$-linear, with respect to the $\mathcal{O}_p$-module structure on $\theta(f)$ coming from composition with f.

We will sometimes use the term $T^1_{\mathscr{A}_e} f$ to denote the quotient $\mathcal{O}_p$-module $\theta(f)/T\mathscr{A}_e f$.

All of the modules and homomorphisms can be understood with the help of the following basic commutative diagram.



(2.2)

Here members of $\theta(f)$ are represented by the dashed diagonal arrow, and the dashed arrows ξ and η represent vector fields on source and target.

The germ f is *stable* if for every deformation f_t there exist deformations φ_t and ψ_t of $\mathrm{id}_{\mathbb{F}^n}$ and $\mathrm{id}_{\mathbb{F}^p}$ such that $f_t = \psi_t \circ g \circ \varphi_t$. Here φ_t and ψ_t are not required to fix 0. It follows that stability implies the condition of "infinitesimal stability": that $T\mathscr{A}_e f = \theta(f)$. The converse is a theorem of J. Mather:

Theorem 2.2.1 (J.Mather, [45]) *Infinitesimal stability implies stability.*

Proof [48, Section 3.5] □

See also the article of Maria Aparecida Ruas in this volume for a discussion of the various notions of stability.

Slightly greater precision is needed.

Definition 2.2.2

1. Let $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ be a germ. A d-parameter **unfolding** of f is a germ $F : (\mathbb{F}^n \times \mathbb{F}^d, (0,0)) \to (\mathbb{F}^p \times \mathbb{F}^d, (0,0))$ of the form $F(x,u) = (f_u(x), u)$, such that $f_0 = f$.
2. The d-parameter unfoldings F and G of f are **equivalent** if there exist d-parameter unfoldings Φ of $\mathrm{id}_{\mathbb{F}^n}$ and Ψ of $\mathrm{id}_{\mathbb{F}^p}$ such that $\Psi \circ F \circ \Phi = G$.
3. The d-parameter unfolding F is **trivial** if F is equivalent to the constant unfolding $f \times \mathrm{id}_{\mathbb{F}^d}$, and f is **stable** if every unfolding is trivial (note that this adds precision to the heuristic definition of stability given above).

4. The k-parameter unfolding G of f is **induced** from the d-parameter unfolding F by the **base-change map** $h : (\mathbb{F}^k, 0) \to (\mathbb{F}^d, 0)$ if $G(x, v) = (f_{h(v)}(x), v)$.
5. The unfolding F of f is $\mathscr{A}_e$**-versal** if every unfolding is equivalent to one induced from F.
6. The germ f is called $\mathscr{A}$**-finite** if $\dim_{\mathbb{F}} \theta(f)/T\mathscr{A}f < \infty$ (or, equivalently, if $\dim_{\mathbb{F}} \theta(f)/T\mathscr{A}_e f < \infty$).

Example 2.2.3

1. H. Whitney showed that up to $\mathscr{A}$-equivalence, the only stable germs $(\mathbb{F}^2, 0) \to (\mathbb{F}^2, 0)$ are the identity map, the fold $f(x, y) = (x, y^2)$ and the cusp $f(x, y) = (x, y^3 + xy)$.

   (i) Let us see that the unfolding $F(x, y, u) = (x, y^3 + xy + uy, u)$ of $f(x, y) = (x, y^3 + xy)$ is trivial. This is easy: the substitutions $\bar{x} = x + u$, $\bar{X} = X + u$ transform F to $f \times \mathrm{id}_{\mathbb{F}}$. In other words, taking $\Phi(x, y, u) = (x + u, y, u)$ and $\Psi(X, Y, u) = (X + u, Y, u)$, we have $\Psi \circ F = (f \times \mathrm{id}_{\mathbb{F}}) \circ \Phi$.

   (ii) For the unfolding $F(x, y) = (x, y^3 + xy + uy^2, u)$, setting $\varphi_u(x, y) = (x - \frac{u^2}{3}, y + \frac{u}{3})$ we have

$$f \circ \varphi_u(x, y) = \left(x - \frac{u^2}{3}, y^3 + xy + uy^2 + \frac{u}{3}x - \frac{2u^3}{27}\right)$$

   and so taking $\Phi = (\varphi_u, u)$ and $\Psi(X, Y, u) = (X + \frac{u^2}{3}, Y + \frac{u}{3}X - \frac{2u^3}{27})$ we have

$$\Psi \circ F = (f \times \mathrm{id}_{\mathbb{F}}) \circ \Phi$$

Let us check infinitesimal stability of f modulo $\mathfrak{m}_2^4\theta(f)$. We work in the category of formal power series: for $\mathscr{A}$-finite germs, Theorem 2.2.6 below ensures that this is enough. In general we will write the members of θ_p and $\theta(f)$ as column vectors with p components, and the members of θ_n as column vectors with n components. Of course in this example n and p are the same, so we write θ_S and θ_T ("S" and "T" for "source" and "target") in place of θ_n and θ_p.

- *Terms of degree 3* For every monomial p of degree 3, $\begin{pmatrix} p \\ 0 \end{pmatrix} = tf \begin{pmatrix} p \\ 0 \end{pmatrix}$ and hence $T\mathscr{A}_e f + \mathfrak{m}_2^4\theta(f) \supset \begin{pmatrix} \mathfrak{m}_S^3 \\ 0 \end{pmatrix}$. Moreover

$$tf \begin{pmatrix} -\frac{1}{3}x \\ \frac{1}{3}y \end{pmatrix} + \omega f \begin{pmatrix} \frac{1}{3}X \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y^3 \end{pmatrix}, \qquad tf \begin{pmatrix} 0 \\ \frac{1}{3}x \end{pmatrix} - \omega f \begin{pmatrix} 0 \\ X^2 \end{pmatrix} = \begin{pmatrix} 0 \\ xy^2 \end{pmatrix}$$

$$tf \begin{pmatrix} x^2 \\ 0 \end{pmatrix} - \omega f \begin{pmatrix} X^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x^2y \end{pmatrix}, \qquad \omega f \begin{pmatrix} 0 \\ X^3 \end{pmatrix} = \begin{pmatrix} 0 \\ x^3 \end{pmatrix}$$

so $T\mathscr{A}_e f + \mathfrak{m}_2^4\theta(f) \supseteq \begin{pmatrix} 0 \\ \mathfrak{m}_S^3 \end{pmatrix}$ also.

- *Terms of degree* < 3 We leave the remaining calculations to the reader.

2. The unfolding $F(u, y) = (y^3 + uy, u)$ of $f(y) = y^3$ is not trivial. This can be seen in two ways: first, for $u \neq 0$, f_u has two critical points and two critical values, whereas f has only one of each; second, one checks easily that $y \notin T\mathscr{A}_e f$.

Exercise 2.2.4

1. Complete the calculation begun in the example.
2. Check infinitesimal stability of the Whitney cusp modulo $\mathfrak{m}_S^5$.
3. Use similar methods to show that the parameterisation of the Whitney umbrella,

$$f(x, y) = (x, y^2, xy)$$

   is infinitesimally stable, modulo m_2^4.
4. Show that the 1-dimensional cusp map $f : x \mapsto (x^2, x^3)$ has $\mathscr{A}_e$-codimension 1, and find a basis for the quotient space $\theta(f)/T\mathscr{A}_e f$.
5. Show that the map germs $b : (x, y) \mapsto (x, xy)$ and $c : (x, y) \mapsto (x, y^2, x^2 y)$ are not $\mathscr{A}$-finite.

Note that in the examples with the Whitney cusp here, if we write Φ and Ψ as (φ_u, u) and (ψ_u, u) then for $u \neq 0$, both φ_u and ψ_u move 0. From the point of view of deformation theory, $\mathscr{A}_e$-equivalence of unfoldings is more natural than $\mathscr{A}$-equivalence. On the other hand, for classification evidently it is $\mathscr{A}$-equivalence that is required—"$\mathscr{A}_e$" is not a group.

Computing $T\mathscr{A}_e f$ precisely requires some more advanced techniques than we have described up to now. The approximations calculated in Example 2.2.3 and Exercise 2.2.4 can be shown to lead directly to the correct answer using a theorem due to Terry Gaffney which involves the auxiliary module $T\mathscr{K}_e f$ defined as

$$T\mathscr{K}_e f = tf(\theta_n) + f^*\mathfrak{m}_p\theta(f). \tag{2.3}$$

This is in fact the extended tangent space for *$\mathscr{K}$-equivalence*, or *contact equivalence*, but we do not need this fact at the moment.[3] Unlike $T\mathscr{A}_e f$, it is an $\mathcal{O}_n$-module, and evidently finitely generated. Denote by J_f the critical ideal of f, generated by the $p \times p$ minors of the matrix of df, and by Σ_f the set of critical points, $V(J_f)$, where $d_x f$ is not surjective. Cramer's rule shows that $tf(\theta_n) \supseteq J_f\theta(f)$; this proves the harder direction in part 1 of Proposition 2.2.16, which characterizes $\mathscr{K}$-finiteness in terms of the ideal $J_f + f^*\mathfrak{m}_p$.

Example 2.2.5 It follows from Nakayama's Lemma that if $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ has multiplicity k (i.e. $\dim_{\mathbb{F}} (\mathcal{O}_n/f^*\mathfrak{m}_p\mathcal{O}_n) = k$—this can happen only when $n \leq p$) then $T\mathscr{K}_e f \supset \mathfrak{m}_n^k\theta(f)$.

[3] See Sect. 4.4 in the article of Ruas in this volume for the definition of the group $\mathscr{K}$.

Theorem 2.2.6 (Terry Gaffney) *Let C be an $\mathcal{O}_p$-submodule of $\theta(f)$, and suppose that*

1. $C \supseteq \mathfrak{m}_n^k \theta(f)$
2. $T\mathscr{K}_e f \supseteq \mathfrak{m}_n^\ell \theta(f)$.

Then

$$C \subseteq T\mathscr{A}_e f \quad \Longleftrightarrow \quad C \subseteq T\mathscr{A}_e f + \mathfrak{m}_n^{k+\ell}\theta(f) + f^*\mathfrak{m}_p C.$$

Exercise 2.2.7

1. Use Theorem 2.2.6 to complete the proof that the Whitney cusp and the parameterised Whitney umbrella are infinitesimally stable.
2. Use Theorem 2.2.6 to derive a statement for $T\mathscr{A}_e f$ from your formal calculation in Exercise 2.2.4(4).
3. Let $f(x, y) = (x, y^2, y^3 + x^3 y)$. Use Theorem 2.2.6 to show that

$$T\mathscr{A}_e f = \begin{pmatrix} \mathcal{O}_2 \\ \mathcal{O}_2 \\ \mathcal{O}_2 \smallsetminus \{y, xy\} \end{pmatrix}$$

Here the last row means the subspace of the ring of smooth germs whose Taylor series has no y nor xy term.

The proof of Theorem 2.2.6, and indeed of most of the theorems in this section, uses the Preparation Theorem, due to Weierstrass in the complex analytic case and to Malgrange in the C^∞ case. We state it in the form given it by John Mather in [43]:

Theorem 2.2.8 *Let $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ be a smooth map-germ, and M a finitely generated $\mathcal{O}_n$-module. Then*

$$M \text{ is finitely generated over } \mathcal{O}_p \text{ via } f^* \iff \dim_{\mathbb{F}} M/f^*\mathfrak{m}_p M < \infty.$$

Proof See e.g. [42]. □

2.2.2 Versal Unfoldings

There is an infinitesimal criterion for the versality of an unfolding:

Theorem 2.2.9 (J. Martinet, [41]) *The unfolding $F : (\mathbb{F}^n \times \mathbb{F}^d, (0,0)) \to (\mathbb{F}^p \times \mathbb{F}^d, (0,0))$ of $f : (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$, with $F(x,u) = (f_u(x), u)$, is $\mathscr{A}_e$-versal if and only if*

$$T\mathscr{A}_e f + Sp_{\mathbb{F}}\left\{ \left.\frac{\partial f_u}{\partial u_1}\right|_{u=0}, \cdots, \left.\frac{\partial f_u}{\partial u_d}\right|_{u=0} \right\} = \theta(f). \tag{2.4}$$

Proof [48, Theorem 5.1]. □

Disentangling the terminology here, one sees that the theorem is in effect the statement that versality of F is equivalent to transversality of the relative jet extension map

$$j^k(F/\mathbb{F}^d) : (\mathbb{F}^n \times \mathbb{F}^d, (0, 0)) \to J^k(\mathbb{F}^n, \mathbb{F}^p), \quad (x, u) \mapsto j^k f_u(x)$$

to the $\mathscr{A}$-orbit of $j^k f(0)$ in a suitably high jet space (see [48, Theorem 5.2]). Here the notation $j^k(F/\mathbb{F}^d)$ is used to distinguish the relative version from the usual jet extension map $j^k F$—it ignores partial derivatives with respect to u. Some indication of just how high k needs to be is given in the next subsection.

Example 2.2.10 By Theorem 2.2.9, $F(x, u) = (x^2, x^3+ux, u)$ is a versal unfolding of the germ $x \mapsto (x^2, x^3)$. Placing in sequence the images of f_u for $u < 0, u = 0$ and $u > 0$, we recognise the first Reidemeister move of knot theory. Of course, this is not a coincidence.

2.2.3 Finite Determinacy

Definition 2.2.11 The germ f is k**-determined** (for $\mathscr{A}$-equivalence) if any other germ g with the same k-jet as f (i.e. with the same partial derivatives of degree $\leq k$ at 0) is $\mathscr{A}$-equivalent to f, and is **finitely determined** if it is k-determined for some finite k.

Theorem 2.2.12 *f is finitely determined if and only if it is $\mathscr{A}$-finite. In fact*

$$\begin{aligned} f \text{ is } \mathscr{A}\text{-finite} &\iff T\mathscr{A}f \supseteq \mathfrak{m}_n^k\theta(f) \text{ for some } k < \infty \\ f \text{ is } k\text{-determined} &\implies T\mathscr{A}f \supseteq \mathfrak{m}_n^{k+1}\theta(f) \\ T\mathscr{A}f \supseteq \mathfrak{m}_n^k\theta(f) &\implies f \text{ is } 2k{+}1\text{-determined} \\ \mathscr{A}\text{-codim} f = d &\implies \mathfrak{m}_n^{(p+d)^2}\theta(f) \subset T\mathscr{A}f \end{aligned}$$

Proof [48, Chapter 6]. □

The second and third implications here are known as the *infinitesimal criteria for finite determinacy*. See Theorem 4.7 in the paper of Ruas in this volume for a slightly different presentation of this result and an overview of the proof.

This theorem, coupled with Theorem 2.2.6, leads to the following useful estimate of the determinacy degree, due to Terry Gaffney.

Theorem 2.2.13 ([15, page 127]) *If $T\mathscr{A}_e f \supset \mathfrak{m}_n^k\theta(f)$ and $T\mathscr{K}_e f \supset \mathfrak{m}_n^\ell\theta(f)$ then f is $k + \ell$-determined for $\mathscr{A}$-equivalence.*

One of the useful consequences of the fact that $\mathscr{A}$-finiteness implies finite determinacy is that one can replace an $\mathscr{A}$-finite C^∞ germ f by a polynomial germ f_1 which is $\mathscr{A}$-equivalent to it, and then complexify, to obtain an $\mathscr{A}$-finite germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ whose real part is f_1, $\mathscr{A}$-equivalent to f. By this means, arguments involving the complexification can be used to deduce geometrical consequences for f_1 and thus for f. In Sect. 2.2.6 we take this approach, and focus on the complex case.

Since the work of Gaffney and du Plessis in the 1980s, improving on Mather's original estimates for the determinacy degree, the subject has essentially been put to bed by the paper [1] of Bruce, du Plessis and Wall on unipotent group actions. There is no space here to explain the results of this paper. We point out only that for unipotent actions, such as the action of the group $\mathscr{A}^{(1)}$ consisting of pairs of germs of diffeomorphisms whose 1-jet coincides with that of the identity, the orbits in jet space are Zariski closed. This leads easily to the conclusion that the necessary condition for k-determinacy, that $\mathfrak{m}_n^{k+1} \subseteq T\mathscr{A}^{(1)}f$, is also sufficient (see [48, §6.3] for a slightly more elementary take than in the original paper [1]). This very often leads to optimal estimates of the $\mathscr{A}$-determinacy degree.

2.2.4 Multi-Germs

For reasons which we encourage the reader to ponder, the three Reidemeister moves R1, R2, R3 of knot theory are, in fact, versal unfoldings of the three $\mathscr{A}_e$-codimension 1 singularities of maps $\mathbb{F} \to \mathbb{F}^2$. In Exercise 2.2.4 and 2.2.7 we suggested the calculation that the singularity at the centre of R1, namely the 1-dimensional cusp $x \mapsto (x^2, x^3)$, has $\mathscr{A}_e$-codimension 1 (see Fig. 2.1). The remaining two moves may be parameterised as

$$R2 \quad \begin{cases} x_1 \mapsto (x_1, x_1^2) \\ x_2 \mapsto (x_2, 0) \end{cases} \qquad R3 \quad \begin{cases} x_1 \mapsto (x_1, 0) \\ x_2 \mapsto (0, x_2) \\ x_3 \mapsto (x_3, x_3) \end{cases} \tag{2.5}$$

We need to set some notational conventions. For a multigerm $f : (\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$ with $S = \{a^{(1)}, \ldots, a^{(r)}\}$, we denote by $f^{(i)} : (\mathbb{F}^n, a^{(i)}) \to (\mathbb{F}^p, 0)$ the i'th constituent monogerm. Then $\theta(f) = \bigoplus_{i=1}^r \theta(f^{(i)})$ and the space of germs at S of sections of $T\mathbb{F}^n$ is a direct sum of the space of germs of sections at the points $a^{(i)}$. Up to now we have used the notation θ_n to indicate the space of germs of sections of $T\mathbb{F}^n$ at 0; to deal with the added complexity of a multi-germ, we denote space of germs at $a^{(i)}$ by $\theta_{n,a^{(i)}}$ and their direct sum by $\theta_{n,S}$. We represent elements of $\theta(f)$ by $p \times r$ matrices, whose i'th column represents elements of $\theta(f^{(i)})$. Similarly, elements of $\theta_{n,S}$ are represented by $n \times r$ matrices, whose i'th column represents elements of $\theta_{n,a^{(i)}}$. Thus, *everything concerning the monogerm at $a^{(i)}$ takes place in the i'th column.* For this reason,*we can safely dispense with the use of different*

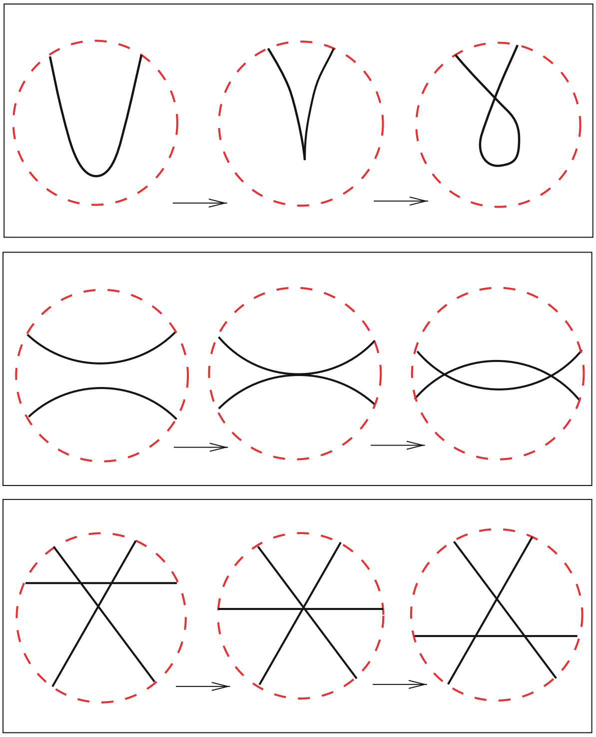


Fig. 2.1 The three Reidemeister moves

labels for the local coordinates around the different points of S. We can represent $R2$ simply as

$$f : \begin{cases} x \mapsto (x, x^2) \\ x \mapsto (x, 0) \end{cases}$$

We find a basis for $T^1_{\mathscr{A}_e} f$ in this case. Working cumulatively and slightly abusing notation,

$$\begin{aligned} tf\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \text{ so that } T\mathscr{A}_e f &\supset \begin{pmatrix} 0 & \mathcal{O}_1 \\ 0 & 0 \end{pmatrix} \\ \omega f\begin{pmatrix} h(Y_1) \\ 0 \end{pmatrix} &= \begin{pmatrix} h & h \\ 0 & 0 \end{pmatrix} \text{ so that } T\mathscr{A}_e f &\supset \begin{pmatrix} \mathcal{O}_1 & 0 \\ 0 & 0 \end{pmatrix} \\ tf\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} r & 0 \\ 2xr & 0 \end{pmatrix} \text{ so that } T\mathscr{A}_e f &\supset \begin{pmatrix} 0 & 0 \\ \mathfrak{m}_1 & 0 \end{pmatrix} \\ \omega f\begin{pmatrix} 0 \\ h(Y_1) \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ h & h \end{pmatrix} \text{ so that } T^1_{\mathscr{A}_e} f &= \mathrm{Sp}_{\mathbb{F}}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \end{aligned} \tag{2.6}$$

We note that the versality theorem 2.2.9 holds, without modification, in the case of multigerms.

Exercise 2.2.14

1. Find a basis for $T^1_{\mathscr{A}_e} f$ for the parameterisation of R3 above.
2. Let f be the bigerm $\mathbb{F}^n \to \mathbb{F}^{n+1}$ given by

$$\begin{cases} x \mapsto (x, 0) \\ x \mapsto (x, h(x)) \end{cases}$$

   where $h \in \mathfrak{m}_n$ has isolated singularity. Show that $T^1_{\mathscr{A}_e} f \simeq T^1_{\mathscr{K}_e} h$.
3. Let f be the tri-germ of map $\mathbb{F}^2 \to \mathbb{F}^4$ consisting of three immersions meeting two-by-two transversely. Parameterise f and calculate a basis for $T^1_{\mathscr{A}_e} f$.

2.2.5 *Construction of Stable Map-Germs as $\mathscr{K}_e$-versal Unfoldings of Rank 0 Germs*

There is a general procedure for finding all stable monogerms as unfoldings of lower-dimensional $\mathscr{K}$-finite germs of rank zero, due to Mather in [46].

1. Given $f: (\mathbb{F}^n, 0) \to (\mathbb{F}^p, 0)$ of rank 0, calculate $T\mathscr{K}_e f$. Because f_0 has rank 0 at 0, $T\mathscr{K}_e f_0 \subset \mathfrak{m}_n \theta(f_0)$.
2. Find a basis for the quotient $\mathfrak{m}_n \theta(f)/T\mathscr{K}_e f$.
3. If $g_1, \dots, g_d \in \theta(f)$ project to this basis, then the unfolding $F: (\mathbb{F}^n \times \mathbb{F}^d, (0, 0)) \to (\mathbb{F}^p \times \mathbb{F}^d, (0, 0))$ defined by

$$F(x, u_1, \dots, u_d) = (f(x) + \sum_j u_j g_j(x), u_1, \dots, u_d) \tag{2.7}$$

   is stable, and is minimal in the sense that it is not a trivial unfolding of a lower-dimensional stable germ.

Mather's construction is explained in [48, Chapter 7], and in Section 4.3 of the paper of Ruas in this volume.

Example 2.2.15 We believe the following example is self-explanatory.

Let $f_0: (\mathbb{F}^2, 0) \to (\mathbb{F}^3, 0)$ be given by $f(x, y) = (x^2, y^3, xy)$. We show how to use SINGULAR [11] to find a stable unfolding following the recipe explained. Our script is deliberately pedestrian—it is of course possible to combine many of the steps into one.

Step 1 We find an $\mathbb{F}$- basis for $T := \theta(f)/T\mathscr{K}_e f$.

```
ring S=0,(x,y),ds;
```

```
ideal i= x2,y3,xy;

module J=jacob(i);

module A=freemodule(3)*i;

module B=J+A;

ideal m=x,y;

module C=freemodule(3)*m;

module T=modulo(C,B);

matrix b=kbase(std(T));

matrix bb=C*b;

print(bb);
```

The last command returns the matrix

$$\begin{pmatrix} x & y^2 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y^2 & y \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2 The columns of the matrix bb give the unfolding monomials;

$$F(x, y, u_1, \ldots, u_6) = (x^2+u_1x+u_2y^2+u_3y,\, y^3+u_4x+u_5y^2+u_6y,\, xy,\, u_1, \ldots, u_6)$$

is a minimal stable unfolding of f.

Mather showed that given n and p, a stable map-germ $(\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$ is determined, up to $\mathscr{A}$-equivalence, by its local algebra $O_{n,S}/F^*\mathfrak{m}_pO_{n,S}$ ([46], [48, Chapter 7]). All germs, whether stable or not, are determined up to $\mathscr{K}$-equivalence by their local algebra, so the procedure described here gives the unique stable map-germ for each algebra type of finite singularity type and each dimension-pair in which it can occur.

2.2.6 Geometrical Criterion for $\mathscr{A}$-Finiteness

A well known geometrical criterion due to Mather and Gaffney says that a holomorphic germ $f: (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is $\mathscr{A}$-finite (i.e. has finite $\mathscr{A}$-codimension) if and only if it has "isolated instability". In the real case this is not true in general and the "isolated instability" is only a necessary condition. We will give a precise statement of this criterion as well as some details of the proof. There is an analogous result for $\mathscr{K}$-equivalence which we will show first in the next proposition.

Let $f\colon (\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$ be a smooth germ. For simplicity in the notation, we will omit the subindex S whenever there is no risk of confusion, that is:

- $\mathcal{O}_n = \mathcal{O}_{\mathbb{F}^n,S}$ = the ring of smooth function germs $g\colon (\mathbb{F}^n, S) \to \mathbb{F}$,
- $\mathfrak{m}_n = \mathfrak{m}_{\mathbb{F}^n,S}$ = the ideal of $\mathcal{O}_n$ of functions g such that $g(S) = \{0\}$.
- $\theta_n = \theta_{\mathbb{F}^n,S}$ = the $\mathcal{O}_n$-module of germs of vector fields on $(\mathbb{F}^n, S)$.

Proposition 2.2.16 *Let $f\colon (\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$ be a smooth germ.*

1. *f is $\mathscr{K}$-finite if and only if $\dim_{\mathbb{F}} \mathcal{O}_n/(J_f + f^*\mathfrak{m}_p) < \infty$.*
2. *When $\mathbb{F} = \mathbb{C}$, f is $\mathscr{K}$-finite if and only if it is finite-to-one on its critical set Σ. When $\mathbb{F} = \mathbb{R}$, only the implication $\Longrightarrow$ holds.*
3. *$\mathscr{A}$-finiteness implies $\mathscr{K}$-finiteness.*

Proof The proof of item 1, which involves little more than Cramer's rule, can be found in [48, Proposition 4.3]. Now by item 1 and Nakayama's Lemma, f is $\mathscr{K}$-finite if and only if $\mathfrak{m}_n^k \subset J_f + f^*\mathfrak{m}_p$, for some $k \in \mathbb{N}$. If $\mathfrak{m}_n^k \subset J_f + f^*\mathfrak{m}_p$, then

$$V(J_f + f^*\mathfrak{m}_p) \subset V(\mathfrak{m}_n^k) = S, \tag{2.8}$$

where $V(I)$ is the zero locus of the ideal $I \subset \mathcal{O}_n$. Since $V(J_f + f^*\mathfrak{m}_p) = \Sigma \cap f^{-1}(0)$, (2.8) implies that f is finite-to-one on Σ.

When $\mathbb{F} = \mathbb{C}$, we also have the converse. In fact, if f is finite-to-one on Σ, then $V(J_f + f^*\mathfrak{m}_p) \subset S = V(\mathfrak{m}_n)$. By the Rückert Nullstellensatz [28, Theorem 3.4.4],

$$\mathfrak{m}_n \subset \sqrt{J_f + f^*\mathfrak{m}_p},$$

where $\sqrt{I}$ is the radical of I. Hence, $\mathfrak{m}_n^k \subset J_f + f^*\mathfrak{m}_p$, for some $k \in \mathbb{N}$.

Let us see item 3. If f is $\mathscr{A}$-finite, then

$$T\mathscr{A}_e f + \mathrm{Sp}_{\mathbb{F}}\{h_1, \ldots, h_r\} = \theta(f),$$

for some $h_1, \ldots, h_r \in \theta(f)$. Since $\omega f(\theta_p) \subset f^*\mathfrak{m}_p\theta(f) + \mathrm{Sp}_{\mathbb{F}}\{\partial/\partial Y_1, \ldots, \partial/\partial Y_p\}$,

$$T\mathscr{K}_e f + \mathrm{Sp}_{\mathbb{F}}\{h_1, \ldots, h_r, \partial/\partial Y_1, \ldots, \partial/\partial Y_p\} = \theta(f),$$

so f is $\mathscr{K}$-finite. □

Map germs f which are $\mathscr{K}$-finite (i.e. such that $\dim_{\mathbb{F}} \mathcal{O}_n/(J_f + f^*\mathfrak{m}_p) < \infty$) are usually called map germs of *finite singularity type*.

Example 2.2.17 Consider the function $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ given by $f(x, y) = (x^2 + y^2)^2$. We have $\Sigma_f = \{0\}$, so obviously f is finite-to-one on Σ_f. However, its complexification $f_{\mathbb{C}}\colon (\mathbb{C}^2,0)\to(\mathbb{C},0)$ is not $\mathscr{K}$-finite, since in this case both the critical set $\Sigma_{f_{\mathbb{C}}}$ and $f_{\mathbb{C}}^{-1}(0)$ are equal to the union of the two lines $x-iy=0$ and $x+iy=0$.

If f was $\mathscr{K}$-finite over $\mathbb{R}$, then $\mathfrak{m}_2^k\subset J_f+(f)$ in $\mathscr{E}_2$, for some $k\in\mathbb{N}$. That is, for each $i=0,\dots,k$ we would have

$$x^i y^{k-i}=a_i\frac{\partial f}{\partial x}+b_i\frac{\partial f}{\partial x}+c_i f,$$

for some $a_i,b_i,c_i\in\mathscr{E}_2$. Passing to their classes modulo $\mathfrak{m}_2^{k+1}$,

$$x^i y^{k-i}+\mathfrak{m}_2^{k+1}=\bar{a}_i\frac{\partial f}{\partial x}+\bar{b}_i\frac{\partial f}{\partial x}+\bar{c}_i f+\mathfrak{m}_2^{k+1},$$

where now $\bar{a}_i,\bar{b}_i,\bar{c}_i$ are polynomials of degree $\le k$. But these relations can be considered also over $\mathbb{C}$, which would give the inclusion in $\mathcal{O}_2$:

$$\hat{\mathfrak{m}}_2^k\subset J_{f_{\mathbb{C}}}+(f_{\mathbb{C}})+\hat{\mathfrak{m}}_2^{k+1},$$

where $\hat{\mathfrak{m}}_2$ is the maximal ideal of $\mathcal{O}_2$. By Nakayama's lemma, $\hat{\mathfrak{m}}_2^k\subset J_{f_{\mathbb{C}}}+(f_{\mathbb{C}})$ and hence, $f_{\mathbb{C}}$ should be also $\mathscr{K}$-finite, giving a contradiction.

Let $f\colon(\mathbb{F}^n,S)\to(\mathbb{F}^p,0)$ be a smooth multi-germ, with $S=\{a_1,\dots,a_r\}$. A natural question is how the stability of f is related to the stability of each branch $f^{(i)}\colon(\mathbb{F}^n,a_i)\to(\mathbb{F}^p,0)$. To answer this question we associate a vector subspace of $T_0\mathbb{F}^p$ as follows:

$$\tau(f)=\mathrm{ev}((\omega f)^{-1}(T\mathscr{K}_e f)),$$

where $\mathrm{ev}:\theta_p\to T_0\mathbb{F}^p$ is the evaluation map $\eta\mapsto\eta(0)$. The following theorem is due to Mather:

Theorem 2.2.18 *f is stable if and only each branch $f^{(i)}\colon(\mathbb{F}^n,s_i)\to(\mathbb{F}^p,0)$ is stable and $\tau(f^{(1)}),\dots,\tau(f^{(r)})$ meet in general position in $T_0\mathbb{F}^p$.*

Proof See [48, Theorem 3.3]. □

By using this theorem, it is easy to prove that f is stable if and only if its restriction $f\colon(\mathbb{F}^n,S\cap\Sigma)\to(\mathbb{F}^p,0)$ is stable, where Σ is the critical set of f. Thus, it makes sense to say that a smooth mapping $f:X\to Y$ between smooth manifolds is *locally stable* if it is finite-to-one on its singular set Σ and for all $y\in Y$, the multi-germ $f\colon(X,f^{-1}(y)\cap\Sigma)\to(Y,y)$ is stable.

Now we make precise the notion of "isolated instability" and the statement of the Mather-Gaffney criterion.

Definition 2.2.19 We say that a smooth germ $f\colon (\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$ has **isolated instability** if it there exists a representative $f\colon X \to Y$ such that $f^{-1}(0) \cap \Sigma = S$ and the restriction $f\colon X \setminus f^{-1}(0) \to Y \setminus \{0\}$ is a locally stable mapping.

Theorem 2.2.20 (Mather-Gaffney Criterion) *An analytic germ* $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *is* $\mathscr{A}$*-finite if and only if it has isolated instability.*

Proof In both cases, when f is $\mathscr{A}$-finite (by Proposition 2.2.16) or when f has isolated singularity, f is finite-to-one on its singular set Σ. We choose a representative $f\colon X \to Y$ such that $f^{-1}(0) \cap \Sigma = S$ and the restriction $f\colon \Sigma \to Y$ is a finite mapping (i.e., it is finite-to-one and closed).

We denote by θ_X and $\theta(f)$ the sheaves of $\mathcal{O}_X$-modules of vector fields on X and of vector fields along f, respectively. Both are locally free of finite rank and hence coherent. We also have a morphism $tf\colon \theta_X \to \theta(f)$ given by $tf(\xi) = df \circ \xi$ whose cokernel, $\theta(f)/tf(\theta_X)$, is also a coherent sheaf on X. Moreover, the support of $\theta(f)/tf(\theta_X)$ is Σ and since $f\colon \Sigma \to Y$ is finite, the sheaf $f_*(\theta(f)/tf(\theta_X))$ is a coherent sheaf of $\mathcal{O}_Y$-modules, by the finite mapping theorem (see for instance [20, I.3.3]).

As in the case of X, the sheaf θ_Y of vector fields on Y is also coherent. We have a morphism $\omega f\colon \theta_Y \to f_*(\theta(f))$ given by $\omega(\eta) = \eta \circ f$, which induces a morphism of coherent sheaves $\theta_Y \to f_*(\theta(f)/tf(\theta_X))$. Its cokernel

$$\mathcal{T}^1_{\mathscr{A}_e} f := \frac{\theta(f)}{tf(\theta_X) + \omega f(\theta_Y)}$$

is a coherent sheaf on Y. Because of the coherence, the stalk $(\mathcal{T}^1_{\mathscr{A}_e} f)_y$ at each point $y \in Y$ is precisely $T^1_{\mathscr{A}_e} f_y$, where f_y is the multi-germ $f\colon (X, f^{-1}(y) \cap \Sigma) \to (Y, y)$. Hence, the support of $\mathcal{T}^1_{\mathscr{A}_e} f$ is the set of points $y \in Y$ such that f_y is not stable. The equivalence between the $\mathscr{A}$-finiteness and the isolated instability condition follows from the following consequence of the Rückert Nullstelensatz:

$$\dim_{\mathbb{C}} (\mathcal{T}^1_{\mathscr{A}_e} f)_0 < \infty \iff 0 \text{ is an isolated point of } \operatorname{supp}(\mathcal{T}^1_{\mathscr{A}_e} f).$$

(see for instance [48, Theorem E.3]). □

The Mather-Gaffney criterion is necessary, but not sufficient, in the real case. The function in Example 2.2.17 has isolated instability (in fact, it has isolated singularity) but it is not $\mathscr{A}$-finite since it is not $\mathscr{K}$-finite. To prove necessity, we need the following statement.

Proposition 2.2.21 *Let* $f\colon (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ *be analytic and let* $f_{\mathbb{C}}\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *be its complexification. Then,* f *is* $\mathscr{A}$*-finite if and only if* $f_{\mathbb{C}}$ *is* $\mathscr{A}$*-finite. Moreover, if* f *is* $\mathscr{A}$*-finite,*

$$\operatorname{codim}_{\mathscr{A}_e}(f) = \operatorname{codim}_{\mathscr{A}_e}(f_{\mathbb{C}}).$$

Proof See [48, Proposition 3.8]. □

Corollary 2.2.22 *Any smooth $\mathscr{A}$-finite germ $f\colon (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ has isolated instability.*

Proof By the Finite Determinacy Theorem 2.2.12, we can assume that f is a polynomial whose complexification $f_{\mathbb{C}}$ is also $\mathscr{A}$-finite by Proposition 2.2.21. Hence, $f_{\mathbb{C}}$ has isolated instability and there exists a representative $f_{\mathbb{C}}\colon \hat{X} \to \hat{Y}$ such that $f_{\mathbb{C}}^{-1}(0) \cap \Sigma_{f_{\mathbb{C}}} = S$ and $f_{\mathbb{C}}\colon \hat{X} \setminus f_{\mathbb{C}}^{-1}(0) \to \hat{Y} \setminus \{0\}$ is locally stable.

Let X and Y be the projections of $\hat{X}$ and $\hat{Y}$ on $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, and consider the representative $f\colon X \to Y$. For $y = 0$, we get $f^{-1}(0) \cap \Sigma_f = S$. For $y \neq 0$, the complexification of the germ $f\colon (\mathbb{R}^n, f^{-1}(y) \cap \Sigma_f) \to (\mathbb{R}^p, y)$ is obtained as the restriction of $f_{\mathbb{C}}\colon (\mathbb{C}^n, f_{\mathbb{C}}^{-1}(y) \cap \Sigma_{f_{\mathbb{C}}}) \to (\mathbb{C}^p, y)$ to the branches with real base-point. Since the restriction of a stable germ is also stable, $f\colon (\mathbb{R}^n, f^{-1}(y) \cap \Sigma_f) \to (\mathbb{R}^p, y)$ is stable, again by Proposition 2.2.21. Hence, $f\colon X \setminus f^{-1}(0) \to Y \setminus \{0\}$ is locally stable. □

2.2.7 Techniques for Calculating $T^1_{\mathscr{A}_e} f$ when $n + 1 \geq p$

It is easy to make definitions; to compute any of the terms involved may be more difficult. Here this is due in large part to the fact that two rings, $\mathcal{O}_n$ and $\mathcal{O}_p$, are involved in the definition of $T\mathscr{A}_e f$. Although $\theta(f)$ is a finite module (i.e. finitely generated) over $\mathcal{O}_n$, if $n > p$ then it is not finite over $\mathcal{O}_p$. Finiteness plays a crucial role in calculation, because it allows us to use Nakayama's Lemma, by which approximate equalities, valid modulo some power of the maximal ideal, can be turned into precise equalities. The double structure complicates life, because we have to work simultaneously over two rings, and the effect of this additional layer of difficulty can be seen in the relative slowness of the development of the theory, and the paucity of raw material in the form of the classification of map-germs. For this reason, we look for ways of embedding $T^1_{\mathscr{A}_e f}$ into the algebra of functions on the target $\mathbb{F}^p$. This can be done most easily when $n + 1 \geq p$.

In this subsection, we work with $\mathbb{F} = \mathbb{C}$, unless otherwise specified. This allows us use results from commutative algebra and complex analytic geometry. Because of Proposition 2.2.21, all the conclusions about $T^1_{\mathscr{A}_e} f$ also hold when $\mathbb{F} = \mathbb{R}$ and f is analytic: we take the complexification $f_{\mathbb{C}}$. By the finite determinacy theorem 2.2.12, they also apply to the C^∞ case.

Let D be the discriminant of f, $D = f(\Sigma)$. When $n < p$, $\Sigma_f = \mathbb{C}^n$ and D is the image of f. By Proposition 2.2.16, if f is $\mathscr{A}$-finite then it is finite on its critical set, and hence D is a germ of analytic subset of the target $\mathbb{C}^p$. When $n \geq p$, a fundamental theorem of Buchsbaum and Rim ([4, Corollary 2.7]) establishes that $\dim \theta(f)/tf(\theta_n) \geq p - 1$, and that when equality is achieved then $\theta(f)/tf(\theta_n)$ is a Cohen-Macaulay $\mathcal{O}_n$-module. A similar theorem of Eagon and Hochster ([24]) gives the same conclusions for $\mathcal{O}_n/J_f$: Σ_f is a "determinantal variety". From this, and the fact that f is finite on its critical set, it follows that D also has dimension $p-1$—so it is an analytic hypersurface, and thus defined by a single equation. This makes it possible to use a homomorphism originally suggested by Theo de Jong and Duco van Straten for the case of germs of maps $\mathbb{C}^n \to \mathbb{C}^{n+1}$ (see [49]). The following theorem was proved for the case $p = n+1$ in [49]; its extension to the case $n \geq p$ is new.

Theorem 2.2.23 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be $\mathscr{A}$-finite, with $n+1 \geq p \geq 2$ (but excluding the case $(n, p) = (1, 2)$), and let g be a reduced defining equation for D. Then the map $f^*(tg) : \theta(f) \to J_g\mathcal{O}_n$, sending $\sum_i \alpha_i \partial/\partial Y_i$ to $\sum_i \alpha_i(\partial g/\partial Y_i \circ f)$, passes to the quotient to give isomorphisms*

$$\frac{\theta(f)}{tf(\theta_n)} \to J_g\mathcal{O}_\Sigma \tag{2.9}$$

$$\frac{\theta(f)}{T\mathscr{A}_e f} \to \frac{J_g\mathcal{O}_\Sigma}{J_g\mathcal{O}_D} \tag{2.10}$$

Remark 2.2.24 The quotient $J_g\mathcal{O}_\Sigma / J_g\mathcal{O}_D$ can be viewed in two ways. First, composition with f induces a monomorphism $f^* : \mathcal{O}_D \to \mathcal{O}_\Sigma$, and thus $J_g\mathcal{O}_D$ may be thought of as a subset (though not an ideal) of $\mathcal{O}_\Sigma$, evidently contained in the ideal generated in $\mathcal{O}_\Sigma$ by the composed partials $(\partial g/\partial Y_i) \circ f$, which is what we mean by $J_g\mathcal{O}_\Sigma$. But, crucially for the practical value of the proposition, $J_g\mathcal{O}_\Sigma$ can also be thought of as an ideal of $\mathcal{O}_D$—or, more precisely, as the image under f^* of an ideal in $\mathcal{O}_D$—so that the right hand side in (2.10) is isomorphic to a quotient of two ideals of $\mathcal{O}_p$. This makes computing with it easier than calculating $\theta(f)/T\mathscr{A}_e f$. We explain this after the proof of the proposition.

Proof of 2.2.23 First, when $n \geq p$ then fold points, where f is equivalent to

$$x \mapsto (x_1, \ldots, x_{n-1}, \pm x_n^2 \pm \cdots \pm x_p^2),$$

are dense in Σ ([39, §4A]). A local calculation shows that if x is a fold point then Σ is smooth at x and $f|_\Sigma$ is an immersion, so that D is smooth at $f(x)$, and $d_x f(T_x\mathbb{C}^n) = T_{f(x)}D$. When $n+1 = p$, replace Σ by $\mathbb{C}^n$ and "fold" by "immersion" and the same conclusions become obvious. Now let $\xi \in \theta_n$. Then $tf(\xi)$ is tangent to D whenever x is a fold point (or immersive point when $n+1 = p$), and it follows by continuity that $f^*(tg)(tf(\xi)) = t(g \circ f)(\xi)$ vanishes on Σ. Thus $f^*(tg) : \theta(f) \to J_g\mathcal{O}_n$ passes to the quotient to define an epimorphism $\theta(f)/tf(\theta_n) \to J_g\mathcal{O}_\Sigma$.

We claim that to show $\theta(f)/tf(\theta_n) \to J_g\mathcal{O}_\Sigma$ is injective, it suffices to see that its kernel K is supported only at 0. In fact, we have an exact sequence

$$\theta_n \longrightarrow \theta(f) \longrightarrow \frac{\theta(f)}{tf(\theta_n)} \longrightarrow 0$$

where θ_n and $\theta(f)$ are free $\mathcal{O}_n$-modules of rank n and p respectively. When $n \geq p$, $\theta(f)/tf(\theta_n)$ has dimension $p-1$ and hence is Cohen-Macaulay, by a theorem of Buchsbaum-Rim [4]. So, it has depth $p-1 > 0$. When $p = n+1$, f is a finite mapping which implies that tf is injective. Thus,

$$0 \longrightarrow \theta_n \longrightarrow \theta(f) \longrightarrow \frac{\theta(f)}{tf(\theta_n)} \longrightarrow 0$$

is exact in this case and by the depth lemma, see e.g. [28, Lemma 6.5.18] or [48, Exercise C.4.5], $\theta(f)/tf(\theta_n)$ must have depth $\geq n-1 > 0$. In both cases, $\theta(f)/tf(\theta_n)$ has depth > 0, so it has no submodules which are supported only at the origin.

To show that K is supported only at 0, we use the geometrical criterion for $\mathscr{A}$-finiteness, that f must be stable outside 0. So consider the case where f is stable. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Der}(-\log D) & \hookrightarrow & \theta_p & \xrightarrow{\overline{\omega f}} & \dfrac{\theta(f)}{tf(\theta_n)} & \longrightarrow & 0 \\ & & \big\downarrow = & & \big\downarrow = & & \big\downarrow f^*(tg) & & \\ 0 & \longrightarrow & \mathrm{Der}(-\log D) & \hookrightarrow & \theta_p & \xrightarrow{tg} & J_g\mathcal{O}_\Sigma & \longrightarrow & 0 \end{array} \tag{2.11}$$

Here $\mathrm{Der}(-\log D)$ is, by definition, the $\mathcal{O}_p$-submodule of θ_p consisting of germs of vector fields tangent to D at its smooth points. It is easy to show that that

$$\mathrm{Der}(-\log D) = \{\eta \in \theta_p : tg(\eta) \in (g)\},$$

which proves exactness of the second row at θ_p. That $\mathrm{Der}(-\log D)$ is the kernel of $\overline{\omega f}$ is just the statement that $\eta \in \theta_p$ is liftable via f (i.e. there exists $\xi \in \theta_n$ such that $tf(\xi) = \omega f(\eta)$) if and only if $\eta \in \mathrm{Der}(-\log D)$. This is well known (see e.g. [48, Proposition 8.8]).

Because the diagram commutes, the snake lemma tells us that the kernel and cokernel of $f^*(tg)$ are equal to 0. This completes the proof that (2.9) is an isomorphism. That (2.10) is also an isomorphism follows, since the image of tg in $J_g\mathcal{O}_\Sigma$ is $J_g\mathcal{O}_D$. □

The proposition will make it possible for us to calculate

1. the $\mathscr{A}_e$-codimension of f,
2. a miniversal unfolding of f, and
3. an estimate of the determinacy degree of f,

by easy algorithms. To understand them, and in particular to justify the statement that the right hand side in (2.10) is a quotient of two ideals of $\mathcal{O}_p$, we first need to know more about the ring extension

$$\mathcal{O}_D \hookrightarrow \mathcal{O}_\Sigma$$

induced by composition with f. The *conductor* of the extension, which we denote by $\mathscr{C}$, is the annihilator $\mathrm{Ann}_{\mathcal{O}_D}(\mathcal{O}_\Sigma/\mathcal{O}_D)$, i.e. the set

$$\{h \in \mathcal{O}_D : h\mathcal{O}_\Sigma \subseteq \mathcal{O}_D\}.$$

Evidently this is an ideal of $\mathcal{O}_D$. It follows immediately from its definition that, viewing $\mathcal{O}_D$ as a subring of $\mathcal{O}_\Sigma$, $\mathscr{C}$ is also an ideal of $\mathcal{O}_\Sigma$—though it is important to note that a set of $\mathcal{O}_\Sigma$- generators of $\mathscr{C}$ will not in general generate it over $\mathcal{O}_D$, even though they lie in $\mathcal{O}_D$.

Lemma 2.2.25 *Let $f : (\Sigma, 0) \to (\mathbb{C}^p, 0)$ be finite and generically 1-to-1, where Σ is a $p-1$-dimensional Cohen-Macaulay space. Then the matrix of a minimal presentation is square. Moreover, for any square presentation matrix Λ (not necessarily minimal),*

1. $g := \det \Lambda$ *is a reduced equation for D.*
2. *If the list of $\mathcal{O}_p$ generators of $\mathcal{O}_\Sigma$ begins with* 1*, and all of the remaining generators lie in the maximal ideal of $\mathcal{O}_\Sigma$, then* $\mathrm{Ann}_{\mathcal{O}_p}(\mathcal{O}_\Sigma/\mathcal{O}_D)$ *is equal to the ideal of maximal minors of the matrix Λ' obtained by deleting the first row of Λ.*
3. *The first Fitting ideal of $\mathcal{O}_\Sigma$ as $\mathcal{O}_p$-module,* $\mathrm{Fitt}_1^{\mathcal{O}_p}(\mathcal{O}_\Sigma)$*, (i.e. the ideal generated by all of the submaximal minors of Λ) is equal to the ideal generated by the maximal minors of Λ', i.e. to* $\mathrm{Ann}_{\mathcal{O}_p}(\mathcal{O}_\Sigma/\mathcal{O}_D)$.
4. $\mathscr{C} = \mathrm{Fitt}_1^{\mathcal{O}_p}(\mathcal{O}_\Sigma)\mathcal{O}_D$.
5. $J_g \subseteq \mathrm{Fitt}_1^{\mathcal{O}_p}(\mathcal{O}_\Sigma)$*, so that* $J_g\mathcal{O}_D \subset \mathscr{C}$*, and* $J_g\mathcal{O}_\Sigma \subset \mathcal{O}_D$.

Proof

1. As $f|_\Sigma$ is finite, $\mathcal{O}_\Sigma$ is also Cohen-Macaulay over $\mathcal{O}_p$, of the same dimension. Hence by the Auslander-Buchsbaum theorem [21], that for a module over a local Cohen-Macaulay ring R,
$$\text{depth} + \text{projective dimension} = \dim\ R,$$
the projective dimension of $\mathcal{O}_\Sigma$ as $\mathcal{O}_p$-module is 1—i.e. it has a free resolution of length 1. Let
$$0 \longrightarrow \mathcal{O}_p^j \xrightarrow{\Lambda} \mathcal{O}_p^k \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0$$
be one such. Tensoring with the field $\mathbb{K}_p$ of meromorphic functions on $\mathbb{C}^p$, we obtain the exact sequence $0 \longrightarrow \mathbb{K}_p^j \xrightarrow{\Lambda} \mathbb{K}_p^k \longrightarrow 0$, so that j must equal k.
2. The support of any module with square presentation Λ is equal to the zero locus of $\det \Lambda$. In our case, a local calculation at a fold point (or immersive point of $f|_\Sigma$) shows that $\det \Lambda$ is a reduced equation.

3. Λ' presents $\mathcal{O}_\Sigma/\mathcal{O}_D$; by a theorem of Buchsbaum and Eisenbud ([3]), the maximal minors of Λ' generate $\mathrm{Ann}_{\mathcal{O}_p}(\mathcal{O}_\Sigma)$, which is the conductor ideal.
4. See [48, Prop. 11.11].
5. follows from 3 and 4.
6. Let det be the determinant function on the space of $k \times k$ matrices. For each entry λ_{ij} let m_{ij} denote the corresponding signed cofactor. Then $\partial \det/\partial\lambda_{ij} = m_{ij}$. Hence

$$\partial g/\partial Y_\ell = \sum_{i,j}(\partial \det/\partial\lambda_{ij})(\partial\lambda_{ij}/\partial Y_\ell) = \sum_{i,j} m_{ij}\partial\lambda_{ij}/\partial Y_\ell \in \mathrm{Fitt}_1(\mathcal{O}_\Sigma).$$

□

Remark 2.2.26 In the case of $\mathscr{A}$-finite germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, one can understand the fact that J_g is contained in the conductor ideal geometrically. The key idea is that in order for a function $s \in \mathcal{O}_n$ to be a composite $r \circ f$, it is necessary and sufficient that $s(x) = s(x')$ whenever $f(x) = f(x')$. The necessity of this condition is obvious. That it is also sufficient is deeper. Let D be the image of f. If x and x' are immersive points of f and no other point has the same image, then by the $\mathscr{A}$-finiteness of f, at $f(x)$, D is a normal crossing of two embedded copies of $\mathbb{C}^n$. It is evident that a pair of functions, one on each of the two sheets, give rise to a well defined analytic function on their union if and only if they agree on the intersection. So our condition at least guarantees the existence of the function r away from the triple points and the image of non-immersive points of f. But since these have codimension at least 2 in X, the function r extends to all of D, by a variant of the Hartogs theorem which is valid for Cohen-Macaulay spaces, and in particular for hypersurfaces.

Now the conductor $\mathscr{C}$ is the set of functions $h \in \mathcal{O}_n$ such that for every s, hs is a composite $r \circ f$. The only way to guarantee that $(hs)(x) = (hs)(x')$ for every s, whenever $f(x) = f(x')$, is for $h(x) = h(x') = 0$. So $\mathscr{C}$ is the ideal of $\mathcal{O}_n$ consisting of functions which vanish on the preimage of the double locus of D. In the case of maps $\mathbb{C}^n \to \mathbb{C}^{n+1}$, the double locus is dense in the singular locus, and hence $\mathscr{C}$ consists of functions vanishing on the preimage in $\mathbb{C}^n$ of the singular set of X. This set evidently contains J_g.

For the case $n \geq p$, the statement is more subtle. A local calculation at cusp points is also necessary, and shows that $\mathscr{C}$ is equal to the ideal of functions vanishing both at double points in Σ, and at points where $f|_\Sigma$ is not an immersion.

Remark 2.2.27 Is there any way we can adapt the homomorphism $f^*(tg)$ to allow us to express $\theta(f)/T\mathscr{A}^{(1)}f$ as a quotient of ideals of $\mathcal{O}_D$? If we could use this procedure to find the lowest k such that $\mathfrak{m}_n^{k+1}\theta(f) \subset T\mathscr{A}^{(1)}f$, then we could get sharp determinacy estimates very quickly, for since $\mathscr{A}^{(1)}$ is unipotent, this condition is necessary and sufficient for k-$\mathscr{A}^{(1)}$-determinacy. The problem here is that $f^*(dg)$ kills all of $tf(\theta_n)$. How can it be modified so that it kills only $tf(\mathfrak{m}_n^2\theta_n)$?

For the sake of future calculations, we describe here a procedure for finding an $\mathcal{O}_n$-generator of the conductor when $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite

Proposition 2.2.28 *In this situation, let g be a reduced equation for the image of f, and let*

$$m_i := \frac{\partial(f_1, \ldots, \widehat{f_i}, \ldots, f_{n+1})}{\partial(x_1, \ldots, x_n)}$$

Then the quotient

$$c := \frac{(\partial g/\partial Y_i) \circ f}{m_i}$$

is, up to sign, independent of i, and is an $\mathcal{O}_n$-generator of $\mathscr{C}$.

The proposition was proved by Ragni Piene in [56] without the hypothesis of $\mathscr{A}$-finiteness, using Grothendieck local duality. An elementary proof in the present case, using a local calculation, was given by Bruce and Marar in [2].

Now we can give the algorithms listed after the proof of 2.2.23. We give them in the language of SINGULAR. Translating them to other computer algebra systems which support local rings is straightforward.

Algorithm 1 The $\mathscr{A}_e$ codimension of f is the vector space dimension of $\theta(f)/T\mathscr{A}_e f$, and thus, by Lemma 2.2.23, of $J_g\mathcal{O}_n/J_g\mathcal{O}_D$. This expression implicitly views $\mathcal{O}_D$ as a subring of $\mathcal{O}_\Sigma$, which it becomes via the monomorphism f^*; but by Lemma 2.2.25, $J_g\mathcal{O}_\Sigma$ is the isomorphic image under f^* of an ideal of $\mathcal{O}_D$, and thus $J_g\mathcal{O}_\Sigma/J_g\mathcal{O}_D$ can be viewed as a quotient of two ideals of $\mathcal{O}_D$. This is a crucial advantage when it comes to algorithmics—it circumvents the "mixed module" structure of $T\mathscr{A}_e f$ and $\theta(f)/T\mathscr{A}_e f$. This observation appears first in [13].

2.2.8 Implementation of the Algorithms in SINGULAR

The three algorithms of the previous subsection can easily be implemented in SINGULAR. SINGULAR has the advantage over MACAULAY2 that all of its procedures function in local rings, which is essential when the germ is not weighted homogeneous.

Example 2.2.29 Consider the germ at 0 of the map $f : \mathbb{C}^2 \to \mathbb{C}^3$ defined by

$$f(x, y) = (x^2, y^2, xy + x^3 + y^3).$$

This expression is not weighted homogeneous. In fact it can be shown, using Theorem 2.3.8, that f is not weighted homogeneous in any coordinate system. The code for Algorithm 1 begins with the ring declarations:

```
ring T = 0, (X, Y, Z), ds;
ring S = 0, (x, y), ds;
```

The choices for the rings in the source and target are $S = \mathbb{Q}[x, y]_{(x,y)}$ and $T = \mathbb{Q}[X, Y, Z]_{(X,Y,Z)}$, respectively. These are the localisations of $\mathbb{Q}[x, y]$ and $\mathbb{Q}[X, Y, Z]$ with respect to their maximal ideals at the origin (x, y) and (X, Y, Z), respectively. They are considered as subrings of the convergent power series $\mathbb{C}\{x, y\}$ and $\mathbb{C}\{X, Y, Z\}$, respectively. The command ds at the end of the ring declarations means that we are considering the "negative degree reverse lexicographical" local ordering, but any other local ordering could be used instead. Then

```
map f = T, x2, y2, xy + x3 + y3;
ideal zero = 0;
setring T;
ideal I = preimage(S, f, zero);
poly g = I[1];
```

returns an expression for the generator g of the ideal defining the image of f,

$$g = X^6 - 2X^3Y^3 + Y^6 - 2X^4Y - 2XY^4 - 8X^2Y^2Z - 2Y^3Z^2 + X^2Y^2 - 2XYZ^2 + Z^4$$

The "preimage" operator gives the preimage under $f^* : T = \mathcal{O}_3 \to \mathcal{O}_2 = S$ of an ideal in $\mathcal{O}_2$. Here preimage(S, f, zero) is just $\ker f^*$.

In what follows, the expression $f(J)$ means the ideal in $\mathcal{O}_2$ generated by $f^*(J)$—what we have referred to above as $J_g\mathcal{O}_n$. The commands

```
ideal J = jacob(g);
setring S;
ideal fJ = f(J);
setring T;
ideal JJ = preimage(S, f, fJ);
module M = modulo(JJ, J + I);
matrix b = kbase(std(M));
print(b);
```

Then return the matrix

```
0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0 0
1 0 0 0 0 0
0 1 0 0 0 0
0 0 Z 1 0 0
0 0 0 0 Z 1
```

showing that the $\mathscr{A}_e$-codimension of f is 6. The seven rows of this matrix correspond to the seven generators that SINGULAR finds for the ideal `JJ`.

The code for Algorithm 2 is:

```
ideal B = matrix(JJ) * b;
setring S;
ideal A = f(B);
matrix H = lift(fJ, A);
print(H);
```

Here the comand `lift(fJ,A)` is used to divide A by $f(J)$. This returns the following 3×6 matrix whose columns provide a basis for $\theta(f)/T\mathscr{A}_e f$:

0	$-\frac{1}{4}y$	0	0	$-\frac{1}{2}y^5$	$-\frac{1}{2}y^2$
$-x$	0	$-\frac{1}{2}x^5$	$-\frac{1}{2}x^2$	0	0
0	0	$\frac{1}{4}x^2y + \frac{1}{2}x^4 + \frac{3}{4}xy^3$ $+\frac{3}{2}x^3y^2 + \frac{1}{2}y^5 + \frac{3}{4}x^2y^4$	$\frac{1}{4}x + \frac{1}{2}y^2$	$\frac{1}{4}xy^2 + \frac{3}{4}x^3y + \frac{1}{2}y^4$ $+\frac{1}{2}x^5 + \frac{3}{2}x^2y^3 + \frac{3}{4}x^4y^2$	$\frac{1}{4}y + \frac{1}{2}x^2$

This expression contains many redundant terms. We discuss how to remove them below, with the help of the output of Algorithm 3.

The code for Algorithm 3 is:

```
ideal fJ4 = fJ * maxideal(4);
setring T;
ideal JJ4 = preimage(S, f, fJ4);
module M4 = modulo(JJ4, J + I);
```

```
matrix b4 = kbase(std(M4));
print(b4);
setring S;
matrix H3 = jet(H, 3);
print(H3);
```

The output of b4 has given the empty matrix, meaning that $\mathfrak{m}_n^4\theta(f) \subset T\mathscr{A}_e f$. Applying the same steps with 3 in place of 4 returns a non-empty matrix, so $\mathfrak{m}_n^3\theta(f) \not\subset T\mathscr{A}_e f$. Because $\mathfrak{m}_n^4\theta(f) \subset T\mathscr{A}_e f$, we use the command `jet(H,3)` to obtain the 3-jet of the matrix H (that is, we remove all monomials of degree ≥ 4 in H). The resulting matrix is now:

0	$-\frac{1}{4}y$	0	0	0	$-\frac{1}{2}y^2$
$-x$	0	0	$-\frac{1}{2}x^2$	0	0
0	0	$\frac{1}{4}x^2y$	$\frac{1}{4}x + \frac{1}{2}y^2$	$\frac{1}{4}xy^2$	$\frac{1}{4}y + \frac{1}{2}x^2$

We can perform additional simplifications by hand. Using

$$\omega f\left(X\frac{\partial}{\partial X}\right), \omega f\left(X\frac{\partial}{\partial Y}\right), \omega f\left(X\frac{\partial}{\partial Z}\right), \omega f\left(Y\frac{\partial}{\partial X}\right), \omega f\left(Y\frac{\partial}{\partial Y}\right), \omega f\left(Y\frac{\partial}{\partial Z}\right),$$

allows us to remove the monomials x^2 and y^2 from all entries of the matrix. After eliminating also the superfluous coefficients, the final basis for $\theta(f)/T\mathscr{A}_e f$ is very simple and symmetric:

0	y	0	0	0	0
x	0	0	0	0	0
0	0	x^2y	x	xy^2	y

The versal unfolding $F : (\mathbb{C}^2 \times \mathbb{C}^6, 0) \to (\mathbb{C}^3 \times \mathbb{C}^6, 0)$ constructed with this basis is

$$F(x, y, a_1, \ldots, a_6) = (x^2+a_1y, y^2+a_2x, xy+x^3+y^3+a_3x+a_4y+a_5x^2y+a_6xy^2, a_1, \ldots, a_6).$$

Example 2.2.30 The following SINGULAR calculation finds the $\mathscr{A}_e$-codimension and discriminant Milnor number μ_Δ (see Sect. 2.3.1 below) of the germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ defined by

$$f(x, y, z) = (x^2 - y^2 + z^2 + xyz, xy - yz + zx).$$

We work modulo a large prime to speed up the computation, and initially work with global monomial orderings in the rings.

```
ring T = 32003, (X, Y), dp;
ring S = 32003, (x, y, z), dp;
ideal ff = x2 − y2 + z2 + xyz, xy − yz + xz;
map f = T, ff;
ideal C = minor(jacob(ff), 2);
setring T;
ideal I = preimage(S, f, C);
poly g = I[1];
ideal J = jacob(g);
setring S;
qring SC = std(C);
map f = imap(S, f);
ideal fJ = f(J);
setring T;
ideal JJ = preimage(SC, f, fJ);
```

Now we change to the local monomial ordering “ds”; the “preimage” commands run more quickly with the global ordering “dp”, but we need a local ordering to compute the correct dimension.

```
ring Ts = 32003, (X, Y), ds;
ideal JJ = imap(T, JJ);
JJ = std(JJ);
ideal I = imap(T, I);
ideal J = imap(T, J);
module M = modulo(JJ, J + I);
```

```
vdim(std(M));
3
module N = modulo(JJ, J);
vdim(std(N));
5
```

So $\mathscr{A}_e - \mathrm{codim}(f) = 3$ and $\mu_\Delta(f) = 5$.

Exercise 2.2.31

1. Use the algorithms of this section to find the $\mathscr{A}_e$-codimension for the following germs of maps. (i) $f(x, y) = (x^2, y^2, x(x^2 + y^2) + y(x^2 - y^2))$ (ii) $f(x, y, z) = (x, y^2 + xz + x^2 y, yz, z^2 + y^3)$ (iii) $f(x, y, u, v, w) = (x^2 + ux + vy, xy, y^2 + wx + uy, u, v, w)$ (iv) $f(w, x, y, z) = (w, x^2 + y^2 + z^2 + xy - 2xz + 3yz, w(x + y + z) + x^3 + y^3 + z^3 + xyz)$
2. Find a basis for $\theta(f)/T\mathscr{A}_e f$ for germs (i)-(iii) in the previous exercise.
3. What is the least power of $\mathfrak{m}_n$ such that $\mathfrak{m}_n^k \theta(f) \subset T\mathscr{A}_e f$ in each case?
4. Use the procedure of Sect. 2.2.5 to show that the simplest corank 2 stable map-germ $(\mathbb{C}^n, 0 \to (\mathbb{C}^{n+1}, 0)$ (i.e. with the smallest value of n) is

$$F(x, y, a, b, c, d) = (x^2 + ay, xy + bx + cy, y^2 + dx, a, b, c, d).$$

5. Exercise 1(iii) above gives a germ with the same local algebra, one dimension lower, from $\mathbb{C}^5$ to $\mathbb{C}^6$. What is the lowest possible $\mathscr{A}_e$ codimension for a germ $(\mathbb{C}^4, 0) \to \mathbb{C}^5, 0)$ of corank 2? At least guess an answer!
5. Apply algorithms 2 and 3 to the germ of Example 2.2.30, to find the lowest k such that $\mathfrak{m}_3^k \theta(f) \subseteq T\mathscr{A}_e f$, and to find a versal unfolding.

2.2.9 Damon's Theory of Sections of Images and Discriminants

In [9], Jim Damon introduced what turned out to be an extremely fruitful way of viewing the deformation theory of map germs. If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is $\mathscr{K}$-finite then following the procedure of Sect. 2.2.5, it has a stable unfolding – that is, an unfolding which is a stable map in its own right. Such an unfolding, F, gives rise to a fibre square

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^d, (0,0)) & \xrightarrow{F} & (\mathbb{C}^p \times \mathbb{C}^d, (0,0)) \\ j \uparrow & & \uparrow i \\ (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}^p, 0) \end{array} \tag{2.12}$$

in which i and j are standard inclusions. Note that as F is an unfolding on parameter space $(\mathbb{C}^d, 0)$, i and F are transverse to one another. Given any fibre product diagram

$$\begin{array}{ccc} (\mathbb{C}^N, 0) & \xrightarrow{F} & (\mathbb{C}^q, 0)\ , \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ (X, 0) & \xrightarrow{f} & (\mathbb{C}^p, 0) \end{array} \tag{2.13}$$

in which $i \pitchfork F$, we say that f is *induced from F by the transverse base change i*, and sometimes write $i^*(F)$ in place of f. The diagram (2.12) shows that every $\mathscr{K}$-finite germ f is induced from a stable germ F by such a transverse fibre product.

Given a diagram (2.13), let D_F be the discriminant of F. The *logarithmic tangent space* to D_F at the point z is the subspace $T_z^{\log} D_F := \{\xi(z) : \xi \in \mathrm{Der}(-\log D_F)_z\}$ of $T_z\mathbb{C}^q$. We say that i is *logarithmically transverse to D_F at $y \in \mathbb{C}^p$* if

$$d_y i(T_y\mathbb{C}^p) + T_{i(y)}^{\log} D_F = T_{i(y)}\mathbb{C}^q. \tag{2.14}$$

By the use of Nakayama's Lemma, this condition can be restated in terms of $\mathcal{O}_p$-modules: (2.14) holds if and only if

$$ti(\theta_p) + i^*(\mathrm{Der}(-\log D_F)) = \theta(i).$$

Theorem 2.2.32 (J.N. Damon, [9]) *If f is induced from the stable map F by transverse base change i, then*

$$\frac{\theta(f)}{T\mathscr{A}_e f} \simeq \frac{\theta(i)}{ti(\theta_p) + i^*(Der(-\log D_F))}. \qquad \square \tag{2.15}$$

In particular, $i^*(F)$ is stable if and only if i is logarithmically transverse to D_F. Briefer proofs than Damon's can be found in [65] and [48, Theorem 8.7].

The denominator on the right hand side in (2.15) is the extended tangent space for the group $\mathscr{K}_{D(F)}$, which is a subgroup of the contact group $\mathscr{K}$, introduced by Damon in [8], but which we will not use here. Just as in Theorem 2.2.23, the isomorphism (2.15) expresses $T^1_{\mathscr{A}_e} f$ as a quotient of finitely generated $\mathcal{O}_p$-modules. However, its great virtue is that it allows the comparison of $T^1_{\mathscr{A}_e} f$ to a module which, when $n \geq p$, computes the discriminant Milnor number of f, as we will see in Sect. 2.3.2.

Exercise 2.2.33 Find a natural homomorphism from the module in the right hand side of (2.15) to the quotient of ideals on the right hand side of (2.10).

2.3 Vanishing Homology in the Image and Discriminant when $n + 1 \geq p$

In the theory of complex hypersurface singularities $(X, 0)$ in $\mathbb{C}^{n+1}$, a key role is played by the Milnor fibre and the Milnor number (when X has isolated singularity). We refer to [38] for a recent survey on the topology of the Milnor fibration. From the topological viewpoint, a small representative X can be chosen so that X is contractible. In fact, we start by taking any representative X as a closed analytic subset of an open neighbourhood Y of the origin and we fix a finite analytic Whitney stratification of X. By the curve selection lemma (see [47]), there exists $\epsilon > 0$ such that for all ϵ' with $0 < \epsilon' \leq \epsilon$, the sphere $S_{\epsilon'}$ of radius ϵ' and centered at the origin is transverse to all the strata of X. Because of the transversality, if B_ϵ is the closed ball, $X \cap B_\epsilon$ is homeomorphic to the cone in its boundary $X \cap S_\epsilon$. The ball B_ϵ is called a *Milnor ball* for $(X, 0)$.

Next, we deform X in the "best possible way", which means that it becomes a smooth manifold. In the hypersurface case, this can be done easily: assume that $X = f^{-1}(0)$, where $f : Y \to \mathbb{C}$ is a reduced holomorphic function. Again by the curve selection lemma, f has isolated critical value at the origin in $\mathbb{C}$, so there exists $\eta > 0$ such that for any $t \in \mathbb{C}$, with $0 < |t| < \eta$, t is a regular value of f, which implies that $X_t = f^{-1}(t)$ is a closed smooth submanifold of Y. Moreover, by reducing η if necessary, we can assume that X_t is also transverse to S_ϵ, so $F_t := X_t \cap B_\epsilon$ is a compact manifold with boundary. The version of the Ehresmann Lemma for manifolds with boundary (see e.g. [38]) implies that the restriction

$$f : B_\epsilon \cap f^{-1}(\mathring{B}^*_\eta) \to \mathring{B}^*_\eta$$

is a locally trivial fibration, where $\mathring{B}_\eta$ is the open disk of radius η centered at the origin in $\mathbb{C}$ and $\mathring{B}^*_\eta = \mathring{B}_\eta \setminus \{0\}$. Since $\mathring{B}^*_\eta$ is connected, the fibre F_t is independent, up to diffeomorphism, of the choice of $t \in \mathring{B}^*_\eta$ and is called the *Milnor fibre* of $(X, 0)$. It is well known that the Milnor fibre is also independent of the choice of ϵ, η, the defining equation f and the smoothing of X (that is, the way in which we deform the equation to obtain a smooth manifold). In general, F_t is no longer contractible and it presents some non trivial homology known as the *vanishing homology* of $(X, 0)$.

When $(X, 0)$ has isolated singularity, Milnor showed that F_t has the homotopy type of a wedge of spheres of real dimension n; the number of such spheres $\mu(X, 0)$ is called the *Milnor number* of $(X, 0)$. The reduced homology $\tilde{H}_*(F_t; \mathbb{Z})$ is concentrated in the middle dimension n. In fact, $H_n(F_t; \mathbb{Z})$ is free of rank $\mu(X, 0)$ and its generators are known as *vanishing cycles*. The Milnor number can be computed algebraically easily as

$$\mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J_f}.$$

2.3.1 The Homotopy Type of the Discriminant of a Stable Perturbation: Discriminant and Image Milnor Number

Here we explain how to adapt the above construction for $\mathscr{A}$-finite map-germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n + 1 \geq p$. We denote by Σ the critical locus of f and $D = f(\Sigma)$ its discriminant . As we saw at the start of Sect. 2.2.7, D is a hypersurface in $(\mathbb{C}^p, 0)$.

By the Mather-Gaffney criterion 2.2.20, f has isolated instability. We take a representative $f\colon X \to Y$ such that $f^{-1}(0) \cap \Sigma = S$ and $0 \in Y$ is the only unstable point of f. It is also convenient to assume that $f\colon \Sigma \to Y$ is a finite mapping, so that D is a closed analytic subset of Y. This is always possible by shrinking the neighbourhoods X and Y is necessary.

We assume from now on in this section that (n, p) are nice dimensions in Mather's sense, or that f has kernel rank one. In both cases, D has a natural finite analytic Whitney stratification whose strata are the *stable types*. This means that two points y, y' in D belong to the same stratum if and only if the multi-germs $f\colon (\mathbb{C}^n, f^{-1}(y) \cap \Sigma) \to (\mathbb{C}^p, y)$ and $f\colon (\mathbb{C}^n, f^{-1}(y') \cap \Sigma) \to (\mathbb{C}^p, y')$ are $\mathscr{A}$-equivalent. We refer to [48, Section 7.3] for the proof that the stratification by stable types is finite, analytic and Whitney regular. We fix a Milnor ball B_ϵ for D as before.

The next step is to deform D in "the best possible way", given that the deformed space is always the discriminant of a mapping. This forces that some singularities of D must be preserved in the deformation, namely, those which correspond to stable points of f. In fact, these are the "rigid" points in our deformation theory.

We say that a 1-parameter unfolding $F(x, t) = (f_t(x), t)$ is a *stabilisation* of f, if there exists a representative $F : \mathcal{X} \to Y \times T$ such that f_t is locally stable for all $t \in T^* = T \setminus \{0\}$. Here, $\mathcal{X}$ and T are open neighbourhoods of $S \times \{0\}$ and the origin in $\mathbb{C}^n \times \mathbb{C}$ and $\mathbb{C}$ respectively, The mapping f_t, for $t \in T^*$ will be called a *stable perturbation* of f. The existence of a stabilisation of f is always guaranteed when (n, p) are nice dimensions or f has kernel rank one (see [48, Section 5.4]).

Since F is also finite-to-one on its critical set, its discriminant $\mathcal{D}$ is a closed analytic subset of $Y \times T$ (after shrinking the neighbourhoods if necessary). The projection onto the parameter space $\pi : \mathcal{D} \to T$ is a holomorphic mapping whose fibre $D_t = \pi^{-1}(t)$ is the discriminant of f_t, for all $t \in T$. The stratification by stable types is also well defined on $\mathcal{D}$ and on each D_t.

Proposition 2.3.1 *There exists $\eta > 0$ such that*

1. *The multi-germ of f_t at any $y \in S_\epsilon$ is stable, for all $t \in \mathring{B}_\eta$.*
2. *D_t is transverse to S_ϵ, for all $t \in \mathring{B}_\eta$.*
3. *The restriction*

$$\pi : \mathcal{D} \cap (B_\epsilon \times \mathring{B}^*_\eta) \to \mathring{B}^*_\eta$$

is a locally C^0-trivial fibration.

Proof See [48, Proposition 8.2]. □

A difference with the previous case is that here $\mathcal{D}$ is not a smooth manifold, but a stratified space. Instead of the Ehresmann lemma, the proof is based on Thom's first isotopy lemma [17] and this is the reason that we get local C^0-triviality instead of local C^∞-triviality. As before, the fibre $D_t \cap B_\epsilon$, for $t \in \mathring{B}_\eta^*$ is independent, up to homeomorphism, of the choice of $t \in \mathring{B}_\eta^*$. Moreover, it can be shown that it is also independent of the choice of ϵ, η and the stabilisation F. We call $D_t \cap B_\epsilon$ the *disentanglement* of f. Item 1 and 2 of the proposition are also important, since they ensure that the origin is the only critical point, in the stratified sense, of the mapping

$$\pi : \mathcal{D} \cap (B_\epsilon \times \mathring{B}_\eta) \to \mathring{B}_\eta.$$

Suppose now that $p \leq n+1$, so $\mathcal{D}$ is a hypersurface in $Y \times T$. Let $G : Y \times T \to \mathbb{C}$ be a reduced holomorphic function such that $\mathcal{D} = G^{-1}(0)$. For each $t \in T$, $g_t : Y \to C$ is the function $g_t(y) = G(y,t)$ and $D_t = g_t^{-1}(0)$.

Theorem 2.3.2 (D. Siersma, [61]) *Under these circumstances, for each $t \in \mathring{B}_\eta^*$, $D_t \cap B_\epsilon$ has the homotopy type of a wedge of spheres of dimension $p-1$ in which the number of spheres is equal to the sum of the Milnor numbers of the critical points of g_t in $B_\epsilon \setminus D_t$.*

Two different proofs of this theorem can be found in [48, Section 8.3] following arguments of Lê [36] and Siersma [61]. The number of spheres in $D_t \cap B_\epsilon$ is called the *discriminant Milnor number* , and denoted by $\mu_\Delta(f)$ when $n \geq p$, or the *image Milnor number*, and denoted by $\mu_I(f)$, when $p = n+1$. As in the case of isolated hypersurface singularities, the reduced homology is concentrated in the middle dimension $p-1$ and the generators of $H_{p-1}(D_t \cap B_\epsilon; \mathbb{Z})$ are also called *vanishing cycles* of f. The algebraic computation of $\mu_\Delta(f)$ or $\mu_I(f)$ is not so easy as in the case of an isolated hypersurface singularity and requires more sophisticated techniques that we will discuss later.

Example 2.3.3 Let $f\colon (\mathbb{C},0) \to (\mathbb{C}^2,0)$ be the germ $f(x) = (x^2,x^3)$ which parameterises the cusp D with equation $X^3 - Y^2 = 0$ in the plane. A simple computation shows that $\mu(D) = 2$, which means that its Milnor fibre F_t, obtained as $X^3 - Y^2 = t$, for $t \neq 0$, is a compact orientable surface with one boundary component and genus 1. A stable perturbation is given by $f_t(x) = (x^2, x^3 - tx)$, with $t \neq 0$. The image D_t has defining equation $g_t(X,Y) = -t^2X+2tX^2-X^3+Y^2$, which has two non degenerate critical points at $(t/3,0)$ and $(t,0)$. Since $(t/3,0) \notin D_t$ and $(t,0) \in D_t$, we see that $\mu_I(f) = 1$. In fact, D_t is a compact orientable surface with one boundary component and genus 0 and with a singular point of Morse type. We show in Fig. 2.2 pictures for D (left), the Milnor fibre F_t (center) and the disentanglement D_t (right).

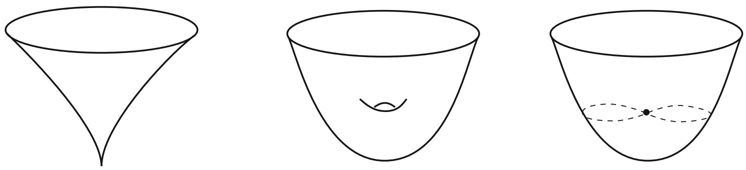


Fig. 2.2 Milnor fibre and disentanglement of the cusp

The topology of $D_t \cap B_\epsilon$ is more complicated when $p \geq n+2$, as can be seen from the first item in Corollary 2.4.9, since it may present homology in several dimensions. Some striking results on the homotopy type in this situation have been obtained by Houston in [25].

In the real case, if $f\colon (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ is $\mathscr{A}$-finite we can assume it is polynomial by the Finite Determinacy Theorem 2.2.12. Then, some of the results of this subsection are still valid, although we find two differences. The first one is that the discriminant D is no longer analytic, but semialgebraic in $(\mathbb{R}^p, 0)$. Anyway, all the arguments from stratification theory apply for semialgebraic sets and mappings, so Proposition 2.3.1 holds word by word also in the real case (see [48, Proposition 8.2]).

The second and more substantial difference is that $\mathring{B}^*_\eta$ has two connected components, so the fibre $D_t \cap B_\epsilon$ can be different whenever $t > 0$ or $t < 0$. But this also implies that the topological type of $D_t \cap B_\epsilon$ depends on the stabilisation of f. In general we must consider a versal d-parameter unfolding $F(x,u) = (f_u(x), u)$ and look at its bifurcation set $\mathscr{B}(F)$, that is, the subset of parameters u in a neighbourhood U of the origin in $\mathbb{R}^d$ such that the mapping f_u is not locally stable. In general $\mathscr{B}(F)$ disconnects U in several connected components and each one these components can give a different topological type for the disentanglement and hence, for the stable perturbation .

Nevertheless, in low dimensional examples it is possible to make drawings of the images and discriminant s of stable perturbations of real $\mathscr{A}$-finite map germs. In some simple cases, by judicious choice of values for the unfolding parameters it is possible to arrange that the real image or discriminant is a deformation retract of the image or discriminant of a stable perturbation of the complexified germ. This is the case for germs $(\mathbb{R}^2, S) \to (\mathbb{R}^3, 0)$ of $\mathscr{A}_e$-codimension 1, whose stable images are shown in Figs. 2.3 and 2.4.

Example 2.3.4 Figure 2.5 shows the discriminant of a stable perturbation of the $\mathscr{A}_e$-codimension 1 bi-germ

$$\begin{cases}(x,y,z) \mapsto (x,y,z^3 - xz),\\ (x,y,z) \mapsto (x,y^3+xy,z).\end{cases}$$

In the unperturbed germ, each component monogerm is a trivial unfolding of the Whitney cusp, and therefore stable; the bigerm is nevertheless unstable, because the

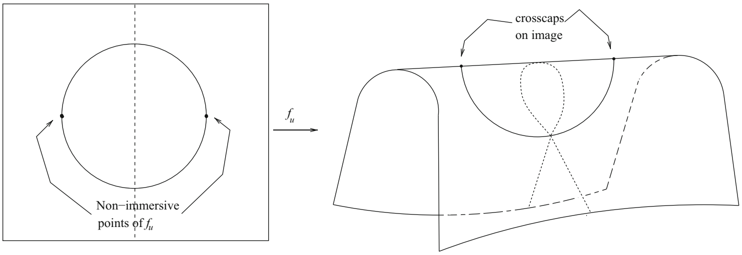


Fig. 2.3 Stable perturbation of $(x, y) \mapsto (x, y^2, y^3 + x^2 y)$

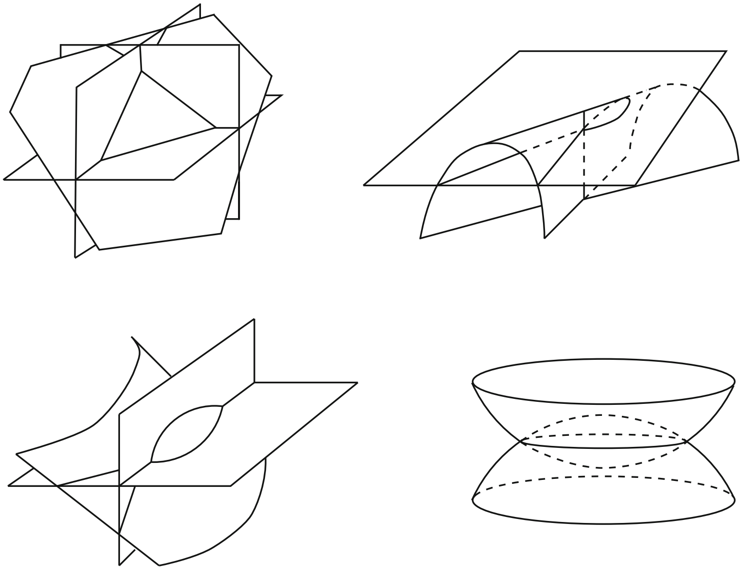


Fig. 2.4 Images of stable perturbations of codimension 1 germs of maps from the plane to 3-space

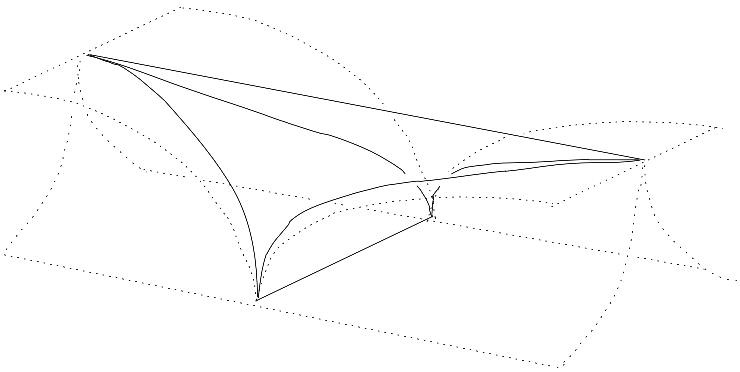


Fig. 2.5 Discriminant of the bi-germ of Example 2.3.4 (after [7])

two analytic strata (the cuspidal edges in the discriminant s of the two monogerms) both pass through $0 \in \mathbb{F}^3$, and therefore do not intersect transversely in 3-space (see Theorem 2.2.18 above). The picture shows, with a dotted line, the discriminant s of the two component mono-germs after they have been perturbed. Each is isomorphic to the product of a first-order cusp $\{(u, v) : u^2 = v^3\}$ and a line. Their intersection, and their cuspidal edges, are drawn with a continuous line. Their union carries a non-trivial 2-cycle, which appears in the drawing as the curvilinear tetrahedron whose edges are made up of the intersection of the two discriminant s, together with their cuspidal edges.

2.3.2 Calculating the Image and Discriminant Milnor Numbers

For map-germs $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n \geq p$, there is a striking "Milnor-Tjurina" relation between discriminant Milnor number and $\mathscr{A}_e$-codimension, which we state and prove below (the result first appeared in [10]). For germs $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, a similar relation holds for $n = 1$ and 2, but the general case remains obstinately conjectural, despite many examples. The difference between the cases $n \geq p$ and $n = p - 1$ resides in the fact that when $n \geq p$, the discriminant s of stable germs are *free divisors*.

Definition 2.3.5 (K.Saito,[58]) The hypersurface D is a **free divisor** if the $\mathcal{O}_p$-module $\mathrm{Der}(-\log D)$ is free.

Theorem 2.3.6 ([39, Theorem 6.13]) *If $f : (\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$, $n \geq p$, is stable, then its discriminant is a free divisor.*

We encourage the reader to use Theorem 2.2.18 to deduce the multi-germ version of this (when $|S| > 1$) from the monogerm version proved by Loooijenga.

Theorem 2.3.7 (Damon-Mond,[10]) *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n \geq p$, is $\mathscr{A}$-finite, and (n, p) are in Mather's range of nice dimensions, then*

$$\mu_\Delta(f) \geq \mathscr{A}_e\text{−}codimension(f) \tag{2.16}$$

with equality if f is weighted homogeneous.

We prove this by a new argument below, which has the advantage over the argument in [10] that it gives a means of calculating $\mu_\Delta(f)$ using only the equation of the discriminant of f and the homomorphism $f^* : \mathcal{O}_p \to \mathcal{O}_n$. The calculation of μ_Δ in [10] requires explicitly finding F and i as in the diagram (2.12) and then finding generators for $\mathrm{Der}(-\log D_F)$.

Theorem 2.3.8 *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite, and $n = 1$ or 2, then*

$$\mu_I(f) \geq \mathscr{A}_e\text{−}codimension(f). \tag{2.17}$$

with equality if f is weighted homogeneous.

This was proved by de Jong and van Straten for $n = 2$ in [30], and by the first author for $n = 1$ in [49]. Proofs can also be found in [48, Chapter 11]. All efforts to prove it in greater generality have so far failed.

For the proof of 2.3.7, we introduce a variant of $\mathrm{Der}(-\log D)$, namely its submodule $\mathrm{Der}(-\log G)$ (where G is a reduced equation for D), defined by

$$\mathrm{Der}(-\log G)_y = \{\xi \in \theta_{p,y} : tG(\xi) = 0\}. \tag{2.18}$$

Whereas $\mathrm{Der}(-\log D)_y$ consists of those germs at y of ambient vector fields tangent to D at its smooth points, sections of $\mathrm{Der}(-\log G)$ are tangent to *all* of the level sets of G. As we will see, this difference allows detection of the critical points of the defining equation of the discriminant of a stabilisation of f which move off the 0-level, whose rôle in the calculation of $\mu_\Delta(f)$ is shown in Theorem 2.3.2.

Evidently $\mathrm{Der}(-\log G)$ is a submodule of $\mathrm{Der}(-\log D)$. For the purposes of the proof of Theorem 2.3.7, we need it to be a direct summand (with $D = D_F$), and this can be arranged by a suitable choice of stable unfolding F. We say that G is a *good defining equation* for the discriminant D_F if there exists a vector field χ such that $dG(\chi) = G$. Any weighted homogeneous defining equation is good; when G is not weighted homogeneous, we can replace F by the trivial unfolding $F \times \mathrm{id}_{\mathbb{C}}$ and G by $e^t G$, where t is the extra parameter, and take $\chi = \partial/\partial t$. Once we have a good defining equation, $\mathrm{Der}(-\log D_F)$ splits as a direct sum of $\mathrm{Der}(-\log G)$ and the submodule generated by χ.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be $\mathscr{A}$-finite, with $n \geq p$ and (n, p) nice dimensions, and let g be a reduced defining equation of the discriminant D_f. Let $F : (\mathbb{C}^n \times \mathbb{C}^d, (0, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, (0, 0))$ be a stable unfolding of f, and let G be a good defining equation for D_F, which restricts to g on $\mathbb{C}^p \times \{0\}$. To reduce clutter, we write $(\mathbb{C}^n, 0)$ as X, $(\mathbb{C}^p, 0)$ as Y, and $(\mathbb{C}^d, 0)$ as U. Recall from the proof of Theorem 2.2.23 and Lemma 2.2.25 that we can think of $J_G\mathcal{O}_{\Sigma_F}$ as an ideal of $\mathcal{O}_{D_F}$, since $J_G|_{D_F}$ is contained in the conductor of $\mathcal{O}_{\Sigma_F}$ into $\mathcal{O}_{D_F}$. Let I_G be the preimage in $\mathcal{O}_{Y\times U}$ of $J_G\mathcal{O}_{\Sigma_F}$ under the quotient projection $\mathcal{O}_{Y\times U} \to \mathcal{O}_{D_F}$. We define the ideal $I_g \subset \mathcal{O}_Y$ analogously, with g in place of G, and the relative module I_G^{rel} as the preimage in $\mathcal{O}_{Y\times U}$ of $J_G^{\mathrm{rel}}\mathcal{O}_{\Sigma_F}$, where J_G^{rel} is the relative jacobian ideal of G, $J_G^{\mathrm{rel}} = \left(\dfrac{\partial G}{\partial Y_1}, \ldots, \dfrac{\partial G}{\partial Y_p}\right)$.

Theorem 2.3.9 *In these circumstances, $\mu_\Delta(f) = \dim_{\mathbb{C}} \dfrac{I_g}{J_g}$.*

Theorem 2.3.7 follows directly, because by Theorem 2.2.23, the $\mathscr{A}_e$-codimension of f is the dimension of the restriction $I_g + (g)/(J_g + (g))$. If $g \in J_g$, and in particular if f is weighted homogeneous, then $\dim I_g + (g)/(J_g + (g)) = \dim I_g/J_g$.

We will use "conservation of multiplicity": we show that $I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$ is a Cohen-Macaulay module over $\mathcal{O}_{Y\times U}$, of dimension d, and from this deduce that its push-forward to U, $\pi_*(I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}})$, is free over $\mathcal{O}_U$.

The Cohen-Macaulay property will follow from the following lemma.

Lemma 2.3.10 *There is an $\mathcal{O}_{Y\times U}$–isomorphism*

$$\frac{\theta(\pi)}{t\pi(Der(-\log G))} \simeq \frac{I_G^{rel}}{J_G^{rel}},$$

where π is the projection $Y \times U \to U$.

Proof

Step 1 We claim that

$$J_G \mathcal{O}_{\Sigma_F} = J_G^{\mathrm{rel}} \mathcal{O}_{\Sigma_F}. \tag{2.19}$$

To see this, recall that in the proof of Theorem 2.2.23, we showed that $dG \circ dF = 0$ on Σ_F. Differentiating with respect to u_i, at points of Σ_F we therefore have

$$0 = \frac{\partial(G\circ F)}{\partial u_i} = \sum_{j=1}^{p} \frac{\partial G}{\partial Y_j} \circ F + \frac{\partial G}{\partial u_j} \circ F,$$

showing that $\dfrac{\partial G}{\partial u_i} \circ F \in J_G^{\mathrm{rel}} \mathcal{O}_{\Sigma_F}$ and thus proving (2.19).

Step 2 Since F is stable we have

$$0 = \frac{J_G \mathcal{O}_{\Sigma_F}}{J_G \mathcal{O}_{D_F}}$$

by Theorem 2.2.23, and so $I_G = J_G + (G)$. This implies

$$I_G^{\mathrm{rel}} = (F^*)^{-1}(J_G^{\mathrm{rel}} \mathcal{O}_{X\times U}) = (F^*)^{-1}(J_G \mathcal{O}_{X\times U}) = J_G + (G).$$

And as G is a good defining equation, $I_G^{\mathrm{rel}} = J_G$. Thus,

$$\frac{I_G^{\mathrm{rel}}}{J_G^{rel}} = \frac{J_G}{J_G^{rel}}.$$

By the exactness of

$$0 \longrightarrow \mathrm{Der}(-\log G) \hookrightarrow \theta_{Y\times U} \xrightarrow{dG} J_G \longrightarrow 0$$

it follows that

$$0 \longrightarrow \left(\mathrm{Der}(-\log G) + \mathcal{O}_{Y\times U}\left\{\frac{\partial}{\partial Y_1}, \cdots, \frac{\partial}{\partial Y_p}\right\}\right) \hookrightarrow \theta_{Y\times U} \longrightarrow \frac{J_G}{J_G^{\mathrm{rel}}} \longrightarrow 0$$

is also exact, showing that

$$\frac{I_G^{\mathrm{rel}}}{J_G^{\mathrm{rel}}} = \frac{J_G}{J_G^{rel}} \simeq \frac{\theta_{Y\times U}}{\mathcal{O}_{Y\times U}\{\frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_{n+1}}\} + \mathrm{Der}(-\log G)} \simeq \frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log G))}.$$

□

Lemma 2.3.11 *Provided* (n, p) *are nice dimensions,* $\dfrac{\theta(\pi)}{t\pi(Der(-\log G))}$ *is Cohen-Macaulay of dimension* d.

Proof Since G is a good defining equation for D_F, $\mathrm{Der}(-\log G)$ is free of rank $p+d-1$. The presentation

$$\mathrm{Der}(-\log G) \xrightarrow{t\pi} \theta(\pi) \longrightarrow \frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log G))} \longrightarrow 0$$

shows that $\dfrac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log G))}$ is Cohen Macaulay, by the Buchsbaum-Rim theorem, provided the codimension of its support takes the maximum value possible, namely p. To show that it takes this value, it is enough that the intersection of this support with $Y \times \{0\}$ consists only of $\{(0,0)\}$. For this we invoke the hypothesis that (n, p) are nice dimensions. We know that outside 0, f is stable, by the geometrical criterion for $\mathscr{A}$-finiteness. In the nice dimensions, all stable germs are weighted homogeneous, with respect to suitable coordinates ([48, §7.4]). Now

$$\left(\mathrm{supp}\frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log G))}\right) \cap (Y \times \{0\}) = \mathrm{supp}\frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log G)) + \mathfrak{m}_{U,0}\theta(\pi)}$$

and

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log G)) + \mathfrak{m}_{U,0}\theta(\pi)} \simeq \frac{\theta(i)}{ti(\theta_Y) + i^*\mathrm{Der}(-\log G)}.$$

At a point y where f is weighted homogeneous, this latter module coincides with the stalk at y of the sheaf $\dfrac{\theta(i)}{ti(\theta_Y) + i^*\mathrm{Der}(-\log D_F)}$, and hence, by Damon's theorem, 2.2.32, with $T^1_{\mathscr{A}_e} f_y$. Outside 0, each germ of f is stable, so $T^1_{\mathscr{A}_e} f = 0$. And, as (n, p) are nice dimensions, each stable germ is weighted homogeneous, in suitable coordinates. This proves that the intersection of the support of $\dfrac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log G))}$ with $Y \times \{0\}$ is just the single point $\{(0, 0)\}$, and thus that the codimension of this support is at least p. □

Proof of Theorem 2.3.9

1. Now that we have shown that $I^{\mathrm{rel}}_G/J^{\mathrm{rel}}_G$ is Cohen Macaulay of dimension d, it follows that $\pi_*(I^{\mathrm{rel}}_G/J^{\mathrm{rel}}_G)$ is a free $\mathcal{O}_U$-module, for the restriction of π to the support of $I^{\mathrm{rel}}_G/J^{\mathrm{rel}}_G$ is just $\{(0, 0)\}$. From this it follows that for u in a suitable neighbourhood of 0,

$$\sum_y \dim_{\mathbb{C}} \left(\frac{I_{g_u}}{J_{g_u}}\right)_y = \dim_{\mathbb{C}} \frac{I_g}{J_g}. \tag{2.20}$$

   The sum on the left hand side can be split into two parts: the sum at points y where $g_u = 0$, and the sum at points where $g_u \neq 0$. As (n, p) are nice dimensions, the first sum vanishes, by the argument involving weighted homogeneity used in the proof of Lemma 2.3.11. At a point y where $g_u \neq 0$, we have $I_{g_u,y} = \mathcal{O}_{Y_y}$ and so $\dim_{\mathbb{C}} \left(\frac{I_{g_u}}{J_{g_u}}\right)_y$ is just the Milnor number of the critical point of g_u. By Siersma's theorem,2.3.2, this sum is equal to $\mu_\Delta(f)$.

□

It would be interesting to prove a counterpart of the theorems of K. Saito in [57] and of H.Vosegaard in [64], and show that, in the nice dimensions at least, weighted homogeneity in some system of coordinates is a *necessary* condition for the equality of $\mathscr{A}_e$-codimension of f and $\mu_\Delta(f)$. This would give, as a special case, the curious empirically observed fact that in the nice dimensions every stable germ is weighted homogeneous in suitable coordinates.

Example 2.3.12 It gives the authors some pleasure to observe that one can see the equality of Theorem 2.3.7 in something we all learned about at school, namely the appearance of a local maximum and local minimum when the function $f(x) = x^3$ is perturbed to $f_t(x) = x^3 - tx$. Here the discriminant of f_t consists just of the pair max, min of critical values, and its vanishing homology is generated by the 0-cycle [max] − [min]. The $\mathscr{A}_e$ codimension of f here is equal to 1.

Remark 2.3.13 The argument given here, with the exception of the conservation of multiplicity (2.20), applies equally to the case of germs of maps $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. We do not know that we always have (2.20) in that case, because the discriminant (i.e. the image) of a stable germ in this case is not in general a free divisor—one can check this easily in the case of the Whitney umbrella. Nevertheless, in all cases we know, (2.20) does hold because the relative module

$I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$ turns out, *post hoc*, to be Cohen Macaulay. We do not know why this is. In fact the argument given above was first given in the case of germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, in [13], with conservation of multiplicity (i.e. Cohen-Macaulayness of the quotient $I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$) as an explicit assumption. Our argument here, for the case $n \geq p$, essentially combines the proof in [13] with a proof of conservation of multiplicity which closely follows the argument in [10]. Using the formula in this case, we have been able to compute image Milnor numbers for germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0), n > 1$, using the formula

$$\mu_I = \dim_{\mathbb{C}} \frac{I_g}{J_g} \tag{2.21}$$

where now g is an equation for the image of f and I_g is the ideal $f^*(J_g \mathcal{O}_n)$ of $\mathcal{O}_{n+1}$, after checking that the quotient $I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$ is Cohen Macaulay. A proof that this formula is always valid would immediately prove the conjectured relation $\mu_I \geq \mathscr{A}_e -$ codimension with equality if f is weighted homogeneous.

Example 2.3.14 We consider again the germ $f(x, y) = (x^2, y^2, xy + x^3 + y^3)$ of Example 2.2.29. The *Singular* procedure for computing the $\mathscr{A}_e$ codimension of f needs only to be altered in one particular to calculate $\mu_I(f)$: instead of

```
module M=modulo(JJ,J+I);
```

we set

```
module N=modulo(JJ,J);
```

and then

```
matrix bb=kbase(std(N));
print(bb);
```

then shows that $\mu_I(F) = 7$.

We leave the reader to check that in this case the relative module is Cohen-Macaulay.

2.4 Multiple Points in the Source

For map-germs $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $p > n+1$, the Morse theory used to define and compute image and discriminant Milnor numbers is not available. The image of an $\mathscr{A}$-finite non-immersive germ in these dimensions is never Cohen-Macaulay, and certainly not a complete intersection. A different technique for calculating the homology of the image of a stable perturbation , the *image computing spectral sequence* (ICSS) was introduced in [18], and further developed in [19, 25], and [6]. The difference between the topology of the image and of the domain of a finite surjective map $f : X \to Y$ is accounted for by the identifications which take place—the gluing. The gluing data is encoded in the multiple point spaces $D^k(f)$ for $k \geq 2$, defined (with one slight wrinkle) as the closure, in X^k, of the set of pairwise distinct ordered k-tuples of points sharing the same image. The homological content of this information is decoded by the ICSS, which has as its E^1 page the array $AH_j(D^k(f))$ of *alternating homology groups* of the spaces $D^k(f)$, defined below, with differential induced by the projection $D^k(f) \to D^{k-1}(f)$ which forgets the last copy of X. Even in the case where we already know the homology of the image, or discriminant, D, when $\dim X + 1 \geq \dim Y$, the ICSS gives more detailed information, in the sense of a natural filtration of the homology $H_*(D)$ whose k'th successive quotient is precisely the alternating homology $AH_{*-k+1}(D^k(f))$.

We begin with some definitions. In order that when F is an unfolding of f on parameter space U, the natural projection $D^k(F) \to U$ should be a deformation of $D^k(f)$, our definition of $D^k(f)$ has an extra step, suggested by Terry Gaffney in [16], which can lead to a minor difference from the approximate definition above when f is not stable. Since we are most interested in applying the ICSS to describe the image or discriminant of a stable perturbation , this extra complication can often be ignored.

Definition 2.4.1 If $f : X \to Y$ is a finite map of topological spaces, then

1. $D^k_{\mathrm{cl}}(f)$ is the closure, in X^k, of the set of ordered k-tuples of pairwise distinct points $(x_1, \ldots, x_k)$ such that $f(x_1) = \cdots = f(x_k)$;
2. $\varepsilon^k : D^k_{\mathrm{cl}}(f) \to D^{k-1}_{\mathrm{cl}}(f)$ is the projection forgetting the last component
3. For a $\mathscr{K}$-finite germ f, following Gaffney in [16], we define $D^k(f)$ (without the suffix) by the fibre diagram

$$\begin{array}{ccc} D^k(f) & \longrightarrow & D^k_{\mathrm{cl}}(F) \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & U \end{array}$$

   where F is a stable unfolding of f and $D^k_{\mathrm{cl}}(F)$ is given its reduced structure.
4. $D^k_\ell(f)$ is the image of $D^k(f)$ in $D^\ell(f)$ under the composite $\varepsilon^{\ell+1} \circ \cdots \circ \varepsilon^k$.

The spaces whose closure we take in this definition are constructible, so the analytic (Zariski) closure is the same as the closure in the Euclidean topology. The definition is independent of the choice of stable unfolding, and the two spaces $D^k_{\mathrm{cl}}(f)$ and $D^k(f)$ differ only in the case where D^k consists of an isolated point ([48, Prop. 9.4]). The extra step in the definition ensures, for example, that the point $(0, 0)$ lies in $D^2(f)$ when $f : \mathbb{F} \to \mathbb{F}^2$ is the cusp map $f(x) = (x^2, x^3)$.

For an $\mathscr{A}$-finite germ, the dimension of $D^k(f)$ is always $p - k(p - n)$, or 0 when this number is negative provided it is not empty, since most k-tuple points in the target will be normal crossings of k immersed sheets. We say that this is the *expected dimension* of $D^k(f)$ in general.

The paper [40] (see also [48, Section 9.5]) provides explicit generators for the ideal defining $D^k(f)$, suggested originally by Mark Roberts, when f has corank 1 and is in the linearly adapted form $f(x) = (x_1, \ldots, x_{n-1}, f_n(x), \ldots, f_p(x))$. They make use of the evident fact that in this case $D^k(f)$ embeds in $\mathbb{C}^{n-1} \times \mathbb{C}^k$, and, with respect to coordinates $u_1, \ldots, u_{n-1}, x_1, \ldots, x_k$, are given by

$$\frac{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{i-1} & f_j(u, x_1) & x_1^{i+1} & \cdots & x_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_1 & \cdots & x_k^{i-1} & f_j(u, x_k) & x_k^{i+1} & \cdots & x_k^{k-1} \end{vmatrix}}{VDM}, \quad j = n, \ldots, p, i = 1, \ldots, k-1 \tag{2.22}$$

where VDM is the Vandermonde determinant of $x_1, \ldots, x_k$.

For map-germs of corank 1, the $D^k(f)$ provide a simple criterion for stability and for $\mathscr{A}$-finiteness.

Theorem 2.4.2 ([40]) *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ $(n \leq p)$ be a map-germ of corank 1.*

1. *f is stable if and only if for each k, $D^k(f)$ is smooth of the expected dimension provided this number is positive, and empty otherwise.*
2. *f is $\mathscr{A}$-finite if and only if for each k $D^k(f)$ is an ICIS (i.e. isolated complete intersection singularity) of the expected dimension provided this number is positive, and* 0*-dimensional otherwise.*

The reader will note that without the extra (Gaffney) step in the definition of $D^k(f)$, statement 1 here would be false, as is shown by the example of the cusp $x \mapsto (x^2, x^3)$.

For germs of higher corank, much less is known. We only have a recipe for defining equations for $D^k(f)$ in terms of f alone in the case $k = 2$. Here the recipe is as follows: in the ring of the product space $\mathbb{C}^n \times \mathbb{C}^n$ with coordinates $x' = (x'_1, \ldots, x'_n)$ and $x'' = (x''_1, \ldots, x''_n)$, we factorise the p-tuple $(f_1(x') - f_1(x''), \ldots, f_p(x') - f_p(x''))$ as

$$\begin{pmatrix} x'_1 - x''_1 & \cdots & x'_n - x''_n \end{pmatrix} \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1p} \\ \vdots & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{np} \end{pmatrix} \tag{2.23}$$

where $\alpha_{ij} = \alpha_{ij}(x', x'')$, and take $I_2(f)$ to be the ideal in $\mathcal{O}_{2n}$ generated by $f_1(x') - f_1(x''), \ldots, f_p(x') - f_p(x'')$ and by the $n \times n$ minors of the $n \times p$ matrix (α_{ij}) in (2.23).

Theorem 2.4.3 *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ $(n \leq p)$ be any germ. If $D^2(f)$ has the expected dimension, $2n - p$, then it is Cohen-Macaulay.*

The proof uses a theorem on the *variety of complexes* proved by G. Kempf—see [48, §9.4] or [54].

2.4.1 Alternating Homology

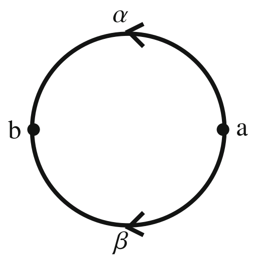
The symmetric group S_k acts on $D^k(f)$, permuting the copies of X, and this action is crucial in what follows. It will be useful to regard X as $D^1(f)$.

Definition 2.4.4 Let D^k be any space on which S_k acts. Then

1. $C_j^{\text{Alt}}(D^k)$ is the subgroup of the usual singular chain group (with integer coefficients) $C_j(D^k)$ consisting of chains c such that $\sigma_\#(c) = \text{sign}(\sigma)c$ for all $\sigma \in S_k$.
2. $AH_j(D^k)$ is the j'th homology of the resulting chain complex $C_\bullet^{\text{Alt}}(D^k)$

We refer to the chains in $C_j^{\text{Alt}}(D^k)$ as *alternating chains* and to the homology of the complex $C_\bullet^{\text{Alt}}(D^k)$ as the *alternating homology* of D^k. For CW complexes, alternating homology can also be calculated using the alternating subcomplex of the cellular chain complex, provided the cellular structure is respected by the group action.

Exercise 2.4.5 Let the symmetric group S_2 act on the circle S^1 by reflection in a diameter. Compute the alternating homology $AH_*(S^1)$ using the CW structure with 0-cells a and b and 1-cells α and β.



2.4.2 The Image Computing Spectral Sequence

The following lemma is proved by an elementary argument—see e.g. [48, Section 10.1].

Lemma 2.4.6 *Let $f : X \to Y$ be any continuous map of topological spaces. Then*

1. $\varepsilon_\#^k(C_j^{Alt}(D^k(f))) \subseteq C_j^{Alt}(D^{k-1}(f))$
2. $\varepsilon_\#^{k-1} \circ \varepsilon_\#^k : C_j^{Alt}(D^k(f)) \to C_j^{Alt}(D^{k-2}(f))$ *and* $f_\# \circ \varepsilon_\#^2 : C_j^{Alt}(D^2(f)) \to C_j(Y)$ *are both equal to 0. Hence,*
3. *the double array* $\{C_j^{Alt}(D^k(f))\}_{j,k}$*, with horizontal differential*

$$\partial : C_j^{Alt}(D^k(f)) \to C_{j-1}^{Alt}(D^k(f))$$

and vertical differential

$$(-1)^j \varepsilon_{\#}^k : C_j^{Alt}(D^k(f)) \to C_j^{Alt}(D^{k-1}(f))$$

is a double complex.

Comparison of the spectral sequences arising from the two standard filtrations on this double complex leads to the following theorem.

Theorem 2.4.7 *If $f : X \to Y$ is a surjective finite triangulable map then there is a spectral sequence with $E^1_{p,q} = AH_p(D^{q-1}(f))$ and $d_1 : E^1_{p,q} \to E^1_{p,q-1}$ equal to $(-1)^p \varepsilon_*^q$, converging to $H_{p+q}(Y)$.*

Finite subanalytic maps are triangulable (by a theorem of Hardt in [23]) so the theorem applies, for example, to stable perturbation s of finite real or complex analytic map-germs.

The ICSS was first constructed in [18] for homology with rational coefficients, when $AH_*(D^k)$ coincides with the part of the ordinary homology $H_*(D^k)$ on which S_k acts by its sign representation. The alternating chain complex was introduced by Goryunov in [19]; he showed that the ICSS converges to $H_*(Y)$ by relating it to the chain complex of a geometric realisation of the semi-simplicial object $D^\bullet(f)$. The statement given here is from [6]. Triangulability allows us to replace singular homology with simplicial homology, which makes possible a "classical" proof of Theorem 2.4.7. For replacing f with a simplicial map, one obtains easily a triangulation of each space $D^k(f)$, from which it follows, in a straightforward way, that for each fixed n, the sequence of alternating simplicial chain groups

$$\cdots \to C_n^{\mathrm{Alt}}(D^k(f)) \xrightarrow{\varepsilon^k_{\#,n}} \cdots \xrightarrow{\varepsilon^3_{\#}} C_n^{\mathrm{Alt}}(D^2(f)) \xrightarrow{\varepsilon^2_{\#}} C_n(X) \xrightarrow{f_\#} C_n(Y) \to 0 \tag{2.24}$$

is exact. This means that the spectral sequence of the first filtration of the double complex C of Lemma 2.4.6 collapses onto the p-axis at E^1, so that

$$E^2_{n,0} \simeq H_n(\mathrm{Tot}(C)) \simeq H_n(Y).$$

([6, Proposition 2.1]). The ICSS is the spectral sequence associated to the second filtration of the double complex C, and since both spectral sequences converge to the homology of the total complex $\mathrm{Tot}(C)$, and we have seen that $H_*(\mathrm{Tot}(C)) = H_*(Y)$, the theorem is proved.

In the context of the initial construction of the ICSS, for rational homology, $AH_*(D^k(f); \mathbb{Q})$ is isomorphic to a subspace of $H_*(D^k(f); \mathbb{Q})$. Thus, vanishing theorems on $H_j(D^k(f); \mathbb{Q})$ imply vanishing of $AH_j(D^k(f); \mathbb{Q})$. When f_t is a stable perturbation of a germ $f_0 : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ $(n \leq p)$ of corank 1, then this is exactly what happens, because $D^k(f_0)$ is an ICIS, of dimension $p - k(p - n)$, and $D^k(f_t)$ is a Milnor fibre , and thus is homotopy-equivalent to a wedge of

$p-k(p-n)$-dimensional spheres. The fact that $AH_*(D^k(f_t))$ is concentrated in a single dimension then means that the ICSS collapses at E^1, giving rise to some rather striking formulas for the filtration on the homology of Y, which we will come to shortly. This conclusion, however, has been significantly strengthened by subsequent work. In [19], Goryunov showed that if D^k is an S_k-invariant ICIS then $AH_*(D^k)$ is isomorphic to a subgroup of $H_*(D^k)$ even with integer coefficients, so that the formulas just alluded to hold also over $\mathbb{Z}$, but still only for stable perturbation s of germs of corank 1. Even more strikingly, in [25], Kevin Houston showed that this last condition can be dispensed with.

Theorem 2.4.8 (K.Houston, [25]) *If f_t is a stable perturbation of an $\mathscr{A}$-finite germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$, then $AH_j(D^k(f_t)) = 0$ for $j \neq \dim D^k(f_t)$.*

The $D^k(f_t)$ may have homology in other dimensions ([51]). It is only the alternating homology that is concentrated in middle dimension. Houston's underlying heuristic here is that away from the diagonals, where two or more of the x_j making up a point $(x_1, \ldots, x_k)$ of $D^k(f)$ are equal, $D^k(f)$ coincides with the k-fold fibre product of X over Y, which is defined by the naive equations $f_\alpha(x_i) = f_\alpha(x_j)$ for $1 \leq i < j \leq k$ and $1 \leq \alpha \leq p$. Thus, at such points $D^k(f)$ is a complete intersection. Alternating homology is the homology of a sheaf supported away from the diagonals—one can show easily that an alternating j-chain can contain no simplex contained in any diagonal $\{x_i = x_j\}$ ([48, Lemma 10.8]). So in some way, the fact that $D^k(f)$ is a complete intersection at the points of the support of the sheaf of alternating chains leads to the familiar conclusion that the alternating homology of $D^k(f_t)$ is concentrated in middle dimension.

From the ICSS, together with Houston's theorem, we derive

Corollary 2.4.9 *Suppose that $f_t : X_t \twoheadrightarrow Y_t$ is a stable perturbation of an $\mathscr{A}$-finite map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+\ell}, 0)$.*

1. If $\ell \geq 2$, then

$$H_q(Y_t) = \begin{cases} AH_{n-(k-1)\ell}(D^k(f_t)) & \text{if } q = n-(k-1)(\ell-1) \text{ for some } k \\ \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. If $\ell = 1$, then $H_q(Y_t) = 0$ if $q \neq 0, n$, and there is an increasing filtration F_k, $0 \leq k \leq n+1$ on $H_n(Y_t)$ with $F_0 = F_1 = 0$ (see below) and

$$F_k/F_{k-1} \simeq AH_{n-k+1}(D^k(f_t)).$$

The term F_1 in the filtration here corresponds to the image of $H_n(X_t)$ in $H_n(Y_t)$, which is 0 because X_t is the contractible domain of a stable perturbation of f.

A dual statement holds for cohomology.

The corollary can be appreciated with the help of a diagram chase. We place ourselves in the situation of Houston's theorem, so that $AH_*(D^k(f_t))$ is concentrated in the (complex) dimension of $D^k(f_t)$.

1. Suppose that $z_{3,\ell} \in C_\ell^{\mathrm{Alt}}(D^3(f))$ represents a homology class in $AH_\ell(D^3(f_t))$, where $\ell = \dim D^3(f_t)$. Since $\dim D^3(f_t) < \dim D^2(f_t)$, $AH_\ell(D^2(f_t)) = 0$, so $\varepsilon^3_\#(z_{3,\ell})$ is an alternating boundary in $D^2(f_t)$—there exists $c_{2,\ell+1} \in C^{\mathrm{Alt}}_{\ell+1}(D^2(f_t)))$ such that $\partial c_{2,\ell+1} = \varepsilon^3_\#(z_{3,\ell})$.
2. Now
$$\partial \varepsilon^2_\#(c_{2,\ell+1}) = \varepsilon^2_\#(\partial c_{2,\ell+1}) = \varepsilon^2_\#(\varepsilon^3_\#(z_{3,\ell}) = 0,$$
because $\varepsilon^2_\# \circ \varepsilon^3_\# = 0$ on alternating chains. As $H_{\ell+1}(X_t) = 0$, $\varepsilon^2_\#(c_{2,\ell+1})$ is a boundary in X_t, and there exists $c_{1,\ell+2} \in C_{\ell+2}(X)$ such that $\partial c_{1,\ell+2} = \varepsilon^2_\#(c_{2,\ell+1})$.
3. Similarly, because $f_\# \circ \varepsilon^2_\# = 0$ on alternating chains, $f_\#(c_{1,\ell+2})$ is a cycle in Y_t.

The argument is summarised by the following diagram, which should be read from top to bottom.

$$
\begin{array}{ccccccccc}
D^3(f) & & 0 & \xleftarrow{\partial} & z_{3,\ell} & & & & \\
\downarrow{\scriptstyle \varepsilon^3} & & & & \downarrow & & & & \\
D^2(f) & & 0 & \xleftarrow{\partial} & \varepsilon^3_\#(z_{3,\ell}) & \xleftarrow{\partial} & \exists\, c_{2,\ell+1} & & \\
\downarrow{\scriptstyle \varepsilon^2} & & & & \downarrow & & \downarrow & & \\
X & & & & 0 & \xleftarrow{\partial} & \varepsilon^2_\#(c_{2,\ell+1}) & \xleftarrow{\partial} & \exists\, c_{1,\ell+2} \\
\downarrow{\scriptstyle f} & & & & & & \downarrow & & \downarrow \\
Y_y & & & & & & 0 & \xleftarrow{\partial} & f_\#(c_{1,\ell+2})
\end{array}
$$

Here, our construction has led from an ℓ-dimensional alternating homology class on $D^3(f_t)$ to a $\ell + 2$-dimensional class in Y_t. There are choices on the way, of course: different choices of $c_{2,\ell+1}$ differ by an alternating cycle on $D^2(f_t)$, so the homology class we end up with in Y_t is only well defined modulo the subgroup of $H_n(Y_t)$ consisting of classes originating in $AH_{\ell+1}(D^2(f_t))$. Thus, the apparent homomorphism $AH_\ell(D^3(f_t)) \to H_{\ell+2}(Y_t)$ is really a homomorphism into the quotient F_3/F_2. Note that because X_t is contractible, the choice of $c_{1,\ell+2}$ has not created additional imprecision. There really is a homomorphism $AH_{\ell+1}(D^2(f_t)) \to H_{\ell+2}(Y_t)$, defined by the same diagram chasing construction given here.

Example 2.4.10 We continue with the germ of $f(x, y) = (x^2, y^2, xy + x^3 + y^3)$ discussed in Examples 2.2.29 and 2.3.14.

- We use the recipe in Sect. 2.4 to find $I_2(f)$. Computing its primary decomposition, we find that $D^2(f)$ has five irreducible components, each of which is a nonsingular curve. It is clear from the calculation (which we urge the reader to undertake) that under the involution $\sigma : D^2(f) \to D^2(f)$ interchanging x' and x'', each branch is mapped to itself. This can also be seen by calculating the primary decomposition of the first Fitting ideal $\mathrm{Fitt}_1(f_*\mathcal{O}_2)$ from the matrix of a presentation—again there are five components (see Examples 2.5.10 and 2.5.15 below).
- An $\mathcal{O}_2$-generator for the conductor ideal, which defines the image $D^2_1(f)$ of $D^2(f)$ under the projection ε^2, may be found by the procedure of Proposition 2.2.28, or by pulling back the principal minor of the presentation—see Corollary 2.5.14 below. It has Milnor number 16.
- If f_t is a stable perturbation of f, then $D^2_1(f_t)$ is a smoothing of $D^2_1(f)$, except for the presence of $3T$ nodes, preimages of the T triple points in the image. In this case (see the calculation in Example 2.5.19 below) $T = 1$. Hence $h_1(D^2_1(f_t)) = \mu(D^2_1(f) - 3) = 13$.
- In $D^2(f_t)$ the three nodes are separated—$D^2(f_t)$ is the normalisation of $D^2_1(f_t)$. Thus $h_1(D^2(f_t)) = 10$. Since $D^2(f_t)$ has five ends, corresponding to the five branches of $D^2_1(f)$, and since for any smooth complex curve C (with at least one end)

$$h_1(C) = 2\,\mathrm{genus}(C) + \#\mathrm{ends} - 1,$$

  it follows that the genus of $D^2(f_t)$ is 3.
- f_t has three non-immersive points. For $\dim_{\mathbb{C}} \mathcal{O}_2/\mathrm{minors}(2, df) = 3$, and this multiplicity is conserved in a perturbation, by a standard Cohen-Macaulay argument. Thus the involution σ has three non-degenerate fixed points. By the Lefschetz fixed point theorem,

$$3 = \text{Lefschetz number}(\sigma) = \mathrm{Trace}\left((\sigma_*|_{H_0(D^2(f_t))}\right) - \mathrm{Trace}\left(\sigma_*|_{H_1(D^2(f_t))}\right)$$

  from which it follows that

$$\mathrm{Trace}\left(\sigma_*|_{H_1(D^2(f_t))}\right) = -2. \tag{2.25}$$

- Let $\bar{\sigma}$ be the extension of σ to the compact genus 3 surface Σ obtained by capping the ends of $D^2(f_t)$ with discs. The homology classes of the cycles around the ends of $D^2(f_t)$ are mapped to themselves by σ_*, since σ maps each end to itself, and σ preserves orientation on $D^2(f_t)$, and hence boundary orientation on $\partial D^2(f_t)$. A calculation with Mayer Vietoris, using (2.25), then shows that on $H_1(\Sigma)$, $\bar{\sigma}_*$ is multiplication by -1. It follows that $\bar{\sigma}$ is conjugate to the rotation through π of the standard embedded genus 3 surface about an axis passing as a diameter through each of the generating 1-cycles. The diagram below illustrates the corresponding involution on $D^2(f_t)$.

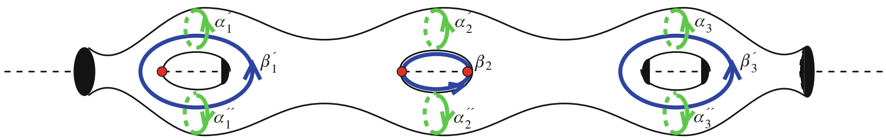


Fig. 2.6 Alternating cycles generating $AH_1(D^2(f_t))$; β_1'' and β_2'' are hidden by the surface and not shown

The three fixed points lie on the axis of rotation; the five ends circle it. We surmise that the picture is as shown in Fig. 2.6, though other configurations of the ends and fixed points are possible.

Here the five ends are represented by black holes in the surface, and the three fixed points by red discs. The involution is rotation through π about the dashed axis.

In this picture, $AH_1(D^2(f_t))$ has rank 6, and is freely generated by $\alpha_i := \alpha_1' + \alpha_i'', i = 1, 2, 3, \beta_i = \beta_i' + \beta_i'', i = 1, 3$ and β_2 (the cycles β_1'' and β_3'' are on the back of the surface, directly behind β_1' and β_3'). With the arrangement of the ends shown here, some of the generators of $H_1(\Sigma)$ have to be doubled to give alternating cycles on $D^2(f_t)$. Only β_1 is a member of a basis for $H_1(D^2(f_t))$. It seems that no matter which arrangement of ends is postulated, the cokernel of $AH_1(D^2(f_t)) \to AH_1(\Sigma)$ is a non-trivial torsion group. Nevertheless, the natural homomorphism from $AH_1(D^2(f_t))$ to the alternating part of $H_1(D^2(f_t))$ is an isomorphism. In [19, Theorem 2.1.2], Goryunov showed that the corresponding homomorphism is always an isomorphism when $D^2(f)$ is an ICIS (which it is not in this case).

Exercise 2.4.11 The diagram in Fig. 2.7 shows the tower of multiple point spaces over the stable perturbation of the germ $f_0(x, y) = (x, y^3, xy + y^5)$ of type H_2. This is one of rather few germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ whose disentanglement can be faithfully drawn over $\mathbb{R}$. A picture of the image, due to Victor Goryunov, is shown on [48, page 254].

1. Find equations for $D^2(f)$ and $D^3(f)$ using the schema shown just before Theorem 2.4.2, and determine the rank of $AH_1(D^2(f_t)$ and $AH_0(D^3(f_t))$ for a stable perturbation f_t of f.
2. The alternating 0-cycle $z_{3,0} = (P, Q, R) - (P, R, Q) + (Q, R, P) - (Q, P, R) + (R, P, Q) - (R, Q, P)$ generates $AH_0(D^3(f_t))$.
   (i) Find an alternating 1-chain in $c_{2,1} \in C_1(D^2(f_t)$ such that $\partial c_{2,1} = \varepsilon^3_\#(z_{3,0})$, and
   (ii) a 2-chain $c_{1,2}$ in U_t such that $\partial c_{1,2} = \varepsilon^2_\#(c_{2,1})$.
   (iii) Find a non-trivial alternating 1-cycle $z_{2,1} \in D^2(f_t)$, and a 2-chain $c'_{1,2}$ in U_t such that $\partial c'_{1,2} = \varepsilon^2_\#(z_{2,1})$.

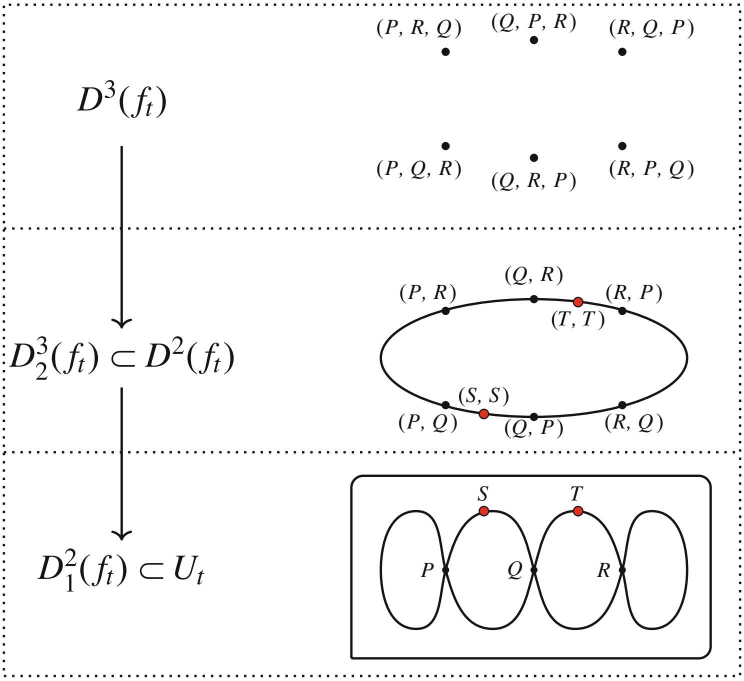


Fig. 2.7 Tower of multiple point spaces of the stable perturbation of H_2

2.4.3 Further Developments

The ICSS was the basis for a plethora of results on the local topology of germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$ for $k > 1$ and the homotopy type of the image of a stable perturbation , proved by Kevin Houston in [25] and [26].

2.5 Multiple Points in the Target: Fitting Ideals

The kth multiple point in the target $M_k(f)$ is the set of points $y \in Y$ such that $f^{-1}(y)$ has at least k preimages, counting multiplicity. It is important to give these spaces an analytic structure which behaves well under deformation. We do this using Fitting ideals. We will see that for $k \leq 3$, $M_k(f)$ is Cohen-Macaulay when it has the expected dimension. If $M_k(f)$ is zero dimensional, then the non reduced structure will predict the number of "genuine" k-tuple points that appear in a stabilisation of f. The construction of $M_k(f)$ based on Fitting ideals appeared in [62] and [52]. Here we give brief account of the main results.

We recall that if R is a ring and M is an R-module with a presentation

$$R^p \xrightarrow{\lambda} R^q \longrightarrow M \longrightarrow 0$$

then the k'th Fitting ideal $\mathrm{Fitt}_k(M)$ is $\min_{q-k}\lambda$, the ideal in R generated by the $q-k$-minors of the matrix λ (here we take the convention that $\min_\ell \lambda = 0$ when $\ell \le 0$ or $\min_\ell \lambda = R$ when $\ell > \min\{p, q\}$). This ideal is independent of the choice of presentation. A presentation always exists when R is Noetherian and M is finitely generated over R. We refer to [21] for basic properties of Fitting ideals.

Definition 2.5.1 Let $f\colon (X, S) \to (Y, y_0)$ be a finite morphism of complex space-germs. By the preparation theorem, $\mathcal{O}_{X,S}$ is finitely generated over $\mathcal{O}_{Y,y_0}$ via $f^*\colon \mathcal{O}_{Y,y_0} \to \mathcal{O}_{X,S}$. The k'**th Fitting ideal** is the ideal in $\mathcal{O}_{Y,y_0}$ defined as

$$\mathscr{F}_k(f) := \mathrm{Fitt}_k(\mathcal{O}_{X,S}).$$

The *k'th target multiple point space* $M_k(f)$ is defined by

$$M_k(f) := V(\mathscr{F}_{k-1}(f)), \quad \mathcal{O}_{M_k(f)} := \frac{\mathcal{O}_{Y,y_0}}{\mathscr{F}_{k-1}(f)}.$$

The following proposition gives the precise statement that, set-theoretically, $M_k(f)$ is the set of points $y \in Y$ such that $f^{-1}(y)$ has at least k preimages, counting multiplicity.

Proposition 2.5.2 *Let $f\colon (X, S) \to (Y, y_0)$ be a finite morphism of complex space-germs. Take a representative $f\colon X \to Y$ which is a finite mapping. As a set-germ, $M_k(f)$ is the germ at y_0 of the set of points $y \in Y$ such that*

$$\sum_{f(x)=y} \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,x}}{f^*\mathfrak{m}_{Y,y}} \ge k.$$

In particular, $M_1(f)$ is the image of f.

Proof See [48, Corollary 11.2]. □

Also the precise statement that target multiple point spaces behave well under deformations is as follows:

Proposition 2.5.3 *Suppose we have a fibre-product diagram of complex space-germs*

$$\begin{array}{ccc} (X, S) & \xrightarrow{F} & (Y, y_0) \\ \uparrow j & & \uparrow i \\ (X \times_Y Z, S_0) & \xrightarrow{f} & (Z, z_0) \end{array} \tag{2.26}$$

where F is finite. Then:

1. *f is also finite.*
2. *For each matrix presentation of $\mathcal{O}_{X,S}$ over $\mathcal{O}_{Y,y_0}$*

$$\mathcal{O}^r_{Y,y_0} \xrightarrow{\lambda} \mathcal{O}^s_{Y,y_0} \xrightarrow{\mu} \mathcal{O}_{X,S} \longrightarrow 0\,,$$

there is an induced matrix presentation of $\mathcal{O}_{X\times_Y Z,S_0}$ over $\mathcal{O}_{Z,z_0}$

$$\mathcal{O}^r_{Z,z_0} \xrightarrow{i^*\lambda} \mathcal{O}^s_{Z,z_0} \xrightarrow{i^*\mu} \mathcal{O}_{X\times_Y Z,S_0} \longrightarrow 0\,.$$

3. *$M_k(f) = i^{-1}(M_k(F))$ as complex space-germs.*

Proof See [48, Proposition 11.6]. □

Example 2.5.4 Let $f\colon (\mathbb{C},0) \to (\mathbb{C}^2,0)$ be the plane curve $f(x) = (x^3, x^4)$. The classes of $1, x, x^2$ give a basis over $\mathbb{C}$ of $\mathcal{O}_1/(x^3, x^4)$. By the preparation theorem, $1, x, x^2$ is a minimal set of generators of $\mathcal{O}_1$ over $\mathcal{O}_2$. For a presentation, we need generators for the relations of $1, x, x^2$ over $\mathcal{O}_2$. If X, Y are coordinates in $\mathbb{C}^2$, the following relations are evident:

$$Y\cdot 1 = y^4 = X\cdot y, \quad Y\cdot y = y^5 = X\cdot y^2, \quad Y\cdot y^2 = y^6 = X^2\cdot 1.$$

It will become clear that these three generate all the relations of $1, x, x^2$ over $\mathcal{O}_2$. This implies that a matrix presentation of $\mathcal{O}_1$ over $\mathcal{O}_2$ is

$$\begin{pmatrix} -Y & 0 & X^2 \\ X & -Y & 0 \\ 0 & X & Y \end{pmatrix}.$$

The non trivial Fitting ideals are $\mathscr{F}_0(f) = (X^3 - Y^4)$, $\mathscr{F}_1(f) = (X^2, XY, Y^2)$ and $\mathscr{F}_2(f) = (X, Y)$. We see that $M_1(f)$ is nothing but the image of f with reduced structure. But $M_2(f)$ is the origin with non reduced structure. We have $\dim_{\mathbb{C}} \mathcal{O}_2/\mathscr{F}_1(f) = 3$, which coincides with δ, the number of double points that appear in a stabilisation of f.

Example 2.5.5 Let $f\colon (\mathbb{C},0) \to (\mathbb{C}^3,0)$ be the space curve $f(x) = (x^3, x^4, x^5)$. Using SINGULAR , it is not difficult to see that $\mathscr{F}_0(f)$ has a primary component of dimension 0, so $M_1(f)$ is neither reduced nor Cohen-Macaulay.

2.5.1 Finding a Presentation

From now on we restrict ourselves to finite map-germs $f\colon (X,S)\to(\mathbb{C}^{n+1},0)$, where (X,S) is a Cohen-Macaulay complex space-germ of dimension n. In practice, we want to use our results in the case that either $X=\mathbb{C}^n$ or $X=\Sigma$, the critical locus of an $\mathscr{A}$-finite map-germ $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, with $n\geq p$ and in both cases we have Cohen-Maculayness. Moreover, if $S=\{x_1,\dots,x_r\}$ and λ_i is a matrix presentation of each $\mathcal{O}_{X,x_i}$ over $\mathcal{O}_{n+1}$, then a matrix presentation of $\mathcal{O}_{X,S}$ over $\mathcal{O}_{n+1}$ is obtained as the block diagonal matrix $\oplus_{i=1}^r\lambda_i$. So, it suffices to consider the mono-germ case $f\colon(X,x_0)\to(\mathbb{C}^{n+1},0)$. Our purpose now is to describe an algorithm to compute a presentation:

$$\mathcal{O}_{n+1}^p \xrightarrow{\lambda} \mathcal{O}_{n+1}^q \longrightarrow \mathcal{O}_{X,x_0}\,. \tag{2.27}$$

In Lemma 2.2.25, we showed that in a minimal presentation, $p=q$.

To find a matrix λ, one can use the following procedure:

1. Choose a projection $\pi:\mathbb{C}^{n+1}\to\mathbb{C}^n$ such that $\pi\circ f$ is finite. Such a projection always exists by the Noether normalization theorem (see [28]). After a linear coordinate change in $\mathbb{C}^{n+1}$ we can assume without loss of generality that $\pi(y_1,\dots,y_{n+1})=(y_1,\dots,y_n)$, so $\pi\circ f$ is given by the first n components of f.
2. By the Auslander-Buchsbaum formula (see e.g. [21]), $\mathcal{O}_{X,x_0}$ is free as an $\mathcal{O}_n$-module via $(\pi\circ f)^*$. Let $g_0,\dots,g_m$ be a basis. By Nakayama's Lemma it is sufficient that the g_i form a $\mathbb{C}$-vector-space basis for $\mathcal{O}_{X,x_0}/(\pi\circ f)^*\mathfrak{m}_n$. One of the g_i at least must be a unit in $\mathcal{O}_{X,x_0}$; we take $g_0=1$.
3. Find the unique $\lambda_j^i\in\mathcal{O}_n$ such that

$$\begin{aligned} f_{n+1} &= \lambda_0^0 g_0 + \cdots + \lambda_0^m g_m\\ g_1 f_{n+1} &= \lambda_1^0 g_0 + \cdots + \lambda_1^m g_m\\ \cdots &= \cdots \;\cdots\cdots\;\cdots\\ g_m f_{n+1} &= \lambda_m^0 g_0 + \cdots + \lambda_m^m g_m \end{aligned} \tag{2.28}$$

Since $f_{n+1}=y_{n+1}\circ f$, (2.28) can be rewritten as

$$\begin{aligned} 0 &= (\lambda_0^0-y_{n+1})g_0 + \cdots + \cdots + \lambda_0^m g_m\\ 0 &= \lambda_1^0 g_0 + (\lambda_1^1-y_{n+1})g_1 + \cdots + \lambda_1^m g_m\\ \cdots &= \cdots \;\cdots\;\cdots\;\cdots\cdots\;\cdots\\ 0 &= \lambda_m^0 g_0 + \cdots + \cdots + (\lambda_m^m-y_{n+1})g_m \end{aligned} \tag{2.29}$$

Thus the columns of the matrix

$$\begin{pmatrix} \lambda_0^0 - y_{n+1} & \lambda_1^0 & \cdots & \lambda_m^0 \\ \lambda_0^1 & \lambda_1^1 - y_{n+1} & \cdots & \lambda_m^1 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_0^m & \lambda_1^m & \cdots & \lambda_m^m - y_{n+1} \end{pmatrix} \tag{2.30}$$

are relations between the g_i.

Proposition 2.5.6 (2.30) *is the matrix of a presentation of* $\mathcal{O}_{X,x_0}$ *over* $\mathcal{O}_{n+1}$*. In other words, the columns of* (2.30) *generate all the relations among the* g_i *over* $\mathcal{O}_{n+1}$.

Proof See [48, Proposition 11.1]. □

The matrix presentation in Example 2.5.4 is the one obtained with this procedure. Here we present one more example:

Example 2.5.7 Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be the germ of Example 2.2.29, $f(x, y) = (x^2, y^2, xy + x^3 + y^3)$. Take $\pi(X, Y, Z) = (X, Y)$; then $\mathcal{O}_2$ (source) is generated over $\mathcal{O}_2$ (target) by $1, x, y, xy$. We have

$$\begin{aligned} f_3 &= xy + x^3 + y^3 &&= 01 + Xx + Yy + 1xy \\ xf_3 &= x^2y + x^4 + xy^3 &&= X^21 + 0x + Xy + Yxy \\ yf_3 &= xy^2 + x^3y + y^4 &&= Y^21 + Yx + 0y + Xxy \\ xyf_3 &= x^2y^2 + x^4y + xy^4 &&= XY1 + Y^2x + X^2y + 0xy \end{aligned}$$

so as matrix of the presentation we obtain

$$\lambda = \begin{pmatrix} -Z & X^2 & Y^2 & XY \\ X & -Z & Y & Y^2 \\ Y & X & -Z & X^2 \\ 1 & Y & X & -Z \end{pmatrix}.$$

The first column shows that the generator xy is a linear combination of the others. Clearing entries 2,3 and 4 from the last row by column operations and deleting column 1, we obtain the minimal presentation

$$\lambda_0 = \begin{pmatrix} X^2 + YZ & Y^2 + XZ & XY - Z^2 \\ -(XY + Z) & Y - X^2 & Y^2 + XZ \\ X - Y^2 & -(XY + Z) & X^2 + YZ \end{pmatrix}$$

2.5.2 Double and Triple Points in the Target

We give here a more detailed description of how to compute double and triple points in the target. In particular, we will show that $M_2(f)$ and $M_3(f)$ are Cohen-Macaulay, provided that they have the expected dimension. Moreover, we will prove some of the statements of Lemma 2.2.25 in a more general setting. Unless otherwise specified, we will assume that $f\colon (X, x_0) \to (\mathbb{C}^{n+1}, 0)$ is a finite morphism, where (X, x_0) is Cohen-Macaulay of dimension n and reduced. We start with a result with describes the space $M_1(f)$.

Proposition 2.5.8 *Assume (X, x_0) is irreducible and $f\colon (X, x_0) \to (\mathbb{C}^{n+1}, 0)$ has degree k onto its image. Then $\mathcal{F}_0(f)$ is generated by h^k, where $h \in \mathcal{O}_{n+1}$ is a reduced equation for the image of f.*

Proof See [48, Proposition 11.7]. □

The case where (X, x_0) is not irreducible can be adapted easily: denote by $(X_1, x_0), \dots, (X_r, x_0)$ the irreducible components of (X, x_0) and assume $f|_{(X_i, x_0)}$ has degree k_i onto its image. Then $\mathcal{F}_0(f)$ is generated by $h_1^{k_1} \dots h_r^{k_r}$, where $h_i \in \mathcal{O}_{n+1}$ is a reduced equation for $(f(X_i), 0)$. As a consequence, $M_1(f)$ is always a hypersurface, so it is Cohen-Macaulay. Moreover, $M_1(f)$ is reduced if and only if f is generically one-to-one.

Lemma 2.2.25(3) says that when the first of the generators of $\mathcal{O}_X$ over $\mathcal{O}_{n+1}$ is 1, then $\mathscr{F}_1(f)$ is generated by the maximal minors of the matrix obtained by deleting the first row of λ. This simplifies the computation of $\mathscr{F}_1(f)$. Moreover, since $M_2(f)$ is defined by $m \times m$ minors of a matrix of size $m \times (m+1)$, then it always has dimension $\geq n-1$. When $M_2(f)$ has dimension equal to $n-1$, then $M_2(f)$ is a determinantal variety and hence Cohen-Macaulay, by the Hilbert-Burch theorem (see for instance [12]). But $\dim M_2(f) = n-1$ if and only if f is generically one-to-one. Thus we have:

Corollary 2.5.9 *Assume $f\colon (X, x_0) \to (\mathbb{C}^{n+1}, 0)$ is generically one-to-one. Then $M_2(f)$ is a determinantal variety of dimension $n-1$, and hence Cohen-Macaulay.*

These results can be used when we consider an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $2 \leq p \leq n+1$. We recall that the critical locus Σ is the set of non submersive points of f, so $\Sigma = (\mathbb{C}^n, 0)$ when $p = n+1$. We saw in Sect. 2.2.7 that $\dim \Sigma = p-1$ and the restriction $f|_\Sigma : \Sigma \to (\mathbb{C}^p, 0)$ is finite. Since Σ is defined by the $p \times p$ minors of the Jacobian matrix, it is determinantal and hence Cohen-Macaulay. Moreover, by the Mather-Gaffney criterion 2.2.20, f has isolated instability which implies that Σ is generically smooth, so Σ must be reduced, by Serre's conditions R0 and S1. All the results of this subsection apply to $f|_\Sigma : \Sigma \to (\mathbb{C}^p, 0)$.

Again by the Mather-Gaffney criterion we see that $f|_\Sigma$ is generically one-to-one, thus $M_1(f|_\Sigma)$ is nothing but the discriminant $D = f(\Sigma)$ with reduced structure and $M_2(f|_\Sigma)$ is Cohen-Macaulay of dimension $p-2$. If $p \geq 3$ and $y \in M_2(f|_\Sigma)$ is a generic point, then it is not difficult to see that it has exactly two preimages $x_1, x_2 \in$

Σ and the multi-germ $f\colon (\Sigma, \{x_1, x_2\}) \to (\mathbb{C}^p, y)$ is an immersion with normal crossings. It follows that $M_2(f|_\Sigma)$ is smooth at y. Again by Serre's conditions R0 and S1, $M_2(f|_\Sigma)$ must be reduced.

Example 2.5.10 Consider the presentation λ of Example 2.5.7. Then $\det(\lambda)$ is a reduced equation for the image of f. It coincides with the equation obtained in 2.2.29:

$$X^6 - 2X^4Y - 2X^3Y^3 - 2X^3Z^2 - 8X^2Y^2Z + X^2Y^2 - 2XY^4 - 2XYZ^2 + Y^6 - 2Y^3Z^2 + Z^4.$$

The ideal $\mathscr{F}_1(f)$ is generated by the 3×3-minors of the matrix λ' obtained from λ by deleting its first row. In fact we need only those containing the first column. They are

$$XY - Z^2 - X^3 - Y^3 - 2XYZ,$$
$$X^2Z + YZ^2 + X^3Y + XY^2Z,$$
$$Y^2Z + XZ^2 + XY^3 + X^2YZ.$$

This ideal defines the curve $M_2(f)$ in $(\mathbb{C}^3, 0)$ which is equal to the singular locus of the image $M_1(f)$, with reduced structure. A primary decomposition of $\mathscr{F}_1(f)$ gives five smooth branches: $(Y - X^2, Z + XY)$, $(Z + Y^3, X - Y^2)$, $(Y + Z, X + Z)$ and two more smooth branches coming from the ideal $(Y^2 - YZ + Z^2, X + Y - Z)$.

We have seen in Lemma 2.2.25 that if λ is the presentation matrix of a finite map germ $f\colon (X, x_0) \to (\mathbb{C}^{n+1}, 0)$ with respect to generators $g_0 = 1, g_1, \ldots, g_m$, then it has the property that $\min_m(\lambda) = \min_m(\lambda')$, where λ' is the matrix obtained from λ by deleting its first row. This property is called the rank condition (RC) in [5] and [52], and the ring condition (RC) in [29]. Surprisingly, the rank condition characterises all the presentation matrices obtained in this way.

Definition 2.5.11 Let λ be an $(m+1) \times (m+1)$-matrix with entries in $\mathcal{O}_{n+1}$ and λ' the matrix obtained from λ by deleting its first row. If $\min_m(\lambda) = \min_m(\lambda')$, then we say that λ satisfies the **rank condition** (RC).

The next theorem appears in [52], based on arguments by de Jong and van Straten, who also gave a clearer proof in [29].

Theorem 2.5.12 *Suppose λ is an $(m+1) \times (m+1)$-matrix with entries in $\mathcal{O}_{n+1}$ satisfying* RC*, and such that* $\det \lambda \neq 0$ *and* $\mathcal{O}_{n+1}/\min_m \lambda$ *has codimension* 2*. There exists a multi-germ of Cohen-Macaulay complex space (X, S) of dimension n and a finite map $f\colon (X, S) \to (\mathbb{C}^{n+1}, 0)$ such that λ is the matrix of a presentation of $\mathcal{O}_{X,S}$ over $\mathcal{O}_{n+1}$.*

Proof See [48, Corollary 11.7]. □

2.5.3 Triple Points

Cohen-Macaulayness of $M_3(f)$ is proved in [52] under the additional assumption that (X, x_0) is Gorenstein. In this case, f admits a symmetric matrix presentation ([5, 52])—see [48, Theorem 11.8].

The case where (X, x_0) is a complete intersection (and in particular, when (X, x_0) is smooth) has the advantage that we can use the results of Scheja and Storch in [59] to find the symmetric matrix λ explicitly. Suppose $\pi \circ f$ is finite, where $\pi(y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_n)$, as in Proposition 2.5.6. Instead of $f : (X, x_0) \to (\mathbb{C}^{n+1}, 0)$ we consider $F : (X \times \mathbb{C}, (x_0, 0)) \to (\mathbb{C}^{n+1}, 0)$ given by

$$F(x, t) = (f_1(x), \ldots, f_n(x), f_{n+1}(x) - t).$$

Since (X, x_0) is a complete intersection of dimension n, $S := (X \times \mathbb{C}, (x_0, 0))$ is a complete intersection of dimension $n + 1$. Assume that S is embedded in some $(\mathbb{C}^N, 0)$ and that $\mathcal{O}_S = \mathcal{O}_N/(G_1, \ldots, G_{N-n+1})$. Let $J \in \mathcal{O}_N$ be the Jacobian determinant of (F, G) and consider the morphism $\eta : \mathcal{O}_S \to \mathcal{O}_{n+1}$ given by $\eta(a) = \mathrm{Tr}(a/J)$, the trace of the $\mathcal{O}_{n+1}$-linear endomorphism of the field of fractions of $\mathcal{O}_S$ given by multiplication by a/J. Then $\eta(a)$ is in fact an element of $\mathcal{O}_{n+1}$, η is $\mathcal{O}_{n+1}$-linear and $\eta(J)$ is a unit in $\mathcal{O}_{n+1}$. Moreover, η induces a symmetric $\mathcal{O}_{n+1}$-bilinear map

$$\langle \cdot | \cdot \rangle : \mathcal{O}_S \times \mathcal{O}_S \to \mathcal{O}_{n+1}.$$

given by $\langle a|b \rangle = \eta(ab)$, which turns out ([59]) to be a perfect pairing.

For each basis $G := \{g_0, \ldots, g_m\}$ for $\mathcal{O}_S$ as $\mathcal{O}_{n+1}$-module there is a dual basis $\check{G} := \{\check{g}_0, \ldots, \check{g}_m\}$ with the property that $\langle \check{g}_i | g_j \rangle = \delta_{ij}$. Let $\lambda := [t]_G^{\check{G}}$ denote the matrix of multiplication by t with respect to $\check{G}$ in the source and G in the target. Then λ is the symmetric matrix presentation of $\mathcal{O}_{X,x_0}$ over $\mathcal{O}_{n+1}$ whose existence was claimed above (see [48, Section 11.7] for more details).

Example 2.5.13 Let $f(x, y) = (x^2, y^2, xy + x^3 + y^3)$ as in Example 2.5.7. The presentation matrix obtained there was not symmetric. But we can use the above procedure to find a symmetric one. We take $\{1, x, y, xy\}$ as a basis of $\mathcal{O}_S$ over $\mathcal{O}_3$. With respect to this basis, multiplication by $1, x, y$ and xy have matrices

$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 0&X&0&0 \\ 1&0&0&0 \\ 0&0&0&X \\ 0&0&1&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&Y&0 \\ 0&0&0&Y \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&XY \\ 0&0&Y&0 \\ 0&X&0&0 \\ 1&0&0&0 \end{pmatrix},$$

respectively. The Jacobian determinant is $J = xy$ (up to a constant factor) and the matrix of multiplication by J^{-1} is the inverse of the last matrix in the above list:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{X} & 0 \\ 0 & \frac{1}{Y} & 0 & 0 \\ \frac{1}{XY} & 0 & 0 & 0 \end{pmatrix}.$$

This gives $\eta(1) = \eta(x) = \eta(y) = 0$ and $\eta(xy) = 4$. The matrix of the pairing $\langle \cdot | \cdot \rangle$ with respect to this basis is

$$\begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

and the basis dual to $\{1, x, y, xy\}$ is $\{xy, y, x, 1\}$ (up a to a constant factor). The corresponding symmetric presentation matrix is

$$\lambda_s = \begin{pmatrix} XY & Y^2 & X^2 & -Z \\ Y^2 & Y & -Z & X \\ X^2 & -Z & X & Y \\ -Z & X & Y & 1 \end{pmatrix}.$$

In Example 2.5.7, we replaced the matrix λ by a minimal presentation λ_0 by means of column operations. In the same way, we can obtain a symmetric minimal presentation by applying simultaneous column and row operations to the matrix λ_s we have obtained here. It is also straightforward to symmetrise λ_0 directly by column operations.

A first consequence of the existence of a symmetric presentation is that the conductor ideal $\mathcal{C}$ is principal in $\mathcal{O}_{X,x_0}$. This was also shown by Piene in [56].

Corollary 2.5.14 *Let $f\colon (X, x_0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically 1-1, where (X, x_0) is an n-dimensional Gorenstein space-germ. Then $\mathcal{C}$ is a principal ideal in $\mathcal{O}_{X,x_0}$.*

Proof Choose a symmetric presentation λ, with respect to generators $g_0 = 1, \ldots, g_m$. Then by Lemma 2.2.25, $\mathcal{F}_1(f)$ is generated by $(m_0^0, \ldots, m_m^0)$, where m_j^i the minor of λ obtained by deleting row i and column j. So $f^*\mathcal{F}_1(f)$ is generated by $f^*(m_0^0), \ldots, f^*(m_m^0)$. A simple argument using Cramer's rule (see [52]) shows that in $\mathcal{O}_{X.x_0}$, $m_j^i = m_j^0 g_i$ and it follows, by the symmetry of λ, that $f^*\mathcal{F}_1(f)$ is generated by $f^*(m_0^0)$. □

We refer to m_0^0 as the *principal minor* of the symmetric presentation matrix λ.

Example 2.5.15 Consider the symmetric presentation λ obtained in Example 2.5.13 for $f(x, y) = (x^2, y^2, xy + x^3 + y^3)$. The principal minor is

$$(x+y)\left(x^2+y\right)\left(x+y^2\right)\left(x^2-xy+y^2\right).$$

This is the equation for the double point curve $D_1^2(f)$. We see it has five smooth branches and Milnor number 16. Since $M_2(f)$ has also five smooth branches, this shows that the involution $\sigma : D^2(f) \to D^2(f)$ sends each branch to itself (see Example 2.4.10).

The second consequence is the following theorem from [52].

Theorem 2.5.16 *Let $f\colon (X, x_0) \to (\mathbb{C}^{n+1}, 0)$ be finite and generically one-to-one, where (X, x_0) is an n-dimensional Gorenstein germ. Let λ be a symmetric presentation of $\mathcal{O}_{X,x_0}$ over $\mathcal{O}_{n+1}$, with respect to generators $g_0 = 1, g_1, \ldots, g_m$. Then $\mathcal{F}_2(f)$ is generated by the $(m-1) \times (m-1)$ minors of the matrix λ'' obtained from λ by deleting its first row and column.*

Proof [48, Theorem 11.9] gives the proof of Kleiman and Ulrich in [33]. □

Kleiman and Ulrich go on to show that also for $k < m - 1$, the radicals of the ideals $\min(\lambda, k)$ and $\min(\lambda'', k)$ are equal. Calculations with examples support the conjecture that these ideals themselves are equal.

By a theorem of Kutz [34] (but see also [31]), the variety of zeros of the ideal of submaximal minors of a symmetric $m \times m$ matrix can have codimension no greater than 3, and if the codimension is 3 then the variety in question is Cohen Macaulay. Thus

Corollary 2.5.17 *In the circumstances of Theorem 2.5.16,* $\operatorname{codim} M_3(f) \leq 3$. *Moreover, if* $\operatorname{codim} M_3(f) = 3$, *then $M_3(f)$ is Cohen-Macaulay.*

By applying 2.5.17 to a stabilisation of f, we deduce the following corollary from the conservation of multiplicity for Cohen-Macaulay spaces.

Corollary 2.5.18 *Let $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ be finite and generically* 1*-to-*1*. If $M_3(f) = \{0\}$, then the number of triple points in the image of a stable perturbation of f is equal to* $\dim_{\mathbb{C}} \mathcal{O}_3/\mathcal{F}_2(f)$.

Example 2.5.19 Consider again the symmetric presentation λ of Example 2.5.13 for $f(x, y) = (x^2, y^2, xy + x^3 + y^3)$. The ideal $\mathscr{F}_2(f)$ is defined by the 2×2-minors of the matrix λ'' obtained from λ by deleting its first row and column. This gives $\mathscr{F}_2(f) = (X, Y, Z)$. By Corollary 2.5.18, the number of triple points in a stabilisation of f is 1. This was used in Example 2.4.10.

2.6 Open Problems

2.6.1 The $\mu \geq \tau$-Conjecture

Conjecture 2.6.1 Assume $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite and either $(n, n+1)$ are nice dimensions or f has corank one. Then

$$\mu_I(f) \geq \mathscr{A}_e\text{-codimension}(f) \tag{2.31}$$

with equality if f is weighted homogeneous.

The statement is analogous to that of Theorem 2.3.7 for $n \geq p$ and $\mu_\Delta(f)$ instead of $\mu_I(f)$. This conjecture is known to be true for $n = 1, 2$ but it still remains open for $n \geq 3$. The case $n = 2$ was solved in [30], and a second proof given in [49]. A proof for $n = 1$ can be found in [50]. A proof of the conjecture in the particular case of map-germs of fold type (i.e. multiplicity two) has been also obtained by Houston in [27].

There is plentiful evidence. In particular, the recent example of Sharland in [60], of a weighted homogeneous germ $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ of corank 3, has $\mu_I = \mathscr{A}_e -$ codimension $= 18,967$. Sharland found the codimension using the algorithm described here, and calculated μ_I using a remarkable formula for weighted homogeneous germs due to Toru Ohmoto in [55].

As we explained in Sect. 2.3.2, the conjecture is true if one can show that $\mu_I(f) = \dim_{\mathbb{C}} I_g/J_g$, where $g \in \mathcal{O}_{n+1}$ is a reduced defining equation of the image of f and $I_g = (f^*)^{-1}(J_g\mathcal{O}_n)$. And the equality $\mu_I(f) = \dim_{\mathbb{C}} I_g/J_g$ follows if the relative version of the quotient, $I_G^{\text{rel}}/J_G^{\text{rel}}$, is Cohen Macaulay of dimension d. Here $G \in \mathcal{O}_{n+1+d}$ is a good defining equation for the image of a stable d-parameter unfolding F, J_G^{rel} is the ideal generated by the partial derivatives of G with respect only to the variables in $\mathbb{C}^{n+1}$ and $I_G^{\text{rel}} = (F^*)^{-1}(J_G^{\text{rel}}\mathcal{O}_{n+d})$.

2.6.2 Does $\mu = \tau$ Imply Weighted Homogeneity?

A natural question related to Theorem 2.3.7 and the $\mu \geq \tau$-conjecture 2.6.1 is whether the equality in (2.16) or in (2.31) respectively implies that f is $\mathscr{A}$-equivalent to a weighted homogeneous map-germ. In the theory of isolated hypersurfaces singularities $(X, 0)$, a well known theorem of K. Saito in [57] states that if $\mu(X, 0) = \tau(X, 0)$, then we can choose coordinates in the ambient space such that $(X, 0)$ is weighted homogeneous. Later, H. Vosegaard proved the corresponding result for ICIS's, in [64]. Thus, our question can be seen as a natural counterpart of these theorems, for singularities of mappings with isolated instability.

Assume $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is $\mathscr{A}$-finite with $p \leq n + 1$, and $g \in \mathcal{O}_p$ is a reduced equation for the discriminant of f (or the image when $p = n + 1$). It follows from Theorem 2.2.23 that the $\mathscr{A}_e$-codimension is equal to $\dim_{\mathbb{C}} I_g/(J_g + (g))$. Hence, the equality

$$\dim_{\mathbb{C}} I_g/(J_g + (g)) = \dim_{\mathbb{C}} I_g/J_g$$

holds if and only if $g \in J_g$. So, it is seems natural to state the conjecture in the following way:

Conjecture 2.6.2 Assume $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is $\mathscr{A}$-finite with $p \leq n+1$ and let $g \in \mathcal{O}_p$ be a reduced equation for the discriminant of f. If $g \in J_g$, then f is $\mathscr{A}$-equivalent to a weighted homogeneous map-germ.

2.6.3 μ-Constant Families

A celebrated theorem by Lê and Ramanujam [37] states that if $\{(X_t, 0)\}$ is an analytic family of isolated hypersurface singularities of dimension $n \neq 2$ such that $\mu(X_t, 0)$ is constant, then $\{(X_t, 0)\}$ has constant topological type. This was improved later by Timourian [63] to show that the family is in fact topologically trivial. The restriction $n \neq 2$ is due to the use of the h-cobordism theorem in the proof and the case $n = 2$ is still open.

It is natural to ask for a Lê-Ramanujam type theorem for analytic families of map germs $\{f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)\}$ with isolated instability and $p \leq n+1$. By an analytic family we mean that we can embed it into an unfolding $F(x,t) = (f_t(x), t)$ which is origin preserving, that is, $f_t(S) = 0$, for all t. We say that the family is *μ-constant* if $\mu_\Delta(f_t)$ is constant when $n \geq p$ or if $\mu_I(f_t)$ is constant when $p = n+1$.

Conjecture 2.6.3 Any μ-constant family of map-germs as above is topologically trivial.

The case of families of plane curves $\{f_t\colon (\mathbb{C}, S) \to (\mathbb{C}^2, 0)\}$ is not difficult: we have $\mu_I(f_t) = \delta(X_t, 0) - |S| + 1$, where $\delta(X_t, 0)$ is the δ-invariant of the image $(X_t, 0)$. Therefore if μ_I is constant then so is $\mu(X_t, 0)$, by Milnor's formula relating μ and δ, and $\{(X_t, 0)\}$ is topologically trivial. A simple argument shows now that $\{f_t\colon (\mathbb{C}, S) \to (\mathbb{C}^2, 0)\}$ is also topologically trivial as a family of map-germs.

For families $\{f_t\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\}$ the conjecture is also true, and follows from the results of [14]: the authors show that the family is topologically trivial if $\mu(D^2_1(f_t))$ is constant. It follows from [40] that

$$\mu_I(f_t) = C_t + T_t + \mu(D^2(f_t)/S_2) - 1, \tag{2.32}$$

where C_t and T_t are the numbers of Whitney umbrellas and triple points respectively which appear in a stable perturbation of f_t and $D^2(f_t)/S_2$ is the quotient of $D^2(f_t)$ under the S_2-action. The upper semicontinuity of the invariants implies that the three numbers C_t, T_t and $\mu(D^2(f_t)/S_2)$ must be constant when $\mu_I(f_t)$ is constant. Again by (2.32),

$$\mu(D^2_1(f_t)) = 2\mu(D^2(f_t)/S_2) + C_t + 6T_t - 1$$

is also constant.

2.6.4 *Defining Equations and Cohen-Macaulayness of Multiple Point Spaces*

The multiple point spaces $D^k(f)$ have been introduced in Definition 2.4.1 for any $\mathscr{K}$-finite map germ f. But the definition is based on the existence of a stable unfolding F of f. In practice, this stable unfolding can have a huge number of parameters and this makes the explicit computation of $D^k(f)$ a hard task. It would be interesting to find explicit defining equations for $D^k(f)$ just in terms of f instead of F.

For $k = 2$, this is solved by the ideal $I_2(f)$ described in (2.23) which gives a simple way to compute $D^2(f)$. Also when f has corank one we have an explicit construction of all the spaces $D^k(f)$ given by the "divided differences" (2.22). But the general case for $k \geq 3$ and corank ≥ 2 is a difficult and mysterious problem which is still open. In the same vein, we know that $D^2(f)$, and $D^k(f)$ when f has corank one, are Cohen-Macaulay spaces, provided they have the expected dimension. As we have seen, this is a nice property when looking at deformations of f. We hope that an explicit description of the spaces $D^k(f)$ in the general case could lead to a proof that there too they are Cohen-Macaulay. Once again, calculations with stable maps of low multiplicity support this conjecture.

A different approach to multiple point spaces is due to Kleiman in [32] and Laksov in [35]. Kleiman's construction is done for general mappings $f: X \to Y$, where X and Y are schemes (possibly with singularities) and is based on the principle of iteration: the double point space $K_2(f)$ is defined as the residual scheme of $X \times_Y X$ along the diagonal ΔX; then $K_3(f)$ is the double point space of the projection $K_2(f) \to X$, $K_4(f)$ is the double point space of the projection $K_3(f) \to K_2(f)$ and so on. The definition is very clean from the categorical point of view but again is difficult to find explicit equations of $K_k(f)$ in a particular case. It is not difficult to see that $K_k(f)$ coincides with $D^k(f)$ when X and Y are smooth and f has corank one, but for higher corank, $K_k(f)$ and $D^k(f)$ are different spaces, indeed $K^k(f)$ is not finite over X when f has corank ≥ 2.

In a recent paper [53] (see also [48, Section 9.8]), the authors give an alternative construction of $K_k(f)$ when X are Y smooth, by embedding it into a smooth space and finding explicit equations which generalise the divided differences. It follows that $K_k(f)$ are local complete intersections when they have the expected dimension. Another interesting property is that the spaces $K_k(f)$ provide a desingularisation of $D^k(f)$ when $k = 2, 3$ and f is stable.

2.6.5 Fitting Ideals

It is also still unknown whether for $k > 3$ the spaces $M_k(f)$ defined by the Fitting ideals $\mathcal{F}_{k-1}(f)$ are Cohen-Macaulay when they have the expected dimension. Gruson and Peskine proved in [22] that it is true for germs of corank 1 (see also [48, §11.6]), and calculations show that it is true for all germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ of multiplicity ≤ 5—see [48, Subsection 11.2.1]. For germs of muliplicity ≤ 5, the strong version of the theorem of Kleiman and Ulrich referred to after Theorem 2.5.16, that $\min(\lambda, k) = \min(\lambda'', k)$, also holds. Does this too hold in general?

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Chapter 3
Topological Equisingularity: Old Problems from a New Perspective (with an Appendix by G.-M. Greuel and G. Pfister on SINGULAR)

Javier Fernández de Bobadilla

Contents

Abstract Among the various notions of equisingularity and equivalence of singularities, topological equisingularity is one of the oldest an easiest to define, but it is far to be well understood and several challenging questions remain open. Important

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examples are Zariski multiplicity conjecture and the charaterization of topological triviality of families of isolated and non-isolated singularities. In this chapter we survey developments in topological equisingularity, some of its relations with other equisingularity notions, and hint new possible approaches to old questions based in algebro-geometric methods, Floer theory and Lipschitz geometry. Topological equsingularity questions, were crucial motivation sources for the development of the Computer Algebra program SINGULAR; this is explained in an appendix by G.-M. Greuel and G. Pfister.

3.1 Introduction

Smooth algebraic and analytic varieties are locally diffeomorphic to Euclidean spaces. On the other hand, the geometry at singular points can be arbitrarily complicated. This motivated the question of defining equivalence relations which say when two singularities are "essentially the same". By a singularity we mean a germ (X, O) of algebraic or analytic varieties. We will deal with complex varieties. Although in several equisingularity questions one compares two germs of varieties, it is more common to study flat families $\sigma : \mathcal{X} \to T$ parametrized over a base, together with a section $s : T \to \mathcal{X}$, and to compare the germs $(\mathcal{X}_t, s(t))$ for different values of $t \in T$. The origin of equisingularity theories goes back to Zariski and Whitney, and the list of contributors is huge (Teissier, Lê, Gaffney, Ramanujam, King, Perron, Wall, Briançon, Speder, Henry, Massey, Villamayor, Luengo, Trotman, Parusiński...). There are several equisingularity notions (Zariski, Whitney, Topological Equisingularity, several versions of equiresolution), and many relations between the notions are established. A quite complete picture have been achieved concerning characterizations of Zariski and Whitney equisingularity in terms of algebraic invariants, and the relations between them. On the other hand no complete algebraic characterization of topological equisingularity have been achieved and many old and classical questions have resisted, already for several decades, the different approaches that the community has tried, and remain open. One of the most important ones is Zariski's multiplicity question.

In this paper we do not aim to summarize the history of the subject, or to give a comprehensive exposition of some of its main results. Rather we are going to focus in a few classical open questions, chosen by the personal taste of the author, highlight some of the approaches, results and methods motivated by them and propose new possible ideas, problems and connections with newer methods (arc and jet spaces, Floer theories, Lipschitz geometry...) that could lead to some answers. The approaches described in this paper lead to several new open problems and conjectures, from several researchers including the author, whose study may foster more beautiful developments. We carefully describe the motivation and state this questions.

As it is not surprising, several of the topological questions concerning the topology of singularities reach its maximal difficulty in complex dimension 2, since this corresponds to the 4-dimensional topology (and 3-dimensional topology of the link). This is the case in the classical question of whether a μ-constant family of hypersurfaces is topologically trivial, which only remains open in the surface case. Beyond hypersurfaces with isolated singularities the algebraic characterization of topological equisingularity is completely open, and we devote much effort in explaining the available results and a conjectural program.

In topological equisingularity the analysis of examples is crucial, and manipulating interesting ones by hand is almost impossible. In fact it was in connection with Zariski's multiplicity question that the computer algebra program SINGULAR was developed. The last section of this paper is an appendix by Greuel and Pfister explaining how this happened. I am very thankful to them for agreeing to contribute with this section.

By looking at the table of contents above the reader may get an idea of what to expect.

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3.2 A Fast Review on Topology of Hypersurface Singularities and their Milnor Fibration

We will briefly recall a collection of classical results and methods that are needed for our discussion. The literature and the set of results available in this direction is so vast that we can not aim for any kind of completeness. However we have not resisted to mention, in passing, some results that are not strictly needed in what follows, but that are close enough to those we need.

Definition 3.2.1 Two complex analytic germs (X_1, O), (X_2, O) have the same *abstract topological type* if there is a homeomorphism $\varphi : (X_1, O) \to (X_2, O)$. If φ extend to the ambient space we say that (X_1, O), (X_2, O) have the same *embedded topological type*. Two holomorphic function germs $f_i : (\mathbb{C}^n, O) \to \mathbb{C}$ are topologically $\mathcal{R}$-equivalent if there exist a homeomorphism germ $\varphi : (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ such that $f_1 \circ \varphi = f_2$. If there are homeomorphism germs $\varphi : (\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$, $\psi : (\mathbb{C}, O) \to (\mathbb{C}, O)$ such that $\psi \circ f_1 \circ \varphi = f_2$ we say that f_1 and f_2 are topologically $\mathcal{RL}$-equivalent.

We have the obvious implications for functions and their zero sets: "topologically $\mathcal{R}$-equivalence" $\Longrightarrow$ "topologically $\mathcal{RL}$-equivalence" $\Longrightarrow$ "same embedded topological type" $\Longrightarrow$ "same abstract topological type".

Milnor's Conical Structure Theorem (Theorem 2.10 of [83]), implies that the abstract topological type of X is determined by the topological type of the intersection $L_X := X \cap \mathbb{S}_\epsilon$, where $\mathbb{S}_\epsilon$ denotes the sphere of sufficiently small radius ϵ. Likewise, the embedded topological type of a germ $(X, O) \subset (\mathbb{C}^n, O)$ is determined by the topological type of the pair $(\mathbb{S}_\epsilon, L_X)$. We call L_X the *abstract link* of X and $(\mathbb{S}_\epsilon, L_X)$ the *embedded link of* X.

The notion of embedded link is most meaningful and studied for hypersurface singularities. In this case the implication "same embedded link up to homeomorphism" $\Longrightarrow$ "topologically $\mathcal{RL}$-equivalence" was proved by King for $n \neq 3$ (Theorem 3 of [57]), and by Perron [108] for $n = 3$. Saeki proved conversely in [114] that having the same abstract or embedded topological type implies having homeomorphic abstract or embedded links; the case $n = 3$ is special in his proof and non-trivial arguments from Perron [108] are used. Putting all these results together we obtain that the notions "same embedded link up to homeomorphism", "same embedded topological type" and "topologically $\mathcal{RL}$-equivalence" are equivalent.

Perhaps the most fundamental source of invariants of hypersurface germs is its Milnor fibration. In [83], Milnor proved that, given a holomorphic function germ $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ for a sufficiently small radius ϵ, the mapping

$$f/|f| : \mathbb{S}_\epsilon \setminus V(f) \to \mathbb{S}^1$$

is a locally trivial fibration. Such a fibration shows that the embedded link of a hypersurface singularity is in fact a fibered link. The typical fibre is called the *Milnor fibre*, and its monodromy the *Milnor monodromy*.

At this point a link characterization of topological $\mathcal{R}$-equivalence can also be described. Two holomorphic function germs f_1 and f_2 are called *strongly link equivalent* if they have the same embedded link up to homeomorphism, by a homeomorphism transforming the Milnor fibration of f_1 into a fibration homotopic to the Milnor fibration of f_2. King [57] ($n \neq 3$) and Perron [108] ($n = 3$) proved that "topological $\mathcal{R}$-equivalence" is equivalent to "strong link equivalence".

Milnor proved in [83] that if $\epsilon > 0$ is small enough and δ is positive and smaller than ϵ^N for sufficiently high N (that is δ is small enough in comparison with ϵ) then Minor's fibration is equivalent to

$$f|_{B_\epsilon \cap f^{-1}(\partial D_\delta)} : B_\epsilon \cap f^{-1}(\partial D_\delta) \to \partial D_\delta,$$

where B_ϵ and D_δ denote the ball in $\mathbb{C}^n$ and disc in $\mathbb{C}$ of radius ϵ and δ, centered at the origin. This fibration is sometimes called the *monodromy fibration*. The action on the monodromy on the (co)-homology of the Milnor fibre and the homology of the Milnor fibre together with its intersection and Seifert form are prominent invariants in the topological setting.

The Milnor fibration can be studied by several methods:

1. Morse theory,
2. polar methods,
3. the resolution approach,
4. the Morsification-deformation method,
5. spaces of arcs and motivic integration.
6. logarithmic geometry.
7. sheaf theoretic, Mixed Hodge theory and D-modules theory and microlocal methods,

which are complementary to each other.

We will not touch the last two items. Some bibliography on log-geometry methods is [18, 19, 56]. The last item condensates in just one line a massive amount of work, from different but related methods, which do not fall within the scope of the present survey. A reasonable account of them would take a much longer article than the present one.

Morse theory was the method employed originally by Milnor in [83], where in particular the following result was proved. Define the Milnor number $\mu(f) := \dim_{\mathbb{C}}(O_{\mathbb{C}^n,O}/J(f))$, where $J(f)$ is the jacobian ideal generated by the partial derivatives of f.

Theorem 3.2.2 (Milnor) *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a holomorphic function germ. Then*

- *the link $L := f^{-1}(0) \cap \mathbb{S}_\epsilon$ is $(n-2)$-connected,*
- *the Milnor fibre of f is a parallelizable manifold which has the homotopy type of a finite CW-complex of dimension at most $n-1$. If f has an isolated singularity at the origin, then the Milnor fibre has the homotopy type of a bouquet of $\mu(f)$ spheres of dimension $(n-1)$.*

The polar method consists in considering a generic linear function $l : \mathbb{C}^n \to \mathbb{C}$ and studying the map $(f, l) : (\mathbb{C}^n, O) \to \mathbb{C}^2$. Its discrimimant is a plane curve, and based on the well known structure of the embedded topology of plane curves, the so-called Carrousel Method, introduced by Lê [67], yields very precise information on the Milnor fibration by induction on the dimension (which is achieved by restricting to the slices $V(l-t)$). It yields the following improvement of the result above:

Theorem 3.2.3 (Lê-Perron [69]) *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a holomorphic function germ with an isolated singularity at the origin. Then the Milnor fibre is a $2(n-1)$ ball with $\mu(f)$ $(n-1)$-handles attached.*

The polar method has many other applications, for example it was used to construct monodromy diffeomorphisms without fixed points in [68], and yields a proof the Monodromy Theorem [67], which states that the eigenvalues of the action of the monodromy on the homology of the Milnor fibre are roots of unity, and the size of the Jordan blocks is at most n. The monodromy theorem is one of the most central results in the theory and admits proofs using very different methods. The polar method proof has the virtue of producing a generalization of the monodromy theorem for isolated complete intersection singularities, a class of singularities to which a good portion of the theory for hypersurfaces extends (see [74] for a quite complete account).

The resolution method uses an embedded resolution $\pi : X \to \mathbb{C}^n$ of $f^{-1}(0)$ which does not modify away from the singular set, and studies the composition $f \circ \pi$. It was used by A'Campo in order to study the Lefschetz number and the zeta function of the monodromy [1, 2], and also yields a proof of the monodromy theorem (see [6], for example).

The Morsification method works primarily for isolated singularities, although it has been extended to special classes of non-isolated ones (see later in this paper). A complete exposition of the method for isolated singularities, including all the results mentioned below can be found in [6] . The idea of the method is to consider a generic deformation f_s of f; then for small $s \neq 0$ the isolated singularity of f splits into finitely many ordinary quadratic points (Morse points); such a deformation is called a *Morsification*. One gains information on the Milnor fibration of f via the function f_s, which have several simpler singularities (Morse points) by means of the map $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ defined by $F(x, s) := (f_s(x), s)$, whose discriminant is a plane curve as in Carrousel Method [6]. For example, each Morsification yields a basis of the homology of the Milnor fibre generated by one embedded sphere of dimension $(n-1)$ for each of the Morse points. Such spheres are called vanishing cycles. Picard-Lefschetz theory allows to reconstruct the *algebraic monodromy* (that is, the action of the monodromy in homology) in terms of the intersection matrix associated with a basis of vanishing cycles. Morsifications also allow to define the monodromy group of a singularity, a subgroup of the mapping class group of the Milnor fibre generated by the Dehn twits associated to a basis of vanishing cycles. Characterizing when a sphere embedded in the Milnor fibre is a vanishing cycle, and giving an intrinsic description of the monodromy group as a subgroup of the mapping class group of the Milnor fibre are hard problems that has been solved only for curves and very recently [112].

Even more precise properties on the Milnor fibration can be gained studying, instead of F, the whole versal deformation of f. For instance, exploiting the irreducibility of the discriminant of the versal deformation the following can be proved. To a basis of vanishing cycles associated to a Morsification we attach a *Dynkin diagram* as follows: draw a vertex for each vanishing cycle, and connect two vertices with an edge if the intersection number between the two corresponding vanishing cycles is non-zero. The following two results are usually called "irreducibility of the monodromy" (see [6], Section 3.2):

Theorem 3.2.4 (Gabrielov [40], Lazzeri [65]) *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a holomorphic function germ with an isolated singularity at the origin. The Dynkin diagram of the basis of vanishing cycles of any Morsification is connected.*

The following result proved independently by Gabrielov, Lazzeri and Lê can be deduced as a corollary:

Theorem 3.2.5 (Gabrielov, Lazzeri, Lê) *Let f_t be a deformation of a function germ f_0 having an isolated singularity. If for every t the function f_t has only one critical value, then the function f_t has only one critical point.*

Several aspects of the Milnor fibration are invariant by topological equivalence. In particular the following result is due to Teissier in the case of isolated singularities and by Lê in general:

Theorem 3.2.6 (Lê [66], Teissier [132]) *If f and g are function germs having the same embedded topological type then their Milnor fibres are homotopy equivalent, in particular they have the same Betti numbers.*

Even if the result stated in the references above is restricted to Betti numbers, their proofs yield the claimed homotopy equivalence.

The theorem above implies the following previous result of A'Campo [1] and Lê [66]:

Theorem 3.2.7 (A'Campo, Lê) *If* $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ *has the same embedded topological type than a smooth germ, then* f *is smooth.*

The Theorem above should be compared with the analogous situation for the abstract topological type (see Mumford's theorem 3.3.4 and the existence of hypersurface singularities with topological sphere links below).

A very recent development by A'Campo, Pe Pereira, Portilla, Sigurðsson and the author express monodromy of curve singularities as a generalized Dehn twist (a so called *tête-à-tête twist*) around a 1-dimensional skeleton which is a strong deformation retract of the Milnor fibre. (see [3, 111]).

3.3 Open Problems in Topological Equisingularity

3.3.1 Zariski's Conjectures

In his influential paper [154] Zariski posed 8 questions that served as an inspiration for the later development of the subject. In this paper we are mainly interested in Questions A and B, which are of topological nature. In order to formulate them let us recall that the multiplicity m of a function is the degree of the lowest non-zero terms in its series expansion $f = f_m + f_{m+1} + ...$ in homogeneous forms, and that the projectivized tangent cone is the projective hypersurface $V(f_m) \subset \mathbb{P}^{n-1}_{\mathbb{C}}$.

Question 3.3.1 (Zariski, [154]) Let $V(f)$ and $V(g)$ be hypersurface germs in $(\mathbb{C}^n, O)$ having the same embedded topological type.

(A) Are the multiplicities of f and g the same?
(B) Are the projectivized tangent cones of $V(f)$ and $V(g)$ homeomorphic?

Question B was answered in the negative, as it will be explained in detail below. Question A is still open, and is to the present date one of the major open problems in Singularity Theory. If one of the hypersurfaces is smooth the conjecture follows from Theorem 3.2.7. We would like to single out also the partial result of Ephraim [25] (independently Trotman [136]), who proves the conjecture under the hypothesis that the homeomorphism is C^1. Conjecture A was also proved by Zariski [153] for plane curve singularities, if one of the germs is smooth (see Theorem 3.2.7), if $n = 3$ and one of the germs has multiplicity 2 (see Navarro-Aznar [87]), if $n = 3$ and f and g are quasihomogeneous with an isolated critical point at the origin (Xu-Yau [149, 150]), if $n = 3$, f and g have an isolated critical point at 0 and the arithmetic genus of $V(f)$ is at most 2 (Yau [151]) and for suspensions of irreducible plane curve singularities (Mendris-Némethi [82]). If f and g are topologically right-left equivalent by bilipschitz homeomorphisms Risler-Trotman [113] proved that they have the same multiplicity. See Sect. 3.6 for further information on the Lipschitz version of multiplicty's conjecture.

The reader may consult Eyral's survey [26] and book [27] for further information on Zariski's multiplicity conjecture.

One may ask to what extent the abstract topology determines either the embedded topology, or analytic invariants like the multiplicity. In dimension 1 having the same abstract topological type is the same that having the same number of irreducible components, since the abstract link is a disjoint union of circles, one for each irreducible components. On the other hand the embedded link is an iterated torus link, and in its topology all the information on Puiseux exponents and contact order between branches is codified. So, in dimension 1 the abstract topology is too simple to determine the embedded one, or to determine the multiplicity.

For the surface case the abstract topology contains already substantial information. For example, in the surface case the topology of the embedding of the exceptional divisor in the minimal resolution of singularities is equivalent to the abstract topology of the link [90], but even in this case this information is not enough to determine the embedded topology or the multiplicity:

Example 3.3.2 (Némethi [91]) The surface singularities $x^3 + y^7 + z^{21} = 0$ and $x^4 + y^5 + z^{20} = 0$ have isomorphic abstract links but different embedded topology. In fact their Milnor numbers are 240 and 228 respectively.

It is remarkable that in [82] the authors not only recovered the multiplicity from the embedded topology. In fact they study functions of the form $f(x, y) + z^N$, for f irreducible, and recover the whole set of Puiseux pairs and N from the topology of the abstract link of $V(f(x, y) + z^N)$ when the abstract link is a rational homology sphere. If the abstract link is not a rational homology sphere, in special cases there are pairs of suspensions with the same link but different Puiseux data and N. However, in these cases too, the link together with the Milnor number determines the Puiseux pairs of f and the integer N.

They proposed the following daring conjecture:

Conjecture 3.3.3 (Mendris, Némethi) Assume that $V(f) \subset \mathbb{C}^3$ is an isolated hypersurface singularity with rational homology sphere abstract link. Then the topology of the abstract link characterizes completely the embedded topological type, the geometric genus and the multiplicity.

The following theorem due to Mumford [86] allows to characterize, among normal surface singularities, the smooth ones in terms of the abstract topology (that is, the abstract topology characterizes when the multiplicity of a normal surface singularity is equal to 1).

Theorem 3.3.4 *A normal surface germ (X, O) is smooth if and only if its abstract link has trivial fundamental group.*

In higher dimension isolated hypersurface germs (X, O) which are non-smooth, but whose link is homeomorphic to a sphere were discovered by Brieskorn, Hirzebruch and Milnor. Even more interestingly, the link of some of such hypersurfaces carries Milnor's exotic differentiable structures (see [14] and [83]). This rules out the chance of recovering the multiplicity (even charaterizing multiplicity = 1) from abstract topological information in the higher dimensional case.

This behavior should be compared with the situation concerning the different notions of topological triviality for families, see next section and Remark 3.3.9.

3.3.2 Topological Triviality Conjectures

The conjectures above have a very important special case: the family versions of these questions. Now we will introduce these questions and relate them with other classical questions as well.

For the kind of problems we are dealing with it is enough to work with families depending holomorphically over a disc. A family of holomorphic functions deforming a germ $f : (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ is a holomorphic function $F : \mathcal{U} \to \mathbb{C}$ defined in an open neighborhood $\mathcal{U}$ of $(O, 0)$ in $\mathbb{C}^n \times T$ such that $f(x) = F(x, 0)$. Here T denotes a disc in $\mathbb{C}$ containing 0. We denote by f_t the restriction $F|_{\mathcal{U}\cap(\mathbb{C}^n\times\{t\})}$. Often it will be needed to consider the map $\bar{F} : \mathcal{U} \to \mathbb{C} \times \mathbb{C}$ defined by $\bar{F}(x, t) := (F(x, t), t)$. Let X be an analytic space. A deformation of X over a base $(T, 0)$ is a flat morphism $\mathcal{X} \to T$ together with an isomorphism $\theta : X \to \mathcal{X}_0$, where $\mathcal{X}_t$ denotes the fibre over t. The notion of deformations of an embedded analytic space $X \subset \mathbb{C}^n$ is defined analogously. A deformation F of a function f induces a deformation $F^{-1}(0) \to T$ of $f^{-1}(0)$, in fact it induces an embedded deformation.

Definition 3.3.5 A family of holomorphic functions is $\mathcal{R}$-trivial if there exists a homeomorphism $\Phi : \mathcal{U} \to \mathcal{U}$ (perhaps after shrinking $\mathcal{U}$ to a smaller neighborhood of $(O, 0)$), such that Φ is of the form $\Phi(x, t) = (\phi_t(x), t)$ and $f_t \circ \phi_t = f_0$ for all $t \in T$.

A deformation $\mathcal{X} \to T$ is *topologically trivial* if there exists a homeomorphism $\Phi : \mathcal{X} \to \mathcal{X}_0 \times T$ which commutes with both projections to T; that is, for any $t \in T$, there is an induced homeomorphism $\phi_t : \mathcal{X}_t \to \mathcal{X}_0$, and the family of homeomorphisms is continuous in t. The notion of *topological triviality for embedded deformations* is defined analogously.

Topological $\mathcal{RL}$-triviality can also be defined, but we will not focus on it in this paper. Let F be a family of holomorphic functions deforming a germ f. Then $\mathcal{X} := F^{-1}(0) \to T$ is a deformation of $X := f^{-1}(0)$, and topological $\mathcal{R}$-triviality of F obviously implies topological triviality of $\mathcal{X} \to T$. In the situations we are dealing with, when we are able to prove topological triviality of $\mathcal{X} \to T$, then we can deduce topological $\mathcal{R}$-triviality of F. This is the reason why we do not consider the intermediate relations "topological" $\mathcal{RL}$-triviality or "embedded topological triviality".

The critical values of any complex analytic function are isolated (this is an easy consequence of Curve Selection Lemma). So, given a family of functions we can assume, after shrinking the domain, that 0 is the only critical value of f_0. Then, if the family is topologically $\mathcal{R}$-trivial, using Theorem 3.2.7 we conclude that for any $t \in T$ the function f_t has a single critical value. Without loosing generality we can assume that 0 is the only critical value of f_t for any t, that is that $f_t^{-1}(0)$ is the only singular fibre of f_t. Working with families of functions with isolated singularities we get a much more precise condition:

Lemma 3.3.6 *If a family F is topologically $\mathcal{R}$-trivial, then (after shrinking the domain $\mathcal{U}$)*

1. *we may assume that 0 is the only critical value of f_t for all $t \in T$. In other words, the discriminant of $\bar{F}$ is $\{0\} \times T$.*
2. *if f_0 has an isolated singularity at the origin we may assume that f_t also has an isolated singularity at the origin and that $\mu(f_t)$ does not depend on t.*

Proof The first assertion has been proved above. The second is a consequence on the invariance of the Betti numbers of the Milnor fibre for topologically $\mathcal{R}$-equivalent functions (Theorem 3.2.6) and the fact that the middle dimension Betti number is the Milnor number. The second assertion can be deduced also from the first as follows: by the first assertion all singular points of f_t are in $f_t^{-1}(0)$. Since the Milnor number is the intersection number of the partial derivatives, by conservation of intersection number by deformation of the equations we have that $\mu(f_0)$ equals the sum of the Milnor numbers of f_t at all singular points. We conclude using Theorem 3.2.5. □

A remarkable result of Lê and Ramanujam [70], combined with work by Timourian and King [58] provides a converse except in the surface case:

Theorem 3.3.7 (Lê-Ramanujam, King, Timourian) *Let $f_t : (\mathbb{C}^n, O) \times \mathbb{C}$ be a family of isolated singularities with constant Minor number at the origin. If $n \neq 3$ then*

1. *For any t we have that $(\mathbb{C}^n, f_0^{-1}(0))$ and $(\mathbb{C}^n, f_t^{-1}(0))$ have the same embedded topological type.*
2. *Furthermore, the family is topologically $\mathcal{R}$-equivalent.*

Proof (Main Steps of the Proof) Lê and Ramanujam proved the first part of the Theorem. They compare the Milnor fibres of f_t and f_0 as follows: fix $t \neq 0$ and let $\epsilon_t < \epsilon_0$ be radii of the Milnor fibrations of f_t and f_0 respectively. Let $\delta > 0$ be sufficiently small so that it serves as a radius for the Milnor disc both for f_0 and f_t. They prove using an easy argument based on Ehresmann Fibration Theorem that $f_t^{-1}(\delta) \cap B_{\epsilon_0}$ is diffeomorphic to the Milnor fibre of f_0. Then we have an inclusion

$$f_t^{-1}(\delta) \cap B_{\epsilon_t} \subset f_t^{-1}(\delta) \cap B_{\epsilon_0}$$

of the Milnor fibre of f_t into the Milnor fibre of f_0. Define the cobordism

$$W_{t,\delta} := f_t^{-1}(\delta) \cap B_{\epsilon_0} \setminus \mathring{B}_{\epsilon_t} \tag{3.1}$$

to be the difference of the two Milnor fibres. Using (with some care) that both Milnor fibres have the homotopy type of a bouquet of $\mu(f_t) = \mu(f_0)$ spheres, and that the links are 1-connected by Theorem 3.2.2 they deduce that the cobordism $W_{t,\delta}$ is homotopically trivial (a h-cobordism). Now, by the dimension assumption the hypotheses of Smale's h-cobordism Theorem are satisfied, and one concludes that the cobordism $W_{t,\delta}$ is trivial. This allows to show that the Milnor fibration associated with f_0 is smoothly equivalent to the Milnor fibration associated with f_t. This is a very important intermediate step in the proof. For $n = 2$ the cobordism is 2-dimensional, and easy to analyze.

They conclude the proof of the first assertion using that the mapping

$$f_t|_{B_{\epsilon_0}\setminus \mathring{B}_{\epsilon_t}\cap f^{-1}(D_\delta)} : B_{\epsilon_0} \setminus \mathring{B}_{\epsilon_t} \cap f^{-1}(D_\delta) \to D_\delta \tag{3.2}$$

is a trivial fibration. This is another easy consequence of Ehresmann Fibration Theorem, but uses in a crucial way that f_t has an isolated singularity at the origin and that $\mu(f_t)$ is constant. □

After this, King [58] and Timourian [135] proved the second part in an independent way. I would like to highlight some steps in King's approach. King works in larger generality, and his proof applies to classes of real singularities. As explained above, a μ-constant family f_t satisfies that the origin is the only critical point of f_t. This is in King's terminology a family with *no coalescing critical points*. Given a μ-constant family for each t sufficiently close to the origin we assign a cobordism as follows. Let ϵ_0 be a Milnor radius for f_0. Since the intersection of $\mathbb{S}_{\epsilon_0}$ with $f_0^{-1}(0)$ is transversal, the same happens for the intersection of $\mathbb{S}_{\epsilon_0}$ with $f_t^{-1}(0)$ for t sufficiently small. Let $\epsilon_t < \epsilon_0$ be a Milnor radius for f_t. The absence of critical points outside the origin show that the following space is a cobordism between $\mathbb{S}_{\epsilon_t} \cap f_t^{-1}(0)$ and $\mathbb{S}_{\epsilon_0} \cap f_t^{-1}(0)$:

$$W_t := f_t^{-1}(0) \cap B_{\epsilon_0} \setminus \mathring{B}_{\epsilon_t}. \tag{3.3}$$

In fact, because of the triviality of the map (3.2) we have that W_t is diffeomorphic to $W_{t,\delta}$.

With these notations King's result implies the following in our setting:

Theorem 3.3.8 (King) *Let $f_t : (\mathbb{C}^n, O) \times \mathbb{C}$ be a family of isolated singularities with constant Minor number at the origin. Then:*

1. *The cobordism W_t is homologically trivial.*
2. *If the cobordism W_t is invertible then f_0 is topologically $\mathcal{R}$-equivalent to f_t.*
3. *The family is topologically $\mathcal{R}$-trivial if and only if W_t is invertible for all t.*

Proof The first assertion was proved already by Lê and Ramanujam in [70] as we have just explained. For the last assertion it is worth to remark that it is enough to prove that W_t is invertible for a single $t \neq 0$ which is sufficiently small. We refer to [58] for a complete proof. □

Remark 3.3.9 Notice that if $f_t^{-1}(0)$ and $f_0^{-1}(0)$ are homeomorphic then it is easy to prove that the cobordism W_t is invertible. We conclude that in the case of families f_t with isolated singularities, constant abstract topological type implies topological $\mathcal{R}$-equivalence. This gives an indication that when dealing with families one needs less hypotheses to conclude triviality statements. Indeed, abstract equisingularity does not imply embedded one even in the curve case when comparing germs that are not in family. However, in case that the germs are in a family abstract equisingularity implies topological $\mathcal{R}$-triviality, which is the strongest possible topological equisingularity notion.

In the case of isolated singularities, a result of Siersma [126] implies that if ϵ_0 and ϵ_t are Milnor radii for f_0 and f_t, for sufficiently small t the intersection $f_t^{-1}(0) \cap B_{\epsilon_0}$ has the homotopy type of a bouquet of $\mu(f_0) - \mu(f_t)$ spheres. Since $f_0^{-1}(0) \cap B_{\epsilon_0}$ is contractible by the Conical Structure Theorem, we have

Remark 3.3.10 The Milnor number must be constant when the cobordism W_t is homologically trivial.

The combination of Zariski multiplicity question with the above discussion leads to the following important open questions:

Question 3.3.11 Let $f_t : (\mathbb{C}^n, O) \to \mathbb{C}$ be a μ-constant family of isolated singularities:

1. Is f_t topologically $\mathcal{R}$-trivial?
2. Is the multiplicity of f_t constant.

One could formulate also the family version of Zariski's Question B, but as we will see below it has a negative answer.

The first question was conjectured by Hironaka, according to Teissier [131], and is open only for $n = 3$ due to Theorem 3.3.7. The reason are low-dimensional topology difficulties that make the analysis of the cobordism W_t very hard.

The second question is open for any $n > 2$. It can be formulated for families of functions defined over every field, since both the hypothesis and the conclusion are of an algebraic nature. For the curve case ($n = 2$), it is worth to notice that there exists no purely algebraic proof. The only proof over the complex numbers consists in proving that a μ-constant family is topologically trivial, and then applying the affirmative answer known for Zariski Multiplicity Question for $n = 2$. For families of functions defined over a field of characteristic 0, an application of Lefschetz's Principle gives the same conclusion. Gabrielov and Kushnirenko [41] answered Question 3.3.11 affirmatively for f_0 homogeneous. Greuel [45] and O'Shea [102] proved it for μ-constant deformations of functions f_0, with the only hypothesis that f_0 is quasi-homogeneous.

Remark 3.3.12 An interesting historical remark is that Question 3.3.11 (2) was one of the main driving motivations for the computer algebra program SINGULAR. An appendix of this paper by Greuel and Pfister summarizes the origin of SINGULAR.

3.3.2.1 Other Results Related to Topological Triviality Questions

Let $f_t : \mathbb{C}^n \to \mathbb{C}$ be a family of functions holomorphically depending on a parameter t such that f_0 has an isolated singularity at the origin. Let $\mathcal{X} \subset \mathbb{C}^n \times \mathbb{C}$ be defined by $(x, t) \in \mathcal{X}$ if and only if $f_t(x) = 0$. As it has been explained above if $\mathcal{X}$ is topologically trivial as a family then $\mu(f_t)$ is constant. Topological triviality is the weakest equisingularity notion for families of analytic spaces. Stronger notions are Zariski's equisingularity for systems of generic projections and Whitney equisingularity (see Parusiński and Trotman contributions to this Handbook for surveys of these equisingularity conditions). In fact the following sequence of implications holds:

Theorem 3.3.13 *"Zariski's equisingularity for systems of generic projections" ⇒ "Whitney equisingularity" ⇒ "topological triviality".*

The first implication is due to Speder [129] and the second is Thom's famous "first isotopy lemma" . The fact that the first assertion implies the third was proved in a stronger form by Varchenko (he did not need to use generic projections, see [140–143] for this and related results). So both Zariski's equisingularity for systems of generic projections and Whitney equisingularity should imply constancy of multiplicity if we want that Question 3.3.11 (2) has a positive answer. In fact these two implications have been confirmed in the literature. For Zariski's equisingularity, even without genericity assumption, it has been shown already by Zariski in [155]. For Whitney equisingularity it was proved by Hironaka in [53]. Briançon and Speder [12] gave counterexamples to both converse implications.

The implication "Whitney equisingularity" ⇒ "constant multiplicity is also a particular case of the following beautiful characterization of Whitney equisingular families of hypersurfaces (which was proved later). Given a function germ $f : (\mathbb{C}^n, 0) \to \mathbb{C}$, its $\mu^*(f)$ sequence is the sequence of Milnor numbers of the restrictions of f to the generic linear sections of any dimension through 0 in $\mathbb{C}^n$.

Theorem 3.3.14 (Teissier, Briançon-Speder) *Let $\{F(x, t) = 0\}$ be a family of isolated hypersurface singularities at the origin, depending holomorphically on a parameter t. Then $\{F(x, t) = 0\}$ is Whitney equisingular along the t-axis if and only if $\mu^*(f_t)$ is independent of t.*

Teissier [131] proved the "if" part and Briançon and Speder [13] the converse. The Milnor number of the restriction of f to a generic line equals the multiplicity minus 1. So, Teissier result implies the invariance of multiplicity under Whitney equisingularity. Another important consequence is that in order to find possible conuterexamples for Question 3.3.11 (1), one should find μ constant families that are not μ^*-constant. This is not extremely hard, but already takes a bit of effort.

Families which are linear in the parameter (of the form $f_0 + tf_1$ for $f_0, f_1 \in \mathcal{O}_{\mathbb{C}^n,O}$ and t the parameter of T) are easier to handle with respect to Question 3.3.11. In fact Parusiński gave an affirmative answer to Question 3.3.11, (1) in [104], and Trotman answered Question 3.3.11, (2) in [136].

An important ingredient in Parusiński's proof is the following characterization due to Lê and Saito [71]:

Theorem 3.3.15 (Lê-Saito) *Let $f_t : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a family of function germs with an isolated singularity at the origin, depending holomorphically on t, with $t \in D$ a disc. Denote by $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ the function $F(x, t) := f_t(x)$. The following are equivalent*

(i) the Milnor number $\mu(f_t)$ is independent of t for t close to the origin.

(ii) Thom A_f condition is satisfied: for any sequence of points $\{(x_k, t_k)\}_{k\in\mathbb{N}}$ converging to $(0, 0)$ and with $x_k \neq O$, the limit of the tangent spaces $T_{(x_k,t_k)}F^{-1}(F(x_k, t_k))$ (assume the limit exists and otherwise take a subsequence) is a hyperplane containing the axis $\{O\} \times \mathbb{C}$.

(iii) for any sequence of points $\{(x_k, t_k)\}_{k\in\mathbb{N}}$ converging to $(0, 0)$ and with $x_k \neq O$ the limit

$$\lim_{k\to\infty} |\partial F/\partial t(x_k, t_k)|/sup\{|\partial F/\partial z_i(x_k, t_k)| : 1 \leq i \leq n\} = 0,$$

where $(z_1, \ldots, z_n)$ is a coordinate system of $\mathbb{C}^n$.

This criterion was used by Greuel in his proof of constancy of multiplicity for μ-constant families of quasi-homogeneous germs [45].

The last two conditions are easy reformulations of each other. The point of listing the condition (iii) is to compare it with the following result due to Teissier [131].

Theorem 3.3.16 (Teissier) *Let $f_t : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a family of function germs with an isolated singularity at the origin, depending holomorphically on t, with $t \in D$ a disc. Denote by $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ the function $F(x, t) := f_t(x)$. The following are equivalent*

(i) $\{F(x, t) = 0\}$ is Whitney equisingular along the t-axis (equivalently the μ^* sequence is constant).

(ii) $\partial F/\partial t$ belongs to the integral closure of the ideal $(z_1, \ldots, z_n) \cdot (\partial F/\partial z_1, \ldots, \partial F/\partial z_n)$.

(iii) There exists a positive constant C such that

$$|\partial F/\partial t| \leq C|z|sup(|\partial F/\partial z_j| : j \in \{1, \ldots, n\}).$$

The second and third conditions are equivalent, and the third condition is obviously stronger then the third condition in Theorem 3.3.15. Teissier's characterization of Whitney equisingularity in terms of integral closure was greatly generalized by Gaffney to a theory of equisingularity in higher codimension, which we will not touch here since it goes away from the main topic of our paper. The reader may consult Wall's survey [148] instead. Also, a characterization of Whitney equisingularity in terms of the conormal space was obtained by Henry, Merle and Sabbah (see [50, 51]).

A consequence of Parusiński and Trotman papers is that finding possible counterexamples for Question 3.3.11 becomes harder: a μ-constant family $f_t = f_0 + tf_1 + t^2 f_2 + \ldots$ is *essentially non linear* if the linear interpolation between a generic member f_{t_0} for a fixed values $t_0 \neq 0$ generic and f_0 is not μ constant. If the family is not essentially non-linear then Parusiński and Trotman Theorems, combined with Theorem 3.3.8 show that the family is topologically $\mathcal{R}$-trivial and equimultiple. On the other hand finding examples of essentially non-linear μ-constant families, which are not μ^*-constant families is not so easy. All known examples have high Milnor numbers, and checking that a candidate family is in fact μ-constant can be computationally very expensive. In fact the computer program SINGULAR was born in the attempt of finding counter-examples to Question 3.3.11 (2). This is explained in detail in an appendix to this paper by Greuel and Pfister.

An important class of functions are those Newton non-degenerate with respect to their Newton polyhedron. Kushnirenko [60] provided a formula for the Milnor number for this kind of functions only in terms of the Newton polyhedron. Oka [99] and Damon-Gaffney [20] proved that families of Newton non-degenerate functions with constant Newton polyhedron are topologically trivial. Oka's method consists in proving the existence of a uniform Milnor radius (see next section). Very recently Leyton, Mourtada and Spivakovsky [73] have posted a preprint in arXiv where μ-constant families of Newton non-degenerate functions are proven to be topologically trivial; their method is to prove simultaneous embedded resolution, a much stronger result (see Sect. 3.3.6).

3.3.3 Vanishing Folds

Given a μ-constant family, if there exists a uniform Milnor radius for f_t for t sufficiently small then the cobordism W_t is automatically trivial, and hence the family is topologically $\mathcal{R}$-trivial by Theorem 3.3.8. This happens for example for quasi-homogeneous μ-constant families. It is remarkable that, given a μ-constant family, it is unknown whether there exists a Hermitian form in $\mathbb{C}^n$ such that there is a uniform radius for the Milnor ball (for the distance associated with that Hermitian metric). This is the viewpoint taken by O'Shea in [101]. Fixed a Hermitian metric H in $\mathbb{C}^n$ let us denote by d_H the distance function to the origin. An H-vanishing fold is a real analytic curve $\alpha : (0, t_0) \to \mathbb{C}^n \times T$ of the form $\alpha(t) = (\alpha_1(t), t)$ such that for any $t \in (0, s)$ the restriction of the function d_H to $f_t^{-1}(0)$ has a critical point in $\alpha_1(t)$, and such that $\lim_{t\to 0} \alpha(t) = 0$. By Curve Selection Lemma a vanishing fold exists if there is no uniform Milnor radius for the distance function d_H. O'Shea formulates the following conjecture, which remains open and implies an affirmative answer to Question 3.3.11, (1).

Conjecture 3.3.17 (O'Shea) If f_t is a family of isolated singularities at the origin, and for any Hermitian metric H the family f_t has an H-vanishing fold then $\mu(f_t) < \mu(f_0)$ for $t \neq 0$ sufficiently small.

One may interpret the approach of O'Shea as the attempt to show that the cobordisms W_t are trivial by finding a function without critical points in them.

The only result I know in this direction is Oka's paper [99] where uniform Milnor radius is proven for Newton non-degenerate functions with constant Newton polyhedron.

3.3.4 The Low Dimensional Topology Approach

Another tempting approach is to try to exploit that the 3-manifolds that can be the links of surface singularities (which are the boundaries of the cobordism W_t) are of a very particular kind: negative-definite plumbing 3-manifolds. These manifolds are simple in the sense that they have no hyperbolic hyperbolic/atoroidal pieces in their geometric/JSJ decomposition, and moreover can be constructed from a negative definite decorated graph via Neumann's plumbing construction. Building on work of Waldhausen [145], Neumann [97] provided a plumbing calculus that completely codifies the topology of such manifolds in terms of graphs, and the operations that can be performed in a graph while keeping the same topology. Plumbing manifolds are also called graph manifolds.

This viewpoint has been pursued by Perron and Shalen [109]. In page 3 of their paper they state:

Remark 3.3.18 "It can be proved that the Lê-Ramanujan theorem is true if the links of f_0 and f_t are homeomorphic. This is a well-known consequence of an argument due to Levine [72]"

Later we will see that the same statement is also a consequence of Laufer work (with algebraic geometry methods).

Exploiting the fact that the links are graph manifolds Perron and Shalen proved the following:

Proposition 3.3.19 (Perron-Shalen) *Let M , N be two compact, connected, closed, irreducible graph manifolds with infinite fundamental group and suppose there exists a cobordism W between M and N such that:*

(i) $\pi_1(N) \to \pi_1(W)$ *is surjective;*
(ii) *W is obtained from N by adding handles of index* 2*;*
(iii) *the inclusions $M \to W$ and $N \to W$ are $\mathbb{Z}$-homology-equivalences.*

Then M and N are homeomorphic.

Conditions (ii) and (iii) are satisfied in the cobordisms appearing in μ-constant families. The second is because $f_f^{-1}(0)$ is Stein and the distance function is strictly pluri-subharmonic, and the third was proved by Lê and Ramanujam. So the above proposition reduces Question 3.3.11 (1) to the proof of condition (i) of Proposition 3.3.19.

The previous theorem and the following remark hint a possible approach to Question 3.3.11 (1) based on the study of fundamental groups of 3-manifolds.

Remark 3.3.20 Except in the case of cyclic quotient and cusp surface singularities, the links of normal surface singularities are sufficiently large in the sense of Waldhausen, and therefore their homeomorphism type is determined by the fundamental group [144]. So, except for a well-known list of classes, that should be possible to handle by different methods, in order to answer Question 3.3.11 (1) one is reduced to study whether the fundamental groups of the links are isomorphic or not. This is still a hard question.

3.3.5 The Topological Counter-Example of Borodzik and Friedl

If f_t is a μ-constant family of surface singularities the associated cobordism W_t satisfies the following properties (see the explanations given above and [10]):

(a) W_t and L_0 and L_t are compact oriented, and L_0, L_t are plumbed 3-manifolds.
(b) The image of $\pi_1(L_t)$ in $\pi_1(W_t)$ normally generates $\pi_1(W_t)$.
(c) W_t can be built from $L_t \times [0, 1]$ by adding handles of indices 0, 1, and 2.
(d) The maps $H_*(L_0; \mathbb{Z}) \to H_*(W_t; \mathbb{Z})$ and $H_*(L_t; \mathbb{Z}) \to H_*(W_t; \mathbb{Z})$ induced by inclusions are isomorphisms.

Notice that Property (b) is a slight weakening of Condition (i) in Proposition 3.3.19. As a measure of the difficulty of proving Lê-Ramanujam conjecture for surfaces by purely topological statements we would like to point out that in [10], the authors construct a non-trivial cobordism satisfying Properties (a)-(d) above. Therefore it does not satisfy Condition (i). No example is known with (a) replaced by the following stronger condition:

- W_t and L_0 and L_t are compact oriented, and L_0, L_t are negative-definite plumbed 3-manifolds.

So, it appears that the negative-definite condition is an essential one. In [10] the authors point out two further conditions that W_t must satisfy when it comes from the Lê-Ramanujam conjecture situation, but that their counter-example does not satisfy: W_t is Stein and symplectic, with one concave and one convex boundaries.

3.3.6 Simultaneous Resolutions

Equiresolution of families of algebraic varieties, and its relation with equisingularity conditions has been studied by many authors from several points of view. I will only present those developments in connection with the possible approach to

Question 3.3.11 (1) that is described below. Throughout this section we will assume basic knowledge on the theory of normal surface singularities. The reader may use Némethi's [90] as a reference.

In [133] Teissier introduced the following simultaneous resolution notions, and posed the problem of relating them with equisingularity criteria. Let $\sigma : \mathcal{X} \to T$ be a family of Gorenstein normal surface singularities. A proper modification

$$\Pi : \mathcal{Y} \to \mathcal{X}$$

from a smooth 3-fold $\mathcal{Y}$ is called a *very weak simultaneous resolution* if

(a) $\mathcal{Y}$ is flat over T,
(b) $\mathcal{Y}_t$ is a resolution of $\mathcal{X}_t$ for every $t \in T$.

Let $s : T \to \mathcal{X}$ be a section of σ and assume that the singular set of $\mathcal{X}_t$ is the point $s(t)$, and that Π is an isomorphism over $\mathcal{X} \setminus s(T)$. Denote by $\mathcal{E} := \Pi^{-1}(s(T))$ the exceptional divisor with its non-reduced structure, and by $(\mathcal{E})_{red}$ the exceptional divisor reduced structure. The map Π is a *weak simultaneous resolution* of the family of germs $(\mathcal{X}_t, s(t))$ if it is a very weak simultaneous resolution and, in addition,

(c) the restriction $\sigma \circ \Pi|_{(\mathcal{E})_{red}} : (\mathcal{E})_{red} \to T$ is a locally trivial fibration.

The map Π is a *strong simultaneous resolution* of the family of germs $(\mathcal{X}_t, s(t))$ if it is a very weak simultaneous resolution and, in addition,

(c') the restriction $\sigma \circ \Pi|_{\mathcal{E}} : \mathcal{E} \to T$ is simple,

where simple is a version of locally trivial which takes into account the non-reduced structure (see [133]) for details.

After work of Laufer [63, 64], Vaquié [139], Kollár and Shepherd-Barron [59] the following characterizations were found. Given a normal surface germ let $\tilde{X}_{min} \to X$ be its minimal resolution, and E be its exceptional divisor. The canonical divisor $K_{\tilde{X}_{min}}$ is numerically equivalent to a $\mathbb{Q}$-divisor with support in E, and hence the intersection number $K_X^2 := K_{\tilde{X}_{min}}^2$ is well defined. When X is a Stein surface which has several singularities K_X^2 is the sum of the contributions of each singular point; observe that, even if $\mathcal{X}_0$ has a single isolated singularity, this singularity may split in several singular points when deforming it..

Theorem 3.3.21 (Laufer) *Let $\sigma : \mathcal{X} \to T$ be a family of Gorenstein normal surface singularities. Then $K_{\mathcal{X}_t}^2$ is constant if and only if $\sigma : \mathcal{X} \to T$ admits a very weak simultaneous resolution after pull-back by a finite base change $T' \to T$.*

In fact this result is a combination of a result of Brieskorn [15] asserting that $\sigma : \mathcal{X} \to T$ admits a simultaneous rational double point resolution if and only if it admits a very weak simultaneous resolution after pull-back by a finite base change, with the result of Laufer that a simultaneous rational double point resolution exists if and only if $K^2_{\mathcal{X}_t}$ is constant. We do not introduce simultaneous rational double point resolution because it is not needed for our discussion.

The theorem above is false without the Gorenstein condition (see for example [59], 2.8). Kollár and Shepherd-Barron used the Minimal Model Program to prove a version of the result above, avoiding the Gorenstein condition, at the expense of imposing projectivity for σ and using the global self-intersection K^2 defined for a projective surface. We do not give more details here since we are concerned with the local case only.

Theorem 3.3.22 (Laufer [63]) *A family $\sigma : \mathcal{X} \to T$ of Gorenstein normal surface singularities together with a section s has a weak simultaneous resolution after a finite base change if and only if all the links of the germs $(\mathcal{X}_t, s(t))$ for $t \in T$ are homeomorphic.*

Theorem 3.3.23 (Laufer [64], Teissier [131]) *A family $\sigma : \mathcal{X} \to T$ of Gorenstein normal surface singularities together with a section $s : T \to \mathcal{X}$ has a strong simultaneous resolution after a finite base change if and only if the family is Whitney equisingular.*

Teissier proved the "only if" part and Laufer the converse.

Now we can provide an alternative argument proving Remark 3.3.18:

Proof (Proof of Remark 3.3.18) If $F : \mathbb{C}^n \times T \to \mathbb{C}$ defines a μ constant family of singularities at the origin of $\mathbb{C}^n$, then the family $\mathcal{X} := V(F)$ with the section $s(t) = (O, t)$ satisfies the hypothesis of Theorem 3.3.22 if and only if the topology of the abstract link is independent of t. In case that this happens there exists a weak simultaneous resolution $\Pi : \mathcal{Y} \to \mathcal{X}$, and the mapping

$$\sigma \circ \Pi : (\mathcal{Y}, \mathcal{E}) \to T$$

is a locally trivial fibration of pairs (this is deduced easily from the local triviality of $\mathcal{E} \to T$). Now since $\mathcal{X}$ is obtained from $\mathcal{Y}$ by collapsing each fibre $\mathcal{E}$ to a point, we obtain that the projection

$$\mathcal{X} \to T$$

is also locally trivial. Thus we obtain that the abstract topological type of the family is constant, and by Theorem 3.3.8 and Remark 3.3.9, the family is topologically $\mathcal{R}$-trivial. □

Let us turn the discussion to Theorem 3.3.21. The quantity K^2 can be computed from the resolution graph of the minimal good resolution, and the information that resolution graph contains is equivalent to the topological type of the abstract link. Therefore if a family $\sigma : \mathcal{X} \to T$ together with a section s is topologically trivial then K_t^2 is independent of t. Therefore Question 3.3.11 (1) splits in the following two questions, which hint an algebraic geometry approach to the Lê-Ramanujam problem.

Question 3.3.24 Let f_t be a μ-constant family of isolated surface singularities.

1. Is it true that K_t^2 is independent of t?
2. If f_t is in addition K_t^2-constant, is the topology of the abstract link independent of t? (hence the family would be topologically $\mathcal{R}$-trivial.

Laufer [62] proved the following formula relating several invariants of normal surface singularities. Let $\mathcal{X}$ be a smoothing of a Gorenstein normal surface singularity $\mathcal{X}_0$ (a deformation of $\mathcal{X}_0$ such that $\mathcal{X}_t$ is smooth). Denote by $\mu(\mathcal{X})$ the second Betti number of $\mathcal{X}_t$ (this is consistent with definition of μ for hypersurfaces). Then we have

$$\mu(\mathcal{X}) = 12p_g + K_{\mathcal{X}_0}^2 + \nu(\mathcal{X}_0) - b_1(L_{\mathcal{X}_0}), \tag{3.4}$$

where ν denotes the number of irreducible components of the exceptional divisor of the minimal resolution of $\mathcal{X}_0$, $b_1(L_{\mathcal{X}_0})$ is the first Betti number of the link of $\mathcal{X}_0$ and $p_g := \dim_{\mathbb{C}}(R^1\pi_*\mathcal{O}_{\mathcal{Y}_0})$ (where $\pi : \mathcal{Y}_0 \to \mathcal{X}_0$ is a resolution of singularities) is the geometric genus.

All the invariants appearing in the formula except K^2 and ν are constant in μ-constant families of hypersurfaces: the case of the first Betti number of the link is clear by the homological triviality of the cobordism W_t. For p_g a possible proof follows realizing p_g as a certain sum of spectral numbers and invoking a proof by M. Saito that the spectrum is constant in μ-constant deformations. Therefore the sum $K_{\mathcal{X}_t}^2 + \nu(\mathcal{X}_t)$ is constant in a μ-constant family.

Remark 3.3.25 By the above arguments the constancy of $K_{\mathcal{X}_t}^2$ in a μ-constant family is equivalent to the constancy of $\nu(\mathcal{X}_t)$. Observe that, since the topology of X is equivalent to the topology of the embedding of the exceptional divisor in the minimal resolution, the quantity ν_t has a straightforward topological meaning, which is certainly much weaker than the whole topology of the link. So, in order to prove the existence of a very weak simultaneous resolution, only weak topological information of the link must be preserved. However, it is still a hard open question how to control $\nu(\mathcal{X}_t)$ in μ-constant families

The previous remark motivates at least two different new lines of research that in my opinion have independent interest.

3.3.6.1 JSJ Decompositions for Groups

Except in the case of cyclic quotient and cusp surface singularities the whole topology, and hence ν is determined by the fundamental group of the link [144], however it is widely open how to read this information in the group. The only development I know pointing to this direction is the JSJ-decomposition for groups due to Scott and Swarup [119, 120], where at least the topological JSJ decomposition of the link is obtained purely algebraically from the fundamental group. It would be very interesting if this trend could be continued to know exactly which information on the fundamental group has to be preserved to deduce that ν is constant, and to determine if this happens in a μ-constant family.

3.3.6.2 Arc Spaces

Due to the positive solution to Nash Conjecture for surfaces [39], given a normal surface singularity (X, O), the number ν coincides with the number of irreducible components of the space of arcs $\mathcal{L}(X, O)$ centered at the singularity. This motivates the following general equisingularity question, which is also being considered by M. Leyton:

Question 3.3.26 Let f_t be a family of functions with an isolated singularity at the origin that satisfies an equisingularity condition (Zariski, Whitney or μ-constant. Let $X_t := V(f_t)$. Which information is preserved in the associated family $\mathcal{L}(X_t, O)$?

Nearly nothing seems to be known on this question. Of course under Zariski or Whitney Equisingularity it is known that the number of irreducible components of the exceptional divisor of the minimal resolution remains constant, but the proof uses the topological triviality, that is known in those cases, and the positive solution to Nash conjecture. I would be very interesting to have a direct proof of the constancy of ν under these equisingularity conditions, in order to build some knowledge for the study of the harder question for μ-constant families.

3.3.7 Connection with the Artin-Laufer Program, Heegaard-Floer and Lattice Homology

The inspiration for the ideas contained in this section come from the connection of developments of very different origin: algebraic geometry of normal surface singularities, and gauge and Floer theoretic invariants invariants of 3-manifolds. As the reader may expect at this stage the connecting point is the abstract link of a normal surface singularity. The purpose of this section is to propose a new conjecture on deformations of surface singularities, and trace back some of its historical roots.

As explained in Sect. 3.3.4, the link L_X of a normal surface singularity X is a negative definite graph manifold. In fact (see [90]) the graph codifying the topology of the link coincides with the graph codifying the embedded topology of the exceptional divisor of a normal crossings resolution $\tilde{X} \to X$. For a fixed topology (i.e. a fixed graph or embedded topology of the exceptional divisor) there is in most of the cases a whole non-discrete family of possible analytic structures on X (and hence in $\tilde{X}$). One of the most important directions in the study of normal surface singularities is the Artin-Laufer program. It can be broadly interpreted as the attempt of understanding the interplay between topological invariants and analytic ones, and how they guide the classification of normal surface singularities. Under certain topological and analytic hypotheses, like having a rational homology sphere link, or being Gorenstein, a priori analytically defined invariants turn out to be topological, or at least bounded by topological invariants. Besides the multiplicity, the analytic invariant that is the most important for our discussion is the geometric genus $p_g(X)$, which seems to be a good measure of the complexity of a normal surface singularity. The reader may consult [90, 91] for an explanation of the program and the involved analytic and topological invariants.

A crucial question is if $p_g(X)$ can be estimated, and in some cases calculated in terms of topological invariants of the link. In this direction Némethi and Nicolaescu [95] conjectured the following inequality.

Conjecture 3.3.27 (Némethi-Nicolaescu) Assume that X is a normal surface singularity with rational homology sphere link. Then

$$sw^0_{L_X}(\sigma_{can}) - (K^2 + \nu)/8 \geq p_g, \tag{3.5}$$

with equality if X is Gorenstein. Here $sw^0_{L_X}(\sigma_{can})$ is the modified Seiberg-Witten invariant of the link associated with the canonical Spin^c structure (defined in [95]), K^2 is the self intersection of the canonical cycle at a resolution and ν the number of irreducible components of the same exceptional divisor of the same resolution.

The quantity $K^2 + \nu$ is independent of the resolution, and the left hand side of the inequality only depends on the topology of L_X.

The conjecture was proved in several important cases [95, 96] but finally disproved by Luengo, Melle and Némethi [76]. Interestingly, the counterexamples belong to the class of super-isolated singularities, introduced by I. Luengo in his proof that the μ-constant stratum is not necessarily smooth [75]. A super-isolated singularity is a surface singularity $(X, O) \subset \mathbb{C}^3$ defined by $f_d + l^{d+1}$, where $\{f_d = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ is a plane curve with isolated singularities and $l = 0$ is a line in $\mathbb{P}^2_{\mathbb{C}}$ not meeting the singularities of $\{f_d = 0\}$. By their definition super-isolated singularities bridge the theory of projective plane curves with the theory of normal surface singularities, that allows to transfer questions between these subjects. A super-isolated singularity has a rational homology sphere link if and only if $\{f_d = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ is a rational cuspidal curve, that is, a curve homeomorphic to $\mathbb{P}^1_{\mathbb{C}}$. The counter-examples found in [76] are super-isolated singularities associated with rational cuspidal curves having more than 1 singularity. However, no counter-example was found among the class of rational unicuspidal plane curves (with just 1 singular point). This motivated the formulation of a purely algebraic conjecture for rational unicuspidal plane curves $\{f_d = 0\}$ (Semigroup Condition Conjecture) by Luengo, Melle Némethi and the author [35] by translating Némethi's Seiberg-Witten conjecture for the super-isolated singularity $\{f_d + l^{d+1} = 0\}$ to a relation between the degree d and the semigroup of the only singularity of $\{f_d = 0\}$.

The above mentioned counterexamples also forced to re-think Némethi-Nicolaescu's conjecture, and a relation with Ozsváth and Szabó's recent theory of Heegaard-Floer homology for 3-manifolds was found by Némethi. Heegaard-Floer homology is a topological invariant for 3-manifolds of Floer theoretic nature introduced by Ozsváth and Szabó [103], which was created with the purpose of providing a more topological description of Seiberg-Witten theory for 3-manifolds. There are several versions of Heegaard-Floer homology. In our case we will be interested in HF^+. Let L be a orientable 3-manifold and σ a Spin^c structure on L (see Turavev [138] for the formulation of Spin^c structures used in Heegaard-Floer theory). Then $HF^+(L, \sigma)$ is a $\mathbb{Z}_2$-graded $\mathbb{Z}[U]$-module, and the Seiberg-Witten invariant $sw^0_{L_X}(\sigma)$ can be computed as a "normalized Euler characteristic" of $HF^+(L, \sigma)$. Némethi sought for a definition of Heegaard-Floer homology for 3-manifolds which are links of singularities, which is closer to the methods of singularity theory, and in his attempt he created Lattice Homology [92]. Let L_X be a singularity link and σ a Spin^c structure on it. The Lattice Homology $\mathbb{H}(L_x, \sigma)$ is a $\mathbb{Z}$-graded $\mathbb{Z}[U]$-module which can be constructed entirely in terms of the combinatorial data contained in the plumbing graph. If L_X is a rational homology sphere the *mod* 2-graded version of $\mathbb{H}(L_x, \sigma)$ conjecturally coincides with $HF^+(L, \sigma)$, beyond that case the coincidence of the two invariants is not so clear. As a supporting evidence for this conjecture Némethi proved that the "normalized Euler characteristic" of $\mathbb{H}(L_x, \sigma)$ also coincides with $sw^0_{L_X}(\sigma)$ [93], and proved the coincidence to be true in several cases together with T. László [61]. In fact the authors define the number of *bad vertices* of a plumbing graph defining a rational homology sphere as the minimal number of vertices whose self-intersections have to be decreased in order to obtain a rational graph (that is, the plumbing graph of a rational surface singularity), and confirm the conjecture for the case up to 2 bad vertices.

Heegaard-Floer and Lattice homologies have had very interesting applications bridging singularity theory with low dimensional topology (see [94] for example), but at this survey we will concentrate only in their conjectural links with equisingularity and simultaneous resolution.

The point is that the Semigroup Condition Conjecture can be re-formulated as follows: let $\{f_d = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ define a unicuspidal rational plane curve, defined by a homogenous polynomial of degree d, and let $X = \{f_d + l^{d+1}\}$ be its associated super-isolated singularity (here l is a generic linear function), let $\{g_d = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a union of d lines meeting in a point, and $Y = \{g_d + l^{d+1}\}$ be the associated super-isolated singularity. Then Némethi noticed that the Semigroup Condition Conjecture is equivalent to each of the following equalities

$$HF^+(L_X, \sigma_{can}) = HF^+(L_Y, \sigma_{can}),$$

$$\mathbb{H}(L_X, \sigma_{can}) = \mathbb{H}(L_Y, \sigma_{can}),$$

where σ_{can} is the canonical Spin^c structure.

The point now is that X and Y sit in the same K^2-constant deformation. This motivated Némethi to formulate the following conjecture, which would imply the Semigroup Condition Conjecture:

Conjecture 3.3.28 Let $\mathcal{X} \to T$ be a K^2 constant deformation of Gorenstein normal surface singularities with $\mathbb{Q}$-homology sphere links. Then we have the equality

$$\mathbb{H}^0(L_{X_0}, \sigma_{can}) = \mathbb{H}^0(L_{X_t}, \sigma_{can}).$$

While the conjecture above remains open, Borodzik and Livinston [11] proved the Semigroup Condition Conjecture by Heegaard-Floer theoretic methods.

The connection point with equisingularity theory is via the following question, which is a converse to the previous conjecture, and if answered positively would mean that Lattice homology is precisely the right topological invariant to control K^2 in flat deformations:

Question 3.3.29 Let $\mathcal{X} \to T$ be a deformation of Gorenstein normal surface singularities with $\mathbb{Q}$-homology sphere links such that $\mathbb{H}^*(L_{X_t}, \sigma_{can})$ is constant. Is it K^2-constant?

It would be very interesting to find formulations not requiring the $\mathbb{Q}$-homology sphere link hypothesis.

3.3.8 On Topological Triviality of μ and K^2-Constant Families

Above we have focused in analyzing problems which are motivated by Question 3.3.24 (1). Now let us concentrate in Question 3.3.24 (2).

3.3.8.1 A Problem of Combinatorial/Arithmetic Nature

In order to study Question 3.3.24 (2) let us consider $f_t : \mathbb{C}^3 \to \mathbb{C}$ a μ and K^2-constant family holomorphically dependent on t. Define $F(x,t) := f_t(x)$, let $\mathcal{X} := V(F) \subset \mathbb{C}^n \times T$ and let $s(t) := (O,t)$ the constant section at the origin. By Theorem 3.3.22 there exists a very weak simultaneous resolution

$$(\mathcal{Y}, \mathcal{E}) \to (\mathcal{X}, s(T))$$

after a finite base change. We assume that the base change has been performed, and that the weak simultaneous resolution exists. Choosing a representative of $\mathcal{X} \subset B_\epsilon \times T$, for ϵ and T small, Ehresmann Fibration Theorem yields that $\mathcal{Y}$ is smoothly locally trivial over T. We have proved that in order to establish topological $\mathcal{R}$-triviality it is enough to show that the projection $\mathcal{E} \to T$ is locally trivial.

Consider the decompositions in irreducible components $\mathcal{E}_0 = \cup_{i=1}^{N} \mathcal{E}_{0,i}$ and $\mathcal{E}_t = \cup_{i=1}^{N} \mathcal{E}_{t,i}$ for $t \neq 0$. Here $N = \nu_0 = \nu_t$, where the second equality is known by Eq. 3.4 and the remarks following it, since K_t^2 is constant. By choosing a sufficiently small representative we may assume that $H_2(\mathcal{Y}_0, \mathbb{Z})$ is free and generated by the homology classes $\{[\mathcal{E}_{0,i}]\}_{i=1}^{N}$. Since $\mathcal{Y} \to T$ is topologically trivial, we may regard the family $\mathcal{Y}_t$ as the same smooth manifolds with a varying family of complex structures J_t. Therefore we can see $[\mathcal{E}_{t,i}]$ as a homology class in $H_2(\mathcal{Y}_0, \mathbb{Z})$. Therefore we have an expression $[\mathcal{E}_{t,i}] = \sum_{j=1}^{N} m_{i,j} [\mathcal{E}_{0,j}]$. The divisor $\mathcal{E}_t$ may be thought as degenerating into $\mathcal{E}_0$, and the number $m_{i,j}$ is the number of intersection points, counted with multiplicity of $\mathcal{E}_{t,i}$ with a holomorphic disc in $\mathcal{X}_0$ which is transversal to $\mathcal{E}_{0,j}$ at a generic point. Since J_t specializes to J_0 we conclude that all the intersection points are positive and that the numbers $m_{i,j}$ are all of them non-negative.

For $t \neq 0$ we denote by $\mathcal{Z}_t$ a small tubular neighborhood of $\mathcal{E}_t$ in $\mathcal{Y}_t = \mathcal{Y}_0$. Then the cobordism W_t coincides with the difference $\mathcal{Y}_0 \setminus \mathring{\mathcal{Z}}_t$. Since the cobordism has been proved homologically trivial by the μ-constant condition in [70], we conclude that the collection of classes $\{[\mathcal{E}_{t,i}]\}_{i=1}^{N}$ provides another basis of $H_2(\mathcal{Y}_0, \mathbb{Z})$. This implies that $M = (m_{i,j})$ is uni-modular (the modulus of its determinant equals 1). .

Denote by A_t the intersection matrix $A_t = (a_{t,ij})$, where $a_{t,ij} := \mathcal{E}_{t,i} \cdot \mathcal{E}_{t,j}$. Then we have the matrix equality

$$A_t = M^t A_0 M.$$

The family is topologically $\mathcal{R}$-trivial if and only if the projection $\mathcal{E} \to T$ is locally trivial, but this happens precisely when the matrix M is a permutation matrix. This motivates the following problem of a combinatorial/arithmetic nature:

Problem 3.3.30 Let M be a uni-modular matrix with non-negative integral entries. Let A_t and A_0 be negative definite symmetric matrices with integral entries and that A_0 have no -1's in the diagonal. If we have the equality $A_t = M^t A_0 M$ then M is a permutation matrix.

We have been able to solve this problem when A_0 is the resolution graph of a minimal singularity (which are characterized by the fact that the valency of a vertex of the good resolution graph is bounded by the absolute value of its self-intersection). Beyond that case the problem seems quite difficult. The closer is A_0 to be non-definite the harder it gets.

A positive solution to the previous problem would show that any μ and K^2-constant family not having divisors with self-intersection -1 at the minimal resolution is topologically $\mathcal{R}$-trivial. Self-intersection -1 components of the exceptional divisor have to be handled in a different way, since the above problem has easy counter-examples without this assumption.

3.3.8.2 Okuma's Work on $(-P^2)$-Constant Deformations

Wahl defined in [137] a characteristic number for the link of a normal surface singularity X as follows. A characteristic number defined for closed oriented 3-manifolds, is, according to Thurston, a real number $\lambda(M)$ associated to each such manifold M such that it is a topological invariant and $\lambda(N) = k\lambda(M)$ for a k-sheeted covering $N \to M$. Let $\tilde{X} \to X$ be a resolution with a strict normal crossings exceptional divisor E. Consider the Zariski decomposition (see [117] for a definition) $K_{\tilde{X}} + E = P + N$. Wahl [137] proved, among other results

Theorem 3.3.31 (Wahl) *$(-P^2)$ is a characteristic number for links of normal surface singularities. It only depends on the fundamental group of the link; in particular it does not depend on the resolution. $(-P^2)$ vanishes for a singularity if and only if it is log-canonical.*

Okuma [100] studied the behavior of $(-P^2)$-constant deformations of Gorenstein normal surface singularities. He proved

Theorem 3.3.32 (Okuma) *Let $\sigma : \mathcal{X} \to T$ be a $(-P^2)$-constant family of Gorenstein normal surface singularities which are not log-canonical. After a finite base change there exists a section $s : T \to \mathcal{X}$ such that $s(t)$ is a non log-canonical point in $\mathcal{X}_t$, and a simultaneous resolution $\Pi : \tilde{\mathcal{X}} \to \mathcal{X}$ with the following properties*

1. *For any $t \in T$ the map $\tilde{\mathcal{X}}_t \to \mathcal{X}_t$ is the minimal resolution with normal crossings divisor.*
2. *Let E be the divisor of $\Pi^{-1}(s(T))$ with reduced structure. There exists a reduced divisor $S \leq E$ such that $\Pi|_S$ is a locally trivial deformation and S_t is the sum of all maximal strings of rational curves at the ends of E_t for any $t \in T$.*

The singularities of $\mathcal{X}_t$ outside $s(t)$ are rational double points of type A_n.

The strong aspect of this theorem is that it not only produces a simultaneous resolution, but it also imposes restrictions on the variation of the topology of the exceptional divisor along the family. For example the following Corollary is stated in [100]:

Corollary 3.3.33 *Assume that $(-P^2)$ is constant and that $\mathcal{X}_0$ has a star shaped resolution graph. Then σ is an equisingular deformation (it is topologically trivial).*

This motivates the analogue of Question 3.3.11, but replacing K^2 by $(-P^2)$.

3.3.9 Newton Non-degenerate Embeddings

A much stronger notion of simultaneous resolution is the following

Definition 3.3.34 (Leyton, Mourtada, Spivakovsky [73]) Let $f_t : \mathbb{C}^n \to \mathbb{C}$, $t \in T$ be a family of isolated singularities with T smooth. Define $F(x,t) := f_t(x)$, let $\mathcal{X} := V(F) \subset \mathbb{C}^n \times T$. A *simultaneous embedded resolution* of $\mathcal{X}$ is a proper modification $\Pi : Y \to \mathbb{C}^n \times T$ such that the strict transform of $\mathcal{X}$ is a very weak simultaneous resolution of $\mathcal{X}$, and the total transform of $\mathcal{X}$ is *normal crossings relative* to T.

Normal crossings relative to T means essentially that each fibre over t is normal crossings in a locally trivial way over T (see [73] for a precise statement). The important point is that simultaneous embedded resolution implies topological triviality of the family.

A result related to the question above is contained in a very recent preprint:

Theorem 3.3.35 (Leyton, Mourtada, Spivakovsky [73]) *Assume that f_t is a deformation such that for f_t is Newton non-degenerate. Then the deformation μ-constant if and only if it admits a simultaneous embedded resolution. In that case it is topologically trivial.*

For the definition of Newton non-degenerate we refer to [73]. Newton non-degenerate hypersurface singularities form a special but very important class of singularities. The theorem above is relevant for two reasons: first it brings hope to the program of resolving the Lê-Ramanujam problem by resolution techniques. The second reason is philosophically more important: as conjectured by Teissier within its program of toric resolution of singularities in arbitrary characteristic after adequate re-embedding, Tevelev [134] proved that any singularity defined over a field of characteristic 0 can be re-embedded in such a way that the existence of toric resolution is guaranteed (this is the effect of the Newton non-degenerate condition for hypersurfaces). So, a natural open question is to

Problem 3.3.36 Generalize Leyton, Mourtada, Spivakovsky to appropriately re-embedded μ-constant families.

3.4 Topological Triviality for Families of Non-isolated Singularities

3.4.1 The Structure of Milnor Fibre of Non-isolated Hypersurface Singularities and Topological Triviality

A natural question is to find a numerical invariant of a holomorphic function germ with non-isolated singularities whose constancy for a family f_t implies, at least conjecturally, topological $\mathcal{R}$-triviality, or constant embedded topological type. In this direction D. Massey introduced the Lê cycles and Lê numbers in [78–80]. All the results due to Massey that I will summarize below were published originally at [78, 79], but the book [80] is an excellent account of all of them.

Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a holomorphic germ. Denote by Σ_f be the critical set of f. For a coordinate system $\mathbf{z} = (\mathbf{z_1}, \ldots, \mathbf{z_n})$ we define the relative polar variety $\Gamma^k_{f,\mathbf{z}}$ to be the closure $\overline{V(\partial f/\partial z_{k+1}, \ldots, \partial f/\partial z_n) \setminus \Sigma_f}$. Notice that in Massey's notation at [78–80], he considers functions from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ and numbers the variables as $z_0, \ldots, z_n$, unlike in this paper that are numbered by $z_1, \ldots, z_n$. This produces an index shifting in our formulae with respect to his.

Given a variety X we denote by $[X]$ the cycle associated with it. We define the k-th *Lê cycle* associated with f and $\mathbf{z}$ to be the difference

$$\Lambda^k_{f,\mathbf{z}} := [\Gamma^{k+1}_{f,\mathbf{z}} \cap V(\partial f/\partial z_{k+1})] - [\Gamma^k_{f,\mathbf{z}}].$$

Massey proved that $\Lambda^k_{f,\mathbf{z}}$ is supported in the critical set Σ_f for any k, and that for what we called *prepolar coordinate system* the cycle $\Lambda^k_{f,\mathbf{z}}$ is k-dimensional. Massey proved that prepolar coordinate systems are generic (a generic linear change of coordinates yields a prepolar coordinate system). For a prepolar coordinate system we define the k-th Lê number as the intersection number

$$\lambda^k_{f,\mathbf{z}} := \Lambda^k_{f,\mathbf{z}} \cdot [V(z_1, \ldots, z_k)].$$

The following properties are important:

1. The Lê numbers depend on the choice of prepolar coordinate system. However there is a Zariski open subset of *generic* coordinate systems for which the Lê numbers coincide.
2. If $\dim(\Sigma_f) = s$ then $\lambda^k_{f,\mathbf{z}} = 0$ if $k \geq s$.
3. If f has an isolated singularity at the origin then the only non-zero Lê number is $\lambda^0_{f,\mathbf{z}} = \mu(f)$, so Lê numbers reduce to the Milnor number for any isolated hypersurface singularity.

D. Massey succeeded to prove the following generalization of Lê-Ramanujam Theorem (see 3.3.7, part (1) above):

Theorem 3.4.1 (Massey) *Let f_t be a family of holomorphic germs depending holomorphically on a parameter t, Let s be the dimension of the critical set of f_0. If there is a coordinate system $\mathbf{z}$ which is prepolar for any t, such that the all Lê numbers $\lambda^k_{f_t,\mathbf{z}}$ are constant, then all the Milnor fibrations of f_t are homologically equivalent (the integral homology of the Milnor fibres together with the monodromy action are conjugate). If $n > s+3$ then the Milnor fibrations are smoothly equivalent (conjugate by a diffeomorphism preserving the projection to the base circle).*

In order to understand why the theorem above generalizes Lê-Ramanujam result an explanation is needed: in the sketch of the proof of the Lê-Ramanujam Theorem 3.3.7 we noticed that a very important intermediate step was the proof of the equivalence of Milnor fibrations. In fact this is what Massey shows. The dimension restriction is the needed one in order that the cobordism (3.1) is an h-cobordism. The direct attempt to generalize Lê-Ramanujam proof of the fact that f_0

and f_t have the same topological type breaks down because the map (3.2) is not a trivial fibration in this case, because of the non-isolated singularities of the 0-fibre.

In fact it is false that constant Lê numbers imply constant topological type in general. Before we explain the counter-examples we will disgress on the structure of the Milnor fibre for non-isolated singularities.

On the positive side we have the connectivity result of Kato and Matsumoto [55]:

Theorem 3.4.2 (Kato, Matsumoto) *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a homolomorphic function germ with s-dimensional critical set. Then the Milnor fibre is $(n-2-s)$-connected. The bound is sharp.*

A proof can be achieved easily by polar methods, yielding an induction based on restriction to successive hyperplane sections. Massey improved the last result with the following handle structure theorem, which generalizes Theorem 3.2.3.

Theorem 3.4.3 (Massey) *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a homolomorphic function germ with s-dimensional critical set ($s < n-1$), and $\mathbf{z}$ be a prepolar coordinate system. Then the Milnor fibre is a $2(n-1)$-dimensional ball with $\lambda^k_{f,\mathbf{z}}$ $(n-1-k)$-handles attached for $k = 0, \ldots, s$.*

Despite these two positive results the homotopy type of the Milnor fibre can be arbitrarily complicated (see [32]):

Theorem 3.4.4 *For any germ $(Z, O) \subset (\mathbb{C}^N, O)$ of complex analytic subset there exists a function germ f whose Milnor fibre has the local homotopy type of $\mathbb{C}^N \setminus Z$ at O. In particular there exist simply connected non-formal Milnor fibres.*

In the theorem above, if we have $Z = V(F_1, \ldots, F_k)$, then we can choose $f = \sum_{i=1}^k F_i y_i : \mathbb{C}^N \times \mathbb{C}^k \to \mathbb{C}$, where $(y_1, \ldots, y_k)$ is a coordinate system for $\mathbb{C}^k$. The second assertion can be deduced from the first given Denham and Suciu's construction of simply connected non-formal complements [24]. Previously Zuber provided the first example of a non-formal, non-simply connected Milnor fibre in [156].

Finding classes of non-isolated singularities where one can find a reasonably complete understanding of the Milnor fibre is not an easy task. A succesful program in this direction was pioneered by Siersma [123, 124, 126–128] and developed by de Jong [23], Zaharia [152], Némethi [88, 89], Shubladze [122], Marco-Buzunariz and the author [28, 29, 37].

We summarize the method very briefly:

Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$, consider a deformation f_s of f, and take the mapping $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ defined by $F(x, s) := (f_s(x), s)$. Let $\epsilon > 0$, $\delta > 0$ be the radii for the Milnor ball and disc of f respectively. Denote by Γ be the critical set and by Δ the discriminant of the map F (the image of its critical set). We say that the deformation f_s is *admissible* if there exists $\eta > 0$ such that the restriction

$$F| : (B_\epsilon \times D_\eta) \cap F^{-1}(D_\delta \times D_\eta \setminus \Delta) \to D_\delta \times D_\eta \setminus \Delta \quad (3.6)$$

is a locally trivial fibration. In this case the Milnor fibration of f can be studied in terms of the restriction $f_s|_{B_\epsilon \cap f_s^{-1}(D_\delta)}$. The critical set of this function equals $\Gamma_s := \Gamma \cap B_\epsilon \times \{s\}$ and the discriminant equals $\Delta_s := \Delta \cap D_\delta \times \{s\}$. The critical set Γ_s splits as a dijoint union of the singular set Σ_s of $f_s^{-1}(0)$, which is non-isolated, and finitely many isolated points, which are of Morse type for a well chosen deformation f_s. Then Sierma's "homology splitting" is satisfied: denote by ν the number of Morse points appearing in the critical set of f_s and let T_s be a (small enough) tubular neighborhood of Σ_s, the homology of the Milnor fibre splits as follows

$$H_{n-1}(f^{-1}(\delta); \mathbb{Z}) \cong H_n(T_s, f_s^{-1}(\delta) \cap T_s); \mathbb{Z}) \oplus \mathbb{Z}^\nu,$$

$$H_k(f^{-1}(\delta); \mathbb{Z}) \cong H_{k+1}(T_s, f_s^{-1}(\delta) \cap T_s); \mathbb{Z})$$

for $k < n - 1$. Usually the critical set Σ_s admits a stratification by "transversal singularity types", which have relatively simple Milnor fibres, the topology of the stratification can be understood, and formulae for the Betti numbers of the pair $((T_s, f_s^{-1}(\delta) \cap T_s)$ in terms of the stratification and of the transversal singularity types can be deduced.

For the purpose of this survey we are interested in a particular situation where the method described above has been successful. Let $g_1, \ldots, g_{n-k}$ be a collection of function germs in $\mathcal{O}_{\mathbb{C}^n,O}$ defining a k-dimensional isolated complete intersection singularity (i.c.i.s.) in $\mathbb{C}^n$. The map $g := (g_1, \ldots, g_k) : (\mathbb{C}^n, O) \to \mathbb{C}^k$ has a Milnor fibration, and the Milnor fibre, as in the hypersurface case, is homotopy equivalent to a bouquet of spheres of dimension $(k-1)$ (see [74] for a proof and a comprehensive account of i.c.i.s. theory). We denote by $\mu(g)$ the number of spheres. Denote by I the ideal generated by $g_1, \ldots, g_{n-k}$. Denote by $\Theta_{I,e}$ the submodule of vector fields tangent to the i.c.i.s. $V(g_1, \ldots, g_{n-k})$. A function $f \in I^2$ is called of finite dimension with respect to I if the number

$$c_{I,e}(f) := \dim_{\mathbb{C}}(I^2/\Theta_{I,e}(f))$$

is finite. Notice that $\Theta_{I,e}(f))$ is a sort of "Jacobian ideal relative to the i.c.i.s.", and in this sense the number above is a generalization of the Milnor number.

Given $f \in I^2$ we express f as

$$f = (g_1, \ldots, g_{n-k})(h_{i,j})(g_1, \ldots, g_{n-k})^t,$$

where $(h_{i,j})$ is a $k \times k$ symmetric matrix of function germs. If $c_{I,e}(f)$ is finite any deformation f_s of the form

$$f = (g_{1,s}, \ldots, g_{n-k,s})(h_{i,j,s})(g_{1,s}, \ldots, g_{n-k,s})^t,$$

where $g_{1,s}, \ldots, g_{n-k,s}$ and $h_{i,j,s}$ are arbitrary deformations of the corresponding functions, is admissible. Taking the deformations $g_{1,s}, \ldots, g_{n-k,s}$ and $h_{i,j,s}$ generic (here generic means that for a certain k the k-th Taylor expansion of the functions

$g_{1,s}, \ldots, g_{n-k,s}$ and $h_{i,j,s}$ are outside a Zariski open subset of the corresponding space of jets) we achieved the situation where Σ_s coincides with the Milnor fibre of the i.c.i.s., the stratification by transversal type in Σ_s coincides with the stratification by rank of the matrix $(h_{i,j})$, and the transversal singularities are of type $D(k, p)$, which has the following normal form (see [106]):

$$\sum_{1\leq i\leq j\leq p} x_{i,j} y_i y_j + \sum_{p+1\leq i\leq n-k} y_i^2 = 0,$$

where $\{x_{i,j}\}_{1\leq i\leq j\leq p} \bigcup \{y_i\}_{1\leq i\leq n-k}$ is an independent system of linear forms in $\mathbb{C}^n$. This was proved for i.c.i.s. of dimension at most two by Siersma [124], Zaharia [152] and Némethi [88], and in general in [28, 29].

The deformation above gives a way to study the homology and the homotopy type of the Milnor fibre of f. This was archived by Siersma [124] in dimension 1, by Némethi [88] and Zaharia [152] in dimension 2, and by Marco-Buzunariz and the author [37] in dimension 3. In all these cases it is proved that the Milnor fibre is a bouquet of spheres of different dimensions (in contrast with the very general homotopy type that the Milnor fibre of a non-isolated singularity can have (see Theorem 3.4.4) Here we only quote the result for dimension 3, and only in the cases that have impact in equisingularity questions (the reader may consult [37] for complete statements):

Theorem 3.4.5 *Let μ_0 and μ_1 be the Milnor numbers of the i.c.i.s. $(g_1, \ldots, g_{n-3})$ and $(\det(h_{i,j}), g_1, \ldots, g_{n-3})$. Denote by $\mathfrak{a}$ the number of points $x \in \Sigma_s$ such that $corank(h_{i,j,s}(x)) \geq 2$. The homology of the Milnor fibre is the following:*

- *If $corank(h_{i,j}(0)) \geq 3$:*

$$H_{n-1}(\mathbf{F}_f; \mathbb{Z}) \cong \mathbb{Z}^{\mu_0+2\mu_1-4\mathfrak{a}+1+\#A_1},$$

$$H_k(\mathbf{F}_f; \mathbb{Z}) = 0$$

if $1 \leq k \leq n-2$,

$$H_0(\mathbf{F}_f; \mathbb{Z}) \cong \mathbb{Z}.$$

- *If $corank(h_{i,j}(0)) = 2$:*

$$H_{n-1}(\mathbf{F}_f; \mathbb{Z}) \cong \mathbb{Z}^{\mu_0+2\mu_1-4\mathfrak{a}+2+\#A_1},$$

$$H_{n-2}(\mathbf{F}_f; \mathbb{Z}) \cong \mathbb{Z},$$

$$H_k(\mathbf{F}_f; \mathbb{Z}) = 0$$

if $1 \leq k \leq n-3$,

$$H_0(\mathbf{F}_f; \mathbb{Z}) \cong \mathbb{Z}.$$

Remark 3.4.6 The formula computing the top Betti number has a "−" sign. So in principle "different geometries" in the central fibre could lead to Milnor fibres with the same Betti numbers. If such a behavior could be exhibited in a family of hypersurface singularities then it could lead to a family with the same Milnor fibration, but not topologically equisingular. With a bit of luck such a family could have constant Lê numbers.

The idea expressed in the remark above was the starting points to the series of counter-examples described in [30] and that we explain now. Choose $l < k$ and consider the family of function germs

$$f_t := y_1^2 x_3 + y_1 y_2 x_2 + y_2^2 (x_1^k + t x_2^l + x_3) : (\mathbb{C}^5, O) \to \mathbb{C}. \tag{3.7}$$

Define $I := (y_1, y_2)$. For any t the number $c_{I,e}(f_t)$ is finite. On the other hand, with the notation of the previous theorem $\mu_0(f_t) = 0$ for any t, $\mu_1(f_t) = 2l$ for $t \neq 0$, $\mu_1(f_0) = 2k$, $a(f_t) = l$ for $t \neq 0$ and $a(f_0) = k$. So according with the previous theorem the Betti numbers of the Milnor fibre are constant in the family. On the other hand if one looks at the geometry of the family of determinants $x_3(x_3+x_1^k+tx_1^l)+x_2$ one sees a deformation from the A_k to the A_l singularity, which is not topologically trivial. A similar family is given by

$$f_t = y_1^2 x_3 + y_1 y_2 x_2 + y_2^2 (t x_1 + x_3), \tag{3.8}$$

in this case f_0 is not finite-dimensional with respect to I, but one may think this family as a "degenerate case" of the previous family. Using the remarks above, in [30] it is proved:

Example 3.4.7 The families (3.7) and (3.8) above have constant generic Lê numbers, the generic Lê numbers of the restrictions to generic linear sections in any dimension are constant, the family is linear in the parameter t, the Milnor fibration for $t = 0$ is smoothly equivalent to the Milnor fibration for $t \neq 0$, but $V(f_0)$ and $V(f_t)$ (for $t \neq 0$) are not homeomorphic.

The above counterexamples do not satisfy the condition $n > dim(\Sigma)+3$, but taking suspensions one obtains counterexamples for which the condition hold. Although the formulae of Theorem 3.4.5 were proved later than the counterexamples above, they were conjecturally known to the author by the moment of finding the counterexamples.

On the other hand, on the positive side in [30] it is proved:

Theorem 3.4.8 *Suppose* $n \geq 4$. *Let* $f_t : (\mathbb{C}^n, O) \to \mathbb{C}$, *let* $\mathbf{z}$ *be a prepolar system for any* t. *Suppose that the Lê numbers* $\lambda^k_{f_t,\mathbf{z}}$ *are independent of* t. *Then the homotopy type of the abstract link of* f_t *is independent of* t.

Combining two results above for the family $f_t = y_1^2 x_3 + y_1 y_2 x_2 + y_2^2(tx_1 + x_3)$, which is homogeneous one obtains examples of homotopically trivial families of real algebraic varieties (the abstract link for a fixed radius) which are not topologically trivial.

Now, notice that the projective hypersurface in $\mathbb{P}^{n-1}$ defined by a homogeneous polynomial f in n variables is the quotient of the abstract link $V(f) \cap \mathbb{S}_\epsilon$ by a free $\mathbb{S}^1$ action. Exploiting this observation one proves (with some work) that for f_t as in formula (3.8) the family of projective hypersurfaces $V(f_t) \subset \mathbb{P}^4$ is homotopically trivial but not topologically trivial. The projective hypersurfaces appearing in this family are not normal. Considerations arising from the study of universal covers of projective varieties motivated the search of normal counterexamples. J. Kollár and the author provided in [36] the following family of homogeneous cubic polynomials:

Example 3.4.9

$$f_t(x_1, x_2, x_3, y_1, y_2, y_3) := (y_1, y_2, y_3) \cdot \begin{pmatrix} tx_1 & x_2 & x_3 \\ x_2 & tx_3 & x_1 \\ x_3 & x_1 & tx_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The induced family of projective hypersurfaces is homotopically trivial around any $t \in \mathbb{C}_t \setminus \{0, -2, -2\xi, -2\xi^2\}$ where ξ is a third root of unity. On the other hand it is not is not topologically locally trivial in any neighborhood of t if $\xi' t^3 - 3t + 2\xi' = 0$ for some third root of unity ξ' (for example $t = 1$ is one such value).

Another indication that the Lê numbers are not so linked to the topology of a hypersurface as the Milnor number in the isolated case is the fact that they are not topological invariants. The first counterexample was found by Gaffney and the author in [34].

Example 3.4.10 The family

$$f_t(x, y, z) := (x^{15} + y^{10} + z^6)^2 - (xy + tz)^{12}$$

is topologically $\mathcal{R}$-trivial, but the generic Lê numbers are not constant in the family.

The idea to construct the example above was: start with the μ-constant family of i.c.i.s. $V(x^{15} + y^{10} + z^6, xy + tz)$, discovered by Henry (appearing in [16]), that has non-constant multiplicity. Combine the functions to create a function germ which is singular at the i.c.i.s. with prescribed transversal type A_5 outside the origin. Then it is easy to see that the top generic Lê number, which equals the transversal Milnor number times the multiplicity of the singular set, is not constant. The topological $\mathcal{R}$-triviality is shown by ad-hoc arguments.

3.4.2 Equisingularity at the Critical Set

While in the case of isolated singularities the Milnor fibration determine completely the embedded topological type, in the case of non-isolated ones the examples above show that the connection between the Milnor fibration and even the abstract topological type is weak. We propose here a new approach in order to characterize topological equisingularity. This approach stems from [31].

Let $f : (\mathbb{C}, O) \to \mathbb{C}$ be a holomorphic function germ with critical set Σ. Let R be a coefficient ring (usually we take $R = \mathbb{Z}, \mathbb{Q}$). Denote by $\phi_{f,R}$ the vanishing cycles with coefficients in R. This is a cohomologically constructible complex supported in the critical set Σ; it may be endowed with the monodromy action. At any point $x \in \Sigma$ the cohomology of the stalk $(\phi_{f,R})_x$ coincides with the cohomology of the local Milnor fibration of the germ of f at x. We stratify Σ_f according with the cohomology of the stalks as follows: let H be a $\mathbb{Z}$-graded R-module. Define

$$\Sigma_f^H := \{x \in \Sigma_f : H^*((\phi_{f,R})_x) \cong H\}.$$

This induces a finite partition of Σ and it is natural to expect that Σ_f^H is locally closed in Σ_f.

Given a family of function germs $f_s : (\mathbb{C}, O) \to \mathbb{C}$, we define the function

$$\overline{F} : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

by $\overline{F}(x, s) := (f_s(x), s)$. The intersection of the critical set $\Sigma_{\overline{F}}$ with the slice $\mathbb{C} \times \{s\}$ is the critical set Σ_s of the function f_s.

Definition 3.4.11 The family f_s is *equisingular at the critical set for the coefficient ring* R if there is a homeomorphism germ

$$\Phi : (\Sigma_{f_0} \times \mathbb{C}, (O, 0)) \to (\Sigma_F, (O, 0))$$

such that for any $\mathbb{Z}$-graded R-module H we have the equality $\Phi : (\Sigma_{f_0}^H \times \{s\}) = \Sigma_{f_s}^H$.

Obviously, topological $\mathcal{R}$-triviality implies equisingularity at the critical set for any coefficient ring. The examples 3.4.7, 3.4.9 are not equisingular at the singular set. I conjecture the following:

Conjecture 3.4.12 (Topological Equisingularity Conjecture) A family of function germs f_s holomorphically dependent on a parameter is topologically $\mathcal{R}$-trivial if and only if it is equisingular at the critical set for the coefficient ring $R = \mathbb{Z}$.

The version of the above conjecture for coefficient ring $\mathbb{Q}$ is also open and very interesting in my opinion.

The whole picture would look better if, in addition, the following natural (and probably too optimistic) conjecture holds:

Conjecture 3.4.13 (Smoothness Conjecture) Choose $R = \mathbb{Z}, \mathbb{Q}$ Let f be a function germ and H a graded R-module. Then Σ_f^H is a smooth locally closed subset of Σ_f.

The smoothness conjecture has proved to be very hard even assuming that Σ_f is irreducible and 1-dimensional. For partial results towards its solution, see [49, 81].

The Topological Equisingularity Conjecture for critical set of dimension 1 has been studied in [31]:

Theorem 3.4.14 *Let $f_s : (\mathbb{C}^n, O) \to \mathbb{C}$ be a family of function germs holomorphically dependent on a parameter s, such that $\dim(\Sigma_{f0}) = 1$, $n \geq 5$, and such that there exists a homeomorphism germ*

$$\Phi : (\Sigma_{f0} \times \mathbb{C}, (O, 0)) \to (\Sigma_F, (O, 0))$$

with the following properties

1. *it preserves the origin of the germ, that is $\Phi(O, s) = (O, s)$,*
2. *Σ_{f_s} is smooth away from the origin for any s,*
3. *for any $x \in \Sigma_{f0} \setminus \{O\}$ and for any s the Milnor number of the restriction of f_0 to a generic hyperplane section through x equals the Milnor number of the restriction of f_s to a generic hyperplane section through $\Phi(x, s)$.*

Then the family f_s is topologically $\mathcal{R}$-trivial. Moreover the restrictions of trivializing homeomorphism at the complement of the critical set and at the critical set minus the origin, can be chosen to be smooth.

The hypothesis $n \geq 5$ comes from the use of h-cobordism theorem. A consequence of this result is that, if the critical set is of dimension 1 then the Smoothness Conjecture implies the Topological Equisingularity Conjecture. Another consequence is that when the critical set is of dimension 1 Massey's expectation that constant Lê numbers imply topological $\mathcal{R}$-triviality becomes true (see [31]):

Theorem 3.4.15 *Let $f_s : (\mathbb{C}^n, O) \to \mathbb{C}$ be a family of function germs holomorphically dependent on a parameter s, such that $\dim(\Sigma_{f0}) = 1$, $n \geq 5$. If one of the following conditions hold*

1. *there is a prepolar coordinate system **z** such that the Lê numbers at the origin of f_t with respect to **Z** are independent of t,*
2. *the generic Lê numbers at the origin of f_t are independent of t,*

then the family f_s is topologically $\mathcal{R}$-trivial. The trivializing homeomorphism can be assumed to have the same smoothness properties than in the previous theorem.

It is worth to mention that for the proof of the two theorems above a new trivializing device, called *Cuts* is introduced. It allows to prove results slightly stronger that those stated above, in the sense that the trivializing homeomorphims are global in the space of parameters of the family f_s, as long as it is contractible. See [31] for details. The technique of cuts was extended, in collaboration with M. Pe

Pereira [38] to prove the following topological triviality result for hypersurfaces in $\mathbb{C}^3$ with "topologically constant normalization". The interest of this result relies on the facts that in the surface case the h-cobordism or s-cobordism theorems are not available, and that controlling the variation of the fundamental group of the link in a family is very hard. So the condition "topologically constant normalization" may be seen as a convenient replacement for the lack of enough topological techniques.

Let $f_s : (\mathbb{C}^3, O) \to \mathbb{C}$ be a family of holomorphic function germs depending holomorphically on a parameter $s \in S$. Let $X = V(F)$, where $F(x, s) : f_s(x)$. Let $n : \hat{X} \to X$ be the normalization, and let $\hat{\Sigma}_F := n^{-1}(\Sigma_F)$, $\hat{S} := n^{-1}(\{O\} \times S)$. The family $f_s : (\mathbb{C}^3, O) \to \mathbb{C}$ is equisingular at the normalisation if there is a homeomorphism

$$\alpha : (\hat{X}, \hat{\Sigma}_F, \hat{S}) \to (\hat{X}_0, (\hat{\Sigma}_F)_0, \hat{S}_0) \times S$$

such that we have the equality $\alpha(\hat{X}_s, (\hat{\Sigma}_F)_s, \hat{S}_s = (\hat{X}_0, (\hat{\Sigma}_F)_0, \hat{S}_0) \times \{s\}$ for any s. It is easy to show that topological $\mathcal{R}$-triviality implies equisingularity at the normalization. The following converse is the main result of [38]:

Theorem 3.4.16 *Equisingularity at the normalisation and equisingularity at the critical set implies topological $\mathcal{R}$-triviality.*

In [38] the reader can find further consequences of this result, in particular in connection with topological triviality for families of maps

$$\varphi_s : (\mathbb{C}^2, O) \to (\mathbb{C}^3, O).$$

In the following example, published originally in [31], we show a topologically $\mathcal{R}$-trivial family whose critical set undergoes drastic analytic changes (it deforms non flatly and gets smoothed). It seems to be an indication that a purely numerical characterization of topological $\mathcal{R}$-triviality is unlikely to exist, or at least hard to find.

Consider the following deformation of a parametrisation in $\mathbb{C}^3$ (with deformation parameter t):

$$x = s^3 \qquad y = s^4 \qquad z = ts. \tag{3.9}$$

The following equations in $\mathbb{C}\{t, x, y, z\}$ define the image $Z \subset \mathbb{C} \times \mathbb{C}^3$ of the family as a set:

$$ty - xz = 0 \qquad tx^3 - y^2z = 0 \qquad y^3 - x^4 = 0 \qquad t^3x - z^3 = 0. \tag{3.10}$$

Example 3.4.17 The family

$$f_t := (ty - xz)^9 + (tx^3 - y^2z)^4 + (y^3 - x^4)^3 + (t^3x - z^3)^{12} \tag{3.11}$$

is equisingular at the critical set and has the following remarkable properties:

- For any t its critical set is the image of the parametrisation (3.9). Therefore, in this example the family of *reduced* critical sets is not flat (the fibre of $\pi : \Sigma_{res} \to \mathbb{C}$ at 0 has an embedded component at the origin).
- The critical set $(\Sigma_t)_{red}$ is smooth for $t \neq 0$ and singular (of multiplicity 3) for $t = 0$.
- The transversal Milnor number is 6 and, hence the first Lê number with respect to a generic coordinate system is 6 for $t \neq 0$ and 18 for $t = 0$.

Taking suspensions and applying Theorem 3.4.14 we obtain topologically $\mathcal{R}$-trivial families whose critical set has the same properties.

3.4.3 Series of Singularities

A phenomenon that was discovered by Arnol'd [4, 5] while working in his classification of low Milnor number singularities, is that isolated hypersurface singularities are naturally grouped into series of singularities, which are families of increasing Milnor number, but that tend to share similar geometric properties. At the limit of a series usually one finds a non-isolated singularity, which governs the geometric behavior of the series.

The Lê-Yomdin series associated with a non-isolated singularity with 1 dimensional critical set is an example of this phenomenon. Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a holomorphic function germ with a 1 dimensional critical set Σ. Let $l : \mathbb{C}^n \to \mathbb{C}$ be a linear function vanishing at the origin, but not vanishing identically at Σ. Then for $k >> 0$ the function

$$f_k := f + tl^k$$

has an isolated singularity for t small enough, and its Milnor number is independent of t. The sequence of functions f_k (or rather the sequence of μ-constant strata), is the Lê-Yomdin series associated with f. The Milnor number, homology of the Milnor fibre, monodromy, and even deeper invariants like the spectrum of f_k or motivic invariants can be recovered from those of f and k, see [42, 116, 125, 130].

The idea of Lê-Yomdin series was extended by Massey to singularities f with critical set of arbitrary dimension [80]

Theorem 3.4.18 (Massey) *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a holomorphic function germ with a k-dimensional critical set Σ. Let $\mathbf{z} = \{\mathbf{z_1}, \dots, \mathbf{z_n}\}$ be a prepolar coordinate system. Consider the coordinate system $\mathbf{z}' = \{\mathbf{z_2}, \dots, \mathbf{z_n}, \mathbf{z_1}\}$. There is $k_0 \geq 0$ such that if $k > k_0$ and $t > 0$, then $f + tz_1^k$ has critical set of dimension $s - 1$, the coordinate system $\mathbf{z}'$ is prepolar for $f + tz_1^k$ and the Lê numbers of f and $f + tz_1^k$*

are related by the following formulae:

$$\lambda^0_{f+tz_1^k,\mathbf{z}'} = \lambda^0_{f,\mathbf{z}} + (k-1)\lambda^1_{f,\mathbf{z}},$$

$$\lambda^i_{f+tz_1^k,\mathbf{z}'} = \lambda^{i+1}_{f,\mathbf{z}}$$

for $1 \leq i \leq s-1$.

An immediate application of the last theorem allows to produce μ-contant families by adding sufficiently high powers of variables to a family with constant Lê numbers. It turns out that often, even if the power of the variable is not bigger than the k_0 predicted in the Theorem above, the predicted equality of Lê numbers still holds for t small enough. This motivated us to consider the following family in [30]:

Example 3.4.19 The family

$$g_t := y_1^2x_3 + y_1y_2x_2 + y_2^2(tx_1 + x_3) + x_1^4 + x_2^4 + x_3^4$$

is μ^*-constant, but the projectivized tangent cones for $t = 0$ is not homeomorphic to the projectivized tangent cone for $t \neq 0$. So g_t is a counterexample to Zariski's Question B (see Question 3.3.1, (B)).

The counterexamples above are of high dimension. Lower dimensional counterxamples were found by Artal, Luengo, Melle and the author in [7], as a consequence of an study of the Milnor number weighted Lê-Yomdin singularities, which represents another generalization of Lê-Yomdin series:

Example 3.4.20 The families

$$h_t := z^{12} + zy^3x + ty^2x^3 + x^6 + y^5,$$

$$h_t + w^5$$

are μ-constant, topologically equisingular, but the projectivized tangent cones for $t = 0$ is not homeomorphic to the projectivized tangent cone for $t \neq 0$. So h_t and $h_t + w^5$ are 2 and 3-dimensional counterexamples to Zariski's Question B (see Question 3.3.1, (B)).

The philosophy underlying the construction of the examples above is: produce a family of function germs f_s with non-isolated singularities with constant Lê number and some exotic behavior. Then adding convenient powers of linear functions one may obtain a μ-constant family that inherits some exotic behavior as well. This can be used for example as a method to produce μ-constant families that are not μ^*-constant (therefore they are not Whitney equisingular). The first example of this behavior was discovered long ago by Briançon and Speder, and one can see that indeed fits in the scheme just described. Some examples found following this inspiration, and checked using the computer program SINGULAR, follow below (the bold-face monomial plays the role of the power of the linear function):

Example 3.4.21 (Briançon-Speder)

$$z^5 + ty^6z + y^7x + \mathbf{x^{15}} = 0$$

The following example was found by Luengo and Melle together with the author:

Example 3.4.22

$$z^7 + y^7 + ty^5x^3 + \mathbf{x^{10}} = 0$$

The following examples, again due to Luengo and Melle and the author are essentially non-linear in the parameter:

Example 3.4.23

$$(x^3 + txy^3 + y^4z + z^A)^2 + xy^B + \mathbf{x^C} = 0$$

Here we checked $(A, B, C) \in \{(9, 9, 6), (9, 8, 6), (9, 8, 7), (10, 8, 6), (11, 8, 6), (11, 8, 7)\}$. Surely many more values work. For each of the values above the sequence of numbers $(\mu, \mu(1)_{t=0}, \mu(1)_{t\neq 0})$ is

$$(761, 45, 42), (707, 43, 41), (782, 43, 42), (785, 43, 41), (863, 43, 41), (960, 43, 42)$$

respectively.

Arnol'd observed the phenomenon of series of singularities, and although it was not possible to give a precise definition of what a series of singularities is, it became clear that series are associated with singularities of infinite codimension (non-isolated singularities). Inspired by Arnol'd remark C.T.C Wall formulated a conjecture (see the Conjecture at the introduction of [146], p. 463), or rather, a guiding principle for classification of singularities. Later D. Mond [84] introduced the idea of singularity stem, as a first step to understand the notion of series of singularities in the context of mappings from $\mathbb{C}^2$ to $\mathbb{C}^3$ (the idea is that a singularity stem is a non-isolated singularity which lies in the limit of a series of singularities). Following a suggestion of Montaldi, Pellikaan [107] defined inductively the notion of stem of degree d, characterized stems of degree 1 as functions with irreducible 1-dimensional critical set and transversal type A_1, and proved that any stem of finite degree is a function with 1-dimensional critical set. They were also able to give bounds on the degree of the stem depending on the number of irreducible components of the critical set of the function and the transversal Milnor number. In [31] we modified the concept of stem and define topological stem as follows:

Definition 3.4.24 Let d be a positive integer. We define topological stems of degree d inductively as follows:

- A holomorphic function germ $f : (\mathbb{C}^n, O) \to \mathbb{C}$, is a *topological stem of degree* 1 if there exists a positive integer N such that for any $g \in \mathbf{m}^N$, and $t \in \mathbb{C}$ sufficiently small, either $f + tg$ has an isolated singularity at the origin, or it is topologically R-equisingular to f.
- A holomorphic function germ $f : (\mathbb{C}^n, O) \to \mathbb{C}$ is a *topological stem of degree* d if there exists a positive integer N such that for any $g \in \mathbf{m}^N$, and $t \in \mathbb{C}$ sufficiently small, either $f + tg$ is a stem of degree strictly smaller than d, or it is topologically R-equisingular to f.

With this notion we proved in [31]:

Theorem 3.4.25 *A holomorphic function germ $f : (\mathbb{C}^n, O) \to \mathbb{C}$, $n \geq 5$, is a topological stem of positive finite degree if and only if its critical set is* 1*-dimensional at the origin. Moreover the degree of the stem is bounded above by the generic first Lê number at the origin of f.*

Our modification of the definition of stem consists essentially in replacing differentiable $\mathcal{R}$-equivalence in Pellikaan definition by topological $\mathcal{R}$-equivalence. This is natural since series in Arnol'd classification of singularities are in fact topological series (series of μ-classes), and thus it is therefore reasonable that the object to find at the limit of the series admits a topological definition as well. It is interesting to notice that Theorem 3.4.25 uses crucially in its proof Theorem 3.4.15.

3.5 Floer Homology of the Milnor Fibration

Consider in $\mathbb{C}^n$ the canonical symplectic structure ω given by the imaginary part of the standard inner product $h(z, z') := \sum_{i=1}^{n} z_i \bar{z}_i$. Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be a function germ with an isolated singularity. The Milnor fibre $f^{-1}(\delta) \cap B_\epsilon$ is a symplectic manifold whose symplectic form is the restriction of ω. It is, furthermore, a Liouville domain: there is a 1-form α_f such that the dual vector field of the restriction of α_f to the Milnor fibre with respect to the symplectic form $\omega|_{f^{-1}(\delta)\cap B_\epsilon}$ points outwards the Milnor fibre near the boundary. There is a monodromy

$$\varphi : f^{-1}(\delta) \cap B_\epsilon \to f^{-1}(\delta) \cap B_\epsilon$$

such that

(a) φ is a exact symplectomorphism, that is $\varphi^*\alpha_f = \alpha_f + dF_\varphi$, for $F_\varphi : f^{-1}(\delta) \cap B_\epsilon \to \mathbb{R}$ smooth,
(b) φ restricts to the identity at the boundary of the Milnor fibre.

In this conditions the Floer cohomology groups $HF^*(\varphi^m, +)$ of any iterate of φ are defined [121]. The chain complex defining Floer cohomology is generated by the fixed points of a suitable perturbation of φ^m. As in usual Floer theory, the Floer cohomology groups refine the count of fixed points provided by easier algebro-topological invariants like the Lefschetz number. Indeed, the Euler characteristic of $HF^*(\varphi^m, +)$ coincides with the Lefschetz number $\Lambda(\varphi^m)$.

One should notice at this point a related result of Denef and Loeser. For each $m \geq 0$, the m-th jet scheme $\mathcal{L}_m(\mathbb{C}^n)$ is the variety parametrizing morphisms

$$\gamma : Spec(\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}^n$$

of schemes over $\mathbb{C}$. We denote by $\gamma(0)$ the center of a jet γ, that is, the image in $\mathbb{C}^n$ of the closed point of $Spec(\mathbb{C}[t]/(t^{m+1})$. For a jet $\gamma \in \mathcal{L}_m(\mathbb{C}^n)$ we denote by $f(\gamma)$ the truncated power series given by the composition $f \circ \gamma$. The *m-th (restricted) contact locus of f at O* is defined to be

$$\mathcal{X}_m(f, O) := \{\gamma \in \mathcal{L}_m(\mathbb{C}^n) \mid \gamma(0) = O \text{ and } f(\gamma) \equiv t^m \pmod{t^{m+1}}\}.$$

Denef and Loeser proved in [22] that the Euler characteristic of $\mathcal{X}_m(f, O)$ coincides with the Lefschetz number $\Lambda(\varphi^m)$ (a second proof avoiding resolution of singularities and based on Hrushovski-Kazhdan motivic integration was provided recently by Hrushovski and Loeser [54]). This motivated the question of Seidel of finding a relation between the cohomology of $\mathcal{X}_m(f, O)$ and the Floer cohomology $HF^*(\varphi^m, +)$.

A crucial step in this direction is McLean's paper [77]. Let us introduce some notation before we explain his result.

Let $h : Y \to \mathbb{C}^n$ be an embedded resolution of the pair $(V(f), O)$, that is, a proper morphism from a smooth variety Y such that $E = h^{-1}(V(f))$ and $h^{-1}(O)$ are divisors with simple normal crossings and the restriction $h\colon Y \setminus h^{-1}(O) \to X \setminus \{O\}$ is an isomorphism. We denote by E_i with i in S, the irreducible components of E. We define

$$m_i = ord_f E_i \quad \text{and} \quad \nu_i = ord_{K_{Y/\mathbb{C}^n}} E_i + 1,$$

where $K_{Y/\mathbb{C}^n}$ is the relative canonical divisor defined by the vanishing of $\det dh$.

We assume that h is *m-separating*. This means, by definition, that $m_i + m_j > m$ if $E_i \cap E_j \neq \emptyset$ for all $i \neq j \in S$. An m-separating resolution exists for any m (see for example [17]). We set

$$A := \{i \in S \mid h(E_i) \subset \Sigma\},$$

$$S_m := \{i \in A \mid m_i \text{ divides } m\},$$

and

$$k_i := m/m_i$$

for each $i \in S_m$. Fix a tuple of integers $w = (w_i)_{i \in S}$ with $w_i \geq 0$ such that the divisor

$$W = -\sum_{i \in S} w_i E_i$$

is relatively very ample for h. Assume that $w_i = 0$ if E_i is not an exceptional divisor. For an integer p, we let

$$S_{m,p} := \{i \in S_m \mid w_i k_i = -p\}.$$

Let $E_i^\circ = E_i \setminus \cup_{j \neq i} E_j$. Then there exists an unramified cyclic cover $\tilde{E}_i^\circ \to E_i^\circ$ of degree m_i, given locally in a neighborhood U in Y of a point in E_i° by

$$\{(z, P) \in \mathbb{C} \times (E_i^\circ \cap U) \mid z^{m_i} = u(P)^{-1}\},$$

where $f \circ h = u \cdot y_i^{m_i}$ with y_i a local equation for E_i and u an invertible regular function on U.

McLean [77, Theorem 1.2] proved:

Theorem 3.5.1 *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$, $h : Y \to \mathbb{C}$ be as above. There exists a spectral sequence*

$$'E_1^{p,q} = \bigoplus_{i \in S_{m,p}} H_{d-1-2k_i \nu_i-(p+q)}(\tilde{E}_i^\circ, \mathbb{Z}) \quad \Rightarrow \quad HF^*(\phi^m, +) \tag{3.12}$$

converging to the Floer cohomology of the m-th iterate of the monodromy ϕ on the Milnor fiber of f.

In particular he obtains that the multiplicity is the lowest integer m such that $HF^*(\phi^m, +)$ does not vanish. This gives a symplectic interpretation of multiplicity, which represents a very promising step towards the solution of Zariski's Question 3.3.1 (A). In fact Zariski's question has a positive answer if any two germs with the same embedded topological type have the same Floer cohomology for any iterate of the monodromy. A interpretation of the log-canonical threshold in terms of Floer cohomology also can be found in [77]. We do not describe it here since it is further away from the main topic of this paper.

N. Budur, Honc Duc Nguyen, Lê Quy Thuong and the author [17] proved the following analogue statement concerning the cohomology of contact loci:

Theorem 3.5.2 *Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$, $h : Y \to \mathbb{C}$ be as above. There is a cohomological spectral sequence*

$$E_1^{p,q} = \bigoplus_{i \in S_{m,p}} H_{2(d(m+1)-k_i \nu_i -1)-(p+q)}(\tilde{E}_i^{\circ}, \mathbb{Z}) \quad \Rightarrow \quad H_c^{p+q}(\mathcal{X}_m(f, O), \mathbb{Z})$$

converging to the cohomology with compact support of the m-th contact locus of f.

We note that E_1 in this case differs from $'E_1$ by a $(2dm + d - 1)$-shift in the total degree $p + q$, hence up to relabelling, the two pages are the same. The proof of the following conjecture, by N. Budur, Honc Duc Nguyen, Lê Quy Thuong and the author, would lead to the sought identification of Floer cohomologies and cohomologies of contact loci:

Conjecture 3.5.3 The two spectral sequences $\{E_r, d_r\}_{r\geq 1}$ and $\{'E_r, 'd_r\}_{r\geq 1}$ are isomorphic, and

$$HF^*(\phi^m, +) \cong H_c^{*+2dm+d-1}(\mathcal{X}_m(f, O), \mathbb{Z})\,.$$

The conjecture is true if m is the multiplicity of f at the singularity [17] .

Proposition 3.5.4 *Let*

$$f = f_m + f_{m+1} + \dots$$

be a polynomial in d variables vanishing and with an isolated singularity the origin, where f_i are the homogeneous components of degree i, and $m > 0$ is the multiplicity of f at the origin. Then,

$$H_c^*(\mathcal{X}_m(f, O), \mathbb{Z}) \cong H_{2(dm-1)-*}(F, \mathbb{Z}),$$

where $F \simeq \{f_m = 1\}$ is the Milnor fiber at the origin of the initial form f_m of f. Moreover we have

$$HF^{*-2dm-d+1}(\phi^m, +,) \cong H_{2(dm-1)-*}(F, \mathbb{Z}).$$

The first assertion is proved by easy geometric considerations, and the second holds because both spectral sequences above degenerate for the at the first page if m is the multiplicity.

Now, if the assertion that any two germs with the same embedded topological type have the same Floer cohomology for any iterate of the monodromy is true, then the first assertion of the following conjecture of N. Budur, Honc Duc Nguyen, Lê Quy Thuong and the author would also be true:

Conjecture 3.5.5 (Corrected Zariski's Question (B)) Let $f, g : (\mathbb{C}^n, O) \to (\mathbb{C}, O)$ be two germs of holomorphic functions. If f and g are embedded topologically equivalent, then the Milnor fibers of their initial forms have the same homology. We also conjecture the stronger statement that the Milnor fibers of their initial forms have the same homotopy type.

The last conjecture can be seen as a corrected Zariski's Question 3.3.1 (B), since we replace the statement about the zero set of the initial form, by an statement about its Milnor fibre. In fact a more optimistic question would be the following

Question 3.5.6 Let $f_t : (\mathbb{C}^n, O) \to \mathbb{C}$ be a family of function germs depending holomorphically on a parameter. If the Milnor fibration along the family is constant (that is the Milnor fibrations of f_0 and f_t are smoothly equivalent for any $t \neq 0$ small enough), is it true that the Milnor fibres of the initial forms of f_0 and f_t are homotopy/homology equivalent?

A positive answer to this question would strength the implications of the fact of having constant Lê numbers.

3.6 Lipschitz Equisingularity

One of the most promising modern directions in the geometric study of singularities deals with Lipschitz geometry. It is a vast and rapidly developing subject, and a complete summary is out of the scope of the present survey. Since in this Handbook there will be other surveys touching Lipschitz Geometry here we will limit ourselves to mention a few selected developments that are very connected with the topological equisingularity questions that we have been discussing up to now.

Consider in $\mathbb{C}^n$ the standard Euclidean distance. Given a subset $X \subset \mathbb{C}^n$ we consider in X the distance obtained restricting the euclidean distance to X. This is usually called the *outer metric*, and is the one that we are considering here. The other metric that is very commonly used in Lipschitz geometry is the inner one, based on measuring lengths of paths inside X. Since it has less applications concerning equisingularity we will not discuss it here.

Studying singularities up to Lipschitz equivalence is interesting since Lipschitz equivalence is in between the topological and analytic equivalence relations. To be precise, a bi-Lipschitz homeomorphism germ $\phi : (X, O) \to (Y, O)$ is a homeomorphism germ that is Lipschitz and that has a Lipschitz inverse with respect to the metric. Given a function germ or a germ of analytic subset in $\mathbb{C}^n$, the notions of *same abstract/embedded outer Lipschitz type and outer Lipschitz $\mathcal{R}/\mathcal{RL}$-equivalence* are defined analogously to Definition 3.2.1. Given a family of function germs f_s or a deformation $\mathcal{X}$ of an analytic subgerm of $\mathbb{C}^n$, the notions of *abstract/embedded outer Lipschitz triviality and outer Lipschitz $\mathcal{R}/\mathcal{RL}$-equivalence* are defined analogously to Definition 3.3.5.

As it is obvious, the Lipschitz equivalence relations preserve more that the topological ones. Which analytic invariants are preserved by the different Lipschitz equivalent relations is an active research area nowadays. In this line a very interesting recent result of Birbrair, Fernandes, Lê and Sampaio is the Lipschitz version of Mumford's Theorem 3.3.4, which is valid for any dimension:

Theorem 3.6.1 (Birbrair, Fernandes, Lê, Sampaio [8]) *If a complex algebraic germ* (X, O) *is abstract outer Lipschitz equivalent to a smooth germ, then* X *is smooth at* O.

Remarkably, Sampaio [118] also proved that the abstract outer Lipschitz version of Zariski's Question 3.3.1 (B) is true in much greater generality:

Theorem 3.6.2 (Sampaio) *Let* (X, O) *and* (Y, O) *be subanalytic germs with the outer distance. If* (X, O) *and* (Y, O) *are abstract outer Lipschitz equivalent then their tangent cones are abstract outer Lipschitz equivalent as well.*

However, when turning to the abstract outer Lipschitz version of Zariski's Question 3.3.1 (A) the answer is positive for surfaces and negative in higher dimension.

Theorem 3.6.3 *Let* (X, O) *and* (Y, O) *be outer Lipschitz equivalent complex analytic germs of dimension at most* 2*, then their multiplicities coincide. There exists pairs* (X, O) *and* (Y, O) *of* 3*-dimensional outer Lipschitz equivalent complex analytic germs with different multiplicity.*

The positive assertion for surfaces was proved by Neumann and Pichon [98] for families of hypersurfaces with isolated singularities, and in general by Fernandes, Sampaio and the author in [33], and the counterexamples in higher dimension were found by Birbrair, Fernandes, Sampaio and Verbitsky in [9].

Concerning Neumann and Pichon result for families of hypersurfaces with isolated singularities, we should remark that their result is a shadow of a stronger result: a family of hypersurfaces of dimension 2 is outer Lipschitz trivial if it is Zariski equisingular. In fact Parusiński and Paunescu [105] obtained recently an even stronger version of this result by different methods: they improved the Lipschitz triviality and constructed a Lipschitz trivialization from the integration of a Lipschitz vector field.

However the embedded Lipschitz version of Zariski's Question 3.3.1 (A) is open in general:

Question 3.6.4 If two complex hypersurface germs are embedded Lipschitz equivalent, do they necessarily have the same multiplicity?

It is worth to remark that, unlike in the topological case, Lipschitz $\mathcal{R}$-equivalence is a very restrictive equivalence relation. This is demonstrated by the following facts:

1. While the embedded Lipschitz version of Zariski's Question 3.3.1 (A) is open, we have already mentioned that the Lipschitz $\mathcal{RL}$-equivalence version of Zariski's multiplicity question was proven affirmatively by Risler and Trotman [113].
2. While Lipschitz classification is a tame one (this is true by Mostowski result on existence of Lipschitz stratifications [85]), the classification of function

germs under Lipschitz $\mathcal{R}$-equivalence presents moduli. The first example was discovered by Henry and Parusiński [52]: the following family of function germs

$$f_s(x, y) = x^3 - 3t^2xy^4 + y^6$$

satisfies that for any sufficiently generic s, s' the germs f_s and $f_{s'}$ are not Lipschitz $\mathcal{R}$-equivalent.

3.7 An Appendix by Gert-Martin Greuel and Gerhard Pfister: History of Singular and Its Relation to Zariski's Multiplicty Conjecture

When you call SINGULAR [21], local on your computer or online, the following heading appears:

SINGULAR
A Computer Algebra System for Polynomial Computations
by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann
FB Mathematik der Universitaet, D-67653 Kaiserslautern

In fact, SINGULAR is nowadays a widely used computer algebra system for polynomial computations with special emphasis on the needs of commutative algebra, algebraic geometry, and singularity theory. However, at the beginning this was never planned, we just wanted to solve mathematical problems. Only later when we had been (partially) successful, we decided to create a system also to be used by others. The development of SINGULAR has been strongly motivated and was for a long period mainly driven by mathematical problems in singularity theory. Even its appreciated computational speed is a consequence of problems in singularity theory, which are theoretically as well as computationally very hard. It is perhaps of interest to the singularities community to see how this all came about.

It started at a time, when symbolic computations was just beginning to emerge and algorithms, in particular for local computations, were practically not existent. Moreover, our cooperation within two Germanies was anything but easy because a visit from East Germany to West Germany was not possible. Anyway, we could meet in East Berlin and we started a cooperation around 1984.

The birth of SINGULAR goes back to our efforts to generalize Kyoji Saito's well known result for hypersurface singularities (cf. [115]):

Theorem 3.7.1 (K. Saito, 1971) *Let $(X, 0)$ be the germ of an isolated complex hypersurface singularity. The following conditions are equivalent:*

1. $(X, 0)$ *is quasi-homogeneous (that is, has a good $\mathbb{C}^*$-action).*
2. $\mu(X, 0) = \tau(X, 0)$.
3. *The Poincaré complex of* $(X, 0)$ *is exact.*

Pfister (left) and Greuel at Humboldt University Berlin, 1984

If $(X, 0)$ is given by $f \in \mathbb{C}\{x_1, \dots, x_n\}$ then $\mu(X, 0) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \dots, x_n\}/j(f)$ is the Milnor number and $\tau(X, 0) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \dots, x_n\}/\langle f, j(f)\rangle$ the Tjurina number, with $j(f) = \langle \partial f/\partial x_1, \dots, \partial f/\partial x_n\rangle$.

Based on results in [43] and [44] we proved in [46] the following generalization of Saito's result to isolated complete intersection curve singularities (the result in [46] was more general for reduced Gorenstein curve singularities, with the Tjurina number replaced by the Deligne number).

Theorem 3.7.2 (G.-M. Greuel, B. Martin, G. Pfister, 1985) *If $(X, 0)$ is a reduced complete intersection curve singularity, then*

$$(X, 0) \text{ quasi-homogeneous } \iff \mu(X, 0) = \tau(X, 0)\,.$$

So we asked ourselves in [46, Problem 1] whether $(X, 0)$ is quasi-homogeneous if the Poincaré complex of $(X, 0)$ is exact (the other direction is clear). At the beginning we actually conjectured that the answer should be positive. However, we did not succeed in proving it and so we started to look for possible counter examples. But the computations by hand were very time consuming and with the small examples at hand we were unable to find any counter example. Nevertheless, we started not to believe in the conjecture.

To compute potential counter examples with a help of a computer, two main problems appeared: First, we needed Teo Mora's tangent cone algorithm (a variation of Buchberger's algorithm for local rings) to compute standard bases for $\mathcal{O}_{X,0}$-modules. However, no package for this existed at that time, not even for ideals. The second problem was more of a theoretical nature. We needed to compute the kernel of the exterior derivation in the Poincaré complex, which is only $\mathbb{C}$-linear but not $\mathcal{O}_X$-linear and hence not directly tractable by standard bases computations. Fortunately, using a result of Reiffen (see below) we had been able in [46] to reformulate the exactness of the Poincaré complex as a question of computing submodule membership and dimensions of $\mathcal{O}_{X,0}$-modules.

The first problem was more serious. There was no computer algebra system available which could compute this kind of examples. In 1984 Neuendorf and Pfister (during vacations at the Baltic sea) started an implementation of Buchberger's Gröbner basis algorithm in Basic on a ZX-Spectrum (an 8 bit home PC from Sinclair UK, 1982). It took Pfister and his student Hans Schönemann two more years of development to obtain a Modula-2 implementation of a package, called *Buchmora* at that time (Buchberger's and Mora's algorithm) for Atari computers. Using this implementation the following counter examples were found (cf. [110]).

Theorem 3.7.3 (G. Pfister, H. Schönemann, 1989) *Let $(X_{lk}, 0)$ be the germ of the unimodal space curve singularity $FT_{k,l}$ of the classification of C.T.C. Wall (cf. [147]) defined by the equations*

$$xy + z^{l-1} = xz + yz^2 + y^{k-1} = 0\,, \quad (4 \leq l \leq k,\ 5 \leq k)\,.$$

Then the Poincaré complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{X_{lk},0} \longrightarrow \Omega^1_{X_{lk},0} \longrightarrow \Omega^2_{X_{lk},0} \longrightarrow \Omega^3_{X_{lk},0} \longrightarrow 0$$

is exact, but $(X_{lk}, 0)$ is not quasi-homogeneous.

Proof To show that $(X_{lk}, 0)$ is not quasi-homogeneous, it suffices to show

$$\mu(X_{lk}, 0) = \tau(X_{lk}, 0) + 1 = k + l + 2$$

by the following formulas. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be the space curve singularity defined by $f = g = 0$, with $f, g \in \mathbb{C}\{x, y, z\}$. Then

- $\mu(X, 0) = \dim_{\mathbb{C}}(\Omega^1_{X,0}/d\mathcal{O}_{X,0}) = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/\langle f, M_1, M_2, M_3\rangle$
  $- \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$,
- $\tau(X, 0) = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/\langle f, g, M_1, M_2, M_3\rangle$,

with M_1, M_2, M_3 the 2-minors of the Jacobian matrix $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$.

On the other hand, a result of Reiffen says:

- The Poincaré complex is exact iff
  1. $\langle f, g\rangle \cdot \Omega^3_{\mathbb{C}^3,0} \subset d(\langle f, g\rangle \cdot \Omega^2_{\mathbb{C}^3,0})$, and
  2. $\mu(X, 0) = \dim_{\mathbb{C}}(\Omega^2_{X,0}) - \dim_{\mathbb{C}}(\Omega^3_{X,0})$.

All these statements could be checked with the Buchmora algorithm. □

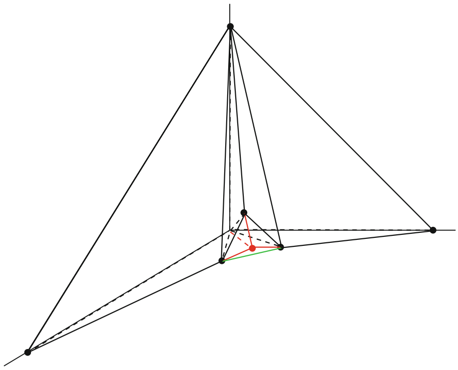
Encouraged by this success and having a computer algebra system that was able to compute in local rings, we tried to find a counter example to Zariski's multiplicity conjecture (Zariski had posed this as a question, which he supposed to have quick answer by topologists, cf. [154]).

Conjecture 3.7.4 (O. Zariski [154]) Two hypersurface singularities (given by convergent power series) with the same topological type have the same multiplicity.

The weaker family-version of this conjecture says:
In a μ–constant deformation of an isolated hypersurface singularity the multiplicity is constant.

The conjecture was already known for reduced plane curve singularities and the weaker conjecture for isolated quasi-homogeneous hypersurface singularities (cf. [45]). The methods of [45] are in principal applicable to any isolated hypersurface singularity, but we failed to prove the weak Zariski's conjecture in general. Due to the many unsuccessful efforts by us and others we were (and are still) convinced that Zariski's conjecture might not be true.

Hence, we tried to find a counterexample. The main problem is the difficulty to construct examples of μ-constant deformations. Since Zariski's conjecture is true in the semi quasi-homogeneous case and for plane curve singularities, the examples to test should be somewhat complicated. We used the Newton diagram to construct families of surface singularities where the multiplicity drops and with Newton diagram becoming degenerate but with rather small degeneracy area, hoping that the Milnor number would stay constant. Among others we tried a series of examples of the following form:



$$F_t = x^a + y^b + z^{3c} + x^{c+2}y^{c-1} + x^{c-1}y^{c-1}z^3 + x^{c-2}y^c(y^2 + tx)^2$$

The multiplicity can be read of from the equation, but for the Milnor number we had to use a computer and the package Buchmora. However, this and other examples took hours to compute. Whenever we met, in East Germany (often in Pfister's dacha close to Berlin) or at conferences outside West Germany, we tried to improve the algorithm by checking different local orderings and trying to optimize the selection strategies during the standard basis computation (producing huge tables with timings[1]). The selection strategies for different orderings, which we finally preferred, are still in use in the present version of SINGULAR.

The place where everything started: Pfister's dacha in the GDR

Among the above series of examples there was unfortunately no counter example. We found e.g. for $(a, b, c) = (40, 30, 8)$:

$$m(F_0) = 17,\ m(F_t) = 16,\ \mu(F_0) = 10661,\ \mu(F_t) = 10655.$$

The computations for μ took many hours (today within a few seconds), but smaller Milnor numbers could be excluded by heuristical arguments. A significant speed up of the computation of standard bases for local orderings was needed and we decided to make a further step towards a more professional development of a computer algebra package.

In 1989 Buchmora was renamed to SINGULAR. It was jointly developed by groups from Berlin (Pfister) and Kaiserslautern (Greuel) within a DFG priority program 1990–1996. Within this program we could hire Hans Schönemann, who moved to Kaiserslautern in 1990, right after the unification of Germany. SINGULAR was ported to Unix (still in Modula-2) and a first user manual was released. In 1993 Pfister moved to Kaiserslautern and we decided to rewrite the code in C/C++, carried out mainly by Schönemann. Within the DFG priority program the SINGULAR programming language was developed and many libraries had been established. Around 1996 Olaf Bachmann joined the team in Kaiserslautern and with his help it was possible to improve the code of SINGULAR significantly, mainly by adapting the data structures and the memory management, which increased the speed drastically.

[1] The following anecdotes illustrate our involvement. During a conference at Mount Holyoke College we were so absorbed in the tests that we were startled by a sudden fire alarm. Only after some time we realized that the alarm was just triggered by our pipe smoke (we were both pipe smokers at the time). Fortunately, the alarm bell stopped after a while and nothing happened.
To check correctness was also difficult at that time. Pfister wanted to compare a Gröbner bases computed by Buchberger with our own implementation. He managed to do this only with the help of his wife by checking the two (many pages) print-outs by hand.

In spite of these improvements, no counter example was found! But by analyzing the above examples a partial solution to Zariski's conjecture was published in [48], including the first publication of a standard basis algorithm for arbitrary mixed monomial orderings (implemented in Singular since 1993):

Proposition 3.7.5 (G.-M. Greuel, G. Pfister [48]) *Let*

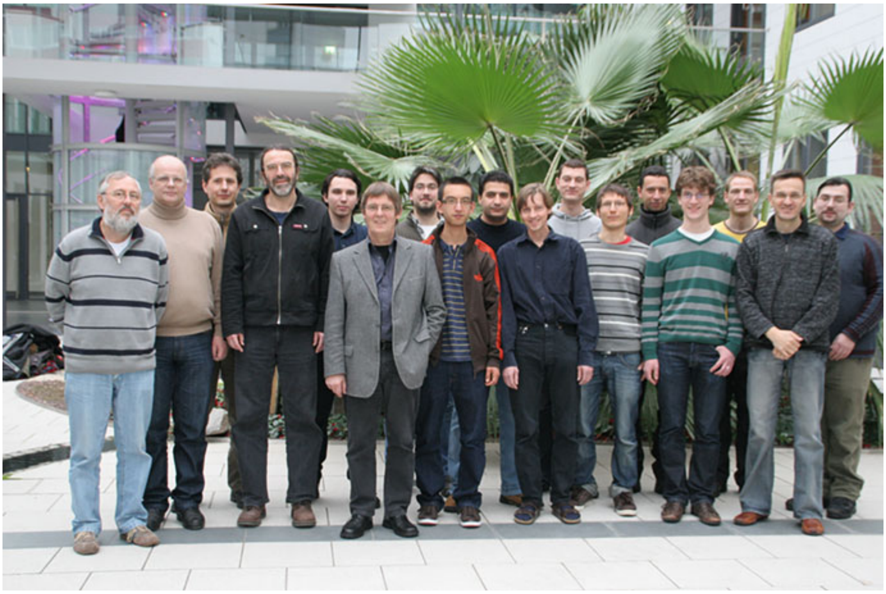
$$F_t(x_1, \dots, x_n) = G_t(x_1, \dots, x_{n-1}) + x_n^2 H_t(x_1, \dots, x_n)$$

be a family of isolated hypersurface singularities. Let G_0 be semiquasihomogeneous or let $n = 3$. If the family has constant Milnor number and the multiplicity of G_t is smaller or equal to the multiplicity of $H_t + 2$ then the multiplicity of F_t is constant.

To conclude, let us remark, that the *failure* to find a counter example to Zariski's conjecture was the most important reason for the development of SINGULAR as it is now. First of all, for many years it was the main motivation to improve its speed, since the possible counter examples were complicated to compute. Secondly, it was a very good theoretical problem that convinced the referees to support the development of SINGULAR for many years.

SINGULAR—Some History

- 1984 Neuendorf/Pfister: Implementation of the Gröbner basis algorithm in Basic on a ZX-Spectrum.
- 1990 Schönemeann moved to KL, porting to Unix
- 1993 Pfister moved to KL, C/C++ version.
- 1996 Bachmann joined the team, improvement of code and speed.
- 1996–2000 Greuel/Pfister: symbolic/numerical algorithms in SINGULAR, joint with electrical engineers and a Mathematic package "Analog Insydes".
- 1997/1998 Singular release 1.0–1.2, with multivariate polynomial factorization, gcd, syzygies, free resolutions, communication links, primary decomposition and normalization.
- 2002 Book: A SINGULAR Introduction to Commutative Algebra [47].
  By G.-M. Greuel and G. Pfister, with contributions by O. Bachmann, C. Lossen and H. Schönemann.
- 2004 First Richard D. Jenks Memorial Prize for Excellence in Software Engineering awarded to SINGULAR at ISSAC in Santander.
- 2004 Greuel/Levandovsky: The subsystem PLURAL for non-commutative polynomial algebras is included in SINGULAR.
- 2008 interface to the computer algebra system "Sage".
- 2009 Decker moves to KL, with Greuel/Pfister/Schönemann one of the leaders of the SINGULAR development.
- 2016 The "Oscar" system includes a Julia package for the Singular library.
- SINGULAR has been supported by Deutsche Forschungsgemeinschaft (DFG), Stiftung Rheinland-Pfalz für Innovation, and Volkswagen Stiftung.
- SINGULAR is free software, available at https://www.singular.uni-kl.de/



SINGULAR-team with Pfister, Schönemann, Lossen, Decker, Greuel, ...

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Chapter 4
Surface Singularities, Seiberg–Witten Invariants of Their Links and Lattice Cohomology

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Contents

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Abstract The present note aims to focus on certain topological and analytical invariants of complex normal surface singularities and wishes to analyse their interferences. The first preliminary part introduces the needed notations, definitions and terminologies: e.g. resolutions, universal abelian coverings, natural line bundles on resolutions, links, spin^c structures on the links. Here we also recall certain vanishing theorems and statements connected with Serre's and Laufer's dualities. The next part presents two multivariable series, a topological one (associated with a dual resolution graph) and an analytic one (associated with the divisorial filtration), then we compare them. Then we introduce several topological invariants, as the Casson and Casson–Walker invariants, Turaev's torsion, the Seiberg–Witten invariant. By the 'Seiberg–Witten Invariant Conjecture' they are compared with the cohomology of the natural line bundles. In this discussion certain 'additivity formulae' will also be crucial. After a preparation (introduction of the weighted cubes) we continue with the presentation of the (topological) lattice cohomology and of the (topological) graded roots associated with rational homology sphere singularity links. They are exemplified by links of superisolated singularities, when the theory is also connected with the classification of irreducible rational cuspidal projective plane curves.

4.1 Introduction

Let (X, o) be a complex analytic normal surface singularity. The main motif of the present work is the following: what are the ties between analytic and topological invariants of (X, o)? Historically this program was started by Mumford, Artin and Laufer. Mumford realized the link as plumbed 3–manifold and proved that if the fundamental group of the link is trivial then the germ is (analytically) smooth [64]. Artin and Laufer characterized topologically the rational and minimally elliptic singularities (respectively), and computed several analytic invariants for them from the resolution graph [5, 6, 49, 50].

Let us exemplify a few pairs of analytic/topological objects, which play a central role in the text.

On the analytic side our fundamental objects are the dimensions of the sheaf cohomologies of line bundles on a resolution (including e.g. the geometric genus) and the multivariable Poincaré series of the divisorial filtration associated with a resolution. If the link of (X, o) is a rational homology sphere then we consider the universal abelian covering $(X_a, o) \to (X, o)$ too and the above listed analytic invariants associated with (X_a, o). These, reinterpreted at the level of (X, o) (and its resolutions) can be related with cohomological properties of the 'natural line bundles' on the resolution spaces $\widetilde{X}$ of (X, o).

On the topological side, the link, as an oriented 3-manifold, carries the Casson invariant whenever the link is an integral homology sphere. In the rational homology sphere case, it carries Casson–Walker invariant, the (refined) Turaev torsion, the Seiberg–Witten invariants, the lattice (co)homology and the graded roots.

Then, the Seiberg–Witten invariant (which agrees with the Euler characteristic of the lattice cohomology) will be compared with the ranks of cohomologies of line bundles (formulated by the Casson Invariant Conjecture of Neumann and Wahl whenever the link is an integral homology sphere, or by the Seiberg–Witten Invariant Conjecture of Nicolaescu and the author in the rational homology sphere case). Moreover, a topological multivariable Poincaré series (a 'zeta' function, associated with the dual graph) will be compared with its analytic counterpart provided by the divisorial filtration (as extensions of Campillo–Delgado–Gusein-Zade identity). The parallelism will be emphasized by several surgery and additivity formulae of a very similar shape present in both analytic and topological sides. (For more on such parallelisms see [77] as well.)

Regarding the topological invariants, the research of the author was greatly influenced by the work of Ozsváth and Szabó on Heegaard Floer theory of 3-manifolds. However, the techniques developed by the author to create a bridge between singularities and the low dimensional topology differ from those used in Heegaard Floer theory. The effort to create such a bridge had as a fruit and culminated in the lattice cohomology. It is defined combinatorially from the graph. Conjecturally it coincides with the Heegaard Floer cohomology. However, its definition and several of its properties resemble sheaf cohomology long exact sequences. Indeed, behind certain definitions and techniques in lattice cohomology theory one experiences certain generalizations of ideas of Laufer regarding computation sequences, used in sheaf cohomological arguments. In the new context these sequences appear as discrete 'homotopy deformation retracts'. Our presentation emphasises this continuity with Laufer's work.

The theory is exemplified by cyclic quotient, weighted homogeneous and superisolated singularities.

The presentation follows rather closely [66]. However, the present work concentrates mostly on the main statements and different connections and ideas behind the results, and basically we omit most of the proofs. The interested reader is invited to consult [66] for more information.

4.2 Resolution of Surface Singularities

4.2.1 Local Resolutions

Definition 4.2.1 Consider the germ (X, o) of a normal complex analytic surface singularity with singular points $o \in X$. Let $\phi : \widetilde{X} \to X$ be a proper analytic map, where X is a sufficiently small representative of (X, o). We also set $E := \phi^{-1}(o)$. We say that ϕ is a local *modification* of (X, o) if the restriction of ϕ induces an isomorphism $\widetilde{X} \setminus E \to X \setminus o$. Additionally, if $\widetilde{X}$ is smooth then we say that ϕ is a *resolution*.

Given two modifications $\phi_i : \widetilde{X}_i \to X_i$ $(i = 1, 2)$ of (X, o), we say that ϕ_1 *dominates* ϕ_2 if after replacing both representatives X_i of (X, o) by some smaller representative X, there exists an analytic map $\psi : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\phi_2 \circ \psi = \phi_1$.

A resolution is called *good* if all the irreducible components of E (with reduced structure) are smooth (in particular, they have no self-intersections), and intersect each other transversally.

A resolution is called *minimal* if it does not dominate (with ψ non-isomorphism) any other resolution. One defines similarly the *minimal good resolutions* as well.

Lemma 4.2.2 (Zariski's Main Theorem, see [120], [34, p. 280] for the Algebraic and [29, 30] for the analytic case) *Assume that (X, o) is a germ of a normal surface singularity and fix a resolution $\phi : \widetilde{X} \to X$, which is not an isomorphism. Then $E = \phi^{-1}(o)$ is connected, compact and one-dimensional.*

Definition 4.2.3 Let (X, o) be a normal surface singularity and ϕ a resolution.

(a) The analytic (reduced) curve E is called the *exceptional set (or curve)* of ϕ. We write $\{E_v\}_{v=1}^s$ (or, $\{E_v\}_{v\in\mathcal{V}}$) for the irreducible components of E and $g_v = g(E_v)$ denotes the geometric genus of (the normalization of) E_v.
(b) The intersection matrix I of ϕ consists of the intersection numbers $(E_v, E_u)_{v,u}$ in $\widetilde{X}$.
(c) Let $f : (X, o) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. Then the divisor $\mathrm{div}(f \circ \phi)$ on $\widetilde{X}$ decomposes into $\mathrm{div}_E(f \circ \phi) + S(f \circ \phi)$, abbreviated as $\mathrm{div}_E(f) + S(f)$, where $\mathrm{div}_E(f)$ is the part supported on E, while $S(f)$ is the *strict transform of the divisor of* f.

Example 4.2.4 Assume that (X, o) is smooth. Then by blowing up o we get a modification with an exceptional curve $E \simeq \mathbb{P}^1$ and $E^2 = -1$.

In general, if C is a curve on a smooth surface $\widetilde{X}$ with $C \simeq \mathbb{P}^1$ and $C^2 = -1$ then C is called *a* (-1)*-curve on* $\widetilde{X}$. By *Castelnuovo's Contractibility Criterion* any (-1)-curve appears as a blow up of a smooth point.

Assume that $\widetilde{X}$ is a smooth surface and C is an irreducible curve on it with $(C, C) < 0$, with genus $g(C)$, and the sum of the delta-invariants of its points is $\delta(C)$. Then by the adjunction formula $(K_{\widetilde{X}}, C) + (C, C) = -2 + 2g(C) + 2\delta(C) \geq -2$. Therefore, C is a (-1)-curve if and only if $(K_{\widetilde{X}}, C) < 0$.

The next statement guarantees the existence of a resolution, cf. [7, 35, 40, 43, 48, 57, 118, 119].

Theorem 4.2.5 *Let* (X, o) *be a normal surface singularity germ. Then the following facts hold.*

1. *A good resolution exists.*
2. *There is a unique minimal resolution and a unique minimal good resolution.*
3. *A resolution is minimal if and only if none of the curves* E_v *is a* (-1)*-curve.*
4. *A good resolution is minimal good if and only if any* (-1)*-curve intersects at least three other components.*

Remark 4.2.6 Since (X, o) is normal, $X \setminus \{o\}$ is smooth. Above, in the definition of the resolution, X was an open representative. However, (in topological discussions) we can assume additionally that X is contractible to $o \in X$ and it is closed with a compact and C^∞ boundary, cf. subsection 4.2.2. In particular, $\widetilde{X}$ has the homotopy type of E and it also has a C^∞ boundary $\partial\widetilde{X}$.

Proposition 4.2.7 (Du Val [16], see also [5, 48, 64]) *Let* (X, o) *be a normal surface singularity and* ϕ *a resolution. Then the intersection matrix* $I := (E_v, E_u)_{v,u=1}^s$ *is negative definite.*

Remark 4.2.8 The converse of Proposition 4.2.7 is also true. By a famous theorem of Grauert [28], any connected collection of (compact) curves on a smooth surface with negative definite intersection form can analytically be contracted to a normal singular point, hence it appears as the exceptional curve of a resolution of some normal surface singularity.

4.2.9 The Lattice Associated with a Resolution Let (X, o) be a complex normal surface singularity and let $\phi : \widetilde{X} \to X$ be a resolution. Here we take X sufficiently small and contractible (see 4.2.20).

Set $L := H_2(\widetilde{X}, \mathbb{Z})$. Since $\widetilde{X}$ has the homotopy type of E, L is freely generated by the classes of $\{E_v\}_v$ (still denoted by the same symbol E_v), and it becomes a lattice with the intersection form I. Define also $L' := H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$. It is dual to L. If for each $v \in \mathcal{V}$ one takes a transversal disc D_v to E_v (at a generic point of E_v), then their classes form a basis of L'. Furthermore, the homological map $L \to L'$ in the bases $\{E_v\}$ and $\{D_v\}$ is exactly the matrix I. Since I is non-degenerate, $L \to L'$ is injective. We write $H := L'/L$. Clearly, $|H| = |\mathrm{coker}(I)| = |\det(I)|$.

We extend the intersection form I of L to $L \otimes \mathbb{Q}$. By the perfect pairing between L and L', L' is identified with $\mathrm{Hom}(L, \mathbb{Z})$. On the other hand, $\mathrm{Hom}(L, \mathbb{Z})$ is also identified with those elements l' of $L \otimes \mathbb{Q}$ for which $(l', l) \in \mathbb{Z}$ for any $l \in L$. In the sequel we will think about L' in this way, as a sublattice of $L \otimes \mathbb{Q}$, and as an overlattice of L, endowed with the (rational) intersection form I.

Effective classes $l = \sum r_v E_v \in L'$ with all $r_v \in \mathbb{Q}_{\geq 0}$ are denoted by $L'_{\geq 0}$, and $L_{\geq 0} := L'_{\geq 0} \cap L$. There is a natural partial ordering in $L \otimes \mathbb{Q}$ associated with the bases $\{E_v\}_v$: we say that $l_1 \geq l_2$ if $l_1 - l_2$ is effective. We write $l_1 > l_2$ if $l_1 \geq l_2$ and $l_1 \neq l_2$. The cycle $\min\{l_1, l_2\}$ is the largest l with $l_1, l_2 \geq l$. If $l' = \sum_v r_v E_v$ is a rational cycle, its *support* $|l'|$ is $\cup_{v\,:\,r_v \neq 0} E_v$. Moreover, we set $\lfloor l' \rfloor := \sum_v \lfloor r_v \rfloor E_v$, and $\{l'\} := l' - \lfloor l' \rfloor$.

4.2.10 The Pontrjagin Dual of H We denote the Pontrjagin dual $\mathrm{Hom}(H, S^1)$ of H by $\widehat{H}$. Let $\theta : H \to \widehat{H}$ be the isomorphism $[l'] \mapsto e^{2\pi i (l', \cdot)}$ of H with $\widehat{H}$.

4.2.11 Lipman's Cones Associated with the Resolution [56] We prefer to replace the classes $[D_v] \in H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$, reinterpreted in L', by their 'opposites', denoted by E_v^*. That is, $E_v^* \in L' \subset L \otimes \mathbb{Q}$ satisfies $(E_v^*, E_w) = -1$ for $v = w$, and 0 otherwise. In particular, the vectors E_v^*, written in the base $\{E_v\}_v$, are exactly the columns of the matrix $-I^{-1}$, and $(I^{-1})_{vw} = (E_v^*, E_w^*)$.

Let $\mathcal{S}_{\mathbb{Q}} := \{l' \in L \otimes \mathbb{Q} \ :\ (l', E_v) \leq 0 \text{ for all } v \in \mathcal{V}\}$ be the anti-nef rational cone, $\mathcal{S}' := \mathcal{S}_{\mathbb{Q}} \cap L'$ and $\mathcal{S} := \mathcal{S}_{\mathbb{Q}} \cap L$. $\mathcal{S}'$ is generated over $\mathbb{Z}_{\geq 0}$ by the elements E_v^*.

The definition of the cone $\mathcal{S}$ is motivated by the following fact:

Lemma 4.2.12 *Let $f : (X, o) \to (\mathbb{C}, 0)$ be a holomorphic function, and ϕ a good resolution of (X, o). Then $\mathrm{div}_E(f) \in \mathcal{S} \setminus \{0\}$.*

The divisor $\mathrm{div}_E(f) = \sum_{w \in \mathcal{V}} m_w E_w$ satisfies $m_w > 0$ for all w. This is a general fact of all the elements of $\mathcal{S}'$ by the next corollary. In particular, $\mathcal{S}'$ is in the first quadrant. (This motivates the sign modification in the definition of E_v^*.)

Corollary 4.2.13

(a) Assume that $l = \sum_v r_v E_v$ with $r_v \in \mathbb{Q}$, $l \neq 0$, and $(l, E_v) \leq 0$ for all $v \in \mathcal{V}$. Then $r_v > 0$ for all $v \in \mathcal{V}$. In particular, all the entries of E_v^* are strictly positive.

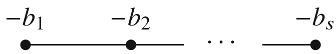
(b) For any fixed $l' \in L'$ the set $\{\tilde{l}' \in \mathcal{S}',\ \tilde{l}' \not\geq l'\}$ is finite.

4.2.14 The Resolution Graph Let (X, o) be a normal surface singularity and let $\phi : \widetilde{X} \to X$ be a *good resolution*. Denote by E the exceptional curve of ϕ with irreducible decomposition $\{E_v\}_{v\in\mathcal{V}}$. We construct a graph Γ as follows. Its *vertices* $\mathcal{V}$ correspond to the irreducible exceptional components. If two irreducible divisors corresponding to $v_1, v_2 \in \mathcal{V}$ have k intersection points then we connect v_1 and v_2 by k edges in Γ. The graph Γ is decorated as follows. Any vertex $v \in \mathcal{V}$ is decorated with the self-intersection $e_v := E_v^2$ and genus g_v of E_v (denoted as $[g_v]$). The valency (number of adjacent edges) of a vertex is denoted by κ_v.

Remark 4.2.15

(a) The graph Γ is connected by Lemma 4.2.2.
(b) The resolution is not unique, e.g. one can blow up a point of the exceptional divisor of a resolution. Accordingly, the graph Γ depends on the choice of ϕ. However, dual resolution graphs associated with different resolutions are connected by a sequence of blow ups and blow downs of vertices associated with (-1)-curves (well–defined modifications at the level of graphs).

Definition 4.2.16 A vertex of a graph with positive genus decoration, or adjacent to at least three edges, is called a *node*. A *string* is a 'linear' (sub)graph (with all genus-decorations zero) of type



Strings can be characterized by continued fractions.

Definition 4.2.17 To any two relative prime positive numbers n and q we associate the following (Hirzebruch, or negative) continued fraction:

$$\frac{n}{q} = [b_1, b_2, \dots, b_s] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_s}}}, \quad b_1 \geq 1,\ b_2, \dots, b_s \geq 2. \tag{4.1}$$

The entries $(b_1, \dots, b_s)$ characterize a string graph with decorations $-b_1, \dots, -b_s$.

For any pair n and q we also consider the *Dedekind sum*

$$\mathbf{s}(q, n) = \sum_{l=0}^{n-1} \left(\!\left(\frac{l}{n}\right)\!\right)\left(\!\left(\frac{ql}{n}\right)\!\right),$$

where $((x))$ is the *Dedekind symbol* (and $\{\cdot\}$ is the 'fractional part'):

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Example 4.2.18 ([7, 35, 48, 105, 106]) For a normal surface singularity, the following conditions are equivalent. If (X, o) satisfies any of them, then it is called *Hirzebruch–Jung* or *cyclic quotient singularity*.

1. (X, o) is isomorphic with one of the 'model spaces' $\{X_{n,q}\}_{n,q}$, where $X_{n,q}$ is the normalization of $(\{xy^{n-q} = z^n\}, 0)$, where $0 < q < n,\ (n, q) = 1$.
2. There is an analytic covering $p : (X, o) \to (\mathbb{C}^2, 0)$ such that the reduced branch locus of p is $\{uv = 0\}$ in some local coordinates (u, v) of $(\mathbb{C}^2, 0)$.
3. The resolution graph Γ_X is a string (with $g_v = 0$ for any $v \in \mathcal{V}$).
4. (X, o) is the quotient singularity $(\mathbb{C}^2, 0)/\mathbb{Z}_n$ of the cyclic group $\mathbb{Z}_n = \{\xi \in \mathbb{C} : \xi^n = 1\}$ of order n, where the action is $\xi * (z_1, z_2) = (\xi z_1, \xi^q z_2)$ for some $0 < q < n$ with $(q, n) = 1$.

4.2.2 The Link

4.2.19 Let (X, o) be the germ of a normal complex analytic surface singularity and U a neighborhood of o. We fix a *real analytic* function $\rho : U \to [0, \infty)$ with $\rho^{-1}(0) = \{o\}$. In the sequel we write X_S for $\rho^{-1}(S)$ for different subsets S of $[0, \infty)$. The next theorem characterizes the local homeomorphism type of (X, o) showing its *conic structure*. For different levels of generality see [14, 18, 32, 54, 58, 59, 63].

Theorem 4.2.20 *There exists a sufficiently small* $\epsilon_0 > 0$ *such that for any* $0 < \epsilon \leq \epsilon_0$ *the inverse image* $X_{\{\epsilon\}} := \rho^{-1}(\epsilon)$ *is a* C^∞ *manifold of dimension three. Its* C^∞ *type is independent of the choice of* ϵ *and* ρ.

Moreover, the homeomorphism type of $(X_{[0,\epsilon]}, X_{\{\epsilon\}})$ *is independent of the choice of* ϵ *and* ρ*, and it is the same as the homeomorphism type of* (*real cone*$(X_{\{\epsilon\}})$, $X_{\{\epsilon\}}$), *where the vertex corresponds to* o.

As $X_{[0,\epsilon]} \setminus \{o\}$ is a C^∞ manifold with a canonical orientation (induced by the complex structure), its boundary $X_{\{\epsilon\}}$ inherits a canonical orientation too.

Definition 4.2.21 The oriented diffeomorphism type of $X_{\{\epsilon\}}$ is called the *link of* X *at* o. It is denoted by $L(X, o)$.

Example 4.2.22

(a) Assume that X is a normal affine surface, which admits a good $\mathbb{C}^*$ action (cf. 4.2.3). Then $L(X, 0)$ is a Seifert 3-manifold.

(b) Consider the situation of Example 4.2.18(4). Set $S^3 = \{|z_1|^2 + |z_2|^2 = \epsilon\}$. Then the $\mathbb{Z}_n$-action preserves S^3, where it *acts freely*. Hence the link $L(X_{n,q}, o)$ is the lens space $L(n, q) = S^3/\mathbb{Z}_n$. Moreover, $L(n, q)$ and $L(m, p)$ are *orientation preserving diffeomorphic* if and only if $m = n$ and $p \in \{q, q'\}$, where $0 < q' < n$ and $qq' \equiv 1$ modulo n.

4.2.23 Links as Plumbed 3-Manifolds To any normal surface singularity (X, o) we associated its link $L(X, o)$ and its resolution graph Γ (well-defined up to blow up/down of (-1)-curves). The point is that they determine each other. Indeed, $L(X, o)$ is recovered from Γ via the *plumbing construction*, by considering Γ as a *plumbing graph*. For more details, see [37, 64, 87]. Note also that different plumbing graphs might produce diffeomorphic 3-manifold (via orientation preserving diffeomorphisms). However, if we restrict the plumbing construction to graphs which are *connected and have negative definite intersection matrix* then $M(\Gamma_1)$ and $M(\Gamma_2)$ are diffeomorphic if and only if the graphs are related by a sequence of (-1) blow ups and/or their inverses.

4.2.24 Homological Properties of the Link Let $\widetilde{X} = \phi^{-1}(\rho^{-1}([0, \epsilon]))$ as above with $0 < \epsilon \ll 1$. Since $i : L = H_2(\widetilde{X}, \mathbb{Z}) \to L' = H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$ is injective (see 4.2.9), the exact sequence of $(\widetilde{X}, \partial\widetilde{X})$ reads as

$$0 \to H_2(\widetilde{X}) \xrightarrow{i} H_2(\widetilde{X}, \partial\widetilde{X}) \to H_1(L_X) \to H_1(E) \to 0. \tag{4.2}$$

Set $g(\Gamma) := \sum_{v \in \mathcal{V}} g_v$ and let $c(\Gamma)$ be the number of independent cycles in Γ.

Proposition 4.2.25 ([37, 64, 107]) $L'/L = \mathrm{coker}(I) = \mathrm{Tors}(H_1(L_X, \mathbb{Z}))$*, and*

$$H_1(L_X, \mathbb{Z}) = \mathrm{coker}\,(I) \oplus H_1(E, \mathbb{Z}) = \mathrm{coker}\,(I) \oplus \mathbb{Z}^{2g(\Gamma)+c(\Gamma)}.$$

Hence, L_X is a rational homology sphere if and only if Γ is a tree with all $g_v = 0$, and L_X is an integral homology sphere when additionally $\det(-I) = 1$.

4.2.3 Example: Weighted Homogeneous Singularities

4.2.26 Definitions[99, 100] Fix some positive integers $(w_1, \ldots, w_n)$. One defines the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ with weights $(w_1, \ldots, w_n)$ by $t \cdot (x_1, \ldots, x_n) = (t^{w_1}x_1, \ldots, t^{w_n}x_n)$. A polynomial $f \in \mathbb{C}[x]$ is called weighted homogeneous of degree ℓ with respect to the weights $(w_1, \ldots, w_n)$ if $f(t \cdot x) = t^{\ell} f(x)$, where $\ell \in \mathbb{Z}_{\geq 0}$.

Let us fix an affine algebraic variety $X \subset \mathbb{C}^n$. X is called weighted homogeneous with weights $\{w_i\}_i$ if it is stable with respect to the above action of $\mathbb{C}^*$. Since the weight are all positive the action on X is *good*, that is, the origin is contained in the closure of any orbit. If additionally we assume that $\gcd_i\{w_i\} = 1$ and $X \not\subseteq \cup_i\{x_i = 0\}$ then the action is *effective* too, that is, if $t \cdot x = x$ for all $x \in X$ then $t = 1$. If X is weighted homogeneous then its defining ideal is generated by weighted homogeneous polynomials. In particular, its affine coordinate ring is $\mathbb{Z}_{\geq 0}$-graded: $R = \oplus_{\ell \geq 0} R_\ell$. In fact, all finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebras correspond to affine varieties with good $\mathbb{C}^*$-action. However, note that the normality of $R = \oplus_{\ell \geq 0} R_\ell$ is not automatically guaranteed.

A normal analytic surface singularity (X_{an}, o) is called weighted homogeneous if there exists a normal affine surface X, which admits a good $\mathbb{C}^*$ action (with $w_i > 0$ and $\gcd_i\{w_i\} = 1$) and a singular point $o \in X$ such that (X_{an}, o) is analytically isomorphic with the (induced analytic germ) (X, o).

4.2.27 The Resolution [99] The dual graph of the minimal good resolution $\widetilde{X}$ of a weighted homogeneous germ is *star-shaped*.

A connected graph Γ is called *star-shaped* if it has a *central vertex* v_0, and $\Gamma \setminus v_0$ consists of $\nu \geq 0$ strings. Each string is connected to v_0 by an edge at one of the end-vertices of the string. In some cases, for a fixed Γ, the choice of the central vertex is not unique; e.g. if Γ itself is a string then any vertex can be central.

Next we recall some of the combinatorial properties of the star-shaped graphs.

We use the following notations: v_0 has self–intersection (Euler) number $-b_0$ and genus $g \geq 0$. The Euler numbers of the vertices v_{ji} of the jth string ($1 \leq j \leq \nu$) are $-b_{j1}, \ldots, -b_{js_j}$, with $b_{ji} \geq 2$, determined by the continued fraction $\alpha_j/\omega_j = [b_{j1}, \ldots, b_{js_j}]$, where $\gcd(\alpha_j, \omega_j) = 1,\ 0 < \omega_j < \alpha_j$. For each j, v_0 is connected with v_{j1} by one edge. Set also $n_{j,i}/q_{j,i} := [b_{ji}, \ldots, b_{js_j}]$ with $\gcd\{n_{j,i}, q_{j,i}\} = 1$.

In such a case the plumbed 3-manifold $M(\Gamma)$ is a *Seifert fibered 3-manifold*, which means that $M(\Gamma)$ is foliated by circles such that any circle has a compact orientable saturated neighbourhood [38, 39, 87, 89, 108]. $M(\Gamma)$ and the foliation is characterized by the collection $(b_0, g; \{(\alpha_j, \omega_j)\}_j)$, called the *Seifert invariants*.

If either $g > 0$ or $\nu \geq 3$ then the choice of the central vertex is unique. In the sequel we assume this fact. The *virtual (or orbifold) Euler number* e and the *virtual Euler characteristic* χ are defined by

$$e := -b_0 + \sum_j \omega_j/\alpha_j, \quad \chi := 2 - 2g - \sum_j (\alpha_j - 1)/\alpha_j. \tag{4.3}$$

Note that for general star–shaped plumbing graphs $e < 0$ if and only if the intersection matrix $I = I(\Gamma)$ is negative definite.

Assume that $g = 0$ and let h_j denote the class $[E^*_{js_j}]$ ($j = 1, \ldots, \nu$) and h_0 the class $[E^*_0]$ in $H = L'/L$. Then H is generated by $\{h_j\}_{j=0}^{\nu}$ with relations $b_0 h_0 = \sum_{j=1}^{\nu} \omega_j h_j$ and $\alpha_j h_j = h_0$ $(j = 1, \ldots, \nu)$. Moreover, if $\mathfrak{o}$ be the order of h_0 in H and $\alpha := \operatorname{lcm}\{\alpha_1, \ldots, \alpha_\nu\}$ then (cf. [88]) $|H| = \alpha_1 \cdots \alpha_\nu |e|$ and $\mathfrak{o} = \alpha|e|$.

4.2.28 The Dolgachev–Pinkham–Demazure Formulae [103] Fix X normal, and let $R = \oplus_{\ell \geq 0} R_\ell$ be the graded algebra of X, and $P_X(t) = \sum_{\ell \geq 0} \dim R_\ell \cdot t^\ell$ the corresponding Poincaré series. Let $p_g = h^1(\mathcal{O}_{\widetilde{X}})$ be the geometric genus of (X, o) Assume next that L_X is a rational homology sphere, that is $g = 0$, and set

$$N(\ell) = \ell b_0 - \sum_j \lceil \ell\omega_j/\alpha_j \rceil. \tag{4.4}$$

Since $e < 0$ one has $\lim_{\ell\to\infty} N(\ell) = \infty$. Moreover, the following formulae hold:

$$P_X(t) = \sum_{\ell\geq 0} \max\{0, N(\ell)+1\}\, t^{\ell}, \quad \text{and} \quad p_g(X,o) = \sum_{\ell\geq 0} \max\{0, -N(\ell)-1\}. \tag{4.5}$$

In particular, P_X and p_g are topological.

4.2.4 Example: Superisolated Singularities

4.2.29 Hypersurface superisolated singularities connect in a tautological way the theory of complex projective plane curves with normal surface singularities. They were introduced by I. Luengo [60]. For different applications see [3, 4, 60–62]. Before we start the definition of superisolated germs we review some basic facts and notations about plane curve singularities.

4.2.30 Invariants of Irreducible Plane Curve Singularities Let us fix first an irreducible plane curve singularity $(C,o) \subset (\mathbb{C}^2, 0)$. We write $\{(p_i, q_i)\}_i$ for its Newton pairs, $\Delta(t)$ for the characteristic polynomial (of the first homology of the Milnor fiber), $\mu = \deg\Delta(t)$ for the Milnor number. Furthermore, its delta-invariant $\delta(C)$ is the codimension of $n^*\mathcal{O}_{C,o} \subset \mathcal{O}_{\mathbb{C},o} = \mathbb{C}\{t\}$, where n is the normalization of (C,o). By Jung/Milnor's formula $\mu(C,o) = 2\delta(C)$ [41, 63].

The semigroup $\mathcal{S}_{C,o} \subset \mathbb{N}$ of (C,o) is the set of all the possible intersection multiplicities $(h,C)_o$, where $h \in \mathcal{O}_{\mathbb{C}^2,0}$. The delta-invariant $\delta(C)$ appears also as the cardinality of the finite set $\mathbb{N}\setminus\mathcal{S}_{C,o}$. The largest element of $\mathbb{N}\setminus\mathcal{S}_{C,o}$ is $\mu-1$, and for $0\leq k\leq \mu-1$ one has the following 'gap-symmetry': $k\in\mathcal{S}_{C,o}$ if and only if $\mu-1-k\notin\mathcal{S}_{C,o}$. Moreover, by Campillo et al. [15]

$$\Delta(t)/(1-t) = \sum_{k\in\mathcal{S}} t^k. \tag{4.6}$$

Since $\Delta(1) = 1$ and $\Delta'(1) = \delta$, one gets $\Delta(t) = 1 + \delta(t-1) + (t-1)^2\cdot Q(t)$ for some polynomial $Q(t) = \sum_{i=0}^{\mu-2}\alpha_i t^i$ with integral coefficients. In fact, all the coefficients $\{\alpha_i\}_{i=0}^{\mu-2}$ are strict positive, and $\delta = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{\mu-2} = 1$. Indeed, by the above identity (4.6), one has $\delta + (t-1)Q(t) = \sum_{k\notin\mathcal{S}} t^k$, or $Q(t) = \sum_{k\notin\mathcal{S}}(t^{k-1}+\cdots+t+1)$. This shows that

$$\alpha_i = \#\{k\notin\mathcal{S} \,:\, k > i\}. \tag{4.7}$$

4.2.31 Definition of Superisolated Singularities [60] A hypersurface singularity $(X,o)\subset(\mathbb{C}^3,0)$ is called superisolated if the modification $\widetilde{X}$ of (X,o), induced by the blow up $0 \in \mathbb{C}^3$, is smooth. The definition guarantees that (X, o) is isolated. In fact, if X is not smooth, this $\widetilde{X}$ is exactly the *minimal* resolution of X.

Assume that (X, o) is the zero set of $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \cdots$, where f_j is homogeneous of degree j, $f_d \not\equiv 0$. Then (X, o) is superisolated if and only if the projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with (isolated) singularities $\{p_i\}_i$, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$. In this case the embedded topological type (and the equisingularity type) of f does not depend on the choice of f_j's for $j > d$, as long as f_{d+1} satisfies the above requirement. Therefore, those invariants of (X, o), which are stable with respect to equisingular deformations, depend only on C.

In the sequel we will assume that C is irreducible. In such a case the minimal resolution $\widetilde{X}$ has only one irreducible exceptional divisor, which is isomorphic to C, and C^2 in $\widetilde{X}$ is $-d$. Hence, the link of (X, o) is a rational homology sphere if and only if C is rational and all the plane curve singularities $(C, p_i) \subset (\mathbb{P}^2, p_i)$ are irreducible. (We use the terminology *cusp* for them.) Such a curve C is called *rational cuspidal plane curve*. We denote by μ_i and Δ_i (with the choice $\Delta_i(1) = 1$) the Milnor number and the characteristic polynomial of the local plane curve singularities $(C, p_i) \subset (\mathbb{P}^2, p_i)$. Then $\sum_i \mu_i = (d-1)(d-2)$.

The minimal good resolution is obtained from $\widetilde{X}$ by resolving the plane curve singularities $(C, p_i) \subset (\widetilde{X}, p_i)$. Note that the embedded topological types $(C, p_i) \subset (\widetilde{X}, p_i)$ and $(C, p_i) \subset (\mathbb{P}^2, p_i)$ agree. Hence, under the condition that C is irreducible and the link L_X is a rational homology sphere, the minimal good resolution graph Γ of (X, o) is the surgery graph described in 4.2.32. That is, the link of (X, o) is the oriented surgery 3-manifold $S^3_{-d}(\#_i K_i)$, where $(K_i \subset S^3)$ are the local knots of $(C, p_i) \subset (\mathbb{P}^2, p_i)$.

4.2.32 The Plumbing Graph of the Surgery Manifold $S^3_{-d}(\#_i K_i)$ with K_i Algebraic and d Arbitrary We fix an integer d and a collection of algebraic knots $\{K_i\}_{i=1}^{\nu}$ in S^3 (determined by irreducible plane curve singularities $(C_i, 0) \subset (\mathbb{C}^2, 0)$). Set the connected sum $K = K_1 \# \cdots \# K_\nu \subset S^3$ of the knots K_i. Then $S^3_{-d}(K)$ is a plumbed 3-manifold whose plumbing graph is constructed as follows. First, let Γ_i be the minimal good embedded resolution graph of $(C_i, 0) \subset (\mathbb{C}^2, 0)$ with a unique -1 vertex v_i which supports the strict transform. One also considers the cycle $Z_i = \mathrm{div}_{E(\Gamma_i)}(f_i) \in L(\Gamma_i)$ given by the local reduced equation f_i of $(C_i, 0)$; let m_i be the multiplicity in Z_i of the -1 curve of Γ_i. Then, in order to get the graph of $S^3_{-d}(K)$ from the disjoint union $\sqcup_i \Gamma_i$, one introduces a new vertex v_+, which is glued to each graph Γ_i via a new edge connecting v_+ and v_i, and one inserts the Euler decoration $-d - \sum_i m_i$ on v_+. The Euler decorations of $\{\Gamma_i\}_i$ stay unmodified. The resulting graph is negative definite if and only if $d > 0$. Furthermore, $|\det(I)| = |d|$.

4.2.33 A Restrictions Satisfied by the Combinatorial Type Consider a superisolated singularity. Let $\mathcal{S}_{C,p_i}$ be a semigroup of the local singularities (C, p_i). Fix an integer $0 \le l < d$. In [24] is proved (via Bézout theorem) the following *Semigroup*

Distribution Inequality:

$$\min_{j_1+\cdots+j_\nu=ld+1}\ \sum_{i=1}^{\nu}\#\{\,\mathcal{S}_{C,p_i}\cap[0,\,j_i)\,\}\geq(l+1)(l+2)/2.$$

Moreover, in [24] the authors conjectured under the name *Semigroup Distribution Property*, that in the above inequality one has equality in any unicuspidal case. The general proof for any cusps was obtained by Borodzik and Livingston based on the d-invariant of Heegaard Floer theory [9]. That is, with the previous notations,

$$\min_{j_1+\cdots+j_\nu=ld+1}\ \sum_{i=1}^{\nu}\#\{\,\mathcal{S}_{C,p_i}\cap[0,\,j_i)\,\}=(l+1)(l+2)/2$$

for any rational cuspidal curve. In the unicuspidal case this reads as

$$\#\{\,\mathcal{S}_{C,p}\cap(\,(l-1)d,ld\,]\,\}=\min\{l+1,d\}\quad(l\geq 0).$$

4.2.5 Local Divisor Class Group

4.2.34 Sheaf Cohomological Properties of $\widetilde{X}$ Let us start this subsection with the following observations.

Let (X,o) be a complex normal surface singularity and let $\phi:\widetilde{X}\to X$ be a good resolution. In cohomological considerations, e.g. in the computation of $H^*(\widetilde{X},\mathbb{Z})$ or $H^*(\widetilde{X},\mathcal{F})$, we might take for $\widetilde{X}$ the space $\phi^{-1}(\rho^{-1}([0,\epsilon]))$, cf. 4.2.20. Therefore, for an analytic coherent sheaf and $q\geq 1$, $H^q(\widetilde{X},\mathcal{F})$ agrees with $(R^q\phi\mathcal{F})_o=\lim_{\to U}H^q(\phi^{-1}(U),\mathcal{F})$, where U runs over open sets $o\in U\subset X$.

By 'Theorem of formal functions', $(R^q\phi\mathcal{F})_o=\lim_{\leftarrow Z}H^q(Z,\mathcal{F}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}_Z)$, where Z runs over (larger and larger) effective cycles supported on E. In fact, for a line bundle $\mathcal{F}$ we have $H^{\geq 2}(\widetilde{X},\mathcal{F})=0$ and $H^1(\widetilde{X},\mathcal{F})=H^1(Z,\mathcal{F}\otimes\mathcal{O}_Z)$ for $Z\gg 0$, hence $\dim H^1(\widetilde{X},\mathcal{F})<\infty$. Furthermore, by Serre duality, for a locally free sheaf $\mathcal{F}$, $H^1_c(\widetilde{X},\mathcal{F})=H^1(\widetilde{X},\mathcal{F}^\vee\otimes\Omega^2_{\widetilde{X}})^*$. Note that for a divisor D supported on E and a locally free sheaf $\mathcal{F}$ on $\widetilde{X}$ we have $H^0(\widetilde{X}\setminus E,\mathcal{F}(D))=H^0(\widetilde{X}\setminus E,\mathcal{F})$ and $H^0(\widetilde{X}\setminus E,\mathcal{F})/H^0(\widetilde{X},\mathcal{F})$ is finite dimensional since it embeds into $H^1_c(\widetilde{X},\mathcal{F})$ [49].

4.2.35 The Picard Group Let $\mathrm{Pic}(\widetilde{X})=H^1(\widetilde{X},\mathcal{O}^*_{\widetilde{X}})$ denote the Picard group of $\widetilde{X}$, the group of isomorphism classes of analytic line bundles on $\widetilde{X}$. Recall also that the geometric genus of (X,o) is $p_g:=h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}})$. (It is independent of the choice of the resolution.)

By duality, L' is isomorphic to $H^2(\widetilde{X},\mathbb{Z})$, hence it is the target of the first Chern class $c_1:\mathrm{Pic}(\widetilde{X})\to H^2(\widetilde{X},\mathbb{Z})$. This morphism is part of the following exact sequence induced by the exponential exact sequence of sheaves $0\to\mathbb{Z}_{\widetilde{X}}\to\mathcal{O}_{\widetilde{X}}\to$

$\mathcal{O}^*_{\widetilde{X}} \to 0$:

$$0 \to H^1(\widetilde{X}, \mathbb{Z}) \longrightarrow H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \xrightarrow{\varepsilon} \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} H^2(\widetilde{X}, \mathbb{Z}) \to 0. \tag{4.8}$$

Set

$$\mathrm{Pic}^0(\widetilde{X}) := \ker(c_1) \simeq H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^1(\widetilde{X}, \mathbb{Z}) \simeq \mathbb{C}^{p_g}/H^1(E, \mathbb{Z}).$$

Since $H^1(\widetilde{X}, \mathbb{Z}) = \lim_{\to U} H^1(U, \mathbb{Z})$ and $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \lim_{\to U} H^1(U, \mathcal{O}_U)$, $E \subset U$, from (4.8) we also have $H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}}) = \lim_{\to U} H^1(U, \mathcal{O}^*_U)$. Furthermore, by Mumford [64], for any line bundle $\mathcal{L} \in H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})$ there exists $E \subset U \subset \widetilde{X}$ sufficiently small such that $\mathcal{L}|_U$ admits a meromorphic section over U. In particular, $\mathrm{Pic}(\widetilde{X})$ can be identified with the group $\mathrm{Cl}(\widetilde{X})$ of *local analytic divisors* near E modulo linear equivalence. More precisely, by a local analytic divisor we mean a sum $\sum_i n_i D_i$ of irreducible analytic divisors defined in a neighbourhood of E. Such a divisor is locally linear equivalent to zero if there exists a neighbourhood U of E, where all D_i are defined, and a meromorphic function on U such that $\mathrm{div}(f) = \sum_i n_i(D_i \cap U)$.

The lattice L embeds into both $L' = H^2(\widetilde{X}, \mathbb{Z})$ and $\mathrm{Pic}(\widetilde{X})$. For L' see 4.2.9, into $\mathrm{Pic}(\widetilde{X})$ by $l \mapsto \mathcal{O}_{\widetilde{X}}(l)$. Similarly to the group $L'/L = \mathrm{Tors}(H^2(X \setminus \{o\}, \mathbb{Z}))$ (cf. 4.2), $\mathrm{Pic}(\widetilde{X})/L$ is also independent of the choice of the resolution $\widetilde{X}$. Indeed, the sequence

$$0 \to L \to \mathrm{Pic}(\widetilde{X}) \xrightarrow{r} \mathrm{Cl}(X, o) \to 0$$

is exact (cf. [64]), where $\mathrm{Cl}(X, o)$ denotes the *local divisor class group* of (X, o). This is the class group of local Weil divisors of (X, o) modulo local Cartier divisors. If D is a local irreducible analytic divisor on $\widetilde{X}$, then its restriction to $\widetilde{X} \setminus E$ can be mapped to $X \setminus \{o\}$ by ϕ, and the class of its closure is $r(\mathcal{O}_{\widetilde{X}}(D))$. [This is exactly the definition of the natural map $\phi_* : \mathrm{Cl}(\widetilde{X}) \to \mathrm{Cl}(X, o)$, a reinterpretation of r.]

Hence we obtain the exact sequence

$$0 \to H^1(L_X, \mathbb{Z}) \to \mathbb{C}^{p_g} \to \mathrm{Cl}(X, o) \xrightarrow{\bar{c}_1} \mathrm{Tors}(H^2(L_X, \mathbb{Z})) \to 0. \tag{4.9}$$

The Chern class morphism $\bar{c}_1$—in the language of divisors and homology—has the form $\bar{c}'_1 : \mathrm{Cl}(X, o) \to \mathrm{Tors}(H_1(L_X, \mathbb{Z}))$, where $\bar{c}'_1$ assigns to a Weil divisor the homological class of its intersection with the link.

$\mathrm{Cl}(X, o)$ coincides with the group of isomorphism classes of divisorial sheaves on (X, o). [If $\mathcal{F}$ is a divisorial sheaf, then $\mathcal{L} = (\phi^*(\mathcal{F}))^{\vee\vee}$ is locally free on $\widetilde{X}$, such that $\mathcal{L}|_{\widetilde{X}\setminus E} = \mathcal{F}|_{X\setminus\{o\}}$. By the above discussion $\mathcal{L}$ has the form $\mathcal{O}_{\widetilde{X}}(D)$, hence $\mathcal{F} = r(\mathcal{O}_{\widetilde{X}}(D))$, that is, $\mathcal{F}$ is associated with a Weil divisor $\phi_*(D)$.]

Example 4.2.36 If $j : X \setminus \{o\} \hookrightarrow X$ is the inclusion, then $\omega_X := j_*(\Omega^2(X \setminus \{o\}))$ is a divisorial sheaf. One can also write it in the form $\mathcal{O}_X(K_X)$ for a certain Weil divisor K_X. If $K_{\widetilde{X}}$ is a canonical divisor on $\widetilde{X}$, then K_X can be takes as $\phi_*(K_{\widetilde{X}})$ (or, $r(\Omega^2_{\widetilde{X}})$).

Definition 4.2.37 A Weil divisor of (X, o) (or its class) is called $\mathbb{Q}$-*Cartier*, if its class in $\mathrm{Cl}(X, o)$ has finite order. Its order is called its *index*.

4.2.6 Canonical Coverings

4.2.38 The germ of an analytic finite map $\pi : (Y, o) \to (X, o)$ (where (Y, o) and (X, o) are normal and $\pi^{-1}(o) = o$) is called o–ramified if the restriction $Y \setminus o \to X \setminus o$ is a regular (topological, unbranched) covering. An o–ramified covering is called G–covering if $Y \setminus o \to X \setminus o$ is Galois with deck transformations G. If $\pi : (Y, o) \to (X, o)$ is o–ramified, then there is a morphism $\widetilde{Y} \to \widetilde{X}$ at the level of (convenient) resolutions, and the pullback $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Y})$ induces a well–defined morphism $c^* : \mathrm{Cl}(X, o) \to \mathrm{Cl}(Y, o)$.

4.2.39 Let us recall a possibility how one can construct a cyclic o–ramified covering topologically. Let (X, o) be as above and let $\pi_1(L(X, o)) \to G$ be an epimorphism. Then, by Stein [110] it determines an o–ramified G–covering. E.g., if $L(X, o)$ is a $\mathbb{Q}HS^3$ link (that is, $H_1(L_X, \mathbb{Z}) = H = L'/L$) and we fix a character $\alpha \in \widehat{H}$, then it determines an epimorphism $\pi_1(L(X, o)) \to H \to \mathbb{Z}_N$ (for some N) and a Galois cyclic o–covering. In particular, if $L(X, o)$ is a $\mathbb{Q}HS^3$, and we start with a cycle $l' \in L'$, such that the order of $[l'] \in H$ is N, and we considered the character $\alpha := \theta([l']) \in \widehat{H}$, then we get a o–ramified $\mathbb{Z}_N$-covering $(X_\alpha, o) \to (X, o)$.

4.2.40 Next we associate a cyclic o–ramified covering $(X_D, o) \to (X, o)$ to any $\mathbb{Q}$-Cartier divisor D (in this case L_X is not necessarily a $\mathbb{Q}HS^3$).

Proposition 4.2.41 *Let D be a $\mathbb{Q}$-Cartier divisor of index N of (X, o). Then it determines a uniquely defined o-ramified Galois $\mathbb{Z}_N$-covering $c : (X_D, o) \to (X, o)$, where (X_D, o) is a normal surface singularity, and $c^*(D) = 0$ in $\mathrm{Cl}(X_D, o)$. The covering $c : (X_D, o) \to (X, o)$ depends only on the class of D in $\mathrm{Cl}(X, o)$.*

(In fact, the kernel of $c^* : \mathrm{Cl}(X, o) \to \mathrm{Cl}(X_D, o)$ is cyclic of order N and it is generated by the class of D.)

Indeed, adding a principal divisors to D we can assume that D is effective. Then $N \cdot D$ is an effective principal divisor of (X, o). Hence $N \cdot D = \mathrm{div}(f)$ for some holomorphic germ $f : (X, o) \to (\mathbb{C}, 0)$. Then define $X_{f,N}$ as the normalization of $\{(x, z) \in (X \times \mathbb{C}, (o, 0)),\ f(x) = z^N\}$. Then a local computation shows that the natural projection $c : (X_{f,N}, (o, 0)) \to (X, o)$ is o–ramified. The second statement claims that $\mathrm{div}(f \circ c)/N$ is an integral principal divisor of (X_D, o). But, indeed, this is exactly $\mathrm{div}(z)$.

Note also that the added principal divisors do not alter the isomorphism class of $X_{f,N}$. Indeed, (the normalized) $X_{fg^N,N}$ and $X_{f,N}$ are isomorphic.

4.2.42 The above facts can be used to define (in an analytic way) a covering associated with any $l' \in L'$. The construction depends on a choice, but it has no ambiguity whenever the link is a rational homology sphere. First, we associate to l' a $\mathbb{Q}$-Cartier divisor as follows. For parts *(a)–(b)* see [96, 112, 113].

Proposition 4.2.43

(a) *Fix a resolution* $\phi : \widetilde{X} \to X$, $l' \in L'$, *and let* N *be the order of its class in* L'/L. *Then there exists a divisor* $D = D(l')$ *on* $\widetilde{X}$ *such that one has a linear equivalence* $N \cdot D \sim N \cdot l'$ *and* $c_1\mathcal{O}_{\widetilde{X}}(D) = l'$ *(where* Nl' *is identified with an integral divisor supported on* E*). In particular,* $\phi_*(D)$ *has finite order* N *in* $\mathrm{Cl}(X, o)$.

(b) *If* $H^1(\widetilde{X}, \mathbb{Z}) = 0$ *then* D *is unique up to a linear equivalence. Hence, in this case, the correspondence* $l' \mapsto \mathcal{O}_{\widetilde{X}}(D(l'))$ *is a section of the exact sequence (4.8).*

(c) *If* $H^1(\widetilde{X}, \mathbb{Z}) = 0$ *then the covering associated with* l' *defined in 4.2.41 via* $D(l')$ *agrees with the covering associated with* l' *defined in 4.2.39 via the character* $\theta([l'])$.

Proof *(a)* Since c_1 is onto, there exists a divisor D_1 on $\widetilde{X}$ with $c_1\mathcal{O}_{\widetilde{X}}(D_1) = l'$. Hence $\mathcal{O}_{\widetilde{X}}(ND_1 - \mathrm{div}(Nl'))$ has the form $\epsilon(\mathcal{L})$ for some $\mathcal{L} \in \mathrm{Pic}^0(\widetilde{X}) = \mathbb{C}^{p_g}/H^1(\widetilde{X}, \mathbb{Z})$. Define D_2 so that $\mathcal{O}_{\widetilde{X}}(D_2) := \frac{1}{N}\mathcal{L} \in \mathrm{Pic}^0(\widetilde{X})$. Then $D := D_1 - D_2$ works. For *(b)* use the fact that $\mathrm{Pic}(\widetilde{X})$ is torsion free. For *(c)* use the definitions. □

Definition 4.2.44

(a) Write $\Omega^2_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ and assume that K_X is $\mathbb{Q}$-Cartier. Then the cyclic covering associated with K_X (as in 4.2.41) is called the *analytic canonical covering* of (X, o).

(b) Assume that the link of (X, o) is a rational homology sphere. The well-defined cyclic covering associated with $c_1(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}))$, constructed in 4.2.39 is called the *topological canonical covering* of (X, o).

If both assumptions are satisfied then the analytic and topological canonical coverings agree. However, if $H_1(\partial\widetilde{X}, \mathbb{Q}) = 0$, then the *topological canonical covering* is well-defined even if K_X is not $\mathbb{Q}$-Cartier.

4.2.7 Natural Line Bundles

4.2.45 Let $\phi : (\widetilde{X}, E) \to (X, o)$ be a good resolution and assume that $L(X, o)$ is a $\mathbb{Q}HS^3$. In the next discussion we identify the homology classes $l \in L$ and the integral divisors supported on E.

In the exact sequence (4.8) c_1 admits a natural group section s_L over the integral cycles $L \subset L'$. Indeed, for any $l \in L$ we can take $\mathcal{O}_{\widetilde{X}}(l) \in \mathrm{Pic}(\widetilde{X})$. Clearly $c_1(\mathcal{O}_{\widetilde{X}}(l)) = l$. In the sequel we extend s_L in a unique way to a natural group section $s : L' \to \mathrm{Pic}(\widetilde{X})$. Its existence is guaranteed by the facts that $H = L'/L$ is finite, while $\mathrm{Pic}^0(\widetilde{X}) \simeq \mathbb{C}^{p_g}$ is torsion free. In fact, we present several constructions of s, which emphasize its different geometrical aspects.

4.2.46 The Construction of s via $\mathrm{Cl}(X, o)$ [96]

For any $l' \in L'$ consider the divisor $D(l')$ provided by Lemma 4.2.43. Since $H^1(\widetilde{X}, \mathbb{Z}) = 0$, $D(l')$ is unique with the required properties of 4.2.43. Therefore one has a well-defined map $l' \mapsto s(l') = \mathcal{O}_{\widetilde{X}}(D(l'))$. By the uniqueness $D(l'_1 + l'_2) \sim D(l'_1) + D(l'_2)$, hence s is a homomorphism and a section of (4.8) as well.

Definition 4.2.47 The line bundles $s(l')$, indexed by $l' \in L'$, and denoted also by $\mathcal{O}_{\widetilde{X}}(l') := s(l')$, will be called *natural line bundles*.

Corollary 4.2.48

(a) *A line bundle* $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ *is natural if and only if some power of it has the form* $\mathcal{O}_{\widetilde{X}}(l)$ *(in its usual classical sense) for an integral cycle* $l \in L$*. Equivalently,* $\mathcal{L}$ *is natural if and only if its projection by* $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{X})/L = \mathrm{Cl}(X, o)$*) has finite order (i.e., if it is* $\mathbb{Q}$*-Cartier).*

(b) *One has a natural isomorphism* $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}^0(\widetilde{X}) \oplus L'$ *given by* $\mathcal{L} \mapsto (\mathcal{L} \otimes s(c_1\mathcal{L})^{-1}, c_1\mathcal{L})$*. This induces a natural isomorphism* $\mathrm{Cl}(X, o) \to \mathrm{Pic}^0(\widetilde{X}) \oplus H$*.*

In particular (since $\mathrm{Pic}^0(\widetilde{X})$ *is torsion free), under this identification* H *is isomorphic with the group of* $\mathbb{Q}$*-Cartier divisor classes of* (X, o)*.*

4.2.49 The Universal Abelian Covering Let $c : (X_a, o) \to (X, o)$ be the universal abelian covering of (X, o). It is the Galois o–covering associated with $\pi_1(L_X) \to H_1(L_X, \mathbb{Z}) = L'/L$ (cf. [110]).

Let $\widetilde{c} : Z \to \widetilde{X}$ be the normalized pullback of c via ϕ. The (reduced) branch locus of $\widetilde{c}$ is included in E, and the Galois action of H extends to Z as well. Since E is a normal crossing divisor, the only singularities what Z might have are cyclic quotient singularities, cf. 4.2.18. Let $r : \widetilde{Z} \to Z$ be a resolution of these singular points such that $(\widetilde{c} \circ r)^{-1}(E)$ is a normal crossing divisor. Set $p := \widetilde{c} \circ r$.

$$\begin{array}{ccccc} \widetilde{Z} & \xrightarrow{r} & Z & \xrightarrow{\psi_a} & (X_a, o) \\ & & \downarrow{\scriptstyle \widetilde{c}} & & \downarrow{\scriptstyle c} \\ & & \widetilde{X} & \xrightarrow{\phi} & (X, o) \end{array} \tag{4.10}$$

4.2.50 The Construction of s via $p^* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Z})$ [71] One has the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & L & \to & L' & \to & H & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^H} & & \\ 0 & \to & L_a & \to & L'_a & \to & H_a & \to & 0 \end{array} \tag{4.11}$$

where the vertical arrows are pullbacks associated with $p = \widetilde{c} \circ r$ (e.g., p^* is the cohomology morphism $H^2(\widetilde{X}, \mathbb{Z}) \to H^2(\widetilde{Z}, \mathbb{Z})$ and the first arrow is the relative cohomology morphism), and the bottom line is the 'lattice exacts sequence' (4.2) associated with the resolution $\widetilde{Z} \to X_a$ of (X_a, o). We claim that:

$$p^H = 0. \tag{4.12}$$

In particular, $p^*(l') \in L_a$ for any $l' \in L'$, hence considering $p^*(l')$ as an integral divisor, the element $\mathcal{O}_{\widetilde{Z}}(p^*(l')) \in \mathrm{Pic}(\widetilde{Z})$ is well-defined.

Theorem 4.2.51 *The line bundle $\mathcal{O}_{\widetilde{Z}}(p^*(l'))$ is a pullback of a unique element $\mathcal{L}$ of* $\mathrm{Pic}(\widetilde{X})$. *This line bundle $\mathcal{L}$ will be denoted by $\mathcal{O}_{\widetilde{X}}(l')$. Moreover, $s : L' \to \mathrm{Pic}(\widetilde{X})$, defined by $l' \mapsto \mathcal{O}_{\widetilde{X}}(l')$, is a group section of c_1 in (4.8), which extends s_L.*

Furthermore, the definition of $\mathcal{O}_{\widetilde{X}}(l')$ is independent of the choice of the resolution $r : \widetilde{Z} \to Z$.

Proof Using the two exponential exact sequences one verifies that $p^* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Z})$ is injective and its image is the subgroup of invariants $(\mathrm{Pic}(\widetilde{Z}))^H$. On the other hand, $\mathcal{O}_{\widetilde{Z}}(p^*(l'))$ is H-invariant. □

4.2.52 The Construction of s via $c_*\mathcal{O}_{X_a,o}$ [42, 71, 96, 97]

Associated with the resolution $\phi : \widetilde{X} \to X$ we consider the 'unit closed-open cube' $Q := \{l' \in L' : \lfloor l' \rfloor = 0\}$. Obviously, for any $h \in H$ there is a unique element $r_h \in Q$, whose class is h. It is the minimal representative of h in the cone $L'_{\geq 0}$.

Theorem 4.2.53 ([71, 96, 97] (for the cyclic case see also [20–22])) *Assume, as above, that $H^1(\widetilde{X}, \mathbb{Z}) = 0$. Consider the finite covering $\widetilde{c} : Z \to \widetilde{X}$. Then $\widetilde{c}_*\mathcal{O}_Z$ is a vector bundle and its H-eigensheaf decomposition has the form:*

$$\widetilde{c}_*\mathcal{O}_Z \simeq \oplus_{\alpha \in \widehat{H}} \mathcal{L}_\alpha, \tag{4.13}$$

where $\mathcal{L}_{\theta(h)} = \mathcal{O}_{\widetilde{X}}(-r_h)$ for any $h \in H$. In particular, $\widetilde{c}_*\mathcal{O}_Z \simeq \oplus_{l' \in Q} \mathcal{O}_{\widetilde{X}}(-l')$.

More generally, for any $l' \in L'$ one has

$$\widetilde{c}_*\mathcal{O}_Z(-\widetilde{c}^*(l')) \simeq \oplus_{h \in H} \mathcal{O}_{\widetilde{X}}(-r_h + \lfloor r_h - l' \rfloor). \tag{4.14}$$

Corollary 4.2.54 *The set of natural line bundles on $\widetilde{X}$ coincides with the set of line bundles of type $\mathcal{L} \otimes \mathcal{O}(l)$, where $\mathcal{L}$ is an eigensheaf of $\widetilde{c}_*\mathcal{O}_Z$ and $l \in L$. Or, via (4.14), the set of natural line bundles coincides with the set of eigensheaf of bundles of type $\widetilde{c}_*\mathcal{O}_Z(-\widetilde{c}^*(l'))$, $l' \in L'$.*

4.2.8 The Canonical Cycle

4.2.55 Fix any resolution $\widetilde{X}$. Let $K_{\widetilde{X}}$ be a *canonical divisor* (defined up to a linear equivalence), $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \Omega^2_{\widetilde{X}}$, and let $K = -Z_K$ be $c_1(\Omega^2_{\widetilde{X}}) \in L'$, the *canonical cycle of the resolution* ϕ. The cycle Z_K can be determined combinatorially from $(L', (\,,\,))$ via the adjunction formula, namely $(-Z_K + E_v, E_v) + 2 \cdot (1 - g(E_v) - \delta(E_v)) = 0$ for all $v \in \mathcal{V}$. (Here $\delta(E_v)$ is the sum of delta invariants of singularities of E_v.) In particular, $Z_K = 0$ if and only if $g(E_v) = \delta(E_v) = 0$ and $E_v^2 = -2$ for al v. In such a case (X, o) is an ADE singularity.

By Laufer [53], if the resolution is minimal, and $Z_K \neq 0$, then all the coefficients of Z_K are positive. Moreover, if $\widetilde{X}$ is a *minimal good resolution* and (X, o) is not of type ADE, then all the coefficients of Z_K are still positive.

Theorem 4.2.56 (Riemann–Roch Formula) *Fix a line bundle* $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ *and set* $c_1(\mathcal{L}) = l' \in L'$ *and* $k := -Z_K - 2l'$. *For any* $l \in L_{>0}$ *we consider the sheaf* $\mathcal{L} \otimes \mathcal{O}_l$ *on* l. *Then its analytic Euler characteristic satisfies*

$$\chi(\mathcal{L} \otimes \mathcal{O}_l) = -(l, l + k)/2. \tag{4.15}$$

We denote the combinatorial term from the right hand side of (4.15) by $\chi_k(l)$, or just by $\chi(l)$ if $k = -Z_K$. This expression motivates the following.

Definition 4.2.57 The set of characteristic elements are defined as

$$\mathrm{Char} = \mathrm{Char}(L) = \{k \in L' \;:\; (l, l + k) \in 2\mathbb{Z} \text{ for any } l \in L\}. \tag{4.16}$$

Note that $-Z_K$ is a characteristic element and $\mathrm{Char} = -Z_K + 2L'$.

The expression (4.15) can be extended to L', that is, for any $k \in \mathrm{Char}$ one defines $\chi_k : L' \to \mathbb{Q}$ by $\chi_k(l') := -(l', l' + k)/2$. If $k = -Z_K$ then we write $\chi := \chi_k$.

4.2.58 The expression $Z_K^2 + |\mathcal{V}|$ of the link behaves like a characteristic class in many index formulae. It is independent of the resolution. We have the following general formula for it.

Proposition 4.2.59 ([78]) $Z_K^2 + |\mathcal{V}|$ *in terms of the graph has the expression*

$$Z_K^2+|\mathcal{V}| = 2-2b_1(L_X)+\sum_v (E_v^2+3)+\sum_{v,w} (2\chi(E_v)-\kappa_v)(2\chi(E_w)-\kappa_w)(E_v^*, E_w^*).$$

Example 4.2.60 ([36]) For the cyclic quotient singularity $X_{n,q}$ we have

$$Z_K^2 + |\mathcal{V}| = 2(n-1)/n - 12 \cdot \mathbf{s}(q, n).$$

Example 4.2.61 ([79]) For a star-shaped graph, with $\mathfrak{r} := \chi/e$, we have

$$Z_K^2 + |\mathcal{V}| = e\mathfrak{r}^2 + e + 5 - 12 \cdot \sum_{j=1}^{\nu} \mathbf{s}(\omega_j, \alpha_j).$$

Example 4.2.62 Assume that $L_X = S^3_{-d}(\#_i K_i)$ (cf. 4.2.32), with $\mu/2 = \delta = \sum_i \delta_i$ (the sum of delta-invariants of K_i) and arbitrary $d > 0$. Then $K^2 + |\mathcal{V}| = 1 - (d - 2 + \mu)^2/d$. If $\mu = (d-1)(d-2)$ (as in the superisolated case), then $K^2 + |\mathcal{V}| = 1 - d(d-2)^2$.

4.2.63 Splice Formula Assume that $L(X, o)$ is an integral homology sphere and let $\mathfrak{G}$ be the splice diagram associated with the plumbing graph Γ [19]. Assume that $\mathfrak{G}$ is obtained by splicing the diagrams $\mathfrak{G}_1$ and $\mathfrak{G}_2$ along the knots $K_1 \subset M(\mathfrak{G}_1)$, $K_2 \subset M(\mathfrak{G}_2)$. Let Γ_i be the plumbing graphs, which correspond to $\mathfrak{G}_i$. Recall also that $K_i \subset M(\mathfrak{G}_i)$ determines an open book decomposition, let μ_i be the first Betti number (Milnor number) of its fiber. Then one has the following.

Theorem 4.2.64 ([92])

$$\big(Z_K^2 + |\mathcal{V}|\big)(\Gamma) = \big(Z_K^2 + |\mathcal{V}|\big)(\Gamma_1) + \big(Z_K^2 + |\mathcal{V}|\big)(\Gamma_2) - 2 \cdot \mu_1 \cdot \mu_2.$$

Definition 4.2.65 The normal singularity (X, o) is called *Gorenstein* if $\Omega^2_{X\setminus\{o\}}$ is a holomorphically trivial line bundle, equivalently, if $Z_K \in L$ and one can choose for $K_{\widetilde{X}}$ the divisor $-Z_K$. Analogously, (X, o) is called *numerically Gorenstein* if $\Omega^2_{X\setminus\{o\}}$ is a topologically trivial complex line bundle.

Though Gorenstein (local) rings can be defined even without normality assumption, see e.g. [13], (e.g. complete intersections are Gorenstein even if they are not normal), here we discuss the Gorenstein property only for normal germs.

Lemma 4.2.66 ([17]) *(X, o) is numerically Gorenstein if and only if $Z_K \in L$.*

4.2.67 $\mathbb{Q}$-Gorenstein Singularities Let K_X be the canonical divisor of (X, o), cf. 4.2.36. Note that (X, o) is Gorenstein if and only if K_X is Cartier (invertible) at $o \in X$, that is, K_X is zero in $\mathrm{Cl}(X, o)$. Furthermore, if (X, o) is Gorenstein then any o-ramified covering (X', o) of (X, o) is Gorenstein. More generally, (X, o) is called $\mathbb{Q}$-Gorenstein, if there exists a positive integer r such that rK_X is a Cartier divisor at o (equivalently, if K_X has finite order in $\mathrm{Cl}(X, o)$). Again, if (X, o) is $\mathbb{Q}$–Gorenstein then any o-ramified covering (X', o) of (X, o) is $\mathbb{Q}$-Gorenstein. If $L(X, o)$ is $\mathbb{Q}HS^3$ then any numerically Gorenstein, $\mathbb{Q}$-Gorenstein singularity is Gorenstein.

4.2.68 Vanishing Theorems Fix a resolution and $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$. Then for $l_1, l_2 \in L_{>0}$ with $l_2 > l_1$ the morphisms $H^1(\widetilde{X}, \mathcal{L}) \to H^1(\mathcal{L} \otimes \mathcal{O}_{l_2})$ and $H^1(\mathcal{L} \otimes \mathcal{O}_{l_2}) \to H^1(\mathcal{L} \otimes \mathcal{O}_{l_1})$ are onto, and by the 'Theorem of formal functions' $H^1(\widetilde{X}, \mathcal{L}) = \varprojlim H^1(\mathcal{L} \otimes \mathcal{O}_l)$.

Theorem 4.2.69 ***Generalized Grauert–Riemenschneider Theorem*** *[31, 49, 104] Consider a line bundle $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ such that $c_1(\mathcal{L}(-K_{\widetilde{X}})) \in \Delta - \mathcal{S}_{\mathbb{Q}}$ for some $\Delta \in L'$ with $\lfloor \Delta \rfloor = 0$. Then for any $l \in L_{>0}$ one has the vanishing $h^1(l, \mathcal{L}|_l) = 0$. In particular, $h^1(\widetilde{X}, \mathcal{L}) = 0$.*

Corollary 4.2.70 *Write $\lfloor Z_K \rfloor$ as $\lfloor Z_K \rfloor_+ - \lfloor Z_K \rfloor_-$ with $\lfloor Z_K \rfloor_+, \lfloor Z_K \rfloor_- \in L_{\geq 0}$ and without common components. If $\lfloor Z_K \rfloor_+ = 0$ then $p_g = 0$. If $\lfloor Z_K \rfloor_+ > 0$ then for any $Z \geq \lfloor Z_K \rfloor_+$, $Z \in L$, $p_g = h^1(\mathcal{O}_Z)$.*

For certain cycles the Grauert-Riemenschneider Theorem 4.2.69 can be improved.

Proposition 4.2.71 (Lipman's Vanishing Theorem [56, Theorem 11.1]) *Take $l \in L_{>0}$ with $h^1(\mathcal{O}_l) = 0$ and $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ for which $(c_1\mathcal{L}, E_v) \geq 0$ for any E_v in the support of l. Then $h^1(l, \mathcal{L}) = 0$.*

4.2.9 The Role of the Monoids $\mathcal{S}$ and $\mathcal{S}'$

4.2.72 The monoids $\mathcal{S}$ and $\mathcal{S}'$ are combinatorially associated with a fixed resolution graph Γ, cf. 4.2.11.

Lemma 4.2.73 *For any fixed $h \in H$ set $L'_h := \{l' \in L' : [l'] = h\}$.*

(a) If $l'_1, l'_2 \in L'_h$ then $l' := \min\{l'_1, l'_2\} \in L'_h$ too.
(b) If $l'_1, l'_2 \in \mathcal{S}' \cap L'_h$ then $\min\{l'_1, l'_2\} \in \mathcal{S}' \cap L'_h$ too.

(For $l'_1, l'_2 \in L'$ it can happen that $\min\{l'_1, l'_2\}$, defined in $L \otimes \mathbb{Q}$, is not in L'.)

Proposition 4.2.74 *Let $\widetilde{X} \to X$ be a resolution of (X, o) as above.*

(a) For any $l' \in L'$ there exists a unique minimal element $e(l') \in L_{\geq 0}$ with $s(l') := l' + e(l') \in \mathcal{S}'$.
(b) $e(l')$ can be found by the following (generalized Laufer's) algorithm. One constructs a 'computation sequence' $z_0, z_1, \dots, z_t \in L_{\geq 0}$ with $z_0 = 0$ and $z_{i+1} = z_i + E_{v(i)}$, where the index $v(i)$ is determined by the following principle. Assume that z_i is already constructed. Then, if $l' + z_i \in \mathcal{S}'$, then one stops, and $t = i$. Otherwise, there exists at least one $v \in \mathcal{V}$ with $(l' + z_i, E_v) > 0$. Take for $v(i)$ one of these v's. Then this algorithm stops after finitely many steps, and $z_t = e(l')$.

Corollary 4.2.75 *For any $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ take $c_1 := c_1(\mathcal{L})$ and $e := e(-c_1)$. Then $c_1(\mathcal{L}(-e)) = -s(-c_1) \in -\mathcal{S}'$ and*

$$h^1(\mathcal{L}(-e)) - h^1(\mathcal{L}) = \chi(\mathcal{O}_e(c_1)) = \chi(e - c_1) - \chi(-c_1) \leq 0.$$

In particular, the computation of any $h^1(\mathcal{L})$ can be reduced, modulo the combinatorics of L, to the computation of some $h^1(\mathcal{L}')$ with $c_1(\mathcal{L}') \in -\mathcal{S}'$.

Example 4.2.76 If $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-l')$ for some $l' \in L'$ then 4.2.75 reads as

$$h^1(\mathcal{O}_{\widetilde{X}}(-s(l'))) - h^1(\mathcal{O}_{\widetilde{X}}(-l')) = \chi(\mathcal{O}_{e(l')}(-l')) = \chi(s(l')) - \chi(l') \leq 0.$$

The next consequence of Proposition 4.2.74 is the existence of the fundamental cycle.

Corollary 4.2.77

(a) *[5, 6]* $\mathcal{S} \setminus \{0\}$ *has a unique minimal element* Z_{min}.
(b) *[49]* Z_{min} *can be found by the following (Laufer's) algorithm. One constructs a computation sequence* $z_1, \ldots, z_t$ *with* $z_1 = E_w$ *(arbitrarily chosen), and* $z_{i+1} = z_i + E_{v(i)}$, *where the index* $v(i)$ *is determined as follows. Assume that* z_i *is already constructed. Then, if* $z_i \in \mathcal{S}$, *then one stops, and* $t = i$. *Otherwise, there exists at least one* $v \in \mathcal{V}$ *with* $(z_i, E_v) > 0$. *Take for* $v(i)$ *one of these* v*'s. Then this algorithm stops after finitely many steps, and* $z_t = Z_{min}$ *(independently of all the choices).*

The cycle $Z_{min} \in L_{>0}$ has several names in the literature: *minimal, fundamental, or Artin cycle*. The sequence from (b) is called the *Laufer's computation sequence for* Z_{min}.

4.2.78 The Representatives r_h and s_h Recall that for any $h \in H$, $r_h \in L'$ is the minimal representative of h in the cone $L'_{\geq 0}$. Replacing the cone $L'_{\geq 0}$ by $\mathcal{S}'$, by 4.2.73 we obtain the following.

Corollary 4.2.79 *For any* $h \in H$ *consider all the representatives* $l' + L \subset L'$ *of* h. *Then* $(l' + L) \cap \mathcal{S}'$ *has a unique minimal element* s_h.

Clearly $s_0 = 0$, and $s_h \geq r_h$. Strict inequality might appear (take e.g. the lens space $L(8, 5)$). $s_h = r_h$ if and only if $r_h \in \mathcal{S}'$, otherwise $s_h = s(r_h)$ in the sense of 4.2.74. Using 4.2.76 we obtain

$$\chi(s_h) \leq \chi(r_h). \tag{4.17}$$

Even at Euler-characteristic level, strict inequality can appear, see 4.2.89.

4.2.10 The Equivariant Geometric Genus and Laufer's Duality

4.2.80 The p_g–Formula of Laufer Let us discuss a different realizations of the geometric genus $p_g = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, where $\widetilde{X} \to X$ is any resolution.

By Serre duality $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}})$. In the exact sequence

$$H^0_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}}) \to H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^1(\widetilde{X}, \Omega^2_{\widetilde{X}})$$

$H^0_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) = 0$ while $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}) = 0$ by 4.2.69. Hence,

Proposition 4.2.81 ([49])

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}), \tag{4.18}$$

where the last vector space is the space of global holomorphic 2-forms on $\widetilde{X} \setminus E$ up to those which can be extended holomorphically across $\widetilde{X}$.

Above, the set of poles can be bounded. Indeed, for any $Z \in L_{>0}$ consider the exact sequence of sheaves

$$0 \to \Omega^2_{\widetilde{X}} \to \Omega^2_{\widetilde{X}}(Z) \to \mathcal{O}_Z(Z + K_{\widetilde{X}}) \to 0.$$

Since $h^1(\Omega^2_{\widetilde{X}}) = 0$ (cf. 4.2.69) we get that

$$H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) = H^0(\mathcal{O}_Z(Z + K_{\widetilde{X}})) = H^1(\mathcal{O}_Z)^*. \tag{4.19}$$

Assume that $p_g \neq 0$. Then from 4.2.70*(a)* $h^1(\mathcal{O}_{\lfloor Z_K \rfloor_+}) = p_g$, hence

$$p_g = \dim \left(H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(\lfloor Z_K \rfloor_+))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})\right). \tag{4.20}$$

This holds if $p_g = 0$ too. Since $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \subset H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(\lfloor Z_K \rfloor_+)) \subset H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, by (4.18) and (4.20) we get that $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(\lfloor Z_K \rfloor_+)) = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$. Hence, the poles of forms from $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$ are bounded by $\lfloor Z_K \rfloor_+$.

If (X, o) is numerically Gorenstein and $Z_K > 0$ then $\chi(Z_K) = 0$ and $h^0(\mathcal{O}_{Z_K}) = h^1(\mathcal{O}_{Z_K}) = p_g$. Hence, from the vanishing $h^1(\widetilde{X}, \mathcal{O}(-Z_K)) = 0$ we obtain

$$p_g = \dim \left(H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})/H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-Z_K))\right). \tag{4.21}$$

If (X, o) is Gorenstein and $Z_K \geq 0$, via the isomorphism $\Omega^2_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-Z_K)$ the p_g formulae from (4.20) and (4.21) agree.

4.2.82 The Geometric Genus of the Universal Abelian Covering Assume that $L(X, o)$ is a $\mathbb{Q}HS^3$.

Let $(X_a, o) \to (X, o)$ be the universal abelian covering of (X, o), and consider the notations of the diagram (4.10). By definition, the geometric genus $p_g(X_a, o)$ of (X_a, o) is $h^1(\widetilde{Z}, \mathcal{O}_{\widetilde{Z}})$. Recall that $r : \widetilde{Z} \to Z$ is the resolution of the cyclic quotient singularities of Z. Note that $r_*(\mathcal{O}_{\widetilde{Z}}) = \mathcal{O}_Z$ (by the normality of Z), and $R^1 r_*(\mathcal{O}_{\widetilde{Z}}) = 0$ since cyclic quotient singularities are rational (have geometric genus zero). Therefore, by Leray spectral sequence $p_g(X_a, o) = h^1(\mathcal{O}_Z)$. Since $\widetilde{c}$ is finite $h^1(\mathcal{O}_Z)$ equals $h^1(\widetilde{c}_*\mathcal{O}_Z)$, and it has an eigenspace decomposition $\oplus_{h\in H} H^1(\widetilde{c}_*\mathcal{O}_Z)_{\theta(h)}$. By Theorem 4.2.53 the dimension of the $\theta(h)$-eigenspace is

$$p_g(X_a, o)_{\theta(h)} := \dim\ H^1(\widetilde{c}_*\mathcal{O}_Z)_{\theta(h)} = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)).$$

By summation:

$$p_g(X_a, o) = \sum_{h\in H} h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)).$$

Clearly, for $h = 0$ we get $p_g(X_a, o)_{\theta(0)} = p_g(X, o)$.

Definition 4.2.83 If $H_1(L_X, \mathbb{Q}) = 0$ we define the equivariant geometric genus of (X, o) associated with $h \in H$ by $p_g(X_a, o)_{\theta(h)} = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h))$.

Via Proposition 4.2.75 it can also be expressed by s_h:

$$p_g(X_a, o)_{\theta(h)} = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-s_h)) + \chi(r_h) - \chi(s_h). \tag{4.22}$$

4.2.84 Laufer's formula (4.18) has the following generalization.

Proposition 4.2.85 *Assume that the link of (X, o) is a rational homology sphere and fix $h \in H$. Let l'_h be either r_h or s_h. Then*

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-l'_h))^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)) = H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}}(l'_h))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h)).$$

Remark 4.2.86 Since $H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}}(r_h)) = H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}}(s_h))$, 4.2.85 gives

$$h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) - h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \dim\ H^0(\Omega^2_{\widetilde{X}}(s_h))/H^0(\Omega^2_{\widetilde{X}}(r_h)).$$

Write $s_h - r_h = \Delta$. Then from the proof of 4.2.85 one has $H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(r_h)) = H^1(\widetilde{X}, \Omega^2_{\widetilde{X}}(s_h)) = H^1(\Omega^2_{\widetilde{X}}(s_h)|_\Delta) = 0$. Hence, the right hand side of the above identity is $\chi(\Omega^2_{\widetilde{X}}(s_h)|_\Delta) = \chi(r_h) - \chi(s_h)$, compatibly with (4.22).

4.2.87 In concrete computations it is always easier to find global sections than to determine higher cohomologies. This is one of the main advantages of the identity from 4.2.85. In several cases one can identify concrete basis for the vector space $H^0(\widetilde{X}\setminus E, \Omega^2_{\widetilde{X}}(l'_h))/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}(l'_h))$, for $l'_h = r_h$ or s_h.

Example 4.2.88 $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h))$ **for weighted homogeneous singularities**, $g = 0$.

Assume that r_h in the dual basis is written as $r_h = a_0E_0^* + \sum_{j,i} a_{ji}E_{ji}^*$. Define also $a_j := \sum_i n_{j,i+1}a_{ji}$ $(1 \le j \le \nu)$ and $N_{r_h}(\ell) = b_0\ell + a_0 - \sum_j \left\lceil \frac{\omega_j\ell - a_j}{\alpha_j} \right\rceil$. Then

$$h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) = \sum_{\ell\ge 0} \max\{0, -N_{r_h}(\ell) - 1\}. \tag{4.23}$$

Example 4.2.89 $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-s_h))$ **for weighted homogeneous singularities,** $g = 0$.

Set $s_h := \bar{a}_0E_0^* + \sum_{j,i} \bar{a}_{ji}E_{ji}^*$ and $\bar{a}_j := \sum_i n_{j,i+1}\bar{a}_{ji}$ $(1 \le j \le \nu)$. Then

$$h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \sum_{\ell\ge 0} \max\{0, -N_{s_h}(\ell) - 1\}, \tag{4.24}$$

where $N_{s_h}(\ell) = b_0\ell + \bar{a}_0 - \sum_j \left\lceil \frac{\omega_j\ell - \bar{a}_j}{\alpha_j} \right\rceil$. Set $\Delta := s_h - r_h$ and let $\Delta_0 \in \mathbb{Z}_{\ge 0}$ be the E_0-coefficient of Δ. Then $N_{s_h}(\ell) = N_{r_h}(\ell + \Delta_0)$, hence

$$h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \sum_{\ell\ge \Delta_0} \max\{0, -N_{r_h}(\ell) - 1\}. \tag{4.25}$$

In particular,

$$h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) - h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = \chi(r_h) - \chi(s_h) = \sum_{0\le\ell<\Delta_0} \max\{0, -N_{r_h}(\ell) - 1\}.$$

This expression can be non-zero. Take e.g. the graph with $b_0 = 2$, and three legs all with invariants $(\alpha_j, \omega_j) = (3, 1)$. Then $s_h = \sum_{j=1}^3 E_{js_j}^*$, $r_h = s_h - E_0$, $\chi(s_h) = h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = 0$, and $\chi(r_h) = h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) = 1$.

Example 4.2.90 For a cyclic quotient germ $h^1(\mathcal{O}_{\widetilde{X}}(-r_h)) = h^1(\mathcal{O}_{\widetilde{X}}(-s_h)) = 0$. (Use 4.2.53 and 4.2.71.)

4.2.11 Spin^c *Structures*

4.2.91 In the next discussion M is a link $L(X, o)$, which is a rational homology sphere.

M admits a spin^c structure. In fact, the set of spin^c structures $\mathrm{Spin}^c(M)$ is an $H^2(M, \mathbb{Z})$ torsor. Furthermore, the restriction $R : \mathrm{Spin}^c(\widetilde{X}) \to \mathrm{Spin}^c(M)$ is onto, where $\mathrm{Spin}^c(\widetilde{X})$ denotes the set of spin^c structures on $\widetilde{X}$. The natural cohomological morphism $H^2(\widetilde{X}, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is the factorization $L' \to L'/L$, $l' \mapsto [l']$. This projects Char onto Char$/L$. Then $c_1 : \mathrm{Spin}^c(\widetilde{X}) \twoheadrightarrow \mathrm{Char} \subset L'$ induces a map $c : \mathrm{Spin}^c(M) \twoheadrightarrow \mathrm{Char/L} \subset L'/L$ such that $c(R(\tilde{\sigma})) = [c_1(\tilde{\sigma})]$.

Moreover, $c([l'] * \sigma) = 2[l'] + c(\sigma)$ for any $[l'] \in L'/L$ and $\sigma \in \mathrm{Spin}^c(M)$.

While c_1 is injective, c in general is not. Its fibers are $H^1(M, \mathbb{Z}_2)$ torsors; $c^{-1}(0) \simeq \mathrm{Spin}(M)$. These facts will be explained next.

We consider the action of L on Char defined by $l*k := k+2l$. Let $\mathrm{Char}/2L$ be its orbit space. Then $\mathrm{Char}/2L$ is an L'/L torsor by the action induced by $l'*k = k+2l'$.

Moreover, the composition $R \circ c_1^{-1} : \mathrm{Char} \to \mathrm{Spin}^c(\widetilde{X}) \to \mathrm{Spin}^c(M)$ factorizes to $\mathrm{Char}/2L \to \mathrm{Spin}^c(M)$. This map is a bijection of L'/L torsors. In the sequel *we identify* $\mathrm{Spin}^c(M)$ *by this bijection*. Then $c : \mathrm{Spin}^c(M) \to \mathrm{Char}/L$ transforms into $c : \mathrm{Char}/2L \to \mathrm{Char}/L$. Its fibers are $\{l' \in L' : 2l' \in L\}/L \simeq H^1(M, \mathbb{Z}_2)$ torsors. The trivial element 0 of L'/L is in Char/L, and

$$c^{-1}(0) = (\mathrm{Char} \cap L)/2L \simeq \mathrm{Spin}(M),$$

where $\mathrm{Spin}(M)$ denotes the set of spin structures of M. (It is an $H^1(M, \mathbb{Z}_2)$ torsor.)

Definition 4.2.92 Let $M = L(X, o)$ be a singularity link. For any $k \in \mathrm{Char}$ we write $\widetilde{\sigma}(k)$ for that spin^c structure of $\widetilde{X}$ for which $c_1(\widetilde{\sigma}(k)) = k$. Similarly, let $\sigma[k] \in \mathrm{Spin}^c(M)$ be the restriction of $\widetilde{\sigma}(k)$ to M. The spin^c structure $\widetilde{\sigma}_{can}$ of $\widetilde{X}$ with $c_1(\widetilde{\sigma}) = K$ will be called *the canonical spin^c structure of* $\widetilde{X}$. Its restriction $\sigma_{can} \in \mathrm{Spin}^c(M)$ will be called the canonical spin^c structure of the link.

Lemma 4.2.93 *There is an involution* $\sigma \mapsto \overline{\sigma}$ *of* $\mathrm{Spin}^c(M)$ *which satisfies:* $c(\overline{\sigma}) = -c(\sigma)$, $\overline{[l'] * \sigma} = [-l'] * \overline{\sigma}$, *and* $\mathrm{Spin}(M) = \{\sigma \in \mathrm{Spin}^c(M) : \sigma = \overline{\sigma}\}$.

In algebraic geometry, by convention, the first Chern class of the 'canonical' line bundle is $K_{\widetilde{X}}$. Nevertheless, in simplectic geometry and differential topology, in the presence of an (almost) complex structure, the 'canonical' spin^c structure is usually defined via $-K_{\widetilde{X}}$. However, in this note we adopt the definition from Definition 4.2.92.

4.2.94 Definition of k_r Assume that the link is a rational homology sphere. Then $\mathrm{Spin}^c(\widetilde{X})$ is identified with the set of characteristic elements k on L', and if k and k' induces the same Spin^c structure on the link, then $k' = k + 2l$ for a certain $l \in L$. In this case $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$, hence the two functions χ_k and $\chi_{k'}$ can be easily compared, and they have identical qualitative properties. Therefore, for each class $[k] = k+2L$ (that is, for each Spin^c structure $\sigma[k]$ of L_X), we choose a representive of $[k]$. Since the set of classes is indexed by H; we define the set of representatives by $k_r := K + 2s_h$, for each $h \in H$. Since $s_0 = 0$, for the trivial class $h = 0$ we get $\chi_{k_r} = \chi$.

Since for any $x \in L$ one has $\chi_{k_r}(x) = \chi(s_h+x)-\chi(s_h)$, the function χ_{k_r} defined on the integral lattice L (up to an additive constant $\chi(s_h)$) can be identified with χ acting on the (rationally) shifted lattice $s_h + L = \{l' \in L' : [l'] = h\}$.

4.3 Multivariable Series

4.3.1 *The Divisorial Filtration*

4.3.1 Let (X, o) be a normal surface singularity, and let $\phi : (\widetilde{X}, E) \to (X, o)$ be an arbitrary fixed resolution of (X, o). We will define an L–filtration of the local ring of (X, o) and a compatible H-equivariant L'–filtration of the local ring of (X_a, o) (where $H = L'/L$). In the whole discussion regarding the universal abelian covering (X_a, o) and the L'–filtration of its local ring we will assume that the link of (X, o) is a rational homology sphere. At the level of the L–filtration of the $\mathcal{O}_{X,o}$ this assumption is not needed.

4.3.2 The Module $\mathbb{Z}[[L']]$ Once a resolution is fixed, hence the natural basis $\{E_v\}_v$ of L is fixed too, $\mathbb{Z}[[L]]$ is identified with $\mathbb{Z}[\mathbf{t}^{\pm 1}] = \mathbb{Z}[[t_1^{\pm 1}, \dots, t_s^{\pm 1}]]$. It is contained in the larger module $\mathbb{Z}[[\mathbf{t}^{\pm 1/d}]] = \mathbb{Z}[[t_1^{\pm 1/d}, \dots, t_s^{\pm 1/d}]]$, the module of formal (Laurent) power series in variables $t_v^{\pm 1/d}$, where $d := |H|$. $\mathbb{Z}[[L']] \subset \mathbb{Z}[[\mathbf{t}^{\pm 1/d}]]$ consists of the $\mathbb{Z}$-linear combinations of monomials of type $\mathbf{t}^{l'} = t_1^{l'_1} \cdots t_s^{l'_s}$ where $l' = \sum_v l'_v E_v \in L'$. $\mathbb{Z}[[L']]$ also admits several $\mathbb{Z}$-submodules corresponding to different cones of L'; e.g. $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$, generated by monomials $\mathbf{t}^{l'}$ with $l' \in L'_{\geq 0}$, or $l' \in \mathcal{S}'$ respectively. Both $\mathbb{Z}[[L'_{\geq 0}]]$ and $\mathbb{Z}[[\mathcal{S}']]$ have natural ring structure.

$\mathbb{Z}[[\mathcal{S}']]$ is a usual formal power series ring in variables $\{\mathbf{t}^{E_v^*}\}_v$: its elements are

$$\Phi(f)(\mathbf{t}) := f(\mathbf{t}^{E_1^*}, \dots, \mathbf{t}^{E_s^*}), \quad \text{where } f(x_1, \dots, x_s) \in \mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, \dots, x_s]]. \tag{4.26}$$

Any series $S(\mathbf{t}) = \sum_{l'} a_{l'} \mathbf{t}^{l'} \in \mathbb{Z}[[L']]$ decomposes in a unique way as

$$S = \sum_{h \in H} S_h, \quad \text{where } S_h = \sum_{[l']=h} a_{l'} \mathbf{t}^{l'}. \tag{4.27}$$

S_h is called the h-component of S. E.g., if $S(\mathbf{t}) := \Phi(f)(\mathbf{t})$ for some $f \in \mathbb{Z}[[\mathbf{x}]]$ as in (4.26) then

$$S_h(\mathbf{t}) = \frac{1}{|H|} \cdot \sum_{\rho \in \widehat{H}} \rho(h)^{-1} \cdot f(\rho([E_1^*])\mathbf{t}^{E_1^*}, \dots, \rho([E_s^*])\mathbf{t}^{E_s^*}). \tag{4.28}$$

4.3.2 *The Analytic Series* $H(\mathbf{t})$ *and* $P(\mathbf{t})$

Consider the diagram and the notations regarding the universal abelian covering from 4.2.49. Set $\phi_a = \psi_a \circ r$ and $p = \widetilde{c} \circ r$.

Recall that by (4.12) $p^*(l')$ is an *integral* cycle for any $l' \in L'$.

Definition 4.3.3 The L'–filtration on the local ring of holomorphic functions $\mathcal{O}_{X_a,o}$ is defined as follows. For any $l' \in L'$, we set

$$\mathcal{F}(l') := \{f \in \mathcal{O}_{X_a,o} \mid \mathrm{div}(f \circ \phi_a) \geq p^*(l')\}. \tag{4.29}$$

Notice that the natural action of H on (X_a, o) induces an action on $\mathcal{O}_{X_a,o}$, which keeps $\mathcal{F}(l')$ invariant. Therefore, H acts on $\mathcal{O}_{X_a,o}/\mathcal{F}(l')$ as well. For any $l' \in L'$, let $\mathfrak{h}(l')$ be the dimension of the $\theta([l'])$-eigenspace $(\mathcal{O}_{X_a,o}/\mathcal{F}(l'))_{\theta([l'])}$. Then one defines the Hilbert series $H(\mathbf{t})$ by

$$H(\mathbf{t}) := \sum_{l' \in L'} \mathfrak{h}(l') \cdot \mathbf{t}^{l'} \in \mathbb{Z}[[L']]. \tag{4.30}$$

Example 4.3.4 The 0-component of $H(\mathbf{t})$ is

$$H_0(\mathbf{t}) = \sum_{l \in L} \dim\left(\mathcal{O}_{X,o} / \{f \in \mathcal{O}_{X,o} : \mathrm{div}_E(f \circ \phi) \geq l\}\right) \cdot \mathbf{t}^l.$$

This is the Hilbert series of $\mathcal{O}_{X,o}$ associated with the divisorial filtration $L \ni l \mapsto \mathcal{F}_0(l) = \{f \in \mathcal{O}_{X,o} : \mathrm{div}_E(f \circ \phi) \geq l\}$ of all irreducible exceptional divisors of ϕ.

4.3.5 Next, we define the Poincaré series $P(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \mathfrak{p}(l')\mathbf{t}^{l'}$ associated with the filtration $\{\mathcal{F}(l')\}_{l'}$.

$$P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1}), \ \text{ or } \ \mathfrak{p}(l') = \sum_{I \subset \{1,\dots,s\}} (-1)^{|I|+1} \mathfrak{h}(l' + E_I), \quad (E_I = \sum_{v \in I} E_v). \tag{4.31}$$

It turns out that the series $P(\mathbf{t})$ is supported in $\mathcal{S}'$, and the following 'inversion identities' hold:

$$\mathfrak{h}(l') = \sum_{l \in L,\, l \ngeq 0} \mathfrak{p}(l' + l). \tag{4.32}$$

Proposition 4.3.6 *Let* $P_0(\mathbf{t}) = \sum_{l \in \mathcal{S}} \mathfrak{p}(l)\mathbf{t}^l$ *be the 0-component of* $P(\mathbf{t})$. *Then for* $l \in L$

$$h^1(\mathcal{O}_{\widetilde{X}}(-l)) = -\sum_{\tilde{l} \in L,\, \tilde{l} \ngeq l} \mathfrak{p}(\tilde{l}) + \chi(l) + p_g. \tag{4.33}$$

If $l \leq 0$, *then the sum on the right hand side is empty.*

If $l \in (-K_{\widetilde{X}} + \mathcal{S}') \cap L$ *then by the vanishing Theorem 4.2.69*

$$\sum_{\tilde{l}\in L,\ \tilde{l}\ngeq l} \mathfrak{p}(\tilde{l}) = \chi(l) + p_g. \tag{4.34}$$

That is, the counting function of the coefficients of $P_0(\mathbf{t})$*, associated with the special truncation* $\{\tilde{l} \in \mathcal{S},\ \tilde{l} \ngeq l\}$*, evaluated in the chamber* $-K + \mathcal{S}'$*, equals the quadratic polynomial* $\chi(l) + p_g$*.*

In particular, $P_0(\mathbf{t})$ *determines completely* p_g *and the functions* $l \mapsto \chi(l)$*,* $l \mapsto h^1(\mathcal{O}_{\widetilde{X}}(l))$ *(*$l \in L$*).*

4.3.7 The Equivariant Version of Proposition 4.3.6 Next, we assume that the link of (X, o) is a rational homology sphere. In particular, the universal abelian covering is well defined with its H-action. Recall that the geometric genus of (X_a, o) is the sum $\sum_h h^1(\mathcal{O}(-r_h))$ (of the equivariant genera of (X, o)) corresponding to the eigenspace decomposition of $H^1(\mathcal{O}_Z)$. Let l'_h be either r_h or s_h. Then for any fixed h the equivariant analogues of the formulae from Example 4.3.6 are the following.

For $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-l')$, where $l' \in L'$, $l' = l + l'_h$ with $l \in L$,

$$\begin{aligned} h^1(\mathcal{O}(-l')) &= -\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathfrak{p}(\tilde{l}') + \chi_{K+2l'_h}(l) + h^1(\mathcal{O}(-l'_h)) \\ &= -\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathfrak{p}(\tilde{l}') + \chi(l') + h^1(\mathcal{O}(-l'_h)) - \chi(l'_h). \end{aligned} \tag{4.35}$$

In particular, when $l' \in -K + \mathcal{S}'$, $l' = l + l'_h$ with $l \in L$,

$$\begin{aligned} \sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathfrak{p}(\tilde{l}') =& \chi_{K+2l'_h}(l) + h^1(\mathcal{O}(-l'_h)) \\ =& \chi(l') + h^1(\mathcal{O}(-l'_h)) - \chi(l'_h). \end{aligned} \tag{4.36}$$

Therefore, $P(\mathbf{t})$ determines completely each $h^1(\mathcal{O}_{\widetilde{X}}(l'))$ ($l' \in L'$).

Remark 4.3.8 The following comment is appropriate. In the above formulae (e.g. in 4.3.6 and 4.3.7) the term consisting of the sum of the coefficients of P can be replaced (via (4.32)) by the corresponding coefficient of the Hilbert series $H(\mathbf{t})$. E.g., (4.34), under the same assumption, reads as $\mathfrak{h}(l) = \chi(l) + p_g$. The corresponding versions in terms of the Hilbert series are simpler (and from the analytic point of view even more conceptual). The reason why we prefer above the summation expressions is the following. Later we will introduce the topological analogues of the above identities. The point is that $P(\mathbf{t})$ will have a topological analogue, namely $Z(\mathbf{t})$ (see subsection 4.3.3), however, the analogue of $H(\mathbf{t})$ will be defined ('merely') as the inversion of $Z(\mathbf{t})$, that is, by the summation of its coefficients. Hence, later we will hunt in the topological side for sum–expressions as above, where the coefficients of P will be replaced by those of Z.

4.3.3 The Topological Series $Z(\mathbf{t})$

4.3.9 We assume that L_X is a $\mathbb{Q}HS^3$ and we fix a good resolution as above.

Definition 4.3.10 We define the rational function $Z(\mathbf{t})$ in variables $x_v = \mathbf{t}^{E_v^*}$ by

$$Z(\mathbf{t}) := \Phi(z)(\mathbf{t}), \quad \text{where} \quad z(\mathbf{x}) := \prod_{v\in\mathcal{V}} (1-x_v)^{\kappa_v-2}. \tag{4.37}$$

Hence $Z(\mathbf{t}) = \prod_v (1-\mathbf{t}^{E_v^*})^{\kappa_v-2}$. By (4.28), its h-component for any $h \in H$ is

$$Z_h(\mathbf{t}) := \frac{1}{|H|} \cdot \sum_{\rho\in\widehat{H}} \rho(h)^{-1} \cdot \prod_{v\in\mathcal{V}} \left(1-\rho([E_v^*])\mathbf{t}^{E_v^*}\right)^{\kappa_v-2}. \tag{4.38}$$

In the sequel we identify the rational function $Z(\mathbf{t})$ with its Taylor expansion at the origin, as an element of $\mathbb{Z}[[\mathcal{S}']]$ (cf. 4.26).

Example 4.3.11 (Splice Quotient Singularities) Splice quotient singularities were introduced by Neumann and Wahl in [91]. From any fixed graph Γ (for which $M(\Gamma)$ is a $\mathbb{Q}HS^3$ and Γ has some additional special arithmetical properties too, see below) one constructs a family of singularities with common equisingularity type, such that any member admits a distinguished resolution, whose dual graph is exactly Γ. The construction suggests that the analytic properties of the singularities constructed in this way are strongly linked with the fixed resolution and with its graph Γ. (Hence, the expectation is that certain analytic invariants might be computable from Γ.)

There are three different approaches how one can define the splice quotient singularities; they are based on different geometric properties: (a) the 'original' construction of Neumann–Wahl [91] (where Γ satisfies the additional *semigroup and the congruence conditions*), (b) the 'modified' version by Okuma [97] (where Γ satisfies the *monomial condition*), and (c) considering resolution of singularities satisfying the *end-curve condition* [93, 98]. It turns out that all these approaches provide the same family of singularities.

Rational singularities (where ϕ is an arbitrary resolution), minimally elliptic singularities, (where ϕ is a resolution in which the support of the minimal elliptic cycle is E), and weighted homogeneous singularities (where ϕ is the minimal good resolution) are splice quotient singularities.

Theorem 4.3.12 ([75]) *Assume that (X,o) admits a resolution ϕ, which satisfies the end curve condition, and $H^1(\widetilde{X},\mathbb{Z})=0$. Then $P(\mathbf{t}) = Z(\mathbf{t})$.*

Conversely, assume that the singularity (X, o) *satisfies* $H^1(\widetilde{X}, \mathbb{Z}) = 0$*, and we fix one of its good resolutions* ϕ*. If associated with* ϕ *one has* $P(\mathbf{t}) = Z(\mathbf{t})$*, then the 'end curve condition' for* ϕ *is also satisfied.*

Corollary 4.3.13 *Assume that* (X, o) *admits a resolution* ϕ*, which satisfies the end curve condition, and* $H^1(\widetilde{X}, \mathbb{Z}) = 0$*. Then* $h^1(\mathcal{O}_{\widetilde{X}}(l'))$ *is topological for any* $l' \in L'$*.*

Indeed, write $Z(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \mathfrak{z}(l')\mathbf{t}^{l'}$*. Then, after the identification* $P(\mathbf{t}) = Z(\mathbf{t})$*, the formulae from 4.3.7 read as follows:*

1. *For* $l' \in -K + \mathcal{S}'$

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}' \ngeq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l' - r_h) + h^1(\mathcal{O}_{\widetilde{X}}(-r_h)); \tag{4.39}$$

2. *More generally, for* $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(-l')$ *with arbitrary* $l' \in L'$*,*

$$h^1(\mathcal{O}_{\widetilde{X}}(-l')) = - \sum_{[\tilde{l}']=[l'],\ \tilde{l}' \ngeq l'} \mathfrak{z}(\tilde{l}') + \chi_{K_{\widetilde{X}}+2r_h}(l' - r_h) + h^1(\mathcal{O}_{\widetilde{X}}(-r_h)). \tag{4.40}$$

4.3.4 Reductions of Variables in the Series $P(\mathbf{t})$ and $Z(\mathbf{t})$

For any fixed resolution ϕ, in the definition of the series $P(\mathbf{t})$ and $Z(\mathbf{t})$ one takes a variable t_v for each exceptional divisor E_v of ϕ. In most of the situations we strongly suspect that some of the variables are superfluous. E.g., if the resolution is not minimal, the non-essential exceptional components carry less information; the same is valid even for some of the exceptional curves of the minimal resolution, e.g. those with $\kappa_v = 2$. Moreover, certain exceptional divisors might have some intrinsic geometric meaning, and sometimes we wish to concentrate only on them.

4.3.14 We fix (X, o) as in 4.3.1 and the resolution ϕ. Let $\mathcal{I}$ be a non-empty subset of $\mathcal{V}$. Associated with it we consider formal series in variables $\{t_v\}_{v \in \mathcal{I}}$, denoted by $\mathbf{t}_{\mathcal{I}}$, and the projection $\pi_{\mathcal{I}} : L' \to L \otimes \mathbb{Q}$, $\pi_{\mathcal{I}}(\sum_{v \in \mathcal{V}} l'_v E_v) = \sum_{v \in \mathcal{I}} l'_v E_v$. We write

$$l'_{\mathcal{I}} := \pi_{\mathcal{I}}(l'), \quad \text{and } \mathbf{t}_{\mathcal{I}}^{l'} = \prod_{v \in \mathcal{I}} t_v^{l'_v} = \mathbf{t}^{l'}|_{t_v=1 \text{ for all } v \notin \mathcal{I}}.$$

Here a word of warning is necessary. In the original case $\mathcal{I} = \mathcal{V}$, from a series $S(\mathbf{t}) = \sum_{l'} a_{l'}\mathbf{t}^{l'}$ we can recover its h-components S_h. Indeed, the monomial $a_{l'}\mathbf{t}^{l'}$ belongs to S_h if and only if $[l'] = h$. However, this property will be lost when we reduce the variable: from the information carried by $\pi_{\mathcal{I}}(l')$ one cannot recover $[l']$. Therefore, *the reduced h-components of a series* $S(\mathbf{t})$ *are defined as the* *reductions of the original h-components* $S_h(\mathbf{t})$ (*and they cannot be recovered from the reduced* S).

Definition 4.3.15 The **reduced series of** Z is defined as $Z_{\mathcal{I}}(\mathbf{t}_{\mathcal{I}}) := Z(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathcal{I}}$. Similarly, for any $h\in H$, $Z_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}}) := Z_h(\mathbf{t})|_{t_v=1 \text{ for all } v\notin\mathcal{I}}$. Equivalently,

$$Z_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}}) := \frac{1}{|H|}\cdot\sum_{\rho\in\widehat{H}}\rho(h)^{-1}\cdot\prod_{v\in\mathcal{V}}\left(1-\rho([E_v^*])\mathbf{t}_{\mathcal{I}}^{E_v^*}\right)^{\kappa_v-2}. \tag{4.41}$$

The substitutions $\{t_v=1\}_{v\notin\mathcal{I}}$ are well-defined since $Z(\mathbf{t})$ is supported on $\mathcal{S}'$, which has the special finiteness property 4.2.13.

4.3.16 Reducing Variables in Series $P(\mathbf{t})$ In the case of the analytic series $P(\mathbf{t})$ we can proceed, a priori, in two different ways. By the first one we reduce $P(\mathbf{t})$ 'blindly', as we did with $Z(\mathbf{t})$ in 4.3.15, via substitutions $t_v=1$ for all $v\notin\mathcal{I}$. Again, this step is well-defined since P too is supported on $\mathcal{S}'$.

On the other hand, we can also repeat the original geometric definition of $P(\mathbf{t})$, as the multivariable Poincaré series associated with the divisorial filtration as in (4.31), however, at this time we will use the 'reduced set of divisors' indexed by $\mathcal{I}$. However, it turns out that the two approaches lead to the same object.

Corollary 4.3.17 *Assume that for a resolution* ϕ *and an element* $h\in H$ *the identity* $P_h(\mathbf{t})=Z_h(\mathbf{t})$ *is valid. Then for the same* ϕ *and* h *and for any non-empty* $\mathcal{I}\subset\mathcal{V}$ *the 'reduced identity'* $Z_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}})=P_{h,\mathcal{I}}(\mathbf{t}_{\mathcal{I}})$ *(in* $\mathbb{Z}[[t_v^{1/\det(I)}, v\in\mathcal{I}]]$*) is valid too.*

In Sects. 4.3.5 and 4.3.6 we exemplify cases when $\mathcal{I}$ contains only one element. Our goal is to compare the analytic reduced series $P_{h,\mathcal{I}}$ with the topological series $Z_{h,\mathcal{I}}$.

4.3.5 Example: P and Z for Weighted Homogeneous Germs

Assume that (X,o) is weighted homogeneous and its minimal good resolution is star-shaped with $\nu\geq 3$. We set $\mathcal{I}=\{\text{central vertex } v_0\}$.

Our plan is to compare three filtrations and to show that they agree.

Firstly, the E_0-divisorial filtration coincides with the filtration given by the $\mathbb{C}^*$ action.

Assume next that $g=0$, hence the universal abelian covering is well-defined, it is a Brieskorn isolated complete intersection singularity. Therefore, one has three equivariant $\mathbb{Z}$–filtrations of $\mathcal{O}_{X_a,o}$: the divisorial filtration $\mathcal{F}_{\mathcal{I}}$ associated with the central divisor E_0, the filtration/grading associated with the $\mathbb{C}^*$-action, and the monomial filtration $\mathcal{G}_{\mathcal{I}}$ associated with v_0.

The monomial filtration is determined by the following grading. If we denote the variables of the Brieskorn equations by $\{z_i\}_{i=1}^{\nu}$, then their degrees are $\deg(z_i)=\deg(E_{s_i}^*)=(\alpha_i|e|)^{-1}$ $(1\leq i\leq\nu)$. The degree of the Brieskorn equations of the universal abelian covering are $|e|^{-1}$ (hence the Brieskorn exponent of z_i is α_i). This coincides exactly with the weights of the $\mathbb{C}^*$-action on (X_a, o). In particular, the monomial filtration and the filtration induced by the $\mathbb{C}^*$-action agree. Similarly as above, the filtration induced by the $\mathbb{C}^*$-action and the divisorial filtrations agree too.

The (common) Poincaré series of the above filtrations agree with the topological series $Z_{h,\mathcal{I}}(t)$ (the variable t corresponds to v_0). This fact can be seen in many different ways (see e.g. [79, 88, 103]). E.g.:

(i) The identity $P = Z$ was proved for any singularity which satisfies the end curve condition. Then the identity $P_{h,\mathcal{I}} = Z_{h,\mathcal{I}}$ follows from 4.3.17 (since the minimal good resolution of a weighted homogeneous germ satisfies the end curve condition).
(ii) If $h = 0$ then the Poincaré series of the graded $\mathcal{O}_{X,o}$ was computed analytically via the Dolgachev–Pinkham–Demazure technique, the output is identical with $Z_{h,\mathcal{I}}(t)$, cf. 4.2.28.

For any fixed $h \in H$, let $l'_h \in L'$ be one of its representatives. If $l'_h = a_0E^*_0 + \sum_{ik} a_{ik}E^*_{ik}$, then $l'_{\text{red}} := a_0E^*_0 + \sum_{ik} a_{ik}n^i_{k+1,s_i}E^*_{is_i}$ is still a representative, and

$$\mathfrak{a} := \pi_{\mathcal{I}}(l') = \pi_{\mathcal{I}}(l'_{\text{red}}) = -(E^*_0, l') = \frac{1}{|e|}\cdot\Big(a_0 + \sum_j \frac{a_j}{\alpha_j}\Big) \in \frac{1}{\mathfrak{o}}\mathbb{Z}.$$

The rational number $\mathfrak{a}$ modulo $\mathbb{Z}$ is independent of the choice of the representative l'_h, it depends only on h (and any integral shift can be realized by different choices). In particular, $\pi_{\mathcal{I}}(L + r_h) = \mathfrak{a} + \mathbb{Z}$.

The common Poincaré series is given by

$$P_{h,\mathcal{I}}(t) = \sum_{\ell\in\mathbb{Z},\ \ell\geq-\mathfrak{a}} \max\Big\{\,0\,,\ 1 + a_0 + \ell b - \sum_j \Big\lceil\frac{\ell\omega_j - a_j}{\alpha_j}\Big\rceil\Big\}\cdot t^{\ell+\mathfrak{a}}.$$

With the choice $l'_h = r_h$ one has $\mathfrak{a} \in [0, 1)$.

This expression can also be compared with another expression obtained via a rather different construction, namely via the universal cycles $x(\ell)$ and their τ-function, cf. 4.7.22.

4.3.6 *Example: P_0 and Z_0 for Superisolated Singularities*

Next, we compute the one-variable $\{v_+\}$–reduced series P_0 and Z_0 for superisolated singularities associated with an irreducible curve C, and we formulate geometric properties and conjectures about their difference. Such properties might serve as combinatorial criteria for the existence of the rational cuspidal curve C with given topology.

4.3.18 Assume that (X, o) is a superisolated singularity with C irreducible and with a rational homology sphere link, cf. subsection 4.2.4. Let ϕ be its minimal good resolution described in 4.2.31 and 4.2.32. We set $\mathcal{I} = \{v_+\}$ (the vertex corresponding to the curve) and $h = 0$.

Set $\Delta(t) := \prod_i \Delta_i$. Then $\Delta(1) = 1$ and $d\Delta/dt(1) = \delta$, where $\delta = \sum_i \delta_i = (\sum_i \mu_i)/2 = (d-1)(d-2)/2$ is the sum of delta-invariants. Hence, Δ can be written as $\Delta(t) = 1+(t-1)\delta+(t-1)^2 Q(t)$ for an integral polynomial $Q(t) = \sum_{j=0}^{2\delta-2} \alpha_j t^j$ (see 4.2.30). For $\nu = 1$ one has $Q(t) = \sum_{s\notin \mathcal{S}_{C,p_1}} (1 + t + \cdots + t^{s-1})$, hence

$$\alpha_j = \#\{s \notin \mathcal{S}_{C,p_1} \ : \ s > j\} \quad (\text{if } \nu = 1). \tag{4.42}$$

Since $s \notin \mathcal{S}_{C,p_1}$ if and only if $2\delta - 1 - s \in \mathcal{S}_{C,p_1}$, we get

$$\alpha_{(d-3-j)d} = \#\{s \in \mathcal{S}_{C,p_1} \ : \ s \leq jd\} \quad (\text{if } \nu = 1,\ 0 \leq j \leq d-3). \tag{4.43}$$

4.3.19 We wish to compare $P_{0,\mathcal{I}}(t)$ and $Z_{0,\mathcal{I}}(t)$. Firstly, $P_{0,\mathcal{I}}(t) = (1-t^d)/(1-t)^3$.

By the definition of $Z_{0,\mathcal{I}}$, and from A'Campo's formula (and using the fact that $H = \mathbb{Z}_d$ is generated by $[E_+]$), we obtain

$$Z_{0,\mathcal{I}}(t) = \frac{1}{d}\sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2}\,.$$

Lemma 4.3.20 *The difference*

$$N(t) := Z_{0,\mathcal{I}}(t) - P_{0,\mathcal{I}}(t) = \frac{1}{d}\sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} - \frac{1-t^d}{(1-t)^3} \tag{4.44}$$

has the following properties:

(a) $N(0) = 0$, *and* $N(t)$ *is a symmetric polynomial:* $N(t) = t^{d-3}\cdot N(1/t)$.
(b)

$$N(t) = \sum_{j=0}^{d-3}\Big(\alpha_{(d-3-j)d} - \frac{(j+1)(j+2)}{2}\Big)\, t^{d-3-j}.$$

Assume that $\nu = 1$. Then 4.3.20*(b)* combined with (4.43) says that the Semigroup Distribution Property guarantees the vanishing of $N(t)$. However, for $\nu \geq 2$, $N(t) \neq 0$ might appear (see [24]). Several examples computed in [loc. cit.] supported the following (hasty) conjecture.

Conjecture 4.3.21 ([24]) All the coefficients of $N(t)$ are non-positive for any rational cuspidal curve.

If $\nu = 1$ then the conjecture is true since $N(t) \equiv 0$. If $\nu = 2$ then the Conjecture is true again, it follows from the Semigroup Distribution Property and certain lattice cohomology formulae of the link of superisolated singularities; the method even provides a conceptual meaning of the coefficients of $-N(t)$ in terms of ranks of certain first lattice cohomology groups. See subsection 4.9.2 for a detailed discussion.

However, the conjecture fails for certain curves with $\nu = 3$ [8].

A 'weaker' version of Conjecture 4.3.21 was formulated in [8], it is a numerical inequality (instead of a polynomial one); in fact, it is more in the spirit of the motivation of the original Conjecture 4.3.21, since it is a reformulation of an inequality between the geometric genus of a superisolated singularity and the normalized Seiberg–Witten invariant of the link (see again subsection 4.9.2 for the complete discussion).

Conjecture 4.3.22 ([8]) $N(1) \leq 0$ for any rational cuspidal curve.

Note that by Lemma 4.3.20*(b)* one has:

$$N(1) = \sum_{j=0}^{d-3} \alpha_{(d-3-j)d} - \frac{d(d-1)(d-2)}{6} = -p_g + \sum_{j=0}^{d-3} \alpha_{(d-3-j)d}. \tag{4.45}$$

Clearly, Conjecture 4.3.21 implies this second one, hence by the above discussion Conjecture 4.3.22 for $\nu \leq 2$ is also true. Moreover, in [8] a case-by-case verification provides its validity for all the 'known' curves (which, conjecturally, provide all the possible combinatorial types with $\nu \geq 3$).

4.3.7 The Periodic Constant of One-Variable Series

Definition 4.3.23 ([82, 3.9], [97]) Let $F(t) = \sum_{i\geq 0} a_i t^i$ be a formal power series. Suppose that there exist a positive integer p and a polynomial $\mathfrak{P}_p(t)$ such that $\sum_{0\leq i<pn} a_i = \mathfrak{P}_p(n)$ for every $n \in \mathbb{Z}_{>0}$. We call the constant term $\mathfrak{P}_p(0)$ the *periodic constant* of F and we denote it by $\mathrm{pc}(F)$. The integer p is called the 'period'. Furthermore, we extend the above definition to expressions of type $t^r \cdot F(t)$ via $\mathrm{pc}(t^r F(t)) := \mathrm{pc}(F(t))$, where F is a power series as above and $r \in \mathbb{Q} \cap [0, 1)$.

If the periodic constant exists then it is independent of the choice of the period p.

If F_1 and F_2 admit periodic constants, then the same is true for the series $F_1 + F_2$, cF_1 (where $c \in \mathbb{C}$), $F_1(t^m)$ (where $m \in \mathbb{Z}_{>0}$). Moreover, $\mathrm{pc}(F_1 + F_2) = \mathrm{pc}(F_1) + \mathrm{pc}(F_2)$, $\mathrm{pc}(cF_1) = c \cdot \mathrm{pc}(F)$, $\mathrm{pc}(F_1(t^m)) = \mathrm{pc}(F_1(t))$.

If $F(t)$ is a finite sum (i.e. it is a polynomial), then $\mathrm{pc}(F)$ exists and equals $F(1)$.

For certain rational functions, one has the following equivalent description. (Here, we identify a rational function R with its Taylor expansion at the origin.) Clearly, any rational function can be written in a unique way as $R = R^+ + R^-$, where R^+ is a polynomial and R^- is a rational function of negative degree.

Lemma 4.3.24 *Let R be a rational function having poles only at infinity or at certain roots of unity. Then R admits a periodic constant and* $\mathrm{pc}(R) = R^+(1)$.

Example 4.3.25 Recall that for **cyclic quotients** (with $s > 1$) $Z(\mathbf{t}) = (1 - \mathbf{t}^{E_1^*})^{-1}(1 - \mathbf{t}^{E_s^*})^{-1}$, which equals also $P(\mathbf{t})$. We fix $\mathcal{I} = \{v_1\}$ and $h = e^{2\pi i a/n}$ $(0 \leq a < n)$. Then $Z_{h,\mathcal{I}}$ equals $t^{a/n} \cdot \sum_{m\geq 0}(1 + \lfloor (a + nm)/q \rfloor)t^m$.

For the period it is convenient to take q, and one can check that $\mathrm{pc}(Z_{h,\mathcal{I}}) = 0$.

Example 4.3.26 Fix a **weighted homogeneous germ** with $g = 0$ and the representative r_h. Take $\mathcal{I}$ consisting of the central vertex E_0. Then, with the above notations (where $\mathfrak{a} \in [0, 1)$ stays for $-(r_h, E_0^*)$)

$$P_{h,\mathcal{I}}(t) = Z_{h,\mathcal{I}}(t) = \sum_{\ell\geq 0} \max\{0, 1 + N_{r_h}(\ell)\}t^{\ell+\mathfrak{a}}.$$

By a computation $Z^+_{h,\mathcal{I}}(t) = \sum_{\ell\geq 0} \max\{0, -1 - N_{r_h}(\ell)\}t^{\ell+\mathfrak{a}}$. Thus, by (4.23),

$$\mathrm{pc}(P_{h,\mathcal{I}}(t)) = \mathrm{pc}(Z_{h,\mathcal{I}}(t)) = \sum_{\ell\geq 0} \max\{0, -1 - N_{r_h}(\ell)\} = h^1(\mathcal{O}_{\widetilde{X}}(-r_h)).$$

4.3.8 Okuma's Additivity Formula

4.3.27 The Setup Consider a normal surface singularity (X, o) and fix one of its resolutions $\phi : \widetilde{X} \to X$. We fix a vertex $v \in \mathcal{V}$. Let $\cup_{j\in J}\Gamma_j$ be the connected components of the graph obtained from Γ by deleting v and its adjacent edges. Assume that v is connected to each Γ_j by exactly one edge. Let X' be the space obtained from $\widetilde{X}$ by contracting (via τ) all irreducible exceptional curves to normal points except E_v. It has $|J|$ normal singular points $\{o_j\}_j$, which are the images of the connected components of $E \setminus E_v$. Let X_j be a small Stein neighbourhood of o_j in X', and $\widetilde{X}_j = \tau^{-1}(X_j)$ its pre-image via the contraction $\tau : \widetilde{X} \to X'$. We denote the local singularities by (X_j, o_j). They are resolved by $\widetilde{X}_j$ with dual graphs Γ_j. Set $\tau(E) = E' \subset X'$. The resolution $\phi : \widetilde{X} \to X$ and the contraction $\tau : \widetilde{X} \to X'$ induce an analytic modification $\phi' : X' \to X$ with (irreducible) exceptional curve E'.

We say that the Assumption (C) is satisfied if

$$(C) \qquad nE' \subset X' \text{ is a Cartier divisor for a certain } n > 0.$$

Theorem 4.3.28 (Additivity for $\mathcal{O}_{\widetilde{X}}$ [97]) *If Assumption (C) is satisfied then $P_{0,\mathcal{I}}(t)$ admits a periodic constant and*

$$p_g(X, o) = \mathrm{pc}(P_{0,\mathcal{I}}(t)) + \sum_j p_g(X_j, o_j).$$

4.3.29 Additivity for Natural Line Bundles Assume that $H^1(\widetilde{X}, \mathbb{Z}) = 0$.

Theorem 4.3.30 *Set $\mathcal{I} = \{v\}$ and fix $h \in H$. Under the Assumption (C)*

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)) = \mathrm{pc}(P_{h,\{v\}}(t)) + \sum_j h^1(\widetilde{X}_j, \mathcal{O}_{\widetilde{X}}(-r_h)|_{\widetilde{X}_j}).$$

4.4 The Seiberg–Witten Invariant Conjecture

4.4.1 The Casson Invariant

4.4.1 The Setup Let M be an oriented integral homology 3–sphere. The original definition of the Casson invariant $\lambda(M)$ given by Casson is based on a Heegaard splitting of M, and on the study of the space of conjugacy classes of SU_2-representations of different fundamental groups of the splitting [2, 26].

Here we will adopt a specific surgery formula of $\lambda(M)$ as starting definition, valid for any plumbed manifold $M(\Gamma)$. It was proved in the PhD thesis of A. Ratiu (Paris VII), and it follows also from the surgery formulae from Lescop's book [55].

Definition 4.4.2 Assume that M is the plumbed manifold of a connected negative definite graph Γ. Then

$$-24 \cdot \lambda(M) = \sum_{v \in \mathcal{V}} (E_v^2 + 3) + \sum_{v \in \mathcal{V}} (2 - \kappa_v)(E_v^*, E_v^*). \tag{4.46}$$

We extend the definition of λ by the same expression for non-connected graphs as well, (i.e., for connected sums of negative definite plumbed 3-manifolds). One verifies that the expression from the right hand side depends only on $M(\Gamma)$, i.e., it is stable to the plumbing calculus of negative definite plumbing graphs.

By a computation $\lambda(S^3) = 0$ and $\lambda(\Sigma(2, 3, 5)) = \lambda(\Sigma(2, 3, 7)) = -1$.

Example 4.4.3 If M is a Seifert 3-manifold, then

$$-24 \cdot \lambda(L_X) = \frac{1}{e}\Big(2 - \nu + \sum_{j=1}^{\nu} \frac{1}{\alpha_j^2}\Big) + e + 3 - 12 \cdot \sum_{j=1}^{\nu} \mathbf{s}(\omega_j, \alpha_j). \tag{4.47}$$

In this case (X, o) is a Brieskorn–Hamm complete intersection

$$\Big\{(z_1, \dots, z_\nu) \,:\, \sum_j a_{ij} z_j^{\alpha_j} = 0 \ \text{ for } \ 1 \le i \le \nu - 2\Big\}$$

with (a_{ij}) of full rank. Hence $L(X, o) = M = \Sigma(\alpha_1, \dots, \alpha_\nu)$. Furthermore, the integers $\{\alpha_k\}_k$ are pairwise relatively prime, and the integers ω_j's are determined from $\{\alpha_k\}_k$ by

$$\omega_j \cdot (\prod_k \alpha_k)/\alpha_j \equiv -1 \ (\mathrm{mod}\ \alpha_j).$$

Hence

$$\mathbf{s}(\omega_j, \alpha_j) = -\mathbf{s}((\prod_k \alpha_k)/\alpha_j, \alpha_j).$$

In this case one also has $e^{-1} = -\prod_k \alpha_k$. Note also that

$$\lambda(\Sigma(\alpha_1, \dots, \alpha_\nu)) = \lambda(\Sigma(\alpha_1, \dots, \alpha_j, \alpha_{j+1} \cdots \alpha_\nu)) + \lambda(\Sigma(\alpha_1 \cdots \alpha_j, \alpha_{j+1}, \dots, \alpha_\nu)). \tag{4.48}$$

In particular, the computation of $\lambda(\Sigma(\alpha_1, \dots, \alpha_\nu))$ can be reduced to the case $\nu = 3$. On the other hand, if $M = \Sigma(\alpha_1, \alpha_2, \alpha_3)$, then one also has

$$\lambda(M) = -\tfrac{1}{2} \cdot \{\text{number of irreducible SU}_2\text{-representations of } \pi_1(M) \text{ up to conjugation}\}. \tag{4.49}$$

Additionally, in [11, 27] is proved that the Casson invariant is additive with respect to the splice decomposition. In particular, $\lambda(L(X,o))$ equals the sum of Casson invariants of the splice components of $L(X,o)$. Since all of them are of type $\Sigma(\alpha_1, \dots, \alpha_\nu)$, we obtain that for any singularity link $\lambda(L(X,o)) \leq 0$, and $\lambda(L(X,o)) = 0$ if and only if $L(X,o) = S^3$.

4.4.2 The Casson Invariant Conjecture of Neumann–Wahl

Based on a result of Fintushel and Stern [26], valid for $\Sigma = \Sigma(\alpha_1, \alpha_2, \alpha_3)$, which identifies the irreducible SU_2-representations of $\pi_1(\Sigma)$ with Brieskorn formula for the signature of the Milnor fiber (cf. 4.49), Neumann and Wahl formulated the following conjecture.

Conjecture 4.4.4 (Casson Invariant Conjecture (CIC) [90]) Assume that (X,o) is an isolated complete intersection singularity of dimension two, whose link $L(X,o)$ is an integral homology sphere. Let $\sigma(F)$ be the signature of its Milnor fiber F. Then $\lambda(L(X,o)) = \sigma(F)/8$. *(Since the intersection form on the Milnor fiber is even, and the intersection form is unimodular, the signature is multiple of 8 by Serre [109, p. 53].)*

The conjecture would imply (via formulae of Durfee $\sigma(F)+8p_g+Z_K^2+|\mathcal{V}|=0$ [17] and Laufer $\mu = 12p_g + Z_K^2 + |\mathcal{V}| - \mathrm{rank}(H_1(L(X,o))))$ [51] that the Milnor number μ and the geometric genus p_g can also be computed from the abstract link.

Neumann and Wahl supported their conjecture by its verification for Brieskorn–Hamm complete intersection singularities and (hypersurface) suspensions [90]. More generally, the CIC was proved for any splice (complete intersection) singularity in [82].

4.4.3 The Casson–Walker Invariant

The Casson invariant defined for integral homology spheres has an extension to rational homology spheres given by Walker [116]. Similarly to the Casson invariant we adopt a working definition, valid for negative definite plumbed 3-manifolds, based on a surgery formula of [55].

Definition 4.4.5 Assume that $H = H_1(M(\Gamma), \mathbb{Z})$ is finite. We define

$$-\frac{24}{|H|}\cdot\lambda(M) = \sum_{v\in\mathcal{V}}(E_v^2+3) + \sum_{v\in\mathcal{V}}(2-\kappa_v)(E_v^*, E_v^*). \tag{4.50}$$

Again, a direct verification shows that the right hand side depends only on M and it is independent of the choice of the negative definite graph Γ.

Example 4.4.6 If M is a Seifert 3-manifold with $\nu \geq 3$ then

$$-\frac{24}{|H|}\cdot\lambda(M) = \frac{1}{e}\Big(2-\nu+\sum_{j=1}^{\nu}\frac{1}{\alpha_j^2}\Big) + e + 3 - 12\cdot\sum_{j=1}^{\nu}\mathbf{s}(\omega_j, \alpha_j). \tag{4.51}$$

Example 4.4.7 For a lens space one has $\lambda(L(n,q)) = n\cdot\mathbf{s}(q,n)/2$.

Remark 4.4.8 The CIC identity $\lambda(L_X) = \sigma(F)/8$, expected in the case $H = 0$, does not extend in the same form to hypersurfaces with rational homology sphere links. For example, in the case of A_{n-1} germs, one has $\lambda(L(X,o)) = \lambda(L(n, n-1)) = -(n-1)(n-2)/24$, while $\sigma/8 = -(n-1)/8$.

4.4.4 Additivity Formulae for λ and $K^2 + |\mathcal{V}|$

In the rational homology sphere case there is no natural splice decomposition, hence there is no analogues for the Casson–Walker invariant of the splice formula valid for integral homology spheres. However, we present another type of 'additivity formula', more in the spirit of Okuma's analytic additivity formulae 4.3.28. We start with some notations.

For $v, w \in \mathcal{V}$ we define $m_{vw} := -(E_v^*, E_w^*) = -(I^{-1})_{vw} \in \mathbb{Q}_{>0}$, and let κ_v be the valency of v in Γ as usual. Then for any fixed $v \in \mathcal{V}$ we set

$$\alpha_v := \sum_{w \in \mathcal{V}} (\kappa_w - 2) m_{vw}, \qquad \beta_v := \sum_{w \in \mathcal{V}} (\kappa_w - 2) m_{vw}^2. \tag{4.52}$$

4.4.9 For a fixed vertex v of Γ, we denote the connected components of $\Gamma \setminus v$ by $\{\Gamma_i\}_i$. We indicate by a subscript i when we consider an invariant in Γ_i, instead of Γ. We regard L_i as a sublattice of L and let $R_i : L' \to L'_i$ be the natural *cohomological restriction*, that is, $R_i(E_w^*) = E_{w,i}^*$ if $w \in \mathcal{V}_i$, and $R_i(E_w^*) = 0$ otherwise. By projection formula $(R_i(x), x_i)_{L'_i} = (x, x_i)_{L'}$ for any $x \in L'$ and $x_i \in L'_i$. Then R_i maps $\mathrm{Char}(\Gamma)$ into $\mathrm{Char}(\Gamma_i)$, and the canonical characteristic element K of $\mathrm{Char}(\Gamma)$ into the canonical characteristic element K_i of $\mathrm{Char}(\Gamma_i)$.

Theorem 4.4.10 *For any* $l' = \sum_w r_w E_w \in L'$

$$((K + 2l')^2 + |\mathcal{V}|) - \sum_i ((K_i + 2R_i(l'))^2 + |\mathcal{V}_i|) = 1 - \frac{(\alpha_v + 1 - 2r_v)^2}{m_{vv}}, \tag{4.53}$$

$$\frac{24}{|H|} \cdot \lambda - \sum_i \frac{24}{|H_i|} \cdot \lambda_i = -3 + \frac{1 - \beta_v}{m_{vv}}. \tag{4.54}$$

Example 4.4.11 Consider the **surgery 3-manifold** $M = S^3_{-d}(\#_i K_i)$ as in 4.2.32 with $d > 0$ and K_i algebraic with Alexander polynomial Δ_i. Let $\Delta(t) = \prod_i \Delta_i(t)$ and $\mu = \sum_i \mu_i = 2\delta$ as in 4.3.6. By a computation

$$24 \cdot \lambda = (d-1)(d-2) + 3\mu(\mu - 2) - 12 \cdot \Delta''(1).$$

If $\mu = (d-1)(d-2)$ then this transforms into $24\lambda = \mu(3\mu - 5) - 12 \cdot \Delta''(1)$.

4.4.5 The Reidemeister–Turaev Torsion: Generalities

For the general definition of the sign-refined torsion associated with spin^c–structures see the books of Turaev and work of Nicolaescu and Ranicki, see [94, 114, 115] and the references therein.

4.4.12 The Case of 3-Manifolds Assume that M is a closed connected 3-manifold without boundary with a fixed orientation. We assume that $H = H_1(M, \mathbb{Z})$ is finite.

Theorem 4.4.13 ([115]) *The 'universal abelian sign-refined torsion'*

$$\tau : \mathrm{Spin}^c(M) \to \mathbb{Q}[H]; \quad \sigma \mapsto \tau_\sigma = \sum_h \mathcal{T}_\sigma(h)h \quad (\mathcal{T}_\sigma(h) \in \mathbb{Q}) \tag{4.55}$$

has the following properties:

(a) ***Duality:*** *Consider the involution* $\mathbb{Q}[H] \to \mathbb{Q}[H]$, *given by* $x = \sum_h a(h)h \mapsto \bar{x} := \sum_h a(h)h^{-1}$. *Then* $\tau_{\bar{\sigma}} = \overline{\tau_\sigma}$, *or* $\mathcal{T}_{\bar{\sigma}}(h^{-1}) = \mathcal{T}_\sigma(h)$.
(b) H***-equivariance:*** $\tau_{h\sigma} = h\tau_\sigma$; *that is, for any* $g, h \in H$ *one has* $\mathcal{T}_{g\sigma}(gh) = \mathcal{T}_\sigma(h)$. *In particular, for fixed* $\sigma_0 \in \mathrm{Spin}^c(M)$ *the coefficients* $\{\mathcal{T}_{\sigma_0}(h)\}_h$, *or, for fixed* $h_0 \in H$, *the coefficients* $\{\mathcal{T}_\sigma(h_0)\}_\sigma$, *determine the whole* τ.
(c) ***Augmentation:*** *Let* $\mathfrak{aug} : \mathbb{Q}[H] \to \mathbb{Q}$ *be the augmentation* $\sum_h a(h)h \mapsto \sum_h a(h)$. *Then, for any* σ *one has* $\mathfrak{aug}(\tau_\sigma) = 0$. *Equivalently,*

$$\sum_\sigma \mathcal{T}_\sigma(h) = 0 \ \textit{for any } h.$$

4.4.14 The Fourier Transform We wish to have a dual description of the torsion in terms of Fourier transform. First we recall the definition of the Fourier transform.

Let H be a finite abelian group and let $\widehat{H} = \mathrm{Hom}(H, S^1)$ be its Pontryagin dual (the group of characters). If $\chi \in \widehat{H}$ then $\bar{\chi}$ denotes its conjugate: $\bar{\chi}(h) = \overline{\chi(h)}$.

The Fourier transform $\widehat{f} : \widehat{H} \to \mathbb{C}$ of a function $f : H \to \mathbb{C}$ satisfies

$$\widehat{f}(\chi) = \sum_{h \in H} f(h)\bar{\chi}(h), \quad f(h) = \frac{1}{|H|}\sum_{\chi \in \widehat{H}} \widehat{f}(\chi)\chi(h).$$

Example 4.4.15 For any σ set $f(h) := \mathcal{T}_\sigma(h)$. Then $\widehat{f}(1) = \widehat{\mathcal{T}_\sigma}(1) = \mathfrak{aug}(\tau_\sigma) = 0$.

Example 4.4.16 By 4.4.13*(a)–(b)* for any σ, χ, h one has

$$(a)\ \widehat{\mathcal{T}_\sigma}(\chi) = \widehat{\mathcal{T}_{\bar{\sigma}}}(\overline{\chi}), \quad (b)\ \widehat{\mathcal{T}_\sigma}(\chi) = \chi(h) \cdot \widehat{\mathcal{T}_{h\sigma}}(\chi). \tag{4.56}$$

4.4.6 The Reidemeister–Turaev Torsion of Graph 3-Manifolds

Let M be an oriented rational homology sphere 3-manifold associated with a connected negative definite plumbing graph Γ.

In 4.4.22 we provide a combinatorial expression in terms of Γ for the refined Reidemeister–Turaev torsion. The equivalence of this expression with the original definition of the refined torsion is proved in [78].

4.4.17 The Fourier Transform of $Z_{h,\mathcal{I}}(t)$ Assume that $\mathcal{I} = \{u\} \subset \mathcal{V}$ is a distinguished vertex, and for each $h \in H$ we consider the reduced series $Z_{h,\mathcal{I}}(t)$, where t is the variable corresponding to u. Set $m_{vu} := -(E_v^*, E_u^*) > 0$. From (4.38)

$$Z_{h,\{u\}}(t) = \frac{1}{|H|} \cdot \sum_{\chi \in \widehat{H}} \chi(h)^{-1} \cdot \prod_{v \in \mathcal{V}} \left(1 - \chi([E_v^*]) t^{m_{vu}}\right)^{\kappa_v - 2}.$$

This shows that the Fourier transform of the map $h \mapsto Z_{h,\{u\}}(t)$ is

$$\widehat{Z_{\{u\}}(t)}(\bar{\chi}) = \prod_{v \in \mathcal{V}} \left(1 - \chi([E_v^*]) t^{m_{vu}}\right)^{\kappa_v - 2}. \tag{4.57}$$

4.4.18 Character Values on Γ Since $\{[E_v^*]\}_v$ generate H, any character $\chi \in \widehat{H}$ is completely characterized by the values $\xi_v := \chi([E_v^*])$, $v \in \mathcal{V}$. These are roots of unity. When we wish to identify the character χ, we put its values $\{\xi_v\}_v$ as decorations on the vertices of the graph Γ. The collection $\{\chi([E_v^*])\}_{v,\chi}$ is a more subtle information than the abstract group $\widehat{H}$ itself: it shows the 'distribution along Γ' of the corresponding values of the characters as well. Since for any $v \in \mathcal{V}$ one has $e_v[E_v^*] + \sum_{(u,v) \text{ edge}} [E_u^*] = [-E_v] = 0$ in H (where $e_v = E_v^2$), for each χ one has

$$\xi_v^{e_v} \cdot \prod_{(u,v) \text{ edge}} \xi_u = 1. \tag{4.58}$$

Conversely, any collection of complex numbers $\{\xi_v\}_{v \in \mathcal{V}}$, $\xi_v \in S^1$, which satisfy (4.58) for any v, determines a character χ defined by $\chi([E_v^*]) = \xi_v$.

Furthermore, for any $\chi \in \widehat{H} \setminus \{1\}$, define the 'extended support' $\text{supp}^e(\chi)$ of χ as the set of those vertices $v \in \mathcal{V}$ for which either $\chi([E_v^*]) \neq 1$, or v has an adjacent vertex w such that $\chi([E_w^*]) \neq 1$.

Lemma 4.4.19 *Fix a character* $\chi \in \widehat{H} \setminus \{1\}$.

(a) For an arbitrary vertex u the limit $\lim_{t \to 1} \widehat{Z_{\{u\}}(t)}(\chi)$ *exists and it is finite.*
(b) This limit is independent of u whenever $u \in \text{supp}^e(\chi)$.

Remark 4.4.20 For $\chi = 1$, the Laurent expansion at 1 of the series $\widehat{Z_{\{u\}}(t)}(1)$ has a non-trivial principal part, hence $\lim_{t \to 1} \widehat{Z_{\{u\}}(t)}(1)$ is not finite.

4.4.21 In the sequel, the torsion $\sigma \in \text{Spin}^c(M) \mapsto \mathcal{T}_\sigma$, $\mathcal{T}_\sigma = \sum_h \mathcal{T}_\sigma(h) h \in \mathbb{Q}[H]$ is defined via the Fourier transform of $h \mapsto \mathcal{T}_\sigma(h)$ in the following way.

Definition 4.4.22

(a) For the trivial character $\widehat{\mathcal{T}_\sigma}(1) = 0$.
(b) If $\chi([E_v^*]) \neq 1$ for every v with $\kappa_v \neq 2$, then we set

$$\widehat{\mathcal{T}_\sigma}(\chi) = (\chi(h_\sigma))^{-1} \cdot \prod_{v \in \mathcal{V}} (1 - \chi([E_v^*]))^{\kappa_v - 2}, \quad \sigma = h_\sigma \sigma[K].$$

(c) If $\chi \neq 1$, but the assumption from (b) does not hold, then the formula from (b) is regularised as follows:

$$\widehat{\mathcal{T}_\sigma}(\chi) = (\chi(h_\sigma))^{-1} \cdot \lim_{t\to 1} \prod_{v\in\mathcal{V}} (1-\chi([E_v^*]t^{m_{vu}}))^{\kappa_v - 2} = (\chi(h_\sigma))^{-1} \cdot \lim_{t\to 1} \widehat{Z_{\{u\}}(t)}(\bar{\chi}),$$

for certain (any) $u = u_\chi \in \mathrm{supp}^e(\chi)$.

Theorem 4.4.23

(a) $\sigma \mapsto \mathcal{T}_\sigma$ *defined in 4.4.22 and the refined Reidemeister–Turaev torsion 4.4.12 coincide.*
(b) $\mathcal{T}$ *defined in 4.4.22 is independent of the choice of the resolution.*

Remark 4.4.24

(a) By Fourier inversion

$$\mathcal{T}_\sigma(h) = \frac{1}{|H|} \cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} \chi(h) \cdot (\chi(h_\sigma))^{-1} \cdot \lim_{t_{u_\chi}\to 1} \widehat{Z_{\{u_\chi\}}(t_{u_\chi})}(\bar{\chi}).$$

One verifies that the Properties (4.56) are valid, hence $\{\mathcal{T}_\sigma(h)\}_{\sigma,h}$ satisfy the duality and H-equivariance properties. Hence

$$\mathcal{T}_\sigma(1) = \mathcal{T}_{\overline{\sigma}}(1), \quad \text{and} \quad \mathcal{T}_\sigma(1) = \mathcal{T}_{h_\sigma\sigma[K]}(1) = \mathcal{T}_{\sigma[K]}(-h_\sigma). \tag{4.59}$$

In particular, $\mathcal{T}_{\sigma[K]}(h)h \in \mathbb{Q}[H]$ contains the same information as $\{\mathcal{T}_\sigma(1)\}_\sigma$.

(b) From part (a),

$$\mathcal{T}_\sigma(1) = \frac{1}{|H|} \cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} (\chi(h_\sigma))^{-1} \cdot \lim_{t_{u_\chi}\to 1} \widehat{Z_{\{u_\chi\}}(t_{u_\chi})}(\bar{\chi}).$$

Usually, for different characters χ one needs different regularization vertices u_χ. However, if $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi) \neq \emptyset$, then any $u \in \cap_{\chi\neq 1}\mathrm{supp}^e(\chi)$ might serve as a *common* regularization vertex (with a *common* variable $t = t_u$). In such a case, via $\widehat{Z_{\{u\}}(t)}(1) = Z_{\{u\}}(t)$,

$$\mathcal{T}_\sigma(1) = \lim_{t\to 1}\Big(\frac{1}{|H|} \cdot \sum_{\chi\in\widehat{H}\setminus\{1\}} (\chi(h_\sigma))^{-1} \cdot \widehat{Z_{\{u\}}(t)}(\bar{\chi})\Big) = \lim_{t\to 1}\Big(Z_{h_\sigma,\{u\}}(t) - \frac{1}{|H|} \cdot Z_{\{u\}}(t)\Big).$$

We rewrite $\{Z_{h,\{u\}}(t)\}_h$ equivariantly as $Z_{H,\{u\}}(t) := \sum_{h\in H} Z_{h,\{u\}}(t)h \in \mathbb{Q}[[t]][H]$, and we set $N := \sum_h h \in \mathbb{Q}[H]$. Then, via $\mathcal{T}_\sigma(1) = \mathcal{T}_{\sigma[K]}(-h_\sigma)$,

$$\mathcal{T}_{\sigma[-K]} = \overline{\mathcal{T}_{\sigma[K]}} = \lim_{t\to 1}\Big(Z_{H,\{u\}}(t) - Z_{\{u\}}(t) \cdot \frac{N}{|H|}\Big) \in \mathbb{Q}[H]. \tag{4.60}$$

The identity (4.60) is not true in general, i.e. when $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)=\emptyset$.

The above formula already shows in this special case that the principal (pole) part of the Laurent series at $t=1$ of $Z_{h,\{u\}}(t)$ is independent of $h\in H$. This statement is true in general, even without the restriction $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)\neq\emptyset$.

(c) If Γ is star-shaped then the central vertex is an element of $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)$. Similarly, if H is cyclic, then again $\cap_{\chi\neq 1}\mathrm{supp}^e(\chi)\neq\emptyset$.

Example 4.4.25 (The Torsion of a Lens Space) We fix $\sigma=h_\sigma\sigma[K]\in\mathrm{Spin}^c(L_X)$. Then for $\chi\neq 1$

$$\widehat{\mathcal{T}_\sigma}(\chi)=\chi(h_\sigma)^{-1}\cdot(1-\chi([E_s^*]))^{-1}(1-\chi([E_1^*]))^{-1}.$$

Assume that $h_\sigma=a[E_s^*]$ for some $0\le a<n$. Set $\xi:=\chi([E_s^*])$. Then,

$$\widehat{\mathcal{T}_\sigma}(\chi)=\frac{\xi^{-a}}{(1-\xi)(1-\xi^q)}\quad(\xi\neq 1),\ \text{and}\ \mathcal{T}_\sigma(1)=\frac{1}{n}\cdot\sum_{\xi^n=1\neq\xi}\frac{\xi^{-a}}{(1-\xi)(1-\xi^q)}. \tag{4.61}$$

4.4.7 Additivity Formula for the Torsion

We fix a graph Γ such that $M(\Gamma)$ is a rational homology sphere. For a vertex $v\in\mathcal{V}$ of Γ let $\{\Gamma_i\}_i$ be the connected components of $\Gamma\setminus v$. For any $\sigma\in\mathrm{Spin}^c(M(\Gamma))$ we define its restrictions $\sigma_i\in\mathrm{Spin}^c(M(\Gamma_i))$ as follows.

Choose $l'=\sum_w r_wE_w\in L'$ such that $r_v\in[0,1)$ so that $[l']=h_\sigma$ satisfies $\sigma=\sigma[2l'+K]=h_\sigma\sigma[K]\in\mathrm{Spin}^c(M(\Gamma))$. Then we set $\sigma_i=\sigma[R_i(2l'+K)]=[R_i(l')]\sigma[K_i]\in\mathrm{Spin}^c(M(\Gamma_i))$. (For R_i see paragraph 4.4.9.)

Theorem 4.4.26 ([12]) *Set $l'=\sum_w r_wE_w$, $r_v\in[0,1)$, $[l']=h_\sigma$ as above. Recall also the notations from (4.52)*

$$\alpha_v:=\sum_{w\in\mathcal{V}}(\kappa_w-2)m_{vw},\quad \beta_v:=\sum_{w\in\mathcal{V}}(\kappa_w-2)m_{vw}^2.$$

Then

$$\mathcal{T}_\sigma(1)(M(\Gamma))-\sum_i\mathcal{T}_{\sigma_i}(1)(M(\Gamma_i))=\mathrm{pc}(Z_{h_\sigma,\{v\}}(t^d))+\frac{1-\beta_v}{24m_{vv}}-\frac{(\alpha_v+1-2r_v)^2}{8m_{vv}}.$$

Corollary 4.4.27 *$\mathcal{T}_\sigma(1)(M(\Gamma))$ is a rational number.*

4.4.8 The Seiberg–Witten Invariant

In this section we fix a plumbed rational homology sphere 3-manifold M associated with a connected negative definite plumbing graph Γ. The Seiberg–Witten invariant of M, $\mathfrak{sw}$, associates to each spin^c structure $\sigma \in \mathrm{Spin}^c(M)$ of M a rational number $\mathfrak{sw}_\sigma$. Here, based on [95], we 'define' it as the refined Turaev torsion modified by the Casson–Walker invariant. Based on the formulae of the previous sections, this provides $\mathfrak{sw}$ combinatorially from Γ.

Definition 4.4.28 We define $\mathfrak{sw} : \mathrm{Spin}^c(M) \to \mathbb{Q}$, $\sigma \mapsto \mathfrak{sw}_\sigma$ by

$$\mathfrak{sw}_\sigma := \mathcal{T}_\sigma(1) - \lambda/|H|.$$

Example 4.4.29 If $H = 0$ then $\mathrm{Spin}^c(M)$ has only one element, and the corresponding Seiberg–Witten invariant is $-\lambda(M)$ (the negative of the Casson invariant).

4.4.30 Additivity Formula for the Seiberg–Witten Invariant The previous additivity formulae imply the following formula.

Theorem 4.4.31 ([12]) *Set $l' = \sum_w l'_w E_w$, $l'_v \in [0, 1)$, as in Theorem 4.4.26. Let $\sigma \in \mathrm{Spin}^c(M(\Gamma))$ be defined as $[l']\sigma[K] = \sigma[K+2l']$, and take also its restrictions $\sigma_i := [R_i(l')]\sigma[K_i] = \sigma[R_i(K+2l')]$ too. Set $h_\sigma = [l']$. Then one has the following identities:*

$$\mathfrak{sw}_\sigma(M(\Gamma)) - \sum_i \mathfrak{sw}_{\sigma_i}(M(\Gamma_i)) = \mathrm{pc}(Z_{h_\sigma,\{v\}}(t)) + \frac{1}{8} - \frac{(\alpha_v + 1 - 2r_v)^2}{8m_{vv}}.$$

and

$$\Big(\mathfrak{sw}_\sigma(M(\Gamma)) - \frac{(K+2l')^2 + |\mathcal{V}|}{8}\Big) - \sum_i \Big(\mathfrak{sw}_{\sigma_i}(M(\Gamma_i)) - \frac{(K_i + 2R_i(l'))^2 + |\mathcal{V}_i|}{8}\Big)$$
$$= \mathrm{pc}(Z_{h_\sigma,\{v\}}(t)).$$

Proof Combine Theorems 4.4.10 and 4.4.26 and use $\mathrm{pc}(S(t^d)) = \mathrm{pc}(S(t))$. □

This additivity formula should be compared with its 'analytic counterpart', namely with Okuma's additivity formula 4.3.30.

4.4.9 The Seiberg–Witten Invariant and the Series $Z(\mathbf{t})$

We prove two key formulae for the Seiberg–Witten invariant of a rational homology sphere link. One of them identifies it with a weighted Euler characteristic of (shifted) weighted cubes in a large rectangle of $L \otimes \mathbb{R}$, the other one with the constant term of the counting function of the coefficients of $Z(\mathbf{t})$. The proofs are based on additivity formulae of the compared invariants.

The similarities with the analytic counterpart (the series $P(\mathbf{t})$ and the equivariant genera) are emphasized.

4.4.32 In the next discussion we will use the weighted cubes, see also 4.6.3. Let us fix an element h of H and write $L'_h = \{l' \in L' : [l'] = h\}$. Recall that the set of 'combinatorial' q–cubes (associated with h) consists of pairs $(l', I) \in L'_h \times \mathcal{P}(\mathcal{V})$, $|I| = q$ ($q \in \mathbb{Z}_{\geq 0}$). (l', I) will be identified with the vertices $\{l' + \sum_{v\in I'} E_v\}_{I'\subset I}$ of an 'Euclidean' cube in $L \otimes \mathbb{R}$. One defines the weight function $w : L' \to \mathbb{Q}$, $w(l') := \chi(l')$, and also the a weight of the q–cubes

$$w((l', I)) = \max_{I'\subset I} \Big\{ w(l' + \sum_{v\in I'} E_v) \Big\}.$$

Assume that a set $A \subset L \otimes \mathbb{R}$ has the following property: if an Euclidean cube (as above) is in A then any face of any dimension of that cube is in A. For such a set A one defines the 'weighted Euler characteristic'

$$Eu_\chi(A) := \sum_{(l', I)\in A} (-1)^{|I|+1} w((l', I)).$$

Such a set A might appear as follows. For the fixed class $h \in L'/L$ one takes two representatives $l'_1, l'_2 \in L'_h$ with $l'_2 \leq l'_1$. Then $R_h = R_h(l'_2, l'_1)$ consists of the union of all combinatorial cubes (l', I), of any dimension, such that $[l'] = h$ and any vertex $l' + \sum_{v\in I'} E_v$ of (l', I) satisfies $l'_2 \leq l' + \sum_{v\in I'} E_v \leq l'_1$. Accordingly to the above identification, $R_h(l'_1, l'_2)$ will also denote the real rectangle $\{x \in L \otimes \mathbb{R} : l'_2 \leq x \leq l'_1\}$, or the union of all Euclidean cubes (with all vertices having class $[h]$) in this real rectangle.

Remark 4.4.33 For a fixed $h \in H$, we can consider two types of rectangles and weighted q–cubes, depending on the geometric situation. First, in the context of lattice cohomology (see e.g. 4.6.3, and in its preparation 4.5.2) we take integral lattice points and rectangles $R(l_2, l_1)$ and cubes with vertices in the lattice L, but we twist the weight function: we take χ_k (which generates w_k) with $k = K + 2l'_h$, for some representative l'_h of h.

Second, when we wish to relate the cubes with the coefficients of $Z(\mathbf{t})$ (as in the previous paragraph), we take shifted rectangles $R_h := R_h(l'_2, l'_1)$ ($[l'_j] = h$) with cubes (l', I) of type $[l'] = h$ in them, together with the usual untwisted Riemann–Roch-function $\chi = \chi_K$.

The two approaches can be compared easily (see also 4.6.3). Indeed, if $k = K + 2l'_h$, $[l'_h] = h$, then for $l \in L$ we have $\chi(l + l'_h) = \chi_k(l) + \chi(l'_h)$. In particular, with the notation $l'_j = l_j + l'_h$ ($l_j \in L$), we have $R_h(l'_2, l'_1) = l'_h + R(l_2, l_1)$ as rectangles, and

$$Eu_\chi(R_h(l'_2, l'_1)) = Eu_{\chi_k}(R(l_2, l_1)) - \chi(l'_h).$$

4.4.34 Via the two incarnations of the weighted cubes (cf. 4.4.33) the next result is the 'pair' of Lemma 4.5.8.

Lemma 4.4.35 *Fix a class h and take a representative l'_0 of h in $-K+\mathcal{S}'$.*

(a) For any $l' \in L'$, $[l'] = h$, $l' > l'_0$, there exists an E_v in the support of $l' - l'_0$ such that $w(l' - E_v) \leq w(l')$.
(b) There exists a computation sequence $\{\ell_i\}_{i\geq 0}$, $\ell_i \in L$, with $\ell_0 = 0$, and $\ell_{i+1} = \ell_i + E_{v(i)}$ for some $v(i) \in \mathcal{V}$ when $i \geq 0$, satisfying:

(i) The coefficients of ℓ_i tend to infinity, that is $\lim_{i\to\infty}(\ell_i, -E^*_v) = \infty$ for all v.
(ii) For any $i \geq 0$ one has $w(l'_0 + \ell_i) \leq w(l'_0 + \ell_{i+1})$.

(c) For any $l' < 0$, with $[l'] = h$, there exists $E_v \in |l'|$ such that $w(l'+E_v) \leq w(l')$.
(d) For any representatives l'_1, l'_2 of h, such that $l'_1 \geq l'_0 > 0 \geq l'_2$, $Eu_\chi(R_h(l'_2, l'_1))$ is independent of the choice of l'_1 and l'_2. In particular, with such choices, $h \mapsto Eu_\chi(R_h(l'_2, l'_1))$ is a numerical invariant of $h \in H = L'/L$.

Definition 4.4.36 The invariant provided by 4.4.35(d) will be denoted by $\overline{\mathfrak{s}}_h$.

4.4.37 Let $Z(\mathbf{t}) = \sum_{l'\in L'} \mathfrak{z}(l')\mathbf{t}^{l'}$ be the combinatorial series defined in Sect. 4.3.3. Since Z is supported on $\mathcal{S}'$, the next sum in (4.62) is finite by 4.2.13.

Theorem 4.4.38 *Fix $h \in H$. For any $l' \in -K + \mathcal{S}'$ with $[l'] = h$, the expression*

$$-\chi(l') + \sum_{l\in L,\, l\not\geq 0} \mathfrak{z}(l'+l) \tag{4.62}$$

depends only on the class h of l', and, in fact, it equals $\overline{\mathfrak{s}}_h$ defined in 4.4.36.

Theorem 4.4.39 ([73]) *For any Γ and $[K+2l'] \in \mathrm{Char}$ one has $\mathfrak{sw}_{\sigma[K+2l']}(M(\Gamma)) = \overline{\mathfrak{s}}_{[l']} + (K^2 + |\mathcal{V}|)/8$, or,*

$$Eu_\chi(R_h(l'_2, l'_1)) = \overline{\mathfrak{s}}_{[l']} = \mathfrak{sw}_{\sigma[K+2l']}(M(\Gamma)) - (K^2 + |\mathcal{V}|)/8. \tag{4.63}$$

The proof is based on the 'additivity formula' 4.4.31 and a similar formula valid for $\overline{\mathfrak{s}}_h$.

Therefore, Theorem 4.4.38 reads as follows.

Theorem 4.4.40 *Assume that $l' \in -K + \mathcal{S}'$ and Let $Z(\mathbf{t}) = \sum_{l'\in L'} \mathfrak{z}(l')\mathbf{t}^{l'}$ be the combinatorial series defined in Sect. 4.3.3. Then*

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\not\geq l'} \mathfrak{z}(\tilde{l}') = \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2 + |\mathcal{V}|}{8}. \tag{4.64}$$

If we write $l' = r_h + l$ (where $h = [l']$ and $l \in L$), then (4.64) transforms into

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l) + \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2 + |\mathcal{V}|}{8}. \tag{4.65}$$

In particular, in the chamber $l' = l + r_h \in -K + \mathcal{S}'$, the sum from the left hand side of the above identities is a multivariable quadratic function in l with constant term $\mathfrak{sw}_{\sigma[K+2r_h]} - ((K+2r_h)^2 + |\mathcal{V}|)/8$.

These formulae should be compared with those from (4.36) valid for the coefficients of the series P. The fact that in (4.36) (associated with the series P) the constant terms are the equivariant geometric genera, is rather natural. However, the fact that the constant terms in the above Theorem 4.4.40 (associated with Z, a rather 'simple' series) is the Seiberg–Witten invariant, is rather surprising. Nevertheless, the above identity provides a very natural, direct and conceptual explanation, how the Seiberg–Witten invariant might appear in the theory of singularity links.

Example 4.4.41 If Γ is numerically Gorenstein and $h = 0$ then (4.65) reads as

$$\sum_{l\in L,\ l\ngeq Z_K} \mathfrak{z}(l) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathcal{V}|}{8}. \tag{4.66}$$

4.4.10 The Seiberg–Witten Invariant Conjecture/Coincidence

In this section we treat a set of potential identities connecting the analytic invariants with the topological ones, namely, the equivariant geometric genera with the Seiberg–Witten invariants of the link. Whenever these identities are valid they provide a topological description of the equivariant geometric genera. The identities are generalizations of the expectation of the Casson Invariant Conjecture to the case of singularities with rational homology sphere links.

Superisolated singularities in general do not satisfy SWIC, their case will be discussed in subsection 4.4.11.

4.4.42 Seiberg–Witten Invariant Conjecture/Coincidence (SWIC) [73, 75, 78] In this section we assume that the link of (X, o) is a rational homology sphere, and we fix a resolution $\widetilde{X} \to X$, and we keep all the notations associated with it. We say that (X, o) satisfies SWIC(r_h) for a certain $h \in H$ if the following identity holds

$$h^1(\widetilde{X}, \mathcal{O}(-r_h)) = \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2 + |\mathcal{V}|}{8}. \tag{4.67}$$

We say that (X, o) satisfies the *equivariant* SWIC if (4.67) holds for every $h \in H$.

We say that (X, o) satisfies the SWIC if it satisfies SWIC(0), that is, if

$$p_g(X, o) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathcal{V}|}{8}. \tag{4.68}$$

The identity SWIC was formulated as a conjecture in [78] (while the equivariant case in [71]): the expectation was that it holds for any $\mathbb{Q}$-Gorenstein singularity. Although the conjecture can be verified for several subfamilies of singularities, since [61] we know that it is not true for the large class of $\mathbb{Q}$-Gorenstein singularities (see also 4.4.11 for the treatment of superisolated singularities, a family which produces several counterexamples). But even in the case of families when it fails, it still indicates interesting 'virtual' properties (e.g., in the superisolated case it has lead to the Semigroup Distribution Property). The limits of the validity of the SWIC are not clarified at this moment. Having in mind the existence of cases when the identities do not hold, one might say that its name as SWI 'Conjecture' is not totally justified, although this was its name in the literature. Hence, the reader might read the abbreviation SWIC as SWI 'Coincidence' too.

Example 4.4.43 Assume that (X, o) is Gorenstein and it admits a smoothing with smooth nearby (Milnor) fiber F. Then the signature satisfies $\sigma(F) + 8p_g + K^2 + |\mathcal{V}| = 0$, hence the SWIC (for $h = 0$) reads as

$$-\sigma(F)/8 = \mathfrak{sw}_{\sigma[K]}. \tag{4.69}$$

In this case, usually, $\sigma(F)/8$ is not an integer, see the germ A_n.

Example 4.4.44 Assume that (X, o) is a complete intersection with integral homology sphere link. Then $\mathcal{T}_{\sigma[K]}(1) = 0$, hence the SWIC reduces to the CIC (see 4.4.2):

$$\sigma(F)/8 = \lambda(L(X, o)).$$

Example 4.4.45 The identity $P(\mathbf{t}) = Z(\mathbf{t})$ (that is, the topological description via Z of the Poincaré series associated with the divisorial filtration) implies the equivariant SWIC. In particular, the identity $P_0(\mathbf{t}) = Z_0(\mathbf{t})$ implies SWIC. Indeed, for any $l' \in -K + \mathcal{S}'$ with $l' = l + r_h$ ($l \in L$), from (4.36) one has

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\not\geq l'} \mathfrak{p}(\tilde{l}') = \chi_{K+2r_h}(l) + h^1(\mathcal{O}(-r_h)). \tag{4.70}$$

On the other hand, from (4.65),

$$\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\not\geq l'} \mathfrak{z}(\tilde{l}') = \chi_{K+2r_h}(l) + \mathfrak{sw}_{\sigma[K+2r_h]} - \frac{(K+2r_h)^2 + |\mathcal{V}|}{8}. \tag{4.71}$$

For $l' \in -K + \mathcal{S}'$ and $l' = l + r_h$, we can regard the evaluation at $l = 0$ of the counting function $\sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathrm{coeff}(\tilde{l}')$ as an operator. It associates with any multivariable series its 'multivariable periodic constant', cf. [45, 46]. In this sense, the above identities say that the periodic constant of P_h is $h^1(\mathcal{O}(-r_h))$, while of Z_h is $\mathfrak{sw}_{[K+2r_h]} - ((K + 2r_h)^2 + |\mathcal{V}|)/8$.

Hence, if $P_h(\mathbf{t}) = Z_h(\mathbf{t})$ then the SWIC(r_h) automatically holds as well.

In fact, in order to have the SWIC(r_h) we need the validity of the above identities for a certain $l' \in -K + \mathcal{S}'$ ($[l'] = h$) only. Indeed, if a certain $l'_0 \in -K + \mathcal{S}'$, $[l'_0] = h$, has the property that $P_h(\mathbf{t}) - Z_h(\mathbf{t})$ is supported on $\{\tilde{l}' : \tilde{l}' \geq l'_0\}$, then by the above identities applied for this l'_0 we obtain SWIC(r_h). In such a case, again by the identities (4.70)–(4.71), even if $P_h(\mathbf{t}) \neq Z_h(\mathbf{t})$, their counting functions $l' \mapsto \sum_{[\tilde{l}']=[l'],\ \tilde{l}'\ngeq l'} \mathrm{coeff}(\tilde{l}')$ in the whole chamber $l' \in -K + \mathcal{S}'$ coincide (independently of the position of l'_0 in this chamber).

For a fixed h, the identity $P_h = Z_h$ is much stronger than the SWIC(r_h): examples when $P_h \neq Z_h$ but the SWIC(r_h) holds can be constructed.

4.4.46 Extension to the Other Natural Line Bundles

Recall that in 4.2.74 we proved that for any $l' \in L'$ there exists a unique minimal $s(l') \in \mathcal{S}'$ such that $s(l') - l' \in L_{\geq 0}$. We wish to compare $h^1(\mathcal{O}(-l'))$ and $h^1(\mathcal{O}(-s(l')))$ via the SWIC property.

We say that $l' \in L'$ satisfies the SWIC identity, denoted by SWIC(l'), if

$$\text{SWIC}(l') : \qquad h^1(\widetilde{X}, \mathcal{O}(-l')) = \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K + 2l')^2 + |\mathcal{V}|}{8}. \tag{4.72}$$

If this holds, then it obviously provides a topological description for $h^1(\widetilde{X}, \mathcal{O}(-l'))$.

By 4.2.76 one has

$$h^1(\widetilde{X}, \mathcal{O}(-s(l'))) - h^1(\widetilde{X}, \mathcal{O}(-l')) = \chi(s(l')) - \chi(l').$$

A computation shows that the right hand side of (4.72) behaves similarly. Hence

Proposition 4.4.47 *The SWIC(l') is valid if and only if SWIC($s(l')$) is valid. In particular, SWIC(r_h) is valid if and only if SWIC(s_h) holds.*

This shows that the validity of SWIC(r_h) implies the validity of SWIC(l') for all $l' \in L'_h$ with $s(l') = s_{[l']}$. This covers exactly those cycles $l' \in L'_h$ with $l' \leq s_{[l']}$ (including all cycles $l' = \sum_v l'_v E_v$ with $l'_v < 1$ for any v).

This topological characterization SWIC(l') of $h^1(\mathcal{O}(-l'))$ (modulo the validity of SWIC) in this 'negative' region $\{l' : l' \leq s_{[l']}\}$ can be compared with the vanishing $h^1(\mathcal{O}(-l')) = 0$ in the 'opposite positive' region $\{l' : l' \in -K + \mathcal{S}'\}$.

It is natural to ask the following question: what can one say in the case of an arbitrary l', which sits outside of these two regions.

Proposition 4.4.48 *If SWIC(r_h) holds then for any $l' \in L'_h$*

$$h^1(\mathcal{O}(-l')) = -\sum_{a\in L,\, a\not\geq 0} \mathfrak{p}(l'+a) + \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2+|\mathcal{V}|}{8}. \tag{4.73}$$

Additionally, if $P_h = Z_h$ (or, at least their counting functions coincide), then one has the following topological characterization of $h^1(\mathcal{O}(-l'))$:

$$h^1(\mathcal{O}(-l')) = -\sum_{a\in L,\, a\not\geq 0} \mathfrak{z}(l'+a) + \mathfrak{sw}_{\sigma[K+2l']} - \frac{(K+2l')^2+|\mathcal{V}|}{8}. \tag{4.74}$$

Remark 4.4.49 Assume that the equivariant SWIC is true for (X, o). Then, taking the sum of the identities SWIC(r_h) from (4.67), and using $\sum_\sigma \mathfrak{T}_\sigma(1) = 0$, we get the following expression for the geometric genus of the universal abelian covering (X_a, o) in terms of the graph Γ:

$$p_g(X_a, o) = -\lambda(M(\Gamma)) - |H| \cdot \frac{K^2+|\mathcal{V}|}{8} + \sum_{h\in H} \chi(r_h).$$

Example 4.4.50 (SWIC is True for Cyclic Quotients) In this case the link is $L(n, q)$, $H = \mathbb{Z}_n$ and the spin^c structures are indexed by $\sigma = \sigma[K + 2aE^*_s]$, where $a \in \mathbb{Z}$ and $0 \leq a < n$. Set also $h = a[E^*_s] \in H$. Then

$$\mathfrak{T}_\sigma(1) = -\mathbf{s}(q, n) + \frac{n-1}{4n} - \frac{a}{2n} - \sum_{i=1}^{a} \left(\left(\frac{iq'}{n}\right)\right).$$

Since $\lambda/n = \mathbf{s}(q, n)/2$, cf. 4.4.7, we also have

$$\mathfrak{sw}_\sigma = -\frac{3}{2} \cdot \mathbf{s}(q, n) + \frac{n-1}{4n} - \frac{a}{2n} - \sum_{i=1}^{a} \left(\left(\frac{iq'}{n}\right)\right).$$

On the other hand, $(K + 2r_h)^2 + |\mathcal{V}|)/8 = (K^2 + |\mathcal{V}|)/8 - \chi(r_h)$ can also be computed explicitly. From 4.2.60 one has $(K^2+|\mathcal{V}|)/8 = (n-1)/4n - 3\mathbf{s}(q, n)/2$.

Furthermore, from 4.2.76 we have $h^1(\mathcal{O}(-s_h)) - h^1(\mathcal{O}(-r_h)) = \chi(s_h) - \chi(r_h)$. But $h^1(\mathcal{O}(-s_h)) = 0$ by the vanishing 4.2.71, while $h^1(\mathcal{O}(-r_h)) = p_g(X_a, o)_{\theta(h)} = 0$ (cf. 4.2.82) since the universal abelian covering (X_a, o) is smooth. Hence $\chi(r_h) = \chi(s_h)$, and its expression is

$$\chi(r_h) = \frac{a}{2n} + \sum_{i=1}^{a} \left(\left(\frac{iq'}{n}\right)\right).$$

In particular, the right hand side of SWIC(r_h) is zero, and the same is true for the left hand side because of the vanishing already mentioned.

Example 4.4.51 The equivariant SWIC is true for splice quotient singularities. In particular, it is true for rational, minimally elliptic and weighted homogeneous singularities (with $\mathbb{Q}HS^3$ link). The SWIC(0) is valid for all elliptic singularities and suspensions $\{z^n + f(x, y) = 0\}$, where f is irreducible (and with $\mathbb{Q}HS^3$ link).

4.4.11 SWIC and Superisolated Singularities

We assume that (X, o) is a superisolated singularity associated with the irreducible projective rational cuspidal curve C of degree d.

Though in many cases (e.g. for weighted homogeneous singularities) we discuss the SWIC together with equivariant SWIC, this is not the case for the superisolated germs. The main obstruction is that in the superisolated case (though $p_g(X, o)$ and $P_{0,\{v_+\}}(t)$ are extremely simple), usually we have very little information about the analytic properties of the universal abelian covering, e.g. about its geometric genus $p_g(X_a, o)$ (see e.g. [111]). Therefore, in this subsection we focus merely on the SWIC (for $h = 0$).

It turns out that for a superisolated singularity the SWIC is valid if and only if $N(1) = 0$, a property which is not always true, cf. subsection 4.3.6. Let us list first the involved invariants.

4.4.52 From Example 4.4.11 we have $K^2 + |\mathcal{V}| = -d(d-2)^2 + 1$ and $24\lambda = \mu(3\mu - 5) - 12 \cdot \Delta''(1)$ ($\mu = 2\delta$). Moreover, the divisorial filtration associated with $\mathcal{I} = \{C\} = \{v_+\}$ agrees with the filtration associated with weights $(1, 1, 1)$, hence $P_{0,\mathcal{I}}(t) = (1 - t^d)/(1 - t)^3$. Since in the good resolution $\Gamma \setminus v_+$ supports only smooth germs, by 4.3.30 $p_g(X, o) = \mathrm{pc}(P_{0,\mathcal{I}}(t))$, which is $d(d-1)(d-2)/6$.

The definition of $Z_{\mathcal{I}}(t)$ compared with A'Campo formula [1] gives

$$Z_{0,\mathcal{I}}(t) = \frac{1}{d} \sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} \quad \text{and} \quad Z_{\mathcal{I}}(t) = \frac{\Delta(t^{1/d})}{(1-t^{1/d})^2}.$$

Since H is generated by $[E_+^*]$, the vertex v_+ (corresponding to C) is a regularization vertex for any character. Therefore, from 4.4.24

$$\mathcal{T}_{\sigma[K]}(1) = \lim_{t\to 1} \Big(Z_{0,\mathcal{I}}(t) - \frac{1}{d} Z_{\mathcal{I}}(t) \Big) = \frac{1}{d} \sum_{\xi^d=1\neq\xi} \frac{\Delta(\xi)}{(1-\xi)^2}.$$

Following 4.3.6 we also consider

$$N(t) := Z_{0,\mathcal{I}}(t) - P_{0,\mathcal{I}}(t) = \frac{1}{d} \sum_{\xi^d=1} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2} - \frac{1-t^d}{(1-t)^3}.$$

Then

$$\lim_{t\to 1} N(t) = \mathcal{T}_{\sigma[K]}(1) + \lim_{t\to 1}\Big(\frac{1}{d}\cdot\frac{\Delta(t^{1/d})}{(1-t^{1/d})^2} - \frac{1-t^d}{(1-t)^3}\Big).$$

If we write $\Delta(t) = 1 + \delta(t-1) + Q(t)(t-1)^2$ as in 4.3.6, then the limit can be computed in terms of d and $Q(1) = \Delta''(1)/2$. The computation provides

Proposition 4.4.53

$$N(1) = \mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathcal{V}|}{8} - p_g.$$

This combined with (4.45) gives (with $Q(t) = \sum_{j=0}^{\mu-2}\alpha_j t^j$)

$$\mathfrak{sw}_{\sigma[K]} - \frac{K^2 + |\mathcal{V}|}{8} = \sum_{j=0}^{d-3}\alpha_{jd}.$$

Corollary 4.4.54

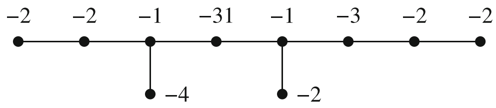
(a) SWIC for $h = 0$ is equivalent to $N(1) = 0$.
(b) The Conjecture 4.3.22 (which predicts that $N(1) \le 0$ for any superisolated singularity) is equivalent to $\mathfrak{sw}_{\sigma[K]} - \frac{K^2+|\mathcal{V}|}{8} \le p_g$.

Corollary 4.4.54 has the following consequences (for some of the arguments see the paragraphs after 4.3.21): via the 'Semigroup Distribution Property' 4.2.33, the SWIC (for $h = 0$) is valid whenever $\nu = 1$. In fact, in this case not only $N(1) = 0$, but even $N(t) \equiv 0$, i.e. $Z_{0,\mathcal{I}}(t) \equiv P_{0,\mathcal{I}}(t)$.

If $\nu = 2$ then the coefficients of $N(t)$ are non-positive, however, it can happen that $N(t) \ne 0$, see. e.g. several examples in [61]. Hence, if $\nu = 2$ and $N(t) \ne 0$ then the SWIC fails and $\mathfrak{sw}_{\sigma[K]} - \frac{K^2+|\mathcal{V}|}{8} < p_g$. (The difference will be interpreted in terms of lattice cohomology in 4.9.2.)

Remark 4.4.55 Though till now we tried to convince the reader that the SWIC, for certain analytic types, is a 'natural' reality, the superisolated case suggests the opposite. Indeed, for such germs, p_g depends only on d, but the topological side depends in a subtle way on the local singularity types of C (see above the formulae of λ and $\mathcal{T}_{\sigma[K]}(1)$). Having in mind this subtle sensitivity to the local singularity data of C, the validity of SWIC (when it holds) is a true marvel.

Example 4.4.56 Let us analyse a particular case with more details. Assume $d = 5$, $\nu = 2$, and the two singularities have multiplicity sequence [3] and $[2_3]$. The graph Γ is presented below, and $N(t) = -2t$, hence SWIC fails: $p_g = 10$, while $-\lambda = 21/2$ and $\mathcal{T}_{\sigma[K]}(1) = 2/5$, hence $\mathfrak{sw}_{\sigma[K]} - (K^2 + |\mathcal{V}|)/8 = 8$.



In fact, we can consider two analytic structures supported on this topological type (given by the graph). They are rather different, though both are very natural. The first is a superisolated hypersurface singularity, as analysed above. On the other hand, this topological type supports also a splice quotient singularity which satisfies SWIC, hence it has $p_g = 8$.

4.5 Weighted Cubes and the Spaces $S_{k,n}$

4.5.1 Weighted Cubes and Generalized Computation Sequences

To any good resolution graph Γ and characteristic element $k \in \text{Char}$, we consider the weight function $\chi_k : L \to \mathbb{Z}$, and a natural cubical decomposition of $\mathbb{R}^s$ associated with the embedding $L \simeq \mathbb{Z}^s \hookrightarrow \mathbb{Z}^s \otimes \mathbb{R} = \mathbb{R}^s$, where $s = |\mathcal{V}|$ and the identification $L \simeq \mathbb{Z}^s$ is given by the base vectors $\{E_v\}_{v\in\mathcal{V}}$. Then, for each $n \geq \min_{l\in L}\{\chi_k(l)\}$, we define the topological space $S_{k,n}$, as the union of all cubes, which have all vertices of weight $\leq n$. We show that the homotopy type of the tower $\{S_{k,n}\}_n$ depends only on the 3-manifold $M(\Gamma)$ and on the spin^c structure associated with k. The tower $\{S_{k,n}\}_n$ carries an extremely deep information about $M(\Gamma)$; the final goal is to determine their homotopy types. Via the spaces $\{S_{k,n}\}_n$ this section prepares the theory of graded roots and lattice cohomology.

4.5.1 Cubes in $L \otimes \mathbb{R}$ and the Spaces $\{S_{k,n}\}_n$ [72] Fix a connected plumbing graph Γ with negative definite intersection form, and *we assume that the plumbed 3-manifold $M(\Gamma)$ is a rational homology sphere*.

We use the standard notations for the lattice L, which has the distinguished base elements $\{E_v\}_{v\in\mathcal{V}}$. Using this basis, one identifies L with $\mathbb{Z}^s$ with its fixed standard basis, still denoted by $\{E_v\}_{v\in\mathcal{V}}$.

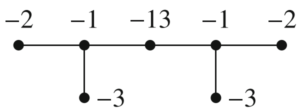
$\mathbb{Z}^s \otimes \mathbb{R} \simeq \mathbb{R}^s$ has a natural decomposition into cubes given by the inclusion $\mathbb{Z}^s \hookrightarrow \mathbb{R}^s$. The zero-dimensional cubes are exactly the lattice points $\mathbb{Z}^s$. Any $l \in \mathbb{Z}^s$ and subset $I \subset \mathcal{V}$ of cardinality q defines a q-dimensional cube $\square_q = (l, I)$, which has its vertices in the lattice points $(l + \sum_{v\in I'} E_v)_{I'}$, where I' runs over all subsets of I.

Next, we fix a characteristic element $k \in \text{Char}$ and we consider the Riemann–Roch function $\chi_k : L \to \mathbb{Z}$, $\chi_k(l) = -(l, l + k)/2$. Here we regard χ_k as a weight function on the set of cubes: the weights of zero-dimensional cubes are defined by $w_0(l) = \chi_k(l)$, while, in general, $w_q((l, I)) := \max\{\chi_k(v) : v \text{ is a vertex of } (l, I)\}$.

Definition 4.5.2 For every $n \in \mathbb{Z}$, define $S_n \subset \mathbb{R}^s$ as the union of all the cubes $\square_q$, of any dimension, with $w(\square_q) \leq n$ (with induced topology of $\mathbb{R}^s$). Clearly, $S_n \neq \emptyset$ exactly when $n \geq m_k$, where $m_k := \min_{l \in \mathbb{Z}^s} \chi_k(l)$. If we wish to emphasize the k-dependence we write $S_{k,n}$.

One has the natural inclusions $S_{m_k} \subset \ldots \subset S_n \subset S_{n+1} \subset \ldots$. It turns out that the topology of the spaces $\{S_n\}_{n \geq m_k}$ might be rather interesting. The tower has a finiteness property: only finitely many S_n have nontrivial topology (are non-contractible), but an S_n with n 'small' might have rather complicated homology groups. In general it is rather hard to solve the corresponding Diophantine equations and to analyse the adjacent positions of the solutions (in order to get the cubes which build up the topological space S_n). However, this combinatorial/artihmetical structure can be extremely rich covering a big amount of deep information.

Example 4.5.3 ([72]) Consider the following graph:



A computation shows that $\chi \geq -1$. S_{-1} consists of two contractible connected components. The space S_0 has three connected components, two of them contractible, and the third has the homotopy type of the circle. The spaces S_n for $n \geq 1$ are contractible.

4.5.4 Assume that k and k' determine the same Spin^c–structure of $M(\Gamma)$, cf. 4.2.94, hence $k' = k + 2l$ for some $l \in L$. Then $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. This means that the transformation $x \mapsto x' := x - l$ realizes an identification of the 'S_n-spaces' associated with k and k': $S_{k,n} = S_{k',n-\chi_k(l)}$. Hence, fixing a representative k from each class $[k] \in \mathrm{Spin}^c(M(\Gamma))$ we can speak about the tower of spaces $\{S_{k,n}\}_n$, indexed by $[k] \in \mathrm{Spin}^c(M(\Gamma))$.

Proposition 4.5.5 ([72]) *The tower of spaces* $\{S_{k,n}\}_n$*, indexed by* $[k] \in \mathrm{Spin}^c(M(\Gamma))$*, up to homotopy equivalence, depends only on* $M = M(\Gamma)$*, it is independent of the choice of the negative definite plumbing graph* Γ*, which provides* M.

Remark 4.5.6 A possible generalization of the set of weighted cubes and spaces S_n is provided via a set of compatible *weight functions*. Let Q_q denote the set of q–cubes. A set of functions $w_q : Q_q \to \mathbb{Z}$ $(0 \leq q \leq |\mathcal{V}|)$ is called a *set of compatible weight functions* if the following hold:

(a) for any integer $n \in \mathbb{Z}$, the set $w_0^{-1}((-\infty, n])$ is finite;
(b) for any $\square_q \in Q_q$ and for any of its faces $\square_{q-1} \in Q_{q-1}$ one has $w_q(\square_q) \geq w_{q-1}(\square_{q-1})$.

Then one can define S_n as $\cup_q \{\square \in Q_q \ : \ w_q(\square_q) \leq n\}$.

4.5.2 The Topology of the Spaces $\{S_{k,n}\}_n$

In order to analyse the topology of the space $S_n = S_{n,k}$ it is convenient to introduce the set of finite rectangles indexed by pairs $l_1, l_2 \in L$ with $l_1 \le l_2$.

Definition 4.5.7 For any such pair $l_1 \le l_2$ set $R(l_1, l_2) := \{x \in \mathbb{R}^s \,:\, l_1 \le x \le l_2\}$. Define also $R(l_1, \infty) := \{x \in \mathbb{R}^s \,:\, l_1 \le x\}$.

The point in the next lemma is that χ-monotone (non-increasing) computation sequences provide strong deformation retracts for the spaces $S_{k,n}$.

Lemma 4.5.8 *Fix* $k \in \mathrm{Char}$ *and write* $S_n = S_{k,n}$.

(I) There exist $l_+ \in L$ *and an increasing infinite sequence of cycles* $\{l_i\}_{i\ge 0}$ $(l_i \in L)$ *with* $l_0 = l_+$, *such that*

(a) for any $i \ge 0$ *one has* $l_{i+1} = l_i + E_{v(i)}$ *for some* $v(i) \in \mathcal{V}$,
(b) if $l_i = \sum_v m_{i,v} E_v$, *then* $\lim_{i\to\infty} m_{i,v} = \infty$ *for all* v,
(c) $\chi_k(l_{i+1}) \ge \chi_k(l_i)$.

Similarly, there exists $l_- \in L$ *and an increasing infinite sequence of cycles* $\{y_i\}_{i\ge 0}$, *satisfying* $y_0 = l_-$, *the analogs of (a)–(b), and (c)* $\chi_k(-y_{i+1}) \ge \chi_k(-y_i)$.

(II) Take l_- *and* l_+ *as in (I). Without loss of generality we can assume that* $-l_- \le l_+$. *Then the inclusion* $R(-l_-, \infty) \cap S_n \subset S_n$ *and* $R(-l_-, l_+) \cap S_n \subset S_n$ *are homotopy equivalences for any* $n \in \mathbb{Z}$.

Corollary 4.5.9 *For any* $k \in \mathrm{Char}$ *the space* S_n *is contractible for any* $n \gg 0$.

Proof Fix $l_- \le l_+$ as in Lemma 4.5.8*(I)*. Let n be so large that $R(-l_-, l_+) \subset S_n$. Then, by Lemma 4.5.8*(II)* S_n has the same homotopy type as $R(-l_-, l_+)$. □

4.5.10 Distinguished Representatives and Their Spaces S_n As we already said in 4.5.4, if $k' = k + 2l$ for some $l \in L$ then $S_{k,n} = S_{k',n-\chi_k(l)}$. Hence, it is natural to choose one representative from each spin^c structure. For several results the choice is irrelevant, however, certain choices have certain advantages. Our preferred choice is the *distinguished representative*, or *distinguished characteristic element* $k_r := K + 2s_h$, cf. 4.2.94, where $s_h \in L'$ is the smallest representative of h in $\mathcal{S}'$, cf. 4.2.78.

A possible motivation for the choice of k_r is the following. Recall that the rationality criterion for graphs is $\chi(l) > 0$ for any $l \in L_{>0}$, hence it is decided in the 'first quadrant' $L_{\ge 0}$ of L. More generally, for arbitrary graphs, the essential properties of $\chi : L \to \mathbb{Z}$ are already coded in the restriction $\chi|L_{\ge 0}$. The choice $k_r = K + 2s_h$ guarantees that the essential properties of $\chi_{k_r} : L \to \mathbb{Z}$ are coded again in $L_{\ge 0}$ (or, equivalently, for a fixed h, the essential information of $\chi_{\mathbb{Q}}|\{l' \in L' : [l'] = h\}$ is coded in $\chi_{\mathbb{Q}}|s_h + L_{\ge 0}$.

Lemma 4.5.11 *Fix $h \in H$ and $k_r = K + 2s_h$ as above. Then the following facts hold.*

(a) In Lemma 4.5.8 one may take $l_- = 0$. This means that $R(0, \infty) \cap S_{k_r,n} \subset S_{k_r,n}$ is a homotopy equivalence for any n. In particular, by Lemma 4.5.8, there exists $l_+ \geq 0$ such that $R(0, l_+) \cap S_{k_r,n} \subset S_{k_r,n}$ is a homotopy equivalence for any n.
(b) Assume that $Z_K \geq 0$ (e.g., as in the minimal good resolution). Then one can take $l_+ = \lfloor Z_K \rfloor$. Hence, $S_{k_r,n}$ has the homotopy type of $R(0, \lfloor Z_K \rfloor) \cap S_{k_r,n}$.
(c) For any $x \geq 0$ one has $\chi_{k_r}(x) \geq \chi_K(x)$. Therefore, $\min \chi_{k_r} \geq \min \chi_K$.
(d) $S_{K,n}$ (i.e. when $h = 0$ and $s_h = 0$) is connected for all $n \geq 1$.

Example 4.5.12 (Characterization of Rational Graphs via the Spaces S_n [70]) Let Γ be a connected, negative definite plumbing graph whose plumbed 3-manifold is a rational homology sphere. Recall that Γ is rational if $\chi(l) > 0$ for any $l \in L_{>0}$. (In this case $p_g(X, o) = 0$ for any analytic type supported on the topological type determined by Γ.) Then the following facts are equivalent:

(a) Γ is rational;
(b) $S_{K,n}$ is contractible for every $n \geq \min \chi$;
(b') $S_{K,n}$ is connected for every $n \geq \min \chi$;
(c) $S_{k,n}$ is contractible for all $k \in \text{Char}$ and $n \geq \min \chi_k$.

Additionally, if Γ is rational and $k_r = K + 2s_h$, then $\min \chi_{k_r} = 0$.

Example 4.5.13 (Characterization of Elliptic Graphs via the Spaces $S_{K,n}$ [70]) Assume again that $M(\Gamma)$ is a $\mathbb{Q}HS^3$. Recall that Γ is elliptic if $\min \chi = 0$ and Γ is not rational. Then Γ is elliptic if and only if $S_{K,n}$ is empty for $n < 0$ and $S_{K,0}$ is not connected.

4.5.3 'Bad' Vertices, Almost Rational Graphs and Lattice Fibrations

We measure how far an arbitrary graph (tree) Γ is from being rational. Recall that decreasing all the self-intersection numbers of a tree, with all the vertices decorated by $g_v = 0$, we obtain a rational graph. The next definition aims to identify those vertices where such a decrease is really necessary. [Such a subset of $\mathcal{V}$ was already considered in different articles like [70, 72, 74, 102], mostly under the name 'bad vertices'. Since the definition of the 'badness' was not uniform here we use the notation SR for them, for several other families see [66].]

Definition 4.5.14 Let Γ be a negative definite connected tree with $M(\Gamma)$ a $\mathbb{Q}HS^3$.

A subset of vertices $\overline{\mathcal{V}} = \{v_1, \ldots, v_\nu\} \subset \mathcal{V}$ is called *SR–set*, if by replacing the Euler numbers $e_v = E_v^2$ indexed by $v \in \overline{\mathcal{V}}$ by some more negative integers $e'_v \leq e_v$ we get a rational graph. A graph is called AR-graph ('almost rational graph') if it admits an SR–set of cardinality ≤ 1.

Example 4.5.15

(a) A possible SR–set can be chosen in many different ways, it is not determined uniquely even if it is minimal with this property.
(b) Usually we allow non-minimal SR–sets as well.
(c) Any rational graph is AR; for rational graphs the empty set is an SR–set. The class of AR graphs is closed while taking subgraphs or/and decreasing the Euler numbers.
(d) The set of nodes is an SR–set. Hence any star-shaped graph (with $g = 0$) is AR with $\overline{\mathcal{V}} = \{v_0\}$.
(e) Any elliptic graph with $H_1(L_X, \mathbb{Q}) = 0$ is AR.
(f) Consider the graph Γ of $S^3_{-d}(K)$ (for $d > 0$ and $K \subset S^3$ algebraic knot). Then Γ is AR: if we modify the -1 decoration of v_1 into -2, we get a sandwiched (hence rational) graph.
(g) Let $\{K_i\}_{i=1}^{\nu}$ be algebraic knots and set $K = \#_i K_i$. For $d > 0$ the negative definite graph Γ of $S^3_{-d}(K)$ is given in 4.2.32. Then the smallest SR–set consists of the set of (-1)-vertices (their number is ν).

4.5.16 Lattice Fibrations: Universal Cycles in the Fibers Let us give some intuition for the next construction.

If Γ is rational, then 0 is a χ_{k_r}–minimal lattice point, and $0 \hookrightarrow S_{k_r,n}$ ($n \geq 0$) admits a strong deformation retraction: there is a χ_{k_r}–non-increasing (combinatorial) flow contracting any $S_{k_r,n}$ (and $L \otimes \mathbb{R}$) to 0.

In general, let us start with the lattice L and a representative $k = K + 2l'_h$. Then (dictated by some 'badness properties' of some of the vertices, indexed by $\overline{\mathcal{V}}$) we will write the set of vertices $\mathcal{V}$ as a disjoint union $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$, such that any sublattice of type $\bar{l} + L(\mathcal{V}^*)$ (where $\bar{l} = \sum_{v \in \overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}})$) behaves as a rational lattice, that is, it can be contracted to one of its lattice points via a χ_k–non-increasing flow. (In other words, 'L, or the spaces S_n, project to $L(\overline{\mathcal{V}})$ with contractible fibers'.) On the other hand, the χ_k–minimal point of $\bar{l} + L(\mathcal{V}^*)$, where $\bar{l} + L(\mathcal{V}^*)$ contracts, depends essentially on $\bar{l}$; it is a crucial universal point $x_{l'_h}(\bar{l})$ of $\bar{l} + L(\mathcal{V}^*)$. The aim of different reduction theorems is to recover different invariants of the weighted lattice (L, χ_k) from $\{\chi_k(x_{l'_h}(\bar{l}))\}_{\bar{l} \in L(\overline{\mathcal{V}})}$.

In this subsection we define and analyse the points $x_{l'_h}(\bar{l})$. If $l'_h = s_h$ then some additional 'positivity' properties hold for them.

4.5.17 The Definition of the Lattice Points $x(\bar{l})$ Let us fix a resolution of a germ (whose link is not necessarily a rational homology sphere). Suppose we have a family of *distinguished* vertices $\overline{\mathcal{V}} := \{v_k\}_{k=1}^{\nu} \subseteq \mathcal{V}$ (usually chosen by a certain geometric property). Then we split the set of vertices $\mathcal{V}$ into the disjoint union $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$. Let $\{m_v(x)\}_v$ denote the coefficients of a cycle $x \in L \otimes \mathbb{Q}$, that is $x = \sum_{v \in \mathcal{V}} m_v(x) E_v$.

We use the notation $\bar{l} := \sum_{v \in \overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}})$, and we fix $h \in H$ and a representative $l'_h \in L'$ with $[l'_h] = h$. Then the cycles $x(\bar{l})$ are defined as follows.

Proposition 4.5.18 ([70, Lemma 7.6], [47]) *For any $\bar{l} \in L(\overline{\mathcal{V}})$ there exists a unique cycle $x(\bar{l}) \in L$ (depending on the choice of l'_h) satisfying the next properties:*

(a) $m_v(x(\bar{l})) = \ell_v$ for any distinguished vertex $v \in \overline{\mathcal{V}}$;
(b) $(x(\bar{l}) + l'_h, E_v) \leq 0$ for every 'non-distinguished vertex' $v \in \mathcal{V}^*$;
(c) $x(\bar{l})$ is minimal with the two previous properties (with respect to $\leq$).

Furthermore, the cycle $x(\bar{l})$ automatically satisfies

$$x(\bar{l}) + \bar{l}_1 \leq x(\bar{l} + \bar{l}_1) \quad \text{for any } \bar{l}_1 \geq 0, \bar{l}_1 \in L(\overline{\mathcal{V}}). \tag{4.75}$$

If we wish to emphasize the dependence on l'_h we write $x_{l'_h}(\bar{l})$.

The cycles $x(\bar{l})$ satisfy the following universal property as well.

Lemma 4.5.19 *Assume that a certain $x \in L$ satisfies $m_v(x) = m_v(x(\bar{l}))$ for all $v \in \overline{\mathcal{V}}$, and $x \leq x(\bar{l})$.*

Then there is a ***generalized Laufer's computation sequence*** *connecting x with $x(\bar{l})$. The sequence $\{z_i\}_{i=0}^t$ is constructed as follows. Set $z_0 = x$. Assume that z_i is already constructed. If for some $v \in \mathcal{V}^*$ one has $(z_i + s_h, E_v) > 0$ then take $z_{i+1} = z_i + E_{v(i)}$, where $v(i)$ is such an index. If z_i satisfies 4.5.18(b), then stop and set $t = i$. Then this procedure stops after finite steps and z_t is exactly $x(\bar{l})$.*

Along the computation sequence $\chi_k(z_{i+1}) \leq \chi_k(z_i)$ for any $0 \leq i < t$. Equality holds if $(z_i + l'_h, E_{v(i)}) = 1$.

In the case of an SR–set we have the following statement.

Proposition 4.5.20 *Let $\overline{\mathcal{V}}$ be an SR–set. Choose l'_h and set $k = K + 2l'_h$. Then $\bar{l} + L(\mathcal{V}^*) = \{x \in L : m_v(x) = m_v(x(\bar{l}))$ for all $v \in \overline{\mathcal{V}}\}$ contracts to $x(\bar{l})$ such that along the contraction χ_k is non-increasing. In particular, $\chi_k(x) \geq \chi_{k_r}(x(\bar{l}))$ for any $x \in \bar{l} + L(\mathcal{V}^*)$.*

4.5.4 Concatenated Computation Sequences of AR Graphs [70]

Assume that Γ is an AR resolution graph, let $\{v_0\}$ be an SR–set. In particular $M(\Gamma)$ is a rational homology sphere.

Theorem 4.5.21 *If Γ is AR, then for any $k \in$ Char and $n \geq m_k = \min \chi_k$ any connected component of $S_{k,n}$ is contractible.*

Note that the statement is independent of the choice of k in its class, cf. 4.5.10. In the sequel we will choose the distinguished representative k_r, and we write S_n for $S_{k_r,n}$. We also write $\mathcal{V} = \overline{\mathcal{V}} \sqcup \mathcal{V}^*$, where $\overline{\mathcal{V}} = \{v_0\}$. For each $\ell \in \mathbb{Z}$ we consider the cycles $\bar{l} := \ell E_{v_0} \in L(\overline{\mathcal{V}})$ and $x(\bar{l}) \in L$ from 4.5.16. We abridge $x(\ell E_{v_0})$ as $x(\ell)$.

In order to prove the theorem we construct an increasing path $\gamma = \{l_i\}_{i\geq 0}$ in L (with $\ell_{i+1} = \ell_i + E_{v(i)}$ for all i), which determines the 1-chain $C_\gamma := \cup_{i\geq 0}[l_i, l_{i+1}]$ of 1-cubes in $L \otimes \mathbb{R}$ (without any loop), such that $C_\gamma \cap S_n \hookrightarrow S_n$ is a homotopy equivalence. The construction and the statement of the theorem constitute the prototype of the more general Reduction Theorem 4.8.2 and also this was the original intuitive motivation and starting point in the definition of the graded roots, cf. 4.7 and 4.7.2.

The construction start as follows. By Lemma 4.5.11*(a)* the inclusion $R(0,\infty) \cap S_n \subset S_n$ admits a strong deformation retract. Hence we can restrict ourself to the intersection with the first quadrant. The path $\gamma = \{l_i\}_{i\geq 0}$ is defined as a series of concatenated computation sequences. It contains, as intermediate terms, all the universal cycles $\{x(\ell)\}_{\ell\geq 0}$ in an increasing order. The first term is $l_0 = x(0) = 0$. The part of the sequence starting from $x(\ell)$ and ending with $x(\ell+1)$ starts with $x(\ell)$ and the next term is $x(\ell) + E_{v_0}$. Then, the continuation is generalized Laufer-type computation sequence connecting $x(\ell) + E_{v_0}$ with $x(\ell+1)$. Indeed, the multiplicity of E_0 in both $x(\ell) + E_{v_0}$ and $x(\ell+1)$ is $\ell + 1$, and by (4.75) $x(\ell+1) \geq x(\ell) + E_{v_0}$. Hence Lemma 4.5.19 gives a computation sequence $\gamma^{(\ell+1)} = \{l_i^{(\ell+1)}\}_i$, which connects them. Then we proceed inductively.

Define $\tau(\ell) := \chi_{k_r}(x(\ell))$. Let o be the order of $E^*_{v_0}$ in L'/L and $p = m_{v_0}(oE^*_{v_0})$.

Lemma 4.5.22

(a) *The path* $\{l_i\}_i$ *is increasing:* $l_{i+1} = l_i + E_{v(i)}$.
(b) *For any* E_v*-coefficient one has* $\lim_{\ell\to\infty} m_v(x(\ell)) = \infty$ *(where* $v \in \mathcal{V}$*).*
(c) *(Quasiperiodicity)* $x(\ell + tp) = x(\ell) + toE^*_{v_0}$.
(d) χ_{k_r} *along each part (subsequence)* $\gamma^{(\ell)}$ *is constant.*
(e) $\tau(\ell+1) = \tau(\ell) + 1 - (x(\ell) + s_h, E_{v_0})$.
(f) *There exists* ℓ_0 *such that* $\tau(\ell+1) \geq \tau(\ell)$ *for* $\ell \geq \ell_0$.

Theorem 4.5.23 *Consider the 1-chain* $C_\gamma := \cup_{i\geq 0}[l_i, l_{i+1}]$ *in* $L \otimes \mathbb{R}$ *as above. Then for any n the inclusion* $C_\gamma \cap S_n \subset S_n$ *is a homotopy equivalence. In particular, since each connected component of* $C_\gamma \cap S_n$ *is contractible, Theorem 4.5.21 follows.*

Remark 4.5.24 In general, it is not easy to find the cycles $x(\ell)$. Fortunately, in several applications (see e.g. 4.7.3) one does not need all the coefficients of these cycles, only the values $\tau(\ell) = \chi_{k_r}(x(\ell))$. In most of the cases they are computed inductively using 4.5.22*(e)*, hence basically one needs only to know $(x(\ell), E_{v_0})$ for any ℓ.

Example 4.5.25 For the determination of the universal cycles $\{x(\ell)\}_\ell$ and the corresponding τ-function in the case of star-shaped graphs and surgery manifolds see 4.7.22, 4.7.4 and Sect. 4.9.

4.6 Lattice Cohomology

We provide two equivalent definitions for the lattice cohomology $\{\mathbb{H}^q\}_{q\geq 0}$ associated with a free $\mathbb{Z}$-module endowed with a fixed basis and with a set of 'compatible weight functions'. The first definition is based on the construction of a cochain complex. The second one involves the spaces $\{S_n\}_n$ introduced in 4.5.2. Once Γ is fixed, any characteristic element $k \in$ Char determines a set of weights (via the RR expression χ_k), hence the lattice cohomology $\mathbb{H}^*(\Gamma, k)$. It turns out that they depend only on $M(\Gamma)$ and $[k] \in \mathrm{Spin}^c(M(\Gamma))$. In 4.6.3 we show that the Euler characteristic of $\mathbb{H}^*(\Gamma, k)$ is the normalized Seiberg–Witten invariant of $M(\Gamma)$.

For more details see e.g. [71–73].

4.6.1 The Lattice Cohomology Associated with a System of Weights

We consider a free $\mathbb{Z}$-module, with a fixed basis $\{E_v\}_{v\in\mathcal{V}}$, denoted by $\mathbb{Z}^s$. It is also convenient to fix a total ordering of the index set $\mathcal{V}$, which in the sequel will be denoted by $\{1,\dots,s\}$. Our goal is to define a graded $\mathbb{Z}[U]$-module associated with the pair $(\mathbb{Z}^s, \{E_v\}_v)$ and a set of weights. First we set some notations regarding $\mathbb{Z}[U]$-modules.

4.6.1 $\mathbb{Z}[U]$-Modules Consider the graded $\mathbb{Z}[U]$-module $\mathcal{T} := \mathbb{Z}[U, U^{-1}]$, and (following [102]) denote by $\mathcal{T}_0^+$ its quotient by the submodule $U \cdot \mathbb{Z}[U]$. This has a grading in such a way that $\deg(U^{-d}) = 2d$ ($d \geq 0$). Similarly, for any $n \geq 1$, the quotient of $U^{-(n-1)} \cdot \mathbb{Z}[U]$ by $U \cdot \mathbb{Z}[U]$ (with the same grading) defines the graded module $\mathcal{T}_0(n)$. Hence, $\mathcal{T}_0(n)$, as a $\mathbb{Z}$-module, is freely generated by $1, U^{-1}, \dots, U^{-(n-1)}$, and has finite $\mathbb{Z}$-rank n.

More generally, for any graded $\mathbb{Z}[U]$-module P with d-homogeneous elements P_d, and for any $r \in \mathbb{Q}$, we denote by $P[r]$ the same module graded (by $\mathbb{Q}$) in such a way that $P[r]_{d+r} = P_d$. Then set $\mathcal{T}_r^+ := \mathcal{T}_0^+[r]$ and $\mathcal{T}_r(n) := \mathcal{T}_0(n)[r]$. Hence, for $m \in \mathbb{Z}$, $\mathcal{T}_{2m}^+ = \mathbb{Z}\langle U^{-m}, U^{-m-1}, \dots\rangle$ as a $\mathbb{Z}$-module.

4.6.2 The Cochain Complex $\mathbb{Z}^s \otimes \mathbb{R}$ has a natural cellular decomposition into cubes (see also 4.5.1). The set of zero-dimensional cubes is provided by the lattice points $\mathbb{Z}^s$. Any $l \in \mathbb{Z}^s$ and subset $I \subset \mathcal{V}$ of cardinality q defines a q-dimensional cube, which has its vertices in the lattice points $(l + \sum_{v\in I'} E_v)_{I'}$, where I' runs over all subsets of I. On each such cube we fix an orientation. This can be determined, e.g., by the order $(E_{v_1}, \dots, E_{v_q})$, where $v_1 < \cdots < v_q$, of the involved base elements $\{E_v\}_{v\in I}$. The set of oriented q-dimensional cubes defined in this way is denoted by $\mathcal{Q}_q$ ($0 \leq q \leq s$).

Let C_q be the free $\mathbb{Z}$-module generated by oriented cubes $\square_q \in Q_q$. Clearly, for each $\square_q \in Q_q$, the oriented boundary $\partial\square_q$ (of 'classical' cubical homology) has the form $\sum_k \varepsilon_k \square^k_{q-1}$ for some $\varepsilon_k \in \{-1,+1\}$. These are the *faces* of $\square_q$. It is clear that $\partial\circ\partial = 0$. But, obviously, the homology of the chain complex (C_*, ∂) (or, of the dual cochain complex $(\mathrm{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z}), \delta)$) is not very interesting: it is the (co)homology of $\mathbb{R}^s$. A more interesting (co)homology can be constructed as follows. For this, we consider a set of compatible *weight functions* $\{w_q\}_q$ as in 4.5.6. In the sequel sometimes we will omit the index q of w_q.

4.6.3 In the presence of any fixed set of compatible weight functions $\{w_q\}_q$ we define $\mathcal{F}^q$ as the set of morphisms $\mathrm{Hom}_{\mathbb{Z}}(C_q, \mathcal{T}^+_0)$ with finite support on Q_q.

Notice that $\mathcal{F}^q$ is a $\mathbb{Z}[U]$-module by $(p * \phi)(\square_q) := p(\phi(\square_q))$ $(p \in \mathbb{Z}[U])$. Moreover, $\mathcal{F}^q$ has a $\mathbb{Z}$-grading: $\phi \in \mathcal{F}^q$ is homogeneous of degree $\deg(\phi) = d \in \mathbb{Z}$ if for each $\square_q \in Q_q$ with $\phi(\square_q) \neq 0$, $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}^+_0$ of degree $d - 2\cdot w(\square_q)$. (In fact, the grading is $2\mathbb{Z}$-valued; hence, the reader interested only in the present construction may divide all the degrees by two. Nevertheless, we prefer to keep the present form in our presentation because of its resonance with the Heegaard Floer homology of the link.)

Next, we define $\delta_w : \mathcal{F}^q \to \mathcal{F}^{q+1}$. For this, fix $\phi \in \mathcal{F}^q$ and we show how $\delta_w\phi$ acts on a cube $\square_{q+1} \in Q_{q+1}$. First write $\partial\square_{q+1} = \sum_k \varepsilon_k \square^k_q$, then set

$$(\delta_w\phi)(\square_{q+1}) := \sum_k \varepsilon_k\, U^{w(\square_{q+1}) - w(\square^k_q)}\, \phi(\square^k_q).$$

Lemma 4.6.4 $\delta_w \circ \delta_w = 0$, *i.e.* $(\mathcal{F}^*, \delta_w)$ *is a cochain complex.*

4.6.5 In fact, $(\mathcal{F}^*, \delta_w)$ has a natural **augmentation** too. Indeed, set $m_w := \min_{l\in\mathbb{Z}^s} w_0(l)$ and choose $l_w \in \mathbb{Z}^s$ such that $w_0(l_w) = m_w$. Then one defines the $\mathbb{Z}[U]$-linear map

$$\epsilon_w : \mathcal{T}^+_{2m_w} \longrightarrow \mathcal{F}^0$$

such that $\epsilon_w(U^{-m_w-s})(l)$ is the class of $U^{-m_w+w_0(l)-s}$ in $\mathcal{T}^+_0$ for any $l \in L$ and $s \geq 0$.

Lemma 4.6.6 ϵ_w *is injective, and* $\delta_w \circ \epsilon_w = 0$.

One verifies that both ϵ_w and δ_w are morphisms of $\mathbb{Z}[U]$-modules and are homogeneous of degree zero.

Definition 4.6.7 The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*(\mathbb{R}^s, w)$. The homology of the augmented cochain complex

$$0 \longrightarrow \mathcal{T}^+_{2m_w} \xrightarrow{\epsilon_w} \mathcal{F}^0 \xrightarrow{\delta_w} \mathcal{F}^1 \xrightarrow{\delta_w} \dots$$

is called the *reduced lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$.

If the pair $(\mathbb{R}^s, w)$ is clear from the context, we omit it from the notation.

For any $q \geq 0$ fixed, the $\mathbb{Z}$-grading of $\mathcal{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$; the homogeneous part of degree d is denoted by $\mathbb{H}^q_d$, or $\mathbb{H}^q_{red,d}$. Moreover, both $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$ admit an induced graded $\mathbb{Z}[U]$-module structure and $\mathbb{H}^q = \mathbb{H}^q_{red}$ for $q > 0$.

It is easy to see that $\mathbb{H}^*(\mathbb{R}^s, w)$ depends essentially on the choice of w.

Lemma 4.6.8 *One has a graded* $\mathbb{Z}[U]$*-module isomorphism* $\mathbb{H}^0 = \mathcal{T}^+_{2m_w} \oplus \mathbb{H}^0_{red}$.

4.6.9 Next, we present another realization of the modules $\mathbb{H}^*$. In 4.5.2 for each $n \in \mathbb{Z}$ we defined $S_n = S_n(w) \subset \mathbb{R}^s$ as the union of all the cubes $\square_q$ (of any dimension) with $w(\square_q) \leq n$. Clearly, $S_n = \emptyset$, whenever $n < m_w$. For any $q \geq 0$, set

$$\mathbb{S}^q(\mathbb{R}^s, w) := \oplus_{n \geq m_w} H^q(S_n, \mathbb{Z}).$$

Then $\mathbb{S}^q$ is $\mathbb{Z}$ (in fact, $2\mathbb{Z}$)-graded, the $d = 2n$-homogeneous elements $\mathbb{S}^q_d$ consist of $H^q(S_n, \mathbb{Z})$. Also, $\mathbb{S}^q$ is a $\mathbb{Z}[U]$-module; the U-action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. Moreover, for $q = 0$, the fixed base-point $l_w \in S_n$ provides an augmentation (splitting) $H^0(S_n, \mathbb{Z}) = \mathbb{Z} \oplus \tilde{H}^0(S_n, \mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$-modules

$$\mathbb{S}^0 = \mathcal{T}^+_{2m_w} \oplus \mathbb{S}^0_{red} = (\oplus_{n \geq m_w} \mathbb{Z}) \oplus (\oplus_{n \geq m_w} \tilde{H}^0(S_n, \mathbb{Z})).$$

Theorem 4.6.10 *There exists a graded* $\mathbb{Z}[U]$*-module isomorphism, compatible with the augmentations:*

$$\mathbb{H}^*(\mathbb{R}^s, w) = \mathbb{S}^*(\mathbb{R}^s, w).$$

4.6.11 Restrictions Assume that $T \subset \mathbb{R}^s$ is a subspace of $\mathbb{R}^s$ consisting of a union of some cubes (from $\mathcal{Q}_*$). Let $C_q(T)$ be the free $\mathbb{Z}$-module generated by q-cubes of T, $\mathcal{F}^q(T)$ be the restriction of $\mathcal{F}^q$ to $C_q(T)$. Then $(\mathcal{F}^*(T), \delta_w)$ is a complex, whose homology will be denoted by $\mathbb{H}^*(T, w)$. It has a natural graded $\mathbb{Z}[U]$-module structure. The restriction map induces a natural graded $\mathbb{Z}[U]$-module homogeneous homomorphism (of degree zero)

$$r^* : \mathbb{H}^*(\mathbb{R}^s, w) \to \mathbb{H}^*(T, w).$$

4.6.2 The Lattice Cohomology Associated with a Plumbing Graph

4.6.12 We consider a connected negative definite plumbing graph Γ and we assume that $M(\Gamma)$ is a $\mathbb{Q}HS^3$. We write $s := |\mathcal{V}|$. We also fix a characteristic element $k \in \mathrm{Char}$.

Note that Γ automatically and naturally provides a free $\mathbb{Z}$-module $L = \mathbb{Z}^s$ with a fixed bases $\{E_v\}_v$, cf. 4.2.9 and 4.5.1. Using Γ and k, we define a set of compatible weight functions w as in 4.5.1: $w_k(\square_q) = \max\{\chi_k(v) \ : \ v \text{ is a vertex of } \square_q\}$.

Definition 4.6.13 The $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\mathbb{R}^s, w)$ and $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$ obtained by these weight functions are called the *lattice cohomologies* associated with the pair (Γ, k) and are denoted by $\mathbb{H}^*(\Gamma, k)$, respectively $\mathbb{H}^*_{red}(\Gamma, k)$.

Proposition 4.6.14

(a) $\mathbb{H}^*_{red}(\Gamma, k)$ *is finitely generated over* $\mathbb{Z}$.
(b) $\mathbb{H}^0_{red,d}(\Gamma, K) = 0$ *for the canonical characteristic element* K *and* $d > 0$.

Remark 4.6.15 There is a symmetry present in the picture. Indeed, the involution $x \mapsto -x$ ($x \in L'$) induces identities $\chi_{-k}(-l) = \chi_k(l)$, hence isomorphisms

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, -k) \ \text{ and } \ \mathbb{H}^*_{red}(\Gamma, k) = \mathbb{H}^*_{red}(\Gamma, -k).$$

The involution $[k] \mapsto [-k]$ corresponds to the natural involution of $\mathrm{Spin}^c(M)$, cf. 4.2.93.

4.6.16 Assume that $[k] = [k']$, hence $k' = k+2l$ for some $l \in L$. Then $\chi_{k'}(x-l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. Therefore, the transformation $x \mapsto x' := x - l$ realizes the following identification:

Lemma 4.6.17 *If* $k' = k + 2l$ *for some* $l \in L$, *then:* $\mathbb{H}^*(\Gamma, k') = \mathbb{H}^*(\Gamma, k)[-2\chi_k(l)]$.

4.6.18 In fact, there is an easy way to choose one module from the multitude $\{\mathbb{H}^*(\Gamma, k)\}_{k\in[k]}$. Indeed, set $m_k = \min_{l\in L} \chi_k(l)$ as above. Since $(k + 2l)^2 = k^2 - 8\chi_k(l)$, we get

$$8m_k = k^2 - \max_{k'\in[k]} (k')^2 \leq 0. \tag{4.76}$$

Set $M_{[k]} := \{k \in [k] : m_k = 0\}$. Hence, if k_0 and $k_0 + 2l \in M_{[k]}$, then $-\chi_{k_0}(l) = 0$. In particular, for any fixed orbit $[k]$, any choice of $k_0 \in M_{[k]}$ provides the same module $\mathbb{H}^*(\Gamma, k_0)$, in the sequel denoted by $\mathbb{H}^*(\Gamma, [k])$. Hence, for any $k \in [k]$

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, [k])[2m_k]. \tag{4.77}$$

Proposition 4.6.19 *For each fixed* $[k] \in \mathrm{Spin}^c(M(\Gamma))$*,* $\mathbb{H}^*(\Gamma, [k])$ *depends only on* $M(\Gamma)$ *and is independent of the choice of the graph* Γ*, which provides* $M(\Gamma)$*.*

Next, consider the distinguished characteristic element k_r, cf. 4.5.10. The following statement follows from Lemma 4.5.11.

Proposition 4.6.20 *The restriction* $\mathbb{H}^*(\Gamma, k_r) \to \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$ *induced by the inclusion* $(\mathbb{R}_{\geq 0})^s \hookrightarrow \mathbb{R}^s$ *is an isomorphism of graded* $\mathbb{Z}[U]$ *modules.*

Remark 4.6.21 Assume that Γ is either rational or elliptic, in particular, $\min(\chi) = 0$. Then by 4.5.11 $\min(\chi_{k_r}) \geq 0$. Hence, by (4.76), in fact, $\min(\chi_{k_r}) = 0$.

Example 4.6.22 (Rational Graphs) Theorem 4.5.12 transforms into the following statement. The following facts are equivalent:

(a) Γ is rational;
(b) $\mathbb{H}^*_{red}(\Gamma, K) = 0$;
(b') $\mathbb{H}^0_{red}(\Gamma, K) = 0$;
(c) $\mathbb{H}^*_{red}(\Gamma, k) = 0$ for every $k \in \mathrm{Char}$.

Additionally, by Remark 4.6.21, if Γ is rational then $\mathbb{H}^0(\Gamma, k_r) = \mathcal{T}_0^+$ for any k_r.

Example 4.6.23 (Elliptic Graphs) Theorem 4.5.13 and Remark 4.6.21 transform into the following statement: Γ is elliptic if and only if $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+ \oplus \mathbb{H}^0_{red}(\Gamma, K)$ with $\mathbb{H}^0_{red}(\Gamma, K) \neq 0$. (In fact, if Γ is elliptic then $\mathbb{H}^0_{red}(\Gamma, K) = \mathcal{T}_0(1)^\ell$, where $\ell > 0$ is the length of the elliptic sequence in the sense of Laufer and Yau).

Example 4.6.24 (Almost Rational Graphs) By 4.5.21, if Γ is almost rational, $\mathbb{H}^q(\Gamma, k) = 0$ for any $q \geq 1$ and $k \in \mathrm{Char}$. (For $\mathbb{H}^0(\Gamma, k)$ see 4.7.3.)

Remark 4.6.25 The author knows no example when $\mathbb{H}^*(\Gamma, k)$ has a non-zero $\mathbb{Z}$-torsion element. *It is a challenge to prove that this cannot occur indeed.*

4.6.3 The Lattice Cohomology and the Seiberg–Witten Invariant

Fix Γ and k as above. Our goal is to identify the 'Euler characteristic' of the lattice cohomology $\mathbb{H}^*(\Gamma, k)$. Recall that by 4.6.14 $\mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^*_{red}(\Gamma, k)) < \infty$.

Definition 4.6.26 The Euler characteristic of $\mathbb{H}^*(\Gamma, k)$ is defined as

$$eu(\mathbb{H}^*(\Gamma, k)) := -m_k + \sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k)).$$

For motivation of the $-m_k$ term see 4.7.6 and the computations from below.

4.6.27 Fix l_- and l_+ and the rectangle $R = R(-l_-, l_+)$ as in Lemma 4.5.8. We define

$$Eu_{\chi_k}(R) := \sum_{\square_q \subset R} (-1)^{q+1} w_k(\square_q) \text{ and } Eu^{pol}_{\chi_k}(\mathfrak{q}) := \sum_{\square_q \subset R} (-1)^q \mathfrak{q}^{w_k(\square_q)} \in \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}].$$

In particular, if we write $Eu^{pol}_{w_k}(\mathfrak{q})/(1-\mathfrak{q})$ as $\sum_{n\geq m_k} a_n \mathfrak{q}^n$ then

$$a_n = \sum_{\square_q \subset R,\, w_k(\square_q)\leq n} (-1)^q = \chi_{top}(S_n \cap R),$$

where χ_{top} is the topological Euler characteristic. But, by 4.5.8, $S_n \cap R \hookrightarrow S_n$ is a homotopy equivalence, hence $a_n = \chi_{top}(S_n)$. This by 4.6.10 reads as

$$\frac{Eu^{pol}_{\chi_k}(\mathfrak{q}) - \mathfrak{q}^{m_k}}{1-\mathfrak{q}} = \sum_{n\geq m_k} (a_n - 1)\mathfrak{q}^n = \sum_{n\geq m_k} \Big(\sum_{q\geq 0} (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red,2n}(\Gamma, k)) \Big) \mathfrak{q}^n.$$

In particular, this expression is independent of the choice of R. Finally, by taking the limit $\lim_{\mathfrak{q}\to 1}$ we get

$$Eu_{\chi_k}(R) + m_k = \sum_{q\geq 0} (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k))),$$

or

$$Eu_{\chi_k}(R) = eu(\mathbb{H}^*(\Gamma, k)). \tag{4.78}$$

The above identity is a generalization to the level of weighted cubes of the classical fact that the Euler characteristic computed at the level of cubes equals the homological Euler characteristic.

4.6.28 Recall from 4.6.2 that if $k' = k + 2l$, $l \in L$, then $\mathbb{H}^*(\Gamma, k') = \mathbb{H}^*(\Gamma, k)[-2\chi_k(l)]$, hence the lattice cohomologies associated with different k's with the same class $[k]$ are equal up to a shift. This has no effect on $\sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k))$, however it has on m_k. This can be remedied either by choosing k from $M_{[k]}$ (cf. 4.6.18), or by taking k_r (cf. 4.6.16). Next we present another way to eliminate the above shift.

Let us replace the weight function $w_k(\square_q) := \{\chi_k(v) : v \text{ is a vertex of } \square_q\}$ by

$$\overline{w}_k(\square_q) := w_k(\square_q) + \mathfrak{d}_k, \text{ where } \mathfrak{d}_k := -\frac{k^2 + |\mathcal{V}|}{8} + \frac{K^2 + |\mathcal{V}|}{8} = \chi\left(\frac{k-K}{2}\right),$$

and denote the corresponding lattice cohomologies by $\overline{\mathbb{H}}^*(\Gamma, k)$. Then

Lemma 4.6.29 $\overline{\mathbb{H}}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k)[\mathfrak{d}_k]$ *is independent of the choice of k from* $[k]$.

Remark 4.6.30 In the spirit of 4.4.33, and with the notation $k = K+2l'_h$, $\overline{\mathbb{H}}^*(\Gamma, k)$ is the lattice cohomology of the cubes of l'_h+L, where the weight function is generated by the restriction of χ on this shifted lattice $l'_h + L$. (Indeed, for $l \in L$, $\chi(l + l'_h) = \chi_k(l) + \mathfrak{d}_k$.)

In particular, Theorem 4.4.39 combined with (4.78) give

Theorem 4.6.31 ([73])

$$eu(\mathbb{H}^*(\Gamma, k)) = \mathfrak{sw}_{\sigma[k]}(M(\Gamma)) - \frac{k^2 + |\mathcal{V}|}{8}.$$

4.6.32 The SWIC Revisited For any $h \in H$ assume that the representative l'_h is either r_h or s_h. Then via the extension 4.4.47 of the SWIC combined with 4.6.31 from above, the SWIC(h) is equivalent to

$$(\text{SWIC}(h)) \qquad h^1(\widetilde{X}, \mathcal{O}(-l'_h)) = eu(\mathbb{H}^*(\Gamma, K + 2l'_h)). \tag{4.79}$$

We wish to emphasize that to some extent this conjectured identity lead to the definition of graded roots and lattice cohomology (at least, of $\mathbb{H}^0$), see e.g. [70]. Indeed, for several singularities with AR graphs (e.g. for the weighted homogeneous germs) the left hand side was computed by a concatenated Laufer computations sequence, and its χ-fluctuation was reformulated as the key topological object at the right hand side too (cf. 4.5.4 and 4.7.3).

4.7 Graded Roots and Their Cohomologies

We introduce abstract graded roots (R, χ) and we define their cohomology $\mathbb{Z}[U]$-module $\mathbb{H}(R, \chi)$. We provide several constructions, which provide graded roots. One of them (cf. 4.7.2) associates a graded root $(R, \chi)_{\Gamma,k}$ with a plumbing graph Γ and a characteristic element k. It turns out that $\mathbb{H}^0(\Gamma, k) = \mathbb{H}((R, \chi)_{\Gamma,k})$. In particular, for any (Γ, k), the associated graded root is a geometrical/topological enhancement of $\mathbb{H}^0(\Gamma, k)$.

4.7.1 The Definition of Graded Roots and Their Cohomologies

In this subsection we follow [70, 71].

Definition 4.7.1 Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with end-vertices u and v. We say that R is a *graded root* with grading $\chi : \mathcal{V} \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$;
(b) $\chi(u) > \min\{\chi(v), \chi(w)\}$ for any $[u, v]$, $[u, w] \in \mathcal{E}$, $v \neq w$;
(c) χ is bounded below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$, and $|\chi^{-1}(k)| = 1$ if $k \gg 0$.

An isomorphism of graded roots is a graph isomorphism, which preserves the gradings.

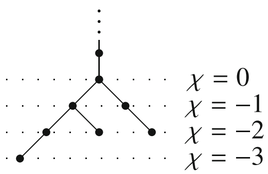
If (R, χ) is a graded root, and $r \in \mathbb{Z}$, then $(R, \chi)[r]$ denotes the same R with the new grading $\chi[r](v) := \chi(v) + r$.

Example 4.7.2

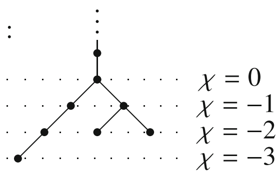
(1) For any integer $n \in \mathbb{Z}$, let $R_{(n)}$ be the tree with $\mathcal{V} = \{v^k\}_{k \geq n}$ and $\mathcal{E} = \{[v^k, v^{k+1}]\}_{k \geq n}$. The grading is $\chi(v^k) = k$.
(2) Let I be a finite index set. For each $i \in I$ fix an integer $n_i \in \mathbb{Z}$; and for each pair $i, j \in I$ fix $n_{ij} = n_{ji} \in \mathbb{Z}$ with the next properties: $n_{ii} = n_i, n_{ij} \geq \max\{n_i, n_j\}$, and $n_{jk} \leq \max\{n_{ij}, n_{ik}\}$ for any $i, j, k \in I$.
For any $i \in I$ consider $R_i := R_{(n_i)}$ with vertices $\{v_i^k\}$ and edges $\{[v_i^k, v_i^{k+1}]\}$, $(k \geq n_i)$. In the disjoint union $\sqcup_i R_i$, for any pair (i, j), identify v_i^k and v_j^k, resp. $[v_i^k, v_i^{k+1}]$ and $[v_j^k, v_j^{k+1}]$, whenever $k \geq n_{ij}$. Write $\bar{v}_i^k$ for the class of v_i^k. Then $\sqcup_i R_i/\sim$ is a graded root with $\chi(\bar{v}_i^k) = k$. It will be denoted by $R = R(\{n_i\}, \{n_{ij}\})$.
(3) Any map $\tau : \{0, 1, \ldots, T_0\} \to \mathbb{Z}$ produces a starting data for construction (2). Indeed, set $I = \{0, \ldots, T_0\}$, $n_i := \tau(i)$ $(i \in I)$, and $n_{ij} := \max\{n_k : i \leq k \leq j\}$ for $i \leq j$. Then $\sqcup_i R_i/\sim$ constructed in (2) using this data will be denoted by (R_τ, χ_τ).

For example, for $T_0 = 4$, take for the values of τ: $-3, -1, -2, 0$ and -2 (respectively $-3, 0, -2, -1$ and -2). Then the two graded roots are:

R^1 :



R^2 :



This construction can be extended to the case of a map $\tau : \mathbb{N} \to \mathbb{Z}$, whenever τ has the property that there exists some $k_0 \geq 0$ such that $\tau(k + 1) \geq \tau(k)$ for any $k \geq k_0$. In this case one can take any $T_0 \geq k_0$ and construct the root associated with the restriction of τ to $\{0, \ldots, T_0\}$. It is independent of the choice of T_0. By definition, this is the root associated with τ.

Definition 4.7.3 (The (cohomology) $\mathbb{Z}[U]$-Modules Associated with a Graded Root) For any graded root (R, χ), let $\mathbb{H}(R, \chi)$ (briefly $\mathbb{H}(R)$) be the set of functions $\phi : \mathcal{V} \to \mathcal{T}_0^+$ with the following property: whenever $[v, w] \in \mathcal{E}$ with $\chi(v) < \chi(w)$, then $U \cdot \phi(v) = \phi(w)$. Clearly $\mathbb{H}(R)$ is a $\mathbb{Z}[U]$-module via $(U\phi)(v) = U \cdot \phi(v)$. Moreover, $\mathbb{H}(R)$ has a $\mathbb{Z}$-grading: the element $\phi \in \mathbb{H}(R)$ is homogeneous of degree $d \in \mathbb{Z}$ if for each $v \in \mathcal{V}$ with $\phi(v) \neq 0$, $\phi(v) \in \mathcal{T}_0^+$ is homogeneous of degree $d - 2\chi(v)$. Since $2\chi(v) + \deg \phi(v) = 2\chi(w) + \deg \phi(w)$, d is well-defined.

Note also that any ϕ as above is automatically finitely supported.

Remark 4.7.4 By the definitions $\mathbb{H}((R, \chi)[r]) = \mathbb{H}(R, \chi)[2r]$ for any $r \in \mathbb{Z}$.

Example 4.7.5

(a) $\mathbb{H}(R_n) = \mathcal{T}_{2n}^+$.
(b) The graded roots R^1 and R^2 constructed in 4.7.2(3) are not isomorphic but their $\mathbb{Z}[U]$-modules are isomorphic. Hence, in general, a graded root carries more information than its $\mathbb{Z}[U]$-module.

One has a natural graded $\mathbb{Z}[U]$ module isomorphism $\mathbb{H}(R, \chi) = \mathcal{T}_{2m}^+ \oplus \mathbb{H}_{red}(R, \chi)$, such that the $\mathbb{Z}[U]$-module $\mathbb{H}_{red}(R)$ has finite $\mathbb{Z}$-rank.

Proposition 4.7.6 *Let (R_τ, χ_τ) be a graded root associated with some function $\tau : \mathbb{N} \to \mathbb{Z}$, cf. 4.7.2(3). Then*

$$\operatorname{rank}_{\mathbb{Z}} \mathbb{H}_{red}(R_\tau, \chi_\tau) = -\tau(0) + \min_{i \geq 0} \tau(i) + \sum_{i \geq 0} \max\{\tau(i) - \tau(i+1), 0\}.$$

The summand $\mathcal{T}_{2m}^+$ of $\mathbb{H}(R_\tau, \chi_\tau)$ has index $m = \min_{i \geq 0} \tau(i) = \min_v \chi_\tau(v)$.

4.7.2 The Graded Root Associated with a Plumbing Graph

4.7.7 The Graded Root Associated with a System of Weigh Functions Fix a free $\mathbb{Z}$-module and a system of weights $\{w_q\}_q$. Consider the sequence of topological spaces (finite cubical complexes) $\{S_n\}_{n \geq m_w}$ with $S_n \subset S_{n+1}$. Let $\pi_0(S_n) = \{C_n^1, \ldots, C_n^{p_n}\}$ be the set of connected components of S_n.

Then we define the graded graph (R_w, χ_w) as follows. The vertex set $\mathcal{V}(R_w)$ is $\sqcup_{n \in \mathbb{Z}} \pi_0(S_n)$. The grading $\chi_w : \mathcal{V}(R_w) \to \mathbb{Z}$ is $\chi_w(C_n^j) = n$, that is, $\chi_w|_{\pi_0(S_n)} = n$.

Furthermore, if $C_n^i \subset C_{n+1}^j$ for some n, i and j, then we introduce an edge $[C_n^i, C_{n+1}^j]$. All the edges of R_w are obtained in this way.

Lemma 4.7.8 *(R_w, χ_w) satisfies all the required properties of the definition of a graded root, except maybe the last one: $|\chi_w^{-1}(n)| = 1$ whenever $n \gg 0$.*

4.7.9 The Graded Roots Associated with a Plumbing Graph Fix a graph and $k \in$ Char, their compatible weight functions and the graded cubes as in 4.6.12. The graded graph associated with this system of weight functions (cf. 4.7.7 and 4.7.8) is denoted by (R_k, χ_k).

For the system of weight functions induced by χ_k the sequence of spaces $\{S_n\}_n$ have a finiteness property: only finitely many S_n are not contractible, cf. 4.5.9.

Corollary 4.7.10

(a) (R_k, χ_k) *is a graded root.*
(b) $\mathbb{H}(R_k, \chi_k)$ *is a finitely generated* $\mathbb{Z}[U]$*-module, and* $\mathbb{H}_{red}(R_k, \chi_k)$ *is a finitely generated* $\mathbb{Z}$*-module.*

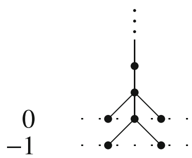
Remark 4.7.11 There are several natural **symmetries** in the picture.

(a) **The** Spin^c**-involution.** The involution $l' \mapsto -l'$ ($l' \in L'$) induces the identity $\chi_{-k}(-l) = \chi_k(l)$, hence an isomorphism of the graded roots $(R_k, \chi_k) \simeq (R_{-k}, \chi_{-k})$. ($[k] \mapsto [-k]$ is the natural involution of Spin$^c(M(\Gamma))$, cf. 4.2.93.)
(b) **The Gorenstein symmetry.** If Γ is numerically Gorenstein then χ_K is stable with respect to the transformation $L \to L$, $x \mapsto Z_K - x$. This shows that (R_K, χ_K) has a $\mathbb{Z}_2$-symmetry.

More generally, if $k \in L$ (that is, k is spin) then $x \mapsto -k - x$ induces a $\mathbb{Z}_2$-symmetry of (R_k, χ_k).

Theorem 4.7.12 *Let* (R_k, χ_k) *be the graded root associated with* Γ *and* k*. Then* $\mathbb{H}(R_k, \chi_k) = \mathbb{H}^0(\Gamma, k)$.

Example 4.7.13 Consider the example from 4.5.3. Those computations show that the graded root (R_K, χ_K) is



Then $\mathbb{H}^0\Gamma, K) = \mathcal{T}^+_{-2} \oplus \mathcal{T}_{-2}(1) \oplus \mathcal{T}_0(1) \oplus \mathcal{T}_0(1)$, $\mathbb{H}^1(\Gamma, K) = \mathcal{T}_0(1)$ and $\mathbb{H}^q(\Gamma, K) = 0$ for $q \geq 2$.

4.7.14 Next, with the notations from 4.6.16, we have the analogues of 4.6.17, 4.77, 4.6.19:

Proposition 4.7.15

(a) If $k' = k + 2l$ *for some* $l \in L$*, then:* $(R_{k'}, \chi_{k'}) = (R_k, \chi_k)[-2\chi_k(l)]$.
(b) $(R_k, \chi_k) = (R_{[k]}, \chi_{[k]})[2m_k]$
(c) The set $(R_{[k]}, \chi_{[k]})$*, indexed by* $[k] \in$ Spin$^c(M(\Gamma))$*, depends only on* $M = M(\Gamma)$ *and is independent of the choice of the plumbing graph* Γ *which provides* M.

Example 4.7.16 (Rational Graphs) The following facts are equivalent:

(a) Γ is rational;
(b) $R_K = R_{(0)}$;

(c) $R_K = R_{(m)}$ for some $m \in \mathbb{Z}$;
(d) For all characteristic elements $k \in \text{Char}$, $R_k = R_{(m_k)}$ for some $m_k \in \mathbb{Z}$;

Recall from 4.6.21 that $\min \chi_{k_r} = 0$ for rational Γ. In particular, if Γ

Example 4.7.17 (Elliptic Graphs) Γ is elliptic; if and only if $(R_K, \chi_K) = R(\{n_i\}, \{n_{ij}\})$ for some index set I, $|I| = \ell + 1 \geq 2$, such that $n_i = 0$ for any $i \in I$ and $n_{ij} = 1$ for any pair $i \neq j$.

4.7.18 The following tasks appear very naturally.

Problem *Determine all the possible canonical* (R_K, χ_K) *(and non-canonical* (R_k, χ_k) *) graded roots.*

The possible resolution graphs are characterized by Grauert Theorem, namely they are connected and negative definite. For each negative definite graph (tree) we construct a canonical graded root in a direct combinatorial way. The problem is to find a combinatorial characterization of all of them.

Problem *Determine all the possible graded* $\mathbb{Z}[U]$*-modules, which might appear as* $\mathbb{H}^*(\Gamma, k)$ *for some* (Γ, k).

4.7.3 Graded Roots of Almost Rational Graphs

4.7.19 Assume that Γ is an AR graph, with SR-set $\{v_0\}$. We fix a distinguished characteristic element $k_r = K + 2s_h$ and we consider the universal cycles $\{x(\ell)\}_{\ell \geq 0}$ associated with (Γ, k_r), and their τ-function $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ defined as $\tau(\ell) := \chi_{k_r}(x(\ell))$, cf. 4.5.4. Associated with this τ-function we consider its graded root (R_τ, χ_τ) as well, cf. 4.7.2(3).

Theorem 4.7.20 *Assume that* Γ *is AR, and set* $k_r = K + 2s_h$ *for some* $h \in H$. *Then*

(a) $\mathbb{H}^q(\Gamma, k_r) = 0$ *for* $q \geq 1$*;*
(b) $\mathbb{H}^0(\Gamma, k_r) = \mathbb{H}(R_{k_r}, \chi_{k_r})$*;*
(c) $(R_{k_r}, \chi_{k_r}) = (R_\tau, \chi_\tau)$*;*
(d) $x(0) = 0$, $\tau(0) = 0$, $\tau(1) = 1 - (s_h, E_{v_0}) \geq 1$, $m_{k_r} = \min_{\ell \geq 0}\{\tau(\ell)\}$ *and*

$$eu(\mathbb{H}^*(\Gamma, k_r)) = -\min_\ell\{\tau(\ell)\} + \text{rank}_{\mathbb{Z}}(\mathbb{H}^0_{red}(\Gamma, k_r)) = \sum_{\ell \geq 0} \max\{\, \tau(\ell) - \tau(\ell+1), 0\,\}.$$

(e) $\tau(\ell) - \tau(\ell+1) = -1 + (x(\ell) + s_h, E_{v_0})$.

Remark 4.7.21

(a) The above theorem shows that for almost rational graphs, any graded tree (R_k, χ_k) is completely determined by the values of χ_k along a very natural (universal) infinite computation sequence (depending on k), which contains the elements $\{x(\ell)\}_{\ell\geq 0}$. (For the construction of the sequence see 4.5.4.) In particular, all the important vertices of R_k can be represented by some special cycles in L, which can be arranged in an increasing linear order (with respect to $\leq$).

(b) The set $\{x(\ell)\}_\ell$ usually is not very economical: only some of the $x(\ell)$'s carry substantial information, which will survive in (R_τ, χ_τ). The others are intermediate steps in some monotone paths. E.g., for rational singularities, $\chi(x(\ell+1)) \geq \chi(x(\ell))$, hence only the information $\chi(x(0)) = 0$ is preserved in R_τ.

Example 4.7.22 (Star-Shaped Graphs) Assume that Γ is star-shaped with ν strings. In the sequel we will use the notations from 4.2.3. We also fix $l'_h = a_0E_0^* + \sum_{j=1}^{\nu}\sum_{t=1}^{s_j} a_{jt}E^*_{jt}$. The coefficients of l'_h also determine the integers $\tilde{a}_{jk} := \sum_{t\geq k} n^j_{t+1,s_j}a_{jt}$ for $1\leq k\leq s_j$. We also write $a_j = \tilde{a}_{j1}$.

Γ is AR, where its SR-set consists of the central vertex, cf. 4.5.15(f). Hence, for any $\bar{l} = \ell E_0$ (and for the fixed l'_h and $k := K + 2l'_h$) we have a cycle $x(\bar{l})$, which will be denoted simply by $x(\ell)$ ($\ell \in \mathbb{Z}$). The next expression describes the cycles $x(\ell)$ in terms of the Seifert invariants and the coefficients of l'_h.

Define the integers $\{v_{jk}\}$ $(1\leq j\leq \nu,\ 1\leq k\leq s_j)$ inductively by

$$v_{j1} := \left\lceil \frac{\ell\omega_j - a_j}{\alpha_j}\right\rceil = \left\lceil \frac{\ell n^j_{2s_j} - \tilde{a}_{j1}}{n^j_{1s_j}}\right\rceil; \quad v_{jk} := \left\lceil \frac{v_{j,k-1}n^j_{k+1,s_j} - \tilde{a}_{jk}}{n^j_{ks_j}}\right\rceil \ (1<k\leq s_j).$$

Then $x(\ell) = \ell E_0 + \sum_{j,k} v_{jk}E_{jk}$.

Assume next that $g=0$ and $l'_h = s_h$, and set $\tau(\ell) := \chi_{k_r}(x(\ell))$ ($\ell\geq 0$). If $\ell = 0$ then $x(0) = 0$, hence $\tau(0) = 0$ too. For $\ell\geq 0$ from 4.5.22 one gets

$$\tau(\ell+1)-\tau(\ell) = 1-(x(\ell)+s_h, E_0) = 1+a_0+\ell b_0 - \sum_j \left\lceil\frac{\ell\omega_j - a_j}{\alpha_j}\right\rceil. \tag{4.80}$$

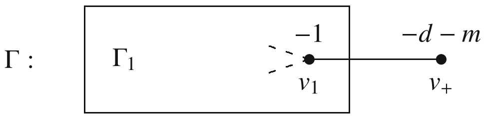
In particular,

$$\tau(\ell) = \sum_{k=0}^{\ell-1}\Big(1 + a_0 + kb_0 - \sum_j \left\lceil \frac{k\omega_j - a_j}{\alpha_j}\right\rceil\Big). \tag{4.81}$$

This can be compared with several similar expressions based on different other studies of weighted homogeneous germs or Seifert 3–manifolds.

4.7.4 Example: The Surgery Manifold $S^3_{-d}(K)$ [69, 71]

4.7.23 The Surgery Manifold $M(\Gamma) = S^3_{-d}(K)$ Fix $d\in\mathbb{Z}_{>0}$ and an *irreducible* plane curve singularity (C, o) with local algebraic knot $(K_1 \subset S^3)$. Several invariants of (C, o) are listed in 4.2.30. For the shape and structure of the surgery 3-manifold $S^3_{-d}(K_1)$ see 4.2.32. If it appears as the link of a superisolated surface singularity associated with an irreducible rational unicuspidal curve (C, o) (cf. 4.2.31) then necessarily $(d-1)(d-2) = \mu(C, o)$. However, in the discussion below we will not assume this additional restriction (in particular, d can be any $d \in \mathbb{Z}_{>0}$). We use the following schematic diagram for Γ:



The basis elements in $L = L(\Gamma)$ corresponding to v_1 and v_+ are denoted by E_1 and E_+. The lattice associated with Γ_1 is L_1, its dual is L'_1. The elements $\{E_v\}_{v \neq v_+}$ of L are identified with the basis elements of L_1.

Recall that Γ is an AR graph with $\overline{\mathcal{V}} = \{v_1\}$, cf. 4.5.15(f). In the sequel we follow [69, 71, 84].

Assume that (C, o) is determined by the function f; denote by Z that part of its divisor which is supported on compact curves. Set m for the E_1-multiplicity of Z. Then, $Z = E^*_1(\Gamma_1)$, hence $-(Z, Z)_{L_1} = m$. This combined with a determinant computation gives $\det(\Gamma) = d$. Since $\det(\Gamma_1) = 1$ the coefficient of E_+ in E^*_+ is $1/d$. Hence $[E^*_+]$ has order d in H, and $H = \mathbb{Z}_d$. We abridge $s_{a[E^*_+]}$ by s_a for $a = 0, 1, \dots, d-1$.

Lemma 4.7.24 *$s_a = aE^*_+$ for any $a = 0, 1, \dots, d-1$.*

4.7.25 Our goal is to determine $\{x_{k_r}(\ell)\}_{\ell \geq 0}$ for Γ and for any spin^c structure. If $k_r = K + 2aE^*_+$ for a certain a then we abridge $x_{k_r}(\ell)$ as $x_a(\ell)$, where $0 \leq a < d$.

Let us write $x_a(\ell)$ as $y_a(\ell) + n_a E_+$, where $n_a \in \mathbb{Z}_{\geq 0}$ and $y_a(\ell) \in L_1$. The inequality $(x_a(\ell) + aE^*_+, E_+) \leq 0$ reads as $n_a(m+d) \geq \ell - a$. Hence $n_a = \lceil (l-a)/(m+d) \rceil$.

On the other hand, for all other vertices $v \in \mathcal{V} \setminus \{v_+, v_1\}$ we have $(x_a(\ell) + aE^*_+, E_v) = (y_a(\ell), E_v)$, hence $y_a(\ell)$ is independent of a; let us denote it by $y(\ell)$. It satisfies the universal property *(a)-(b)-(c)* from 4.5.18 for the graph Γ_1, vertex v_1 and $l'_h = 0$. Namely, $y(\ell)$ is minimal with *(a)* $m_{v_1}(y(\ell)) = \ell$ and *(b)* $(y(\ell), E_v) \leq 0$ for any $v \neq v_1$. For example, $y(0) = 0$.

Proposition 4.7.26 *Let $Z = \mathrm{div}_{E(\Gamma_1)}(f) = E^*_1(\Gamma_1)$ be the cycle as above. Then*

(a) if $\ell = tm + \ell_0$ with $t \geq 0$ and $0 \leq \ell_0 < m$, then $y(\ell) = tZ + y(\ell_0)$;
(b) for any $\ell < m$ one has

$$(y(\ell), E_1) = \begin{cases} 1 & \textit{if} \quad \ell \notin \mathcal{S}_{C,o}; \\ 0 & \textit{if} \quad \ell \in \mathcal{S}_{C,o}. \end{cases}$$

Corollary 4.7.27 *Fix* $0 \leq a < d$ *and write* $\ell = tm + \ell_0$ *for some* $t \geq 0$ *and* $0 \leq \ell_0 < m$. *Then*

$$x_a(\ell) = t \cdot Z + y(\ell_0) + \left\lceil \frac{\ell - a}{m + d} \right\rceil E_+.$$

In particular,

$$(x_a(\ell), E_1) = -t + \left\lceil \frac{\ell - a}{m + d} \right\rceil + (y(\ell_0), E_1).$$

Furthermore, $\chi_{k_r}(x_a(0)) = 0$ *and for any* $\ell \geq 0$ *one has*

$$\chi_{k_r}(\,x_a(\ell+1)\,) - \chi_{k_r}(\,x_a(\ell)\,) = t+1-\left\lceil \frac{\ell - a}{m + d} \right\rceil - \begin{cases} 1 & \text{if} \quad \ell_0 \notin \mathcal{S}_{C,o} \\ 0 & \text{if} \quad \ell_0 \in \mathcal{S}_{C,o}. \end{cases} \tag{4.82}$$

4.7.28 The τ-Function τ_a According to 4.5.4 we set $\tau_a(\ell) := \chi_{k_r}(x_a(\ell))$. Then in (4.82) one has

$$\frac{\ell - a}{m + d} \leq t + 1,$$

hence $\tau_a(\ell + 1) - \tau_a(\ell) \geq -1$ for any ℓ, and $= -1$ only if

$$\frac{tm + \ell_0 - a}{m + d} > t \quad \text{and} \quad \ell_0 \notin \mathcal{S}_{C,o}. \tag{4.83}$$

In order to analyze the cases when this holds, we will consider sequences $Seq(t) := \{tm + \ell_0 : 0 \leq \ell_0 < m\}$ for fixed $t \geq 0$. In such a sequence, notice that the very last element of $\mathbb{N} \setminus \mathcal{S}_{C,o}$, namely $\mu - 1 = 2\delta - 1$, is strictly smaller than $m - 1$, hence the complete set $\mathbb{N} \setminus \mathcal{S}_{C,o}$ sits in $\{0, \ldots, m - 1\}$. Therefore, in $Seq(t)$ there exists an ℓ_0 satisfying (4.83) if and only if

$$\frac{tm + 2\delta - 1 - a}{m + d} > t.$$

This is equivalent to $t \leq t_a$, for $t_a := \lfloor (2\delta - 2 - a)/d \rfloor$. In other words, if $\ell \geq T_0 := (t_a + 1)m$, then $\tau_a(\ell + 1) \geq \tau_a(\ell)$, hence those values of τ_a provide no contribution in the graded root. Moreover, for $t \in \{0, \ldots, t_a\}$, in $Seq(t)$ one has:

$$\Delta(\ell_0) := \tau_a(tm + \ell_0 + 1) - \tau_a(tm + \ell_0) = \begin{cases} 0 & \text{if } \ell_0 \leq td + a, \quad \text{and } \ell_0 \notin \mathcal{S}_{C,o}; \\ +1 & \text{if } \ell_0 \leq td + a, \quad \text{and } \ell_0 \in \mathcal{S}_{C,o}; \\ -1 & \text{if } \ell_0 > td + a, \quad \text{and } \ell_0 \notin \mathcal{S}_{C,o}; \\ 0 & \text{if } \ell_0 > td + a, \quad \text{and } \ell_0 \in \mathcal{S}_{C,o}. \end{cases}$$

In particular, $\Delta(\ell_0) \geq 0$ for any ℓ_0 with $0 \leq \ell_0 \leq td + a$, and $\Delta(\ell_0) \geq 0$ takes the value $+1$ exactly

$$A_t := \#\{s \in \mathcal{S}_{C,o} : s \leq td + a\}$$

times, otherwise it is zero. Furthermore, $\Delta(\ell_0) \leq 0$ for any $\ell_0 > td + a$ and it takes value -1 exactly

$$B_t := \#\{s \notin \mathcal{S}_{C,o} : s > td + a\}$$

times, otherwise it is zero. Recall that in 4.2.30 we rewrote $\Delta(t)$ as $1 + \delta(t-1) + (t-1)^2 Q(t)$, where $Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i$. The above B_t compared with (4.7) reads as $B_t = \alpha_{td+a}$.

Notice that both A_t and B_t are strictly positive (since $0 \in \mathcal{S}_{C,o}$, respectively $2\delta - 1 \notin \mathcal{S}_{C,o}$ and $2\delta - 1 > td + a$). This shows that

$$M_t := \max_{0 \leq \ell_0 < m} \tau_a(tm + \ell_0) = \tau_a(tm) + A_t = \tau_a((t+1)m) + B_t \tag{4.84}$$

and

$$M_t > \max\{\, \tau_a(tm), \tau_a(tm+m)\,\}.$$

Therefore, the graded root associated with the values $\{\tau_a(\ell)\}_{0 \leq \ell \leq (t_a+1)m}$ is the same as the graded root associated with the values

$$\tau_a(0), M_0, \tau_a(m), M_1, \tau_a(2m), M_2, \ldots, \tau_a(t_a m), M_{t_a}, \tau_a(t_a m + m).$$

Finally, since $\#\{s \notin \mathcal{S}_{C,o}\} = \delta$, one has $\delta - B_t = \#\{s \notin \mathcal{S}_{C,o} : s \leq td + a\}$, hence $\delta - B_t + A_t = td + a + 1$. Thus, by (4.84),

$$\tau_a((t+1)m) - \tau_a(tm) = td + a + 1 - \delta.$$

Since $\tau_a(0) = 0$, this gives $\tau_a(tm)$ inductively.

Clearly, the graded root associated with τ_a is the same as the graded root associated with $\tilde{\tau}_a : \{0, 1, 2, \ldots, 2t_a + 2\} \to \mathbb{Z}$, where $\tilde{\tau}_a(2t) := \tau_a(tm)$ and $\tilde{\tau}_a(2t+1) := M_t$.

The above discussion gives the following statement.

Theorem 4.7.29 *For each fixed $a = 0, 1, \dots, d-1$,—corresponding to the d different spin^c-structures of M—one defines the following objects:*

- $t_a := \left\lfloor \frac{2\delta-2-a}{d} \right\rfloor$, *($t_a \geq -1$ automatically);*
- *a function $\tau_a : \{0, 1, \dots, 2t_a+2\} \to \mathbb{Z}$ by*

$$\begin{cases} \tau_a(2t) = d \cdot \frac{t(t-1)}{2} - t(\delta-1-a), & (t = 0, \dots, t_a+1); \\ \tau_a(2t+1) = \tau_a(2t+2) + \alpha_{td+a}, & (t = 0, \dots, t_a). \end{cases}$$

- *and the graded root $(R_{\tau_a}, \chi_{\tau_a})$ associated with τ_a.*

Then $(R_{\tau_a}, \chi_{\tau_a})$ is the graded root of M associated with (Γ, k_r).
Note also that $\min \tau_a = \tau_a(2\lceil t_a/2 \rceil)$.

Remark 4.7.30

(a) Since for any $t \in \{0, \dots, t_a\}$, $\tau_a(2t+1) > \max\{\tau_a(2t), \tau_a(2t+2)\}$, the above representation of the graded root is the most 'economical': all the values are essential. This also shows that $(R_{\tau_a}, \chi_{\tau_a})$ has exactly t_a+2 local minimum points, and they correspond to the values $\tau_a(2t)$, $t = 0, 1, \dots, t_a+1$.
(b) The values $\tau_a(2t)$, $t = 0, 1, \dots, t_a+1$ depend only on t, d and δ, that is, for these values no other information is needed from the semigroup $\mathcal{S}_{C,o}$.

Corollary 4.7.31

(a) $eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{t=0}^{t_a} \alpha_{td+a}$
(b) $\mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) = \sum_{t=0}^{t_a} \alpha_{td+a} + \frac{1}{8}(1 - \frac{(d+2\delta-2-2a)^2}{d})$.

Proof Use 4.7.6 for (a) and 4.6.31 and the identity $k_r^2 + |\mathcal{V}| = 1 - (d+2\delta-2-2a)^2/d$ for (b). □

Example 4.7.32 Assume $d = 1$. In this case M is an integral homology sphere; $a = 0$ and $t_0 = 2\delta - 2 = \mu - 2$. Moreover, $-(K^2 + |\mathcal{V}|)/4 = \delta(\delta-1)$ and $\tau_0(2t) = t(t-2\delta+1)/2$. The reader is invited to draw the graded root and verify that

$$\mathbb{H}^0(\Gamma, K) = \big(\mathcal{T}_0^+ \oplus \mathcal{T}_0(\alpha_{\delta-1}) \oplus \bigoplus_{i=1}^{\delta-1} \mathcal{T}_{i(i+1)}(\alpha_{i-1+\delta})^{\oplus 2}\big)[-\delta(\delta-1)].$$

4.7.5 Superisolated Singularities with One Cusp

4.7.33 In the sequel we will consider a superisolated singularity as in 4.2.31. For different invariants see 4.2.4, whose notations we will adopt. We will assume that C is a rational unicuspidal curve. We invite the reader to review the 'Semigroup

Distribution Inequality' from 4.2.33 and the 'Semigroup Distribution Property' from 4.2.33. The reinterpretations in terms of reduced Poincaré series can be found in 4.3.6, and the connection with the Seiberg–Witten Invariant Conjecture (as the basic motivation and source of the Semigroup Distribution Property) is presented in 4.4.11. Here we present further connections with the graded roots. We follow [25].

4.7.34 In this part we will compare the invariants of the link $M = S^3_{-d}(K)$ of the superisolated singularity with the corresponding invariants of the Seifert 3-manifold $\Sigma(d, d, d+1)$, the link of the hypersurface Brieskorn singularity $x^d+y^d+z^{d+1} = 0$. Before we state the next theorem, we recall that the plumbing graph of $S^3_{-d}(K)$ contains complete information about the embedded link $K \subset S^3$. Moreover, by the statements of 4.7.29, the graded root or lattice cohomology still preserves essential data about the Alexander polynomial. However, the Seifert 3-manifold $\Sigma(d, d, d+1)$ has information only about the degree μ of Δ via $(d-1)(d-2) = \mu$. The point is that the algebraic realizability of C (that is, the existence of an analytic superisolated singularity with link $S^3_{-d}(K)$) imposes the following very surprising necessary topological obstructions.

Theorem 4.7.35 ([25]) *The following facts are equivalent:*

(a) The Seiberg–Witten Invariant Conjecture is true for the superisolated germ.
(b) The Semigroup Distribution Property is true.
(c) The canonical graded roots of $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ are the same.
(d) The canonical lattice homologies of $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ are the same.
(e)

$$\Big(\mathfrak{sw}_{\sigma[K]}(M) - \frac{K^2+\#\mathcal{V}}{8}\Big)\Big|_{M=S^3_{-d}(K)} = \Big(\mathfrak{sw}_{\sigma[K]}(M) - \frac{K^2+\#\mathcal{V}}{8}\Big)\Big|_{M=\Sigma(d,d,d+1)}.$$

Recall that, in fact, the Semigroup Distribution Property *is true* by Borodzik and Livingston [9] (cf. 4.2.33), hence all the statements of 4.7.35 *are true* as well. However, we formulated above a weaker statement, only the *equivalence* of the above statements, whose proof is independent of the Heegaard Floer theory based proof of [9].

The proof of 4.7.35 is given in several steps. The starting point is that both 3-manifolds $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ are almost rational. In particular, in both cases, the canonical graded root can be determined via the τ-function, cf. 4.7.3. In the first case this is done explicitly in 4.7.29, while for the second case see 4.7.22.

Fact 1 Let us rewrite 4.7.29 for $S^3_{-d}(K)$ and for the canonical spin^c structure $a = 0$. Set $c_l := \alpha_{(d-3-l)d}$ and define $\tau : \{0, 1, \dots, 2d-4\} \to \mathbb{Z}$ by

$$\tau(2l) = \frac{l(l-1)}{2}d - l(\delta - 1), \quad \tau(2l+1) = \tau(2l+2) + c_{d-3-l}. \tag{4.85}$$

Then $(R_{can}, \chi_{can}) = (R_\tau, \chi_\tau)$.

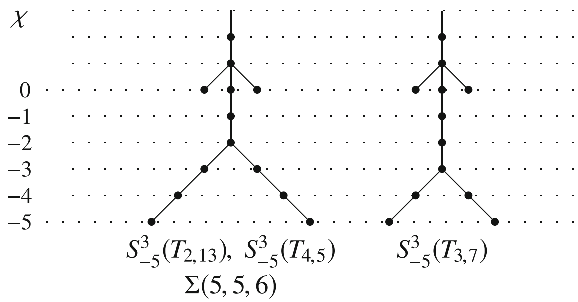
Fact 2 Consider next the Seifert manifold $\Sigma(d, d, d+1)$. Its canonical graded root is the following. For any $0 \leq l \leq d-3$ write $c_l^u := (l+1)(l+2)/2$, and $2\delta := (d-1)(d-2)$ and define $\tau^u : \{0, 1, \ldots, 2d-4\} \to \mathbb{Z}$ by

$$\tau^u(2l) = \frac{l(l-1)}{2}d - l(\delta - 1), \quad \tau^u(2l+1) = \tau^u(2l+2) + c^u_{d-3-l}. \tag{4.86}$$

Then $(R_{can}, \chi_{can}) = (R_{\tau^u}, \chi_{\tau^u})$.

Next we compare 4.85 and 4.86: the graded roots associated with $S^3_{-d}(K)$ and $\Sigma(d, d, d+1)$ coincide exactly when $c_l = c_l^u$ for every l. However, by the Semigroup Distribution Inequality (a consequence of the Bézout's Theorem, cf. 4.2.33) $c_l \geq c_l^u$ for every l. Hence $c_l = c_l^u$ for every l if and only if $\sum_l c_l = \sum_l c_l^u$. But this is exactly the vanishing of $N(1)$, cf. (4.3.20)*(b)*, hence 4.4.54 applies.

Example 4.7.36 Assume that $d = 5$ and C is unicuspidal and its singular point has only one Puiseux pair (a, b) with $a < b$. Then by the genus formula the possible values of (a, b) are (4, 5), (3, 7) and (2, 13). It turns out that the first and the third cases can be realized, while the second case not. This fact is compatible with the above Theorem 4.7.35. Indeed, the corresponding canonical graded roots (together with the root of $\Sigma(5, 5, 6)$) are shown in the next picture.



Remark 4.7.37 As we already mentioned in 4.2.33, the Semigroup Distribution Property (in the unicuspidal case) was partially verified in [24] and proved in [9]. The first approach is based on a case-by-case verification of the families of cuspidal rational projective curves which appear in the classification theorems. The second approach is based on the Heegaard Floer theory. The discussion from 4.7.39 traces a possible third approach, which would lead to a different proof, and would open a new chapter in the deformation theory of surface singularities.

Corollary 4.7.38 *The Seiberg–Witten Invariant Conjecture is true for superisolated germs associated with rational unicuspidal curves.*

4.7.39 Why $\Sigma(d, d, d+1)$? At the first glance the pairing of $S^3_{-d}(K)$ with $\Sigma(d, d, d+1)$ in Theorem 4.7.35 looks very unmotivated. In the next paragraphs we wish to convince the reader that this is not the case, and conjecturally a very deep structure might exist behind the scene.

Assume that the rational unicuspidal curve is given by $f_d(x, y, z) = 0$ in $\mathbb{P}^2$ (for notations see 4.2.31). We can fix the homogeneous coordinates in $\mathbb{P}^2$ in such a way that $z = 0$ intersects C generically. A possible choice for the superisolated singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is $f = f_d + z^{d+1}$. Write f_d as $\sum_{i=0}^{d} g_{d-i}(x, y)z^i$. Then g_d is a product of d linear factors corresponding to the points $C \cap \{z = 0\}$, hence the germ $g_d : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is equisingular with $(x, y) \mapsto x^d + y^d$.

Next, consider the following deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ of isolated hypersurface germs, given by $f_t(x, y, z) = f_d(x, y, tz) + z^{d+1} = \sum_i g_{d-i}(x, y)z^i t^i + z^{d+1}$. For $t \neq 0$ the deformation is μ-constant, the embedded topological type stays constant, and it is equivalent (up to such equivalences) to the type of f. However, for $t = 0$ it is equivalent (in similar sense) to the germ $x^d + y^d + z^{d+1}$.

Along this deformation not only does the embedded topological type jump (e.g. the Milnor number), but even the (non-embedded abstract) link as well: for $t \neq 0$ it is $S^3_{-d}(K)$, while for $t = 0$ it is $\Sigma(d, d, d + 1)$.

However, both graphs are AR and several key invariants stay stable. For example, in both cases $p_g = d(d - 1)(d - 2)/6$. On the other hand, if we compute the (resolution independent) invariant $K^2 + |\mathcal{V}|$ we realize that they are different. However, if we denote by K^2_{min} the self-intersection of K in the *minimal resolution*, then it turns out that in both cases it is $-d(d - 2)^2$. Hence we are dealing with a Gorenstein K^2_{min}-constant deformation. By a result of Laufer [52] such deformations admit a *very weak simultaneous resolution* (possible after a finite base change). This gives the possibility to compare the lattices associated with their minimal resolutions. Indeed, $S^3_{-d}(K)$ and $\Sigma(d, d, d + 1)$ admit certain non-minimal resolution graphs with lattices $L_{t\neq 0}$ and $L_{t=0}$ and a homological map $\iota : L_{t\neq 0} \to L_{t=0}$, which preserves the intersection matrices, the canonical classes, the χ-expression.

We formulate the next conjecture, whose positive answer would produce an extremely strong test for the existence of certain analytic deformations.

Conjecture 4.7.40 Along a K^2_{min}-constant deformation X_t of Gorenstein surface singularities, such that the links of $X_{t=0}$ and $X_{t\neq 0}$ are both rational homology spheres, the graded roots associated with the canonical spin^c structure of $X_{t=0}$ and of $X_{t\neq 0}$ are the same.

Note that along a deformation as in 4.7.40 we *cannot* expect the stability of the whole module $\{\mathbb{H}^q\}_q$. Indeed, for the deformation described in 4.7.39 valid for superisolated germs, for $t = 0$ we have an AR case with $\mathbb{H}^{\geq 1} = 0$. However, for $t \neq 0$, for certain superisolated germs with $\nu \geq 2$ we might have $\mathbb{H}^{\geq 1} \neq 0$. In fact, for any superisolated germ which produced a counterexample for the SWIC, along the above deformation the canonical Seiberg–Witten invariant is non-constant too.

4.8 The Reduction Theorem

4.8.1 Reduction Theorem for Lattice Cohomology

We consider a graph Γ as in 4.6.2. We also fix a distinguished class $k_r \in \mathrm{Char}$ and the corresponding lattice cohomology $\mathbb{H}^*(\Gamma, k_r)$. Recall that there is an isomorphism of graded $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\Gamma, k_r) \simeq \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$, where the second module is generated by weighted cubes in $(\mathbb{R}_{\geq 0})^s$, cf. 4.6.20. Here $s := |\mathcal{V}|$.

This $\mathbb{Z}[U]$-module was drastically simplified in the case of AR graphs, basically the cubes from $(\mathbb{R}_{\geq 0})^s$ were replaced by 0 and 1 dimensional cubes along an infinite increasing path (starting with $0 \in L$), cf. Theorem 4.7.20. Here the AR-assumption is really necessary: such a reduction to a 1-dimensional path (simplicial complex) cannot be done for any graph (e.g. when $\mathbb{H}^1 \neq 0$). In this subsection we discuss the analogue of this statement for an arbitrary graph.

Recall that the definition of an SR-set does not involve any $k \in \mathrm{Char}$, hence such a set can be uniformly used for any k_r. In this section we fix such an SR-set $\overline{\mathcal{V}} \subset \mathcal{V}$ as in 4.5.14, and any $k_r \in \mathrm{Char}$. Then, for each $\bar{l} = \sum_{v \in \overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}})$, with every $\ell_v \geq 0$, we define the universal cycle $x(\bar{l})$ associated with $\bar{l}$ and s_h (where $k_r = K + 2s_h$) as in 4.5.18. For several properties of the cycles $x(\bar{l})$ and of the values $\chi_{k_r}(x(\bar{l}))$ see 4.5.16. Let $\bar{s}$ be the cardinality of $\overline{\mathcal{V}}$. In the next paragraphs we follow [47].

4.8.1 Preparation for the Lattice Reduction Our goal is to replace the cubes of the lattice $\mathbb{R}^s$ (or from $(\mathbb{R}_{\geq 0})^s$) with cubes from $(\mathbb{R}_{\geq 0})^{\bar{s}}$. In order to run the theory we need to define the new weights. Define the function $\overline{w}_0 : (\mathbb{Z}_{\geq 0})^{\bar{s}} \to \mathbb{Z}$ by

$$\overline{w}_0(\bar{l}) := \chi_{k_r}(x(\bar{l})). \tag{4.87}$$

Then $\overline{w}_0$ defines a set $\{\overline{w}_q\}_{q=0}^{\bar{s}}$ of compatible weight functions by $\overline{w}_q(\square) = \max\{\overline{w}_0(v) \ : \ v \text{ is a vertex of } \square\}$, similarly as in 4.6.12. This system is denoted by $\overline{w}[k_r]$.

Here some comments are appropriate. We wish to emphasize that in the definition of the lattice cohomology the *lattice* (that is, the *linear*) *structure* in not used, it is not essential. The important structure consists of the weight-levels of the *lattice points* in some regions (e.g. quadrants, rectangles) and their neighboring properties. Note that in the new situation we do not use the linear structure of $\mathbb{Z}^{\bar{s}}$ either, and we do not even define the weights of the lattice points outside the first quadrant. Furthermore, $\bar{l} \mapsto \chi_{k_r}(x(\bar{l}))$ is a complicated arithmetical function (definitely not quadratic or polynomial).

Let us denote the associated lattice cohomology by $\mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r])$.

Theorem 4.8.2 (Reduction Theorem [47]) *There exists a graded* $\mathbb{Z}[U]$*-module isomorphism*

$$\mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w}[k_r]). \tag{4.88}$$

Corollary 4.8.3 *Fix an arbitrary graph* Γ. *If it admits an SR-set of cardinality* $\bar{s}$ *then* $\mathbb{H}^q(\Gamma, k) = 0$ *for any* $q \geq \bar{s}$ *and* $k \in \mathrm{Char}$.

This vanishing can be proved by surgery exact sequences of lattice cohomology as well, see [74].

4.8.2 Reduction Theorem for $\mathbf{Z(t)}$

The Reduction Theorem has its effect on the relation of the lattice cohomology with the counting function of the coefficients of topological Poincaré series $Z(\mathbf{t})$ as well. Let us consider first the series $Z(\mathbf{t})$ written in terms of weighted cubes (cf. 4.4.33 and 4.4.40).

Theorem 4.8.4 *Fix* h, s_h *and* $k_r = K + 2s_h$ *as above. Let* $w = w[k_r]$ *be the system of weight associated with* k_r. *Then the following facts hold.*

(1)

$$Z_h(\mathbf{t}) = \sum_{l\in L} \Big(\sum_{I\subseteq \mathcal{V}} (-1)^{|I|+1} w((l, I))\Big)\, \mathbf{t}^{l+s_h}.$$

(2) Fix some $l \in L$ *with* $l + s_h \in -K + \mathcal{S}'$. *Then*

$$\sum_{x\in L,\, x\ngeq l} \mathfrak{z}(x + s_h) = \chi_{k_r}(l) + eu(\mathbb{H}^*(\Gamma, k_r)).$$

4.8.5 The Reduced Series Let us return to the SR-set $\overline{\mathcal{V}}$, write $\mathcal{V}$ as $\overline{\mathcal{V}} \sqcup \mathcal{V}^*$, and let $\pi : L' \to L(\overline{\mathcal{V}}) \otimes \mathbb{Q}$ be the projection to the $\overline{\mathcal{V}}$-coordinates. As usual, we also write $\mathbf{t}_{\overline{\mathcal{V}}} = \{t_v\}_{v\in\overline{\mathcal{V}}}$ for the variables of $L(\overline{\mathcal{V}})$, and $\mathbf{t}^{\bar{l}}_{\overline{\mathcal{V}}} = \prod_{v\in\overline{\mathcal{V}}} t_v^{\ell_v}$ for $\bar{l} = \sum_{v\in\overline{\mathcal{V}}} \ell_v E_v \in L(\overline{\mathcal{V}}) \otimes \mathbb{Q}$. For any $h \in H$ set $Z_{h,\overline{\mathcal{V}}}(\mathbf{t}_{\overline{\mathcal{V}}}) = Z_h(\mathbf{t})|_{t_v=1 \text{ for all } v\in\mathcal{V}^*}$. It is supported on the projection of $\mathcal{S}' \cap (s_h + L)$. Write

$$Z_{h,\overline{\mathcal{V}}}(\mathbf{t}_{\overline{\mathcal{V}}}) = \sum_{\bar{l}\in L(\overline{\mathcal{V}})} \bar{\mathfrak{z}}_{\bar{l}+\pi(s_h)} \mathbf{t}^{\bar{l}+\pi(s_h)}_{\overline{\mathcal{V}}}.$$

Theorem 4.8.6 ([47]) *With the above notations (and* $\overline{w} = \overline{w}[k_r]$*)*

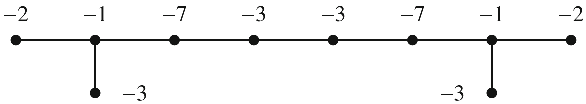
(1)

$$Z_{h,\overline{\mathcal{V}}}(\mathbf{t}_{\overline{\mathcal{V}}}) = \sum_{\bar{l}\in L(\overline{\mathcal{V}})} \Big(\sum_{\bar{I}\subseteq\overline{\mathcal{V}}} (-1)^{|\bar{I}|+1} \overline{w}((\bar{l}, \bar{I}))\Big) \mathbf{t}^{\bar{l}+\pi(s_h)}_{\overline{\mathcal{V}}}.$$

(2) There exists $\bar{l}_0 \in \pi(\mathcal{S})$ such that for any $\bar{l} \in \bar{l}_0 + \pi(\mathcal{S})$

$$\sum_{\bar{x} \ngeq \bar{l}} \bar{\mathfrak{z}}_{\bar{x}+\pi(s_h)} = \overline{w}(\bar{l}) + eu(\mathbb{H}^*((\mathbb{R}_{\geq 0})^{\bar{s}}, \overline{w})).$$

Example 4.8.7 Consider the following graph Γ



It is the minimal good resolution graph of the hypersurface singularity $x^{13} + y^{13} + x^2y^2 + z^3 = 0$. In particular, Z_K is integral.

In the sequel we will calculate the lattice cohomology of $M(\Gamma)$ associated with $k_r = K$. We choose the two nodes as an SR-set. Then Reduction Theorem 4.8.2 implies that $\mathbb{H}^*(\Gamma, K) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^2, \overline{w})$, where $\overline{w}(i, j) := \chi(x(i, j))$ for any $(i, j) \in (\mathbb{Z}_{\geq 0})^2$. It turns out that

$$\overline{w}(i+1, j) - \overline{w}(i, j) = 1 + i - \lceil (53i + j)/351 \rceil - \lceil i/2 \rceil - \lceil i/3 \rceil$$

$$\overline{w}(i, j+1) - \overline{w}(i, j) = 1 + j - \lceil (i + 53j)/351 \rceil - \lceil j/2 \rceil - \lceil j/3 \rceil.$$

Since $\pi(Z_K) = (14, 14)$, the projection of the rectangle $R(0, Z_K)$ is $\pi(R(0, Z_K)) = R((0, 0), (14, 14))$. Hence by Lemma 4.5.11*(b)* the rectangle $R((0, 0), (14, 14)) = \{(i, j) \in (\mathbb{R}_{\geq 0})^2 \ : \ (i, j) \leq (14, 14)\}$ contains all the needed information. The values $\overline{w}(i, j)$ are given in the next diagram. ((0, 0) is at the lower left corner.)

1	1	0	0	0	0	0	1	0	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
0	0	−1	−1	−1	−1	−1	0	−1	−1	−1	−1	−1	0	0
1	1	0	0	0	0	0	1	0	0	0	0	0	1	1
0	1	0	0	0	0	0	1	0	0	0	0	0	1	1

The large frames illustrate the generators of $H^0(S_{-1}, \mathbb{Z})$, the small ones the generators of $H^0(S_0, \mathbb{Z})$ in degree 0 and the circle shows the generator of $H^1(S_0, \mathbb{Z})$. Hence,

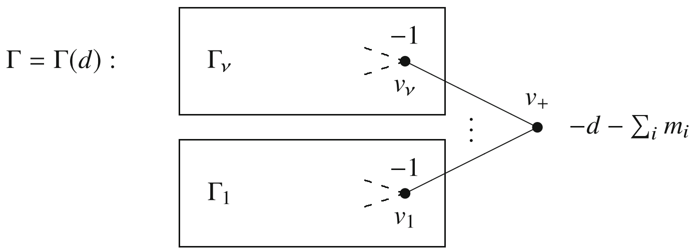
$$\mathbb{H}^0(\Gamma, K) = \mathcal{T}^+_{-2} \oplus \mathcal{T}^3_{-2}(1) \oplus \mathcal{T}^2_0(1) \quad \text{and} \quad \mathbb{H}^1(\Gamma, K) = \mathcal{T}_0(1) \quad \text{and} \quad eu(\mathbb{H}^*(\Gamma, K)) = 5.$$

For several concrete formulae and other explicit examples when Γ has two nodes, and $\overline{\mathcal{V}} = \mathcal{N}$, see [44].

4.9 $\mathbb{H}^*$ of the Surgery Manifold $S^3_{-d}(\#_i K_i)$

4.9.1 Invariants of $M(\Gamma) = S^3_{-d}(\#_i K_i)$ for Any $d > 0$ and for All Spin^c Structures [84]

4.9.1 Consider the notations of 4.2.32, or of 4.4.11 with $d > 0$. Here we do not assume that $\mu = (d-1)(d-2)$ (as in the superisolated link case 4.2.4 or 4.3.6). In this subsection we follow [84]. By 4.2.32



The group H is $\mathbb{Z}_d$ and it is generated by the class of the dual of $E_+ := E_{v_+}$. Furthermore, as in Lemma 4.7.24 one has $s_{[aE^*_+]} = aE^*_+$ for any $a = 0, 1, \ldots, d-1$. We will use the notations $h := [aE^*_+] \in H$ and $k_r := K + 2aE^*_+ \in \text{Char}$. With $\mathcal{I} = \{v_+\}$ one has (cf. 4.4.11)

$$Z_{\mathcal{I}}(t) = \frac{\Delta(t^{1/d})}{(1-t^{1/d})^2} \quad \text{and} \quad Z_{h,\mathcal{I}}(t) = \frac{1}{d} \cdot \sum_{\xi^d=1} \xi^{-a} \frac{\Delta(\xi t^{1/d})}{(1-\xi t^{1/d})^2}. \tag{4.89}$$

Using $\Delta(t) = 1 + (t-1)\delta + (t-1)^2 Q(t)$ and $Q(t) = \sum_{n=0}^{\mu-2} \alpha_n t^n$, by a computation

$$Z_{h,\mathcal{I}}(t) = \frac{t^{a/d}(a+1) + t^{1+a/d}(d-a-1)}{(t-1)^2} + \frac{\delta \cdot t^{a/d}}{t-1} + \sum_{n \equiv a \pmod d} \alpha_n t^{n/d}. \tag{4.90}$$

Since the polynomial part $Z^+_{h,\mathcal{I}}(t)$ of this expression is $\sum_{n\equiv a \pmod d} \alpha_n t^{n/d}$, we get

$$\mathrm{pc}(Z_{h,\mathcal{I}}(t)) = \mathrm{pc}(Z_{h,\mathcal{I}}(t^d)) = \sum_{n\equiv a \pmod d} \alpha_n. \tag{4.91}$$

Next we apply the surgery formula from Theorem 4.4.31 for $v = v_+$ and $l' = aE^*_+$. Then $l'_{v_+} = a/d \in [0,1)$. Furthermore, $R_i(aE^*_+) = 0$, hence all the contributions $\mathfrak{sw}_{\sigma[K_i]}(M(\Gamma_i)) - (K_i^2 + |\mathcal{V}_i|)/8$ vanish (since SWIC is valid for smooth germs). Therefore, from 4.4.31,

$$\mathfrak{sw}_{\sigma[k_r]}(M(\Gamma)) - \frac{k_r^2 + |\mathcal{V}|}{8} = \sum_{n\equiv a \pmod d} \alpha_n. \tag{4.92}$$

This combined with Theorem 4.6.31 give

$$eu(\mathbb{H}^*(\Gamma, k_r)) = \sum_{n\equiv a \pmod d} \alpha_n. \tag{4.93}$$

4.9.2 The Lattice Reduction In the next pages we follow closely [84].

The set $\overline{\mathcal{V}} := \{v_1, \dots, v_\nu\}$ of the (-1)-vertices form an SR-set, cf. 4.5.15(i). Set $E_1, \dots, E_\nu$ for the corresponding elements of L. Next we apply the Reduction Theorem from Sect. 4.8, whose notations we will adopt. Write $\bar{l} = \sum_{i=1}^{\nu} \ell_i E_i \in L(\overline{\mathcal{V}}) = \overline{L}$, and let $x_{k_r}(\bar{l})$ be the universal cycle associated with k_r and $\bar{l}$ as in 4.5.18 and Sect. 4.8. Set $\overline{w}(\bar{l}) := \chi_{k_r}(x(\bar{l}))$ as in (4.87). Then, by the Reduction Theorem 4.8.2 one has a graded $\mathbb{Z}[U]$-module isomorphism:

$$\mathbb{H}^*(\Gamma, k_r) \cong \mathbb{H}^*((\mathbb{R}_{\geq 0})^\nu, \overline{w}). \tag{4.94}$$

For each $\ell_i \geq 0$ consider the cycle $y_i(\ell_i)$ determined in the graph Γ_i as in 4.7.25 and 4.7.26. Set $\Sigma m := \sum_i m_i$ and $\Sigma\ell := \sum_i \ell_i$ (and, in general, $\Sigma x := \sum_i x_i$ for $x \in \mathbb{R}^\nu$). Then the E_+-coefficient of $x_{k_r}(\bar{l})$ is $m_+(\bar{l}) = \lceil (\Sigma\ell - a)/(\Sigma m + d) \rceil$ and

$$x_{k_r}(\bar{l}) = \sum_i y_i(\ell_i) + \left\lceil \frac{\Sigma\ell - a}{\Sigma m + d} \right\rceil \cdot E_+. \tag{4.95}$$

Write $\ell_i = p_i m_i + \ell_{i,0}$ with $p_i \in \mathbb{Z}_{\geq 0}$ and $0 \leq \ell_{i,0} < m_i$. Let Z_i be the cycle $\mathrm{div}_{E(\Gamma_i)}(f_i) = E_i^*(\Gamma_i)$. Then $y_i(\ell_i) = p_i Z_i + y_i(\ell_{i,0})$ (cf. 4.7.26). Furthermore, if for any $i = 1, \dots, \nu$ we take $1_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at entry i) then $\overline{w}(0) = 0$, and

$$\overline{w}(\bar{l} + 1_i) - \overline{w}(\bar{l}) = p_i + 1 - \left\lceil \frac{\Sigma\ell - a}{\Sigma m + d} \right\rceil - \begin{cases} 1 & \text{if } \ell_{i,0} \notin \mathcal{S}_i \\ 0 & \text{if } \ell_{i,0} \in \mathcal{S}_i. \end{cases} \tag{4.96}$$

Here $\mathcal{S}_i$ is the abbreviation for the semigroup $\mathcal{S}_{C,p_i}$.

Next, we reduce $(\mathbb{R}_{\geq 0})^\nu$ to a finite multi-rectangle. We write $\mathbf{m}$ for the vector $(m_1, \dots, m_\nu)$, and $R(\bar{l}_1, \bar{l}_2)$ denotes the rectangle $\{x \in \mathbb{R}^\nu \,:\, \bar{l}_1 \leq x \leq \bar{l}_2\}$, as usual. Set also $R_p := R(p\mathbf{m}, (p+1)\mathbf{m})$.

Lemma 4.9.3

(a) Set $\tilde{p}_0 := \lceil (\mu - a - 1)/d \rceil$. *Then*

$$\mathbb{H}^*\big((\mathbb{R}_{\geq 0})^\nu, \overline{w}\big) \cong \mathbb{H}^*\big(R(0, \tilde{p}_0\,\mathbf{m}), \overline{w}\big) \cong \mathbb{H}^*\big(\cup_{0\leq p<\tilde{p}_0} R_p, \overline{w}\big).$$

(b) $\overline{w}(p\,\mathbf{m}) = p(1 + a - \delta) + dp(p-1)/2$ *for any* $0 \leq p \leq \tilde{p}_0$.

(c) Fix $0 \leq p < \tilde{p}_0$. *Then, for any* $\bar{l} \in R_p \cap \overline{L}$, $\ell_i = pm_i + \ell_{i,0}$, *with* $\Sigma\ell \leq p(\Sigma m + d) + a + 1$ *one has:*

$$\overline{w}(\bar{l}) - \overline{w}(p\,\mathbf{m}) = \sum_i \#\{s \in \mathcal{S}_i \,:\, s \leq \ell_{i,0} - 1\}. \tag{4.97}$$

(d) Fix $0 \leq p < \tilde{p}_0$. *Then, for any* $\bar{l} \in R_p \cap \overline{L}$, $\ell_i = pm_i + \ell_{i,0}$, *with* $\Sigma\ell \geq p(\Sigma m + d) + a + 1$ *one has:*

$$\overline{w}(\bar{l}) - \overline{w}((p+1)\,\mathbf{m}) = \sum_i \#\{s \notin \mathcal{S}_i \,:\, s \geq \ell_{i,0}\}. \tag{4.98}$$

Consider the notation

$$T_p^- := \big\{\, x \in (\mathbb{R}_{\geq 0})^\nu \,:\, (\Sigma x - a - 1)/(\Sigma m + d) = p - 1 \big\}.$$

From the above facts we obtain the following.

Theorem 4.9.4 *Set* $\tilde{p}_0 := \lceil (\mu - a - 1)/d \rceil$ *as above and for any* $0 \leq p < \tilde{p}_0$ *consider*

$$\min T_{p+1}^- := \min\{\, \overline{w}(\bar{l}) \,:\, \bar{l} \in T_{p+1}^- \cap R_p \cap \overline{L}\}.$$

Then the following facts hold:

(a) $\overline{w}(p\,\mathbf{m}) \leq \min T_{p+1}^-$, $\overline{w}((p+1)\,\mathbf{m}) \leq \min T_{p+1}^-$.

(b) $m_{k_r} := \min \chi_{k_r} = \min_{0\leq p\leq \tilde{p}_0}\{\, \overline{w}(p\,\mathbf{m})\,\}$.

(c) Let $p_{\min}$ *be the smallest integer satisfying* $\overline{w}(p_{\min}\mathbf{m}) = m_{k_r}$. *Then*

$$\mathbb{H}^0_{red}(\Gamma, k_r) = \bigoplus_{0\leq p<p_{\min}} \mathcal{T}_{2\overline{w}(p\,\mathbf{m})}\big(\min T_{p+1}^- - \overline{w}(p\,\mathbf{m})\big)$$

$$\oplus \bigoplus_{p_{\min}\leq p<\tilde{p}_0} \mathcal{T}_{2\overline{w}((p+1)\,\mathbf{m})}\big(\min T_{p+1}^- - \overline{w}((p+1)\,\mathbf{m})\big).$$

(d) $\mathrm{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(\Gamma, k_r)$ *equals*

$$\sum_{0\leq p<p_{\min}} \big(\min T^-_{p+1} - \overline{w}(p\,\mathbf{m})\big) + \sum_{p_{\min}\leq p<\tilde{p}_0} \big(\min T^-_{p+1} - \overline{w}((p+1)\,\mathbf{m})\big),$$

or

$$-m_{k_r} + \mathrm{rank}_{\mathbb{Z}}\mathbb{H}^0_{red}(\Gamma, k_r) = \sum_{0\leq p<\tilde{p}_0} \big(\min T^-_{p+1} - \overline{w}((p+1)\,\mathbf{m})\big).$$

(e) *For any* $q > 0$ *one has*

$$\mathbb{H}^q(\Gamma, k_r) = \bigoplus_{0\leq p<\tilde{p}_0} \mathbb{H}^q(R_p, \overline{w}).$$

4.9.5 The Structure of $\mathbb{H}^{\geq 1}(R_p, \overline{w})$ The cohomology $\mathbb{H}^{\geq 1}(R_p, \overline{w})$ depends only on the $\overline{w}$-values at $p\,\mathbf{m}$, at $(p+1)\,\mathbf{m}$ and along T^-_{p+1}. Indeed, for any $n \in \mathbb{Z}$ consider S_n as in 4.5.2. Then for $n < \min T^-_{p+1}$ the space $S_n \cap R_p$ has the same homotopy type as the intersection of S_n with the two-element set $\{p\,\mathbf{m}, (p+1)\,\mathbf{m}\}$; while for $n \geq \min T^-_{p+1}$ it has the homotopy type of the suspension of $S_n \cap T^-_{p+1}$. In particular, all the nontrivial homogeneous elements of $\mathbb{H}^{\geq 1}(R_p, \overline{w})$ have degree $\geq \min T^-_{p+1}$, and one has the graded $\mathbb{Z}[U]$-module isomorphism

$$\mathbb{H}^q(R_p, \overline{w}) = \mathbb{H}^{q-1}_{red}(T^-_{p+1}, \overline{w}) \quad \text{for } q > 0. \tag{4.99}$$

4.9.6 The Structure of $\mathbb{H}^*(T^-_{p+1}, \overline{w})$**. The Modules** $\mathbb{H}^*(\mathbb{T}^-_n, \overline{W})$ In most of the notations above, we have omitted the symbol a codifying the characteristic element k_r. In fact, for any $p \geq 0$ and $a \in \{0, \dots, d-1\}$, T^-_{p+1} is

$$T^-_{p+1,a} := \{\bar{l} \,:\, \ell_i = pm_i + \ell_{i,0};\ \sum_i \ell_{i,0} = pd + a + 1\}.$$

Note that when p runs over $\mathbb{Z}_{\geq 0}$ and $a \in \{0, \dots, d-1\}$, the integer $n = pd + a$ runs over $\mathbb{Z}_{\geq 0}$. This motivates to consider for any $n \in \mathbb{Z}_{\geq 0}$

$$\mathbb{T}_n := \{(\ell_{1,0}, \dots, \ell_{\nu,0}) \in [0, m_1] \times \cdots \times [0, m_\nu] \,:\, \sum_i \ell_{i,0} = n + 1\}. \tag{4.100}$$

Then, for d and a fixed, $T^-_{p+1,a} = \mathbb{T}_{pd+a} + p\,\mathbf{m}$. If $p < \tilde{p}_0$ then $pd + a \leq \mu - 2$, hence the relevant index set of the hyperplanes is $0 \leq n \leq \mu - 2$ (this can be compared with the index set $\{\alpha_n\}_{n=0}^{\mu-2}$ of the coefficients of $Q(t)$). The form $\mathbb{T}_{pd+a} + p\,\mathbf{m}$ shows also how they intersect the small rectangles: when we run a, an element of the set $\{\mathbb{T}_n + \lfloor n/d \rfloor \mathbf{m}\}_{0\leq n\leq \mu-2}$ intersects R_p if and only if $\lfloor n/d \rfloor = p$.

Up to the shift $\overline{w}(p\,\mathbf{m})$, which is constant on each $\mathbb{T}_n$, but otherwise depends on $p = \lfloor n/d \rfloor$, the weights on $\mathbb{T}_n \cap \mathbb{Z}^\nu$ are given by the right hand side of (4.97). Or, up to a shift $\overline{w}((p+1)\,\mathbf{m})$, the weights are given by (4.98). Following this second version we set the following weights for any $\mathbb{T}_n$:

$$\overline{W}((\ell_{1,0}, \dots, \ell_{\nu,0})) = \sum_i \#\{s \notin \mathcal{S}_i \; : \; s \geq \ell_{i,0}\}. \tag{4.101}$$

That is, $\overline{W}|_{\mathbb{T}_n}(\bar{l} - p\,\mathbf{m}) = \overline{w}(\bar{l}) - \overline{w}((p+1)\,\mathbf{m})$, where $p = \lfloor n/d \rfloor$.

The weight function $\overline{W}$ restricted on all the level sets $\{\mathbb{T}_n\}_{n\geq 0}$ of $(\mathbb{Z}_{\geq 0})^\nu$ measures the very subtle distribution properties of the semigroups $\{\mathcal{S}_i\}_i$. Furthermore, up to a well-identified shift in degrees, the collection $(\mathbb{T}_n, \overline{W})$ provides all the lattice cohomologies $\mathbb{H}^*(\Gamma(d), k_r)$ for all the possible values d and a. Here, and in the next discussion, we denote the dependence of Γ on d by $\Gamma(d)$.

More precisely, for any d and $a \in \{0, \dots, d-1\}$ and $q > 0$ one has:

$$\mathbb{H}^q(\Gamma(d), K + 2aE_+^*) = \bigoplus_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} \mathbb{H}^{q-1}_{red}(\mathbb{T}_n, \overline{W})[s_{n,d}], \tag{4.102}$$

where $s_{n,d}$ is the value of the shift $2\overline{w}((p+1)\,\mathbf{m}) = 2(p+1)(1+a-\delta)+d(p+1)p$ (with $p = \lfloor n/d \rfloor$). Moreover, the values $\{\min \overline{W}\,|\mathbb{T}_n\}_n$ and $s_{n,d}$ determine all the cohomology groups $\mathbb{H}^0(\Gamma(d), k_r)$ too. The second identity of (4.9.4)(d) together with (4.98) reads as:

$$-m_{k_r} + \mathrm{rank}\ \mathbb{H}^0_{red}(\Gamma(d), K + 2aE_+^*) = \sum_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} \min\{\overline{W}|_{\mathbb{T}_n}\}. \tag{4.103}$$

In particular, for any fixed $d > 0$ and $a \in \{0, \dots, d-1\}$ one has:

$$\begin{aligned} eu(\mathbb{H}^0(\Gamma(d), K + 2aE_+^*)) &= \sum_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} \min\{\overline{W}|_{\mathbb{T}_n}\}, \\ eu(\mathbb{H}^*(\Gamma(d), K + 2aE_+^*)) &= \sum_{n\equiv a \ (\mathrm{mod}\ d),\ 0\leq n\leq \mu-2} -eu(\mathbb{H}^*(\mathbb{T}_n, \overline{W})). \end{aligned} \tag{4.104}$$

Example 4.9.7 For any $d > 0$ and $q > 0$ the summation of (4.102) over a gives

$$\mathbb{H}^q(\Gamma(d)) = \bigoplus_{a=0}^{d-1} \mathbb{H}^q(\Gamma(d), K + 2aE_+^*) = \bigoplus_{0\leq n\leq \mu-2} \mathbb{H}^{q-1}_{red}(\mathbb{T}_n, \overline{W})[s_{n,d}]. \tag{4.105}$$

On the right hand side of (4.105) the numbers $s_{n,d}$ depend on d, but the rank of the right hand side is independent of d. In particular, up to shifts of different direct sum blocks, $\oplus_{q>0}\mathbb{H}^q(\Gamma(d), k_r)$ is independent of the choice of the integer d. (This can also be deduced from the surgery exact sequences from [74].)

Example 4.9.8

(a) Assume that for a certain d and a one gets $\tilde{p}_0 = 0$. Then $\mathbb{H}^*_{red}(\Gamma, k_r) = 0$, and $\mathbb{H}^0(\Gamma, k_r) = \mathcal{T}_0^+$.
(b) Assume that for a certain d and a one gets $\tilde{p}_0 = 1$. Then $\mathbb{H}^*(\Gamma, k_r) = \mathbb{H}^*(R_0, \overline{w})$, hence everything is determined by $T^-_{1,a}$. Indeed,

$$\begin{aligned}\min T^-_{1,a} &= \min\Big\{\sum_i \#\{s\in\mathcal{S}_i \,:\, s\leq \ell_i-1\},\ \text{where}\ \sum_i \ell_i = a+1\Big\}\\ &= \min\Big\{\sum_i \#\{s\notin\mathcal{S}_i \,:\, s\geq \ell_i\},\ \text{where}\ \sum_i \ell_i = a+1\Big\} + 1 + a - \delta,\end{aligned}$$

$m_{k_r} = \min\{0, 1+a-\delta\}$, $\mathbb{H}^0_{red}(\Gamma, k_r)$ is generated by one element of degree $2\max\{0, 1+a-\delta\}$, rank $\mathbb{H}^0_{red}(\Gamma, k_r) = \min T^-_{1,a} - \max\{0, 1+a-\delta\}$, and finally for $q>0$ one has $\mathbb{H}^q(\Gamma, k_r) = \mathbb{H}^{q-1}_{red}(T^-_{1,a}, \overline{w}) = \mathbb{H}^{q-1}_{red}(\mathbb{T}_a, \overline{W})[2(1+a-\delta)]$, ($T^-_{1,a} = \mathbb{T}_a + \mathbf{m}$).
(c) If $d \geq \mu-1$ then $\tilde{p}_0 = 1$ for $a < \mu - 1$, and $\tilde{p}_0 = 0$ for $a \geq \mu-1$.

Remark 4.9.9 Assume that we know all the cohomology groups $\{\mathbb{H}^*(\Gamma(d), k_r)\}_{k_r}$ for some specific d with $d \geq \mu-1$. Then using them, and also the values $\overline{w}(p\mathbf{m}) = p(1+a-\delta) + dp(p-1)/2$ for all p, a and d, we can recover all the lattice cohomologies $\{\mathbb{H}^*(\Gamma(d), k_r)\}_{k_r}$ for any $d>0$. [For this, use Example 4.9.8 and (4.102).]

Corollary 4.9.10 *For any $n\geq 0$ the coefficients of $Q(t) = \sum_n \alpha_n t^n$ satisfy*

$$\alpha_n = -eu\,(\mathbb{H}^*(\mathbb{T}_n, \overline{W})). \tag{4.106}$$

Proof Use the identities (4.93) and (4.104) for $d \gg 0$, cf. 4.9.9. □

Remark 4.9.11 Above we reduced several computations to the weight function $\overline{W}|_{\mathbb{T}_n}$. It was connected with the weight function provided by the reduction formula via $\overline{W}|_{\mathbb{T}_n}(\bar{l} - p\,\mathbf{m}) = \overline{w}(\bar{l}) - \overline{w}((p+1)\,\mathbf{m})$, where $p = \lfloor n/d \rfloor$. Since each $\overline{w}(p\,\mathbf{m})$ is computable from d, a, δ, cf. 4.9.3*(b)*, the lattice cohomology $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ is computable from d, a, δ and $\{\overline{W}|_{\mathbb{T}_n}\}_n$. On the other hand, by (4.101) $\overline{W}((\ell_{1,0}, \dots, \ell_{\nu,0}))$ equals $\sum_i \#\{s \notin \mathcal{S}_i : s \geq \ell_{i,0}\} = \sum_i \big(\delta_i - \#\{s_i \notin \mathcal{S}_i : s_i < \ell_{i,0}\}\big) = \sum_i(\delta_i - \ell_{i,0}) + \sum_i \#\{s_i \in \mathcal{S}_i : s_i < \ell_{i,0}\}$. Hence

$$\min\{\overline{W}|_{\mathbb{T}_n}\} = \delta - n - 1 + \min_{\sum_i \ell_{i,0} = n+1} \#\{s_i \in \mathcal{S}_i : s_i < \ell_{i,0}\}. \tag{4.107}$$

This motivates the replacement of the semigroup $\mathcal{S}_i$ with an equivalent object of it, with its 'counting function' $j \mapsto H_i(j)$,

$$H_i(j) := \#\{s \in \mathcal{S}_i \, : \, s < j\}. \tag{4.108}$$

From analytic point of view, $H_i(j)$ is the coefficient of t^j in the Hilbert function of the local singularity (C, p_i), associated with the filtration given by its normalization.

The above min-expression can be reformulated formally as follows. Consider any two functions H_1 and H_2 defined on integers and bounded from below. Then we define their *'minimum convolution'* (cf. [9, 5.3]), denoted by $H_1 \diamond H_2$ as $(H_1 \diamond H_2)(j) = \min_{j_1+j_2=j}\{H_1(j_1) + H_2(j_2)\}$.

Then from the counting functions $\{H_i\}_{i=1}^{\nu}$ associated with $\{\mathcal{S}_i\}_{i=1}^{\nu}$ we construct

$$H := H_1 \diamond H_2 \diamond \cdots \diamond H_\nu. \tag{4.109}$$

Since the operator $\diamond$ is associative and commutative, the function H is well-defined.

From the above discussion $\mathbb{H}^0(S^3_{-d}(\#_i K_i))$ is computable from d, a, δ and H.

Remark 4.9.12 In the above discussion (e.g. in 4.9.5–4.9.6), the space $\mathbb{T}_n$—intersection of a simplex with a rectangle—can be replaced by the supporting simplex. Indeed, set

$$\Sigma_n := \{(\ell_{1,0}, \ldots, \ell_{\nu,0}) \in (\mathbb{R}_{\geq 0})^\nu \, : \, \sum_i \ell_{i,0} = n + 1\}. \tag{4.110}$$

A verification shows that $H^*_{red}(\mathbb{T}_n, \overline{W})$ is isomorphic with $H^*_{red}(\Sigma_n, \overline{W})$ for every $n \geq 0$. Furthermore, if $n > \mu - 2$ then $H^*_{red}(\mathbb{T}_n, \overline{W}) = 0$ automatically, hence in several formulae above (e.g. in the summations from (4.102) and (4.105)) the restrictions $n \leq \mu - 2$ can be safely neglected.

4.9.2 Superisolated Singularities with More Cusps

In this subsection we consider a superisolated singularity associated with an irreducible rational cuspidal curve. For different notations and statements regarding the analytic and topological type see Sects. 4.2.4, 4.3.6, 4.4.11, 4.7.4, and 4.9. In this subsection we follow [8].

Our goal is to discuss Conjectures 4.3.21 and 4.3.22 from the point of view of lattice cohomology. Let us recall the two statements. Set (cf. 4.3.20*(b)*)

$$N(t) = \sum_{l=0}^{d-3} \left(\alpha_{(d-3-l)d} - \frac{(l+1)(l+2)}{2}\right) t^{d-3-j}. \tag{4.111}$$

- Conjecture 4.3.21: all the coefficients of $N(t)$ are non-positive. We will refer to this as '**Conjecture C**' ('Conjecture regarding the coefficients of $N(t)$').
- Conjecture 4.3.22: $N(1)$ is non-positive. We will refer to this as the '**Conjecture I**' (we regard $N(1)$ as an 'index type invariant').

Clearly Conjecture C implies Conjecture I.

We will compare these statements with the Semigroup Distribution Property based on the properties of counting function H_i of the semigroups and also on a subtle connection with lattice cohomology.

We consider the counting functions H_i of the semigroups $\mathcal{S}_i$ (cf. (4.108)) and their minimum convolution H as in (4.109). Recall also (cf. 4.2.33) that the Semigroup Distribution Property (SDP) reads as $H(ld+1) = (l+1)(l+2)/2$ for any $l = 0, 1, \dots d-3$.

Example 4.9.13 (The case $\nu = 1$) In this case $\alpha_j = \#\{s \notin \mathcal{S}_1 : s > j\}$, cf. (4.42). From (4.43) $\alpha_{2\delta-2-j} = H_1(j+1)$ for $j = 0, \dots, 2\delta - 2$. Hence, the α-coefficient needed in (4.111) is $\alpha_{(d-3-l)d} = \#\{s \in \mathcal{S}_1 : s \le ld\} = H_1(ld+1)$. Recall that 4.2.33 (Bézout's Theorem) implies $\alpha_{(d-3-l)d} = H_1(ld+1) \ge (l+1)(l+2)/2$. This inequality and (4.111) show that for $\nu = 1$ Conjecture C is equivalent to $N(t) \equiv 0$. But, they are also equivalent to Conjecture I, since if $N(1) \le 0$ then necessarily $N(t) \equiv 0$. Finally, the validity of all these statements follow from SDP.

However, for $\nu \ge 2$ the relationships are more subtle.

Theorem 4.9.14 ([8]) *With the above notations one has:*

1. *If $\nu = 2$, then $q_{2\delta-2-j} \le H(j+1)$ for any $j = 0, 1, \dots 2\delta - 2$. Therefore, for bicuspidal curves the SDP implies Conjecture C (hence Conjecture I too).*
2. *If $\nu \ge 3$, then the inequality $q_{2\delta-2-j} \le H(j+1)$ does not hold in general, not even for $j = ld$ ($l = 0, 1, \dots, d-3$), needed for Conjectures C and I. Moreover, Conjecture C is not true in general, and Conjecture I behaves independently from SDP. (Conjecture I remains as a conjecture, though its validity is verified directly for all 'known' curves.)*

For a direct elementary proof of part *(1)* see [65].

4.9.15 Combinatorial Reformulations The next discussion aims to clarify the similarities and differences between the polynomial Q and the function H.

Let us start with ν semigroups $\{\mathcal{S}_i\}_{i=1}^{\nu}$ associated with local irreducible plane curve singularities. However, in the next discussion we will not require their realizability as singularities of a projective rational curve. [Regarding the realizability, we use the following terminology. If the sum δ of delta-invariants of the local singularity types is of form $2\delta = (d-1)(d-2)$ for some integer d, then we say that these ν local topological types are *combinatorial candidates* for the ν singularities of a rational cuspidal plane curve of degree d. If such a curve really exists then (SDP) is valid for the corresponding local data and d.], The semigroups determine their counting functions H_i by (4.108) and the minimal convolution H of the functions $\{H_i\}_i$ by (4.109). For convenience, define also the sequences $\{h_j^{(i)}\}_{j=0}^{\infty}$ by $h_j^{(i)} := H_i(j+1)$.

For any sequence $a = \{a_j\}_{j=0}^{\infty}$ denote by ∂a its *difference sequence*, i.e. $(\partial a)_j = a_j - a_{j-1}$ with the convention $a_{-1} = 0$. Similarly, we will denote by Σa the *sequence of partial sums*, i.e. $(\Sigma a)_j = a_0 + \cdots + a_j$. Of course, $\Sigma\partial a = a$ and $\partial\Sigma a = a$ for any sequence a.

By (4.108) and $\Delta_i(t) = (1-t)\cdot\sum_{s\in\mathcal{S}_i} t^s$ (cf. (4.6) the coefficient $c_j^{(i)}$ of t^j in $\Delta_i(t)$ can be written as $c_j^{(i)} = (\partial\partial h^{(i)})_j$. The coefficient sequence of a polynomial product is the *usual convolution* of coefficient sequences of the factors. Hence, the coefficient c_j of t^j in $\Delta(t) = \prod_i \Delta_i(t)$ is $c_j = \sum_{j_1+\cdots+j_\nu=j} c_{j_1}^{(1)}\cdots c_{j_\nu}^{(\nu)}$. Denoting the convolution of two sequences $a = \{a_j\}_{j=0}^{\infty}$ and $b = \{b_j\}_{j=0}^{\infty}$ by $a * b$, i.e. $(a*b)_j = \sum_{k=0}^{j} a_k b_{j-k}$, we get $c_j = (\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)})_j$. Let us define:

$$F(j) := (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j. \tag{4.112}$$

Before we identify F, let us recall some symmetry properties. From the symmetry of $\Delta = 1 + (t-1)\delta + (t-1)^2 Q(t)$ (and from $\delta = \sum_i \delta_i$)

$$\alpha_{2\delta-2-j} = \alpha_j + j + 1 - \delta \quad \text{for} \quad 0 \le j \le 2\delta - 2. \tag{4.113}$$

This (or the symmetry of each semigroup) implies also $H_i(j_i) = H_i(2\delta_i - j_i) + j_i - \delta_i$, from which one also obtains

$$H(2\delta - 2 - j + 1) = H(j+1) - j - 1 + \delta \quad \text{for every } j \in \mathbb{Z}. \tag{4.114}$$

Next, if $A(t) = \sum_j a_j t^j$ and $B(t) = \sum_j b_j t^j$ satisfy $A(t) = A(1) + (t-1)B(t)$, then $(\Sigma a)_j = A(1) - b_j$. This applied twice for Δ gives $(\Sigma\Sigma c)_j = j + 1 - \delta + \alpha_j$. Hence, then the definition of Q and (4.113) provide

$$\alpha_{2\delta-2-j} = (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j = F(j) \quad \text{for} \quad 0 \le j \le 2\delta - 2. \tag{4.115}$$

In other words, the H-values are obtained from $\{h^{(i)}\}_i$ by minimal convolution (shifted by one), while the F-coefficients (or α-coefficient in opposite order) are obtained by the composition of $\partial\partial$, the usual convolution, and the $\Sigma\Sigma$ operator.

Then one has the following reinterpretations in terms of F and H.

Let $C \subset \mathbb{C}P^2$ be a rational cuspidal curve of degree d with ν cusps of given topological types (in particular, $d(d-3) = 2\delta - 2$). Set $F(j) := (\Sigma\Sigma(\partial\partial h^{(1)} * \cdots * \partial\partial h^{(\nu)}))_j$, where $h_j^{(i)} = H_i(j+1)$, and H_i is the semigroup counting function of the i-th singularity. Set $H := H_1 \diamond \cdots \diamond H_\nu$. Then

$$\text{(Conjecture C)} \qquad F(ld) \le \frac{(l+1)(l+2)}{2} \quad \text{for all } l = 0, 1, \ldots, d-3. \tag{4.116}$$

$$\text{(Conjecture I)} \quad \sum_{l=0}^{d-3} F(ld) \leq \sum_{l=0}^{d-3} \frac{(l+1)(l+2)}{2} = \frac{d(d-1)(d-2)}{6}. \qquad (4.117)$$

$$\text{(SDP)} \qquad H(ld+1) = \frac{(l+1)(l+2)}{2} \quad \text{for all } l = 0, 1, \dots, d-3. \qquad (4.118)$$

Let us summarize the combinatorial situation. Starting from the semigroups of ν local singularities we define H and F.

If $\nu = 1$ (since $\Sigma\Sigma\partial\partial(h) = h$) then $F(j) = H(j+1)$ for each $j \in \mathbb{Z}_{\geq 0}$ (independently of realizability, hence not just for $j \in d \cdot \mathbb{Z}_{\geq 0}$).

On the other hand, for $\nu > 1$ the values $F(j)$ and $H(j+1)$ become different. Nevertheless, cf. Theorem 4.9.14*(1)* $F(j) \leq H(j+1)$ remains true for $\nu = 2$ and every integer $j \geq 0$, again by combinatorial (lattice cohomology) argument (independently of realizability and d).

With these facts in mind, it is tempting to conjecture that maybe *the inequality* $F(j) \leq H(j+1)$ *is always true—as a property of local singularity types—*, which would make Conjecture C a combinatorial corollary of SDP. But, for $\nu \geq 3$ there is no such relation between the local functions F and H.

4.9.16 Lattice Cohomological Reinterpretation Consider the combinatorial situation from 4.9.15. The semigroups $\mathcal{S}_i$ determine links $K_i \subset S^3$ of the corresponding (topological types) of plane curve singularities. Consider an arbitrary $d > 0$ and the surgery 3-manifold $S^3_{-d}(\#_i K_i)$ as in Sect. 4.9.

The next statements show a remarkable common feature of the functions F and H.

Theorem 4.9.17 *For any $d > 0$ and $0 \leq a < d$ the following facts hold:*

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} (H(j+1) + \delta - 1 - j),$$
$$= \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} H(2\delta - 2 - j + 1); \qquad (4.119)$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} (F(j) + \delta - 1 - j)$$
$$= \sum_{\substack{j \equiv a (\text{mod } d) \\ 0 \leq j \leq 2\delta - 2}} F(2\delta - 2 - j). \qquad (4.120)$$

Proof We will use the identities from (4.104). In the first one, note that by (4.101), (4.100), and (4.107) $\min(\overline{W}|_{\mathbb{T}_j})$ is $\delta - j - 1 + H(j+1)$ and (4.119) follows (for its second identity use (4.114)).

For the second identity, note that $-eu(\mathbb{H}^*(\mathbb{T}_j, \overline{W})$ equals α_j by (4.106), which is $F(2\delta - 2 - j)$ by (4.115). Then use again the symmetry (4.113). □

Remark 4.9.18 In fact, by Theorem 4.9.4, the integer d, the sum of delta-invariants δ and the function H completely determine the whole $\mathbb{H}^0$ as a graded $\mathbb{Z}[U]$-module (and not just its Euler characteristic).

Corollary 4.9.19 *Assume that $d(d-3) = 2\delta - 2$ (that is, d and $\{\mathcal{S}_i\}_i$ constitute a package of combinatorial candidates for algebraic realizability). Then*

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j \equiv -a (\mathrm{mod}\ d) \\ 0 \leq j \leq 2\delta - 2}} H(j+1),$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K + 2aE^*_+)\right) = \sum_{\substack{j \equiv -a (\mathrm{mod}\ d) \\ 0 \leq j \leq 2\delta - 2}} F(j).$$

This for $a = 0$ reads as

$$eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) = \sum_{0 \leq l \leq d-3} H(ld+1),$$

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) = \sum_{0 \leq l \leq d-3} F(ld).$$

Since by 4.2.33 $H(ld+1) \geq (l+1)(l+2)/2$ for any $l = 0, \ldots, d-3$, $\sum_{l=0}^{d-3} H(ld+1) = \sum_{l=0}^{d-3}(l+1)(l+2)/2$ is equivalent to SDP for every l (cf. (4.118)). In particular, in the presence of the algebraic realization, the valid SDP reads as:

$$\text{(SDP)} \quad eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) = d(d-1)(d-2)/6. \tag{4.121}$$

Furthermore, under the same realizability assumption, Conjecture I reads as:

$$eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) \leq d(d-1)(d-2)/6. \tag{4.122}$$

They combined:

$$\text{(Conjecture I)} \quad eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) \leq eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right). \tag{4.123}$$

4.9.20 Proof of Conjecture I for $\nu = 2$ (via SDP)
First note that $\mathbb{H}^q(S^3_{-d}(\#_i K_i), k_r) = 0$ for any $q \geq \nu$ and any k_r (cf. 4.8.3). Then, for $\nu = 2$, one has $eu\left(\mathbb{H}^*(S^3_{-d}(\#_i K_i), K)\right) = eu\left(\mathbb{H}^0(S^3_{-d}(\#_i K_i), K)\right) - \mathrm{rank}_{\mathbb{Z}}\mathbb{H}^1(S^3_{-d}(\#_i K_i), K)$, hence (4.123) follows.

For $\nu \geq 3$ the similar argument does not work. From this point of view, it is even more surprising that in all the known cases Conjecture I still holds, cf. 4.9.14.

4.10 Lattice Cohomology and Heegaard Floer Homology

The Seiberg–Witten invariant is the (normalized) Euler-characteristic of the Seiberg–Witten monopole Floer homology of Kronheimer–Mrowka, or equivalently, of the Heegaard Floer homology of Ozsváth and Szabó. These theories had an extreme influence on the modern mathematics, solving (or disproving) a long list of old conjectures (e.g. Thom Conjecture, or conjectures regarding classification of 4-manifolds, or famous old problems in knot theory); see the long list of distinguished articles of Kronheimer–Mrowka or Ozsváth–Szabó. In [102] Ozsváth and Szabó provided a computation of the Heegaard Floer homology for some special plumbed 3-manifolds. This computation resonated incredibly with the theory of computation sequences used in Artin–Laufer program (see e.g. [50, 67, 68]). These two facts influenced considerably the definition of the lattice cohomology.

4.10.1 The Conjecture Connecting Lattice Cohomology and Heegaard Floer Theory

4.10.1 Short Review of Heegaard Floer Homology $HF^+(M)$ We assume that M is an oriented rational homology 3–sphere, and we restrict ourselves to the +–theory of Ozsváth and Szabó. The Heegaard Floer homology $HF^+(M)$ is a $\mathbb{Z}[U]$–module with a $\mathbb{Q}$–grading compatible with the $\mathbb{Z}[U]$–action, where $\deg(U) = -2$. Additionally, $HF^+(M)$ has another $\mathbb{Z}_2$–grading; $HF^+(M)_{even}$, respectively $HF^+(M)_{odd}$ denote the graded parts. Moreover, $HF^+(M)$ has a natural direct sum decomposition of $\mathbb{Z}[U]$–modules (compatible with all the gradings): $HF^+(M) = \oplus_\sigma HF^+(M, \sigma)$ indexed by the spin^c structures σ of M. For any $\sigma \in \mathrm{Spin}^c(M)$ one has

$$HF^+(M, \sigma) = \mathcal{T}^+_{d(M,\sigma)} \oplus HF^+_{red}(M, \sigma),$$

a graded $\mathbb{Z}[U]$–module isomorphism, and $HF^+_{red}(M,\sigma)$ has finite $\mathbb{Z}$–rank and an induced $\mathbb{Z}_2$–grading. One also considers

$$\chi(HF^+(M,\sigma)) := \operatorname{rank}_{\mathbb{Z}} HF^+_{red,even}(M,\sigma) - \operatorname{rank}_{\mathbb{Z}} HF^+_{red,odd}(M,\sigma).$$

Then the Seiberg–Witten invariant of (M,σ) equals $\chi(HF^+(M,\sigma)) - d(M,\sigma)/2$.

By changing the orientation we have $\chi(HF^+(M,\sigma)) = -\chi(HF^+(-M,\sigma))$ and $d(M,\sigma) = -d(-M,\sigma)$.

4.10.2 The Predicted Connection In [72] the author formulated the following

Conjecture 4.10.3 For any plumbed rational homology sphere associated with a connected negative definite graph Γ, and for any $k \in$ Char, one has

$$d(M,[k]) = \max_{k'\in[k]} \frac{(k')^2+|\mathcal{V}|}{4} = \frac{k^2+|\mathcal{V}|}{4} - 2\cdot\min\chi_k.$$

Furthermore,

$$HF^+_{red,even}(-M,[k]) = \bigoplus_{p\ even} \mathbb{H}^p_{red}(\Gamma,[k])[-d],$$

and

$$HF^+_{red,odd}(-M,[k]) = \bigoplus_{p\ odd} \mathbb{H}^p_{red}(\Gamma,[k])[-d].$$

Both parts of the Conjecture were verified for almost rational graphs in [72], for two bad vertices in [101], see [72, 8.4] too. Otherwise, the Conjecture is still open.

Note that (conjecturally) $\mathbb{H}^*$ has a richer structure: its q–filtration $\mathbb{H}^* = \oplus_q \mathbb{H}^q$ collapses at the level of HF^+ to a $\mathbb{Z}_2$ odd/even filtration.

The fact that both theories have the same Euler characteristic support the above conjecture as well. Another supporting evidence is the following fact.

4.10.4 Coincidence of the Vanishing of the Reduced Theories By 4.6.22 the graph Γ is rational if and only if $\mathbb{H}^*_{red}(\Gamma) = 0$. On the other hand, following Ozsváth and Szabó, by definition, M is an L–space if and only if $HF^+_{red} = 0$. Their equivalence is predicted by Conjecture 4.10.3. This 'tip of the iceberg' statement was proved in [76]:

Theorem 4.10.5 *The following facts are equivalent for a connected negative definite graph* Γ*:*

(i) Γ *is a rational graph,*
(ii) $M = M(\Gamma)$ *is an* L*–space.*

$(i) \Rightarrow (ii)$ follows from lattice cohomology theory [70, 72], while $(ii) \Rightarrow (i)$ uses partly the following equivalence $(ii) \Leftrightarrow (iii)$, where (iii) means that $\pi_1(M)$ is not a left-orderable group. [A non trivial group G is said to be left-orderable if there exist a total order $<$ on G such that if $a < b$ then $ga < gb$ for every $g \in G$.] The equivalence $(ii) \Leftrightarrow (iii)$ was proved in [33] for any graph–manifold. For arbitrary 3–manifolds it was conjectured by Boyer, Gordon and Watson [10], for different developments and other references see [33, 76].

Problem 4.10.6 Characterize elliptic singularities (or other non-rational families of singularities) by a certain property of the fundamental group of the link.

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Chapter 5
Characteristic Classes

Jean-Paul Brasselet

In memory of Roberto Callejas-Bedregal
who died from covid19 on April 6, 2021.
We will never forget your joy, your laughter
and your enthusiasm for mathematics.

Contents

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Abstract The theory of characteristic classes for singular varieties experiences a huge development, in quantity and in quality, since the work of Marie-Hélène Schwartz, Wu Wen-Tsün and Robert MacPherson. An impressive number of researchers are extending the field of applications of characteristic classes and their ingredients, including in applied mathematics and physics.

In 1537, in Messina, Francesco Maurolico observed, for the five Platonic polyhedra, the formula that has been called later Euler formula. The Poincaré-Hopf Theorem says that the Euler-Poincaré characteristic is the obstruction to the construction of continuous non-vanishing vector fields tangent to a compact manifold. That opened the door for the construction of characteristic classes by obstruction theory: for manifolds, by Eduard Stiefel and Hassler Whitney in the real case and by Shiing-shen Chern in the complex case, then, in the singular framework, by Marie-Hélène Schwartz. The functorial definition by Robert MacPherson is the starting point of a huge development of the theory and applications of Chern-Schwartz-MacPherson classes and their ingredients: local Euler obstruction, Wu-Mather classes, Milnor classes, Segre classes, bivariant theory, motivic characteristic classes, etc.

This survey intentionally includes a brief history of the creation of characteristic classes from their very beginning as well as the detailed definition, by obstruction theory, of Schwartz classes giving rise to Chern-Schwartz-MacPherson classes.

5.1 Introduction

It is not easy to define the starting point of the notion of characteristic classes; Is it Pythagoras of Samos who described the first three regular polyhedra? Is it Theaetetus of Athens who described the two last ones? Is it Francesco Maurolico who observed that for these five “Platonic” polyhedra, the number of faces added to the one of vertices exceeds by two the number of edges? Is it René Descartes who wrote his famous theorem concerning the sum of angles of a convex polyhedron? Is it Leonhard Euler who sent a letter to his friend Goldbach with the now called “Euler formula” ? Is it Henri Poincaré who wrote that, for compact smooth surfaces, the Euler-Poincaré characteristic is a measure of the obstruction to the construction of a tangent vector field? Is it Heinz Hopf who completed the result for any dimension and recommended his student Eduard Stiefel to study the case of frames?

All of them, recognized or not, and many other, unknown (or forgotten), provided a stone to the basis of the notion of characteristic classes as we know now.

Hopf recommended his student Stiefel to study obstruction for the construction of r-frames tangent to a manifold. In 1935 and independently, Eduard Stiefel and Hassler Whitney defined characteristic classes in cohomology for real manifolds. In 1947, Lev Pontryagin defined another type of classes of a manifold M, by obstruction theory. Contribution of Wu Wen Tsün[1] in the history of characteristic classes is important. showing that the Stiefel-Whitney classes are the Steenrod squares of the Wu (real) classes he defined in 1955. In 1948, independently, Chern and Wu proved the product formula for Stiefel-Whitney classes.

In the complex situation, Shiing-shen Chern gave, in his fundamental 1946 paper, several constructions of characteristic classes for Hermitian Manifolds. The paper provides foundations for the relationship between obstruction theory, Schubert varieties, differential forms and transgression. Wu proved the product formula for Chern classes.

The Hirzebruch theory (Sect. 5.9) provides a way to unify, in the case of manifolds, three theories of characteristic classes: the Chern class, the Todd class and the Thom-Hirzebruch class. Using multiplicative series and Chern roots, Hirzebruch defines the Todd-Hirzebruch classes which are, according to values of the parameter y, the Chern classes ($y = -1$), the Todd classes ($y = 0$) and the Thom-Hirzebruch L-classes ($y = +1$). Three theorems, namely Poincaré-Hopf Theorem, Hirzebruch-Riemann-Roch Theorem and Hirzebruch signature Theorem become particular cases of a "general Hirzebruch Riemann-Roch Theorem".

The interested reader will find all wished references for characteristic classes of manifolds in the books by Milnor, Steenrod, Hirzebruch, and for historical viewpoint, by Dieudonné.

In the singular case, if there are various constructions, even combinatorial, for Stiefel-Whitney classes of (real) singular varieties, it took a long time before providing definition of characteristic Chern classes for complex singular varieties. The problem is that in the case of singular varieties the tangent bundle is not defined and the previous constructions of characteristic classes do not apply.

In the same year 1965, two constructions of Chern classes for complex singular varieties were published. One was published by Marie-Hélène Schwartz, in French [289], the other was published by Wu Wen-Tsün, in Chinese [342]. Apparently, nobody (or at any rate, few people) noticed these publications.

In 1966, in an unpublished lecture of his seminar, Alexander Grothendieck conjectured the existence and uniqueness of Chern classes in the schematic framework, in the Chow ring (see Sect. 5.15.2). The conjecture, outlined by Pierre Deligne to Denis Sullivan [307], was proved by Robert MacPherson in the year 1973, under the name of Deligne-Grothendieck conjecture [203]. MacPherson's construction is performed in the framework of algebraic complex varieties, and in homology. One of its fundamental ingredients is the Mather class, defined using the Nash transformation. The MacPherson class is a combination of Mather classes, with coefficients defined by means of the Local Euler obstruction.

MacPherson classes are defined for constructible functions on the singular variety. In 1979, Brasselet and Schwartz proved [59] that MacPherson classes (for

[1] There are several ways to write Wu Wen-Tsün name in Latin characters. We use the one he used to sign his articles during his French period, that is the one of main papers cited here.

the constructible function with everywhere value 1) are the same as the previously defined Schwartz classes via the Alexander isomorphism.

Jianyi Zhou [371] showed that the Nash transformation corresponds to the previously described Wu transformation and that the Mather classes are equal to the (complex) Wu classes.

There are in fact various definitions of characteristic classes for singular varieties. In the real case of Stiefel-Whitney classes, there is a combinatorial definition, which simplifies the construction. The (real) Wu classes were used by Goresky and Pardon, for the study of cobordism theory in the case of singular spaces.

In the complex case, the situation is more complicated (and certainly more interesting!), due to the fact that there is no combinatorial definition of Chern classes. The obstruction theory point of view, in the smooth case, is based on the existence of the tangent bundle. If one wants to use the obstruction theory point of view in the singular case, one has to find a substitute to the tangent bundle. There are various candidates to replace the tangent bundle and each of them leads to a different definition of Chern class for singular varieties. Considering a singular variety X embedded in a manifold M, one has (at least) the following three possibilities.

1. the union E_X of tangent spaces to the strata of a stratification of X. M.-H. Schwartz considers the sections of TM whose images are in E_X. She shows that if one wants to use obstruction theory in the singular case, one has to use special vector fields obtained by what she named radial extension.
2. the set of all possible limits of tangent spaces to sequences of points in the regular part of X. That is the (Wu)-Nash transformation and the Nash bundle on it, leading to the notion of (Wu)-Mather classes, one of the ingredients used by MacPherson.
3. the virtual bundle. That is the viewpoint used by Fulton. If X is smooth, one has an exact sequence

$$0 \to TX \to TM|_X \to N_X M \to 0$$

   where $N_X M$ is the normal bundle of X in M. In the case of a singular variety such that the normal bundle $N_X M$ exists (for instance hypersurfaces or local complete intersections), one can define the virtual bundle in the Grothendieck group as

$$\tau_X = TM|_X - N_X M.$$

   This last viewpoint was generalized by Fulton and Johnson, using Segre classes.

Various authors were interested in comparing the diverse viewpoints. The Schwartz and MacPherson classes agree and are now called Chern-Schwartz-MacPherson classes. When all defined, these classes differ from the Fulton (and Fulton-Johnson) classes by classes named Milnor classes. That is the subject of the Callejas-Bedregal, Morgado, and Seade article [74] in this volume, providing several different definitions, applications and examples (also see [2]).

A remarkable point is that the natural transformation c_* defined by MacPherson from constructible functions to homology, allows to express a number of classes depending on the chosen constructible function (and on the situation). Besides the classes of Chern-Schwartz-MacPherson, obtained for the constructible function **1**, that is also the case, in suitable situations (see Sect. 5.23), for the Mather classes, Fulton classes, Milnor classes, and more specific classes such as the weighted Chern-Mather classes and the stringy Chern classes.

Segre classes play an important role in the development of characteristic classes: the local Euler obstruction is expressed in terms of Segre classes (Gonzalez-Sprinberg and Verdier). The Chern-Mather classes, the Chern-Schwartz-MacPherson classes, Fulton and Fulton-Johnson classes, and Milnor classes can be written, with nice expressions, in terms of Segre classes depending on particular cases. That is the subject of the article by Paolo Aluffi in this volume. showing how Segre classes provide a powerful viewpoint and fruitful developments for all these classes [13].

In the same way that the MacPherson Chern natural transformation generalizes the Chern class, the Todd class and the Thom-Hirzebruch class were generalized in the singular framework as natural transformations respectively by Baum-Fulton-MacPherson and by Cappell-Shaneson. Brasselet, Schürmann and Yokura show that the motivic framework allows to unify these three generalizations (Sect. 5.21). The theory is the subject of many generalizations and applications bringing important new developments. For a more complete description see Yokura [362].

Robert MacPherson and William Fulton developed the formalism of bivariant theories. These are simultaneous generalizations of covariant group valued "homology-like" theories and contravariant ring valued "cohomology-like" theories. They showed existence and uniqueness of Stiefel-Whitney classes in this formalism and conjectured the same for Chern classes. Several authors have partially proved the conjecture, bringing important results related to the other mentioned theories (Sect. 5.24.1). For a more complete description see Yokura [362].

Characteristic classes appear in many aspects of mathematics and physics. The present article does not intend to be complete and important topics are not present or only briefly mentioned, for instance (the list of topics is far from complete as well as the list of references):

- Chern-Weil theory and Čech-de Rham cohomology (see Suwa, in this volume [312] and [310, 311], and see [67]),
- Development of Segre classes (see Aluffi [13] in this volume)
- Development of Milnor classes (see Callejas-Bedregal, Morgado, and Seade [74] in this volume and [67])
- Lê cycles (see Massey [210] and Callejas-Bedregal, Morgado and Seade [72, 73])
- Developments of motivic and bivariant theories (see Schürmann and Yokura [285] and Yokura to appear in volume IV [362])
- Positivity questions (see Aluffi, Mihalcea, Schürmann and Su [22, 23] and Jones [175, §6])

– The vast topic of characteristic classes of foliations (see for instance Feigin [123], Pittie [255] Suwa [309], Corrêa and Soares [91]),
– Thom polynomials (see Ohmoto [236, 237, 239]).
– Chern characters (see Baum-Fulton-MacPherson [34], M.-H. Schwartz [290], Kwieciński [189], Suwa [310] and [67, Chapter 13]).
– ...

Some Indications
Although an effort has been made to recall the important concepts evoked in this survey, it is preferable that the reader has elementary knowledge of algebraic topology and algebraic geometry. Thus, the basic ingredients used in the recipes which follow are, in particular, the concepts of homology, cohomology, homotopy, Chow group, manifolds, Grassmannian, fiber bundle, tangent vector fields, pseudomanifolds, stratifications. But don't be frightened by this "long list", some of these notions are recalled or a suitable reference is provided.

Complementary articles to this survey are the notes of two courses given by Paolo Aluffi [10] and by Jörg Schürmann [278].

The First Part (Sects. 5.2 to 5.9) treats the smooth case; the Second Part (Sect. 5.10 to the end) treats the singular case. We use an unusual notation, and write upper indices for cohomology classes and lower indices for homology classes. Hopefully that will be more convenient for the reader.

I thank José Luis Cisneros-Molina, Lê Dũng Tráng and José Seade, editors of the Handbook of Geometry and Topology of Singularities for the invitation to write this survey and for the help to juggle the TEX. I thank UNESP (São José do Rio Preto), USP (São Carlos) and IMPA (Rio de Janeiro) for hospitality during previous work on the subject.

A lot of thanks to all colleagues who supported me by their valuable comments, in particular P. Aluffi, G. Barthel, K.-H. Fieseler, L. Kaup, Thủy Nguyễn T.B., M.A.S. Ruas, J. Schürmann, J. Seade, S. Yokura and a lot of thanks to the anonymous referee for the precious, unvaluable and fantastic report.

5.2 First Part: The Smooth Case, From Pythagoras to Chern

The five convex regular polyhedra were discovered by Pythagoras of Samos (∼ 570–495 B.C.) for the tetrahedron, hexahedron (cube) and octahedron and by Theaetetus of Athens (∼415–365 B.C.) for the icosahedron and dodecahedron. Described by Plato (∼428–348 B.C.) in his philosophical dialogue "Timaeus", they are known as the five Platonic poyhedra or the five Platonic solids.

The Euler-Poincaré characteristic is recognized as the first, if not the embryo of the characteristic classes. This is not the place to enter into the priority controversy of the so-called "Euler formula". Let us only give here the facts corresponding to known manuscripts (an extended history appears for instance in [45]).

Francesco Maurolico (1494–1575), an Italian priest, who lived in Messina, Sicily was interested in planar representations of the five Platonic poyhedra. In his manuscript "*Compaginationes solidorum regularium*" (1537), Maurolico observed the so-called "Euler formula" for Platonic solids (see the very documented thesis by Claudia Addabbo [1, Pages 291 and 295]).

Formula (Maurolico, December 26, 1537) *"Item manifestum est in unoquoque regularium solidorum, numerum basium coniunctum cum numero cacuminum conflare numerum, qui binario excedit numerum laterum."*
"In the same way it is evident that, in each regular solid, the number of faces added to that of the vertices exceeds by two the number of edges," *i.e. Consider a Platonic solid with V vertices, E edges and F faces, then*

$$V - E + F = 2. \tag{5.1}$$

Descartes died in Stockholm on February 11, 1650 without having published his manuscript *De solidorum elementis*. The manuscript suffered many mishaps (see [26]) and finally the original was lost. Fortunately Leibniz made a copy of it during a stay in Paris (1672–1676) with the (unfinished) project of publishing the works of Descartes.

In the year 1883, Foucher de Careil, France's Ambassador to Austria-Hungary and author of several articles on Descartes and Leibniz discovered in Hanover, between Leibniz's documents the copy of Descartes manuscript "under the ancient dust that covered them". The Descartes' Theorem says:

Theorem 5.2.1 (Descartes) *The sum of the angles of all faces of a convex polyhedron is equal to $2(V-2)\pi$ where V is the number of vertices.*

The book by Pierre Costabel [92] provides a critical, annotated and documented edition of the Descartes manuscript.

On November 14, 1750, in a letter to his friend Christian Goldbach, Leonhard Euler mentioned his discovery:

Formula (Euler, 1750) In a convex polyhedron with V vertices, E edges and F faces, one has

$$V - E + F = 2. \tag{5.2}$$

The proof of the formula, given by Euler suffers a flaw (it was corrected in Francese and Richeson [124]). Adrien-Marie Legendre (1752–1833) gave the first correct proof using a projection of the polyhedron on a sphere [198]. In 1811, Augustin-Louis Cauchy provided the first combinatorial proof of the formula [83, 84]. The Cauchy's proof was criticized [192, 199]. A proof using tools known at the Cauchy's time is given in Brasselet and Thủy Nguyễn [57].

In fact, it is easy to see that Descartes' Theorem is equivalent to Euler Formula (see for instance [57]).

In 1885, Poincaré generalized the Euler characteristic: For any finite CW-complex (or triangulated space) X of dimension n, the Euler-Poincaré characteristic is defined as the alternating sum

$$\chi(X) = k_0 - k_1 + k_2 + \cdots + (-1)^n k_n$$

where k_i is the number of i-dimensional cells (or simplexes). The sum is independent of the cell decomposition (or of the triangulation) of X.

5.3 Poincaré-Hopf Theorem

In 1899 Henri Poincaré [256] for surfaces and in 1927, Heinz Hopf [167] for higher dimensions, show that the Euler-Poincaré characteristic of a compact smooth manifold M is the obstruction to the construction of a vector field tangent to M.

Theorem 5.3.1 (Poincaré-Hopf) *Let M be a compact manifold with boundary ∂M, and let v be a continuous vector field tangent to M with isolated singularities. Denote by $a_i \in Sing(v)$ the singularities of v and $I(v, a_i)$ their indices. Then, if v is pointing outwards of M along ∂M,*

$$\chi(M) = \sum_{a_i \in Sing(v)} I(v, a_i)$$

and if v is pointing inwards of M along ∂M,

$$\chi(M) - \chi(\partial M) = \sum_{a_i \in Sing(v)} I(v, a_i).$$

Many proofs of the Poincaré-Hopf Theorem appear in the literature. One will mention the one in Milnor's book [223], using non-degenerated singularities. This proof can be slightly simplified (for any indices) using the method by Marie-Hélène Schwartz (see Sect. 5.12.3 and Brasselet and Thủy Nguyễn [58]).

5.4 Poincaré and Alexander Duality Theorems

Poincaré and Alexander duality Theorems are useful in the generalization of characteristic classes to singular varieties (Sect. 5.14.1). In particular the dual cell decomposition used by Poincaré to prove his famous duality Theorem [256] will be used in Sect. 5.12.1 (also see in this volume [312, §1.2.2 and 1.9.3]).

5.4.1 Poincaré Duality Theorem

One denotes by (K) a triangulation of an n-dimensional triangulated manifold M. A dual cell decomposition of M is obtained in the following way: Considering a barycentric subdivision (K') of (K), the barycenter of a simplex $\sigma \in K$ is denoted by $\widehat{\sigma}$. Every simplex in (K') can be written as

$$(\widehat{\sigma}_{i_1}, \widehat{\sigma}_{i_2}, \dots, \widehat{\sigma}_{i_p})$$

where $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_p}$, the symbol $\sigma < \sigma'$ meaning that σ is a face of σ'.

The dual cell of a simplex σ, denoted by $d(\sigma)$, is the union of all (closed) simplexes τ in (K') such that $\tau \cap \sigma = \{\widehat{\sigma}\}$. That is the union of (geometric) simplexes on the form $(\widehat{\sigma}, \widehat{\sigma}_{k_1}, \dots, \widehat{\sigma}_{k_i})$ with $\sigma < \sigma_{k_1} < \cdots < \sigma_{k_i}$.

In a manifold, the dual cells satisfy the nice properties (see for instance [231]).

Lemma 5.4.1

1. *The dual cell of an i-simplex is an $(n-i)$-cell, homeomorphic to the unit ball $\mathbb{B}^{n-i} \subset \mathbb{R}^{n-i}$ and its boundary is homeomorphic to the corresponding sphere $\mathbb{S}^{n-i-1}$.*
2. *The set of dual cells provide a cell decomposition (D) of M, called dual cell decomposition associated to the barycentric subdivision (K') of (K).*

The unique intersection point $\widehat{\sigma} = d(\sigma) \cap \sigma$ is the barycenter of $d(\sigma)$. It is also the barycenter of σ and will be denoted by

$$\widehat{d} = \widehat{d}(\sigma). \tag{5.3}$$

The cellular decomposition (D) of the manifold M provides the manifold a structure of CW-complex and allows to compute homology or cohomology.

Let us assume $M = |K|$ oriented, that is all n-simplexes are given a compatible orientation. Other simplexes are arbitrarily oriented. One gives to every cell $d(\sigma)$ the orientation such that orientation of $d(\sigma)$ followed by orientation of σ is orientation of M (see Brasselet [40] and Suwa [309]).

- The elementary (D)-cochain whose value is 1 at the cell $d(\sigma)$ and 0 at other cells of (D) is denoted by $d^*(\sigma)$
- The groups of i-dimensional simplicial (K)-chains with integer coefficients are denoted by $C_i^{(K)}$ and the groups of k-dimensional simplicial (D)-cochains with integer coefficients by $C^k_{(D)}$.

Let M be a compact oriented n-dimensional manifold, then one has, for every i, a chain isomorphism:

$$D : C^{n-i}_{(D)}(M; \mathbb{Z}) \longrightarrow C_i^{(K)}(M; \mathbb{Z}), \tag{5.4}$$

that is defined on the elementary elements as

$$d^*(\sigma) \quad \mapsto \quad \sigma.$$

Let c^p a p-dimensional (D)-cell, the coboundary of the elementary cochain c^p is the sum

$$\delta(c^p) = \sum [c^p, c_i^{p+1}]\, c_i^{p+1},$$

where the sum is over the $(p+1)$ cells c_i^{p+1} whose boundary contains the cell c^p and the incidence sign $[c^p, c_i^{p+1}]$ is $+1$ if c^p appears in the boundary of c_i^{p+1} with orientation of the boundary and -1 otherwise.

The chain isomorphism 5.4 commutes with coboundary and boundary, and this provides one of the possible forms of Poincaré duality:

Theorem 5.4.2 (Poincaré Isomorphism) *[256] Let M be a compact oriented n-dimensional manifold, the morphism (5.4) induces, for every i, $0 \leq i \leq n$, an isomorphism*

$$H^{n-i}(M; \mathbb{Z}) \longrightarrow H_i(M; \mathbb{Z}),$$

which is the cap-product with the fundamental class $[M] \in H_n(M; \mathbb{Z})$.

5.4.2 Alexander Duality Theorem

Let M be an n-dimensional triangulated manifold. We consider a triangulation (K) of M compatible with a compact subspace X.

Definition 5.4.3 A *cellular tube* $\mathcal{T}$ around X in M is the union of (closed) cells (D) which are dual of (K)-simplexes situated in X.

This notion generalizes the concept of tubular neighbourhood of a submanifold X. If X is a submanifold, then $\mathcal{T}$ is a bundle around X, whose fibers are discs. In general (in the singular situation), that is not the case.

Remark 5.4.4 A cellular tube $\mathcal{T}$ around X has the following properties:

(i) $\mathcal{T}$ is a compact neighbourhood of X, containing X in its interior and its boundary $\partial\mathcal{T}$ is a retract of $\mathcal{T} \setminus X$.
(ii) $\mathcal{T}$ is a *regular* neighbourhood of X, thus $\mathcal{T}$ retracts to X.

The Alexander isomorphism

$$H^{n-i}(M, M \setminus X) \to H_i(X)$$

is defined in the following way (see [40], also see [312]): By definition, $\mathcal{T}$ is the union of all cells $d(\sigma)$ which are duals of simplexes σ in X. The boundary $\partial\mathcal{T}$ of $\mathcal{T}$ is the union of the dual cells $d(\tau)$ in $\mathcal{T}$ such that τ is not a simplex in X, that implies $d(\tau) \cap X = \emptyset$.

Elements of the relative group of cochains

$$C^{n-i}_{(D)}(\mathcal{T}, \partial\mathcal{T}) \tag{5.5}$$

are linear combinations (with integer coefficients) of elementary $(n-i)$-dimensional (D)-cochains $d^*(\sigma)$ whose value is 1 on the cell $d(\sigma)$, and 0 on other cells.

The correspondence

$$C^{n-i}_{(D)}(\mathcal{T}, \partial\mathcal{T}) \to C^{(K)}_i(X) \tag{5.6}$$

which associates to an elementary $(n-i)$-dimensional (D)-cochain $d^*(\sigma)$ the i-dimensional K-chain σ is an isomorphim and induces the isomorphism

$$H^{n-i}(\mathcal{T}, \partial\mathcal{T}) \to H_i(X).$$

Considering the isomorphisms

$$H^{n-i}(\mathcal{T}, \partial\mathcal{T}) \cong H^{n-i}(\mathcal{T}, \mathcal{T} \setminus X) \cong H^{n-i}(M, M \setminus X) \tag{5.7}$$

the first one obtained by retraction of $\mathcal{T} \setminus X$ on $\partial\mathcal{T}$ and the second by excision, the Alexander isomorphism is the resulting composition:

$$H^{n-i}(M, M \setminus X) \longrightarrow H_i(X). \tag{5.8}$$

5.5 Stiefel-Whitney Classes

5.5.1 Stiefel Manifolds

In the real case, the Stiefel manifold, denoted by $V_r(\mathbb{R}^n)$ is the set of r-frames in $\mathbb{R}^n$, that is the set of ordered r-uples $(v_1, \ldots, v_r)$ of r linearly independent vectors in $\mathbb{R}^n$. (see Steenrod [304] where this manifold is denoted by $V'_{r,n}$). The Stiefel manifold $V_r(\mathbb{R}^n)$ is homotopic to

$$V_{r,n} = O(n)/O(n-r).$$

The natural map

$$V_{r,n} \to G_r(\mathbb{R}^n) = O(n)/(O(n-r) \times O(r))$$

is a principal fiber bundle.

The Grassmannian manifold $G_r(\mathbb{R}^n)$ is the set of r-dimensional linear subspaces in $\mathbb{R}^n$. The *tautological bundle* (also called canonical bundle) η_r^n over $G_r(\mathbb{R}^n)$ is the set of all pairs $\{(P, v)\}$ where P is an element of $G_r(\mathbb{R}^n)$ and v a vector in P. The bundle

$$\eta_r^n \to G_r(\mathbb{R}^n) \tag{5.9}$$

is a vector bundle with rank r, associated to the bundle $V_{r,n} \to G_r(\mathbb{R}^n)$, and with fiber $\mathbb{R}^r$. The bundle is also the *universal bundle* for vector bundles of rank r.

The bundle

$$\eta = \eta_1^2 \to G_1\mathbb{R}^2) = \mathbb{RP}^1 \cong \mathbb{S}^1 \tag{5.10}$$

is the tautological line bundle.

The bundle $V_r(TM)$ of r-frames tangent to a n-differentiable manifold M, is the set of all pairs $(x, (v_1, \dots, v_r))$ where x is a point of M and $(v_1, \dots, v_r)$ is a r-frame in the fiber $T_x M$ over x. That is the fiber bundle over M whose fiber at x is the manifold $V_r(T_x M)$ of all r-frames in $T_x M$. The "typical" fiber is the Stiefel manifold $V_r(\mathbb{R}^n)$. Its homotopy groups are [304, §25.6]:

$$\pi_i(V_r(\mathbb{R}^n)) = \begin{cases} 0 & \text{for } p < n - r + 1 \\ \mathbb{Z} & \text{for } p = n - r + 1 \text{ odd or } p = n \text{ if } r = 1 \\ \mathbb{Z}_2 & \text{for } p = n - r + 1 \text{ even and } r > 1. \end{cases} \tag{5.11}$$

5.5.1.1 Stiefel Manifolds in Engineering and Other Sciences

Stiefel manifolds are used in engineering, image and video-based recognition, econometrics, statistical signal processing etc. The main problem studied in this context consists of estimating the state of a stochastic differential equation in a Stiefel manifold. A partial list of references is [85, 205, 206, 321, 326, 345].

5.5.2 Stiefel-Whitney Classes

The Stiefel-Whitney classes were defined in 1935, at the same time and independently, using obstruction theory, by Stiefel [305] (see [304, §39]) and by Whitney [329, 330] (see [304, §38]). Stiefel used the tangent bundle of a manifold and obtained important results, including the fact that a closed orientable manifold of dimension 3 is parallelizable. Whitney used the same strategy as Stiefel, applying it to arbitrary sphere bundles. The notion of sphere bundle, due to Whitney, leads to algebraic operations on bundles (also see Wu [337, 339]). We follow the description given by Steenrod ([304], part III).

The pth (cohomology) Stiefel-Whitney class of M, denoted by $w^p(M)$, is defined as the obstruction to constructing a tangent r-frame over M, that is a section of $V_r(TM)$ with $r = n - p + 1$.

Using the result in formula (5.11) one can construct such an r-frame by choosing any r-frame $v^{(r)}$ on the 0-skeleton of the cell decomposition (D), then extending it without zeroes till the obstruction dimension $p = n - r + 1$. Then $v^{(r)}$ has no singularity on the $(p-1)$-skeleton and isolated singularities on the p-skeleton of (D). Given the r-frame $v^{(r)}$ on the boundary of each p-cell d, one extend $v^{(r)}$ on d with a singularity at the barycenter $\widehat{d}$ of index

$$I(v^{(r)}, \widehat{d}) = [(v^{(r)})_{p-1}|_{\partial d^p}] \in \pi_{p-1}(V_r(\mathbb{R}^n)).$$

Since $\pi_{p-1}(V_r(\mathbb{R}^n))$ is either infinite-cyclic or isomorphic to $\mathbb{Z}_2$, the coefficients can be reduced modulo 2 obtaining $I(v^{(r)}, \widehat{d}) \in \mathbb{Z}_2$. For alternative definitions of index, see [74, 113]. That defines a p-cochain $\sum I(v^{(r)}, \widehat{d})\, d^*$ in $C^p(D, \mathbb{Z}_2)$, by prescribing that its value on each p-cell d is $I(v^{(r)}, \widehat{d})$. According to general obstruction theory [170, 304], the cochain is a cocycle and defines an element $w^p(M)$ in $H^p(M; \mathbb{Z}_2)$.

Definition 5.5.1 The p-th *Stiefel-Whitney class* of the differentiable manifold M, denoted by $w^p(M) \in H^p(M; \mathbb{Z}_2)$ is the class of the primary obstruction cocycle corresponding to constructing an r-frame tangent to M.

By the general obstruction theory [170, 304], the resulting classes do not depend on the choices made in the construction.

Interesting comments about Stiefel classes [304, §39] and Whitney classes [304, §38] were provided by Steenrod, as well as comments on the contributions by Thom and Wu.

In the particular case $r = 1$, one can use integer coefficients. The evaluation of $w^n(M) \in H^n(M; \mathbb{Z})$ on the fundamental class $[M]$ of M is the Euler-Poincaré characteristic of M.

5.5.3 Combinatorial Definition

A combinatorial definition of the Stiefel-Whitney classes was already conjectured by E. Stiefel [305]. Then H. Whitney wrote a proof for a book which did not appear. G. Cheeger (1968) provided a sketch of proof using different techniques [87] and the complete proof appeared in a paper by Halperin and Toledo [162].

Parts of the proof have been obtained independently by Sullivan [307] who uses the results in a more general situation (also see [139, 140]).

Let M be a differentiable n-manifold without boundary and K a differentiable triangulation of M. Let K' denote the first barycentric subdivision of K. Each K'-simplex τ is written in an unique way as $\tau = \langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle$ where $\sigma_0 < \cdots < \sigma_k \in K$ (see Sect. 5.4.1). Each simplex τ is given the orientation for which $\langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle$ is a positive ordering of the vertices.

An infinite integral simplicial k-chain on M is a formal infinite integral combination $\sum \lambda_\sigma \sigma$ where the sum runs over the k-simplexes of K', oriented with the previous order.

Theorem 5.5.2 *[162] The infinite chain*

$$w_k(M) = \sum_{\sigma_0 < \cdots < \sigma_k} \langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle \tag{5.12}$$

is a (mod 2)*-cycle. It represents the* k^{th} (mod 2) *Stiefel-Whitney homology class of* M.

In [129], J.H.G. Fu and C. McCrory give a new proof of the combinatorial formula for Stiefel-Whitney classes.

5.5.4 Grassmannian and Schubert Cycles

Lev Pontryagin [257, 258, 341] introduced the idea of defining Stiefel-Whitney classes as images of the cohomology classes of Grassmannian manifolds with coefficients in $\mathbb{Z}_2$. He considered cellular decomposition of the "special" Grassmannian $\widetilde{G}_{r,n}$ of oriented vector subspaces of dimension r in $\mathbb{R}^n$. The Pontryagin cellular decomposition goes onto the one of $G_r(\mathbb{R}^n)$ constructed by Ehresmann [115].

The Ehresmann cellular decomposition uses the algebraic subvarieties of the Grassmannian variety introduced by H. Schubert in 1889 [274] (see [102, Part 2, Chapter V, §4 B]). For every $k < n$ a Schubert symbol (μ) of order k is a sequence of k integers μ_i such that

$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq n. \tag{5.13}$$

Denoting by $\mathbb{R}^h$ the vector subspace of $\mathbb{R}^n$ spanned by the first h vectors of the canonical basis, to each Schubert symbol (μ), one associates the subset $e(\mu)$ of $G_r(\mathbb{R}^n)$ consisting of the r-dimensional vector subspaces L such that

$$\dim(L \cap \mathbb{R}^{\mu_i}) = i, \qquad \dim(L \cap \mathbb{R}^{\mu_i - 1}) = i - 1 \qquad \text{for} \quad 1 \leq i \leq k.$$

Ehresmann shows that the homology groups $H_k(G_r(\mathbb{R}^n); \mathbb{Z}_2)$ are free $\mathbb{Z}_2$-modules whose basis consists of homology classes of the Schubert varieties $\overline{e(\mu)}$, which are the closure of the subsets $e(\mu)$ such that $(\mu_1 - 1) + (\mu_2 - 2) + \cdots + (\mu_k - k) = k$.

Wu Wen-Tsün, in his French thesis [338], presented in a simpler way the Pontryagin construction. Dieudonné, in [102, Part 3, Chapter IV, §1 C], provides a nice account of the Wu Wen-Tsün construction. Wu provides also a computation of Steenrod squares in Grassmannian varieties (Sect. 5.6.1, also see [43, 336, 340]).

Chern considered the tautological bundle η_r^n with basis $G_r(\mathbb{R}^n)$ (see formula 5.9). Using differential forms, Chern determined in 1947 [89] the cohomology algebra $H^*(G_r(\mathbb{R}^n); \mathbb{Z}_2)$ and proved that the Stiefel-Whitney classes of the tautological bundle η_r^n form a system of generators for that algebra.

5.5.5 Axiomatic Definition

The Stiefel-Whitney classes of a manifold were defined as obstruction classes of the tangent bundle $E = TM$ and the associated bundles of frames $V_r(TM)$. The obstruction theory applies as well to any real vector bundle E over a triangulated space X. Note that X does not need to be smooth and can be a CW-complex.

In the same way as for the tangent bundle of a manifold, we construct r everywhere independent sections of the bundle E with n-dimensional fiber, without obstruction on the $(n-r)$—skeleton of the given CW-structure of X and with singularities of index $I(v^{(r)}, \widehat{d}) \in \pi_{p-1}(V_r(\mathbb{R}^n))$ on the $p = n - r + 1$ cells d. The data

$$d \mapsto I(v^{(r)}, \widehat{d})$$

define a cocycle in $C^p(X; \pi_{p-1}(V_r(\mathbb{R}^n)))$. and a class $\widehat{w}^p(E)$ in the p-th simplicial (or cellular) cohomology of X with twisted coefficients, the coefficient system being the homotopy group $\pi_{p-1}(V_r(\mathbb{R}^n))$.

$$\widehat{w}^p(E) \in \begin{cases} H^p(X; \mathbb{Z}) & \text{if } p \text{ is odd or } p = n, \\ H^p(X; \mathbb{Z}_2) & \text{if } p \text{ is even and } p < n. \end{cases}$$

Whitney proved that $\widehat{w}^p(E) = 0$ if and only if E, when restricted to the p-th skeleton of X, admits $r = (n - p + 1)$ linearly-independent sections.

Definition 5.5.3 The Stiefel-Whitney classes of the real vector bundle E with n-dimensional fiber, on the triangulated (or CW) space X are the reduced classes modulo 2 of $\widehat{w}^p(E)$,

$$w^p(E) \in H^p(X; \mathbb{Z}_2).$$

In his 1940 paper, Whitney states (for sphere bundles) the formula providing classes $w^p(E \oplus E')$ of the sum of two bundles E and E' over the same base space B.

$$w^p(E \oplus E') = \sum_{i+j=p} w^i(E) \smile w^j(E') \tag{5.14}$$

Whitney writes that, for $p \geq 4$, the proof is very hard, and gives little information on the proof. He proved, in 1941, the formula for line bundles.

In 1948, Chern Shiing Shen and Wu Wen-Tsün published the first complete proofs of the formula (5.14) for vector bundles, both in the same volume of Annals of Mathematics [88, 334]. The Wu's proof is very well summarized in Dieudonné [102, p. 424].

The formula (5.14) is one of the axioms entering in the axiomatic definition of Stiefel-Whitney classes (see [166, 224]).

Definition 5.5.4 Axiomatic definition of (cohomology) Stiefel-Whitney classes.

Let E be a real vector bundle of (finite) rank n over a (paracompact) space X. The (total) Stiefel-Whitney characteristic class $w(E) \in H^*(X; \mathbb{Z}_2)$ of a finite rank real vector bundle E is the unique class such that the following axioms are fulfilled.

1. One has $w(E) = 1 + w^1(E) + \cdots + w^n(E)$, where $w^i(E) \in H^i(X; \mathbb{Z}_2)$ and $w^i(E) = 0$ if $i > n$.
2. (Naturality) If $f : Y \to X$ is a continuous map, then $f^*(w(E)) = w(f^*(E))$ where $f^*(E)$ is the "pull-back" vector bundle on Y.
3. (Whitney-Wu sum) If E and E' are two bundles over X, then
$$w(E \oplus E') = w(E) \cup w(E').$$
4. Let η (see formula 5.10) be the tautological line bundle over $\mathbb{RP}^1 = \mathbb{S}^1$, then $w^1(\eta)$ is the non zero element in $H^1(\mathbb{S}^1; \mathbb{Z}_2)$.

Note that, by definition, $w^0(E) = 1$.

5.5.6 *Stiefel-Whitney Class and Thom Class*

Let E be a rank-n-dimensional vector bundle over a paracompact space X with projection map $\pi : E \to X$. Let $E^* = E \setminus s_0(X)$ be the complement of the zero section in E. Then there exists a unique cohomology class $u_E \in H^n(E, E^*; \mathbb{Z}_2)$, called the Thom class, such that $u_E|_{(F_x, F_x \setminus \{0\})} \neq 0$ for all fibers F_x, (also see [312, §1.9]).

The Thom isomorphism

$$\Phi : H^i(X; \mathbb{Z}_2) \to H^{i+n}(E, E^*; \mathbb{Z}_2)$$

is defined by $\Phi(x) = \pi^* x \cup u_E$. One has $\Phi(1) = u_E$.

Let $1 \in H^0(X;\mathbb{Z}_2)$, then the i-th Stiefel-Whitney class $w^i(E) \in H^i(X;\mathbb{Z}_2)$ is equal to

$$w^i(E) = \Phi^{-1} Sq^i \Phi(1). \tag{5.15}$$

where the Steenrod squares $Sq^i : H^k(A,B;\mathbb{Z}_2) \to H^{k+i}(A,B;\mathbb{Z}_2)$ are defined in [303] (see [82, exp. 14 and 15]). That is

$$\Phi(w^i(E)) = \pi^* w^i(E) \cup u_E = Sq^i(u_E).$$

5.5.6.1 Application: The Thom Theorem

Two manifolds M and N are called cobordant if there is a compact manifold W whose boundary is the disjoint union of M and N, i.e. $\partial W = M \sqcup N$. All manifolds cobordant to a fixed given manifold M form the cobordism class of M. Cobordism is a fundamental equivalence relation on the class of compact manifolds of the same dimension.

The Stiefel-Whitney numbers of an (unoriented) closed n-dimensional manifold M are defined as

$$\left\langle w^{i_1}(M) \cup \cdots \cup w^{i_k}(M), [M] \right\rangle \in \mathbb{Z}_2 \tag{5.16}$$

for any collection $(i_1, \cdots, i_k)$ of integers such that $i_1 + \cdots + i_k = n$.

These numbers are known to be cobordism invariants. It was proved by Lev Pontryagin [258] that if M is the boundary of a smooth compact $(n+1)$-dimensional manifold, then the Stiefel-Whitney numbers of M are all zero. Later on, the converse was proved by René Thom [318].

Theorem 5.5.5 *[318] A smooth compact manifold M is the boundary of some smooth compact (unoriented) manifold if and only if all the Stiefel-Whitney numbers of M vanish.*

Stong in [306] introduced and studied a notion of cobordism for maps $f : X \to Y$ of closed smooth manifolds. He defined Stiefel-Whitney numbers for a map and presented a formula using cohomology groups with Z_2 coefficients to prove that two maps are cobordant if and only if they have the same characteristic numbers.

5.6 (Real) Wu Classes

Let X be a topological space with fundamental class $[X] \in H_n(X;\mathbb{Z}_2)$ and such that one has a Poincaré isomorphism $H^{n-i}(M;\mathbb{Z}_2) \longrightarrow H_i(M;\mathbb{Z}_2)$ given by $z \mapsto$

$z \cap [X]$. That is the case of n-dimensional compact manifold and more generally $\mathbb{Z}_2$-homology manifolds.

A $\mathbb{Z}_2$-homology manifold is a locally compact topological space X such that, for all $x \in X$, the link[2] of x has the same $\mathbb{Z}_2$ homology as an $(n-1)$-sphere, or equivalently,

$$H_p(X, X \setminus \{x\}; \mathbb{Z}_2) = \begin{cases} 0 & \text{if } p \neq n, \\ \mathbb{Z}_2 & \text{if } p = n. \end{cases}$$

Via the Kronecker pairing

$$\langle \cdot, \cdot \rangle : H^i(X; \mathbb{Z}_2) \times H_i(X; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2,$$

there is an isomorphism

$$\mathrm{Hom}(H^{n-i}(X; \mathbb{Z}_2), \mathbb{Z}_2) \cong H_{n-i}(X; \mathbb{Z}_2) \cong H^i(X; \mathbb{Z}_2).$$

Under this isomorphism, the homomorphism $x \mapsto \langle Sq^i(x), [X] \rangle$ from $H^{n-i}(X; \mathbb{Z}_2)$ to $\mathbb{Z}_2$ corresponds to a well defined cohomology class $v^i(X) \in H^i(X; \mathbb{Z}_2)$, such that

$$Sq^i(x) = v^i(X) \cup x, \qquad \text{for any} \quad x \in H_c^{n-i}(X; \mathbb{Z}_2) \tag{5.17}$$

(cohomology with compact supports).

The class $v^i(X)$ is called the i-th Wu class of X. In the original papers by Wu [333, 334] the class was denoted U^i (also see Milnor [224, §11]). One says that the Wu class v^i is the class that represents Sq^i under the cup product.

According to a "result of Wu" (Wu wrote and confirmed to me that this result comes from Cartan), in the case of an n-dimensional orientable manifold M, then $v^{2k+1}(M) = 0$ for all k.

5.6.1 Siefel-Whitney Classes and Wu Classes

Let X be an n-dimensional $\mathbb{Z}_2$-homology manifold, Wu defined the classes

$$\tilde{w}^i(X) = \sum_{k=0}^{i} Sq^k(v^{i-k}(X)), \qquad \text{for } 0 \leq i \leq n. \tag{5.18}$$

[2] Given a triangulation of X for which x is a vertex, the link of x is the union of simplexes τ which are faces of simplexes σ whose x is a vertex but such that x is not a vertex of τ.

These classes are denoted by W^i and called W-classes in the original paper by Wu [333].

Theorem 5.6.1 (Wu [333]) *Let M be a compact n-dimensional manifold. Then the classes $\tilde{w}^i(M)$ coincide with the Stiefel-Whitney classes $w^i(M)$ of the tangent bundle to M. One has*

$$w^i(M) = \tilde{w}^i(M).$$

Let X be an n-dimensional $\mathbb{Z}_2$-homology manifold, the Steenrod squares of Stiefel-Whithey classes are given by the famous Wu's formula [335]

$$Sq^k(w^i) = \sum_{t=0}^{k} \binom{i-k+t-1}{t} w^{k-t} \cup w^{i+t}. \tag{5.19}$$

Wu himself defined other characteristic classes for cohomology with coefficients in $\mathbf{F}_p = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for p an odd prime. These classes played an important role in the developments of the theory of fibrations.

Atiyah and Hizebruch [25] (also see [185]) generalized the notion of Wu classes. The Yoshida-Stong formulae [344] provide an expression of the "Universal Wu class" v^i in terms of the Stiefel-Whitney classes modulo an ideal.

5.6.2 Wu Classes in Mathematical Physics

The first mention of Wu classes in Mathematical Physics appears in the paper of Hopkins and Singer [168] where Wu-structures are defined. Also see the paper by Belov-Moore [36] where the authors extend the work of Witten [332] which uses the fact that on eight-dimensional Spin varieties there is a degree 4 characteristic class which lifts the Wu class in degree 4. However, Witten does not mention explicitly the Wu class.

From there, the notion of Wu-structure has been defined in Mathematical Physics by various authors (see [168, 230, 272]). Roughly speaking, the degree 2 Wu-structures are spin structures on oriented manifolds. They often appear in Physics but their generalization in higher degree is not so frequent.

In the work of Hisham Sati [272], the Wu classes and Wu structures appear in string theory. In the work of Samuel Monnier [230], the Wu structures appear in order to generalize the theories of Chern-Simons spin in high degree.

Wu classes and Wu structures relative to a local system are defined and studied in [230]. The fact that on $(8k+2)$-spaces, the Wu class v^{4k+2} always vanishes is useful for the study of M5-branes as well as type IIB-string theory [273].

5.7 Chern Classes

5.7.1 Complex Stiefel Manifolds

In the complex case, the complex Stiefel manifold, denoted by $V_r(\mathbb{C}^n)$ is the set of r-frames in $\mathbb{C}^n$, that is the set of ordered r-uples $v^{(r)} = (v_1, \dots, v_r)$ of r linearly independent vectors in $\mathbb{C}^n$. The Stiefel manifold $V_r(\mathbb{C}^n)$ is homotopic to

$$W_{r,n} = U(n)/U(n-r).$$

The fiber bundle $W_{r,n} \to G_r(\mathbb{C}^n)$ is a principal bundle with fiber and structural group $U(r)$.

The tautological bundle (also called canonical bundle) γ_r^n over the complex Grassman manifold $G_r(\mathbb{C}^n)$ is the set of all pairs $\{(P, v)\}$ where P is an r-plane in $\mathbb{C}^n$ and v a vector in P. One has the bundle projection

$$\gamma_r^n \to G_r(\mathbb{C}^n).$$

If $r = 1$, then $G_1(\mathbb{C}^n)$ is the projective space $\mathbb{CP}^{n-1}$ and the tautological bundle is a line bundle

$$\gamma_1^n \to \mathbb{CP}^{n-1}, \tag{5.20}$$

also denoted by $\mathcal{O}(-1)$.

In the complex case, one defines $V_r(TM)$, the bundle of complex r-frames tangent to the complex n-manifold M, i.e. the set of all pairs $(x, (v_1, \dots, v_r))$ where x is a point of M and $v^{(r)} = (v_1, \dots, v_r)$ is a complex r-frame in the fiber T_xM over x. That is the fiber bundle whose fiber at x is the manifold $V_r(T_xM)$ consisting of all complex r-frames in T_xM. The "typical" fiber is the complex Stiefel manifold $V_r(\mathbb{C}^n)$.

5.7.2 Chern Classes by Obstruction Theory

In his fundamental paper [88], Chern provides several equivalent definitions of Chern classes for complex Hermitian manifolds, among them the definition by obstruction theory. As Chern wrote, the definition of Chern classes by obstruction theory in the complex case is similar to the real case, even simpler.

Let M denote an analytic complex manifold of (complex) dimension n and TM the complex tangent bundle to M. The pth Chern class of M, denoted by $c^p(M)$, is defined as the obstruction to constructing a complex r-frame over M, that is a section of $V_r(TM)$ or a set of r linearly independent complex vector fields tangent to M, with $p = n - r + 1$.

Using the computation of the homotopy groups of the Stiefel complex manifold [304, §25.7],

$$\pi_i(V_r(\mathbb{C}^n)) = \begin{cases} 0 & \text{for } i < 2n - 2r + 1 \\ \mathbb{Z} & \text{for } i = 2n - 2r + 1, \end{cases} \tag{5.21}$$

one can construct an r-frame by choosing any r-frame $v^{(r)}$ on the 0-skeleton of the cell decomposition (D), then extending it without zeroes till the obstruction dimension

$$2p = 2(n - r + 1). \tag{5.22}$$

That means that $v^{(r)}$ has no singularity on the $(2p - 1)$-skeleton and isolated singularities on the $2p$-skeleton of (D). The r-frame $v^{(r)}$ defined on the boundary of each $2p$-cell d, can be extended on d with a singularity at the barycenter $\widehat{d}$ of index

$$I(v^{(r)}, \widehat{d}) = [(v^{(r)})_{2p-1}|_{\partial d^{2p}}] \in \pi_{2p-1}(V_r(\mathbb{C}^n)) = \mathbb{Z}.$$

For alternative definitions of index, see [74, 113, 312]. The generators of $\pi_{2p-1}(V_r(\mathbb{C}^n))$ being consistent (see [304, §41.3]), one can define a $2p$-cochain $\sum I(v^{(r)}, \widehat{d})\, d^*$ in $C^{2p}(D, \mathbb{Z})$ whose value on each $2p$-cell d is $I(v^{(r)}, \widehat{d})$. According to general obstruction theory, the cochain is a cocycle and defines an element $c^p(M)$ in $H^{2p}(M; \mathbb{Z})$.

Definition 5.7.1 The p-th *Chern class* of M, denoted by $c^p(M) \in H^{2p}(M; \mathbb{Z})$ is the class of the obstruction cocycle corresponding to the construction of a complex r-frame tangent to M.

By the general obstruction theory, the resulting classes do not depend on the choices made in the construction.

In the particular case $r = 1$, the evaluation of $c^m(M)$ on the fundamental class $[M]$ of M yields the Euler-Poincaré characteristic of M.

If the cell decomposition (D) is obtained by duality of a triangulation (K) of M, each $2p$-cell $d = d(\sigma)$ in (D) is dual of an $2(r-1)$-simplex σ in (K). By Poincaré duality (Sect. 5.4.1),

$$H^{2(n-r+1)}(M; \mathbb{Z}) \longrightarrow H_{2(r-1)}(M; \mathbb{Z})$$

the image of d^* is σ and image of $c^p(M)$ is the $2(r - 1)$-homology Chern class, denoted by $c_{r-1}(M)$. A cycle representing $c_{r-1}(M)$ is given by

$$\sum_{\dim \sigma = 2(r-1)} I(v^{(r)}, \widehat{d}(\sigma))\sigma. \tag{5.23}$$

5.7.3 More Definitions

5.7.3.1 By Chern

In his fundamental article [88], Chern gave, in particular, the definitions of classes in terms of Schubert cycles, differential forms, obstruction cocycles, differential forms of transgression.

The context of his first definition is the one of the complex sphere bundle. The context of his second definition is the one of fiber bundles in which Chern considers sections which are ordered sets of r linearly independent complex vectors (our context in Sect. 5.7.2). The context of the third definition is the one of sections which are ordered sets of r mutually perpendicular vectors of the sphere. Chern's observation (see [88, p. 101 and 103]) is that the three contexts are equivalent for the definition of characteristic classes of a complex manifold.

The first definition given by Chern uses two results, the first one is the construction of Charles Ehresmann (Chern's Theorem 3) describing Schubert varieties as the basis of cycles for complex Grassmannian manifolds. The second one (proved by Chern in the Theorems 1 and 2) shows that the Grassmannian of suitable dimension is a classifying space for (sphere) bundles of given rank.

Chern considers $H(n, N)$ the Grassmannian manifold of complex n planes in $\mathbb{C}^{n+N}$ which is therefore the one previously denoted by $G_n(\mathbb{C}^{n+N})$.

Chern provides a first definition of classes (Chern's Theorem 5), considering suitable Schubert varieties Z_r of dimension $2(Nn-n+r-1)$ to which he associates invariant differential forms Φ_r of degree $2p = 2(n-r+1)$ such that, for any cycle ζ of dimension $2p$ one has

$$KI(\zeta, Z_r) = \int_\zeta \Phi_r \tag{5.24}$$

where the Lefschetz's notation KI means the intersection Kronecker index, for transverse cycles of complementary dimensions.

Let M be a complex manifold of complex dimension n, if $f : M \to H(n, N)$ is the classifying map (Chern's Theorem 1), then the Chern classes of M are image, by the map $f^* : H^{2p}(H(n, N)) \to H^{2p}(M)$, of the classes of the cocycles defined by the invariant differential forms Φ_r.

The second Chern's definition [88, Theorem 7] is the one given in (§ 5.7.2, using obstruction theory.

The third Chern's definition [88, Theorem 8] introduces, for a bundle of complex spheres $S(n)$ over the complex manifold M, the associated fiber bundles $\mathcal{F}^{(r)*}$ over M whose fiber at each point is the manifold $U^*(n, r)$ of all ordered sets of $r(1 \leq r \leq n)$ mutually perpendicular complex vectors of $S(n)$.

Theorem 5.7.2 (Chern's Theorem 8) *Each of the cocycles γ of the Chern $2p = 2(n-r+1)$-cohomology class of M, has the following property. Under the projection $\pi : \mathcal{F}^{(r)*} \to M$, the cocycle $\gamma^* = \pi^*(\gamma)$ satisfy*

- *there exists on* $\mathcal{F}^{(r)*}$ *a* $(2n-2r+1)$*-cochain* β^**, such that* $\delta\beta^* = \gamma^*$
- *on each fiber of* $\mathcal{F}^{(r)*}$ *over a point* $x \in M$*, one has, for each* $(2n-2r+1)$*-cycle* λ*,*

$$\beta^*(\lambda) = I(\lambda) \tag{5.25}$$

where $I(\lambda)$ *is the index of the cycle* λ *in* $H_{2n-2r+1}(\mathcal{F}^{(r)*}|_x) \cong \mathbb{Z}$*.*

In Chap. 3, Sect. 3.3 of his article, Chern provides a version of the third definition where γ^* and β^* are explicit transgression differential forms. This property is very useful in M.-H. Schwartz's work.

5.7.3.2 By Gamkrelidze

In 1953, the Georgian mathematician Revaz Valerianovic Gamkrelidze, published, in Russian, "Computation of Chern cycles of Algebraic Manifolds" mainly using the Ehresmann decomposition of Grassmannian manifolds in terms of Ehresmann cycles (see Sect. 5.5.4). The set

$$G_{d,m} = \{(x, P) | x \in P, P \text{ is a } d\text{-plane in } \mathbb{CP}^m\} \tag{5.26}$$

is an algebraic manifold of dimension $n = m + d(m-d)$. In [135, 136], Gamkrelidze provides formulae for homological Chern classes of $G_{d,m}$ in terms of Ehresmann cycles

$$c_{n-k}(G_{d,m}) = \sum_{i=0}^{k} (-1)^i \binom{n-i+1}{n-k+1} [k-i/0, \cdots, n-i, n-i+2, \cdots, n+1], \tag{5.27}$$

from which he deduces Chern cycles for algebraic manifolds (also see Sects. 5.15.4.1 and 5.15.4.2).

5.7.4 Axiomatic Definition of Chern Classes

Let E be a complex vector bundle of (complex) rank n over a space X. In the same way as in the real case, Chern classes $c^p(E) \in H^{2p}(X; \mathbb{Z})$, for $p = 1, \ldots, n$ can be defined by obstruction theory. The total Chern class of E is denoted

$$c(E) = 1 + c^1(E) + \cdots + c^n(E).$$

In his thesis, Wu Wen-Tsün extended the product formula (5.14) to Chern classes,

$$c(E \oplus E') = c(E) \smile c(E'), \tag{5.28}$$

The formula 5.28 is one of the axioms entering in the axiomatic definition of Chern classes, due to Grothendieck [155, Theorem 1] (see also Hirzebruch [166]).

Definition 5.7.3 Axiomatic definition of Chern classes.

Let E be a complex vector bundle of rank n over a space X. There is a unique class $c(E) \in H^*(X; \mathbb{Z})$ satisfying the following properties,

1. One has $c(E) = 1 + c^1(E) + \cdots + c^n(E)$, where $c^i(E) \in H^{2i}(X; \mathbb{Z})$ and $c^i(E) = 0$ if $i > n$.
2. (Naturality) If $f : Y \to X$ is a continuous map, then $f^*(c(E)) = c(f^*(E))$ where $f^*(E)$ is the "pull-back" complex vector bundle on Y.
3. (Whitney-Wu) If E and E' are two bundles over X, then
$$c(E \oplus E') = c(E) \cup c(E').$$
4. Let γ be the tautological line bundle over $\mathbb{CP}^n$ (cf 5.20). Then $c^1(\gamma) = -[a]$.

Here, the class $[a] \in H^2(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$ is the one whose image by Poincaré isomorphism $H^2(\mathbb{CP}^n) \to H_{2n-2}(\mathbb{CP}^n)$ is the homology class of the hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ oriented with the orientation induced from the complex structure.

The Chern classes of a complex analytic manifold M are Chern classes of the tangent bundle TM. By the Poincaré isomorphism $H^*(M) \to H_{2n-*}(M)$, one obtains homology Chern classes $c_{2n-*}(M) = c^*(M) \cap [M]$.

Lemma 5.7.4 *The Stiefel-Whitney homology classes of a complex analytic manifold are the reduction modulo 2 of the homology Chern classes.*

5.7.5 Definitions by Čech-de Rham and Chern-Weil Theories

Tatsuo Suwa provides in this volume [312, §1.5.2] the definition of Chern classes via Čech-de Rham theory [312, §1.6.2] Chern-Weil theory adapted to Čech-de Rham cohomology [312, §1.6]. and coincidence of definitions [312, Theorem 1.8.7].

5.7.6 Applications of Chern Classes in Mathematics and Mathematical Physics

Chern classes have many applications in mathematics, for instance, in knot theory, in Chern-Weil and Chern-Simons theories, theory of Calabi-Yau manifolds, and in physics, for instance, in string theory, quantum field theory etc.

The Chern-Weil theory considers topological invariants of vector bundles on a smooth manifold M in terms of connections and curvature. Characteristic classes are represented in the de Rham cohomology ring of M (see Suwa [311]). The powerful theory leads to a proof of a Chern-Gauss-Bonnet theorem.

In theoretical physics, Chern classes appear mainly through the notion of Calabi-Yau manifolds, important in particular in Hall effect, particle physics, superstring theory, brane models, gauge theory, condensed matter physics, topological quantum field theories, etc.

Chern-Simons theory [90] is applied in mathematics to knot invariants and three-manifold invariants. In theoretical physics, the theory leads to a three-dimensional topological quantum field theory mainly developed by Edward Witten [331]. The Chern-Simons invariant is a secondary characteristic class induced by the second Chern class of a principal bundle. This invariant allows to construct interesting topological quantum fields in Mathematical Physics. On four-dimensional Spin varieties, the second Chern class is necessarily even; this leads to the construction of a "Spin" Chern-Simons invariant in dimension 3. This "Spin" Chern-Simons invariant appears mainly in the modeling of the quantum Hall effect, in condensed matter physics, and also in string theory. The generalization of the theory of Spin Chern-Simons in higher degrees requires manifolds equipped with a Wu structure of corresponding degree (see 5.6.2). These theories are defined on manifolds with too high dimension for being relevant to condensed matter physics, but they appear in String Theory [230, 272].

5.8 Pontryagin Classes

Pontryagin classes were defined in several ways, using obstruction theory (see [258]), using Schubert cells decomposition (see [338, 341]), using relation with Chern classes (see [224, section 15]).

The original Pontryagin definition [258] by obstruction theory was, on the n-dimensional manifold M to consider $(n-2i)+2$ vector fields (sections) in general position. The set of points a, where they span a subspace of dimension less or equal to $n-2i$ in $T_a(M)$, form a $n-4i$-cycle. The cohomology dual class (by Poincaré duality) is the Pontryagin class $p^i(M) \in H^{4i}(M;\mathbb{Z})$.

Note that, using different numbers of sections and dimensions, the Segre classes can be defined in a similar way [116, Proposition 10.2].

The idea of considering characteristic classes defined by particular Schubert varieties was introduced by Wu Wen Tsün in his thesis [338, 339]. Pontryagin considered the special Grassmannian manifold of oriented vector subspaces (see 5.5.4). One can define Pontryagin classes p_i of $\widetilde{G}_{n,r}$ as classes of particular Schubert cycles (see [102, Part III, Chap. IV, §1,C]), then for every vector bundle E over M which is pullback of the Grassmannian vector bundle by $g : M \to \widetilde{G}_{n,r}$, one defines the cohomology class

$$p^i(E) = g^*(p_i).$$

The definition of Pontryagin classes, given in Milnor [224, section 15] coincide with the obstruction theory one, up to second order class. Considering the complexification $E \otimes \mathbb{C}$ of a real vector bundle E, one defines

$$p^i(E) = (-1)^i c^{2i}(E \otimes \mathbb{C}) \in H^{4i}(M; \mathbb{Z}).$$

The relation with Stiefel-Whitney classes is the following, if $\overline{p}^i$ is the image of p^i by the natural homomorphism $H^{4i}(M; \mathbb{Z}) \to H^{4i}(M; \mathbb{Z}_2)$, then [102, Part III, Chap. IV, §1,C]

$$\overline{p}^i = w^{2i} \cup w^{2i}.$$

Chern-Weil theory provides a formula for the Pontryagin classes of a Riemannian manifold, they are represented by differential forms that measure certain types of curvature of the manifold. In [137] Gelfand and MacPherson provide a combinatorial formula for the Pontryagin classes of a triangulated manifold.

5.9 Hirzebruch Theory

The Hirzebruch theory provides a way to unify, in the case of manifolds, three theories of characteristic classes, the Chern class, the Todd class and the Thom-Hirzebruch class.

5.9.1 Arithmetic Genus

Let g_i be the $\mathbb{C}$-dimension of the space of holomorphic differential i-forms on the n-dimensional complex algebraic manifold M.

- g_0 is the dimension of the space of holomorphic functions, i.e. the number of connected components of M,
- g_n is called *geometric genus* of M,
- g_1 is called *irregularity* of M,

Definition 5.9.1 (Arithmetic Genus) [154, 166]. The *arithmetic genus* of M, denoted by $\chi_a(M)$ is defined as:

$$\chi_a(M) = \sum_{i=0}^{n} (-1)^i g_i$$

Example 5.9.2 In the case of a complex algebraic curve, i.e. a compact (connected) Riemann surface, then M is homeomorphic to a sphere with g handles. Then $g_0 = 1$ and $g_1 = g_n = g$. The arithmetic genus of M is $\chi_a(X) = 1 - g$.

5.9.2 Todd Genus

The *Todd genus* $\tau(M)$ [166, 319] was defined (by Todd) in terms of Eger-Todd fundamental classes (polar varieties—see 5.15.8), using results by Severi. The Eger-Todd classes are homological Chern classes of M.

Todd "proved" that

$$\tau(M) = \chi_a(M).$$

In fact, the Todd's proof uses a lemma by Severi which was never completely proved. The Todd result was proved by Hirzebruch, using other methods.

5.9.3 Signature

Definition 5.9.3 (Thom-Hirzebruch) Let M be a (real) compact oriented $4k$-dimensional manifold. Then the map

$$H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \longrightarrow \mathbb{R}, \qquad (x, y) \mapsto \langle x \cup y, [M]\rangle \in \mathbb{R}$$

defines a symmetric bilinear form on the vector space $H^{2k}(M;\mathbb{R})$, therefore, a quadratic form Q.

The *index (or signature) of M*, [224] denoted by sign(M), is defined as the index of Q, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.

René Thom (1954) showed that the signature of a manifold is a cobordism invariant, and in particular is given by some linear combination of its Pontryagin numbers [318]. Friedrich Hirzebruch (1954) found an explicit expression for this linear combination as the L-genus of the manifold. Hirzebruch showed that the L genus of M in dimension $4k$ evaluated on the fundamental class $[M]$ of M is equal to the signature sign(M) of M.

5.9.4 Hirzebruch Theory

The Hirzebruch theory (see [53]) uses multiplicative series and Chern roots.

Definition 5.9.4 The Hirzebruch multiplicative series) are defined, for y a parameter, by

$$Q_y(\alpha) = \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]]$$

Three important cases are

- $Q_{-1}(\alpha) = 1 + \alpha$ $\quad y = -1$
- $Q_0(\alpha) = \frac{\alpha}{1 - e^{-\alpha}}$ $\quad y = 0$
- $Q_1(\alpha) = \frac{\alpha}{\tanh \alpha}$ $\quad y = 1$

Definition 5.9.5 [130, Remark 3.2.3] The *Chern roots* α_i of the n-dimensional complex manifold M are formal indeterminates satisfying the relation

$$\sum_{j=0}^{n} c^j(M)\, t^j = \prod_{i=1}^{n} (1 + \alpha_i t),$$

where $c^j(M) = c^j(TM) \in H^{2j}(M; \mathbb{Z})$ are the Chern classes of the n-dimensional complex manifold M with (complex) tangent bundle TM.

Note that Fulton, in [130, Remark 3.2.3] provides several applications of the use of Chern roots, in particular regarding Chern classes of the Whitney sum (see formula 5.28 above) and of the tensor product of vector bundles.

Definition 5.9.6 The *Todd-Hirzebruch classes*

$$\widetilde{td_{(y)}}(TM) = \prod_{i=1}^{n} Q_y(\alpha_i)$$

are defined for each value y of the parameter.

In the three previous cases the Todd-Hirzebruch classes are

$$\widetilde{td_{(y)}}(TM) = \begin{cases} c^*(TM) & = \prod_{i=1}^{n}(1+\alpha_i) \qquad y=-1 \\ & \text{Chern class,} \\ td^*(TM) & = \prod_{i=1}^{n}(\frac{\alpha_i}{1-e^{-\alpha_i}}) \qquad y=0 \\ & \text{Todd class,} \\ L^*(TM) & = \prod_{i=1}^{n}(\frac{\alpha_i}{tanh \alpha_i}) \qquad y=1 \\ & \text{Thom-Hirzebruch } L\text{-class.} \end{cases}$$

5.9.5 χ_y-Characteristic

Definition 5.9.7 For each value of the parameter y, the χ_y-characteristic of a complex projective manifold M is defined by

$$\chi_y(M) = \sum_{p\geq 0}\left(\sum_{i\geq 0}(-1)^i \dim_{\mathbb{C}} H^i(M, \bigwedge^p T^*M\right)\cdot y^p.$$

In the three previous cases the χ_y-characteristics are

- $y=-1$ $\quad \chi_{-1}(M)=\chi(M)$, Euler-Poincaré characteristic of M (by Hodge theory),
- $y=0$ $\quad \chi_0(M)=\chi_a(M)$, arithmetic genus of M (by definition)
- $y=1$ $\quad \chi_1(M)=\text{sign}(M)$, signature of M (by Hodge theory)

5.9.6 Hirzebruch Riemann-Roch Theorem

Theorem 5.9.8 (Hirzebruch Riemann-Roch Theorem) *One has*

$$\chi_y(M) = \int_X \widetilde{td_{(y)}}(TM)\cap [M] \qquad \in \mathbb{Q}[y].$$

In the three particular cases, the theorem specializes in

- $\chi(M) = \int_X c^*(TM) \cap [M]$ $\qquad y = -1$
  Euler—Poincaré characteristic of M
  Poincaré-Hopf Theorem
- $\chi_a(M) = \int_X td^*(TM) \cap [M]$ $\qquad y = 0$
  arithmetic genus of M
  Hirzebruch-Riemann-Roch Theorem
- $\mathrm{sign}(M) = \int_X L^*(TM) \cap [M]$ $\qquad y = 1$
  signature of M
  Hirzebruch signature Theorem

5.10 Second Part: Singular Varieties: To Schwartz-MacPherson

In [220], M. Merle provides the main results as well as a historical survey at that time (1983). The book by Jörg Schürmann [277] provides useful information on singular spaces.

Definition 5.10.1 Let X be a topological space. We denote by $\mathcal{X}$ a *filtration* of X by closed subsets

$$\mathcal{X} \qquad \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X = X_n \tag{5.29}$$

A *topological stratification* of X is the data of a filtration $\mathcal{X}$ of X such that each difference $V_\alpha = X_\alpha - X_{\alpha-1}$ is either empty or a topological manifold of pure dimension α. The connected components of the V_α are called the *strata*.

One refers to Trotman [320] for information on the different types of stratifications.

Definition 5.10.2 (See [315]) We says that the Whitney conditions are satisfied for a stratification if, for any pair of strata (V_α, V_β) such that V_α is in the closure of V_β, one has

(a) if (x_n) is a sequence of points in V_β with limit $y \in V_\alpha$ and if the sequence of tangent spaces $T_{x_n}(V_\beta)$ admits a limit T (in the suitable Grassmannian space) when n goes to $+\infty$, then $T_y(V_\alpha)$ is included in T.
(b) if (x_n) is a sequence of points in V_β with limit $y \in V_\alpha$ and if (y_n) is a sequence of points in V_α with limit y, such that the sequence of tangent spaces $T_{x_n}(V_\beta)$ admits a limit T for n going to $+\infty$ and such that the sequence of directions $\overline{x_n\, y_n}$ admits a limit λ when n goes to $+\infty$, then λ lies in T.

5.11 Stiefel-Whitney Homology Classes

Let X be a locally compact n-dimensional polyhedron and K any triangulation of X with first barycentric subdivision K'. Sullivan and Akin [162, 307] showed that the infinite simplicial chains (see 5.12)

$$w_k(K') = \sum_{\sigma_0 < \cdots < \sigma_k} (-1)^{|\sigma_0| + \cdots + \cdots |\sigma_k|} \langle \widehat{\sigma}_0, \cdots, \widehat{\sigma}_k \rangle, \qquad 0 \le k \le n$$

where the sum runs over the k-simplexes of K', are mod 2-cycles if and only if X is a $\mathbb{Z}_2$-Euler space, that is, for any point $x \in X$, the Euler number of the pair $(X, X \setminus \{x\})$ satisfies

$$\chi(X, X \setminus \{x\}) = 1 \mod 2. \tag{5.30}$$

For a subspace Z of a space Y, the Euler number of the pair (Y, Z) is defined by $\chi(Y, Z) = \sum_k (-1)^k \dim H_k(Y, Z; \mathbb{Q})$ whatever the sum makes sense.

The combinatorial expression of Stiefel-Whitney classes (5.12), is still valid for $\mathbb{Z}_2$-Euler spaces (see [162, 307]).

J.H.G. Fu and C. McCrory ([126, 129]) provide a geometric (not involving triangulations) definition of Stiefel-Whitney classes using a suitable definition of the conormal cycle of an embedding of X in a smooth variety M (see the Definition 5.16.4 given in Sect. 5.16.1). The resulting Stiefel-Whitney classes are mod 2 reductions of the Chern-Schwartz-MacPherson classes (see 5.7.4 and 5.16). Fu and McCrory show that the Stiefel-Whitney classes defined in this way satisfy axioms similar to the Deligne-Grothendieck axioms for the Chern classes 5.15.11 (see the axioms in Theorem [203]).

In a series of articles A. Matsui [212–215] and A. Matsui and H. Sato [216, 217] studied intensively generalizations and alternative descriptions of Stiefel-Whitney homology classes.

5.12 Poincaré-Hopf for Singular Varieties – Marie-Hélène Schwartz

The first proof of Poincaré-Hopf Theorem for singular varieties and the first definition of Chern classes for singular varieties were given in 1964 by Marie-Hélène Schwartz in the preprint [288] (Lille University), published in 1965 in two "Notes aux CRAS" [289].

This section and the following one provide an extended survey on Marie-Hélène Schwartz classes, their construction being less known but helpful in understanding the meaning of the Chern classes of singular varieties. For a general overview see [44] and for a complete exposition see [47].

In the case of a stratified singular variety embedded in a manifold, the first notion to consider is the one of stratified vector field. That is a vector field v such that $v(x) \in T_x(V_\alpha)$ for every point x in a stratum V_α. The index of a stratified vector field with isolated singularities can be defined by computing the index at a singular point either in the stratum of the given point, or in the ambient manifold. Unfortunately, in general, none of these definitions provide a Poincaré-Hopf Theorem. In the following section, we will discuss explicit counterexamples. The main reason for being a counterexample is that the index computed in the stratum and the index computed in the ambient manifold are different.

5.12.1 The Use of Dual Cells Decomposition

Let M be an m-dimensional real analytic manifold equipped with a real semi-analytic Whitney stratification $\{V_\alpha\}$ and $X \subset M$ an n-dimensional real analytic compact subset stratified by $\{V_\alpha\}$. Denote by (K) a triangulation of M compatible with the stratification, i.e. each open simplex is contained in a stratum (see [201, 202]).

The first observation of Marie-Hélène Schwartz concerns the triangulations.

The obstruction dimension for the construction of a vector field tangent to M is equal to m (5.22). In the same way, s is the obstruction dimension for the construction of a vector field tangent to the s-dimensional strata. That means that if one intends to construct a stratified vector field tangent to X using the triangulation (K), then one will use simplexes of different dimensions according to the dimension of the considered strata. An obstruction cocycle in that way will have different dimensions according to the strata. That is an obstacle for the use of the triangulation (K) in order to obtain a global Poincaré-Hopf Theorem.

The way M.-H. Schwartz used to overcome this obstacle is the following. Let (D) be the dual cell decomposition of (K) associated to a barycentric subdivision (K') (see Sect. 5.4). The triangulation (K) is compatible with the stratification, then each (D)-cell is transverse to the strata. In particular, if d is an m-dimensional (D)-cell and if V_α is a stratum of dimension s, then the dimension of the cell $d \cap V_\alpha$ is

$$\dim(d \cap V_\alpha) = \dim(V_\alpha)$$

that is precisely the obstruction dimension for the construction of a vector field tangent to V_α.

This observation leads naturally to the construction of a stratified vector field by induction on the dimension of the strata, using the dual cell decomposition (D) and not the triangulation (K).

5.12.2 The Use of Radial Vector Fields

Example 5.12.1 below shows that the method explained in Sect. 5.12.1 is not sufficient to obtain a Poincaré-Hopf Theorem. The second observation of M.-H. Schwartz, based on that example and example 5.12.2, is that one has to consider stratified vector fields which are radial in a sense to be made clear. That is precisely M.-H. Schwartz's construction of radial extensions of vector fields explained below.

Let v be a stratified vector field on M with isolated singularities. The index of the stratified vector field v at a singular point a situated in the stratum V_α can be defined as the index of the restriction $I(v|_{V_\alpha}, a)$. The natural generalization of the Poincaré-Hopf Theorem to singular varieties would be the following formula

$$\chi(X) \overset{?}{=} \sum_{a_k} I(v|_{V_{\alpha(a_k)}}, a_k), \tag{5.31}$$

where $V_{\alpha(a_k)}$ is the stratum of X containing the singularity a_k of v.

In general, the formula (5.31) is not true. Counterexamples are very useful in order to understand the situation. The following counterexample has been given by M.-H. Schwartz in [293, 6.2.1].

Example 5.12.1 As a first step, in $\mathbb{R}^2$ with coordinates (x, y), one considers the (closed) balls centered at the origin, B with radius 1 and B' with radius 2 (see Fig. 5.1 (i)). We have $\chi(B') = +1$.

Inside the ball B, one consider the continuous vector field $v_1(x, y) = (|x|, y)$. One has $v_1(0) = 0$, the point 0 is an isolated singularity of v_1 with index $I(v_1, 0) = 0$.

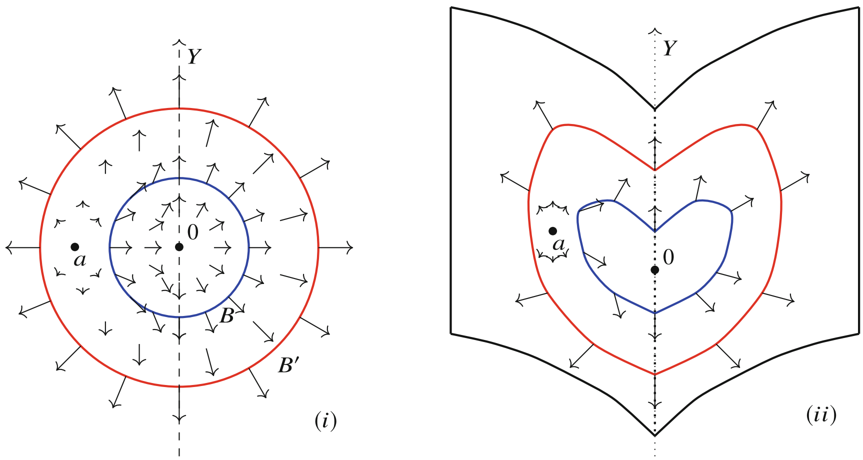


Fig. 5.1 M.-H. Schwartz's counter-example

On the boundary $\partial B'$, one consider the vector field $v_2(x, y) = (x, y)$ pointing outwards. The vector field v_2 can be extended inside B' as a continuous vector field v which is v_2 along $\partial B'$, v_1 inside B and which is tangent to the y-axis Y along Y. For instance, consider (see Fig. 5.1 (i)).

$$v(x, y) = \begin{cases} \left(2|x| - x + (x - |x|)\sqrt{x^2 + y^2}, y\right) & \text{on } B' \setminus B \\ v_1(x, y) = (|x|, y) & \text{inside } B \end{cases}$$

The vector field v has an isolated singular point of index 0 at 0 and another isolated singular point at $a = (-3/2, 0) \in B' \setminus B$. By the Poincaré-Hopf Theorem with boundary (5.3.1), we have

$$\chi(B') = +1 = I(v, 0) + I(v, a),$$

that implies $I(v, a) = +1$.

Remark that while $I(v, 0) = 0$, one has $I(v|_Y, 0) = +1$.

In a second step, fold the picture along the y-axis, in order to obtain a singular surface $x^2 - z^3 = 0$ in $\mathbb{R}^3$ (see picture 5.1 (ii)). Then B' becomes a singular variety X, with boundary and stratified by Y and $X \setminus Y$. The vector field v in B' defines a stratified vector field, still denoted by v on X. It has two isolated singular points: 0 and a with indices $I(v|_Y, 0) = +1$ and $I(v, a) = +1$. Then

$$\chi(X) = +1 \neq I(v, a) + I(v|_Y, 0) = 1 + 1 = 2.$$

So, the formula (5.31) is not true.

The vector field v is not "radial" at the singular point 0, in the sense that it is not pointing outwards the unit ball centered at 0 in $\mathbb{R}^3$.

The following example shows that formula (5.31) cannot be written by using the indices in the ambient manifold and that one cannot take any vector field in order to prove a Poincaré-Hopf Theorem for singular varieties.

Example 5.12.2 The pinched torus X in $\mathbb{R}^3$ is obtained from the two-dimensional torus T by collapsing a meridian S_a to the point a. The pinched point a is a singular point of X, that is the singular set of the pinched torus.

A small ball $\mathbb{B}^3(a) \subset \mathbb{R}^3$ centered at $\{a\}$ intersects the pinched torus along two meridians. The surface joining the two meridians, inside the ball, can be considered either as a cylinder (in that case, we obtain the torus T), or as a double cone, to obtain the pinched torus.

The vector field v can be defined in the small ball $\mathbb{B}^3(a) \subset \mathbb{R}^3$ with an isolated singularity at $\{a\}$ and such that its restriction to $X \setminus \{a\}$ is tangent to $X \setminus \{a\}$. On the one hand, such a vector field is non singular on the boundary $\partial\mathbb{B}^3(a)$. On the other hand, such a vector field can be obtained from a continuous vector field tangent to the torus T.

Let us consider two examples of such a vector field:

(a) Firstly, consider the unit vector field on the torus T, tangent to the parallels of the torus, it has no singular point on the torus. This vector field can be extended in a neighbourhood of the torus, by parallel extension, in order to be defined on the boundary $\partial\mathbb{B}^3(a)$ of the ball $\mathbb{B}^3(a)$. The vectors $v(x)$ for $x \in \partial\mathbb{B}^3(a)$ are all unit and "parallel" vectors, so the index $I(v, a)$ is zero.

Now, pinch the torus along S_a. The vector field v does not change outside the ball $\mathbb{B}^3(a)$. Inside the ball, the length of the vector goes to zero with the distance to the point a. We obtain a vector field on the pinched torus with only one singular point with index $I(v, a) = 0$ (see Fig. 5.2 (i)).

In this case, the Poincaré-Hopf Theorem is not satisfied, indeed one has

$$\chi(X) = 1 \neq 0 = I(v, a).$$

(b) Consider now a radial vector field ρ, i.e. a vector field with an isolated singularity at $\{a\}$, pointing outwards the ball $\mathbb{B}^3(a)$ along $\partial\mathbb{B}^3(a)$ and tangent to the pinched torus X along the intersection $X \cap \partial\mathbb{B}^3(a)$. On the one hand, the vector field ρ has index $I(\rho, a) = +1$ at a. On the other hand, ρ can be extended on the pinched torus as a continuous vector field without other singularities. Indeed, one can define an extension of ρ in $X \setminus \mathbb{B}^3(a)$ such that the angle of $\rho(x)$ with the tangent line to the meridian containing x decreases with the distance to a untill being 0 for the meridian opposed to a. This angle is $\pi/2$ on $X \cap \partial\mathbb{B}^3(a)$ (see Fig. 5.2 (ii)). In that case, the formula given in the

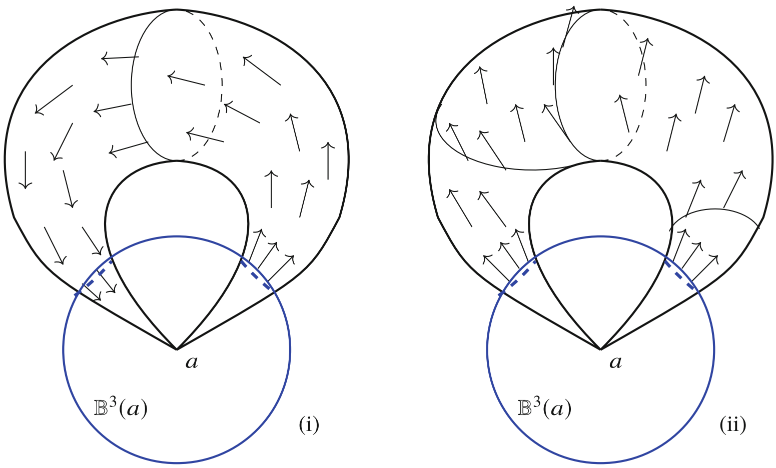


Fig. 5.2 Vector fields on the pinched torus

Poincaré-Hopf theorem still holds true.

$$\chi(X) = 1 = \sum_{a_k \in \mathrm{Sing}(\rho)} I(\rho, a_k) = I(\rho, a)$$

The vector field ρ is the first example of M.H. Schwartz radial vector field.

5.12.3 Radial Vector Fields

The idea of the construction of radial vector fields is very simple. The construction has to be made in suitable tubes, that makes M.-H. Schwartz's proof delicate. The details of the construction of the tubular neighbourhoods are provided in [293].

Theorem 5.12.3 *Let M be a real analytic manifold equipped with a real semi-analytic Whitney stratification $\{V_\alpha\}$ and $X \subset M$ a real analytic compact subset stratified by $\{V_\alpha\}$. It is possible to construct on M stratified vector fields $\widetilde{v}$ with isolated singularities satisfying the following properties:*

1. *the vector fields $\widetilde{v}$ are pointing outwards of small tubes $\mathcal{T}(X)$ along $\partial\mathcal{T}(X)$,*
2. *If $a \in V_\alpha$ is a singularity of $\widetilde{v}$, then the index of $\widetilde{v}$ at a as a vector field tangent to the stratum V_α is the same than the index of $\widetilde{v}$ at a as a vector field tangent to the ambient manifold M:*

$$I(v, a; V_\alpha) = I(\widetilde{v}, a; M). \tag{5.32}$$

The construction of a radial vector field goes in two steps: the local (5.12.3.1) and the global (5.12.3.2) construction. One obtains the Poincaré-Hopf Theorem for singular varieties (Theorem 5.12.5).

5.12.3.1 Radial Vector Fields – Local Construction

Denote by $U_\alpha \subset V_\alpha$ a neighbourhood of a point a in a stratum V_α and by v a vector field tangent to V_α on U_α with possibly an isolated singularity at a. The local radial extension of the vector field v is obtained as the sum of two vector fields defined in a suitable tubular neighbourhood of V_α in the ambient manifold M: the parallel extension and the transverse vector field. This construction has been developed at the same time and independently by M.H. Schwartz for the proof of Poincaré-Hopf Theorem for singular varieties, with any index and by Milnor [223] for smooth manifold, and for non-degenerated vector fields.

In the case of a stratified singular variety, the construction must be made in such a way as to obtain a stratified vector field section of TM. This is possible thanks to the Whitney conditions (a) and (b).

(a) The *parallel extension* of the vector field v in a neighbourhood of U_α in M is defined in the following way. If y is a point on the fiber at x of a sufficiently small tube $\mathcal{T}_\varepsilon(U_\alpha)$, of "ray" ε, then the Whitney condition (a) implies that the vector $v_p(y)$ parallel to $v(x)$ can be projected perpendicularly as a non-zero vector $\widetilde{v_p}(y)$ on the tangent space at y to the stratum containing the point y. The vector field v is extended in that way inside the tube $\mathcal{T}_\varepsilon(U_\alpha)$ as a stratified vector field "parallel" to v. Of course, if v admits (isolated) singular points, the vector field $\widetilde{v_p}$ will have "disks" of singular locus corresponding to singularities of v ([288], §3, [293], Théorème 1.1).

(b) The *transverse vector field* ρ is defined in the following way. The vector ρ gradient of the "distance to V_α" (relatively to a suitable metric) vanishes on V_α, it is transverse to the boundary of every sufficiently small "geodesic" tube $\mathcal{T}'_\varepsilon(U_\alpha)$ composed of the geodesic rays in M normal to V_α. The Whitney condition (b) guarantees that for every point $y \in \mathcal{T}'_\varepsilon(U_\alpha)$, the vector $\rho(y)$ can be projected as a non-zero vector $\widetilde{\rho}(y)$ on the tangent space at y to the stratum containing y, providing a stratified vector field in $\mathcal{T}'_\varepsilon(U_\alpha)$ ([293], Théorème 2.3.1).

It is clear that the vector fields $\widetilde{v}_p(y)$ and $\widetilde{\rho}(y)$ are not continuous, as vector fields tangent to M. To overcome this drawback, Marie-Hélène Schwartz considers "tapered" neighbourhoods of the strata in which she modifies the vector fields $\widetilde{v}_p(y)$ and $\widetilde{\rho}(y)$ so as to obtain vector fields tangent to the strata and also continuous.

In order to provide an idea of the construction, let us consider the case of the field $\widetilde{\rho}(y)$ and the case where the stratum. V_α is a singleton $\{x\}$, the stratum V_β is a curve and the plane is a two-dimensional stratum V_γ (Fig. 5.3 (i)). The general case (for higher dimensional strata) follows the same principle [293, §3], [294, §4].

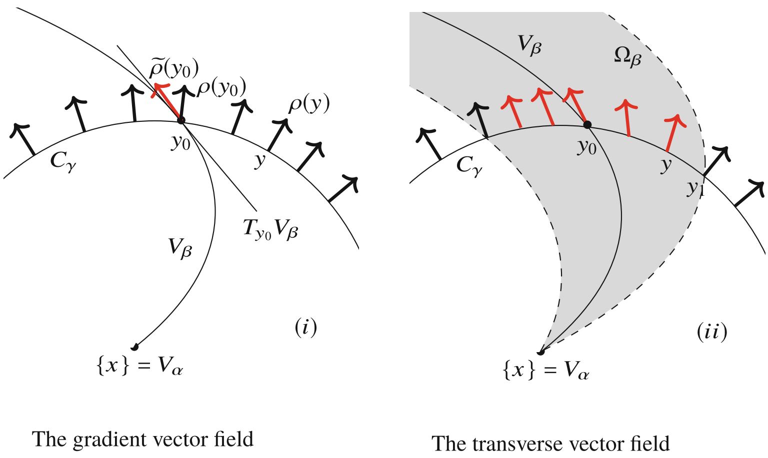


Fig. 5.3 A "tapered" neighbourhood

Consider a point y_0 of V_β, intersection of V_β with a small sphere $\mathbb{S}$ centered at x. In order to obtain a vector field tangent to V_β, the vector $\rho(y_0)$ is replaced by its projection $\widetilde{\rho}(y_0)$ on $T_{y_0}(V_\beta)$. Then the obtained vector field is not continuous: $\widetilde{\rho}(y_0)$ is not limit of the vectors $\rho(y)$ as y approaches y_0 along the curve $C_\gamma = \mathbb{S} \cap V_\gamma$.

The idea of Marie-Hélène Schwartz is to modify the vector field $\rho(y)$ in a "tapered" neighbourhood Ω_β of the stratum V_β, that is a tubular neighbourhood in M whose the radius decreases when approaching to the strata V_α of the boundary of V_β (see Fig. 5.3 (ii)).

The "transverse" vector field, tangent to the strata and also continuous is built as follows. Denote by $\lambda \in [0, 1]$ a parameter of the portion of the curve C_γ going from y_0 to the point y_1 intersection of the curve C_γ with the boundary of Ω_β. At the point y of the curve, of parameter λ, the field $\widetilde{\rho}(y)$ is the projection on $T_y(V_\gamma)$ of the vector

$$\lambda\rho(y) + (1-\lambda)\widetilde{\rho}_y(y_0) \tag{5.33}$$

where $\widetilde{\rho}_y(y_0)$ is the vector equipollent to $\widetilde{\rho}(y_0)$ at the point y. (see Fig. 5.3 (ii)).

Similarly, in the "tapered" neighbourhoods of the strata, one build a "parallel" vector field $\widetilde{v_p}(y)$ tangent to the strata and also continuous, from the field $v_p(y)$.

While this construction is rather technical, it allows M.-H. Schwartz [293] to consider a neighbourhood, $\mathcal{T}_\varepsilon(U_\alpha)$, the two systems of neighbourhoods $\mathcal{T}_\varepsilon$ and $\mathcal{T}'_\varepsilon$ defined above being equivalent. The (local) radial extension of the vector field v is the vector field defined in $\mathcal{T}_\varepsilon(U_\alpha)$ by:

$$\widetilde{v}(y) = \widetilde{v_p}(y) + \widetilde{\rho}(y).$$

Proposition 5.12.4 (Local Radial Extension of a Vector Field) *The local radial extension $\widetilde{v} = \widetilde{v_p} + \widetilde{\rho}$ satisfies the following properties:*

1. *the vector field $\widetilde{v}$ is pointing outwards of the tube $\mathcal{T}_\varepsilon(U_\alpha)$ along $\partial\mathcal{T}_\varepsilon(U_\alpha) \setminus \mathcal{T}_\varepsilon(\partial U_\alpha)$,*
2. *If $a \in U_\alpha$ is an isolated singularity of v, it is also an isolated singularity of $\widetilde{v}$ and the index of $\widetilde{v}$ at a as a vector field tangent to the ambient manifold M is the same than the index of v at a, computed as a vector field tangent to V_α:*

$$I(v, a; V_\alpha) = I(\widetilde{v}, a; M). \tag{5.34}$$

3. *if two vector fields v_1 and v_2, tangent to V_α are homotopic as sections of $T(V_\alpha)$ over U_α, then their extensions $\widetilde{v}_1 = \widetilde{v_1}_p + \widetilde{\rho}$ and $\widetilde{v}_2 = \widetilde{v_2}_p + \widetilde{\rho}$ are homotopic as sections of TM over $\mathcal{T}_\varepsilon(U_\alpha)$.*

5.12.3.2 Radial Vector Fields – Global Construction

The M.-H. Schwartz's "global" construction of vector fields by radial extension goes as follows. The stratification of X (see 5.29) is denoted by

$$\emptyset = X_{-1} \subset X_{\alpha_0} = V_{\alpha_0} \subset X_\beta \subset X_\gamma \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X = X_n \tag{5.35}$$

where the lowest dimensional stratum can be a zero-dimensional one or a stratum V_{α_0} of dimension $2s > 0$. If V_{α_0} is zero-dimensional, i.e. a set V_0 of finitely many points a_i, then one considers a radial vector field v in balls $B_\varepsilon(a_i)$ centered at each of these points. If the lowest dimensional stratum is a stratum of dimension $2s > 0$, then we construct a vector field v on V_{α_0} with finitely many isolated singularities a_i. The stratum V_{α_0} is a manifold and it has to be compact if X is compact. In this case the total Poincaré-Hopf index of v on V_{α_0} is $\chi(V_{\alpha_0})$. Denote by $\varepsilon(\alpha_0) = \inf \varepsilon_i$ where $\mathcal{T}_{\varepsilon_i}(U_{\alpha_0})$ is the local tubular neighbourhood we constructed around the singular point a_i. The vector field v is well defined by radial extension in the tubular neighbourhood $\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$ of V_{α_0} with same singularities a_i and their indices satisfy the formula 5.34.

The radial vector field $\widetilde{v}$ is defined in a tubular neighbourhood $\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$ of the lowest dimensional stratum V_{α_0} and it is pointing outwards from $\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$.

The vector field $\widetilde{v}$ is now extended in the next strata in $X_\beta \setminus X_{\alpha_0}$ i.e. the lowest dimensional strata V_β such that $V_{\alpha_0} \subset \partial V_\beta$. The (finite) union of these strata is denoted by V_β and

$$W_\beta = V_\beta \setminus \mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0}).$$

Then W_β is a manifold such that the vector field $\widetilde{v}$ is well defined and pointing inwards of W_β on the boundary $\partial W_\beta = V_\beta \cap \partial \mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0})$. The vector field $\widetilde{v}$ can be extended inside V_β with finitely many isolated singular points b_j. The Poincaré-Hopf Theorem with boundary (Theorem 5.3.1) implies

$$\chi(W_\beta) - \chi(\partial W_\beta) = \sum_{b_j \in V_\beta} I(\widetilde{v}, b_j),$$

where

$$\chi(\partial W_\beta) = \sum_{a_i \in V_{\alpha_0}} I(\widetilde{v}, a_i).$$

Then:

$$\chi(X_\beta) = \sum_{a_i \in V_{\alpha_0}} I(\widetilde{v}, a_i) + \sum_{b_j \in V_\beta} I(\widetilde{v}, b_j).$$

The strata V_β admit a tubular neighbourhood $T_{\varepsilon(\beta)}(V_\beta)$ in which a radial extension of $\widetilde{v}$ can be constructed.

The process continues by increasing dimension of the strata, knowing that, for the next dimensional strata V_γ one has to consider

$$W_\gamma = V_\gamma \setminus \left(\mathcal{T}_{\varepsilon(\alpha_0)}(V_{\alpha_0}) \cup \mathcal{T}_{\varepsilon(\beta)}(V_\beta)\right).$$

At the end of the process, the construction provides a "tubular neighbourhood"

$$\mathcal{T}_\varepsilon(X) = \bigcup \mathcal{T}_{\varepsilon(\kappa)}(V_\kappa) \tag{5.36}$$

where κ describes all indices of strata, and a radial vector field $\widetilde{v}$ defined on $\mathcal{T}_\varepsilon(X)$. The radial vector field satisfies the Theorem 5.12.3. Then:

$$\chi(X) = \sum_{a_k \in X} I(\widetilde{v}, a_k)$$

for all singularities a_k of $\widetilde{v}$, with $I(\widetilde{v}, a_k) = I(\widetilde{v}, a_k; V_{\alpha(a_k)}) = I(\widetilde{v}, a_k; M)$.

5.12.4 Poincaré-Hopf Theorem for Singular Varieties

Theorem 5.12.5 (Poincaré-Hopf for Singular Varieties) [293, Théorème 6.2.2] *Let X be an analytic subset of the analytic manifold M and $\{V_\alpha\}$ a Whitney stratification of the pair (M, X). Let $\widetilde{v}$ be a radial vector field (i.e. obtained by radial extensions) defined on X. There is a finite number of zeroes a_k of $\widetilde{v}$ whose index $I(\widetilde{v}, a_k)$ is the same, computed in the stratum of a_k or in M. Then:*

$$\chi(X) = \sum_{a_k \in X} I(\widetilde{v}, a_k) \tag{5.37}$$

where, if $\dim V_{i(a)} = 0$*, then by construction* $I(\widetilde{v}|_{V_{i(a)}}, a) = +1$.

5.13 Poincaré-Hopf for Singular Varieties – Generalizations

Marie-Hélène Schwartz generalized the notion of radial vector field by the notion of "preradial" vector fields [291, 292]. H. King and D. Trotman [181] extend her theory for more general stratified vector fields on more general stratified sets with boundary. Vector fields need not be going inwards or outwards, but may be tangent to the boundary of the given set. They prove a corresponding Poincaré-Hopf theorem, defining a virtual index for such vector fields and the Euler characteristic of the set is shown to be the sum of the virtual indices. In [302], S. Simon gives another proof of a stratified Poincaré-Hopf formula, using a suitable definition of index. For alternative definitions and complete descriptions of indices, including virtual, GSV, see the R. Callejas-Bedregal, M.F.Z. Morgado, and J. Seade article in this volume [74].

W. Ebeling and S. Gusein-Zade [110] compare notions of indices for vector fields and differential forms. In particular, they define a notion of radial index for 1-forms [109] (Sect. 5.17.6). N. G. Grulha Jr., M. S. Pereira and H. Santana [161] prove a Poincaré-Hopf Theorem for Isolated Determinantal Singularities, in the vein of Ebeling and Gusein-Zade.

5.14 Schwartz Classes

The construction of Schwartz classes follows the same general principle as that of the construction of radial vector fields for the Poincaré-Hopf Theorem. However, the context is now the complex situation and no longer real analytic varieties and one considers complex r-frames and no longer vector fields.

Let X be an analytic subset of the analytic manifold M and $\{V_\alpha\}$ a Whitney stratification of the pair (M, X). The *complex* dimensions of M and X will be denoted by m and n.

As for the Poincaré-Hopf Theorem, the first idea of Marie-Hélène Schwartz is to consider stratified (and now complex) vector fields for a (complex) Whitney stratification. That means that, when X is a singular variety whose V_α are the strata, she considers the space (no longer a bundle)

$$\bigcup_{V_\alpha \subset X} T(V_\alpha) \subset T(M)$$

as a substitute to the tangent bundle to X.

5.14.1 Radial Extension of Frames

The main observation of Marie-Hélène Schwartz concerns the obstruction dimensions (see Sect. 5.12.1).

On the one hand, the obstruction dimension to the construction of an r-frame tangent to M is equal to $2p = 2(m - r + 1)$. The obstruction dimension to the construction of an r-frame tangent to a stratum V_α of complex dimension s is equal to $2q = 2(s - r + 1)$. As in Sect. 5.12.1, this property allows one to consider the cell decomposition (D) dual of a triangulation (K) compatible with the given stratification. In that case, the dimension of the intersection of a dual $2p$-cell with V_α is equal to the obstruction dimension for the construction of an r-frame tangent to V_α, i.e. $2q = 2(s - r + 1)$.

On the other hand, the obstruction dimension to the construction of an $(r-1)$-frame tangent to M is equal to $2p+2 = 2(m-r+2)$. This means that it is possible to construct an $(r-1)$-frame $v^{(r-1)} = (v_1, v_2, \ldots, v_{r-1})$ without singularities on the $2p$-cells in (D). In this case, the $(r-1)$ vectors in $v^{(r-1)}$ are $\mathbb{C}$-linearly independent on the $2p$-cells d_i^{2p}. The singularities of an r-frame $v^{(r)} = (v^{(r-1)}, v_r)$ in a $2p$-cell d_i^{2p} will be isolated points at which the last vector v_r either vanish or belongs to the $(r-1)$ complex plane generated by the vectors in $v^{(r-1)}$.

More precisely, if V_α is a stratum of complex dimension s, the $(r-1)$-frame $v^{(r-1)}$ is constructed without singularities on the $2q$-cells $d_i^{2p} \cap V_\alpha$ in $(D)^{2p} \cap V_\alpha$, where d_i^{2p} is dual of the (K)-simplex $\sigma_i^{2(r-1)} \subset V_\alpha$. Then v_r will be a vector field not in the $\mathbb{C}$-span of $v^{(r-1)}$, with an isolated singularity at the barycenter $\widehat{d}_i^{2p}$ which is situated in $d_i^{2p} \cap V_\alpha$ and is also the barycenter of the (K)-simplex $\sigma_i^{2(r-1)}$. One has $2m - 2p = 2s - 2q = 2(r-1)$ (see formula (5.3)).

The intersection $(D)^{2p} \cap V_\alpha$ is a chain Δ_α^{2q} relatively to the barycentric subdivision (K') of (K) defining the dual decomposition (D). The (K')-cell $d_i^{2p} \cap V_\alpha$, dual (in V_α) of the simplex $\sigma_i^{2(r-1)} \subset V_\alpha$ is denoted by δ_i^{2q}.

5.14.1.1 Local Radial Extension of r-Frames

Let us consider a stratified r-frame $v^{(r)} = (v^{(r-1)}, v_r)$, section of $V_r(TM)$ over the $2q$-skeleton $\Delta^{2q} \subset V_\alpha^{2s}$ (with $q = s - r + 1$), with isolated singularities which are zeroes of the last vector v_r. The parallel extension $(\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r)_p)$ of $v^{(r)}$, is defined in the tube $\mathcal{T}_\varepsilon(\delta_i^{2q})$ by the same method as in Sect. 5.12.3.1 and the transverse vector field $\widetilde{\rho}$ is defined as in Formula 5.33.

Proposition 5.14.1 (Local Radial Extension for a Frame) *If ε is sufficiently small, the radial extension of $v^{(r)}$, defined by $\widetilde{v}^{(r)} = (\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r)_p + \widetilde{\rho})$ satisfies the following conditions:*

(i) the radial extension $\widetilde{v}_r = (\widetilde{v}_r)_p + \widetilde{\rho}$ of v_r satisfies Proposition 5.12.4,

(ii) if the $(r-1)$-frame $v^{(r-1)}$ has no singularities on $\delta_i^{2q} = d_i^{2p} \cap V_\alpha^{2s}$ and if $v^{(r)}$ admits an isolated singularity at the barycenter $a \in \delta_i^{2q}$ which is a zero of v_r, then $\widetilde{v}^{(r)} = (\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r))$ satisfies the same properties in $\mathcal{T}_\varepsilon(\delta_i^{2q})$.
In that case, if the $(r-1)$-complex plane generated by $v^{(r-1)}(a)$ is linearly independent of the tangent plane $T_a(\Delta^{2q})$ in $T_a(V_\alpha^{2s})$, then the index $I(\widetilde{v}^{(r)}, a; M)$ of the extension $\widetilde{v}^{(r)}$ at a, considered as an r-frame tangent to M is equal to the index $I(v^{(r)}, a; V_\alpha)$ of $v^{(r)}$ at a considered as an r-frame tangent to V_α^{2s}.

(iii) In the same hypothesis as in (ii), if $q = 0$ (i.e; $s = r - 1$), and if a is a zero of v_r, then the index of $\widetilde{v}^{(r)}$ in a is $+1$.

The index of $\widetilde{v}^{(r)}$ at the isolated singularity a is denoted by $I(v^{(r)}, a)$.

5.14.1.2 Global Radial Extension of r-Frames

As in the case of the Poincaré-Hopf Theorem, the r-field $v^{(r)}$ is constructed over the subsets $\Delta_\alpha^{2q} = (D)^{2p} \cap V_\alpha^{2s}$, by increasing dimensions of the strata V_α. The r-field $v^{(r)}$ is constructed over $\overline{\Delta_\alpha}$ and a tube $\mathcal{T}_\varepsilon(\overline{\Delta_\alpha})$, neighbourhood of $\overline{\Delta_\alpha}$ in $D^{(2p-1)}$.

(i) If V_α^{2r-2} is a stratum whose real dimension is $2r-2 = 2(m-p)$, the obstruction dimension for the construction of a section of $V_r(TV_\alpha)$ is zero. One takes any $(r-1)$-frame $v^{(r-1)}$ tangent to V_α^{2r-2} at the vertices $a_j = \Delta_j^0$ of Δ located in $(D)^{2p} \cap V_\alpha^{2r-2}$ and the last vector v_r zero at these points.

The radial extension of the r-frame is constructed in tubes $\mathcal{T}_\varepsilon(\Delta_j^0)$ as an r-frame still denoted by $v^{(r)}$. According to Proposition 5.14.1 (iii), one has $I(v^{(r)}, a_j) = +1$.

(ii) If $s > r-1$, assume that the construction has already been performed on all strata V_α whose dimension is less than $2s$. That means that the construction has been performed on the sets $\overline{\Delta_\beta}$ and the tubes $\mathcal{T}_\varepsilon(\overline{\Delta_\beta})$. The constructed r-frame is pointing outwards of the $2p$-skeleton of a tubular neighbourhood of V_β^{2t} for all strata V_β with dimension $2t < 2s$.

Let V_β be a $2s$-dimensional stratum that contains a stratum V_α^{2t} in its closure. The r-frame is constructed on a tubular neighbourhood of the boundary of V_β. The r-frame can be extended inside V_β, more precisely on the $2q$-skeleton of Δ_β^{2q}, with $2q = 2(s-r+1)$ and with isolated singularities at the barycenters of cells δ_i^{2q} which are zeroes of the last vector v_r.

In summary, an r-frame already known on a neighbourhood of the boundary of a stratum is extended with isolated singularities inside (a suitable skeleton) of the stratum and then extended with property (ii) of the Proposition 5.14.1 to the $2p$-skeleton of a regular neighbourhood of this stratum.

The number of singularities of $\widetilde{v}$ is finite. We consider a "sufficiently small" triangulation K of M compatible with the stratification and such that

(i) The singularities of $\widetilde{v}$ are barycenters of simplexes of K,
(ii) The cellular tube $\mathcal{T}$ around X, consisting of the (D)-cells which meet X, lies in the tube $\mathcal{T}_\varepsilon(X)$ (see 5.4.3).

The constructed r-frame satisfies:

Theorem 5.14.2 ([59, 289, 294]) *Let X be an analytic subset of the analytic manifold M and $\{V_\alpha\}$ a Whitney stratification of the pair (M, X). We can construct, on the $2p$-skeleton $(D)^{2p}$ of M, a stratified r-frame $v^{(r)}$, called radial frame, whose singularities satisfy the following properties:*

(i) $v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r. On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point and on $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.
(ii) Let $a \in V_\alpha \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the $2s$-dimensional stratum V_α. If $s > r-1$, the index (in M) of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same

as the index of the restriction of $v^{(r)}$ *to* $V_\alpha \cap (D)^{2p}$ *considered as an* r*-frame tangent to* V_α*. If* $s = r - 1$*, then* $I(v^{(r)}, a) = +1$.

(iii) Inside a $2p$*-cell* d *which meets several strata, the only singularities of* $v^{(r)}$ *are inside the lowest dimensional one (in fact located at the barycenter of* d*).*

(iv) The r*-frame* $v^{(r)}$ *is pointing outwards of a regular (cellular) neighbourhood* $\mathcal{T}$ *of* X *in* M*. It has no singularity on* $\partial\mathcal{T}$*.*

5.14.2 Schwartz Classes

Denoting by d^* the elementary (D)-cochain whose value is 1 at d and 0 at all other cells, the $2p$-dimensional (D)-cochain

$$\widehat{c} = \sum_{\substack{d(\sigma)\subset\mathcal{T} \\ \dim d(\sigma)=2p}} I(v^{(r)}, \widehat{\sigma})\, d^*(\sigma)\ . \tag{5.38}$$

is defined in $C^{2p}_{(D)}(\mathcal{T}, \partial\mathcal{T})$ (see formula 5.5). This cochain actually is a cocycle (obstruction cocycle [304, §32]) whose class $c^p(X)$ lies in

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2p}(\mathcal{T}, \mathcal{T}\setminus X) \cong H^{2p}(M, M\setminus X),$$

where the first isomorphism is given by retraction along the rays of $\mathcal{T}$ and the second by excision (by $M \setminus \mathcal{T}$) (see formula (5.7) in Sect. 5.4.2).

Definition 5.14.3 ([289, 294]) The p-th Schwartz class of X is the class $c^p(X)$ denoted by

$$c^p_S(X) \in H^{2p}(M, M\setminus X).$$

It was proved "by hand" [288] by M.H. Schwartz, that the Schwartz classes do not depend of any of the choices: stratification, triangulation, r-frame... However, the proof of independence is facilitated as soon as the coincidence of Schwartz classes and MacPherson classes is established (Theorem 5.16.1).

5.15 MacPherson Classes and (Wu)-Mather Classes

MacPherson's construction of classes [204] answers a conjecture that he named "Deligne-Grothendieck" conjecture, which associates homology classes to constructible functions on algebraic complex varieties, satisfying suitable properties.

The key ingredients are the use, in the singular case, of the Nash bundle as a substitute to the tangent bundle and local Euler obstruction.

5.15.1 Constructible Sets and Functions

A *constructible set* in a complex algebraic variety X is a subset obtained by finitely many unions, intersections and complements of subvarieties. A *constructible function* $\varphi : X \to \mathbb{Z}$ is a function such that $\varphi^{-1}(n)$ is a constructible set for all n. The constructible functions on X form a group denoted by $\mathcal{F}(X)$. If $A \subset X$ is a subvariety, $\mathbf{1}_A$ is the characteristic function whose value is 1 over A and 0 elsewhere.

A complex algebraic variety X being triangulable [165], φ is a constructible function if and only if there is a triangulation (K) of X such that φ is constant on the interior of each simplex of (K). Such a triangulation of X is called φ-adapted [131, §6.1.1], [41, §2.1].

The correspondence $\mathcal{F} : X \to \mathcal{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ for a morphism $f : X \to Y$. It can be made also a covariant functor when considering the pushforward defined on characteristic functions by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A), \qquad y \in Y \tag{5.39}$$

for a morphism $f : X \to Y$, and linearly extended to elements of $\mathcal{F}(X)$.

5.15.2 The "Deligne-Grothendieck" Conjecture

In his book "Récoltes et Semailles" Alexander Grothendieck explains (in French) the genesis of the conjecture [156, Note $(87)_1$].

Grothendieck writes that he gave his conjecture in the last lecture of his seminar SGA 5, in 1966, "surely one of the most interesting..." but unfortunately, this last lecture is not published. In this note, Grothendieck presents and details the conjecture as a "Riemann-Roch" Theorem type, in the schematic framework, "with discrete coefficients instead of coherent coefficients."

In [307] (1971), Dennis Sullivan provides an historical note in which he writes that "[Deligne has] outlined a general conjectural theory of Chern classes for singular varieties based on ideas of Grothendieck and this [Hironaka's] resolution idea."

In his article published in 1974, Robert MacPherson named "Deligne-Grothendieck conjecture" the following conjecture in the framework of algebraic complex varieties.

Conjecture 5.15.1 Let $\mathcal{F}$ be the covariant functor of constructible functions and let $H_*(\ ; \mathbb{Z})$ be the usual covariant $\mathbb{Z}$-homology functor. Then there exists a unique natural transformation

$$c_* : \mathcal{F} \to H_*(; \mathbb{Z})$$

satisfying $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

Returning to the book "Récoltes et Semailles", in the Note $(87)_1$ (6 juin)–(7 juin), Grothendieck writes that "in reading Mac Pherson's paper…I find there, under the name of "Deligne-Grothendieck" conjecture, one of the main conjectures that I introduced in this [last 1966] talk…Compared to my initial conjecture, however, the form presented and proven by MacPherson differs in two ways. One is a "minus", because it fits, not in Chow's ring, but in …the homology group with integer coefficients, transcendentally defined. The other is a "plus"—…That's that for the existence and uniqueness of a map[3]

$$c_{X/S} : \mathrm{Cons}(X) \to A(X)$$

we do not need to restrict ourselves to regular X schemes, provided that $A(X)$ is replaced by the whole homology group …"

The interested reader on history of the conjecture will also see, in [156], the notes (164) and $(169)_{iii}$ Episode 1.

5.15.3 Nash Transformation

Let M be a complex analytic manifold, of complex dimension m. Let X be an n-dimensional semi-analytic complex variety, $X \subset M$. We denote by $\Sigma = X_{\text{sing}}$ the singular part of X and by $X_{\text{reg}} = X \setminus \Sigma$ its regular part.

The Grassmann bundle of n (complex) planes in TM is denoted by $G_n(TM)$. The fiber $G_n(T_xM)$ over $x \in M$ is the set of n-planes in $T_x(M)$ and is isomorphic to $G_n(\mathbb{C}^m)$. An element of $G_n(TM)$ is denoted by (x, P) where $x \in M$ and $P \in G_n(T_xM)$.

On the regular part of X, the Gauss map is defined by

$$\gamma : X_{\text{reg}} \longrightarrow G_n(TM) \qquad\qquad \gamma(x) = (x, T_x(X_{\text{reg}})).$$

Definition 5.15.2 The *Nash transformation* $\widetilde{X}$ [204] is defined as the closure of the image of γ in $G_n(TM)$.

[3] The map is labelled by (6) in [156]. Here $\mathrm{Cons}(X)$ is the set of constructible functions, that we denote by $\mathcal{F}(X)$.

$$\begin{array}{ccc} & & G_n(TM) \\ & \overset{\gamma}{\nearrow} & \downarrow \\ X_{\text{reg}} & \hookrightarrow & M \end{array} \qquad \begin{array}{ccc} \widetilde{X} = \overline{\text{Im}\gamma} & \hookrightarrow & G_n(TM) \\ \downarrow \nu & & \downarrow \\ X & \hookrightarrow & M \end{array} \tag{5.40}$$

In general, $\widetilde{X}$ is not smooth, nevertheless, it is an analytic variety and the restriction $\nu : \widetilde{X} \to X$ of the bundle projection $G_n(TM) \to M$ is analytic.
Examples of Nash transformations.

Some examples are given in the survey by G. Gonzalez-Sprinberg [145]:

1. Let X be the union of two 2-planes in $\mathbb{C}^4$ intersecting in only one point, which is singular. Then the Nash blow-up simply separates the two planes, and the fiber over the singular point is given by two points. This is an example of an algebraic set that is not a complete intersection variety.
2. If X is the affine cone over a smooth projective plane curve C, then the singularity at the vertex is isolated, and the exceptional fiber in the Nash blow-up is the dual curve of C.
3. Let X be the "Whitney-Cartan umbrella" defined by the equation $x^2 = y^2 z \in \mathbb{C}^3$ (see Fig. 5.4 and Sect. 5.17.3.2). The Nash transformation is a smooth surface $\widetilde{X}$. The Nash fiber over the origin is a smooth rational curve corresponding to a pencil of planes with a common axis, and with an immersed point given by the dual to the tangent cone, not reduced since it is a double plane. The fiber over each other singular point of X has only two points corresponding to the two planes of the tangent cone.

The tautological bundle over $G_n(TM)$ is denoted by E. The fiber E_P at a point $(x, P) \in G_n(TM)$ is the set of the vectors v in the n-plane $P \in G_n(T_x M)$.

$$E_P = \{v(x) \in T_x M \ : \ v(x) \in P\}$$

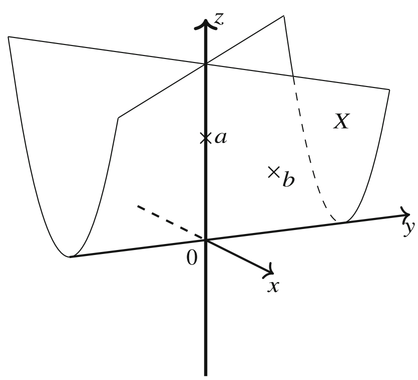


Fig. 5.4 The figure is the real part in $\mathbb{R}^3$ of the Whitney umbrella $X \subset \mathbb{C}^3$ whose equation is $x^2 - y^2 z = 0$. One has $\text{Eu}_0(X) = 1, \text{Eu}_a(X) = 2, \text{Eu}_b(X) = 1$

The restriction of E to $\tilde{X}$ is denoted by $\widetilde{E} = E|_{\tilde{X}}$ and called Nash bundle. One has:

$$\widetilde{E} = E \times_{G_n(TM)} \widetilde{X} = \{(v(x), \tilde{x}) \in E \times \widetilde{X} \; : v(x) \in \tilde{x}\}$$

where $\tilde{x} \in \widetilde{X}$ is an n-complex plane in $T_x(M)$ and $x = \nu(\tilde{x})$. On the inverse image $\widetilde{X}_{\text{reg}} = \nu^{-1}(X_{\text{reg}}) \cong X_{\text{reg}}$ the restriction $\widetilde{E}|_{\tilde{X}_{\text{reg}}}$ can be identified with $T(X_{\text{reg}})$.

One has a diagram:

$$\begin{array}{ccc} \widetilde{E} & \hookrightarrow & E \\ \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & G_n(TM) \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array}$$

An element in $\widetilde{E}$ is written (x, P, v) where $x \in X$, P is an n-plane in $\nu^{-1}(x)$ and v is a vector in P. If $x \in X_{\text{reg}}$, then $P = T_x(X_{\text{reg}})$.

5.15.3.1 Wu Transformation

In 1965, Wu Wen-Tsün defined, in [342, 343], a transformation $\widehat{X}_W$ in the projective situation and using Ehresmann cycles [115]. Let X be a n-dimensional algebraic complex projective variety in $\mathbb{CP}^m$. Consider the algebraic manifold of (complex) dimension $k = m + n(m - n)$ (see 5.26 in Sect. 5.7.3.2).

$$G_{n,m} = \{(x, P)|x \in P, P \text{ is a } n\text{-plane in } \mathbb{CP}^m\}.$$

The Wu transformation of X is the closed subvariety $\widehat{X}_W$ of $G_{n,m}$ obtained as the closure of the set of points $(x, T_x(X_{reg}))$.

Theorem 5.15.3 (Zhou) *[371, 373] Let X be an n-dimensional algebraic complex projective variety in $\mathbb{CP}^m$. The Wu transformation and the Nash transformation are isomorphic.*

5.15.4 *Mather Classes*

Chern-Mather classes were defined by MacPherson in his fundamental article [204]. We provide the original definition, as well as formulations due to Wu and Piene. Alternative definitions and applications are given in two articles in this volume, by Aluffi [13] and by Callejas-Bedregal, Morgado and Seade [74].

The first approach to the proof of the Deligne-Grothendieck's conjecture is to think to the Nash bundle as a substitute to the tangent bundle, in the case of singular varieties. That approach leads to the construction of Mather classes. Let X a possibly singular algebraic complex variety embedded in a smooth one M. We define the Nash transformation $\widetilde{X}$ of X, and the Nash bundle $\widetilde{E}$ on $\widetilde{X}$ as in Sect. 5.15.3.

Definition 5.15.4 The (total) Chern-Mather class (or Mather class) of X is defined [204] by:

$$c_*^{Ma}(X) = \nu_*(c^*(\widetilde{E}) \cap [\widetilde{X}]) \in H_*(X) \tag{5.41}$$

where $c^*(\widetilde{E})$ denotes the usual (total) Chern class of the bundle $\widetilde{E}$ in $H^*(\widetilde{X})$ and the cap-product with $[\widetilde{X}]$ is the Poincaré duality homomorphism (in general not an isomorphism).

The Chern-Mather classes do not satisfy the Deligne-Grothendieck's conjecture (see [143, Contre-exemple, Page 11]).

5.15.4.1 Complex Wu Classes

In 1965, Wu constructed "Chern"-Wu classes. The paper [342], written in Chinese did not have the success it deserved. Jianyi Zhou [371, 373] showed that the Mather classes, defined by MacPherson [204] (Sect. 5.15.4) are the same as the Wu's classes (also see [43]).

Wu used the group $\mathcal{A}_*(X)$ of algebraic equivalence classes (Weil) of X and morphisms [342] (also see [200, 371, 373]). Wu defined a composition of maps $W_k : \mathcal{A}_k(G_{n,m}) \longrightarrow \mathcal{A}_{n-k}(X)$ (see 5.15.3.1) that uses the Ehresmann cycles and corresponds to the Piene expression (see (5.43)).

Definition 5.15.5 [342] The Wu classes are defined by:

$$c^W_{n-k}(X) = \sum_{i=0}^{k} (-1)^i \binom{n-i+1}{n-k+1} W_k\left([k-i/0, \cdots, n-i, n-i+2, \cdots, n+1]\right), \tag{5.42}$$

where $[a/b_0, b_1, \cdots, b_d]$ denote the Ehresmann cycles.

Compare with the Gamkrelidze definition (for manifolds: 5.7.3.2) and the Piene Definition 5.15.4.2, also see Aluffi [11].

Theorem 5.15.6 (Zhou) *[200, 371, 373] The homology Wu classes and Mather classes of an irreducible algebraic projective variety coincide.*

Theorem 5.15.7 (Wu, Zhou) *[342] Let X be a smooth n-dimensional algebraic manifold in $\mathbb{CP}^m$. The Wu classes are the homology Chern classes:*

$$c^W_{n-k}(X) = P_X(c^k(X))$$

where $c^k(X)$ are the classical Chern (cohomology) classes of X and P_X denotes the Poincaré duality isomorphism (see 5.23).

5.15.4.2 Piene's Expression

Ragni Piene [251–254] provides an expression of Mather classes of projective varieties in terms of polar varieties (see Łê Dũng Tráng and Bernard Teissier [197]).

Definition 5.15.8 The k-th polar variety of X relative to a linear subspace L_k of codimension $n-k+2$ in $\mathbb{CP}^n$ is defined by

$$M_k = \text{closure of } \{x \in X_{\text{reg}} | \dim(T_x(X_{\text{reg}}) \cap L_k) \geq k-1\}.$$

For a linear subspace L_k in general position, M_k represents a class of rational equivalence of codimension k in X, denoted by $[M_k]$ and called polar class of X.

Theorem 5.15.9 *[252, Théorème 3] Let us denote by $\mathcal{L}$ the restriction of the hyperplane bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ to X, the Mather classes are equal to:*

$$c_k^{Ma}(X) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1-k+i}{i} c_1(\mathcal{L})^i \cap [M_{k-i}]. \tag{5.43}$$

Reciprocally, the polar classes of X satisfy

$$[M_k] = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1-k+i}{i} c_1(\mathcal{L})^i \cap c_{k-i}^{Ma}(X). \tag{5.44}$$

Mather classes have been computed in various situations, for instance in relation with conormal spaces (Definition 5.16.4) of Schubert varieties by L. C. Mihalcea and R. Singh [222].

Note that Martin Helmer wrote a package "ToricInvariants" [163], using Macaulay2 by Dan Grayson and Mike Stillman [153], in order to compute the Chern-Mather class of a projective toric variety X, pushedforward to the Chow ring of the ambient projective space, without assuming that X is normal.

5.15.5 Weighted Chern-Mather Classes

The "weighted Chern-Mather" classes were defined by P. Aluffi in [3]. in order to extend the notion of Chern-Mather classes to possibly nonreduced schemes Y, taking care of nilpotents. Weighted Chern-Mather classes are suitable weighted sums of the classical Mather classes of subvarieties of Y. The subvarieties are the supports of the components of the (intrinsic) normal cone of Y, and the weights are the lengths of the components of this cone.

If Y is a local complete intersection, then the weighted Chern-Mather classes coincide with Chern-Mather classes.

One of the main applications of weighted Chern-Mather classes is to the computation of the difference between MacPherson's class of a singular hypersurface (in a manifold) and the class of its virtual tangent bundle. This difference is related to the μ-class defined by Aluffi in the context of singularity subscheme of a hypersurface [5, 6].

5.15.6 MacPherson Classes

MacPherson's idea [204] is to give a different weight to the contribution of strata in the Mather construction, depending on the local Euler obstruction. The construction uses both the constructions of Mather classes and local Euler obstruction. Local Euler obstruction has been the subject of many equivalent definitions, examples and applications. It deserves an entire section of its own: the following Sect. 5.17 (see Definition 5.17.1.1).

Using the properties of local Euler obstruction Eu_x, in particular that local Euler obstruction is a constructible function (Proposition 5.17.5), MacPherson proves

Proposition 5.15.10 (MacPherson) [204, Lemma 2] *There is an isomorphism T between the groups of algebraic cycles on X and of constructible functions, given by*

$$T\left(\sum n_i V_i\right)(x) = \sum n_i \mathrm{Eu}_x(\overline{V_i})$$

The ambient complex algebraic manifold M is equipped with a Whitney stratification such that X is union of strata $\{V_\alpha\}$. The proposition implies that there are integers n_α such that, for every point $x \in X$, we have:

$$\sum_\alpha n_\alpha \mathrm{Eu}_x(\overline{V_\alpha}) = 1. \tag{5.45}$$

MacPherson defines the natural transformation

$$c_* : \mathcal{F} \to H_*(\ ;\mathbb{Z}) \quad \text{by} \quad c_*(\mathbf{1}_X) = \sum_\alpha n_\alpha\, i_* c_{Ma}(\overline{V_\alpha})$$

where i denotes the inclusion $\overline{V_\alpha} \hookrightarrow X$, then, by linearity for all constructible functions φ on X. He deduces the "Deligne-Grothendieck" conjecture (Sect. 5.15.2).

Theorem 5.15.11 *[203, 204] There exists a natural transformation from the functor $\mathcal{F}$ to homology*

$$c_* : \mathcal{F} \to H_*(\, ; \mathbb{Z})$$

which, on a nonsingular variety X, assigns to the constant function $\mathbf{1}_X$ the Poincaré dual of the total Chern class of X.

In other words, *to any constructible function φ on a compact complex algebraic variety X, we can assign an element $c_*(\varphi)$ satisfying the following three conditions:*

1. $c_*(\varphi + \psi) = c_*(\varphi) + c_*(\psi)$ *for φ and ψ in $\mathcal{F}(X)$,*
2. $c_*(f_*\varphi) = f_*(c_*(\varphi))$ *for $f : X \to Y$ morphism of complex algebraic varieties and $\varphi \in \mathcal{F}(X)$ (see formule (5.39)),*
3. $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ *if X is a manifold.*

The total Chern-MacPherson class of any compact variety X is defined by

$$c^M(X) = c_*(\mathbf{1}_X). \tag{5.46}$$

MacPherson observes that the compactness restriction may be dropped with minor modifications of his proof if all maps are taken to be proper and if Borel-Moore homology (homology with locally finite supports) is used.

In the Sect. 5.23 we provide some examples of classes depending of the choice of the constructible function, in particular Wu-Mather classes.

5.16 The Chern-Schwartz-MacPherson Classes

Returning to (Sect. 5.4.2), the Alexander isomorphism $H^{2p}(M, M \setminus X) \to H_{2r-2}(X)$ is induced by the isomorphism:

$$C^{2p}_{(D)}(\mathcal{T}, \partial\mathcal{T})) \to C^{(K)}_{2r-2}(X)$$

which associates to a (D)-cochain $(d_i^{2p})^*$ such that $d_i^{2p} \cap X \neq \emptyset$ the (K)-chain σ_i^{2r-2} such that $d_i^{2p} = d(\sigma_i^{2r-2})$.

Theorem 5.16.1 (Brasselet-Schwartz [59]) *The MacPherson class $c^M(X)$ is the image of the Schwartz class $c_S(X)$ by Alexander duality isomorphism (Sect. 5.4.2)*

$$H^{2(m-r+1)}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

We will denote by $c^{SM}(X)$ and call Chern-Schwartz-MacPherson classes the defined (total) class in $H_*(X)$. Many authors write $c_{SM}(X)$ this class however, our upper notation allows to write explicitly the degree of classes.

The proof given in [59] has been simplified in [16] by a new expression of notion of functoriality of Chern classes obtained by P. Aluffi [11].

The following corollaries provide explicite cycles representing the Chern-Schwartz-MacPherson classes (see formula 5.38).

Corollary 5.16.2 *[59] Let (K) be a simplicial triangulation of M compatible with a Whitney stratification of the pair (M, X) and $v^{(r)}$ an r-radial frame defined on the $2p$-skeleton $D^{(2p)}$ of a cellular decomposition (D) dual of (K). The $(r-1)$-st homology Chern-Schwartz-MacPherson class $c^{SM}_{r-1}(X)$ is represented by the cycle*

$$\sum_{\sigma \in X} I(v^{(r)}, \hat{d}(\sigma))\, \sigma$$

with $\dim \sigma = 2(r-1)$.

Corollary 5.16.3 *[59] Let (K) be a simplicial triangulation of M compatible with a Whitney stratification of the pair (M, X) and adapted to a constructible function φ on X. Let $v^{(r)}$ be an r-radial frame defined on the $2p$-skeleton $D^{(2p)}$ of a cellular decomposition (D) dual of (K). The $(r-1)$-st MacPherson class $c^M_{r-1}(\varphi)$ is represented by the cycle*

$$\sum_{\sigma \in X} \varphi(\sigma)\, I(v^{(r)}, \hat{d}(\sigma))\, \sigma,$$

with $\dim \sigma = 2(r-1)$ *and where $\varphi(\sigma)$ is the value of φ on the interior of σ.*

Various authors proposed alternative definition of the Chern-Schwartz-MacPherson cycles for constructible functions, see in particular J. Schürmann and M. Tibar [284].

5.16.1 Alternative Definitions of Chern-Schwartz-MacPherson Classes

The extension from homology to Chow groups was given, in 1984, by Fulton in the context of complex schemes [130, Example 19.1.7, page 355]. In 1990, using the Chow functor instead of homology functor, G. Kennedy generalized the result of MacPherson to algebraically closed fields of characteristic 0 [180].

In [11, 15], P. Aluffi observes that while both functors: constructible functions and Chow are defined for all varieties, the Chow functor A_* is only functorial with respect to proper morphisms. For a variety X, the author defines the pro-Chow group $\widehat{A}_*$ to be the limit of ordinary Chow groups over the inverse system of maps from X to complete varieties. Then $\widehat{A}_*$ is a functor on the category of (not necessary complete) varieties, covariant with respect to arbitrary morphisms, and agreeing with the ordinary Chow functor A_* on complete varieties and proper maps. Aluffi defines a transformation $\mathcal{F} \to \widehat{A}_*$ and illustrates his construction to provide short proofs of two known results on Chern-Schwartz-MacPherson classes: Kwieciński's product formula [190, 191] and the Ehlers, Barthel-Brasselet-Fieseler computation of Chern-Schwartz-MacPherson classes of toric varieties [28, 114].

Alternative definitions provide new insights on the Chern-Schwartz-MacPherson classes. They are either using alternative definition of Chern-Mather classes, or using alternative definition of local Euler obstruction (Sect. 5.17).

In particular, in [197, 317], Lê Dũng Tráng and Bernard Teissier define Chern-Mather classes and local Euler obstruction in terms of polar varieties (see Definition 5.15.8 and Sect. 5.17.1.4). In [4], Paolo Aluffi uses differential forms with logarithmic poles.

In [268], Claude Sabbah provides an alternative definition in terms of conormal cycles and expression of the local Euler obstruction.

Definition 5.16.4 (Sabbah, [268]) Let M a complex analytic manifold and X an irreducible closed analytic subset in M. Let $X_{\text{reg}} = X^0$ the regular part of X and $T^*_{X^0}M$ the conormal bundle of X^0 in T^*M. The closure $T^*_X M = \overline{T^*_{X^0}M} \subset T^*M$

$$T^*_X M = \text{ Closure of } \{(x, \xi) \in T^*X \mid x \in X^0, \xi|_{T_x X^0} = 0\}$$

is the *conormal space* of X in M. The *conormal cycle*, called "classe fondamentale" in [268], is defined by $[T^*_X M]$.

The conormal space is a reduced analytic space, conic on X. The projectivized space $C(X, M) = \mathbb{P}(T^*_X M) \subset \mathbb{P}(T^*M)$ is also named conormal space by some authors.

Linear combinations of the cycles $[T^*_X M]$ form an abelian group (the group of Lagrangian cycles) $L(M)$. Using the Nash transformation (Sect. 5.15.3) and the Gonzalez-Sprinberg–Verdier expression of the local Euler obstruction (formula 5.47)

$$\text{Eu}_a(X) = \int_{\nu^{-1}(a)} c(\widetilde{E}) \cap s(\nu^{-1}(a), \widetilde{X}),$$

one defines

$$\text{Ch}(\text{Eu}(X)) = (-1)^{\dim X}[T^*_X M].$$

The constructible functions $\text{Eu}(X)$ span $\mathcal{F}(X)$, one obtains a homomorphism

$$\text{Ch} : \mathcal{F}(X) \to L(X).$$

The cycle $\text{Ch}(\varphi)$ corresponding to a constructible function φ is called *characteristic cycle*. Note that $\text{Ch}(\mathbf{1}_X)$ is a combination not only of $[T^*_X M]$ but also of conormal cycles to subvarieties of X, according to the singularities of X.

In his article [268], Sabbah shows that operations on $\mathcal{F}(X)$ correspond to natural operations on the group of homogeneous Lagrangian cycles of the cotangent bundle T^*M. Sabbah writes that "En particulier cela montre que la théorie des classes de Chern de $[M]$ se ramène à une théorie de Chow sur T^*M, qui ne fait intervenir que des classes fondamentales."

Chern-Mather classes can be computed in terms of conormal cycles and A. Parusiński and P. Pragacz [246, formula (12)] show that the MacPherson class of the constructible function φ is

$$c_*(\varphi) = (-1)^{(\dim M-1)} c^*(TM) \cap \pi_*(c(\mathcal{O}(1)))^{-1} \cap [\mathbb{P}Ch(\varphi)]).$$

where π is the projection $\mathbb{P}Ch(\varphi) \to M$ and $\mathcal{O}(1)$ is the hyperplane bundle.

Using the Kashiwara expression of local Euler obstruction (Sect. 5.17.1.6), it was shown by Brylinski, Dubson, and Kashiwara [70] that the MacPherson Chern classes of a singular variety X admit a simple expression involving the characteristic cycle of X from the theory of D-modules (also see Ginsburg [138]). J.H.G. Fu provides a geometric insight for the characteristic Kashiwara's cycle [127]. In [128] using this geometric meaning of the construction Fu provides an intuitively proof of the Deligne-Grothendieck axioms (see Theorem 5.15.11).

In a series of lectures, [278], J. Schürmann discusses the calculus of characteristic classes associated with constructible functions. The point of view of characteristic classes of Lagrangian cycles is emphasized.

The relation between Stiefel-Whitney and Chern-Schwartz-MacPherson classes is given by J. Schürmann in [282].

In [283], J. Schürmann compares different notions of transversality for possible singular complex algebraic or analytic subsets of an ambient complex manifold and proves a refined intersection formula for their Chern-Schwartz-MacPherson classes. While the result is known for complex Whitney stratified sets (see [320]), the author extend the result for splayed divisors. The result was conjectured (and proven in some cases) by Aluffi and Faber [19]. The obtained transversality result is based in particular on the multiplicativity of Chern-Schwartz-MacPherson classes with respect to cross products (see Kwieciński and Kwieciński-Yokura product formula [190, 191]).

5.16.2 Thom Polynomials

The definition and properties of Thom polynomials use basic ones regarding singularities of mappings, such as stability, finite determinacy, versal unfoldings, etc. Some of these are recalled. For a complete presentation we refer to the book [227] by David Mond and Juan José Nuño-Ballesteros and their article in this Handbook [228], see also the article by Maria Aparecida Soares Ruas [267] in the same Handbook or Ohmoto [237], [239, Chapter 2]. Thom polynomials will be used in Sect. 5.17.5.4 in relation with the image Milnor number.

5.16.2.1 The Equivalence Relations $\mathcal{A}$ and $\mathcal{K}$ [228, section 1.2] and [267, section 2 and §4.1]

The group of biholomorphic germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ is denoted by $\mathrm{Diff}(\mathbb{C}^m, 0)$. The natural equivalence relation $\mathcal{A}$ classifies map-germs up to isomorphisms of source and target as follows. The direct product $\mathcal{A} = \mathrm{Diff}(\mathbb{C}^m, 0) \times \mathrm{Diff}(\mathbb{C}^n, 0)$ acts on

$$\mathcal{E}_0(m,n) = \{f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0) | f \text{ is holomorphic }\}$$

by $(\sigma, \tau) \cdot f = \tau \circ f \circ \sigma^{-1}$

The natural equivalence relation $\mathcal{K}$ (contact equivalence). measures the tangency of the graphs $y = f(x)$ and $y = 0$ in $\mathbb{C}^m \times \mathbb{C}^n$. The contact group $\mathcal{K}$ consists of pairs (σ, Φ) where $\sigma \in \mathrm{Diff}(\mathbb{C}^m, 0)$ and $\Phi : (\mathbb{C}^m, 0) \to GL(n, \mathbb{C})$, which acts on $\mathcal{E}_0(m,n)$ by $((\sigma, \Phi) \cdot f)(x) = \Phi(x) f(\sigma(x))$.

Example 5.16.5 $f = (x^3 + yx, y)$ and $f = (x^3, y)$ in $\mathcal{E}_0(2,2)$ are $\mathcal{K}$–equivalent but not $\mathcal{A}$-equivalent.

5.16.2.2 Unfoldings [228, § 1.2.1] [267, Definition 4.10]

Let $f : (\mathbb{C}^m, S) \to (\mathbb{C}^n, 0)$ be a map-germ. A *k-parameter unfolding* of f is a map-germ

$$F : (\mathbb{C}^m \times \mathbb{C}^k, S \times \{0\}) \to (\mathbb{C}^n \times \mathbb{C}^k, \{0\} \times \{0\}) \qquad F(x,u) = (\tilde{f}(x,u), u),$$

such that $\tilde{f}(x, 0) = f(x)$. We denote the map $x \mapsto \tilde{f}(x,u)$ by f_u and call f_u a k-parameter deformation of f. Two unfoldings G, F of f with k parameters are equivalent if there are unfoldings of identity maps id_m of $\mathbb{C}^m$ and id_n of $\mathbb{C}^n$

$$\begin{aligned} &\Phi : (\mathbb{C}^m \times \mathbb{C}^k, \{0\} \times \{0\}) \to (\mathbb{C}^m \times \mathbb{C}^k, \{0\} \times \{0\}) \\ \text{and} \quad &\Psi : (\mathbb{C}^n \times \mathbb{C}^k, \{0\} \times \{0\}) \to (\mathbb{C}^n \times \mathbb{C}^k, \{0\} \times \{0\}) \end{aligned}$$

respectively, so that $F \circ \Phi = \Psi \circ G$. An unfolding of f is trivial if it is equivalent to the product $(f \times \mathrm{id}_k)(x,u) = (f(x), u)$.

Given a map $h : (\mathbb{C}^\ell, 0) \to (\mathbb{C}^k, 0)$, the induced unfolding h^*F from F via the base-change h is defined by the unfolding $h^*F(x, v) = (f_{h(v)}(x), v)$. We say that F is an *$\mathcal{A}_e$-versal unfolding* of f if any unfolding of f is equivalent to some unfolding induced from F.

A map-germ $f : (\mathbb{C}^m, S) \to (\mathbb{C}^n, 0)$ is *stable* if every unfolding of f is trivial, i.e. is equivalent to the constant unfolding $(x, u) \mapsto (f(x), u)$.

Two map-germs $f : (\mathbb{C}^{m+k}, 0) \to (\mathbb{C}^{n+k}, 0)$ and $g : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ are stably $\mathcal{A}$-equivalent if f is $\mathcal{A}$-equivalent to the trivial unfolding $g \times id_{(\mathbb{C}^k, 0)}$

We denote by η and call $\mathcal{A}$-singularity type an equivalence class under relation $\mathcal{A}$.

Example 5.16.6 The $\mathcal{A}$-class of $f = (x^3 + yx, y)$ is called an ordinary cusp or stable A_2-singularity.

5.16.2.3 Thom Polynomials

Given a stable singularity type η of holomorphic map-germs from $\mathbb{C}^m$ to $\mathbb{C}^n$, we denote

$$\eta(f) = \{x \in M \mid \text{ the germ } f \text{ at } x \text{ is stably } \mathcal{A}\text{-equivalent to } \eta\}$$

and by $\overline{\eta}(f)$ its closure, called the singular locus of f of type η.

For a holomorphic map $f : M \to N$ between complex manifolds, and a singularity η, In [318], René Thom noticed that the cohomology class represented by the set of points in M where the map has singularity η can be calculated by a universal polynomial depending only on the singularity.

Theorem 5.16.7 *[239, Theorem 4.1] –The Thom polynomial. –For a stable singularity type η, there exists a unique polynomial* $\mathrm{tp}(\eta) \in \mathbb{Z}[c^1, c^2, \ldots]$ *so that for any stable map $f : M \to N$ the singular locus of type η is expressed by the polynomial evaluated by the quotient Chern class $c^i(f) = c^i(f^*TN - TM)$:*

$$\text{Dual } [\overline{\eta}(f)] = \mathrm{tp}(\eta)(c(f)) \in H^{2\,\mathrm{codim}(\eta)}M).$$

Example 5.16.8 The Thom polynomial for the A_2-singularity is $\mathrm{tp}(A_2) = c_1^2 + c_2$.

A major problem is to determine the precise form of $\mathrm{tp}(\eta)$ for a given contact type η (see the discussion in Toru Ohmoto [239, Remark 4.3]).

As an advanced version, the theory of Thom polynomials for stable multi-singularities has been developed by M. Kazarian, [178, 179] that merges multiple point formulas (developed by Kleiman) [183, 184] and the above Thom polynomials for mono-singularities.

In the case that codimension of η is equal to $\dim(M)$, then $\mathrm{tp}(\eta)$ for f counts the number of η-singular points.

5.16.2.4 Thom Polynomials and Chern-Schwartz-MacPherson Classes

Theorem 5.16.9 *[239, p. 193] The higher (Chern-Schwartz-MacPherson) Thom polynomial* tp^{SM} *is introduced so that it universally expresses the Chern-Schwartz-MacPherson class of the η-type singular point locus* $[\overline{\eta}(f)]$*:*

$$\text{Dual } c^{SM}(\overline{\eta}(f)) = c(TM) \cdot \mathrm{tp}^{SM}(\eta)(\overline{\eta}).$$

Here $\mathrm{tp}^{SM}(\eta)(\overline{\eta})$ *is a power series in* $c^i = c^i(f)$ *whose leading term is the Thom polynomial* $\mathrm{tp}(\eta)$.

The Ohmoto strategy is to incorporate not only the Chern-Schwartz-MacPherson classes, but also the Segre-Schwartz-MacPherson classes [13] of the closed embedding $\iota : X \hookrightarrow M$

$$s^{SM}(X, M) = c(\iota^* TM)^{-1} \cap c^{SM}(X) \in H_*(X)$$

into the theory of Thom polynomials, see [239, Section 4.3] for effective computations, using methods developed by Richard Rimányi [261].

Theorem 5.16.10 *[235], [239, Theorem 4.4] For a stable singularity type* η, *there exists a unique power series* $\mathrm{tp}^{SM}(\overline{\eta}) \in \mathbb{Z}[[c^1, c^2, \ldots]]$ *so that for any stable map* $f : M \to N$ *the singular locus of type* η, *one has*

$$\mathrm{Dual}\ s^{SM}(\overline{\eta}(f), M) = \mathrm{tp}^{SM}(\eta)(c(f)) \in H^* M).$$

In particular, if M *is compact, the Euler characteristic of the* η*-type singular locus is given by the degree of* $c_*(\mathbf{1}_{\overline{\eta}(f)})$ *(see formula 5.46), which has a universal expression*

$$\chi(\overline{\eta}(f)) = \int_M c(TM) \cdot \mathrm{tp}^{SM}(\eta)(c(f)).$$

5.16.3 Examples of Chern-Schwartz-MacPherson Classes

Chern-Schwartz-MacPherson classes have been studied and explicitly computed by many authors, providing many examples and applications, for instance in the following cases:

- Definition in terms of Chern-Weil and Čech-de Rham cohomology (see Suwa [310–312] and Brasselet-Seade-Suwa [67]).
- Local complete intersections (Suwa [308] and Yokura [352]).
- Thom spaces (in particular Thom spaces associated to Segre and Veronese embeddings) (Brasselet and Gonzalez-Sprinberg [49, 50]).
- Toric varieties (Barthel, Brasselet, Fieseler and Kaup [28, 30, 31], Ehlers [114], Maxim and Schürmann [219], A. Weber [327]). ,
- Schubert varieties (Aluffi, Mihalcea, Schürmann and Su [22, 23], Jones [175, §5.2], Kumar, Rimányi and Weber [188, 263]).
- Orbit stratifications (Fehér, Rimányi and Weber [121]).
- Determinantal varieties (Nuño Ballesteros, Oréfice-Okamoto and Tomazella [233] and Zhang [363–365]).
- Hypersurfaces (Aluffi [2, 3, 6], Parusiński and Pragacz [246]).

- Degeneracy loci (Parusiński and Pragacz [243–245], Fehér and Rimányi [120], Promtapan and Rimányi [260]), see also [239, Theorem 3.13].
- Embeddable scheme (Aluffi [14]).
- Projective schemes (Aluffi [7]).
- Case of DM-stacks (Jiang [173] also see [99]).
- Applications to Physics (Aluffi and Marcolli [20], Aluffi and Esole [17, 18]).
- Algorithmic expression (Helmer [164]).
- Relation with maximum likelihood degree (Rodriguez and Wang [265, 266]).
- etc. (The list is far from being exhaustive).

5.16.4 The Equivariant Case

Generalizing the study of equivariant characteristic classes in the case of manifolds, equivariant Chern-Schwartz-MacPherson classes have been described by T. Ohmoto [235], by Cappell, Maxim, Schürmann and Shaneson [76] and by Rimányi and Varchenko [262]. In [235, 238] (see also [239, section 3.4]), Ohmoto uses an equivariant version of Thom polynomials to describe equivariant Chern-Schwartz-MacPherson classes.

Equivariant Chern-Schwartz-MacPherson classes of symmetric and of skew-symmetric determinantal varieties were computed explicitly by means of the Fehér-Rimányi method and in terms of Schur polynomials by Sutipoj Promtapan [259].

An equivariant formula for Chern-Schwartz-MacPherson classes of hypersurfaces of projective varieties is provided by Xiping Zhang [366].

5.17 Local Euler Obstruction

The local Euler obstruction, defined by MacPherson is one of the main ingredients of the construction of the MacPherson natural transformation and classes. The local Euler obstruction is the subject of several equivalent definitions as well as of calculus in several special cases and many generalizations (see the surveys [42] and [51] and the book [67]). This is the reason why it deserves a separate chapter.

Note: The notation of local Euler obstruction is, unfortunately, not uniformized. For instance it can be $\mathrm{Eu}_a(X)$ as well as $\mathrm{Eu}_X(a)$ according to the authors. As there is no risk of confusion, and so that the reader can find the same notation here and in the references, it is the latter that we use in each case, even if it means not being consistent.

5.17.1 Definitions

5.17.1.1 MacPherson's Original Definition

The local Euler obstruction was first defined by MacPherson [204] (also see [143]) using differential forms. MacPherson's definition is the following.

Let $z_1, \ldots, z_m$ be local coordinates in M such that $z_i(a) = 0$ and let $\|z\| = \sqrt{z_1\overline{z_1} + \cdots + z_m\overline{z_m}}$. Since $\|z\|^2$ is a real-valued function, $d\|z\|^2$ may be considered as a section of TM^* where $*$ denotes the real dual bundle retaining only its orientation from the complex structure. The section $d\|z\|^2$ pulls back and restricts to a section ρ of TX^*. Considering the Nash transformation $\nu : \widetilde{X} \to X$ (cf. diagram 5.40), by Whitney condition (a) (see Definition 5.10.2), for small enough ε, the section ρ is nonzero over $\nu^{-1}(z)$ with $0 < \|z\| \leq \varepsilon$.

By the Bertini-Sard theorem, (see for instance [323]) the sphere $\mathbb{S}_\varepsilon$ of radius ε, boundary of the ball $\mathbb{B}_\varepsilon$, centered at 0 is transverse to the strata V_α if ε is small enough. The obstruction to extending ρ as a nonzero section of TX^* from $\nu^{-1}(\mathbb{S}_\varepsilon(0))$ to $\nu^{-1}(\mathbb{B}_\varepsilon(0))$, denoted by $\mathrm{Eu}(TX^*, \rho)$ in [204], lies in $H^n(\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon); \mathbb{Z})$. The local Euler obstruction of X at a is defined as the evaluation of $\mathrm{Eu}(TX^*, \rho)$ on the orientation class $O_{\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon)}$ in $H_n(\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon); \mathbb{Z})$, that is:

$$\mathrm{Eu}_a(X) = \langle \mathrm{Eu}(TX^*, \rho),\ O_{\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon)} \rangle.$$

5.17.1.2 Brasselet-Schwartz Definition

The equivalent dual definition given in [59] uses vector fields. Whitney condition (a) (see Definition 5.10.2) implies that a stratified vector field v defined on $A \subset X$ admits a canonical lifting $\tilde{v}$ on $\nu^{-1}(A)$ as a section of $\widetilde{E}$ [59, Proposition 9.1].

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\nu_*} & TM|_X \\ \tilde{v}\uparrow\downarrow & & v\uparrow\downarrow \\ \widetilde{X} & \xrightarrow{\nu} & X \end{array} \qquad \nu_*(x, \tilde{x}, v(x)) = (x, v(x))$$

Consider a *radial* stratified vector field v in a neighbourhood of the point $\{0\} \in X$ so that there exists $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, the vector $v(x)$ is pointing outwards of the ball $\mathbb{B}_\varepsilon$ over the boundary $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$. If ε is small enough, the sphere $\mathbb{S}_\varepsilon$ is transverse to the strata V_α.

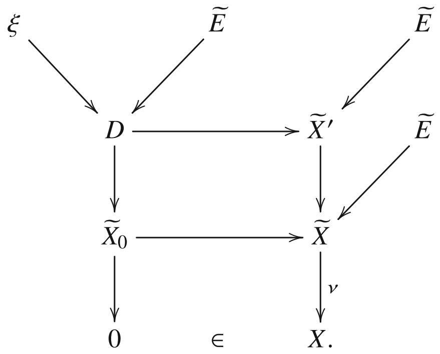
Theorem 5.17.1 [59] *Let v be a stratified vector field pointing outwards $\mathbb{B}_\varepsilon$ along $\mathbb{S}_\varepsilon$ and $\tilde{v}$ the lifting of v on $\nu^{-1}(X \cap \mathbb{S}_\varepsilon)$. The local Euler obstruction* $\mathrm{Eu}_0(X)$ *is the obstruction to extend $\tilde{v}$ as a nowhere zero section of $\tilde{E}$ over $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$, evaluated on the orientation class $O_{\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon)}$:*

$$Eu_a(X) = Obs(\tilde{v}, \widetilde{E}, \nu^{-1}(X \cap \mathbb{B}_\varepsilon)).$$

5.17.1.3 Gonzalez-Sprinberg – Verdier Definition

The Gonzalez-Sprinberg–Verdier definition [143, 144] extends the MacPherson definition to algebraically closed fields, using Segre classes.

The construction of local Euler obstruction being local, one can assume that $X \subset \mathbb{C}^m$ and $0 \in X$. Let $\widetilde{X}_0 = \nu^{-1}(0)$ the fiber of the Nash transformation at 0. Let $\widetilde{X}'$ be the blow-up of $\widetilde{X}_0$ in $\widetilde{X}$. Let D be the exceptional divisor in $\widetilde{X}'$, inverse image of $\widetilde{X}_0$. On D there are two vector bundles: the restriction to D of the inverse image of $\widetilde{E}$, still denoted by $\widetilde{E}$ (denoted by $\widetilde{T}$ in [143, §4.3]), and the normal bundle to D in $\widetilde{X}'$, denoted by ξ. One has the following diagram.



Theorem 5.17.2 *[143, §4.3] The local Euler obstruction of X at a is equal to:*

$$Eu_a(X) = \int_D c_{n-1}(\widetilde{E} - \xi) \cap [D],$$

where $\dim D = \dim X - 1 = n - 1$, $c(\widetilde{E} - \xi) = c(\widetilde{E})/c(\xi)$ *and* $c_i(\xi) = 0$ *for* $i \geq 2$.

The Gonzalez-Sprinberg–Verdier's formula can be written in terms of Segre class [143]

Corollary 5.17.3 *The local Euler obstruction of X at a is equal to:*

$$Eu_a(X) = \int_{\nu^{-1}(a)} c(\widetilde{E}) \cap s(\nu^{-1}(a), \widetilde{X}), \tag{5.47}$$

where s is the (relative) Segre class of the normal cone of $\nu^{-1}(a)$ (for a complete statement see Aluffi [13] and Fulton [130]).

Note that, in [298], Marcos Sebastiani provides a simpler proof of the formula than the one by Gonzalez-Sprinberg.

Recently, this viewpoint of the local Euler obstruction plays an important role in Donaldson-Thomas theory by the work of Behrend [35] in the context of Deligne-Mumford stacks. Following the method of Gonzalez-Sprinberg, Yunfend Jiang [173, 174] gives a similar formula as Corollary 5.17.3 in this framework.

5.17.1.4 Lê-Teissier's Definition

The Lê-Teissier definition uses the notion of polar varieties 5.15.8. We consider, in $\mathbb{C}^m$ a flag of vector subspaces

$$(\mathcal{D}) \qquad D_n \subset D_{n-1} \subset \cdots \subset D_2 \subset D_1 \subset D_0 = \mathbb{C}^m.$$

where i is the codimension of D_i in $\mathbb{C}^m$. The set

$$S_k(\mathcal{D}) = \{V \in G \mid \dim(V \cap D_{d-k+1} \geq k\}$$

is an irreducible algebraic subvariety of G with codimension k, called Schubert variety associated to the flag $(\mathcal{D})$ (see Sect. 5.5.4). It is denoted by $c_k(\mathcal{D})$ in [197, 315]) (also see [252, 253]).

Assuming $X \subset \mathbb{C}^m$, the Nash transformation can be written in the following way. Let $G = G_n(\mathbb{C}^m)$ be the Grassmannian manifold of n-planes in $\mathbb{C}^m$. Let $\gamma^0 : X_{\text{reg}} \to G$ be the analytic map $x \mapsto T_x(X_{\text{reg}})$. The restriction of the projection $\text{pr}_1 : X \times G \to X$ to the closure $\widetilde{X}$ in $X \times G$ of the graph of γ^0 is isomorphic to the Nash transformation $\widetilde{X}$ of X. The Gauss map $\gamma : \widetilde{X} \to G$ is defined as the restriction to $\widetilde{X}$ of the projection $\text{pr}_2 : X \times G \to G$. Lê and Teissier deduce from a theorem by Kleiman [182, 2. Theorem] that if $(\mathcal{D})$ is a sufficiently general flag, then $\gamma^{-1}(S_k(\mathcal{D}))$ is empty or has codimension k in $\widetilde{X}$.

For a suitable Zariski open and dense subset in the space of flags (see [197, (2.2.2)]) the reduced analytic subspace corresponding to $\nu(\gamma^{-1}(S_k(\mathcal{D})))$ is well defined, called the polar variety of X of codimension k and denoted by $P_k(\mathcal{D})$ (see M_k in the Sect. 5.15.4.2).

Theorem 5.17.4 (Lê-Teissier) *[197, Corollaire 5.1.2] For every sufficiently general flag $\mathcal{D} \in \mathbb{C}^m$, one has*

$$Eu_a(X) = \sum_{i=0}^{n-1} (-1)^{n-1-i} m_0(P_{n-1-i}(\mathcal{D})). \tag{5.48}$$

where $m_0(C)$ is the multiplicity of C at 0 *(see [197, Corollaire 4.1.9]).*

The particular case of surfaces is described by Lê D. T. in [193].

In [71], R. Callejas-Bedregal, M.J. Saia, and J.N. Tomazella apply the Lê-Teissier formula to compute the polar multiplicities of a germ at zero of an analytic variety $Y \subset \mathbb{C}^p$ which is the image by a finite morphism $f : Z \to Y$ of an isolated complete intersection singularity, (ICIS) $Z \subset \mathbb{C}^n$.

The Lê-Teissier method has been implemented for effective computation by Tajima and Nabeshima [314].

5.17.1.5 Sabbah's Definition

We note that the idea to consider the (complex) dual Nash bundle was already present in [269], where Sabbah introduces a local Euler obstruction defined by $\check{\mathrm{Eu}}_V(0)$ that satisfies $\check{\mathrm{Eu}}_V(0) = (-1)^d \mathrm{Eu}_V(0)$. Also see [281, sec. 5.2].

Claude Sabbah provides a "dual" version in [268, 269], using the conormal space (Sect. 5.16.1) and associating to each irreducible subanalytic space Z in the manifold M the conormal space $T^*_Z(M)$. Then, operations on $\mathcal{F}(X)$ (in particular intersection and specialization) are translated into operations on homogeneous Lagrangian cycles of the cotangent space T^*M. Applications are provided by Sabbah, in particular to characteristic cycles of holonomic differential systems [268], also see Goresky [146, §5.10.2].

5.17.1.6 Kashiwara's Definition

M. Kashiwara [176] introduced a local invariant of singular complex spaces in relation to his famous local index theorem for holonomic $\mathcal{D}$-modules. It was later observed by Dubson to be the same as MacPherson's local Euler obstruction [70, 103, 104] (also see Ginsburg [138]).

5.17.1.7 Aluffi's Definition

In [13, §1.3.2] provides various expressions of local Euler obstruction in terms of Segre classes. In order not to repeat, we refer to that article in this volume.

5.17.1.8 Dutertre's Definition

In [105], by applying a local Gauss-Bonnet formula for closed subanalytic sets to the complex analytic case, N. Dutrertre obtains characterization of the local (and also the global) Euler obstruction of a complex analytic germ in terms of the Lipschitz-Killing curvatures and the Chern forms of its regular part.

5.17.2 *Main Properties of the Local Euler Obstruction*

The local Euler obstruction satisfies the following properties:

(i) $Eu_x(X) = 1$ if x is a regular point of X.
(ii) Constructibility:

Proposition 5.17.5 ([59, 204], and Many Authors) *The local Euler obstruction is a constructible function, constant along the strata of a Whitney stratification of X.*

iii) Proportionality Theorems ([59], Théorème 11.1):

Theorem 5.17.6 (Proportionality Theorem for Vector Fields) *Let v be any radial vector field with an isolated singularity at the point $a \in V_\alpha$, with index $I(v, a) = I(v|_{V_\alpha}, a)$, and let $\mathbb{B}_\varepsilon$ a small ball centered at a without other singularity of v, then*

$$Obs(\tilde{v}, \widetilde{E}, \nu^{-1}(\mathbb{B}_\varepsilon \cap X)) = Eu_a(X) \cdot I(v, a) \tag{5.49}$$

The bundle on $\widetilde{X}$ associated to $\widetilde{E}$ whose fiber on the point $\widetilde{x}$ is the set of r-frames whose vectors belong to $\widetilde{E}|_{\widetilde{x}}$ is denoted by $\widetilde{E}^r$.

Theorem 5.17.7 (Proportionality Theorem for Frames) *Let $v^{(r)}$ be a radial r-frame with isolated singularities on the $2p$-cells d_i^{2p} with index $I(v^{(r)}, \hat{\sigma}_i)$ at the barycenter $\{\hat{\sigma}_i\} = d_i^{2p} \cap \sigma_i$* (see Theorem 5.14.2 (ii)). *Then the obstruction to the extension of $\tilde{v}^{(r)}$ as a section of $\widetilde{E}^r$ on $\nu^{-1}(d_i^{2p} \cap X)$ is*

$$Obs(\tilde{v}^{(r)}, \widetilde{E}^r, \nu^{-1}(d_i^{2p} \cap X)) = Eu_{\hat{\sigma}_i}(X) \cdot I(v^{(r)}, \hat{\sigma}_i). \tag{5.50}$$

5.17.3 Some Examples of the Local Euler Obstruction

5.17.3.1 Curves

The local Euler obstruction of a curve X at a point x is the multiplicity of the curve at that point ([204, 3.2], [143, §4.5 2], [252, §6, a)]).

5.17.3.2 The Whitney Umbrella

Gerardo Gonzalez-Sprinberg gives the example of local Euler obstruction at points of the Whitney umbrella. [143, §4.5 3]. The computation can be made also using the Lê-Teissier formula 5.48.

5.17.3.3 Singular Point in a Hypersurface

In [143, §5], G. Gonzalez-Sprinberg provides a general formula for two-dimensional hypersurfaces.

Ragni Piene [252] shows that if $x \in X$ is an isolated point in a hypersurface, then the local Euler obstruction at x is given by

$$\mathrm{Eu}_x(X) = 1 + (-1)^n \mu_x^{(n-1)},$$

where $\mu_x^{(n-1)}$ is the Milnor number of a generic hyperplane section of X at x.

As an example, one recovers the MacPherson's example of a cone X over a smooth plane curve of degree d (see [204, 3.2], [143, §4.5 5]). The local Euler obstruction at the vertex v of the cone is given by

$$\mathrm{Eu}_v(X) = 2d - d^2.$$

5.17.3.4 Toric Varieties

G. Gonzalez-Sprinberg in [142] and [143, §4.5] gives some examples of local Euler obstruction. In particular for toric surfaces [142]. The local Euler obstruction at 0 of the toric surface X_σ associated to a cone $\sigma = (e_2, pe_1 - qe_2)$ in $\mathbb{R}^2$ with $0 < q < p$ and p, q coprimes, is $\mathrm{Eu}_{X_\sigma}(0) = 3 - k$ where k is the minimum embedded dimension of X_σ.

The formula was generalized by Yutaka Matsui and Kiyoshi Takeuchi [218] for normal toric varieties, using Newton's polyhedra.

In the case of toric varieties, many authors contributed to provide explicit formulae of local Euler obstruction. The formulae of Yutaka Matsui and Kiyoshi Takeuchi have inspired in particular Ragni Piene [254] and Berndt Ivar Utstøl Nødland [232]. Ragni Piene gives a new formulation equivalent to the one by of Matsui and Takeuchi.

In [93, 94], Dalbelo and Grulha introduce the notion of multitoric surfaces, whose irreducible components are toric surfaces. They generalize the Gonzalez-Sprinberg result in this situation and provide interesting examples of explicit computations of local Euler obstruction for some families of determinantal surfaces.

5.17.3.5 Determinantal Varieties

The case of codimension two determinantal varieties with isolated singularities (IDS) is described in [250], relating the Milnor number to the Ebeling–Gusein-Sade index (Sect. 5.17.6).

Definition 5.17.8 Let $n, k, s \in \mathbb{Z}$, $n \geq 1$, $k \geq 0$ and let $\mathrm{Mat}_{(n,n+k)}(\mathbb{C})$ be the set of all $n \times (n + k)$ matrices with complex entries. The subset $\Sigma^s \subset \mathrm{Mat}_{(n,n+k)}(\mathbb{C})$ formed by matrices that have rank less than s, with $1 \leq s \leq n$ is called *generic determinantal variety*.

In [112] (2009) Ebeling and Gusein–Zade introduced the notion of a determinantal variety with an *essentially isolated determinantal singularity* (EIDS) ([112, Section 1]), as a generalization of isolated singularity.

In [86] (2018) Nancy Chachapoyas–Siesquén computes the local Euler obstruction of EIDS. The author obtains explicit formulae to calculate the local Euler obstruction for the determinantal varieties whose singular set is an isolated complete intersection singularity (ICIS).

In [367] (2018) Xiping Zhang gives explicit formulae computing the Chern-Mather class and the Chern-Schwartz-MacPherson class of generic determinantal varieties. He also obtain formulae for the conormal cycles and the characteristic cycles of these varieties (Sect. 5.16.1). For some small values of n, k and s, Zhang uses Macaulay2 [153] to exhibit examples of the considered classes.

In [134] (2019) Terence Gaffney, Nivaldo G. Grulha Jr. and Maria A. S. Ruas compute the local Euler obstruction of generic determinantal varieties and apply this result to compute the Chern-Schwartz-MacPherson class of such varieties. In a second part they compute the Euler characteristic of the stabilization of an essentially isolated determinantal singularity (EIDS). The formula is given in terms of the local Euler obstruction and Gaffney's m_d multiplicity [132].

Theorem 5.17.9 *[134] Let* $\Sigma^s \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ *be a generic determinantal variety. The local Euler obstruction of* Σ^s *at* 0 *is*

$$\mathrm{Eu}_{\Sigma^s}(0) = \binom{n}{s-1}, \qquad \textit{for } 1 \leq s \leq n. \tag{5.51}$$

The authors provide explicit formulae, in different particular situations and, in particular, they recover the Chachapoyas formula [86].

In [369] Xiping Zhang, using different methods and working over the general framework of arbitrary algebraically closed fields of characteristic 0 shows that the formula 5.51 also holds in this context

In [368], Xiping Zhang finds explicit formulae for Chern-Schwartz-MacPherson classes and Chern-Mather classes of EIDS via Schubert calculus. As corollaries the author obtains formulae for their generic sectional Euler characteristics, characteristic cycles and polar classes.

In [27] Grazielle F. Barbosa, Nivaldo G. Grulha and Marcelo J. Saia apply the theory developed by Gaffney which shows how to determine a Whitney stratification of discriminants of any finitely determined holomorphic map germ in the nice dimensions of Mather [211] [267, Section 5]. The authors compute the local Euler obstruction at 0 in a class of discriminants of finitely determined map germs from $\mathbb{C}^{n+p}$ to $\mathbb{C}^p$ with $n \geq 0$ and with only A_k singularities.

5.17.3.6 Ruled Surfaces

Ruled surfaces are also interesting objects: a germ of ruled surface in $\mathbb{C}^3$ is image of the application $f : D \times \mathbb{C} \to \mathbb{C}^3$

$$f(t, u) = \alpha(t) + u\beta(t),$$

where $D \subset \mathbb{C}$ is a disk centered at the origin and α and β are complex spatial curves with $\beta \not\equiv 0$. We call $\alpha : D \to \mathbb{C}^3$ the *base curve* and $\beta : D \to \mathbb{C}^3$ the *steering curve*. In [160] N. Grulha, M. Escudeiro Hernandes and R. Martins compute the local Euler obstruction of the ruled surface in terms of multiplicities of the pair (α, β). As a consequence of this result, for any positive integer n, it is possible to produce a germ of ruled surface $(X, 0)$ such that $Eu_X(0) = n$.

5.17.4 Generalizations of Local Euler Obstruction

5.17.4.1 Local Euler Obstruction and Hyperplane Sections

The idea of studying the local Euler obstruction “à la” Lefschetz, using hyperplane sections, appears in the work of Dubson [103] and Kato [177]. The approach in Brasselet, Lê and Seade [53] is topological.

Theorem 5.17.10 *[53] Let $(X, 0)$ be a germ of an equidimensional complex analytic space in $\mathbb{C}^m$. Let V_α, $\alpha = 1, \dots, \ell$, be the (connected) strata of a Whitney stratification of a small representative X of $(X, 0)$ such that 0 is in the closure of every stratum. There is a non-empty Zariski open set Ω in the space of complex linear forms on $\mathbb{C}^m$ such that, for each $l \in \Omega$ there is ε_0 such that for any ε, $\varepsilon_0 > \varepsilon > 0$ and $t_0 \neq 0$ sufficiently small, the local Euler obstruction of $(X, 0)$ is equal to:*

$$\mathrm{Eu}_X(0) = \sum_{\alpha=1}^{\ell} \chi(V_\alpha \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_\alpha), \qquad (5.52)$$

where χ denotes the Euler-Poincaré characteristic and $\mathrm{Eu}_X(V_\alpha)$ is the value of the Euler obstruction of X at any point of V_α, $\alpha = 1, \dots, \ell$.

The result is a Lefschetz type formula for the local Euler obstruction and the proof uses the M.-H. Schwartz technics developed in Sect. 5.12.The result shows that the local Euler obstruction, as a constructible function, satisfies the Euler condition relative to generic linear forms (see Definition 5.62).

Theorem 5.17.10 was proved in [53]. A (simpler) alternative proof is given by Schürmann in [276]. Notice that the formula above is somehow in the spirit of the formula by Lê-Teissier in [197].

The theorem has some interesting consequences. The generic slice $X \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)$ in formula (5.52) is by definition (see [149]) the *complex link* of 0 in X. In the case of an isolated singularity the complex link is smooth and there is only one stratum appearing in the sum in Theorem 5.17.10. In this case the theorem gives:

Corollary 5.17.11 *[53] Let X be an equidimensional complex analytic subspace of $\mathbb{C}^m$ with an isolated singularity at 0. The Euler obstruction of X at 0 equals the Gómez-Mont–Seade–Verjovsky index (GSV index) (see [141]) of the radial vector field on a general hyperplane section $X \cap H$.*

Corollary 5.17.12 *[103, 197] Let X be an equidimensional complex analytic space of dimension d in $\mathbb{C}^m$ whose singular set $\mathrm{Sing}(X)$ is one-dimensional at 0. Let l be a general linear form defined on $\mathbb{C}^m$ and denote by $\mathbf{F}_t$, the local Milnor fiber at 0 of the restriction of l to X. The singularities of $\mathbf{F}_t$ are the points $\mathbf{F}_t \cap \mathrm{Sing}(X) = \{x_1, \dots, x_k\}$. Then,*

$$\mathrm{Eu}_X(0) = \chi(\mathbf{F}_t) - k + \sum_1^k \mathrm{Eu}_X(x_i).$$

5.17.4.2 The Local Euler Obstruction of a Function

The local Euler obstruction of a function is defined by J.-P. Brasselet, D. Massey, A. J. Parameswaran and J. Seade in [56], in order to measure how far the equality given in Theorem 5.17.10 is from being true if we replace the generic linear form l by some other function on X with at most an isolated stratified critical point at 0. Let $(X, 0)$ be a complex analytic germ contained in an open subset U of $\mathbb{C}^m$ and endowed with a complex analytic Whitney stratification $\{V_\alpha\}$ such that every stratum contains 0 in its closure.

Using ideas of Thom and M.-H. Schwartz (see Sect. 5.12), it is possible to construct a stratified vector field, taking, for each stratum V_α of X, the gradient vector field of the restriction of f to V_α, and then, using the M.-H. Schwartz techniques, gluing all these vector fields together, obtaining a stratified vector field $\overline{\nabla}_X f$ (see [56, 209] for details).

Definition 5.17.13 Let $\nu : \widetilde{X} \to X$ be the Nash transformation of X. The *local Euler obstruction of f on X* at 0, denoted $\mathrm{Eu}_{f,X}(0)$, is the local Euler obstruction $\mathrm{Eu}(\overline{\nabla}_X f, X, 0)$ of the stratified vector field $\overline{\nabla}_X f$ at $0 \in X$.

These definitions and constructions also work when f is the restriction to X of a real analytic function on the ambient space. For instance, if f is the function distance to 0 on X, then $\overline{\nabla}_X f$ is a radial vector field and the invariant $\mathrm{Eu}_{f,X}(0)$ is the usual local Euler obstruction of X at 0.

The following result [56] compares the local Euler obstruction of the space X with that of a function on X.

Theorem 5.17.14 *[56, 209] Let* $f : (X, 0) \to (\mathbb{C}, 0)$ *have an isolated singularity at* $0 \in X$. *One has:*

$$\mathrm{Eu}_{f,X}(0) = \mathrm{Eu}_X(0) - \left(\sum_\alpha \chi(V_\alpha \cap \mathbb{B}_\varepsilon \cap f^{-1}(t_0)) \cdot \mathrm{Eu}_X(V_\alpha) \right). \tag{5.53}$$

In other words, the invariant $\mathrm{Eu}_{f,X}(0)$ can be regarded as the "defect" for the local Euler obstruction of X to satisfy the Euler condition with respect to the function f. In this way one can generalize the definition of the local Euler obstruction to functions with non-isolated singularities and one gets the *Euler defect* introduced in [56]. This arises as a natural application of Massey's work in [207, 208] on derived categories and intersections of characteristic cycles.

The relative local Euler obstruction, defined in [56] was discussed, with equivalent definitions, properties and interesting examples in [209]. Provided that p is a "stratified isolated critical point" of $f : X \to \mathbb{C}$, D. Massey extends the Definition 5.17.13 to possibly non-isolated critical points of functions on spaces which need not be pure-dimensional [209].

In [107] Dutertre and Grulha present an alternative proof of the formula 5.53 using Ebeling and Gusein-Zade's results on the radial index and the local Euler obstruction of 1-forms (see Sect. 5.17.6 and Definition 5.17.23).

An example of computation of $\mathrm{Eu}_{f,X}(0)$ is given in [297, Example 4.1]. In [24], Ament, Nuño-Ballesteros, Oréfice-Okamoto and Saia compute the local Euler obstruction of a function on a determinantal variety and on a curve.

Stability of the local Euler obstruction of a function has been established by N. Grulha in [159].

In Dalbelo, Grulha Jr. and Pereira ([95]), the authors compute the local Euler obstruction of polynomials on a family of determinantal surfaces. In the same direction, in [97] Dalbelo and Pereira give a formula to compute the local Euler obstruction of a function $f : (X, 0) \to (\mathbb{C}, 0)$ where X is a multitoric surface.

5.17.4.3 Local Euler Obstruction of Map-Germs

The local Euler obstruction of map-germs was studied in relation with Whitney equisingularity by V.H. Pérez and D. Levcovitz and M.J. Saia [247] (case $\mathbb{C}^n \to \mathbb{C}^n$), V.H.Pérez, E.C. Rizziolli and M.J. Saia [248] and E.C. Rizziolli and M.J. Saia [264] (case $\mathbb{C}^n \to \mathbb{C}^3$, with $n > 3$), V.H. Pérez and M. Saia [249] (case $\mathbb{C}^n \to \mathbb{C}^p$, with $n < p$).

5.17.4.4 The Local Euler Obstruction via Morse Theory

The relation between local Euler obstruction of f and the number of Morse points of a Morsification of f is described, for particular germs of singular varieties, in [296] by J. Seade, M. Tibar and A. Verjovsky.

Stratified Morse theory (Goresky-MacPherson [149, p. 52] and Goresky [146]) yields a clear understanding of what the invariant $\mathrm{Eu}_{f,X}(x)$ is for arbitrary functions with an isolated singularity. These results can also be deduced from Schürmann's [275], and also from the work of D. Massey [207, 208].

Definition 5.17.15 Let V_α be a Whitney stratification of X and let $f : X \to \mathbb{C}$ be the restriction to V of a holomorphic function $F : \mathbb{C}^m \to \mathbb{C}$, with $f(x) = 0$. One says that $f : (X, x) \to (\mathbb{C}, 0)$ has a *stratified Morse critical point* at $x \in X$ if the dimension of the stratum V_α that contains x is ≥ 1, the restriction of f to V_α has a Morse singularity (non degenerate critical point) at x and f is general with respect to all other strata containing x in its closure, i.e., $\mathrm{Ker}\, dF(x)$ is transverse in $\mathbb{C}^m$ to every limit of tangent spaces $T_{x_i}(V_\beta)$, for every stratum V_β such that $V_\alpha \subset \overline{V}_\beta$ and every sequence $x_i \in V_\beta$ converging to x.

Every map-germ f on $(X, 0)$ with an isolated singularity can be "morsified", i.e., approximated by Morse singularities [195].

Theorem 5.17.16 *[296] Let f be a holomorphic function germ on $(X, 0)$ with an isolated singularity (stratified critical point) at 0, restriction of a function F on an open subset in $\mathbb{C}^m$. Let $V_\alpha \subset X$ be the stratum that contains 0. Then:*

1. *If $\dim V_\alpha < \dim X$ and $\mathrm{Ker}\, dF$ does not vanish on any generalized tangent space of the regular stratum (in particular if f is Morse at 0), then $\mathrm{Eu}_{f,X}(0) = 0$.*
2. *If f has a stratified Morse singularity at $0 \in V_\alpha$ and $\dim V_\alpha = \dim X = n$, then $\mathrm{Eu}_{f,X}(0) = (-1)^n$.*
3. *In general, the number of critical points of a Morsification of f in the regular part of X is $(-1)^{n+1}\mathrm{Eu}_{f,X}(0)$.*

In [296], Seade, Tibar and Verjovsky show that the local Euler obstruction of f is closely related to the number of Morse points of a Morsification of f.

Proposition 5.17.17 ([296] Proposition 2.3) *Let $f : X \to \mathbb{C}$ be the an analytic function with isolated singularity at the origin. Then:*

$$\mathrm{Eu}_{f,X}(0) = (-1)^d n_{\mathrm{reg}},$$

where n_{reg} is the number of Morse points on X_{reg} in a stratified Morsification of f lying in a small neighbourhood of 0.

5.17.5 Comparison with Generalizations of Milnor Numbers

5.17.5.1 Milnor-Lê Number

In [196] Lê D. T. proposes a new notion of Milnor number, that is a generalization of the Milnor number for analytic functions defined on singular analytic spaces such that the rectified homotopical depth of X at 0, denoted $\mathrm{rhd}(X, 0)$ (see [194, 196]) satisfies $\mathrm{rhd}(X, 0) = \dim_{\mathbb{C}}(X, 0)$.

Let X be a sufficiently small representative of the germ $(X, 0)$. The Milnor fiber of the complex analytic function f, defined on X, with an isolated singularity at 0 (in the stratified way), has the homotopy type of a bouquet of spheres. Lê's Milnor number, denoted by $\mu_L(f)$, is defined as the number of spheres in the bouquet.

The relations between this invariant and the local Euler obstruction of f were obtained by Seade, Tibar and Verjovsky [297]. In particular:

Theorem 5.17.18 *Let X be a sufficiently small representative of the germ $(X, 0)$ of a complex analytic space. Consider a complex analytic function defined on X with a stratified isolated singularity at 0. If $\mathrm{rhd}(X, 0) = \dim_{\mathbb{C}}(X, 0)$, then*

$$\mu_L(f) \geq (-1)^{\dim_{\mathbb{C}}(X,0)} Eu_{f,X}(0).$$

The condition $\mathrm{rhd}(X, 0) = \dim(X, 0)$ is satisfied for a complete intersection with isolated singularity (ICIS). In this case the following holds (see [297, Formule (3)]):

Theorem 5.17.19 *Let X be a sufficiently small representative of an ICIS germ $(X, 0)$, f an analytic function on X with stratified isolated singularity at 0, and l a generic linear form. Then,*

$$Eu_{f,X}(0) = (-1)^{\dim_{\mathbb{C}}(X,0)}[\mu_L(f) - \mu_L(l)].$$

5.17.5.2 Bruce and Roberts' Milnor Number

Bruce and Roberts gave in [69] an alternative generalization for the notion of Milnor number for a function on a singular variety. One of the main goals is to characterize germs of diffeomorphisms preserving X. The technique is the integration of germs of vector fields tangent to X. An important result is that Bruce and Roberts's Milnor number is a topological invariant for families of functions with isolated singularities defined on hypersurfaces with isolated singularities [158].

Let Ω be an open subset of $\mathbb{C}^m$ containing the origin. We denote by $\mathcal{O}$ the sheaf of germs of holomorphic germs of functions $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ at 0 and by $\mathrm{Der}(\mathcal{O})$ the sheaf of $\mathcal{O}$-module of the vector fields germs on Ω. If X is a sufficiently small representative of the germ $(X, 0)$, the $\mathcal{O}$-subsheaf of $\mathrm{Der}(\mathcal{O})$ of vector field germs tangent to X is denoted by $\mathrm{Der}(X)$. The Saito's logarithmic stratification $\{X_\lambda\}$ of X into connected manifolds X_λ such that the tangent space $T_x(X_\lambda)$ coincides with the subspace $\mathrm{Der}_x(X) \subset T_x(\Omega)$ and satisfying the frontier condition [270, §3].

For a logarithmic stratification of X, the associated logarithmic characteristic variety $LC(X)$ is the union of the conormal spaces T^*X_λ to the strata X_λ. They are subspaces of $T_0^*(\mathbb{C}^m)$ of vanishing forms on TX_λ. The multiplicity of T^*X_λ in $LC(X)$ is denoted by m_λ.

In [158] N. Grulha proves the following result that relates the local Euler obstruction and the Bruce-Roberts' Milnor number.

Theorem 5.17.20 *Let $(X, 0)$ be the germ of a reduced equidimensional analytic variety and $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ a function with an isolated singularity at the origin such that f has also an isolated singularity in the stratified way. If $LC(X)$ is Cohen-Macaulay then,*

$$\mu_{BR}(f) = \sum_\lambda m_\lambda (-1)^{\dim_{\mathbb{C}} X_\lambda} Eu_{f,\overline{X}_\lambda}(0).$$

An important class of examples is the case of the discriminant X of an analytic stable map-germ $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n \geq p$, with (n, p) are nice dimensions of Mather [211] [267, Section 5]. N. Grulha obtains relations of constancy for families f_u between $\mu_{BR}(f_u)$, $\mu(f_u)$, $\mu_L(f_u)$ and $Eu_{f_u,X}(0)$.

In [234], J.J. Nuño-Ballesteros, B. Oréfice and J.N. Tomazella consider a weighted homogeneous germ of hypersurface $(X, 0) \subset (\mathbb{C}^n, 0)$ with isolated singularity and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of function finitely determined with respect to X. They show that

$$\mu_{BR}(f) = \mu(f) + \mu(X, f),$$

where $\mu(f)$ and $\mu(X, f)$ denote the Milnor numbers of f and of the fiber $X \cap f^{-1}(0)$ respectively. They show that the logarithmic characteristic subvariety $LC(X)$ is Cohen-Macaulay and provide relations between the Bruce-Roberts number and the local Euler obstruction.

5.17.5.3 Goryunov, Mond and van Straten Milnor Number

In [297], Seade, Tibar and Verjovsky compare $Eu_{f,X}(0)$ with the generalization of the Milnor number, due to V. Goryunov [152] and D. Mond and D. van Straten [229], defined for functions on curve singularities, and generalized by T. Izawa and T. Suwa [171] for functions defined on complete intersections in general. In the case of curves, if the curve singularity $(X, 0)$ is an ICIS, defined by some application $g : (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$ on an open set in $\mathbb{C}^N$, and F is an extension of f to the ambient space, then $\mu_G(f)$ counts the number of critical points (with their multiplicities) of the restriction of F to a Milnor fiber of g.

As noted in the introduction of [56] and in [171, 297], this definition makes sense in any dimension. In [297, Formule (4)] it is proved that if $(X, 0)$ is an ICIS, then:

$$\mu_G(f) = \mu_L(f) + \mu(X, 0).$$

In this case, from Theorem 5.17.19, we get:

$$Eu_{f,X}(0) = (-1)^{\dim X}[\mu_G(f) - \mu_G(l)],$$

where l is a generic linear form.

5.17.5.4 Image Milnor Number

The image Milnor number is a generalization of Milnor number defined for stable unfoldings by D. Mond (see [98, 225][228, Section 1.3]). The bridge between image Milnor number and Thom polynomials and Chern-Schwartz-MacPherson classes was provided by Toru Ohmoto in [239].

The definition and properties of image Milnor number use definition and properties of singularities of mappings, see Sect. (5.16.2.2) and the book [227].

One considers germs with finite $\mathcal{A}_e$-codimension, that is the minimal number of parameters in a versal unfolding of f [225, 226].

The image Milnor number is defined in the framework of stable unfoldings.

A stabilization of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a 1-parameter unfolding $F(x, u) = (f_u(x), u)$ of f with the property that f_u is a locally stable mapping for all $u \neq 0$ close to the origin. The mapping f_u is called a stable perturbation of f, its image X_u is called the disentanglement of f [100, 225]. A stabilization of f always exists when $(n, n+1)$ are Mather nice dimensions ([211] and the presentation by Maria Aparecida Soares Ruas [267, Section 5] in the Handbook, vol. IV) or when f has corank 1. Note that, outside the range of Mather's dimensions, some germs do not admit a stabilization.

Let us provide an example. Assume that M, N are compact complex manifolds of dimension 2, 3, respectively, and $f : M \to N$ is a holomorphic map which admits only stable singularities. Denote

$$A_0 = \text{ the set of regular points of } f, \quad A_1 = \text{ the set of critical points of } f$$
$$A_0^2 = \{x \in A_0 |\ \exists x' \in A_0, x' \neq x,\ f(x) = f(x')\}$$
$$A_0^3 = \{x \in A_0 |\ \exists x', x'' \in A_0 \cap f^{-1}f(x), x, x', x'' \text{distinct.}\}$$

Toru Ohmoto proved that the formula obtained by Izumiya and Marar [172] in the real case, is valid for the complex singularities as well. The constructible function, combination of characteristic functions

$$\alpha_{\text{image}} = \mathbf{1}_M - \frac{1}{2}\mathbf{1}_{\overline{A_0^2}} - \frac{1}{6}\mathbf{1}_{A_0^3} + \frac{1}{2}\mathbf{1}_{A_1}, \tag{5.54}$$

satisfy

$$\mathbf{1}_{f(M)} = f_*(\alpha_{\text{image}}). \tag{5.55}$$

Extending the same procedure to more general case involving mono- and multi-singularities, of higher codimension, Ohmoto obtains, for stable maps between complex manifolds $f : M^n \to N^{n+1}$ $(n \geq 1)$, a constructible function α_{image} on the source space M satisfying the formula (5.55). Taking the image by the MacPherson transformation c_* (Theorem 5.15.11), one has

Theorem 5.17.21 *[239, Theorem 6.5] There is a polynomial* $\text{tp}^{SM}(\alpha_{\text{image}})$ *in the quotient Chern classes* $c^i = c^i(f^*TN - TM)$ *and the Landweber-Noviknov classes* s^I *so that*

$$\text{Dual } c_*(\alpha_{\text{image}}) = c(TM) \cdot \text{tp}^{SM}(\alpha_{\text{image}}) \in H^*(M)$$

for any proper stable maps $f : M^n \to N^{n+1}$, $(1 \leq n \leq 5)$.

The Landweber-Noviknov class s^I corresponding to the multi-index $I = (i_1, i_2, \cdots)$ is defined by $s^I(f) = f^* f_*(c_1(f)^{i_1} c_2(f)^{i_2} \cdots)$.

An analytic map-germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, $f = (f_1, \cdots, f_n)$ is *weighted homogeneous* if there are positive integers $w_1, w_2, \ldots, w_m$, the weights, and positive integers $d_1, d_2, \ldots, d_n$ the degrees, such that, $f_i(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \ldots, \lambda^{w_m} x_m) = \lambda^{d_i} f_i(x)$ for all $x \in \mathbb{C}^m$, $\lambda \in \mathbb{C}$, $i = 1, \ldots, n$.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathcal{A}$-finitely determined weighted homogeneous map-germ which is not equivalent to any trivial unfolding of map-germ of smaller dimensions. Using results of Lê [194] and Siersma [301], David Mond [225] showed that the image (disentanglement) X_u of a stabilization of f has the homotopy type of a wedge of n-spheres and that the number of such spheres is independent of the stabilization. Mond called this number, denoted by $\mu_I(f)$, the *image Milnor number* by its analogy with the classical Milnor number $\mu(X, 0)$ of a hypersurface $(X, 0)$ with isolated singularity (see for instance the formula in [74, page 34, line 2]).

Definition 5.17.22 [225] The image Milnor number of f is defined as

$$\mu_I(f) = (-1)^n(\chi(\text{Im}(f_u) - 1). \tag{5.56}$$

The image Milnor number $\mu_I(f)$is well defined, that is, it is independent of the choice of the parameter u, of the representatives and of the stable unfolding F.

In case of $n = 1, 2$, Mond proved that

$$\mathcal{A}_e\text{–codim}\,(f) \le \mu_I(f)$$

and the equality holds if f is weighted homogeneous.

The original Mond's Conjecture, [225], says that the same is true for any n for which the pair $(n, n+1)$ is in Mather's nice dimensions ([211] and [267, Section 5]).

This conjecture remains open for $n \ge 3$.

The strategy used by Ohmoto provides a general formula [239, Theorem 6.20] for $\chi(\mathrm{Im}(f_u))$ in terms of $\mathrm{tp}^{SM}(\alpha_{\mathrm{image}})$ (or $\mathrm{tp}^{SM}(\mathbf{1}_{f(M)}$ via formula 5.55)

Toru Ohmoto determined $\mathrm{tp}^{SM}(\alpha_{\mathrm{image}})$ and the image Minor number μ_I up to degree three. In [239, Example 6.21]. Ohmoto obtains, for weighted homogeneous map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$,

$$\mu_I = -1 + \left[\frac{1}{w_1 w_2}(1 + w_1 a)(1 + w_2 a)\ \mathrm{tp}^{SM}(\alpha_{\mathrm{image}})(f) \right]_2$$

where α_{image} is given by formula (5.54) and the notation in numerators $[\omega]_2$ means the coefficient of a^2 in $\omega \in \mathbb{Q}[[a]]$. Then Ohmoto recovers the formula for $n = 2$ due to Mond:

$$\begin{aligned}\mu_I = \frac{1}{6w_1^3 w_2^3}\big[& d_1^2(d_2^2 d_3^2 - w_1^2 w_2^2) - w_1^2 w_2^2\{d_2^2 + d_3^2 + 5w_1^2 + 9w_1 w_2 + 5w_2^2 \\ & - 6d_3(w_1 + w_2) + 3d_2(d_3 - 2(w_1 + w_2))\} - 3d_1 w_1 w_2\{w_1 w_2(d_3 - 2(w_1 + w_2)) \\ & + d_2(w_1 w_2 + d_3(w_1 + w_2))\}\big].\end{aligned}$$

Ohmoto provides the formula for $n = 3$ (with much more lines !) in [239] (also see [169]). Irma PallarésTorres and Guillermo Peñafort Sanchis [240, 241] obtain the formulae for $n = 4$ and $n = 5$. Now the formulae cover pages.

In the same way as image Milnor number, the discriminant Milnor number defined by Damon and Mond [98] is described in [228, Section 1.3], see also [239, Definition 6.23].

5.17.6 Local Chern Obstruction of Collections of 1-Forms

In various papers, W. Ebeling and S.M. Gusein-Zade developed MacPherson's idea to define local Euler obstruction using differential forms (see [110, 111]) also see [67]. That notion is closely related to the one introduced by C. Sabbah in [269].

In [109] the definition of the local Euler obstruction of a function was adapted to the case of a 1-form. Let $(X, 0) \subset (\mathbb{C}^m, 0)$ be the germ of a purely n-dimensional reduced complex analytic variety at the origin and $\{V_\alpha\}$ a Whitney stratification of

$(X, 0)$. Let ω be a 1-form on a neighbourhood of the origin in $\mathbb{C}^m$ with an isolated singular point on X at the origin. Let $\varepsilon > 0$ be small enough such that the 1-form ω has no singular points on $X \setminus \{0\}$ inside the ball B_ε. The 1-form ω gives rise to a section $\widetilde{\omega}$ of the dual Nash bundle $\widetilde{T}^*$ over the Nash transform $\widetilde{X}$ without zeros outside of the preimage of the origin.

Definition 5.17.23 The local Euler obstruction $\mathrm{Eu}_{X,0}(\omega)$ of the 1-form ω on X at the origin is the obstruction to extend the non-zero section $\widetilde{\omega}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to the preimage of its interior, more precisely its value, as an element of the cohomology group $H^{2n}(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$, on the fundamental class of the pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.

Just as for vector fields, (see [59]), one can define the Poincaré-Hopf index of a 1-form at a singular point. In this situation, one has the proportionality theorem, similar to Theorem 5.17.6.

Theorem 5.17.24 *[67] Let $V_\alpha \subset X$ be the stratum containing* 0, $\mathrm{Eu}_X(0)$ *the local Euler obstruction of X at* 0 *and ω a (real or complex) 1-form on V_α with an isolated singularity at* 0*. Then the local Euler obstruction of the radial extension ω' of ω and the Poincaré-Hopf index of ω at* 0 *are related by the following proportionality formula:*

$$\mathrm{Eu}_X(\omega', 0) = \mathrm{Eu}_X(0) \cdot \mathrm{Ind}_{\mathrm{PH}}(\omega, 0; X)\,.$$

where $\mathrm{Ind}_{\mathrm{PH}}$ *is the usual Poincaré-Hopf index.*

In the same way that the Euler class corresponds to the obstruction to the construction of a vector field, and the Chern classes to that of frames, collections of vector fields, W. Ebeling and S.M. Gusein-Zade called "local Chern obstruction" the generalization of local Euler obstruction to the case of collection of differential forms [110, 111]. More precisely, the authors perform the following construction.

Let $(X^n, 0) \subset (\mathbb{C}^m, 0)$ be the germ of a purely n-dimensional reduced complex analytic variety at the origin. Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(\mathbb{C}^m, 0)$ with s fixed, $i = 1, \cdots, s$, k_i are integers such that $\sum k_i = n$, $j = 1, \cdots, n - k_i + 1$. Let $\varepsilon > 0$ be small enough so that there is a representative X of the germ $(X, 0)$ and representatives $\{\omega_j^{(i)}\}$ of the germs of 1-forms inside the ball $B_\varepsilon(0) \subset \mathbb{C}^m$.

Definition 5.17.25 [111] A point $x \in X$ is called a *special* point of the collection $\{\omega_j^{(i)}\}$ of 1-forms on the variety X if there exists a sequence x_ℓ of points on the non-singular part X_{reg} of the variety X such that the sequence $T_{x_\ell}X_{reg}$ of the tangent spaces at the points x_ℓ has a limit L (in $G_n(\mathbb{C}^m)$) and the restriction of the 1-forms $\omega_1^{(i)}, \cdots, \omega_{n-k_i+1}^{(i)}$ to the subspace $L \subset T_x\mathbb{C}^m$ are linearly dependent for each $i = 1, \cdots, s$. The collection $\{\omega_j^{(i)}\}$ of 1-forms has an *isolated special point* on $(X, 0)$ if it has no special point on X in a punctured neighbourhood of the origin.

In the discussion following [111, Corollary 3] in section 7, Ebeling and Gusein-Zade consider the Nash transformation $\widetilde{X}$ (they denote it by $\widehat{X}$) and define a fiber bundle $\widehat{\mathbb{T}} \setminus \widehat{\mathbb{D}} \to \widehat{X}$ adapted to the situation and similarly to the Nash bundle. On the one hand the $(2n-1)$-homology group of its fiber is isomorphic to $\mathbb{Z}$. On the other hand, the collection of forms $\{\omega_j^{(i)}\}$ defines a section $\widehat{\omega}$ without singularity on a neighbourhood of the sphere S_ε. One can use obstruction theory and define.

Definition 5.17.26 [111] Let $\{0\}$ be a special point of the collection $\{\omega_j^{(i)}\}$. *The local Chern obstruction* $Ch_{X,0}\{\omega_j^{(i)}\}$ of the collection of germs of 1-forms $\{\omega_j^{(i)}\}$ on $(X, 0)$ at the origin is the obstruction to extend the section $\widehat{\omega}$ of the fiber bundle $\widehat{\mathbb{T}} \setminus \widehat{\mathbb{D}}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to $\widetilde{X}$. More precisely it is the value of the obstruction cocycle (as an element of the cohomology group $H^{2n}(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon), \mathbb{Z}))$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.

The following result is one of the main results in [111].

Proposition 5.17.27 *The local Chern obstruction $Ch_{X,0}\{\omega_j^{(i)}\}$ of a collection $\{\omega_j^{(i)}\}$ of germs of holomorphic 1-forms is equal to the number of special points on X of a generic deformation of the collection.*

The local Chern obstruction can be characterized as an intersection number. In [133] T. Gaffney and N. Grulha compute the local Chern obstruction of a collection of 1-forms on a variety with isolated singularity, not necessarily ICIS using the Gaffney multiplicity polar theorem [132].

5.17.6.1 More Generalizations of the Local Euler Obstruction

The natural generalization of the notion of local Euler obstruction of a function is the one of local Euler obstruction of a map $f : (X, 0) \to (\mathbb{C}^k, 0)$, where $(X, 0)$ is a germ of an equidimensional complex analytic variety with dimension $n \geq k$. Such a notion can be defined using the local Euler obstruction associated to a k-frame on an analytic variety, as defined and studied by J.-P. Brasselet, J. Seade and T. Suwa in [66]. That is performed by N. Grulha in [157], where the notion of local Euler obstruction for maps $f : (X, 0) \to (\mathbb{C}^k, 0)$ defined on singular varieties is introduced. The notion depends, a priori, on a particular choice of a cell $\sigma \subset X$, It is denoted by $Eu_{f,X}(\sigma)$.

In [52], J.-P. Brasselet, N. Grulha and M. Ruas showed the link between the Chern obstruction of Ebeling–Gusein-Zade [111] and the Grulha's local Euler obstruction for maps:

Theorem 5.17.28 *[52] Let $(X, 0)$ as above and $f : (X, 0) \to (\mathbb{C}^k, 0)$ be a map-germ defined on X. Then there exists a collection $\{\omega_j^{(i)}\}$ as in Sect. 5.17.6 such that*

$$Ch_{X,0}\{\omega_j^{(i)}\} = (-1)^{d-p+1} Eu_{f,X}(\sigma).$$

As a consequence of the theorem, the local Euler obstruction of a map, defined in [157] is in fact independent of a generic choice of the cell σ.

In the hypothesis of a good stratification of X relative to f (see [208], p.971), Dutertre and Grulha define the Brasselet number as follows.

Definition 5.17.29 ([106], Definition 3.18) Let $\mathcal{V}$ be a good stratification of X relative to f. We define $\mathrm{B}_{f,X}(0)$ by:

$$\mathrm{B}_{f,X}(0) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i)$$

where $0 < |\delta| \ll \epsilon \ll 1$.

In [106] Dutertre and Grulha proved that the Brasselet number satisfies a Lê-Greuel type formula, which relates this invariant with the number of Morse critical points. The result has been generalized by H. Santana [271, Theorem 3.2].

In [107] Dutertre and Grulha present an alternative proof of the Brasselet, Massey, Parameswaran and Seade formula [28] for the local Euler obstruction of a function using Ebeling and Gusein-Zade's results on the radial index and the local Euler obstruction of 1-forms.

In [96] Dalbelo and Hartmann present a formula to compute the Brasselet number of $f : (Y, 0) \to (\mathbb{C}, 0)$ where $Y \subset X$ is a non-degenerate complete intersection in a toric variety X. As application, in [96, Theorem 4.1] the authors provide sufficient conditions to obtain invariance of the local Euler obstruction for explicit families of ICIS.

In [158] N. Grulha uses the local Euler obstruction in order to investigate Saito free divisors.

In [370] X. Zhang defines "reflective projective varieties" for which the Chern-Schwartz-MacPherson classes of the strata determine the local Euler obstructions and the polar degrees. The author proposes an algorithm to compute the local Euler obstructions when such varieties are formed by group orbits.

5.17.7 Global Euler Obstruction

In [297], Seade, Tibăr and Verjovsky defined the global Euler obstruction: In the same notations as above, let B_R be a ball centered at origin and o sufficiently large radius R.

Definition 5.17.30 ([297], Definition 2.3) Let $\tilde{v}$ be the lifting to a section of the Nash bundle $\tilde{T}$ of a radial-at-infinity stratified vector field v over $X \setminus B_R$. We call global Euler obstruction of X, and denote it by $\mathrm{Eu}(X)$, the obstruction for extending $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ within $\nu^{-1}(X \cap B_R)$.

The obstruction to extend $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ within $\nu^{-1}(X \cap B_R)$ is in fact a relative cohomology class

$$o(\tilde{v}) \in H^{2n}(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R)) \simeq H_c^{2n}(\widetilde{X}).$$

The global Euler obstruction of X is the evaluation of $o(\tilde{v})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R))$. Thus $\mathrm{Eu}(X)$ is an integer and does not depend on the radius of the sphere defining the link at infinity of X. Since two radial-at-infinity vector fields are homotopic as stratified vector fields, it does not depend on the choice of v either.

Remark 5.17.31 The global Euler obstruction has the following properties (see [297] p. 396):

1. if X is non-singular, then $\mathrm{Eu}(X) = \chi(X)$,
2. in general, $\mathrm{Eu}(X) = \chi(X; \mathrm{Eu}_X)$.

Here, the *weighted Euler characteristic* for the constructible function Eu_X is defined by (see 5.63)

$$\chi(X; \mathrm{Eu}_X) = \sum_{n \in \mathbb{Z}} n \cdot \chi(\mathrm{Eu}_X^{-1}(n)). \tag{5.57}$$

In [108], Dutertre and Grulha defined the global Brasselet number of f at a point c and at infinity. They relate these numbers with the number of critical points of a Morsification of a polynomial function f on an algebraic set X. When $X = \mathbb{C}^n$, similar formulas have already appeared in the literature in the work of many authors such as Artal, Luengo, Melle, Tibar, Parusiński, Siersma, Suzuki and others.

Dutertre and Grulha prove in [108] a Brylinski-Dubson-Kashiwara type formula for the global Brasselet number at infinity. In [271], Santana shows that the Brasselet number of a function f with nonisolated singularities describes numerically the topological information of its generalized Milnor fiber.

5.18 Characteristic Classes and Intersection Homology

5.18.1 Some Properties of Intersection Homology

For (compact) m-dimensional oriented manifolds, the Poincaré isomorphism

$$H^{m-i}(M) \longrightarrow H_i(M)$$

sends cohomology characteristic classes to homology ones and the cup-product of cocycles corresponds to intersection product of cycles. It is then possible to define

Stiefel-Whitney numbers and Chern numbers either in cohomology or homology (see for instance formula (5.16) in Sect. 5.5.6.1).

In the case of singular varieties, there is a cup-product in cohomology but no intersection product in homology. On the other hand, characteristic classes (Stiefel-Whitney in the real case and Chern-Schwartz-MacPherson in the complex case) live in homology, not in cohomology. They cannot be multiplied and, a priori, there seems to be no hope to define characteristic numbers.

Fortunately, the intersection homology theory discovered by M. Goresky and R. MacPherson [147, 148] provides a bridge between cohomology and homology, bringing cup product to intersection product for suitable cycles (also see [46]).

Considering a stratification 5.29 of the singular variety such that $X_{n-2} = X_{n-1}$, (see pseudomanifolds [46, 1.1.2] in Handbook Volume II)

$$\mathcal{X} \qquad \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-2} = X_{n-1} \subset X = X_n \tag{5.58}$$

If a chain ξ with support $|\xi|$ meets transversely an element $X_{n-\alpha}$ of the filtration, then one has

$$\dim(|\xi| \cap X_{n-\alpha}) = i - \alpha.$$

The "intersection allowed" chains and cycles are those which meet each element $X_{n-\alpha}$ of the filtration with a controlled and fixed transversality defect p_α. This defect, called the *perversity*, is an integer value function

$$\bar{p} : [0, \dim X] \cap \mathbb{Z} \to \mathbb{N}, \qquad p_\alpha := \bar{p}(\alpha)$$

such that $p_0 = p_1 = p_2 = 0$ and $p_\alpha \leq p_{\alpha+1} \leq p_\alpha + 1 \quad \text{for} \quad \alpha \geq 2.$

Example 5.18.1 Examples of perversities are

- the zero perversity $\bar{0} = (0, 0, \ldots, 0)$,
- the maximal (or top) perversity $\bar{t} = (0, 0, 0, 1, 2, \ldots, n-2)$,
- for n even, $n \geq 4$, the upper middle $\bar{n} = (0, 0, 0, 1, 1, 2, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} - 1)$ and the lower middle perversities $\bar{m} = (0, 0, 0, 0, 1, 1, \ldots, \frac{n}{2} - 2, \frac{n}{2} - 1)$.

Let $\bar{p} = (p_0, p_1, p_2, \ldots, p_n)$ be a perversity, the complementary perversity $\bar{q} = (q_0, q_1, q_2, \ldots, q_n)$ is defined by $p_\alpha + q_\alpha = t_\alpha$ for all $\alpha \geq 2$.

Definition 5.18.2 The *intersection homology* groups $IH_*^{\bar{p}}(X; G)$ are the homology groups of the complex $(IC_*^{\bar{p}}(X; G), \partial_*)$ where

$$IC_i^{\bar{p}}(X; G) = \left\{ \xi \in C_i(X; G) \,\middle|\, \begin{matrix} \dim(|\xi| \cap X_{n-\alpha}) \leq i - \alpha + p_\alpha \\ \\ \dim(|\partial\xi| \cap X_{n-\alpha}) \leq (i-1) - \alpha + p_\alpha \end{matrix} \quad \forall \alpha \geq 2 \right\}$$

Here ∂ is the usual boundary and G can be $\mathbb{Z}$, $\mathbb{Z}/2$ or $\mathbb{Q}$ (even a local system).

The condition means that the perversity is the maximum admissible defect of transversality.

Main properties of intersection homology are:

Proposition 5.18.3 *[147, §2.3] Let X a compact oriented pseudomanifold and let $\bar{p}$, $\bar{q}$ and $\bar{r}$ perversities such that $\bar{p} + \bar{q} \leq \bar{r}$, one has canonical bilinear pairings*

$$IH_i^{\bar{p}}(X; G) \times IH_j^{\bar{q}}(X; G) \to IH_{i+j-n}^{\bar{r}}(X; G).$$

In particular, Goresky and MacPherson prove the generalized Poincaré duality:

Theorem 5.18.4 *[147, §3.3] Let X be a compact, oriented pseudomanifold and let $\bar{p}$ and $\bar{q}$ be two complementary perversities, then the pairing*

$$IH_i^{\bar{p}}(X; \mathbb{Z}) \times IH_{n-i}^{\bar{q}}(X; \mathbb{Z}) \to IH_0^{\bar{t}}(X; \mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}$$

followed by the evaluation map ε (which counts points with their multiplicity order) is non-degenerate, when tensorised by the rationals $\mathbb{Q}$.

Theorem 5.18.5 (Factorization of the Poincaré Homomorphism) *[147, §1.4] [46, §1.1.5]. One has, for each perversity $\bar{p}$ a factorization of the Poincaré homomorphism, cap-product by the fundamental class $[X]$:*

$$\begin{array}{ccc} H^{n-k}(X) & \xrightarrow{\cap[X]} & H_k(X) \\ & \searrow^{\alpha_X} \quad {}^{\omega_X}\nearrow & \\ & IH_k^{\bar{p}}(X) & \end{array} \tag{5.59}$$

It was then hoped that the characteristic classes could be lifted canonically from homology (in $H_k(X)$) to intersection homology (in $IH_k^{\bar{p}}(X)$) where their products could be defined.

5.18.2 Stiefel-Whitney Classes, Wu Classes and Intersection Homology

As pointed out by Goresky and Pardon, "this approach has (so far) failed completely, except in the "extreme" cases where there is a single characteristic number. For example, in Sullivan's theory of mod 2 Euler spaces (formula (5.30) in Sect. 5.11) where the Euler characteristic is the only cobordism invariant, or in P. Siegel's theory of mod 2 Witt spaces where the intersection homology Euler characteristic is the only cobordism invariant."

The main property of G-Witt spaces ([148, §5.6.1], [300]) is that the intersection homology groups of the two middle perversities coincide:

$$IH_*^{\bar{m}}(X;G) \cong IH_*^{\bar{n}}(X;G).$$

There was various work in this direction, in particular Friedman [125], Goresky and Pardon [150], Siegel [300], Sullivan [307] and Szucs [313]. Siegel [300] described the class of $\mathbb{Q}$-Witt spaces and computed the cobordism groups of such spaces, showing that in non trivial cases they are equal to the Witt groups. Pardon [242] computed the cobordism groups of the "Poincaré duality spaces" defined by Goresky and Siegel [151]. Friedman [125] follows Siegel by computing the bordism groups of oriented K-Witt spaces for any coefficient field K as well as identifying the resulting generalized homology theories.

Goresky and Pardon [150] exhibit four interesting classes of singular spaces for which various (cobordism invariant) characteristic numbers can be constructed, and for which these characteristic numbers completely determine the cobordism groups. The authors construct characteristic numbers by lifting Wu classes to intersection homology, and multiplying them, rather than lifting and multiplying Stiefel-Whitney classes. C. McTague in [221] provide also Stiefel-Whitney numbers for singular spaces.

In the same direction than Stong [306] (see Sect. 5.5.6.1), and using the Goresky-Pardon's lifting of Wu classes in intersection homology, J.-P. Brasselet, A. Libardi, E. Rizziolli and M. Saia defined in [54, 55] Wu numbers associated to maps and showed that, with suitable hypotheses, if a map is a coboundary, then the corresponding Wu numbers vanish.

5.18.3 Chern-Schwartz-MacPherson Classes and Intersection Homology

Alberto S. Dubson conjectured that the fundamental class (in homology) of an algebraic cycle in a complex algebraic variety X is the image of a middle intersection class by ω_X (see diagram (5.59)). In [347] Shoji Yokura gives a counterexample for integral coefficients and proves the conjecture in the case when the variety has only isolated singularities and for rational coefficients.

More counter-examples for integral coefficients, due to J.L. Verdier and M. Goresky are provided in [49, 50]. They concern projective cones on the images of Segre and Veronese embeddings. The idea to consider iterated projective cones, i.e. cone over the cone…over a smooth projective variety, was already considered by S. Yokura. In [48] J.-P. Brasselet, K.-H. Fieseler and L. Kaup show that if X is a rational homology manifold, then the Chern-Schwartz-MacPherson classes of iterated cones over X are in the image of intersection homology for rational coefficients and for perversities $\bar{p} \geq \bar{m}$.

More generally, G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber and L. Kaup show in [29] that Chern-Schwartz-MacPherson classes of algebraic varieties lift in intersection homology with rational coefficients and for the middle perversity. However, the lifting is not unique so that (in general) it is not possible to define Chern numbers.

5.19 Fulton Classes and Milnor Classes

The Schwartz classes use a generalization of the tangent bundle in the singular case, it is the union of tangent bundles to the strata of a Whitney stratification (and no longer a bundle). The Mather classes, introduced by MacPherson use the Nash bundle. The Fulton method is another way to generalize the tangent bundle and obtain characteristic classes in the singular situation.

The definition of Fulton classes (1984)

$$c^F(X) = c(TM|_X) \cap s(X, M)$$

uses the Segre classes $s(X, M)$ of the proper subvariety X of the manifold M (see [13, 130]). In the case of local complete intersections, the normal bundle of the regular part X_{reg} canonically extends to X as a vector bundle $N_X M$. The virtual tangent bundle of X is then defined as $\tau_X = TM|_X - N_X M$ (defined in the Grothendieck group of vector bundles on X) and one has

$$c^F(X) = c(\tau_X) \cap [X].$$

The difference between the Schwartz-MacPherson classes and the Fulton classes was (and continue to be) the subject of many papers.

The starting point is the Paolo Aluffi paper [2] which provides, for hypersurfaces, a formula in the schematic framework and the context of Segre classes (also see the documented article [13, Section 4]).

If W is a scheme supported on a Cartier divisor X of a nonsingular variety M, then the Segre class of W in M can be written in terms of the Segre class of X and the Segre class of the residual scheme J to X in W [130, Proposition 9.2] and [2, §2].

Theorem 5.19.1 *[2, Theorem 1] Let X be a section of a very ample line bundle on a nonsingular complex variety M, and let J be its singular subscheme. Then*

$$c^{SM}(X) = c^F(X \setminus J)$$

The theorem means that Fulton's class equals Schwartz-MacPherson's after the scheme is "corrected" for the presence of singularities. As written by Aluffi "At the moment we take this corrected [term] purely as a formal object, although we wonder whether a more concrete geometric meaning can be attached to it." Explicit computation of an example is provided in [2].

A first approach to expression of the difference between Fulton's class and Schwartz-MacPherson's class in terms of "Milnor class" is given in [2, Lemma 3].

Then Tatsuo Suwa [308] and José Seade and Tatsuo Suwa [295] proved that if X is a compact local complete intersection with isolated singularities then the difference

$$\mu_*(X) = (-1)^n (c^F(X) - c^{SM}(X))$$

is localized in degree 0 and is the sum of Milnor numbers at the singular points.

In general, the difference μ_* between Schwartz-MacPherson class and Fulton class has been called Milnor class of X and many authors studied this class providing different characterizations and equivalent definitions, using different notions of indices of vector fields at singular points, for example the GSV-index. Among them: Paolo Aluffi, Jean-Paul Brasselet, Daniel Lehmann, Toru Ohmoto, Adam Paruziński, Piotr Pragacz, José Seade, Tatsuo Suwa, Shoji Yokura…(see [74]).

The R. Callejas-Bedregal, M.F.Z. Morgado, and J.Seade article in this volume [74] provides a complete survey on Fulton, Fulton-Johnson and Milnor classes. We refer to this article concerning these classes. More information is also provided in the Yokura article [362] to appear in Volume IV.

5.20 Segre Classes

As we have seen, sometimes hidden, sometimes in broad daylight, Segre's classes play a very important role in the development of characteristic classes. The article by Paolo Aluffi on Segre classes in this volume [13] shows importance and implication of Segre classes not only for characteristic classes but for singularity theory in general (also see [22]).

According to David Mumford, "The Italian school, and notably Severi, Todd, Eger, and Segre developed a general theory of Chern classes in the algebraic case" (in [68]) Bernard Teissier [317] shows the relation between polar classes and Segre classes, roughly speaking, replacing tangents by secants. The relations between polar classes and Segre classes for singular projective varieties have been also described by S. Yokura in [346, 348].

For a general survey see the article by Aluffi [13].

5.21 Motivic and Hirzebruch Characteristic Classes

In [60, 63] Jean-Paul Brasselet, Jörg Schürmann and Shoji Yokura use motivic theory to obtain a generalization of the result of Hirzebruch (see Sect. 5.9) to the case of singular varieties. They unify the theories of Chern-Schwartz-MacPherson classes and generalizations of Todd classes and L_* classes in the singular case. The interested reader will find all information and details in the very good survey by Jörg Schürmann and Shoji Yokura [285] and Shoji Yokura article [362] (also see [65]).

In particular generating series formulae have been generalized for twisted characteristic classes of symmetric products of a singular complex quasi-projective variety [77].

5.21.1 Motivic Chern Classes: Hirzebruch Theory for Singular Varieties

In the same way as the MacPherson's natural transformation generalises the Chern class to singular varieties, the Todd class and the Thom-Hirzebruch class have been generalized as natural transformations respectively by Baum-Fulton-MacPherson [34] and by Cappell-Shaneson [80, 81, 299]. The motivic theory allows to unify the three generalizations in the spirit of Hirzebruch (§ 5.9).

Definition 5.21.1 (Chern Transformation (MacPherson)) (Theorem 5.15.11 and Formula 5.46)
There is a unique natural transformation

$$c_* : \mathcal{F}(X) \to H_*(X)$$

from the group of constructible functions $\mathcal{F}(X)$ to homology, satisfying the Deligne and Grothendieck conjecture. In particular for the constructible function $\mathbf{1}_X$, one defines $c^{SM}(X) := c_*(\mathbf{1}_X)$ *the Chern-Schwartz-MacPherson class of X.*

Definition 5.21.2 (Todd Transformation (Baum-Fulton-MacPherson)) [34]
There is a unique natural transformation

$$td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q}$$

from the Grothendieck group of coherent sheaves on X, satisfying suitable axioms. In particular, for the structure sheaf $\mathcal{O}_X$ on a smooth variety, $td_*(\mathcal{O}_X)$ is the Todd class of X. In general, one defines $td_*(X) := td_*[\mathcal{O}_X]$ to be the *Baum-Fulton-MacPherson Todd class of X.*

Definition 5.21.3 (L-Transformation (Cappell-Shaneson)) [80, 81, 349] There is a unique natural transformation

$$L_* : \Omega(X) \to H_{2*}(X; \mathbb{Q})$$

from the group of constructible self-dual sheaves on X, satisfying suitable axioms, in particular, for the intersection sheaf $\mathcal{IC}_X$ on a smooth variety, $L_*[\mathcal{IC}_X]$ is the L-class of X. In general, one defines $L_*(X) := L_*([\mathcal{IC}_X])$ to be the *Cappell-Shaneson L-class of X*.

Note that relation of L-classes with intersection homology (also see 5.18) was studied by S.E. Cappell and J.L. Shaneson in [78, 79].

In short, one has the following table:

X manifold		X singular variety
Number	Cohomology classes	Homology classes
$\chi(X)$	Chern	Chern-Schwartz-MacPherson
$\chi_a(X)$	Todd	Baum-Fulton-MacPherson
$\mathrm{sign}(X)$	Thom-Hirzebruch	Cappell-Shaneson

The problem is that the three transformations are defined on different spaces:

$$\mathcal{F}(X), \qquad G_0(X) \qquad \textit{and} \qquad \Omega(X)$$

and one asks for the possibility of unifying them in the same way as the Hirzebruch theory in the smooth case. The problem was solved by Brasselet, Schürmann and Yokura [60, 63] using the motivic framework. Some ingredients will be useful:

Definition 5.21.4 Let X be an algebraic variety. The *Grothendieck relative group of algebraic varieties over X* denoted by

$$K_0(var/X)$$

is the quotient of the free abelian group of isomorphy classes of algebraic maps $Y \longrightarrow X$, modulo the "additivity relation":

$$[Y \longrightarrow X] = [Z \longrightarrow Y \longrightarrow X] + [Y \setminus Z \longrightarrow Y \longrightarrow X]$$

for closed algebraic subvarieties Z in Y.

In [63], the authors prove the following 4 theorems:

Theorem 5.21.5 *The map*

$$e : K_0(var/X) \longrightarrow \mathcal{F}(X) \quad \textit{defined by} \quad e([f : Y \to X]) := f_*(\mathbf{1}_Y)$$

(see (5.39 in Sect. 5.15.1)) is the unique group homomorphism which commutes with direct images for proper maps and such that

$$e([id_X]) = \mathbf{1}_X \quad \textit{for } X \textit{ smooth and pure dimensional.}$$

Theorem 5.21.6 *There is a unique group homomorphism*

$$mC : K_0(var/X) \longrightarrow G_0(X)$$

which commutes with direct images for proper maps and such that

$$mC([id_X]) = [\mathcal{O}_X] \quad \textit{for } X \textit{ smooth and pure dimensional.}$$

Theorem 5.21.7 *[63] There is a unique group homomorphism*

$$sd : K_0(var/X) \longrightarrow \Omega(X)$$

which commutes with direct images for proper maps, such that

$$sd([f : Y \to X]) := [Rf_* \mathbb{Q}_Y[\dim_{\mathbb{C}}(Y) + \dim_{\mathbb{C}}(X)]]$$

for Y smooth pure-dimensional and f proper and such that

$$sd([id_X]) = [\mathbb{Q}_X[2\dim_{\mathbb{C}}(X)]] = [\mathcal{IC}_X] \quad \textit{for } X \textit{ smooth and pure dimensional.}$$

Theorem 5.21.8 *[63] (Definition 5.9.7); There is a unique group homomorphism*

$$T_y : K_0(var/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y]$$

which commutes with direct images for proper maps and such that

$$T_y([id_X]) = \widetilde{td_{(y)}}(TX) \cap [X] \quad \textit{for } X \textit{smooth and pure dimensional.}$$

In particular, one has: $T_{-1}([id_X]) = c_*(X)$.

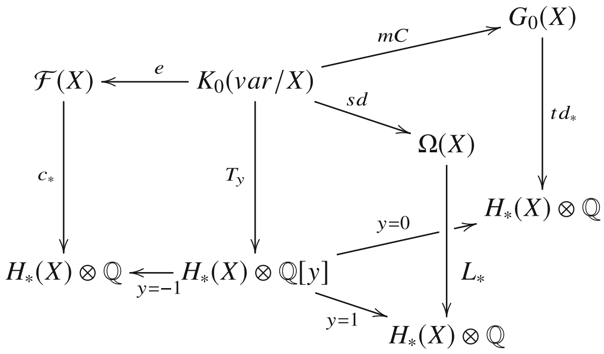
Remark 5.21.9 If a complex algebraic variety X has only rational singularities (for example if X is a toric variety), then:

$$mC([id_X]) = [\mathcal{O}_X] \in G_0(X) \quad \text{and in this case} \quad T_0([id_X]) = td_*(X).$$

That is not true in general !

The main result is the following:

Theorem 5.21.10 *One has a commutative "tripode" diagram:*



5.21.2 Verdier Riemann-Roch Formula

Theorem 5.21.11 *Let $f : X' \to X$ be a smooth morphism of constant relative dimension, then one has*

$$\widetilde{td_{(y)}}(T_f) \cap f^* T_y([Z \longrightarrow X]) = T_y f^*([Z \longrightarrow X]).$$

Here T_f is the bundle over X' of tangent spaces to fibers of f.

Proposition 5.21.12 (Factorisation of T_y) *Defining*

$$td_{(1+y)}([\mathcal{F}]) = \sum_{i=0}^{<\infty} \widetilde{td}_i([\mathcal{F}]) \cdot (1+y)^{-i},$$

then one has:

$$T_y = td_{(1+y)} \circ mC : K_0(var/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y].$$

Conjecture [63] The Hirzebruch homology class $T_{1,*}$ coincides with the Goresky-MacPherson L_*-class for compact complex algebraic varieties that are rational homology manifolds.

The conjecture has been proven by J. Fernández de Bobadilla and I. Pallarés in the projective case [37] and, with a different proof, in the general case, by J. Fernández de Bobadilla, I. Pallarés and M. Saito, [38].

The Saito theory of algebraic mixed Hodge modules allows also S.E. Cappell, A. Libgober, L. Maxim and J.L. Shaneson to produce Hodge theoretic formulae of Atiyah-Meyer type for genera and characteristic classes of complex algebraic varieties [75].

A classical result of Verdier [324] says that the MacPherson Chern class transformation commutes with specialization, which for constructible functions means the corresponding nearby cycles. In [280] J. Schürmann shows in particular that the motivic Chern- and Hirzebruch class transformations defined above commute with specialization defined in terms of nearby cycles.

The motivic Chern classes have been studied in various special cases (the list is far from being complete):

– Motivic Chern classes of Schubert cells by P. Aluffi, L. C. Mihalcea, J. Schürmann and C. Su [21],
– Motivic and derived motivic Hirzebruch classes by J.-P. Brasselet, J. Schürmann, S. Yokura, [64], also see Yokura [362].
– Motivic Chern classes and K-theoretic stable envelope by L. Fehér, R. Rimányi, and A. Weber [122],
– Motivic Chern classes of configuration spaces by J. Koncki [186],
– Twisted motivic Chern classes by J. Koncki and A. Weber [187],
– Specialization of motivic Hodge-Chern classes by J. Schürmann [279],
– Motivic bivariant characteristic classes by J. Schürmann and S. Yokura [286, 287],
– Motivic characteristic classes by S. Yokura [359, 361],
– Motivic Milnor classes by S. Yokura [360],
– Equivariant Hirzebruch classes by A. Weber [328].

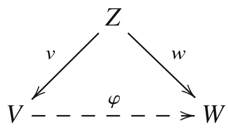
In the list may be mentioned the stringy Chern classes, detailed in the following section.

5.22 Stringy Chern Classes

The reader interested in stringy Chern classes will find a complete presentation in the very informative introductions of articles by P. Aluffi [9, 12] and by T. de Fernex, E. Lupercio, T. Nevins, and B. Uribe [99].

In [32], Batyrev showed that if two nonsingular varieties V and W are birational, and their canonical bundles agree after pull-back to a resolution of indeterminacies of a birational map between them, then the Betti numbers of V and W coincide. In [8] P. Aluffi showed an analog of Victor Batyrev's result, concerning the total Chern class of the tangent bundle (in the Chow group of the variety). More precisely, Aluffi states the

Theorem 5.22.1 *[8, Theorem 1.1] Let $\varphi : V \dashrightarrow W$ be a birational morphism of nonsingular algebraic varieties over an algebraically closed field of characteristic* 0. *Assume that there is a resolution of indeterminacies of φ,*



such that v and w are proper and birational, and the Jacobian ideals of v and w coincide. Then there exists a class $C \in (A_*Z)_{\mathbb{Q}}$ that

$$c_*(TV) = v_*(C) \quad \text{and} \quad c_*(TW) = w_*(C)$$

in $(A_*V)_{\mathbb{Q}}$ and $(A_*W)_{\mathbb{Q}}$ respectively.

As a corollary, the push-forward of the total Chern class of a crepant resolution of a singular variety is independent of the resolution (see [8]). The theorem [8, Theorem 3.1] shows that the previous theorem is true for singular varieties and Chern-Schwartz-MacPherson classes.

However, as shown by simple examples such as a surface with rational double points, compared to its minimal resolution the push-forward of the Chern class of a resolution is not necessarily the Chern-Schwartz-MacPherson class (see [99]).

In order to recover this property in the singular setting, independently and using different methods, P. Aluffi in [12] and T. de Fernex, E. Lupercio, T. Nevins, and B. Uribe in [99] defined stringy Chern classes in the Chow group of a variety X (with at worst log-terminal singularities) whose degree is the Batyrev's stringy Euler number.

The two approaches have points of contact, and produce the same class. In their introductions, both papers explain very well the respective methods they use (also see the introduction in [9]).

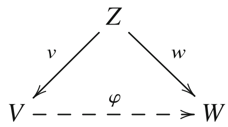
De Fernex et al. make use of motivic integration, a fruitful framework for defining and studying stringy invariants (see the surveys [101, 325]). They provide an explicit formula for quotient varieties and show that the stringy Chern classes may also be obtained by specializing formulas of Lev Borisov and Anatoly Libgober for their orbifold elliptic class [39]. The motivic framework in [99] is close to the one developed by Brasselet, Schürmann and Yokura in [63]. Restricting to the case of normal varieties X with a $\mathbb{Q}$-Cartier canonical divisor and having at most log-terminal singularities (for instance, the singularities of the quotient of a smooth variety by the action of a finite group) in general these varieties form a natural class of singularities in birational geometry

De Fernex et al. encode information coming from resolution of singularities into a ($\mathbb{Q}$-valued) constructible function Φ_X. By combining the ($\mathbb{Q}$-version of) MacPherson transformation with motivic integration (with its natural change-of-variables formula), the authors obtain a direct construction of the cycle class $c_*(\Phi_X)$ with wished birational invariance properties. They define:

$$c_{str}(X) = c_*(\Phi_X) \in A_*(X)_{\mathbb{Q}}. \tag{5.60}$$

and obtain:

Theorem 5.22.2 *Let V and W birational (possibly singular) varieties in the same K-equivalence class and*



a common resolution, then there exists a class $C \in (A_* Z)_{\mathbb{Q}}$ such that

$$c_{str}(V) = v_*(C) \quad \text{and} \quad c_{str}(W) = w_*(C)$$

for some $C \in A_*(Z)_{\mathbb{Q}}$.

In [9, 12], P. Aluffi developed a new very fruitful tool, the "celestial integrals" with several applications. In particular, the celestial integrals may be used to compare Chern classes of birational varieties: celestial integrals are formal integral on the system of varieties mapping properly and birationally to a given one, with value in an associated Chow group. The main property of the celestial integrals is that they satisfy a change of variable formula with respect to proper birational morphisms. Using the change-of-variable formula the corresponding theorem can be proved and then stringy Chern classes are defined.

In some situations, the stringy classes coincide with the Wu-Mather classes, for instance in the case of Schubert varieties (see [175]).

In [33], Batyrev and K. Schaller determine the stringy Chern classes of singular toric varieties as an application of a formula expressing the total stringy Chern class of a generic complete intersection in a normal projective $\mathbb{Q}$-Gorenstein variety X with at worst log-terminal singularities via the total stringy Chern class of X.

5.23 The Different Chern Classes in Terms of Constructible Functions

The majority of the described "Chern type" classes can be defined as classes images of the MacPherson's natural transformation

$$c_* : \mathcal{F}(X) \to A_*(X)$$

for suitable constructible functions.

(a) The Chern-Schwartz-MacPherson class.
The Chern-Schwartz-MacPherson class of X is defined by

$$c_{SM}(X) = c_*(\mathbf{1}_X)$$

(see formula (5.46)).

(b) The Wu-Mather class.
The local Euler obstruction is a constructible function (Proposition 5.17.5). According to the MacPherson's construction, the Wu-Mather classes satisfy

$$c^{Ma}(X) = c_*(\mathrm{Eu}_X).$$

(c) The Chern-Fulton class.
If $X \subset M$ is a hypersurface defined by $X = f^{-1}(0)$ where $f : M \to D$ is a holomorphic function defined into an open disk D around 0 in $\mathbb{C}$. the Chern-Fulton classes verify (see [67, Theorem 11.3.2]),

$$c_F(X) = c_*(\sigma_X)$$

where σ_X is the constructible function whose value at x is the Euler-Poincaré characteristic $\chi(F_x)$ of the local Milnor fiber, i.e., the intersection of a nearby fiber of f with a small ball in M centered at x (see [74, p. 34] and the Verdier specialization [324]).
(d) The Milnor class.
If X is an n-dimensional local complete intersection in a complex manifold M, the Milnor classes have been considered by various authors, from different viewpoints. (see § 5.19 and the article by R. Callejas-Bedregal, M.F.Z. Morgado, and J. Seade article [74] in this volume). The corresponding constructible function

$$\mu_X = (-1)^{n-1}(\mathbf{1}_X - \sigma_X)$$

was defined by ([244, 350], [67, Definition 12.1.1]).
(e) The weighted Chern-Mather class
In [35], K. Behrend considers the "Donaldson-Thomas" invariant, or "virtual count" of stable sheaves on Calabi-Yau threefolds (see [35] for definition). Behrend defines a constructible function ν_X on any scheme X over $\mathbb{C}$. If X is proper and embeddable, then the value of the Donaldson-Thomas invariant is equal to $\chi(X; \nu_X)$.
Here, the *weighted Euler characteristic* for the constructible function ν_X is defined by (see formula (5.57) in 5.17.31)

$$\chi(X; \nu_X) = \sum_{n \in \mathbb{Z}} n \cdot \chi(\nu_X^{-1}(n)).$$

If X is the critical scheme of a regular function f on a smooth scheme M, i.e. $X = Z(df)$, then the constructible function ν_X coincide with μ_X. The class

$$c_*(\nu_X)$$

coincide (up to sign) with the Aluffi weighted Chern-Mather class (Sect. 5.15.5).

(f) The stringy Chern class
In [99] (see formula (5.60) in Sect. 5.22) T. de Fernex, E. Lupercio, T. Nevins, and B. Uribe define a constructible function Φ_X such that the stringy Chern class is equal to:

$$c_{str}(X) = c_*(\Phi_X) \in A_*(X)_{\mathbb{Q}} \tag{5.61}$$

A stringy constructible function $I_X(0, C_X)$ satisfying also (5.61) has been defined by P. Aluffi [9, Definition 5.2] using the notion of modification system C_X that he introduced in [9, Definition 2.1 and Theorem 5.3].

5.24 Bivariant Classes

In 1981, Robert MacPherson and William Fulton [131] developed a formalism called bivariant theories. These are simultaneous generalizations of covariant group valued "homology-like" theories and contravariant ring valued "cohomology-like" theories. The aim of bivariant theories is to define parametrized objects such as characteristic classes: For instance, to a map $f : X \to Y$ associate a "class" such as for each $y \in Y$, one has the class of $f^{-1}(y)$. MacPherson and Fulton proved the existence and uniqueness of Stiefel-Whitney classes in this formalism and conjectured the same for Chern classes.

More information is provided in the Yokura article in the volume IV of the Handbook [362].

5.24.1 Bivariant Theories

Bivariant theories [131] are simultaneous generalizations of covariant (such as homology) and contravariant (such as cohomology) theories.

A bivariant theory **B** on a category C with values in the category of graded abelian groups is an assignment to each morphism $X \xrightarrow{f} Y$ in the category C an abelian group $\mathbf{B}(X \xrightarrow{f} Y)$, equipped with the following three basic operations:

- Product: For morphisms $f : X \to Y$ and $g : Y \to Z$, one has the product operation

$$\bullet : \mathbf{B}(X \xrightarrow{f} Y) \otimes \mathbf{B}(Y \xrightarrow{g} Z) \to \mathbf{B}(X \xrightarrow{gf} Z).$$

- Pushforward: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f proper, one has the pushforward operation

$$f_* : \mathbf{B}(X \xrightarrow{gf} Z) \to \mathbf{B}(Y \xrightarrow{g} Z).$$

- Pullback: For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

one has the pullback operation

$$g^* : \mathbf{B}(X \xrightarrow{f} Y) \to \mathbf{B}(X' \xrightarrow{f'} Y').$$

These three operations are required to satisfy seven axioms [131, Part I, §2.2]: product is associative, pushforward and pullback are functorial, the three operations commute two by two, and they commute with usual projection formula.

The data $X \mapsto B_*(X) := \mathbf{B}(X \to pt)$ becomes a covariant functor and $X \mapsto B^*(X) := \mathbf{B}(X \xrightarrow{\mathrm{id}} X)$ becomes a contravariant functor.

Given two bivariant theories $\mathbf{B}$ and $\mathbf{B}'$ on a category C, a *Grothendieck transformation* from $\mathbf{B}$ to $\mathbf{B}'$

$$\mathbf{G} : \mathbf{B} \to \mathbf{B}'$$

is a collection of homomorphisms

$$\mathbf{B}(X \to Y) \to \mathbf{B}'(X \to Y)$$

one for each morphism $X \to Y$ in the category C, which commute with the above three basic operations.

5.24.2 Bivariant Constructible Functions (Mod 2)

For $A \subset X$ and $\alpha : X \to \mathbb{Z}_2$ a constructible function on X, the *weighted Euler characteristic* of A is defined (see formula (5.57) in 5.17.31) as:

$$\chi(A;\alpha) = \chi(A \cap \alpha^{-1}(1)) = \sum_i (-1)^i \operatorname{rank} H_c^i(A \cap \alpha^{-1}(1)) \qquad (\text{mod } 2).$$

If (K) is an α-adapted triangulation of X, and A is a subcomplex of (K), then one has:

$$\chi(A;\alpha) = \sum_i (-1)^i \sum_{\sigma \in A^i} \alpha(\sigma) \qquad (\text{mod } 2),$$

where A^i is the set of i-simplexes in A and $\alpha(\sigma)$ is the value of α in the interior of σ.

The open star of a simplex σ, union of the interior of all simplexes meeting the interior of σ, is denoted by $\mathrm{St}^\circ(\sigma)$.

Definition 5.24.1 ([41, 131, 269]) Let us consider triangulations (K) and (L) of X and Y respectively such that (K) is α-adapted and f is a simplicial map. The constructible function $\alpha \in \mathcal{F}(X)$ satisfies the *local Euler condition* at $x \in X$, relatively to f, if

$$\alpha(x) = \chi(\mathrm{St}^\circ(\sigma) \cap f^{-1}(y); \alpha) \tag{5.62}$$

for all $y \in \mathrm{St}^\circ(f(\sigma))$, where σ is the simplex of (K) containing x in its interior.

The local Euler condition is independent of the choice of the triangulations (K) and (L). It can be reformulated in the following way: for any local embedding $(X, x) \to (\mathbb{C}^N, 0)$, the following equality holds:

$$\alpha(x) = \chi(\mathbb{B}_\varepsilon \cap f^{-1}(y); \alpha)$$

where $\mathbb{B}_\varepsilon$ is a sufficiently small open ball of the origin with radius ε and y any point close to $f(x)$.

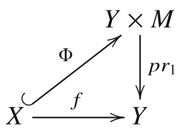
Definition 5.24.2 Let $f : X \to Y$ be a morphism of (algebraic varieties), the group $\mathcal{F}(X \to Y)$ of bivariant constructible functions is defined as the subgroup of constructible functions $\alpha : X \to \mathbb{Z}_2$ satisfying the *local Euler condition* for each point $x \in X$.

The group operations on $\mathcal{F}(X \to Y)$ are defined in an obvious way and satisfy the axioms of a bivariant theory [131].

5.24.3 Bivariant Homology Theory

The bivariant homology groups are defined by Fulton and MacPherson [131, §1.1.3] in the following way.

For a morphism $f : X \to Y$ of real (or complex) algebraic varieties choose a factorization as a closed embedding $\Phi := (f, \phi) : X \to Y \times M$ into $Y \times M$ where M is an oriented manifold M of real dimension n, followed by the projection $pr_1 : Y \times M \to Y$ onto Y.



The i-th bivariant homology group $\mathbf{H}^i(X \xrightarrow{f} Y)$ is defined by

$$\mathbf{H}^i(X \xrightarrow{f} Y) := H^{i+n}(Y \times M, (Y \times M) \setminus \Phi(X)).$$

This definition does not depend of the choice of the factorization. In particular, $\mathbf{H}(X \overset{\text{id}}{\to} X)$ is the cohomology of X and if Y is non singular, $\mathbf{H}(X \to Y)$ is isomorphic to the homology $H_*(X)$ of X by Alexander duality isomorphism (Sect. 5.4.2).

5.24.4 Bivariant Stiefel-Whitney Classes

Let us consider the bivariant homology theory $\mathbf{H}(X \to Y)$ with modulo 2 coefficients, Fulton and MacPherson proved the following result:

Theorem 5.24.3 *There is one and only one Grothendieck transformation ω from $\mathcal{F}$ to the bivariant homology theory with modulo 2 coefficients, such that if X is a manifold, then*

$$\omega(1_X) = w(TX) \cdot [X]$$

where the product of the Stiefel-Whitney class $w^*(TX) \in \mathbf{H}(X \to X) = H^*(X)$ with the fundamental class $[X] \in \mathbf{H}(X \to \{\text{pt}\}) = H_*(X)$ is the usual cap-product giving $\omega(1_X) \in H_*(X)$.

Definition 5.24.4 An Euler map $f : X \to Y$ is a map for which the constructible function whose value is 1 on all of X satisfies the local Euler condition for all $x \in X$. One denotes by 1_f that function.

As an example, the map $X \to \{pt\}$ is an Euler map if and only if X is a modulo 2 Euler space [162] (see formula 5.30 in Sect. 5.11), that is a space such that for all $x \in X$, one has

$$\chi(X, X - x) \cong 1 \mod 2.$$

The fibers of an Euler map are all modulo 2 Euler spaces. Any fibration with modulo 2 Euler spaces as fibers is an Euler map.

If $f : X \to Y$ is a proper Euler map, then $\chi(f^{-1}(y))$ is a locally constant function on Y, modulo 2. The reason is that in the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & f \searrow \quad \swarrow \text{id} & \\ & Y & \end{array}$$

one has $f_*(\mathbf{1}_f) \in \mathcal{F}(Y \stackrel{\text{id}}{\to} Y)$. The local Euler condition is precisely the right local condition on X to guarantee this result. The following definition agrees with the definition in [162, 307] (see also [117]) of the Whitney class of a modulo 2 Euler space.

Definition 5.24.5 If $X \to \text{pt}$ is an Euler map, one defines $w_*(X) \in \mathbf{H}(X \to \text{pt})$ to be $\omega(1_X)$.

5.24.5 Bivariant Chern Classes

In the case of integer coefficients, the bivariant constructible functions are written in the following way. Let $\alpha : X \to \mathbb{Z}$ be a constructible function on X and $A \subset X$, the *weighted Euler characteristic* of A is defined as:

$$\chi(A; \alpha) = \sum_{n \in \mathbb{Z}} n \cdot \chi(A \cap \alpha^{-1}(n)). \tag{5.63}$$

If (K) is an α-adapted triangulation of X, and A is a subcomplex of (K), then:

$$\chi(A; \alpha) = \sum_i (-1)^i \sum_{\sigma \in A^i} \alpha(\sigma),$$

where A^i is the set of i-simplexes in A and $\alpha(\sigma)$ is the value of α in the interior of σ.

Let $f : X \to Y$ a morphism (of algebraic complex varieties), the group $\mathcal{F}(X \to Y)$ of bivariant constructible functions is defined as the subgroup of $\mathcal{F}(X)$ of functions satisfying the following *local Euler condition* for each point $x \in X$:

Definition 5.24.6 Considering triangulations (K) and (L) of X and Y respectively such that (K) is α-adapted and f is a simplicial map. The constructible function $\alpha \in \mathcal{F}(X)$ satisfies the local Euler condition at $x \in X$, relatively to f, if

$$\alpha(x) = \chi(\mathrm{St}^\circ(\sigma) \cap f^{-1} y; \alpha)$$

for all $y \in \mathrm{St}^\circ(f(\sigma))$, where σ is the simplex of (K) containing x in its interior.

The bivariant homology theory $\mathbf{H}$ and the bivariant constructible functions theory $\mathcal{F}$ are defined in a similar way as in the real case, but with integer coefficients. In [131] was conjectured the following result, about existence and uniqueness of bivariant Chern classes:

Conjecture *There is one and only one Grothendieck transformation* γ *from* $\mathcal{F}$ *to* $\mathbf{H}$ *such that if* X *is a manifold, then*

$$\gamma(1_X) = c(TX) \cdot [X]$$

where $[X]$ *is the fundamental class of* X*.*

The existence part of the conjecture was proved by Brasselet, in [41] in 1983, in the framework of cellular maps using construction of relative radial r-frames, following M.H. Schwartz methods. Any analytic map is conjecturally cellular and no counterexample has been found so far. From a result of Teissier [316], an analytic map to a smooth curve is cellular (see [372],[373, 2.2.5 Lemme]).

Another proof of existence was given by Claude Sabbah, in 1986, in [269], using relative conormal space and relative bivariant cycles. Jianyi Zhou [373] proved that the two definitions coincide when Y is a smooth curve.

The problem of uniqueness of the bivariant Chern classes is still open. The approach by Yokura [351], considering a generalization of Schwartz-MacPherson classes and extension to a Verdier-type Riemann-Roch theorem [322], gives hope to solve the problem. Of course, one of the applications of the uniqueness of bivariant Chern classes would be a bivariant Riemann-Roch theorem similar to the one proved by W. Fulton and R. MacPherson in the real case. The specialization theorem was already proved by J.L. Verdier [324].

It is showed by Shoji Yokura, in [355, 357] that in the case when the target variety Y is non singular, the bivariant Chern class is uniquely determined. More precisely, if there exists a bivariant Chern class $\gamma : \mathcal{F} \to \mathbf{H}$, then for a morphism $f : X \to Y$ with Y non singular and for a bivariant constructible function α the following holds:

$$\gamma(\alpha) = f^* s(TY) \cap c_*(\alpha)$$

where $s(TY) := c^*(TY)^{-1}$ is the total Segre class of the tangent bundle TY.

The result uses the Fulton-MacPherson's notion of strong orientation [131].

Definition 5.24.7 An element $\theta \in \mathbf{B}(X \xrightarrow{f} Y)$ is called a strong orientation for the morphism $f : X \to Y$ if, for all morphisms $h : W \to X$, the homomorphism

$$\mathbf{B}(W \xrightarrow{h} X) \xrightarrow{\bullet\theta} \mathbf{B}(W \xrightarrow{f \circ h} Y)$$

is an isomorphism.

Proposition 5.24.8 *[62, Proposition 4.2] Let* Y *a possibly singular analytic variety such that the morphism* $c : Y \to \mathrm{pt}$ *has a strong orientation* $\theta \in \mathbf{H}(Y \to \{\mathrm{pt}\}) = H_*(Y)$ *which is contained in the image of the Schwartz-MacPherson class* $c_* : \mathcal{F}(Y) \to H_*(Y)$*. Then, for any morphism* $f : X \to Y$ *a bivariant Chern class*

$$\gamma_f : \mathcal{F}(X \xrightarrow{f} Y) \to \mathbf{H}(X \xrightarrow{f} Y)$$

is uniquely determined.

Theorem 5.24.9 *[62, Theorem 4.4] Let Y be a complex analytic variety which is an oriented A-homology manifold. If there exists a bivariant Chern class* $\gamma : \mathcal{F} \to \mathbf{H}$, *then for any morphism* $f : X \to Y$ *the bivariant Chern class*

$$\gamma_f : \mathcal{F}(X \xrightarrow{f} Y) \otimes A \to \mathbf{H}(X \xrightarrow{f} Y) \otimes A$$

is uniquely determined and it is described as

$$\gamma_f(\alpha) = f^* c^*(Y)^{-1} \cap c_*(\alpha).$$

As applications of this theorem, there are in particular, a specialization of Chern classes, following the Verdier results [324] and a "bivariant Riemann-Roch formula" à la Verdier [322].

The interested reader will find more information in the complete and nice survey by Jörg Schürmann and Shoji Yokura [285] see also Yokura [362].

Also see publications by Ernström and Yokura [118, 119] concerning Chern-Schwartz-MacPherson classes with values in Chow groups, by Brasselet, Schürmann and Yokura [61], by Yokura [354, 356, 358] about Ginzburg's bivariant Chern Classes and concerning oriented bivariant theory, about bivariant theory and Milnor classes [353], and...

The history of characteristic classes is not finished !

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Chapter 6
Segre Classes and Invariants of Singular Varieties

Paolo Aluffi

Contents

Abstract *Segre classes* encode essential intersection-theoretic information concerning vector bundles and embeddings of schemes. In this paper we survey a range of applications of Segre classes to the definition and study of invariants of singular spaces. We will focus on several numerical invariants, on different notions of characteristic classes for singular varieties, and on classes of Lê cycles.

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We precede the main discussion with a review of relevant background notions in algebraic geometry and intersection theory.

6.1 Introduction

Segre classes are an important ingredient in Fulton-MacPherson intersection theory: the very definition of intersection product may be given in terms of these classes, as we will recall below. It is therefore not surprising that important invariants of algebraic varieties may be expressed in terms of Segre classes. The goal of this paper is to survey several invariants specifically arising in singularity theory which may be defined or recast in terms of Segre classes. Many if not all of these invariants first arose in complex geometry; the fact that they can be expressed in purely algebraic terms by means of Segre classes extends their definition to arbitrary algebraically closed fields of characteristic zero. Tools specific to the theory of Segre classes yield new information on these invariants, or clarify the relations between them. On the whole, the language of Segre classes offers a powerful perspective in the study of these invariants.

We will begin with a general introduction to Segre classes and their role in intersection theory, in Sect. 6.2; a hurried reader can likely skim through this section at first and come back to it as needed. The survey itself will focus on the following themes:

- Numerical invariants (Sect. 6.3);
- Characteristic classes (Sect. 6.4);
- Lê cycles (Sect. 6.5).

One central result will be an expression for the Chern-Schwartz-MacPherson class of a (possibly singular) subvariety of a fixed ambient nonsingular variety, in terms of the Segre class of an associated scheme: see the discussion in Sect. 6.4.5 and especially Theorem 6.4.30. For example, the topological Euler characteristic of a scheme embedded in a nonsingular compact complex variety may be computed in terms of this Segre class. In the case of hypersurfaces, or more generally local complete intersections, this result implies concrete formulas for (generalized) Milnor numbers and classes. These formulas are explicit enough that they can be effectively implemented in computer algebra systems such as Macaulay2 for subschemes of e.g., projective space. Characteristic classes of singular varieties are also treated in detail in other contributions to this 'Handbook of Geometry and Topology of Singularities'; see especially the papers by Jean-Paul Brasselet [21] and by Roberto Callejas-Bedregal, Michelle Morgado, and José Seade [29]. The relation between Segre classes and David Massey's Lê cycles discussed in Sect. 6.5 is the result of joint work with Massey. Lê cycles are the subject of Massey's contribution to this Handbook, [67].

The role of Segre classes in singularity theory is certainly more pervasive than this survey can convey; because of limitations of space (and of our competence) we had to make a rather narrow selection, at the price of passing in silence many important topics. Among these omissions, we mention:

- The careful study of multiplicities and Segre numbers by Rüdiger Achilles, Mirella Manaresi, and collaborators, see e.g., [1];
- Work on the Buchbaum-Rim multiplicity, particularly by Steven Kleiman and Anders Thorup, [61, 62];
- Work by Terry Gaffney and Robert Gassler on Segre numbers and cycles, [43], briefly mentioned in Sect. 6.5;
- Seminal work by Ragni Piene on Segre classes and polar varieties, [81], also only briefly mentioned;
- Alternative uses of Segre classes in defining characteristic classes of singular varieties, as developed by Kent Johnson [57] and Shoji Yokura [96];
- Toru Ohmoto's work on Segre-SM classes and higher Thom polynomials [76];
- Equivariant aspects and positivity questions, which have recently come to the fore in the study of characteristic classes for Schubert varieties, see e.g., [17, 18].

Each of these topics would deserve a separate review article, and this list is in itself incomplete.

6.2 Segre Classes

In this section we review the general definition of *Segre class* used in the rest of the article, and place it in the context of Fulton-MacPherson intersection theory. The reader can safely skim through this section, coming back to it as it is referenced later in the survey. We also introduce a notion that will be frequently used in the rest, that is, the 'singularity subscheme' of a hypersurface; Sect. 6.2.5 is an extended example revolving around the Segre class of this subscheme for hyperplane arrangements.

We work over an algebraically closed field k; in later considerations, k will be assumed to have characteristic 0. Schemes are assumed to be separated of finite type over k. A *variety* is a reduced irreducible scheme; a *subvariety* of a scheme is a closed subscheme that is a variety. By 'point' we will mean *closed* point. An effective *Cartier divisor* (or, slightly abusing language, a *hypersurface*) is a codimension-1 subscheme that is locally defined by a nonzero divisor. Cartier divisors are zero-schemes of sections of line bundles. A *cycle* in a scheme is a formal integer linear combination of subvarieties. Two cycles are *rationally equivalent* if (loosely speaking) they are connected by families parametrized by $\mathbb{P}^1$. The *Chow group* of dimension-l cycles of a scheme X modulo rational equivalence is denoted $A_l(X)$; the direct sum $\oplus_l A_l(X)$ is denoted $A_*(X)$. We recall that a proper morphism $f: X \to Y$ determines a covariant push-forward homomorphism $f_*: A_*(X) \to A_*(Y)$ preserving dimension, while a flat or l.c.i. morphism f determines a contravariant pull-back/Gysin homomorphism f^*. If X is complete, that is, the structure morphism $X \to \operatorname{Spec} k$ is proper, then the push-forward of a class α via $A_*(X) \to A_*(\operatorname{Spec} k) = \mathbb{Z}$ is the *degree* of α, denoted $\int \alpha$ or $\int_X \alpha$. Intuitively, $\int \alpha$ is the 'number of points' in the zero-dimensional component of α. Vector bundles determine *Chern classes,* which act as operators on the Chow group, and satisfy various compatibilities (such as the 'projection formula') with morphisms. The Chern class $c_i(E) \cap -$ of a vector bundle E on X defines group homomorphisms $A_l(X) \mapsto A_{l-i}(X)$. The 'total' Chern class of E is the operator

$$c(E) = 1 + c_1(E) + \cdots + c_{\operatorname{rk} E}(E)\,.$$

For $i > \operatorname{rk} E$, $c_i(E) = 0$. If $\mathscr{O}(D)$ is the line bundle corresponding to a Cartier divisor D, the action of the operator $c_1(\mathscr{O}(D))$ amounts to 'intersecting by D': if $V \subseteq X$ is a variety not contained in D, $c_1(\mathscr{O}(D)) \cap [V]$ is the class of the Cartier divisor obtained by restricting D to V; we write $c_1(\mathscr{O}(D)) \cap \alpha = D \cdot \alpha$. Every vector bundle $E \to X$ determines an associated projective bundle 'of lines', which we denote $\pi : \mathbf{P}(E) \to X$. This bundle is endowed with a tautological subbundle $\mathscr{O}_E(-1)$ of $\pi^* E$; its dual $\mathscr{O}_E(1)$, which restricts to the line bundle of a hyperplane in each fiber of π, plays a distinguished role in the theory.

Our reference for these notions is William Fulton's text, [39]; Chapters 3–5 of the survey [40] offer an efficient and well-motivated summary. A reader who is more interested in topological aspects will not miss much by assuming throughout that $k = \mathbb{C}$ and replacing the Chow group with homology. The constructions in intersection theory are compatible with analogous constructions in this context, as detailed in Chapter 19 of [39].

6.2.1 Segre Classes of Vector Bundles, Cones, and Subschemes

Let $V \subseteq \mathbb{P}^n$ be any subvariety. The degree of V may be expressed as the intersection number of V with a general linear subspace of complementary dimension:

$$\deg V = \int_{\mathbb{P}^n} H^{n-\dim V} \cdot V\,, \tag{6.1}$$

where $H = c_1(\mathscr{O}(1))$ is the hyperplane class in $\mathbb{P}^n$ and, as recalled above, $\int_{\mathbb{P}^n} \gamma$ denotes the degree of the zero-dimensional component of a rational equivalence class $\gamma \in A_*(\mathbb{P}^n)$. In fact, by definition $\int_{\mathbb{P}^n} \gamma$ denotes the integer m such that $\pi_*\gamma = m[p]$, where $\pi : \mathbb{P}^n \to p = \operatorname{Spec} k$ is the constant map to a point. With this in mind, we can rewrite (6.1) as

$$(\deg V)[p] = \pi_* \left(\sum_{i \geq 0} c_1(\mathscr{O}(1))^i \cap [V] \right) \in A_*(p)\,: \tag{6.2}$$

the only nonzero term on the right is obtained for $i = n - \dim V$, for which it equals $(H^{n-\dim V} \cdot V)[p]$.

The right-hand side of (6.2) may be viewed as the prototype of a *Segre class,* for the trivial projective bundle $\pi : \mathbb{P}^n \to p$. More generally, let X be a scheme and let E be a vector bundle over X. Denote by $\pi : \mathbf{P}(E) \to X$ the projective bundle of lines in E, i.e., let

$$\mathbf{P}(E) = \mathrm{Proj}(\mathrm{Sym}^*_{\mathscr{O}_X}(\mathscr{E}^\vee)) . \tag{6.3}$$

where $\mathscr{E}^\vee$ is dual of the sheaf $\mathscr{E}$ of sections of E. Then for every class $G \in A_*(\mathbf{P}(E))$ we may consider the class

$$\mathrm{Segre}_E(G) := \pi_* \left(\sum_{i \ge 0} c_1(\mathscr{O}_E(1))^i \cap G \right) \in A_*(X) ; \tag{6.4}$$

this defines a homomorphism $A_*(\mathbf{P}(E)) \to A_*(X)$, which we loosely call a *Segre operator.* Even if G is pure-dimensional, $\mathrm{Segre}_E(G)$ will in general consist of components of several dimensions. As in the simple motivating example presented above, however, its effect is to encode intersection-theoretic information on G in terms of a class in $A_*(X)$.

Example 6.2.1 Let $X = \mathbb{P}^m$ and let $E = k^{n+1} \times X$ be a free bundle. Then $\mathbf{P}(E) \cong \mathbb{P}^m \times \mathbb{P}^n$, and the morphism $\pi : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ is the projection on the first factor. If $G \in A_m(\mathbb{P}^m \times \mathbb{P}^n)$ is a class of dimension m (to fix ideas), then

$$G = \sum_{i=0}^{m} g_i H^{n-i} h^i \cap [\mathbb{P}^{m+n}] ,$$

where h, H denote the (pull-backs of the) hyperplane classes from $\mathbb{P}^m$, $\mathbb{P}^n$, respectively, and $g_i \in \mathbb{Z}$ are integers. Then $H = c_1(\mathscr{O}_E(1))$, hence

$$\mathrm{Segre}_E(G) = \pi_* \left(\sum_{i \ge 0} c_1(\mathscr{O}_E(1))^i \cap G \right) = \sum_{i=0}^{m} g_i h^i \cap [\mathbb{P}^m]$$

recovers the information of the coefficients g_i determining the class G. ⌟

Applying Segre_E to classes $G = \pi^*(\gamma)$ obtained as pull-backs of classes from the base defines the total *Segre class* of E as an operator on $A_*(X)$:

$$s(E) \cap \gamma := \mathrm{Segre}_E(G) = \pi_* \left(\sum_{i \ge 0} c_1(\mathscr{O}_E(1))^i \cap \pi^*(\gamma) \right) . \tag{6.5}$$

It is a fundamental observation that $s(E)$ is *inverse to the Chern class* operator, in the sense that

$$c(E) \cap (s(E) \cap \gamma) = \gamma \tag{6.6}$$

for all $\gamma \in A_*(X)$. (Since the intersection product is commutative, it follows that $c(E)$, $s(E)$ are two-sided inverses to each other.) Indeed, consider the tautological sequence

$$0 \longrightarrow \mathscr{O}_E(-1) \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0 .$$

By the Whitney formula,

$$c(\pi^* E) c(\mathscr{O}_E(-1))^{-1} \cap \pi^* \gamma = c(Q) \cap \pi^* \gamma ;$$

by the projection formula,

$$c(E) \cap \pi_*(c(\mathscr{O}_E(-1))^{-1} \cap \pi^* \gamma) = \pi_*(c(Q) \cap \pi^* \gamma) .$$

Since Q has rank $\operatorname{rk} E - 1$, that is, equal to the relative dimension of π,

$$\pi_*(c(Q) \cap \pi^* \gamma) = m\gamma$$

for some integer m. Restricting to a fiber shows that $m = 1$, and (6.6) follows.

In fact, these considerations may be used to *define* Chern classes of vector bundles: Chern classes of line bundles may be defined independently in terms of their relation with Cartier divisors (as mentioned above); once Chern classes of line bundles are available, (6.5) may be used to define Segre classes of vector bundles; and then one may define the Chern class of a vector bundle E as the inverse of its Segre class, and proceed to prove all standard properties of Chern classes. This is the approach taken in [39], Chapters 2 and 3.

Other choices in (6.4) also lead to interesting notions: whenever a tautological line bundle $\mathscr{O}(1)$ is defined, one may define a corresponding Segre class. For example, we could apply the expression in (6.4) to

$$\operatorname{Proj}(\operatorname{Sym}^*_{\mathscr{O}_X}(\mathscr{F}))$$

to define a Segre class for any coherent sheaf $\mathscr{F}$; one instance will appear below, in Sect. 6.4.1. More generally, the definition may be applied to every *projective cone.* A *cone* over X is a scheme

$$C = \operatorname{Spec}(\mathscr{S}^*) = \operatorname{Spec}(\oplus_{k\ge 0} \mathscr{S}^k)$$

where $\mathscr{S}^*$ is a sheaf of graded $\mathscr{O}_X$ algebras and we assume (as is standard) that there is a surjection $\mathscr{S}^0 \twoheadrightarrow \mathscr{O}_X$, $\mathscr{S}^1$ is coherent, and $\mathscr{S}^*$ is generated by $\mathscr{S}^1$ over $\mathscr{S}^0$. It is useful to enlarge cones by a trivial factor: with notation as above, we let

$$C \oplus 1\!\!1 := \operatorname{Spec}(\mathscr{S}^*[t]) = \operatorname{Spec}(\oplus_{k\geq 0}(\oplus_{i=0}^{k}\mathscr{S}^i t^{k-i}))\,, \tag{6.7}$$

so that C may be viewed as a dense open subset of its 'projective completion' $\mathbf{P}(C \oplus 1\!\!1) = \operatorname{Proj}(\mathscr{S}^*[t])$; in fact, C is naturally identified with the complement of $\mathbf{P}(C) = \operatorname{Proj}(\mathscr{S}^*)$ in $\mathbf{P}(C \oplus 1\!\!1)$. Cones over X are endowed with a natural projection π to X and with a tautological line bundle $\mathscr{O}(1)$, so we may defined the *Segre class* of C in the style of (6.4):

$$s(C) := \pi_*\left(\sum_{i\geq 0} c_1(\mathscr{O}_{C\oplus\mathbf{1}}(1))^i \cap [\mathbf{P}(C \oplus 1\!\!1)]\right) \in A_*(X)\,.$$

If C is a subcone of a vector bundle E (as is typically the case), then

$$s(C) = \operatorname{Segre}_{E\oplus\mathbf{1}}([\mathbf{P}(C \oplus 1\!\!1)])\,. \tag{6.8}$$

A case of particular interest is the cone associated with sheaf of $\mathscr{O}_X$ algebras

$$\oplus_{k\geq 0}\mathscr{I}^k/\mathscr{I}^{k+1}$$

where $\mathscr{I}$ is the ideal sheaf defining X as a closed subscheme of a scheme Y. The corresponding cone $\operatorname{Spec}(\oplus_{k\geq 0}\mathscr{I}^k/\mathscr{I}^{k+1})$ is the *normal cone* of X in Y, denoted $C_X Y$.

Definition 6.2.2 Let $X \subseteq Y$ be schemes. The *Segre class of X in Y* is the Segre class of the normal cone of X in Y:

$$s(X, Y) := s(C_X Y) = \pi_*\left(\sum_{i\geq 0} c_1(\mathscr{O}(1))^i \cap [\mathbf{P}(C_X Y \oplus 1\!\!1)]\right)\,, \tag{6.9}$$

an element of $A_*(X)$. ⌟

Remark 6.2.3 The addition of the trivial factor $1\!\!1$ is needed to account for the possibility that e.g., $\mathbf{P}(C)$ may be *empty*. For instance, this is the case if $X = Y$, i.e., $\mathscr{I} = 0$: then $C_X X = \operatorname{Spec}(\mathscr{O}_X) = X$, $\mathbf{P}(C_X X \oplus 1\!\!1) = X$, and $s(X, X) = [X]$.

If X does not contain any irreducible component of Y, then

$$s(X, Y) = \pi_*\left(\sum_{i\geq 0} c_1(\mathscr{O}(1))^i \cap [\mathbf{P}(C_X Y)]\right)\,.$$

In general, it is easy to check that $s(C) = s(C \oplus 1\!\!1)$; in particular, the notation is compatible with the notation $s(E)$ for vector bundles used above. ⌟

6.2.2 Properties

A closed embedding $X \subseteq Y$ is *regular,* of codimension d, if the ideal $\mathscr{I}$ of X is locally generated by a regular sequence of length d. In this case, one can verify that

$$\oplus_{k\geq 0}\mathscr{I}^k/\mathscr{I}^{k+1} \cong \mathrm{Sym}^*_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2)\,,$$

so that the normal *cone* C_XY is a rank-d *vector bundle,* denoted N_XY. From the definitions reviewed in Sect. 6.2.1 it is then clear that the Segre class of X in Y equals the inverse Chern class of its normal bundle:

$$s(X,Y) = s(N_XY)\cap[X] = c(N_XY)^{-1}\cap[X]\,.$$

Example 6.2.4 Let $D \subseteq Y$ be an effective Cartier divisor. Then N_DY is the line bundle $\mathscr{O}(D)$, so that

$$s(D,Y) = c(\mathscr{O}(D))^{-1}\cap[D] = (1+D)^{-1}\cap[D]\,.$$

Abusing notation (writing D for $[D]$), we may write

$$s(D,Y) = \frac{D}{1+D} = D - D^2 + D^3 - \cdots\,.$$

For instance, if H is a hyperplane in $\mathbb{P}^n$, then

$$s(H,\mathbb{P}^n) = H - H^2 + \cdots = [\mathbb{P}^{n-1}] - [\mathbb{P}^{n-2}] + [\mathbb{P}^{n-3}] - \cdots + (-1)^{n-1}[\mathbb{P}^0]$$

viewed as a class on $H = \mathbb{P}^{n-1}$.

More generally, if $X = D_1 \cap \cdots \cap D_r$ is a complete intersection of r Cartier divisors, then $X \subseteq Y$ is a regular embedding, with $N_XY = \mathscr{O}(D_1)\oplus\cdots\oplus\mathscr{O}(D_r)$, and we may write

$$s(X,Y) = \frac{[X]}{(1+D_1)\cdots(1+D_r)} \in A_*(X)\,.$$

Individual components of this Segre class may be written as symmetric polynomials in the classes $D_1,\dots,D_r$. ⌟

By definition, the blow-up of Y along X is $B\ell_XY := \mathrm{Proj}(\oplus_k\mathscr{I}^k)$; the *exceptional divisor* E of this blow-up is the inverse image of X, so it is defined by the ideal

$$\mathscr{I}\oplus\mathscr{I}^2\oplus\mathscr{I}^3\oplus\cdots \quad\subseteq\quad \mathscr{O}_Y\oplus\mathscr{I}\oplus\mathscr{I}^2\oplus\cdots\,. \tag{6.10}$$

it follows that

$$E = \mathrm{Proj}(\oplus_k \mathscr{I}^k/\mathscr{I}^{k+1}) = \mathbf{P}(C_X Y) \xrightarrow{\pi} X\,.$$

That is, the exceptional divisor is a concrete realization of the projective normal cone of X in Y. Further, (6.10) shows that the ideal sheaf of E in $B\ell_X Y$ is the twisting sheaf $\mathscr{O}(1)$. It follows that $c_1(\mathscr{O}(1)) = -E$, and therefore

$$\sum_{i\ge 0} c_1(\mathscr{O}(1))^i \cap [\mathbf{P}(C_X Y)] = E - E^2 + E^3 - \cdots \in A_*(E)\,.$$

If X does not contain irreducible components of Y, it follows (cf. Remark 6.2.3) that

$$s(X,Y) = \pi_*(E - E^2 + E^3 - \cdots)\,. \tag{6.11}$$

This observation (and various refinements and alternatives) may be used to construct algorithms to compute Segre classes; see [6, 33, 48, 50, 52] for a sample of approaches and applications. The algorithms in the recent paper [50] by Corey Harris and Martin Helmer are implemented in the powerful package `SegreClasses` [49] available in the standard implementation of Macaulay2 [46].

The assumption that X does not contain irreducible components of Y is not a serious restriction: as we have noted that $s(C) = s(C \oplus 1\!\!1)$ for a cone C (cf. Remark 6.2.3), it follows that

$$s(X,Y) = s(X, Y\times \mathbb{A}^1)\,,$$

where on the right we view $X \cong X \times \{0\}$ as a subscheme of $Y \times \mathbb{A}^1$. Thus, (6.11) may be used to compute $s(X,Y)$ in general, by employing the exceptional divisor E of the blow-up of $Y \times \mathbb{A}^1$ along X.

The construction of normal cones is functorial with respect to suitable types of morphisms. This leads to the following useful result.

Proposition 6.2.5 ([39, Proposition 4.2]) *Let Y, Y' be pure-dimensional schemes, $X \subseteq Y$ a closed subscheme, and let $f : Y' \to Y$ be a morphism, and $g : f^{-1}(X) \to X$ the restriction. Then*

- *If f is flat, then $s(f^{-1}(X), Y') = g^* s(X,Y)$.*
- *If Y and Y' are varieties and f is proper and onto, then $g_* s(f^{-1}(X), Y') = (\deg f) s(X,Y)$.*

Here, f realizes the field of rational functions on Y' as an extension of the field of rational functions on Y, and $\deg f$ is the degree of this extension if $\dim Y = \dim Y'$, and 0 otherwise. In particular, if Y' and Y are varieties and $f : Y' \to Y$ is proper, onto, and *birational,* then

$$s(X,Y) = g_*(s(f^{-1}(X), Y'))\,.$$

This *birational invariance* of Segre classes is especially useful.

Example 6.2.6 We have verified a particular case of this fact already. Indeed, let $X \subsetneq Y$ be a proper subscheme of a variety, and let $f : Y' = B\ell_X Y \to Y$ be the blow-up of Y along X. Then $f^{-1}(X) = E$ is the exceptional divisor, a Cartier divisor of $B\ell_X Y$, therefore (Example 6.2.4)

$$s(f^{-1}(X), Y') = E - E^2 + E^3 - \cdots .$$

The birational invariance of Segre classes implies that, letting $g = f|_E : E \to X$, we must have

$$s(X, Y) = g_*(E - E^2 + E^3 - \cdots) ;$$

we have verified this above in (6.11) (where g is denoted π). ⌟

The Segre class $s(X, Y)$ depends crucially on the scheme structure of X; in general, $s(X, Y) \neq s(X_{\text{red}}, Y)$. On the other hand, different scheme structures may lead to the same Segre class, and this is occasionally useful. For instance, assume that the ideals $\mathscr{I}_{X,Y}$ and $\mathscr{I}_{X',Y}$ of two subschemes X, X' of Y have the same integral closure. Then $s(X, Y) = s(X', Y)$. Indeed, we may assume $\mathscr{I}_{X,Y}$ is a reduction of $\mathscr{I}_{X',Y}$; then we have a finite morphism $B\ell_{X'} Y \to B\ell_X Y$ preserving the exceptional divisors [92, Proposition 1.44], so the equality follows from (6.11) and the projection formula. See Example 6.3.4 below for a concrete example of this phenomenon.

Summary (and Shortcut) A reader who may not be too comfortable with the algebro-geometric language of Proj and cones employed so far may use the following as a characterization (and hence an alternative definition) of Segre classes.

Let Y be a variety. Every closed embedding $X \subseteq Y$ determines a *Segre class* $s(X, Y) \in A_*(X)$. This class is characterized by the following properties:

- If $X \subseteq Y$ is a regular embedding, with normal bundle $N_X Y$, then

$$s(X, Y) = c(N_X Y)^{-1} \cap [X] ;$$

- if $f : Y' \to Y$ is proper, onto, birational morphism of varieties, and $g : f^{-1}(X) \to X$ is the restriction of f, then

$$s(X, Y) = g_* s(f^{-1}(X), Y') .$$

Indeed, by blowing up Y along X, the second property reduces the computation of Segre class to the case of Cartier divisors, which is covered by the first property.

Unlike this characterization, the definition given in Sect. 6.2.1 does not require the ambient scheme Y to be a variety. In our applications, this more general situation will not be important. In any case we note that if Y is pure-dimensional, with irreducible components Y_i (taken with their reduced structure) one can in fact show [39, Lemma 4.2] that

$$s(X, Y) = \sum_i m_i s(X \cap Y_i, Y_i)\,, \tag{6.12}$$

where m_i is the *geometric multiplicity* of Y_i in Y, and the classes on the right-hand side are implicitly pushed forward to X. Each $s(X \cap Y_i, Y_i)$ is the Segre class of a subscheme of a variety, thus it is determined by the characterization given above.

6.2.3 A Little Intersection Theory

Segre classes play a key role in Fulton-MacPherson's intersection theory; indeed, the very definition of intersection product may be expressed in terms of Segre classes. By way of motivation for the formula giving an intersection product, consider a vector bundle

$$p : E \to X$$

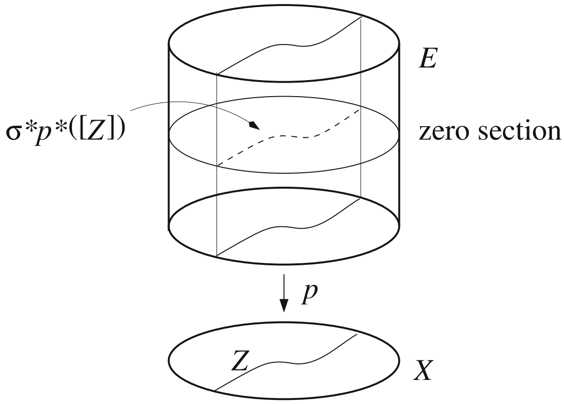
on a scheme X. Then it may be verified [39, Theorem 3.3(a)] that the pull-back $p^* : A_*(X) \to A_*(E)$ is an *isomorphism.*

Remark 6.2.7 The fact that p^* is *surjective* may seem counterintuitive, as it implies that a vector bundle over X has no nonzero rational equivalence classes of codimension larger than the dimension of X. See [39, §1.9], particularly Proposition 1.9 and Example 1.9.2. This fact can be viewed as a generalization of the observation that affine space $\mathbb{A}^n$ has no nonzero classes of dimension $< n$. ⌟

We may therefore define a 'Gysin homomorphism' $\sigma^* : A_*(E) \to A_*(X)$, as the inverse of p^*. That fact that for any subvariety $Z \subseteq X$,

$$\sigma^*([p^{-1}(Z)]) = \sigma^*(p^*[Z]) = [Z]$$

(and linearity) suggests that $\sigma^*(\alpha)$ should be interpreted as the 'intersection class' of α with the zero-section of E.



We can get an explicit expression for $\sigma^*(\alpha)$ in terms of the Segre homomorphism from (6.4). For this, consider E as a dense open subset of its projective completion $\mathbf{P}(E \oplus 1\!\!1)$, and let $\pi : \mathbf{P}(E \oplus 1\!\!1) \to X$ be the projection. If $\alpha \in A_k(E)$, then $\alpha = p^*(\sigma^*(\alpha))$ is the restriction to E of $\pi^*(\sigma^*(\alpha))$. An expression for $\sigma^*(\alpha)$ may be given in terms of *any* class $\overline{\alpha} \in A_k(\mathbf{P}(E \oplus 1\!\!1))$ restricting to α on E.

Lemma 6.2.8 *Let $\alpha \in A_k(E)$. With notation as above,*

$$\sigma^*(\alpha) = \left\{c(E) \cap \mathrm{Segre}_{E\oplus\mathbf{1}}(\overline{\alpha})\right\}_{k-\mathrm{rk}\,E}$$

where $\{\cdots\}_\ell$ is the term of dimension ℓ in the class within braces, and $\overline{\alpha}$ is any class in $A_k(\mathbf{P}(E \oplus 1\!\!1))$ restricting to α on E.

This statement is equivalent to [39, Proposition 3.3]. We sketch a verification. As we argued in (6.6) (note $c(E \oplus 1\!\!1) = c(E)$),

$$\begin{aligned}\sigma^*(\alpha) &= c(E) \cap (s(E \oplus 1\!\!1) \cap \sigma^*(\alpha))\\ &= c(E) \cap \pi_*\left(\sum_{i\geq 0} c_1(\mathscr{O}(1))^i \cap \pi^*(\sigma^*(\alpha))\right)\\ &= c(E) \cap \mathrm{Segre}_{E\oplus\mathbf{1}}(\pi^*(\sigma^*(\alpha))) \in A_{k-\mathrm{rk}\,E}(X)\,.\end{aligned}$$

Now note that if $\overline{\alpha}$ is any class in $A_k(\mathbf{P}(E \oplus 1\!\!1))$ restricting to α on E, then

$$\beta = \overline{\alpha} - \pi^*(\sigma^*(\alpha))$$

is supported on the complement $\mathbf{P}(E)$ of E in $\mathbf{P}(E \oplus 1\!\!1)$. It follows easily that all components of the class

$$c(E) \cap \pi_*\left(\sum_{i\geq 0} c_1(\mathscr{O}(1))^i \cap \beta\right)$$

have dimension $\geq k - (\text{rk}\, E - 1)$. Thus, the component of dimension $k - \text{rk}\, E$ of

$$c(E) \cap \text{Segre}_{E\oplus\mathbf{1}}(\overline{\alpha})$$

agrees with the component of dimension $k - \text{rk}\, E$ of

$$c(E) \cap \text{Segre}_{E\oplus\mathbf{1}}(\pi^*(\sigma^*(\alpha))) = \sigma^*(\alpha)$$

and the statement follows.

A deformation argument reduces to the template of intersecting a class with the zero-section all intersection situations satisfying the following requirements. Let X and V be closed subschemes of a scheme Y. We assume that V is a variety of dimension m, and that $X \subseteq V$ is a regular embedding of codimension d. We have the fiber diagram

$$\begin{array}{ccc} X \cap V & \hookrightarrow & V \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X & \hookrightarrow & Y \end{array} .$$

The pull-back $i^*\mathscr{I}_{X,Y}$ of the ideal of X in Y generates the ideal of $X \cap V$ in V. This induces a surjection

$$i^*\text{Sym}^*_{\mathscr{O}_Y}(\mathscr{I}_{X,Y}/\mathscr{I}^2_{X,Y}) = \oplus_{k\geq 0} i^*(\mathscr{I}^k_{X,Y}/\mathscr{I}^{k+1}_{X,Y}) \twoheadrightarrow \oplus_{k\geq 0}\mathscr{I}^k_{X\cap V,V}/\mathscr{I}^{k+1}_{X\cap V,V}$$

and consequently realizes $C_{X\cap V}V$ as a closed, m-dimensional subscheme of the pull-back j^*N_XY of the normal bundle of X in V. William Fulton and Robert MacPherson (cf. [42], [39, Chapter 6]) define the intersection product $X \cdot V \in A_{m-d}(X \cap V)$ to be the intersection of $[C_{X\cap V}V]$ with the zero section of the bundle j^*N_XY, defined as above by means of the Gysin morphism:

$$X \cdot V := \sigma^*([C_{X\cap V}V])\,.$$

As shown in [39], this definition implies all expected properties of an intersection product. Applying Lemma 6.2.8, we see that

$$X \cdot V = \left\{c(j^*N_XY) \cap \text{Segre}_{j^*N_XY\oplus\mathbf{1}}([\mathbf{P}(C_{X\cap V}V \oplus 1\!\!1)])\right\}_{m-d}$$

since $[\mathbf{P}(C_{X\cap V}V \oplus 1\!\!1)]$ restricts to $[C_{X\cap V}V]$ on j^*N_XY

$$= \left\{c(j^*N_XY) \cap s(X \cap V, V)\right\}_{m-d}$$

(cf. (6.8) and (6.9)). This definition, which we rewrite here for emphasis:

$$X \cdot V := \left\{ c(j^* N_X Y) \cap s(X \cap V, V) \right\}_{\dim V - \operatorname{codim}_X Y} \tag{6.13}$$

is of foundational importance in intersection theory. Note that it assigns an explicit contribution to $X \cdot V$ to every connected component Z of $X \cap V$:

$$\text{contribution of } Z \text{ to } X \cdot V: \quad \{c(N_X Y|_Z) \cap s(Z, V)\}_{\dim V - \operatorname{codim}_X Y} \,. \tag{6.14}$$

It can be shown that the right-hand side of (6.13) preserves rational equivalence in the evident sense, so that it defines Gysin homomorphisms $A_k V \to A_{k-d}(X \cap V)$. More generally, it defines a homomorphism $A_k Y' \to A_{k-d}(X \times_Y Y')$ for every morphism $Y' \to Y$. (See [39, Chapter 6].)

Example 6.2.9 A particular case of (6.13) gives the self-intersection formula of a regularly embedded subscheme X of Y. For this, consider the fiber diagram

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

and apply (6.13) to obtain

$$X \cdot X = \{c(N_X Y) \cap s(X, X)\}_{\dim X - \operatorname{codim}_X Y} = c_d(N_X Y) \cap [X] \,.$$

For instance, the self-intersection of the zero-section of a vector bundle E on a variety W equals $c_{\operatorname{rk} E}(E) \cap [W]$: indeed, the zero-section is regularly embedded, with normal bundle E.

It follows that if σ is any section of a vector bundle E, then writing W for the image of the zero-section of E,

$$\iota_*(W \cdot \sigma(W)) = c_{\operatorname{rk} E}(E) \cap [W] \,,$$

where $\iota : Z(\sigma) \to W$ is the embedding of the zero-scheme of σ. Indeed, $\sigma(W)$ is rationally equivalent to the zero-section. Again using (6.13), we can identify the contribution of a union of connected components Z of $Z(\sigma)$ to $c_{\operatorname{rk} E}(E) \cap [W]$ as

$$\{c(E|_Z) \cap s(Z, W)\}_{\dim W - \operatorname{rk} E} \,, \tag{6.15}$$

'localizing' the top Chern class along the zeros of a section. (See [39, §14.1].) ⌟

The requirement that X be *regularly* embedded in Y is nontrivial. It can be bypassed if the ambient scheme Y is a nonsingular variety, say of dimension m. Indeed, in this case the diagonal embedding $Y \to Y \times Y$ is regular with normal bundle TY, and we can interpret the intersection of any two subvarieties Z, W of Y as the intersection of the diagonal Δ with the product $Z \times W$. The fiber diagram

$$\begin{array}{ccc} Z \cap W & \longrightarrow & Z \times W \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ Y = \Delta & \longrightarrow & Y \times Y \end{array}$$

suggests the definition

$$[Z]\cdot[W] := \Delta\cdot(Z\times W) = \{c(j^*TY)\cap s(Z\cap W, Z\times W)\}_{\dim Z+\dim W-m} . \qquad (6.16)$$

Note that neither Z nor W need be regularly embedded in Y. This definition passes to rational equivalence and extends by linearity to a product $A_*(Y) \times A_*(Y) \to A_*(Y)$ making the Chow group $A_*(Y)$ into a *commutative ring*. It can be shown [39, Proposition 8.1.1(d)] that (6.16) is compatible with the previous definition, in the sense that if Y is nonsingular, $Z \subseteq Y$ is a regular embedding, and $W \subseteq Y$ is any subvariety, then $[Z] \cdot [W]$ agrees with the definition of $Z \cdot W$ given earlier.

Example 6.2.10 Sometimes this intersection product may be used to obtain information about a Segre class. For example, consider the three singular quadrics $Q_1, Q_2, Q_3 \subseteq \mathbb{P}^3$ obtained as unions of two out of three planes in general position. For example, Q_1 could be defined by the ideal (x_2x_3), Q_2 by (x_1x_3), and Q_3 by (x_1x_2). The intersection $J = Q_1 \cap Q_2 \cap Q_3$ is the reduced union of three lines through a point. It follows (cf. Example 6.3.3) that

$$\iota_* s(J, \mathbb{P}^3) = 3[\mathbb{P}^1] + m[\mathbb{P}^0] \qquad (6.17)$$

for some integer m, where ι is the embedding of J in $\mathbb{P}^3$.

By Bézout's theorem, the intersection product $Q_1 \cdot Q_2 \cdot Q_3$ equals 8. On the other hand, we may view this intersection product as arising from the diagram

$$\begin{array}{ccc} J = Q_1 \cap Q_2 \cap Q_3 & \longrightarrow & \mathbb{P}^3 \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \delta} \\ Q_1 \times Q_2 \times Q_3 & \longrightarrow & \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \end{array}$$

where δ is the diagonal embedding. Using (6.13), we get (omitting an evident pullback)

$$\left\{c(N_{Q_1\times Q_2\times Q_3}\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3) \cap s(J, \mathbb{P}^3)\right\}_0 = 8[\mathbb{P}^0] ,$$

that is, denoting by H the hyperplane class in $\mathbb{P}^3$,

$$\left\{(1+2H)^3 \cap (3[\mathbb{P}^1] + m[\mathbb{P}^0])\right\} = 8[\mathbb{P}^0]\,,$$

which implies $18 + m = 8$. This determines $m = -10$, and hence

$$\iota_* s(J, \mathbb{P}^3) = 3[\mathbb{P}^1] - 10[\mathbb{P}^0]\,.$$

This agrees (as it should) with the result obtained by using the `SegreClasses` package [49]:

```
i1 : load("SegreClasses.m2")

i2 : R=QQ[x0,x1,x2,x3]

i3 : I=ideal(x1*x2,x1*x3,x2*x3)

i4 : segre(I,ideal(0_R))

               3      2
o4 = - 10H   + 3H
          1      1
```

(omitting some additional output; and note that the package chooses to call H_1 the hyperplane class).

Remark 6.2.11 We could have chosen the quadrics Q_1, Q_2, Q_3 to be the generators of the ideal of a twisted cubic C, and the same argument would show that the push-forward of $s(C, \mathbb{P}^3)$ also equals $3[\mathbb{P}^1] - 10[\mathbb{P}^0]$. In this case, the negative coefficient of $[\mathbb{P}^0]$ reflects the fact that the normal bundle to a twisted cubic in $\mathbb{P}^3$ is positive. So we could interpret the negative coefficient of $[\mathbb{P}^0]$ in $\iota_* s(J, \mathbb{P}^3)$ as a measure of 'positivity' for the normal cone to the scheme J in $\mathbb{P}^3$. ⌟

The 'reverse engineering' technique illustrated above may be used to compute Segre classes in broad generality. The approach to the computation of Segre classes in projective space developed in [33] is based on an extension of similar methods. ⌟

For every class $\overline{\alpha} \in A_k(\mathbf{P}(E \oplus 1\!\!1))$, Lemma 6.2.8 gives an interpretation for the class

$$\left\{c(E) \cap \mathrm{Segre}_{E\oplus \mathbf{1}}(\overline{\alpha})\right\}_{k-\mathrm{rk}\,E} \; : \tag{6.18}$$

this class encodes the class of the restriction of $\overline{\alpha}$ to E. The other components of the class within braces have an equally compelling interpretation. If E is a vector bundle of rank e over a scheme X, and $\pi : \mathbf{P}(E) \to X$ is its projectivization, the Chow group $A_*(\mathbf{P}(E))$ is described by a precise structure theorem: for every class

$G \in A_k(\mathbf{P}(E))$, there exist e unique classes $g_j \in A_j(X)$, $j = k-e+1,\dots,k$ such that

$$G = \sum_{i=0}^{e-1} c_1(\mathscr{O}_E(1))^i \cap \pi^*(g_{k-e+1+i})\,.$$

(Cf. [39, Theorem 3.3(b)].) We call the sum $g_{k-e+1}+\cdots+g_k \in A_*(X)$ the *shadow* of G. Note that G may be reconstructed from its shadow and its dimension. The following elementary result relates the shadow of G to its Segre class.

Lemma 6.2.12 ([7, Lemma 4.2]) *With notation as above, the shadow of G is given by*

$$\sum_{i=0}^{e} g_{k-e+1+i} = c(E) \cap \mathrm{Segre}_E(G)\,.$$

With this understood, we see that the class

$$c(E) \cap \mathrm{Segre}_{E\oplus\mathbf{1}}(\overline{\alpha})$$

within braces in Sect. 6.18 is simply the shadow of $\overline{\alpha}$. From this point of view, the intersection product $X \cdot V$ is one component of the shadow of $[C_{X\cap V}V \oplus 1\!\!1] \in A_*(\mathbf{P}(j^*N_XY \oplus 1\!\!1))$. Several classes we will encounter will have natural interpretations as shadows of classes in suitable projective bundles.

6.2.4 'Residual Intersection', and a Notation

Let V be a variety, let $X \subseteq V$ be a subscheme, and let $\mathscr{L}$ be a line bundle defined on X. We introduce the following notation: if α is a class in $A_*(X)$, and $\alpha = \oplus_j \alpha^{(j)}$, with $\alpha^{(j)}$ of codimension j *in* V, we let

$$\alpha \otimes_V \mathscr{L} := \sum_{j\geq 0} s(\mathscr{L})^j \cap \alpha^{(j)} = \sum_{j\geq 0} \frac{\alpha^{(j)}}{c(\mathscr{L})^j}\,. \tag{6.19}$$

This definition was introduced in [2]. Its notation is motivated by the following property relating the definition to the ordinary operation of tensor product: if E is a vector bundle on X, or more generally any element in the K-group of vector bundles on X, then for all $\alpha \in A_*(X)$ we have

$$(c(E)\cap\alpha)\otimes_M \mathscr{L} = \frac{c(E\otimes\mathscr{L})}{c(\mathscr{L})^{\mathrm{rk}\,E}} \cap (\alpha \otimes_M \mathscr{L})\,. \tag{6.20}$$

See [2, Proposition 1]; the proof of this fact is elementary. Equally elementary is the observation that the notation gives an action of Pic on the Chow group: if $\mathscr{L}$ and $\mathscr{M}$ are line bundles on X, then for all $\alpha \in A_*(X)$ we have

$$(\alpha \otimes_V \mathscr{L}) \otimes_V \mathscr{M} = \alpha \otimes_V (\mathscr{L} \otimes \mathscr{M}) \,. \tag{6.21}$$

See [2, Proposition 2].

The notation introduced above often facilitates computations involving Segre classes. One good example is a formula for the Segre class of a scheme supported on a Cartier divisor, along with 'residual' (possibly embedded) components. Let $D \subseteq V$ be an effective Cartier divisor, and let $R \subseteq V$ a closed subscheme. The scheme-theoretic union of D and R is the closed subscheme $Z \subseteq V$ whose ideal sheaf is the product of the ideal sheaves of D and R. We say that R is the 'residual' scheme to D in Z. The task is to express the Segre class of Z in V in terms of the Segre classes of D and of the residual scheme R.

Proposition 6.2.13 *With notation as above,*

$$s(Z, V) = s(D, V) + c(\mathscr{O}(D))^{-1} \cap (s(R, V) \otimes_V \mathscr{O}(D)) \,.$$

This is [39, Proposition 9.2], written using the notation given above; see [2, Proposition 3]. An equivalent alternative formulation is

$$s(Z, V) = ([D] + c(\mathscr{O}(-D)) \cap s(R, V)) \otimes_V \mathscr{O}(D) \,. \tag{6.22}$$

Along with definition (6.13) and a blow-up construction, Proposition 6.2.13 may be used to assign a contribution to intersections products due to residual schemes, with important applications; see [39, Chapter 9]. In this article, the residual formula (6.22) will have applications in the theory of characteristic classes for singular varietes, cf. especially Sect. 6.4.4.

6.2.5 Example: Hyperplane Arrangements

In the rest of this article we will focus on the relation between Segre classes and invariants of (possibly) singular spaces. Typically, we will extract information about a variety X by considering a Segre class of a scheme associated with the singular locus of X. In many cases we will deal with the case of hypersurfaces of nonsingular varieties, so we formalize the following definition.

Definition 6.2.14 Let X be a hypersurface in a nonsingular variety M, defined by the vanishing of a section s of $\mathscr{O}(X)$. Then the *singularity subscheme* JX of X is defined as the zero-scheme of the section ds of $\Omega^1_M \otimes \mathscr{O}(X)$ determined by s. We will denote by ι the embedding $JX \hookrightarrow X$ or $JX \hookrightarrow M$, as context will dictate. ⌟

Thus, if $z_1, \dots, z_n$ are local parameters for M at a point p, and f is a local equation of X, the ideal of JX at p as a subscheme of M is the jacobian/Tyurina ideal

$$\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, f\right).$$

In characteristic 0, if $M = \mathbb{P}^n$ and $F(x_0, \dots, x_n)$ is a homogeneous polynomial defining a hypersurface X, then JX is globally defined by the ideal

$$\left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}\right)$$

(in characteristic 0, a homogeneous polynomial belongs to the ideal of its partials).

In order to illustrate the type of information encoded by this subscheme, we present the case of hyperplane arrangements. Let $\mathscr{A}$ denote a hyperplane arrangement in (complex) projective space $\mathbb{P}^n$, consisting of d (not necessarily distinct) hyperplanes, and consider the hypersurface A given by the union of these hyperplanes. More precisely, let $L_i(x_0, \dots, x_n)$, $i = 1, \dots, d$ be linear forms whose vanishing defines the hyperplanes; then the hypersurface A is defined by the polynomial

$$F(x_0, \dots, x_n) := \prod_{i=1}^{d} L_i(x_0, \dots, x_n).$$

Max Wakefield and Masahiko Yoshinaga prove [94] that an essential arrangement of distinct hyperplanes in $\mathbb{P}^n$, $n \geq 2$, may be reconstructed from the corresponding singularity subscheme. The following result proves that the *ranks of the cohomology* of the complement are determined by the Segre class of the singularity subscheme of the arrangement.

Theorem 6.2.15 *For an arrangement $\mathscr{A}$ of d hyperplanes, define integers σ_i, $i = 0, \dots, n$, such that*

$$[\mathbb{P}^n] - \iota_* s(JA, \mathbb{P}^n) = \sum_{i \geq 0} \sigma_i [\mathbb{P}^{n-i}].$$

Then

$$\operatorname{rk} H^k(\mathbb{P}^n \smallsetminus A, \mathbb{Q}) = \sum_{i=0}^{k} \binom{k}{i} (d-1)^{k-i} \sigma_i \tag{6.23}$$

for $k = 0, \dots, n$.

This statement is given in [11, Theorem 5.1]; we will sketch a proof in Sect. 6.4.4 (see Example 6.4.24). In fact, in *loc. cit.,* the result is stated for hyperplane arrangements consisting of distinct hyperplanes. Remarkably, this hypothesis is not needed: if any of the hyperplanes appear with a multiplicity, the effect on the Segre class of the singularity subscheme precisely compensates for these multiplicities.

Example 6.2.16 Consider the arrangement in $\mathbb{P}^3$ consisting of the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. The corresponding hypersurface has equation $x_1x_2x_3 = 0$; the singularity subscheme is defined by the ideal

$$(x_1x_2, x_1x_3, x_2x_3)\,.$$

We have computed the corresponding Segre class in Example 6.2.10:

$$\iota_* s(JA, \mathbb{P}^3) = 3[\mathbb{P}^1] - 10[\mathbb{P}^0]\,.$$

We have $d = 3$ and $(\sigma_0, \dots, \sigma_3) = (1, 0, -3, 10)$, therefore Theorem 6.2.15 gives

$$\operatorname{rk} H^k(\mathbb{P}^3 \smallsetminus A, \mathbb{Q}) = \begin{cases} 2^0 \cdot 1 = \mathbf{1} & k = 0 \\ 2^1 \cdot 1 + 2^0 \cdot 0 = \mathbf{2} & k = 1 \\ 2^2 \cdot 1 + 2 \cdot 2^1 \cdot 0 + 2^0 \cdot (-3) = \mathbf{1} & k = 2 \\ 2^3 \cdot 1 + 3 \cdot 2^2 \cdot 0 + 3 \cdot 2^1 \cdot (-3) + 2^0 \cdot 10 = \mathbf{0} & k = 3 \end{cases}$$

as it should.

Now assume the same planes appear with multiplicities 2, 3, 5 respectively. The ideal of A is generated by $x_1^2x_2^3x_3^5$, therefore JA is defined by the ideal

$$\left(x_1x_2^3x_3^5, x_1^2x_2^2x_3^4, x_1^2x_2^3x_3^4\right)$$

and the package `SegreClasses` evaluates its Segre class as

$$\iota_* s(JA, \mathbb{P}^3) = 7[\mathbb{P}^2] - 46[\mathbb{P}^1] + 270[\mathbb{P}^0]\,.$$

In this case $d = 10$ and $(\sigma_0, \dots, \sigma_3) = (1, -7, 46, -270)$, therefore

$$\operatorname{rk} H^k(\mathbb{P}^3 \smallsetminus A, \mathbb{Q}) = \begin{cases} 9^0 \cdot 1 = \mathbf{1} & k = 0 \\ 9^1 \cdot 1 + 9^0 \cdot (-7) = \mathbf{2} & k = 1 \\ 9^2 \cdot 1 + 2 \cdot 9^1 \cdot (-7) + 9^0 \cdot 46 = \mathbf{1} & k = 2 \\ 9^3 \cdot 1 + 3 \cdot 9^2 \cdot (-7) + 3 \cdot 9^1 \cdot 46 + 9^0 \cdot (-270) = \mathbf{0} & k = 3 \end{cases}$$

according to Theorem 6.2.15, with the same result since the support of the arrangement is the same as in the previous case. ⌟

In general, the fact that multiplicities do not affect the right-hand side of (6.23) is a consequence of the residual formula of Proposition 6.2.13, as the reader may enjoy verifying.

Note that the Segre class appearing in Theorem 6.2.15 is the Segre class $s(JA, \mathbb{P}^n)$ of the singularity subscheme *in the ambient space* $\mathbb{P}^n$. The singularity subscheme JA is also contained in the hypersurface A. It is natural to ask what type of information the Segre class $s(JA, A)$ may encode; a full answer to this question will be given in Sect. 6.4.3. Here we point out that, in the case of reduced arrangements (that is, if the hyperplanes are all different), this Segre class is in fact *determined by the number d of hyperplanes.*

Proposition 6.2.17 *Let $\mathscr{A}$ be a reduced arrangement of d hyperplanes in $\mathbb{P}^n$, and let $\iota : JA \hookrightarrow \mathbb{P}^n$ be the corresponding singularity subscheme. Then*

$$\iota_* s(JA, A) = d \sum_{i=2}^{n} (-1)^i (d-1)^{i-1} [\mathbb{P}^{n-i}] . \tag{6.24}$$

Proof Let $H_1, \dots, H_d$ be the hyperplanes of the arrangement, and let $L_k(x_0, \dots, x_n)$ be a generator of the homogeneous ideal of H_k. By (6.12),

$$s(JA, A) = \sum_k s(JA \cap H_k, H_k) . \tag{6.25}$$

The ideal of $JA \cap H_k$ is given by

$$\left(\sum_{j=1}^{d} \prod_{\ell \neq j} L_\ell \frac{\partial L_j}{\partial x_i}, L_k \right)_{i=0,\dots,n} = \left(\prod_{\ell \neq k} L_\ell \frac{\partial L_k}{\partial x_i}, L_k \right)_{i=0,\dots,n}$$

and this is the ideal

$$\left(\prod_{\ell \neq k} L_\ell, L_k \right)$$

since at least one of the derivatives of L_k is nonzero.

It follows that $JA \cap H_k$ is the subscheme of H_k traced by the union of the other hyperplanes; that is, it is a Cartier divisor of class $(d-1)H$ in H_k, where H denotes the hyperplane class. Therefore

$$\iota_* s(JA \cap H_k, H_k) = (d-1)[\mathbb{P}^{n-2}] - (d-1)^2[\mathbb{P}^{n-3}] + (d-1)^3[\mathbb{P}^{n-4}] + \cdots$$

and the statement follows from (6.25). □

Therefore, while the Segre class $s(JA, \mathbb{P}^n)$ detects nontrivial combinatorial information about the arrangement (as Theorem 6.2.15 shows), the Segre class

$s(JA, A)$ is blind to any information but the degree of the arrangement (assuming that the arrangement is reduced).

In particular, note that $s(JA, \mathbb{P}^n)$ is *not* determined by $s(JA, A)$; we will come back to this point in Sect. 6.4.1, Example 6.4.3.

In Sect. 6.4 we will learn that if X is a hypersurface in a nonsingular variety, then the two classes $s(JX, X)$ and $s(JX, M)$ are closely related to different 'characteristic classes' for X, and this will provide a further explanation for the behavior observed in this example (see Examples 6.4.13 and 6.4.24).

6.3 Numerical Invariants

6.3.1 Multiplicity

The most basic numerical invariant of a singularity is its *multiplicity*. Let X be a hypersurface of $\mathbb{A}^n$, and let p be the origin. Write the equation F of X as a sum of homogeneous terms:

$$F(x_1, \dots, x_n) = \sum_{i \geq 0} F_i(x_1, \dots, x_n)$$

with $F_i(x_1, \dots, x_n)$ homogeneous of degree i. By definition, the multiplicity $m_p X$ of X at p is the smallest m such that $F_m(x_1, \dots, x_n) \neq 0$. Thus, $p \in X$ if and only if $m_p X \geq 1$. The 'initial' homogeneous polynomial $F_{m_p X}$ defines the *tangent cone* to X at p; therefore, $m_p X$ is the degree of the tangent cone to X at p.

There is a natural identification of the tangent cone to X at p defined in the previous paragraph with the *normal cone* $C_p X$ introduced in Sect. 6.2.1:

$$C_p X = \mathrm{Spec}(\oplus_{k \geq 0} \mathfrak{m}^k / \mathfrak{m}^{k+1}) \tag{6.26}$$

where $\mathfrak{m}$ is the maximal ideal in the local ring of X at p. We can projectivize this cone, or rather consider the projective completion $\pi : \mathbf{P}(C_p X \oplus 1\!\!1) \to p$ (cf. (6.7); this accounts for the possibility $X = p$, see Remark 6.2.3), and observe that the degree $m_p X$ of this projective cone satisfies

$$(m_p X)[p] = \pi_* \left(\sum_{i \geq 0} c_1(\mathscr{O}(1))^i \cap [\mathbf{P}(C_p X \oplus 1\!\!1)] \right)$$

(cf. (6.2)). In other words, we have verified that the multiplicity of X at p is precisely the information carried by the Segre class of p in X:

$$s(p, X) = (m_p X)[p] \,. \tag{6.27}$$

Of course these considerations are not limited to the case in which X is an affine hypersurface. The tangent cone to a point p of any scheme X is defined to be the normal cone C_pX, that is, the spectrum of the corresponding associated graded ring, as in (6.26). A standard computation shows that if $U = \mathbb{A}^N$ is an affine space centered at p, and $X \cap U$ is defined by an ideal I, then the ideal defining $\oplus_{k\geq 0}\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is generated by the initial forms of the polynomials in I; so this is indeed a straightforward generalization of the situation for hypersurfaces. We can *define* m_pX to be the degree of the projective completion $\mathbf{P}(C_pX \oplus 1\!\!1)$; and then (6.27) holds in this generality. To avoid certain pathologies, it is common to assume that X be pure-dimensional. For example, this hypothesis implies that the multiplicity of X at p equals the sum of the multiplicities of its irreducible components, by (6.12).

The degree of the projective completion of C_pX can also be computed by means of the Hilbert function defined for all integers $t > 0$ by

$$\mathfrak{h}(t) := \dim_k(\oplus_{i=0}^{t-1}\mathfrak{m}^i/\mathfrak{m}^{i+1}) : \tag{6.28}$$

for $t \gg 0$, $\mathfrak{h}(t)$ agrees with a polynomial with leading term $(m_pX)\frac{t^d}{d!}$, where d is the dimension of X.

More generally, we can consider a subvariety V of a (pure-dimensional) scheme X. Samuel [84] defines the multiplicity m_VX of the local ring $\mathscr{O}_{X,V}$ in terms of the leading term of (6.28), where now $\mathfrak{m}$ is taken to be the maximal ideal of $\mathscr{O}_{X,V}$. This amounts to taking the fiberwise degree of the projective completion $\mathbf{P}(C_VX \oplus 1\!\!1) \to V$ of the normal cone C_VX, hence it determines the dominant term of the Segre class:

$$s(V, X) = (m_VX)[V] + \text{lower dimensional terms}\,. \tag{6.29}$$

Example 6.3.1 Let V be a proper subvariety of codimension d of a variety X, and let $\pi : E \to V$ be the exceptional divisor in the blow-up $B\ell_VX$. Then

$$\pi_*(E^{d-1}) = (-1)^d(m_VX)[V]\,.$$

Indeed, this is the dominant term of the Segre class $s(V, X)$ by (6.11). ⌟

Summarizing, we can take (6.29) as the *definition* of multiplicity of a variety along a subvariety, and this agrees with Samuel's algebraic notion of multiplicity. The agreement can be extended by the additivity formula (6.12) to arbitrary pure-dimensional schemes X. It can also be extended to the case in which V is an irreducible component of a subscheme Z of X, leading to the following interpretation of Samuel's multiplicity.

Definition 6.3.2 The multiplicity of a pure-dimensional scheme X along a subscheme Z at an irreducible component V of Z is the coefficient of $[V]$ in $s(Z, X)$.

Example 6.3.3 If X is nonsingular and Z is reduced, then the multiplicity of X along Z is 1 at every component of Z. For instance, each line in Example 6.2.10 appears with multiplicity 1 in the Segre class $s(J, \mathbb{P}^3)$, and this is the reason why the dominant term in (6.17) is $3[\mathbb{P}^1]$. ⌟

Example 6.3.4 If Z is (locally) a complete intersection in X, and its support V is irreducible, then

$$s(Z, X) = m[V] + \text{lower dimensional terms}$$

where m is the *geometric multiplicity* of V in Z, that is, the length of the local ring $\mathscr{O}_{Z,V}$. Indeed, in this case the Segre class is the inverse Chern class of the normal bundle (Sect. 6.2.2): $s(Z, X) = c(N_Z X)^{-1} \cap [Z] = [Z] + \cdots$, and $[Z] = m[V]$ [39, §1.5]. So the multiplicity of X along Z at V equals the geometric multiplicity of V in Z for complete intersections.

This is not true in general, even if X is nonsingular. For example, let Z be the 'triple point' defined by the ideal (x^2, xy, y^2) in the plane. Then $s(Z, \mathbb{A}^2) = 4[p]$, where $V = p$ is the origin, while the geometric multiplicity is 3. Indeed, let Z' be the scheme defined by (x^2, y^2). Then $s(Z', \mathbb{A}^2) = 4[p]$, since Z' is a complete intersection, and $s(Z, \mathbb{A}^2) = s(Z', \mathbb{A}^2)$ since (x^2, xy, y^2) is the integral closure of (x^2, y^2) (see Sect. 6.2.2). ⌟

Example 6.3.5 Let D be the *discriminant* of a line bundle $\mathscr{L}$ on a nonsingular complete variety M, i.e., the subset of $\mathbb{P}H^0(M, \mathscr{L})$ parametrizing singular sections of $\mathscr{L}$. For $X \in D$, consider the integer

$$m_X D = \int c(\mathscr{L}) c(T^\vee M \otimes \mathscr{L}) \cap s(JX, M) \tag{6.30}$$

where JX is the singularity subscheme of X (Definition 6.2.14) and $T^\vee M$ is the cotangent bundle of M. Under reasonable hypotheses, if D is a hypersurface, then $m_X D \neq 0$ and in this case $m_X D$ is the multiplicity of D at X, as the notation suggests. (See [16] for the precise statement of a more general result. A different formula not using Segre classes may be found in [79].)

To see this, one can realize the discriminant D as the image of the correspondence

$$\hat{D} := \{(p, X) \in M \times \mathbb{P}(M, \mathscr{L}) \mid p \in \mathrm{Sing}(X)\}\,.$$

The fiber of X in this correspondence is (isomorphic to) JX, and $\hat{D}$ maps birationally to D under mild hypotheses. The birational invariance of Segre classes implies then that $s(X, D)$ is the push-forward of $s(JX, \hat{D})$, and the latter is computed by making use of Theorem 6.4.1, which we will discuss later.

For instance, let X consist of a d-fold hyperplane in $M = \mathbb{P}^n$. Then JX is a $(d-1)$-fold hyperplane, and consequently

$$s(JX, \mathbb{P}^n) = (1 + (d-1)H)^{-1} \cap (d-1)[\mathbb{P}^{n-1}]\,,$$

where H denotes the hyperplane class. (Example 6.2.4.) View X as a point of the discriminant D of $\mathscr{O}(X)$ over $\mathbb{P}^n$. Then according to (6.30) the multiplicity of D at X is

$$\begin{aligned} m_D X &= \int (1+dH)\,\frac{(1+(d-1)H)^{n+1}}{1+dH} \cap s(JD,\mathbb{P}^n) \\ &= \int (1+(d-1)H)^n \cap (d-1)[\mathbb{P}^{n-1}] \\ &= n(d-1)^n\,. \end{aligned}$$

At the opposite extreme, assume that X has isolated singularities. Then we will verify that $m_D X$ equals the sum of their *Milnor numbers,* see Sect. 6.3.3. ⌟

Several more refined notions of 'multiplicity' may be defined by means of Segre classes; see [61] and [1] for two particularly well-developed instances.

6.3.2 Local Euler Obstruction

The *local Euler obstruction* $\mathrm{Eu}_X(p)$ of a possibly singular variety X (or more generally a reduced pure-dimensional scheme) at a point $p \in X$ is another numerical invariant, in some ways analogous to the multiplicity; indeed, if X is a curve, then $\mathrm{Eu}_X(p)$ *equals* the multiplicity $m_p X$. We first summarize the original transcendental definition, due to MacPherson [66, §3].

We will assume that X is a subvariety of a nonsingular variety M. If X has dimension n, there is a rational map

$$X \dashrightarrow \mathrm{Gr}_n(TM)|_X$$

associating with each nonsingular $x \in X$ the tangent space $T_x X \subseteq T_x M$, viewed as a point in the Grassmann bundle $\mathrm{Gr}_n(TM)$. The closure of the image of this rational map is the *Nash blow-up* $\hat{X}$ of X; it comes equipped with

- a proper birational map $\nu : \hat{X} \to X$; and
- a rank-n vector bundle $\hat{T}$, the pull-back of the tautological subbundle on $\mathrm{Gr}_n(TM)$.

Over the nonsingular part X° of X, ν is an isomorphism and $\hat{T}$ agrees with the pull-back of TX°. Thus, the Nash blow-up is a modification of X that admits a natural vector bundle $\hat{T}$ restricting to TX° on the nonsingular part of X. The fiber of $\hat{X}$ over a point $x \in X$ parametrizes 'limits' of tangent spaces to X° as one approaches x. At a point $\hat{x} \in \nu^{-1}(x)$, the fiber of $\hat{T}$ over $\hat{x}$ is just this limit tangent space.

The Nash blow-up and the tautological bundle $\hat{T}$ are independent of the chosen embedding of X in a nonsingular variety.

Let $p \in X$. As we will work in a neighborhood of p, we may assume that X is affine, $M = \mathbb{A}^m$, and p is the origin. Over $\mathbb{C}$, MacPherson considers the differential form $d||z||^2$, a section of the real dual bundle TM^*. By construction $\hat{T}$ is a subbundle of $\nu^*(TM)$; we denote by r the pull-back of this form to the real dual $\hat{T}^*$.

Next, consider the ball B_ϵ and the sphere S_ϵ of radius ϵ centered at p. For small enough ϵ, r is nonzero over $\nu^{-1}(z)$, $0 < ||z|| \le \epsilon$ [66, Lemma 1]. By definition, *the local Euler obstruction* $\mathrm{Eu}_X(p)$ *is the obstruction to extending* r *as a nonzero section of* $\hat{T}^*$ *from* $\nu^{-1}(S_\epsilon)$ *to* $\nu^{-1}(B_\epsilon)$.

For curves, the local Euler obstruction equals the multiplicity. The local Euler obstruction of a cone over a plane curve of degree d at the vertex equals $2d - d^2$ [66, p. 426]. In particular, note that (unlike the multiplicity) $\mathrm{Eu}_X(p)$ may be negative.

The following algebraic alternative to the transcendental definition is due to G. Gonzalez-Sprinberg and J.-L. Verdier.

Theorem 6.3.6 ([45]) *With notation as above,*

$$\mathrm{Eu}_X(p) = \int c(\hat{T}|_{\nu^{-1}(p)}) \cap s(\nu^{-1}(p), \hat{X}) . \tag{6.31}$$

The proof of this equality is quite delicate. The section r may be replaced with a section σ_s of $\hat{T}$ obtained by projecting the 'radial' section of $\nu^* T\mathbb{A}^m$ by means of a Hermitian form s. Viewing $\nu^{-1}(p)$ as a union of components of the zero-scheme of this section, the local Euler obstruction is then interpreted as the contribution of $\nu^{-1}(p)$ to the intersection product of $\sigma_s(\hat{X})$ with the zero-section of $\hat{T}$, that is, the localized contribution to the degree of the top Chern class $c_{\dim X}(\hat{T}) \cap [\hat{X}]$. This gives (6.31) as we have seen in Example 6.2.9, particularly (6.15). The main problem with this sketch is that the section σ_s is *not* algebraic. This is handled in [45] by applying this argument to a variety dominating $\hat{X}$ and such that the pull-back of σ_s *is* algebraic; (6.31) then follows by the projection formula.

Theorem 6.3.6 yields an interpretation of the local Euler obstruction that does not depend on complex geometry, so may be adopted over arbitrary fields. The use of the Nash blow-up is not necessary: any proper birational map $\nu : \hat{X} \to X$ such that $\nu^*\Omega^1_X$ surjects onto a locally free sheaf $\hat{\Omega}$ of rank $n = \dim X$ will do, with $\hat{T} = \hat{\Omega}^\vee$. (This follows from the birational invariance of Segre classes; see [39, Example 4.2.9].)

Claude Sabbah [83] recasts the algebraic definition of $\mathrm{Eu}_X(p)$ it in terms of the *conormal space* of X. Recall that if $W \subsetneq M$ is an embedding of nonsingular varieties, then the *conormal bundle* $N^\vee_W M$ of W in M is the kernel of the natural morphism of cotangent bundles $T^\vee M|_W \to T^\vee W$:

$$0 \longrightarrow N^\vee_W M \longrightarrow T^\vee M|_W \longrightarrow T^\vee W \longrightarrow 0 .$$

The conormal *space* $N^\vee_X M$ of a possibly singular subvariety X of M is the closure of the conormal bundle of its nonsingular part X°:

$$N^\vee_X M := \overline{N^\vee_{X^\circ} M}\,.$$

The projectivization $\mathbf{P}(N^\vee_X M) \subseteq \mathbf{P}(T^\vee M|_X)$ is equipped with

- a morphism $\kappa : \mathbf{P}(N^\vee_X M) \to X$; and, letting $m = \dim M$,
- a rank-$(m-1)$ vector bundle $\overline{T}$, the pull-back of the tautological subbundle on $\mathbf{P}(T^\vee M|_X) = \mathrm{Gr}_{m-1}(TM|_X)$.

Proposition 6.3.7 ([60, Lemma 2])

$$\mathrm{Eu}_X(p) = (-1)^{m-n-1} \int c(\overline{T}|_{\kappa^{-1}(p)}) \cap s(\kappa^{-1}(p), \mathbf{P}(N^\vee_X M))\,. \tag{6.32}$$

This result may be established as a corollary of Theorem 6.3.6, by means of a commutative diagram

$$\begin{array}{ccc} J & \longrightarrow & \mathbf{P}(N^\vee_X M) \\ \big\downarrow & & \big\downarrow{\scriptstyle \kappa} \\ \hat{X} & \xrightarrow[\nu]{} & X \end{array}$$

where J is the unique component of the fiber product dominating X; see [60] for details.

Example 6.3.8 Again let D be the discriminant of a line bundle $\mathscr{L}$ on a non singular complete variety (Example 6.3.5), and let $X \in D$ be a singular section of $\mathscr{L}$. Then under mild hypotheses (implying that D is a hypersurface) we have

$$\mathrm{Eu}_D(X) = \int c(T^\vee M \otimes \mathscr{L}) \cap s(JX, M) \tag{6.33}$$

[4, Theorem 3]. Indeed, one can verify that the correspondence $\hat{D}$ mentioned in Example 6.3.5 is the Nash blow-up of D, and JX is isomorphic to the fiber of the point $X \in D$ under $\hat{D} \to D$. Then (6.33) follows from the Gonzalez-Sprinberg–Verdier formula (6.31), after manipulations expressing $s(JX, \hat{D})$ in terms of $s(JX, M)$ and a computation of the Chern class of the tautological bundle.

For a concrete instance, consider (as in Example 6.3.5) the case of a d-fold hyperplane in $\mathbb{P}^n$. According to (6.33), the local Euler obstruction of the discriminant of $\mathscr{O}(X)$ at the corresponding point is

$$\begin{aligned}\mathrm{Eu}_D(X) &= \int \frac{(1+(d-1)H)^n}{1+dH} \cap (d-1)[\mathbb{P}^{n-1}] \\ &= (d-1)\cdot\frac{(d-1)^n-1}{d}\,.\end{aligned}$$

The reader should compare the formulas for the multiplicity of a discriminant at a singular hypersurface X, (6.30), and for the local Euler obstruction at X, (6.33). We do not know if the similarity between these formulas can be extended to more general cases, e.g., discriminants of complete intersections. ⌟

A classical result of Lê Dũng Tráng and Bernard Teissier expresses the local Euler obstruction as an alternating sum of multiplicities of polar varieties, [64, Corollaire 5.1.2].

6.3.3 Milnor Number

Segre classes provide a natural algebraic framework to treat Milnor numbers. Here we work over $\mathbb{C}$; the formulas we will obtain could be taken as alternative algebraic definitions extending the notions to arbitrary algebraically closed fields of characteristic 0.

Let X be a hypersurface in a nonsingular variety M, and let p be an isolated singularity of X. Again consider the singularity subscheme JX of X, Definition 6.2.14. In this case p is the support of one component of JX, which we denote $\hat{p}$. As a subscheme *of* M, the ideal of $\hat{p}$ at p is

$$\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}, f\right)$$

where the ideal of X is locally generated by f and $z_1,\dots,z_n$ are local parameters for M at p.

Proposition 6.3.9 *The Milnor number $\mu_X(p)$ of X at p equals the coefficient of p in $s(\hat{p}, M)$:*

$$s(\hat{p}, M) = \mu_X(p)[p]\,.$$

From this observation and (6.30), it follows that if X only has isolated singularities $p_1,\dots,p_r$, then (under mild hypotheses) the multiplicity of the discriminant of $\mathscr{O}(X)$ at X equals the sum of the Milnor numbers $\sum_i \mu_X(p_i)$. For an earlier proof of this fact, at least in the context of dual varieties, cf. [30].

Proof In characteristic 0, f is integral over the ideal generated by its partials (see e.g., [92, Example 1.43]), therefore $s(\hat{p}, M) = s(p', M)$, where p' is the scheme defined by $(\partial f/\partial z_1, \dots, \partial f/\partial z_n)$. Now (Example 6.3.4) $s(p', M) = m[p]$, where m is the geometric multiplicity of p in p'. By definition,

$$m = \dim \mathscr{O}_{M,p}/(\partial f/\partial z_1, \dots, \partial f/\partial z_n)\,,$$

and this equals the Milnor number μ [73, p. 115]. □

As an alternative, one can verify that $s(\hat{p}, M)$ evaluates the effect on the Euler characteristic of X due to the presence of the singularity p, cf. [39, Example 14.1.5(b)].

Adam Parusiński [78] defines a generalization of the Milnor number to hypersurfaces with arbitrary (compact) singular locus. A section s of $\mathscr{O}(X)$ defining X determines a section ds of $T^\vee M \otimes \mathscr{O}(X)$ in a neighborhood of X, of which JX is the zero-scheme (Definition 6.2.14). By definition, Parusiński's generalized Milnor number $\mu(X)$ equals the contribution of the singular locus to the intersection number of the image of this section and the zero section of $T^\vee M \otimes \mathscr{O}(X)$.

Proposition 6.3.10 ([3, Proposition 2.1]) *With notation as above,*

$$\mu(X) = \int c(T^\vee M \otimes \mathscr{O}(X)) \cap s(JX, M)\,. \tag{6.34}$$

Proof Let U be a neighborhood of JX where ds is defined, and consider the fiber diagram

$$\begin{array}{ccc} JX & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle ds} \\ U & \xrightarrow{\;0\;} & T^\vee M \otimes \mathscr{O}(X)|_U \end{array}$$

The normal bundle of the zero-section equals $T^\vee M \otimes \mathscr{O}(X)|_{JX}$, and $s(JX, U) = s(JX, M)$ as open embeddings are flat, cf. Proposition 6.2.5. The stated formula follows then from (6.13). □

In other words, Parusiński's Milnor number equals the localized contribution of JX to the degree of the top Chern class of $T^\vee M \otimes \mathscr{O}(X)$. (But note that in general the section s does not extend to an algebraic section defined on the whole of M, so this number does *not* equal the degree of the top Chern class.)

If M is compact and X_{gen} is a nonsingular hypersurface linearly equivalent to X, then

$$\mu(X) = (-1)^{\dim X}(\chi(X_{\text{gen}}) - \chi(X)) \tag{6.35}$$

where χ denotes the topological Euler characteristic ([78, Corollary 1.7], and cf. [39, Example 14.1.5(b)]). Thus this generalization of the Milnor number can also be interpreted as the effect on the Euler characteristic of X due to its singular locus. This observation is at the root of the definition of the 'Milnor class', see Sect. 6.4.6.

Comparing (6.33) and (6.34), we see that, under reasonable hypotheses, this generalized Milnor number equals the local Euler obstruction of the discriminant of $\mathscr{O}(X)$ at X. The class $c(T^\vee M \otimes \mathscr{O}(X)) \cap s(JX, M)$ appearing in these formulas is the 'μ-class' studied in [3]. Even when JX or M are not compact, this class carries interesting information.

6.4 Characteristic Classes

The formalism of Segre classes provides a unifying point of view on several 'characteristic classes' for singular varieties. We refer here to various generalizations to (possibly) singular varieties of the basic notion of total Chern class of the tangent bundle of a nonsingular variety:

$$c(TV) \cap [V] \in A_*(V)\,.$$

This is class of evident importance in the nonsingular case. The codimension-1 term $c_1(TV) \cap [V]$ is the canonical class of V, up to a sign. For compact complex varieties, the degree of the dimension 0 term equals the topological Euler characteristic, as a consequence of the classical Poincaré-Hopf theorem. The total Chern class is effective if the tangent bundle is suitably ample. For nonsingular toric varieties, the class has a compelling combinatorial interpretation: it is the sum of the classes of the torus orbit closures, which are determined by the cones of the corresponding fan. In any case, the sheaf of differentials is in a sense the 'only' canonically determined sheaf on a scheme, and the total Chern class of the (co)tangent bundle is correspondingly the 'only' canonically defined class in the Chow group of a nonsingular variety.

It is natural to explore generalizations of this notion to singular varieties, and in this section we will review different alternatives for such an extension, as they relate to Segre classes. We remark that there are several other notions of 'characteristic class' associated to a variety (for example the Todd and L classes), and modern unifications of these notions, such as the *Hirzebruch* and *motivic Chern class* of Brasselet-Schürmann-Yokura [23]. While analogues of Segre classes may be defined in these different contexts, we will limit ourselves to the characteristic classes defined in the Chow group and having a direct relation with the classical notion of Segre classes discussed in Sect. 6.2. We will also not deal with germane notions such as Johnson's or Yokura's Segre classes (see [57, 96]).

6.4.1 Chern-Fulton and Chern-Fulton-Johnson Classes

Let X be a scheme that can be embedded as a closed subscheme of a nonsingular variety M. Here no restrictions on the characteristic of the ground field are needed. We let

$$\begin{aligned} c_{\mathrm{F}}(X) &:= c(TM|_X) \cap s(X, M) \\ c_{\mathrm{FJ}}(X) &:= c(TM|_X) \cap s(\mathscr{N}_X M)\,. \end{aligned} \tag{6.36}$$

Here $\mathscr{N}_X M = \mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is the ideal sheaf of X in M; the Segre class $s(\mathscr{N}_X M)$ is obtained by applying the basic construction of Segre classes to the cone $\mathrm{Sym}^*_{\mathscr{O}_X}(\mathscr{N}_X M)$ (see Sect. 6.2.1). We call $c_{\mathrm{F}}(X)$ the 'Chern-Fulton class' of X, and $c_{\mathrm{FJ}}(X)$ the 'Chern-Fulton-Johnson' class of X. The following result shows that these classes are canonically determined by X.

Theorem 6.4.1 ([39, Example 4.2.6], [41]) *The classes $c_F(X)$, $c_{FJ}(X)$ defined above are independent of the ambient nonsingular variety M.*

This is proved by relating classes determined by different embeddings by means of 'exact sequences of cones' (cf. [39, Examples 4.1.6, 4.1.7]). For instance, if $X \hookrightarrow M$ and $X \hookrightarrow N$ are distinct embeddings in nonsingular varieties, then we have a diagonal embedding $X \subseteq M \times N$, and there is a corresponding exact sequence of cones

$$0 \longrightarrow TN|_X \longrightarrow C_X(M \times N) \longrightarrow C_X M \longrightarrow 0$$

implying

$$s(X, M \times N) = s(TN|_X) \cap s(X, M)\,.$$

The independence of $c_{\mathrm{F}}(X)$ follows.

Theorem 6.4.1 is useful in the computation of Segre classes; it is employed in the computations leading to the formulas presented in Example 6.3.5 and 6.3.8. It has the following consequence.

Corollary 6.4.2 *Let $X = V$ be a nonsingular variety. Then*

$$c_F(V) = c_{FJ}(V) = c(TV) \cap [V]\,.$$

Proof By Theorem 6.4.1, we can use $X = M = V$ to compute $c_{\mathrm{F}}(V)$ and $c_{\mathrm{FJ}}(V)$; and $s(V, V) = s(\mathscr{N}_V V) = [V]$. □

Therefore these two classes are generalizations of the notion of total Chern class from the nonsingular case. They both satisfy formal properties analogous to the nonsingular case. For example, both classes satisfy expected adjunction formulas for sufficiently transversal intersections with smooth subvarieties: for the Chern-Fulton class, this follows from [48, Theorem 3.2]; and see [41, §3] for the Chern-Fulton-Johnson class. On the other hand, some simple relations in the nonsingular case do *not* hold for these classes.

Example 6.4.3 If $W \subseteq X \subseteq M$, with both X and M nonsingular, we may compute $c_{\mathrm{F}}(W)$ using either embedding, and Theorem 6.4.1 implies that

$$s(W, X) = c(N_X M|_W) \cap s(W, M) . \tag{6.37}$$

For instance, if X is a *nonsingular* hypersurface, then

$$s(W, X) = c(\mathscr{O}(X)) \cap s(W, M) .$$

Such appealingly simple formulas do *not* hold in general if X is singular, even if it is regularly embedded in M (so that $N_X M$ is defined, and the terms in the formulas make sense). Indeed, (6.37) fails already for W = a singular point of a curve X in $M = \mathbb{P}^2$.

Without additional hypotheses on W and X guaranteeing that the corresponding sequence of cones is exact, the Segre class of W in X is not determined by the class of W in M. Sean Keel [59] proved that (6.37) does hold if X is regularly embedded, provided that the embedding $W \subseteq X$ is 'linear'.

It would be useful to have precise comparison results relating the difference between the two sides of (6.37) to the singularities of X. We will encounter below (Remark 6.4.16) one case in which this difference has a clear significance. ⌟

The classes $c_{\mathrm{F}}(X)$ and $c_{\mathrm{FJ}}(X)$ differ in general. The discrepancy is a manifestation of the difference between the associated graded ring of an ideal I of a commutative ring R, that is, $\oplus_k I^k/I^{k+1}$, and the symmetric algebra of I/I^2 over R/I. The former is in a sense closer to the ring R: for example, in the geometric context and if R is an integral domain, the Krull dimension of the associated graded ring equals the dimension of R, while the dimension of the symmetric algebra of a module is bounded below by the number of generators of the module [56, Corollary 2.8]. The difference is analogous to the difference between the tangent *cone* of a scheme at a point and the tangent *space* of the same: the former may be viewed as an analytic approximation of the scheme at the point, while the latter only records the minimal embedding dimension. Accordingly, $c_{\mathrm{F}}(X)$ is perhaps a more natural object of study than $c_{\mathrm{FJ}}(X)$. The triple planar point X defined by the ideal (x^2, xy, y^2) in the affine plane $\mathbb{A}^2$ gives a concrete example for which $c_{\mathrm{F}}(X) \neq c_{\mathrm{FJ}}(X)$ (see [8, §2.1]).

One class of ideals for which the associated graded ring is isomorphic to the symmetric algebra is given by ideals generated by regular sequences (cf. [72], [90, Theorem 1.3]). Thus, $c_{\mathrm{F}}(X) = c_{\mathrm{FJ}}(X)$ *if* X *is a local complete intersection.* In this case the embedding $X \subseteq M$ is regular, $s(X, M) = c(N_X M)^{-1} \cap [X]$ (Sect. 6.2.2), and therefore

$$c_{\rm F}(X) = c_{\rm FJ}(X) = c(T_{\rm vir}X) \cap [X]\,, \tag{6.38}$$

where $T_{\rm vir}X$ is the class $TM|_X - N_X M$ in the Grothendieck group of vector bundles on X (so $c(T_{\rm vir}) = c(TM|_X)c(N_X M)^{-1}$). We can view $T_{\rm vir}X$ as a 'virtual tangent bundle' for X; it is well-defined for local complete intersections, i.e., independent of the ambient nonsingular variety M. We note that, more generally,

$$c_{\rm F}(Z) = c(T_{\rm vir}X) \cap s(Z, X)$$

if Z is *linearly* embedded in a local complete intersection X, cf. Example 6.4.3.

If X is a local complete intersection, we will denote the class (6.38) by $c_{\rm vir}(X)$, the 'virtual' Chern class of X. For instance, if $X = D$ is a hypersurface in a nonsingular variety M, then

$$c_{\rm vir}(D) = c_{\rm F}(D) = c_{\rm FJ}(D) = c(TM|_D) \cap \frac{[D]}{1+D}\,.$$

This implies the following useful interpretation of the Chern-Fulton / Fulton-Johnson class of a hypersurface.

Proposition 6.4.4 *Let $i : D \hookrightarrow M$ be a hypersurface in a nonsingular variety M, and let $i' : D_{gen} \to M$ be a nonsingular hypersurface such that $[D] = [D_{gen}]$. Then*

$$i_* c_{vir}(D) = i'_*(c(TD_{gen}) \cap [D_{gen}])\,.$$

In particular, over $\mathbb{C}$ and if M is compact, then $\int c_{\rm F}(D) = \chi(D_{\rm gen})$ is the topological Euler characteristic of a smoothing of D, when a smoothing is available. Barbara Fantechi and Lothar Göttsche prove that in fact $\int c_{\rm F}(X)$ is constant along lci deformations if X is a local complete intersection [34, Proposition 4.15].

These results may be seen as indicating that a class such as $c_{\rm F}(X)$ is *not* useful in the study of singularities, precisely because (at least in the lci case) it is blind to the singularities of X. This feature is balanced by the sensitivity of $c_{\rm F}(X)$ to the scheme structure of X; as we will see below (Proposition 6.4.20) this can be used to encode in a Chern-Fulton class substantial information on the singularities of X.

Bernd Siebert obtains a formula for the 'virtual fundamental class' in Gromov-Witten theory in terms of the Chern-Fulton class, [87, Theorem 4.6]. Siebert also argues that $c_{\rm F}(X)$ could be considered as the Segre class of the Behrend-Fantechi *intrinsic normal cone* of X [20].

6.4.2 The Deligne-Grothendieck Conjecture and MacPherson's Theorem

A functorial theory of Chern classes arose in work of Alexander Grothendieck and Pierre Deligne. Here we will assume that the ground field is algebraically closed, of characteristic 0.

For an algebraic variety X, we denote by $\mathrm{F}(X)$ the group of integer-valued constructible functions on X. These are integer linear combinations of indicator functions for constructible subsets of X; equivalently, every constructible function $\varphi \in \mathrm{F}(X)$ may be written

$$\varphi = \sum_W m_W 1\!\!1_W$$

where W ranges over subvarieties of X, $1\!\!1_W(p) = 1$ or 0 according to whether $p \in W$ or $p \notin W$, and $m_W \in \mathbb{Z}$ is nonzero for only finitely many subvarieties W.

For a proper morphism $f : X \to Y$, we can define a push-forward of constructible functions $f_*\colon \mathrm{F}(X) \to \mathrm{F}(Y)$. By linearity, this is determined by the push-forward $f_*(1\!\!1_W)$ of the indicator function of a subvariety W of X; we set

$$f_*(1\!\!1_W)(p) := \chi(f^{-1}(p) \cap W)$$

for $p \in Y$. Here, χ is the topological Euler characteristic if $k = \mathbb{C}$, and a suitable generalization for more general algebraically closed fields of characteristic 0 (see e.g., [10, §2.1]).

With this push-forward, the assignment $X \mapsto \mathrm{F}(X)$ is a covariant functor from the category of algebraic k-varieties, with proper morphisms, to the category of abelian groups ([66, Proposition 1] for the complex case; the argument generalizes to more general fields).

The Chow group is *also* a covariant functor between the same categories. The following statement, whose conjectural formulation is attributed to Deligne and Grothendieck, gives a precise relationship between these two functors. It was proved by MacPherson [66].

Theorem 6.4.5 *There exists a natural transformation $c_* : \mathrm{F} \Rightarrow A_*$ which, on a nonsingular variety V, assigns to the constant function $1\!\!1_V$ the total Chern class $c(TV) \cap [V]$.*

MacPherson's statement and proof was for complex varieties, in homology; Fulton [39, Example 19.1.7] places the target in the Chow group. Gary Kennedy [60] extended the result to arbitrary algebraically closed fields of characteristic 0. An alternative argument in this generality (and an alternative construction of c_*) is given in [9].

The natural transformation c_* is easily seen to be unique if it exists, as its value is determined by the normalization requirement by resolution of singularities.

MacPherson provides a different construction, not relying on resolutions; and then proves that this construction satisfies the covariance requirement. The ingredients in MacPherson's construction are the *local Euler obstruction* Eu_X, reviewed above in Sect. 6.3.2, and the *Chern-Mather class* $c_{\mathrm{Ma}}(X)$, which will be discussed below in Sect. 6.4.3. MacPherson defines c_* by prescribing that

$$c_*(\mathrm{Eu}_X) = c_{\mathrm{Ma}}(X)\,,$$

and is able to prove that this assignment determines a natural transformation. Since $\mathrm{Eu}_V = 1\!\!1_V$ if V is nonsingular, this definition satisfies the normalization requirement in Theorem 6.4.5. Any choice of a constructible function on varieties X which takes the constant value $1\!\!1_V$ for nonsingular varieties V will then provide us with a 'characteristic class' in the Chow group $A_*(X)$ agreeing with the total Chern class of the tangent bundle when $X = V$ is a nonsingular variety, as prospected in the leader to this section.

Example 6.4.6 Let D be a hypersurface in a nonsingular complex variety. Assume that D may be realized as the central fiber of a flat family over a disk, such that the general fiber D_{gen} is nonsingular. Verdier [93] defines a 'specialization' of constructible functions from the general fiber to D, and proves that this specialization operation is compatible with MacPherson's natural transformation and specialization of Chow classes. As a consequence, if $\sigma(1\!\!1)$ denotes the specialization of the constant function $1\!\!1$, we have

$$c_{\mathrm{vir}}(D) = c_*(\sigma(1\!\!1))$$

(cf. Proposition 6.4.4). If D is itself nonsingular, then $\sigma(1\!\!1) = 1\!\!1_D$, and $c_{\mathrm{vir}}(D) = c(TD) \cap [D]$. We do not know whether the Chern-Fulton or Chern-Fulton-Johnson classes admit a similar description for more general varieties. ⌟

A formula for the Chern-Mather class due to Sabbah, [83, Lemma 1.2.1], leads to a useful alternative description of the image of a constructible function φ via MacPherson's natural transformation c_*. In recalling this description, we essentially follow the lucid account given in [80, §1].

Let X be a proper subvariety of a nonsingular variety M. Every constructible function $\varphi \in \mathrm{F}(X)$ may be written uniquely as a finite linear combination of local Euler obstructions of subvarieties of X:

$$\varphi = \sum_{W \subseteq X} n_W \, \mathrm{Eu}_W$$

([66, Lemma 2]). Now recall (6.3.2) that the conormal *space* $N^\vee_W M$ of a possibly singular subvariety W of M is the closure of the conormal bundle of its nonsingular part W°: $N^\vee_W M := \overline{N^\vee_{W^\circ} M}$. We associate with the local Euler obstruction of a subvariety W of M the cycle of the projectivization of its conormal space, up to a sign recording the parity of the dimension of W:

$$\mathrm{Eu}_W \mapsto (-1)^{\dim W}[\mathbf{P}(N^\vee_W M)]\,. \tag{6.39}$$

By linearity, every constructible function on X is then associated with a cycle in the projectivized cotangent bundle of the ambient nonsingular variety M, $\mathbf{P}(T^\vee M)$, and in fact of the restriction $\mathbf{P}(T^\vee M|_X)$ to X.

Definition 6.4.7 The *characteristic cycle* of the constructible function φ is the linear combination

$$\mathrm{Ch}(\varphi) := \sum_{W\subseteq X} n_W(-1)^{\dim W}[\mathbf{P}(N^\vee_W M)]\,,$$

where $\varphi = \sum_{W\subseteq X} n_W\,\mathrm{Eu}_W$. ⌟

(We have chosen to view $\mathrm{Ch}(\varphi)$ as a cycle in $\mathbf{P}(T^\vee M|_X)$. It is also common in the literature to avoid the projectivization, and consider characteristic cycles as cycles in $T^\vee M|_X$.)

In keeping with the theme of this paper, we will formulate the alternative description of c_* stemming from Sabbah's work in terms of a Segre operator (cf. [7, Lemma 4.3]). For this, it is convenient to adopt the following notation. If $A = \sum_i a_i$ is a rational equivalence class, where a_i is the component of dimension i, we will let

$$A_\vee := \sum_i (-1)^i a_i$$

be the class obtained by changing the sign of all odd-dimensional components of A. Note that if E is a vector bundle, then

$$(c(E)\cap A)_\vee = c(E^\vee)\cap A_\vee\,.$$

Later on, it will also be convenient to use the notation

$$A^\vee := (-1)^{\dim M} A_\vee = \sum_i (-1)^{\dim M-i} a_i\,, \tag{6.40}$$

where M is the fixed ambient nonsingular variety.

Theorem 6.4.8 *The class $c_*(\varphi)_\vee$ is the shadow of the characteristic cycle* $\mathrm{Ch}(\varphi)$. *That is,*

$$c_*(\varphi) = c(TM|_X)\cap \mathrm{Segre}_{T^\vee M|_X}(\mathrm{Ch}(\varphi))_\vee\,. \tag{6.41}$$

Indeed, (6.41) is equivalent to

$$c_*(\varphi) = (-1)^{\dim M-1} c(TM|_X) \cap \pi_*(c(\mathscr{O}(1))^{-1} \cap \mathrm{Ch}(\varphi)) , \tag{6.42}$$

where $\pi_* : \mathbf{P}(T^\vee M|_X) \to X$ is the projection; this is [80, (12)], and the right-hand side is the shadow of $\mathrm{Ch}(\varphi)$ by Lemma 6.2.12, up to changing the sign of every other component. By linearity, (6.42) follows from

$$c_{\mathrm{Ma}}(W) = c_*(\mathrm{Eu}_W) = (-1)^{\dim M-\dim W-1} c(TM|_W) \cap \pi_*(c(\mathscr{O}(1))^{-1} \cap [\mathbf{P}(N_W^\vee M)])$$

(where π is now the projection to W). This formula is (equivalent to) [83, Lemma 1.2.1]; also cf. [60, Lemma 1].

Formula (6.41) should be compared with the formulas (6.36) defining the Chern-Fulton and Chern-Fulton-Johnson classes. The Segre term

$$\mathrm{Segre}_{T^\vee M|_X}(\mathrm{Ch}(\varphi))_\vee$$

plays for MacPherson's natural transformation precisely the same rôle played by the 'ordinary' Segre classes $s(X, M)$, resp., $s(\mathscr{N}_X M)$ for $c_{\mathrm{F}}(X)$, resp., $c_{\mathrm{FJ}}(X)$. We will come back to this term below, see (6.51).

The strength of Theorem 6.4.8 is that (as Sabbah puts it, [83, p. 162]) '*…cela montre que la théorie des classes de Chern de [66] se ramène à une théorie de Chow sur $T^\vee M$, qui ne fait intervenir que des classes fondamentales.*' Indeed, the Segre term is determined by the characteristic cycle $\mathrm{Ch}(\varphi)$; this is a linear combination of $(\dim M - 1)$-dimensional fundamental classes of projectivized conormal spaces. These characteristic cycles (and the local Euler obstruction itself) arise naturally in the theory of holonomic D-modules; this aspect is also treated in [83], as well as in work of Masaki Kashiwara, Victor Ginzburg, and others (see e.g., [25, 44]). The characteristic cycles $\mathrm{Ch}(\varphi)$ are projectivizations of *Lagrangian* cycles in $T^\vee M$, and various functoriality properties admit a compelling geometric description in terms of Lagrangian cycles. Thus, the functor F of constructible functions may be replaced by a 'Lagrangian functor' associating with X the group of integer linear combinations of conormal cycles. See [83] and [60] for more information.

From this point of view, defining a characteristic class for arbitrary varieties that generalizes the total Chern class of the tangent bundle from the nonsingular case amounts to identifying ways to define Lagrangian cycles which, in the nonsingular case, associate a variety with the cycle of its conormal bundle (up to sign). We will focus on two specific choices:

- The conormal *space* of a (possibly singular) variety X, corresponding to the Chern-Mather class $c_{\mathrm{Ma}}(X) = c_*(\mathrm{Eu}_X)$ (Sect. 6.4.3); and
- The 'characteristic cycle' of X, that is, $\mathrm{Ch}(1\!\!1_X)$, corresponding to the 'Chern-Schwartz-MacPherson class' of X (Sect. 6.4.4).

One of the challenges will be to find (more) explicit expressions for the corresponding Segre terms $\mathrm{Segre}_{T^\vee M|_X}(\mathrm{Ch}(\mathrm{Eu}_X))_\vee$, $\mathrm{Segre}_{T^\vee M|_X}(\mathrm{Ch}(1\!\!1_X))_\vee$.

6.4.3 Chern-Mather Classes

A key ingredient in MacPherson's construction of the natural transformation c_* is the *Chern-Mather* class of a variety X, $c_{\mathrm{Ma}}(X)$. MacPherson gives a definition of this class in [66, §2], attributing it to Mather. We note that the definition of an equivalent notion was given earlier by Wu Wen-Tsün [95]; the equivalence was proved later by Zhou Jianyi [100].[1]

Let X be a reduced subscheme of a nonsingular variety M of pure dimension n, and let X° be the nonsingular part of X. Recall (Sect. 6.3.2) that the *Nash blow-up* $\hat{X}$ of X is the closure of the image of the natural rational map $X \dashrightarrow \mathrm{Gr}_n(TM)|_X$ associating with a nonsingular $x \in X^\circ$ the tangent space $T_x X^\circ \subseteq T_x M$. The projection from the Grassmannian restricts to a proper birational map $\nu : \hat{X} \to X$, and the tautological subbundle restricts to a rank-n vector bundle $\hat{T}$ on $\hat{X}$ extending the pull-back of TX°. The local Euler obstruction $\mathrm{Eu}_X(p)$ equals

$$\int c(\hat{T}|_{\nu^{-1}(p)}) \cap s(\nu^{-1}(p), \hat{X})$$

(Theorem 6.3.6). Following MacPherson, we define the Chern-Mather class of X to be the push-forward of the Chern class of $\hat{T}$.

Definition 6.4.9 With notation as above, the *Chern-Mather class* of X is

$$c_{\mathrm{Ma}}(X) = \nu_* \left(c(\hat{T}) \cap [\hat{X}] \right) , \tag{6.43}$$

an element of $A_*(X)$. ⌟

As we discussed in Sect. 6.4.2, we have the following alternative expression for the Chern-Mather class:

$$c_{\mathrm{Ma}}(X) = c(TM|_X) \cap (-1)^{\dim X}\, \mathrm{Segre}_{T^\vee M|_X}([\mathbf{P}(N_X^\vee M)])_\vee\,. \tag{6.44}$$

This is (6.41) for $\varphi = \mathrm{Eu}_X$, as $\mathrm{Ch}(\mathrm{Eu}_X) = (-1)^{\dim X}[\mathbf{P}(N_X^\vee M)]$ (see (6.39)). The equivalence of (6.43) and (6.44), due to Sabbah, may be verified by the same techniques proving Proposition 6.3.7; cf. [60, Lemma 1].

If X is a hypersurface, the Segre term can be expressed directly in terms of ordinary Segre classes. Recall that for a rational equivalence class A of a subvariety

[1] We also note that in his review of [47], Raoul Bott credits Wu with an approach to the algebraic construction of characteristic classes similar to and preceding Grothendieck's.

of a fixed ambient variety M, we let

$$A^\vee := (-1)^{\dim M} A_\vee \,.$$

Theorem 6.4.10 *Let X be a hypersurface of a nonsingular variety M. Then*

$$(-1)^{\dim X}\,\mathrm{Segre}_{T^\vee M|_X}([\mathbf{P}(N^\vee_X M)])_\vee = \big([X] + \iota_* s(JX, X)^\vee\big) \otimes_M \mathscr{O}(X)\,.$$

(Cf. [5, Lemma I.2] and [14, Proposition 2.2].) In this statement, JX is the singularity subscheme of X (Definition 6.2.14), $\iota\colon JX \to X$ is the embedding, and we use the notation $\otimes_M$ recalled in Sect. 6.2.4.

Proof The left-hand side of the stated formula equals

$$\pi_*\left(c(\mathscr{O}_{T^\vee M|_X}(1))^{-1} \cap [\mathbf{P}(N^\vee_X M)]\right)\,, \tag{6.45}$$

where $\pi : \mathbf{P}(T^\vee M|_X) \to X$ is the projection. As X is a hypersurface, the projectivized conormal space $\mathbf{P}(N^\vee_X M)$ may be realized as the closure of the image of the rational map

$$X \dashrightarrow \mathbf{P}((T^\vee M \otimes \mathscr{O}(X))|_X) \cong \mathbf{P}(T^\vee M|_X)$$

associating with every $x \in X^\circ$ the hyperplane $T_x X^\circ$ of $T_x M$, viewed as a point of $\mathbf{P}(N^\vee_x M)$. This closure is isomorphic to the blow-up of X along the base scheme of the rational map, and the base scheme is JX by definition. For another point of view on this observation, recall that the Nash blow-up of a hypersurface X is isomorphic to its blow-up along JX, see e.g., [74, Remark 2]; for hypersurfaces, the conormal space is isomorphic to the Nash blow-up. Now we have

$$\mathscr{O}_{T^\vee M|_X}(1) \cong \mathscr{O}_{(T^\vee M \otimes \mathscr{O}(X))|_X}(1) \otimes \pi^*\mathscr{O}(X)\,; \tag{6.46}$$

therefore (6.45) may be rewritten

$$\pi_*\left(c(\mathscr{O}_{(T^\vee M\otimes\mathscr{O}(X))|_X}(1) \otimes \pi^*\mathscr{O}(X))^{-1} \cap [B\ell_{JX} X]\right)\,,$$

and as $\mathscr{O}_{(T^\vee M\otimes\mathscr{O}(X))|_X}(1)$ restricts to $c(\mathscr{O}(-E))$ on the blow-up, where E denotes the exceptional divisor, this class equals

$$\pi_*\left(c(\pi^*\mathscr{O}(X) \otimes \mathscr{O}(-E))^{-1} \cap [B\ell_{JX} X]\right) = \pi_*\left(c(\mathscr{O}(-E))^{-1} \cap [B\ell_{JX} X]\right) \otimes_M \mathscr{O}(X)\,,$$

where now π denotes the projection from the blow-up and we made use of (6.21) and of the projection formula. This last expression equals the right-hand side of the formula given in the statement, by (6.11). □

Remark 6.4.11 To parse the expression obtained in Theorem 6.4.10, note that as X is a hypersurface,

$$\begin{aligned} c_{\mathrm{F}}(X) &= c(TM|_X) \cap s(X, M) = c(TM|_X) \cap (1+X)^{-1} \cap [X] \\ &= c(TM|_X) \cap ([X] \otimes_M \mathscr{O}(X)) \ , \end{aligned}$$

while (6.44) and Theorem 6.4.10 imply that

$$c_{\mathrm{Ma}}(X) = c(TM|_X) \cap \left(\left([X] + \iota_* s(JX, X)^\vee\right) \otimes_M \mathscr{O}(X)\right) . \tag{6.47}$$

What this is saying is that the Chern-Mather class of a hypersurface X is the Chern-Fulton class of a virtual object whose fundamental class is

$$[X] + \iota_* s(JX, X)^\vee \ , \tag{6.48}$$

a perturbation of the fundamental class of X, determined by the Segre class of the singularity subscheme of X *in* X.

Enforcing the analogy with the Chern-Fulton class, we could formally write

$$c_{\mathrm{Ma}}(X) = c(TM|_X) \cap s_{\mathrm{Ma}}(X, M) \ ,$$

for a 'Segre-Mather class' $s_{\mathrm{Ma}}(X, M)$. Thus $s_{\mathrm{Ma}}(X, M) = s(X, M)$ if both X and M are nonsingular, and Theorem 6.4.10 gives an explicit expression for the Segre-Mather class if X is a hypersurface in a nonsingular variety M.

We do not know a similarly explicit expression of the Segre-Mather class for more general varieties X. ⌟

As there are implementations for the computation of Segre classes (see Sect. 6.2.2), Chern-Mather classes of hypersurfaces in e.g., nonsingular projective varieties can also be computed by making use of (6.47). See [48] for concrete examples.

Remark 6.4.12 We note that the relation between the Segre class of the singularity subscheme of a hypersurface X *of projective space* and the Chern-Mather class of X may also be obtained as a corollary of results of Piene: the *polar classes* of a hypersurface $X \subseteq \mathbb{P}^n$ can be computed in terms of the Segre class $s(JX, X)$ ([81, Theorem 2.3]) and the Chern-Mather class may be expressed in terms of polar classes [82, Théorème 3].

In fact, for projective varieties, the fact that (6.47) only holds for hypersurfaces is tempered by another result of Piene, [82, Corollaire, p. 20], showing that Chern-Mather classes are preserved by general projections. Thus, the computation of the degrees of the components of the Chern-Mather class of a projective variety may be reduced to the hypersurface case. ⌟

In any case, it would be interesting to extend Theorem 6.4.10 beyond the hypersurface case. It is conceivable that even if X is not a hypersurface, the Segre term in (6.44) may admit an equally transparent expression in terms of the Segre class of a scheme naturally associated with the singularities of X.

Example 6.4.13 It follows easily from the definition that if $X = X_1 \cup X_2$ is the union of two closed reduced subschemes of the same pure dimension and with no irreducible components in common, then $c_{\text{Ma}}(X) = c_{\text{Ma}}(X_1) + c_{\text{Ma}}(X_2)$ (where the classes on the right-hand side are viewed as classes in $A_*(X)$). Indeed, the Nash blow-up of X is simply the union of the Nash blow-ups of X_1 and X_2. (This also implies that $\text{Eu}_X = \text{Eu}_{X_1} + \text{Eu}_{X_2}$; cf. [66, p. 426].)

For a hyperplane arrangement $\mathscr{A}$ consisting of d distinct hyperplanes H_i in $\mathbb{P}^n$, this implies that the Chern-Mather class of the corresponding hypersurface A is

$$c_{\text{Ma}}(A) = \sum_i c_{\text{Ma}}(H_i) = \sum_i c(TH_i) \cap [H_i]$$

and therefore if $i : A \to \mathbb{P}^n$ is the embedding, and H denotes the hyperplane class,

$$i_* c_{\text{Ma}}(A) = d \cdot (1+H)^n \cap [\mathbb{P}^{n-1}] .$$

Let's verify that this is compatible with the formula (6.24) for $s(JA, A)$ obtained in Sect. 6.2.5:

$$\begin{aligned}
\iota_* s(JA, A) &= d \sum_{i=2}^{n} (-1)^i (d-1)^{i-1} [\mathbb{P}^{n-i}] \\
&= d \left((d-1)[\mathbb{P}^{n-2}] \otimes_A \mathscr{O}((d-1)H) \right) \\
&= d(d-1)(1+(d-1)H) \cap \left([\mathbb{P}^{n-2}] \otimes_{\mathbb{P}^n} \mathscr{O}((d-1)H) \right) ,
\end{aligned}$$

where we have used the notation in Sect. 6.2.4. It follows that the 'perturbed fundamental class' (6.48) is

$$\begin{aligned}
&[A] + \iota_* s(JA, A)^\vee \\
&\quad = d \left([\mathbb{P}^{n-1}] + (d-1)(1-(d-1)H) \cap \left([\mathbb{P}^{n-2}] \otimes_{\mathbb{P}^n} \mathscr{O}(-(d-1)H) \right) \right)
\end{aligned}$$

and therefore the push-forward of the Segre-Mather class to $\mathbb{P}^n$ equals (using (6.20) and (6.21))

$$\begin{aligned}
&i_* \left(([A] + \iota_* s(JA, A)^\vee) \otimes_{\mathbb{P}^n} \mathscr{O}(A) \right) \\
&= d \left([\mathbb{P}^{n-1}] + (d-1)(1-(d-1)H) \cap \left([\mathbb{P}^{n-2}] \otimes_{\mathbb{P}^n} \mathscr{O}(-(d-1)H) \right) \right) \otimes_{\mathbb{P}^n} \mathscr{O}(dH) \\
&= d \left([\mathbb{P}^{n-1}] \otimes \mathscr{O}(dH) + (d-1) \frac{1+H}{1+dH} \cap ([\mathbb{P}^{n-2}] \otimes_{\mathbb{P}^n} \mathscr{O}(H)) \right)
\end{aligned}$$

$$= d\left(\frac{1}{1+dH} + (d-1)\frac{1+H}{1+dH}\frac{H}{(1+H)^2}\right) \cap [\mathbb{P}^{n-1}]$$
$$= d \cdot (1+H)^{-1} \cap [\mathbb{P}^{n-1}]\,.$$

In conclusion,

$$\begin{aligned} i_*\left(c(T\mathbb{P}^n|_A) \cap s_{\mathrm{Ma}}(A, \mathbb{P}^n)\right) &= d \cdot (1+H)^{n+1}(1+H)^{-1} \cap [\mathbb{P}^{n-1}] \\ &= d \cdot (1+H)^n \cap [\mathbb{P}^{n-1}] \end{aligned}$$

as it should.

More generally, let $X = \cup_{i=1}^r X_i$ be the union of r distinct irreducible (possibly singular) hypersurfaces in a nonsingular variety M. Denote by X_{-i} the union of the hypersurfaces other than X_i. Then, omitting evident push-forwards:

$$s(JX, X) = \sum_i X_i \cdot s(X_{-i}, M) + s(JX_i, X_i) \otimes_M \mathscr{O}(X_{-i})\,. \tag{6.49}$$

This may be proved by the same technique used in the proof of Proposition 6.2.17, using (6.22) (that is, 'residual intersection') to account for the singularity subschemes of the individual components X_i. The reader should have no difficulty verifying that (6.49) is compatible with the fact that $c_{\mathrm{Ma}}(X) = \sum_i c_{\mathrm{Ma}}(X_i)$. ⌟

6.4.4 Chern-Schwartz-MacPherson Classes of Hypersurfaces

Again all our schemes will be subschemes of a fixed nonsingular variety M, and we work in characteristic 0. We do not need to assume that schemes are reduced or pure-dimensional.

Choosing the function $\mathbb{1}_X$ for every scheme is trivially the simplest way to define a constructible function generalizing $\mathbb{1}_V$ for nonsingular varieties V. Thus, this defines a characteristic class trivially generalizing $c(TV) \cap [V]$.

Definition 6.4.14 Let X be a scheme as above. The *Chern-Schwartz-MacPherson (CSM) class* of X is the class

$$c_{\mathrm{SM}}(X) := c_*(\mathbb{1}_X) \in A_*(X)\,.$$

More generally (abusing language) we let

$$c_{\mathrm{SM}}(W) := c_*(\mathbb{1}_W) \in A_*(X)$$

for any constructible subset W of X; the context will determine the Chow group where $c_{\mathrm{SM}}(W)$ is meant to be taken. Note that as $\mathbb{1}_W$ only depends on the support

$W_{\rm red}$ of W, we have $c_{\rm SM}(W) = c_{\rm SM}(W_{\rm red})$. (Cf. Remark 6.4.21 below for relevant comments on this point.)

Definition 6.4.14 is given in [66] (for compact complex varieties, and in homology); MacPherson attributes it to Deligne. In [24], Brasselet and Marie-Hélène Schwartz proved that the class agrees via Alexander duality with the classes defined earlier by Schwartz in relative cohomology [85, 86].

One way to compute $c_{\rm SM}(X)$ is to express the constant function $1\!\!1_X$ as a linear combination of local Euler obstructions:

$$1\!\!1_X = \sum_i m_i \,{\rm Eu}_{W_i}$$

for a choice of finitely many subvarieties W_i of X. It then follows that

$$c_{\rm SM}(X) = c_*(1\!\!1_X) = \sum_i m_i c_*({\rm Eu}_{W_i}) = \sum_i m_i c_{\rm Ma}(W_i)\,.$$

The proof in [24] relies on establishing precise relations between indices of radial vector fields and local Euler obstructions, and hence between Schwartz's classes and Chern-Mather classes. It is also possible to prove that the classes defined by Schwartz satisfy enough of the functoriality properties of the classes defined by MacPherson to guarantee that they must agree [15]; this approach avoids the use of local Euler obstructions or Chern-Mather classes.

One motivation in Schwartz's work was to obtain a class generalizing the classical Poincaré-Hopf theorem to singular varieties. This incorporated in MacPherson's approach as an implication of the naturality of c_*. Assume that X is complete, so that the constant map $\kappa : X \to pt = {\rm Spec}\, k$ is proper. The fact that c_* is a natural transformation implies that the following diagram is commutative:

$$\begin{array}{ccccc} {\rm F}(X) & & \xrightarrow{c_*} & & A_*(X) \\ \downarrow{\scriptstyle \kappa_*} & & & & \downarrow{\scriptstyle \kappa_*} \\ {\rm F}(pt) & = & \mathbb{Z} & = & A_*(pt) \end{array}$$

If $W \subseteq X$ is any constructible subset, the commutativity of the diagram

$$\begin{array}{ccc} 1\!\!1_W & \longmapsto & c_{\rm SM}(W) \\ {\scriptstyle \kappa_*}\downarrow & & \downarrow{\scriptstyle \kappa_*} \\ \chi(W) & \longmapsto & \int c_{\rm SM}(W) \end{array}$$

amounts to the equality

$$\int c_{\rm SM}(W) = \chi(W) : \tag{6.50}$$

the degree of the CSM class equals the topological Euler characteristic (or a suitable generalization over fields other than $\mathbb{C}$). This can be viewed as an extension to possibly singular, possibly noncompact varieties of the Poincaré-Hopf theorem, holding over arbitrary algebraically closed fields of characteristic 0.

By Theorem 6.4.8, we have

$$c_{\rm SM}(X) = c(TM|_X) \cap \text{Segre}_{T^\vee M|_X}(\text{Ch}(1\!\!1_X))_\vee \,.$$

Just as in Sect. 6.4.3, it is natural to ask for a more explicit and computable expression for the Segre term

$$s_{\rm SM}(X, M) := \text{Segre}_{T^\vee M|_X}(\text{Ch}(1\!\!1_X))_\vee \,, \tag{6.51}$$

which we view as a 'Segre-Schwartz-MacPherson' class. In Sect. 6.4.5 we will argue that this task can be reduced to the case of hypersurfaces; in this section we focus on the hypersurface case. The following result is the CSM version of Theorem 6.4.10.

Theorem 6.4.15 *Let X be a hypersurface in a nonsingular variety M. Then*

$$\text{Segre}_{T^\vee M|_X}(\text{Ch}(1\!\!1_X))_\vee = \big([X] + \iota_*(c(\mathscr{O}(X) \cap s(JX, M)))^\vee\big) \otimes_M \mathscr{O}(X)\,.$$

This is [5, Lemma I.3]; cf. [14, Proposition 2.2]. It can be interpreted as stating that if X is a hypersurface of a nonsingular variety M, then the Chern-Schwartz-MacPherson class of X is the Chern-Fulton class of an object whose 'fundamental class' is

$$[X] + \iota_*(c(\mathscr{O}(X)) \cap s(JX, M))^\vee \,. \tag{6.52}$$

Remark 6.4.16 The reader should compare (6.48) and (6.52), that is, the perturbations of the fundamental class corresponding to the different characteristic classes we have encountered, in the case of hypersurfaces:

Chern-Fulton:	$[X]$
Chern-Mather:	$[X] + \iota_* s(JX, X)^\vee$
Chern-Schwartz-MacPherson:	$[X] + \iota_*(c(\mathscr{O}(X)) \cap s(JX, M))^\vee$.

The difference between the Chern-Mather class and the Chern-Schwartz-MacPherson class is captured precisely by the difference between

$$s(JX, X) \quad \text{and} \quad c(\mathscr{O}(X)) \cap s(JX, M)\,.$$

As we have observed in Example 6.4.3, it is natural to compare the classes $s(W, X)$ and $c(\mathscr{O}(X)) \cap s(W, M)$, for any subscheme W of a hypersurface X. The case $W = JX$ provides one instance in which the difference has a transparent and interesting interpretation. ⌟

Different proofs are known for Theorem 6.4.15. One approach consists of proving that the class

$$c(TM|_X) \cap \left(\left([X] + \iota_*(c(\mathscr{O}(X) \cap s(JX, M)))^\vee\right) \otimes_M \mathscr{O}(X)\right) \tag{6.53}$$

has the same behavior under blow-ups along nonsingular subvarieties of JX as the class $c_{\text{SM}}(X)$. By resolution of singularities, we may then reduce to the case in which X is a divisor with normal crossings and nonsingular components, and in this case one can verify that (6.53) does equal $c_{\text{SM}}(X)$. It follows that (6.53) must equal $c_{\text{SM}}(X)$ in general. This approach is carried out in [5].

A perhaps more insightful argument consists of a concrete realization of the characteristic cycle $\text{Ch}(1\!\!1_X)$. For this, view the singularity subscheme JX of X as a subscheme *of* M. Consider the blow-up

$$\pi : B\ell_{JX} M \to M$$

of M along JX. This is naturally embedded as a subscheme of $\mathbf{P}(\mathscr{P}^1_M(\mathscr{O}(X)))$, the projectivization of the *bundle of principal parts* of $\mathscr{O}(X)$. The inverse image $\mathscr{X} := \pi^{-1}(X)$ is contained in $\mathbf{P}((T^\vee M \otimes \mathscr{O}(X))|_X) \subseteq \mathbf{P}((\mathscr{P}^1_M(\mathscr{O}(X)))|_X)$, and contains the exceptional divisor $\mathscr{E} = \pi^{-1}(JX)$ of the blow-up. Thus, we have $(\dim M - 1)$-dimensional cycles $[\mathscr{X}]$, $[\mathscr{E}]$ of $\mathbf{P}((T^\vee M \otimes \mathscr{O}(X))|_X) \cong \mathbf{P}(T^\vee M|_X)$.

The reader may find it helpful to recall that $B\ell_{JX}X$ may also be realized as a subscheme of $\mathbf{P}(T^\vee M|_X)$; the proof of Theorem 6.4.10 relies on the identification of this subscheme with the projectivized conormal space $\mathbf{P}(N^\vee_X M)$, whose cycle is $(-1)^{\dim X}\,\text{Ch}(\text{Eu}_X)$.

Lemma 6.4.17 *The characteristic cycle* $\text{Ch}(1\!\!1_X)$ *equals* $(-1)^{\dim X}([\mathscr{X}] - [\mathscr{E}])$.

This statement implies Theorem 6.4.15, by an argument similar to the proof of Theorem 6.4.10. (Cf. e.g., [5, Theorem I.3].) Lemma 6.4.17 is proved in [80, Corollary 2.4], along with a thorough discussion of characteristic cycles of other constructible functions naturally associated with a hypersurface. An earlier description of the characteristic *variety* of a hypersurface is given in [63, Theorem 3.3].

Theorem 6.4.15 is equivalent to the following formula, which we state as a separate result for ease of reference.

Theorem 6.4.18 *Let X be a hypersurface in a nonsingular variety M. Then*

$$c_{SM}(X) = c(TM|_X) \cap \left(\left([X] + \iota_*(c(\mathscr{O}(X)) \cap s(JX, M))^\vee\right) \otimes_M \mathscr{O}(X)\right) .$$

Remark 6.4.19 Xiping Zhang has generalized this result to the equivariant setting, [99]. ⌟

We have already observed that the formula in Theorem 6.4.18 may be viewed as expressing $c_{\mathrm{SM}}(X)$ as the Chern-Fulton class of a virtual object with a similar behavior to a hypersurface, but with a fundamental class modified to include lower dimensional terms. There is a perhaps more compelling intepretation of this object as a Chern-Fulton class, obtained by applying residual intersection as follows.

Recall that the Chern-Fulton class of a scheme is not just determined by its support; the specific scheme structure affects the class. For a hypersurface X of a nonsingular variety M, we consider the Chern-Fulton class of the scheme obtained by 'thickening' X along its singularity subscheme JX: that is, for $k \geq 0$ we consider the scheme $X^{(k)}$ whose ideal sheaf in M is

$$\mathscr{I}_{X,M} \cdot (\mathscr{I}_{JX,M})^k .$$

Thus $X = X^{(0)}$. The residual formula in Proposition 6.2.13 yields an expression for the Segre class of this scheme in M. According to (6.22),

$$s(X^{(k)}, M) = \left([X] + c(\mathscr{O}(-X)) \cap s((JX)_k, M)\right) \otimes_M \mathscr{O}(X) ,$$

where $(JX)_k$ is the subscheme of M defined by the ideal $(\mathscr{I}_{JX,M})^k$. (Thus $(JX)_0 = \emptyset$, $(JX)_1 = JX$, etc.) Accordingly, we have an expression for the Chern-Fulton class of $X^{(k)}$:

$$c_{\mathrm{F}}(X^{(k)}) = c(TM) \cap \left([X] + c(\mathscr{O}(-X)) \cap s((JX)_k, M)\right) \otimes_M \mathscr{O}(X) . \quad (6.54)$$

This expression makes sense for all nonnegative integers k, and by definition

$$c_{\mathrm{vir}}(X) = c_{\mathrm{F}}(X^{(0)}) .$$

Now we observe that $s((JX)_k, M)$ is determined by $s(JX, M)$ for all $k \geq 0$: indeed, the component of dimension ℓ of this class is given by

$$s((JX)_k, M)_\ell = k^{\dim M - \ell} s(JX, M)_\ell .$$

Indeed, if $\mathscr{E}$ denotes the exceptional divisor of the blow-up $B\ell_{JX}M$, then the inverse image of $(JX)_k$ in the blow-up is $k\mathscr{E}$, so the assertion follows from (6.11).

As a consequence, (6.54) expresses $c_{\mathrm{F}}(X^{(k)})$ as a *polynomial* in k, and as such this class can be given a meaning for every integer k.

Proposition 6.4.20 *Let X be a hypersurface in a nonsingular variety M. With notation as above,*

$$c_{SM}(X) = c_F(X^{(-1)}) .$$

This is of course just a reformulation of Theorem (6.4.18). It identifies the Chern-Schwartz-MacPherson class of X with the Chern-Fulton class of a virtual (fractional?) scheme obtained from X by simply 'removing' its singular locus. The Segre-Schwartz-MacPherson class of a hypersurface X in a nonsingular variety M is simply

$$s_{\rm SM}(X, M) = s(X^{(-1)}, M) .$$

Remark 6.4.21 There is one case in which the virtual scheme $X^{(-1)}$ is *not* virtual. Let V be a *nonsingular* hypersurface of a nonsingular variety M, and let X be the non-reduced hypersurface whose ideal is the r-th power of the ideal of V:

$$\mathscr{I}_{X,M} = \mathscr{I}_{V,M}{}^r .$$

Then (as the characteristic is 0), JX has ideal $\mathscr{I}_{V,M}{}^{r-1}$, hence $X^{(k)}$ has ideal $\mathscr{I}_{V,M}{}^{r+k(r-1)}$ for $k \geq 0$. This ideal makes sense for $k = -1$, giving $X^{(-1)} = V$. Therefore

$$c_{\rm SM}(X) = c_{\rm SM}(V) = c(TV) \cap [V] = c_{\rm F}(V) = c_{\rm F}(X^{(-1)})$$

as it should.

Using Proposition 6.2.13, it is not hard to verify that if X is a possibly non-reduced effective Cartier divisor in a nonsingular variety M, then

$$c_{\rm F}(X^{(-1)}) = c_{\rm F}({X_{\rm red}}^{(-1)}) ,$$

even if the support $X_{\rm red}$ is singular. This is compatible with our definition of the Chern-Schwartz-MacPherson class of a possibly non-reduced scheme X, which guarantees that it only depends on the support of X. ⌟

Example 6.4.22 The *polar degree* of a hypersurface X of $\mathbb{P}^n$ defined by a homogeneous polynomial F is the degree of the gradient map $\mathbb{P}^n \to \mathbb{P}^n$,

$$p \mapsto \left(\frac{\partial F}{\partial x_0} : \cdots : \frac{\partial F}{\partial x_n} \right) . \tag{6.55}$$

A hypersurface is 'homaloidal' if this map is birational, that is, if its polar degree is 1. Igor Dolgachev [32, p. 199] conjectured that a hypersurface X is homaloidal if and only if $X_{\rm red}$ is homaloidal.

Now, the graph of the map (6.55) is isomorphic to the blow-up of the zero-scheme of the partials, that is, to $B\ell_{JX}\mathbb{P}^n$. Therefore, it is straightforward to express the polar degree in terms of the degrees of the components of the Segre class of JX in $\mathbb{P}^n$, and therefore in terms of the degrees of the components of the Chern-Schwartz-MacPherson class of X. The result of this computation is the following (see [10, §3.1] for more details).

Proposition 6.4.23 *Let $X \subseteq \mathbb{P}^n$ be a hypersurface. Denote by $\deg c_i(X)$ the degree of the dimension-i component of $c_{SM}(X)$. Then the polar degree of X equals*

$$(-1)^n - \sum_{i=0}^{n} (-1)^{n-i} \deg c_i(X) .$$

Since $c_{\mathrm{SM}}(X) = c_{\mathrm{SM}}(X_{\mathrm{red}})$, it follows that the polar degree of X equals the polar degree of X_{red}, verifying Dolgachev's conjecture. (To our knowledge, the first proof of the conjecture appeared in [31, Corollary 2], over $\mathbb{C}$. The argument sketched above holds over any algebraically closed field of characteristic 0.) ⌟

Example 6.4.24 We return once more to a hyperplane arrangement $\mathscr{A}$ in $\mathbb{P}^n$ and its corresponding hypersurface A. We will sketch a proof of Theorem 6.2.15, which relies on the computation of $c_{\mathrm{SM}}(A)$. We will assume that A is reduced, but as we just observed, $c_{\mathrm{SM}}(A_{\mathrm{red}}) = c_{\mathrm{F}}(A_{\mathrm{red}}^{(-1)}) = c_{\mathrm{F}}(A^{(-1)})$, and it follows that the result holds without changes for non-reduced arrangements (as stated in Sect. 6.2.5).

The arrangement $\mathscr{A}$ corresponds to a central arrangement $\widehat{\mathscr{A}}$ in $\mathbb{A}^{n+1}$. We let $\chi_{\widehat{\mathscr{A}}}(t)$ be the characteristic polynomial of $\widehat{\mathscr{A}}$; see e.g., [77, Definition 2.5.2]. (For arrangements corresponding to graphs, this is essentially the same as the *chromatic polynomial* of the graph.) We define $\chi_{\mathscr{A}}(t)$ to be the quotient $\chi_{\widehat{\mathscr{A}}}(t)/(t-1)$; this is also a polynomial in $\mathbb{Z}[t]$, of degree n.

Now consider the Chern-Schwartz-MacPherson class of the *complement* of A:

$$c_{\mathrm{SM}}(\mathbb{P}^n \smallsetminus A) = c_*(1\!\!1_{\mathbb{P}^n} - 1\!\!1_A) \in A_*(\mathbb{P}^n) .$$

As an element of $A_*(\mathbb{P}^n)$, this class may be written as an integer linear combination of the classes $[\mathbb{P}^i]$ for $i = 0, \dots, n$.

Theorem 6.4.25 ([11, Theorem 1.2]) *The class $c_{SM}(\mathbb{P}^n \smallsetminus A)$ equals the class obtained by replacing t^i with $[\mathbb{P}^i]$ in $\chi_{\mathscr{A}}(t+1)$.*

This may be proved by a combinatorial argument, using 'Möbius inversion'. Alternately, one may use the deletion-contraction property of the characteristic polynomial and the fact that Chern-Schwartz-MacPherson classes satisfy an inclusion-exclusion property: if W_1 and W_2 are locally closed subsets of a variety V, then

$$\begin{aligned} c_{\mathrm{SM}}(W_1 \cap W_2) &= c_*(1\!\!1_{W_1 \cap W_2}) = c_*(1\!\!1_{W_1} + 1\!\!1_{W_2} - 1\!\!1_{W_1 \cup W_2}) \\ &= c_{\mathrm{SM}}(W_1) + c_{\mathrm{SM}}(W_2) - c_{\mathrm{SM}}(W_1 \cup W_2) \end{aligned} \tag{6.56}$$

in $A_*(V)$. June Huh extracts an expression of the characteristic polynomial from these considerations, see [55, Remark 26].

The information carried by the characteristic polynomial of an arrangement is equivalent to the information in its *Poincaré polynomial*

$$\pi_{\mathscr{A}}(t) := (-t)^n \cdot \chi_A(-t^{-1}) .$$

As the reader can verify, Theorem 6.4.25 is equivalent to the following formula:

$$i_* c_{\mathrm{SM}}(A) = c(T\mathbb{P}^n) \cap \left(1 - \frac{1}{1+H}\pi_{\mathscr{A}}\left(\frac{-H}{1+H}\right)\right) \cap [\mathbb{P}^n]$$

where H is the hyperplane section and $i : A \to \mathbb{P}^n$ is the inclusion. Therefore

$$i_* s_{\mathrm{SM}}(A, \mathbb{P}^n) = \left(1 - \frac{1}{1+H}\pi_{\mathscr{A}}\left(\frac{-H}{1+H}\right)\right) \cap [\mathbb{P}^n]$$

Using Theorem 6.4.15 and simple manipulations, it follows that

$$\pi_{\mathscr{A}}\left(\frac{-H}{1+H}\right) \cap [\mathbb{P}^n] = \frac{1+H}{1+dH}\left(1 - \iota_* s(JA, \mathbb{P}^n)^\vee \otimes_{\mathbb{P}^n} \mathscr{O}(dH)\right) \cap [\mathbb{P}^n]$$

where d is the number of hyperplanes in the arrangement. Letting

$$\iota_* s(JA, \mathbb{P}^n) = \sum_i s_i [\mathbb{P}^i] = \sum_i s_i H^{n-i} \cap [\mathbb{P}^n]$$

we get an equality of power series in h modulo h^{n+1}:

$$\pi_{\mathscr{A}}\left(\frac{-h}{1+h}\right) \equiv \frac{1+h}{1+dh}\left(1 - \sum_{i=0}^n \frac{s_i \cdot (-h)^{n-i}}{(1+dh)^{n-i}}\right) \qquad \mathrm{mod}\ h^{n+1}$$

or equivalently

$$\pi_{\mathscr{A}}(t) \equiv \frac{1}{1-(d-1)t}\left(1 - \sum_{i=0}^n s_i \cdot \left(\frac{t}{1-(d-1)t}\right)^{n-i}\right) \qquad \mathrm{mod}\ t^{n+1} . \tag{6.57}$$

By a classical result of Peter Orlik and Louis Solomon [77, Theorem 5.93],

$$\pi_{\mathscr{A}}(t) = \sum_{i=0}^n \mathrm{rk}\, H^k(\mathbb{P}^n \smallsetminus A, \mathbb{Q}) t^i .$$

Reading off the coefficients of t^i, $i = 0, \ldots, n$ in (6.57) yields Theorem 6.2.15. ⌟

6.4.5 Chern-Schwartz-MacPherson Classes, General Case

Formulas in the style of Theorem 6.4.18 are useful: they have been applied to concrete computations of Chern-Schwartz-MacPherson classes, and they are amenable to implementation in systems such as Macaulay2 since Segre classes are (Sect. 6.2.2). One may expect that there should be a straightforward generalization of Theorem 6.4.15 to higher codimension subschemes X of a nonsingular variety, based on the Segre class of a subscheme defined by a suitable Fitting ideal, generalizing the singularity subscheme JX. One could also expect a generalization of the interpretation of the Chern-Schwartz-MacPherson class as the Chern-Fulton class of a suitable virtual scheme, along the lines of Proposition 6.4.20. With the exception of results for certain types of complete intersections [36, 38], we do not know of explicit results along these lines.

However, a formula for the Chern-Schwartz-MacPherson class of an arbitrary subscheme of a nonsingular variety in terms of the Segre class of a related scheme *can* be given. This is the most direct extension of Theorem 6.4.15 currently available, and it will be presented below (Theorem 6.4.30). Before discussing this result, we note that, for computational purposes, the case of arbitrary subschemes can already be treated by organizing a potentially large number of applications of the hypersurface case.

Proposition 6.4.26 *Let X be a subscheme of a nonsingular variety M, and assume that X is the intersection of r hypersurfaces $X_1, \dots, X_r$. Then*

$$c_{SM}(X) = \sum_{s=1}^{r}(-1)^{s-1} \sum_{i_1<\cdots<i_s} c_{SM}(X_{i_1} \cup \cdots \cup X_{i_s})\,.$$

This is clear from inclusion-exclusion, which holds for CSM classes since it holds for constructible functions, cf. (6.56). Since the classes appearing in the right-hand side are all CSM classes of hypersurfaces, they can be computed by applying Theorem 6.4.18. This approach yields an algorithm for computing Chern-Schwartz-MacPherson classes of subschemes of $\mathbb{P}^n$ and more general varieties, based on the computation of Segre classes (cf. [6, 48, 52, 53, 58]). The current Macaulay2 distribution includes the package `CharacteristicClasses` [54], by Helmer and Christine Jost, which implements this observation.

Example 6.4.27 Let X be the scheme defined by the ideal $(xz^2 - y^2w, xw^2 - yz^2, x^2w - y^3, z^4 - yw^3)$ in $\mathbb{P}^3$. The following Macaulay2 commands compute the push-forward to $\mathbb{P}^3$ of its Chern-Schwartz-MacPherson class.

```
i1 : load("CharacteristicClasses.m2")

i2 : R=QQ[x,y,z,w]

i3 : I=ideal(x*z^2-y^2*w, x*w^2-y*z^2, x^2*w-y^3, z^4 -y*w^3)

i4 : CSM I

          3      2
o4 = 2h   + 6h
       1      1
```

(The package uses h_1 to denote the hyperplane class.) This shows that the locus is a sextic curve with topological Euler characteristic equal to 2. (It is in fact an irreducible rational sextic with one singular point.) As the ideal has four generators, the computation requires 15 separate applications of Theorem 6.4.18, including one for a degree-13 hypersurface. ⌟

One intriguing aspect of this approach via inclusion-exclusion is that the same subscheme may be represented as an intersection of hypersurfaces in many different ways; and extra features such as embedded or multiple components do not affect the result, since the Chern-Schwartz-MacPherson class only depends on the support of the scheme. Massive cancellations involving the Segre classes underlying such computations must be at work. To our knowledge, more direct proofs of such cancellations are not available.

One obvious drawback of Proposition 6.4.26 is the large number of computations needed to apply it: $2^r - 1$ distinct Segre class computations for the intersection of r hypersurfaces. As we will see next, the same input—for example, a set of generators for the homogeneous ideal of a projective scheme $X \subseteq \mathbb{P}^n$—may be used to obtain an expression that is a more direct generalization of Theorem 6.4.18, in the sense that it gives an expression for $c_{\mathrm{SM}}(X)$ in terms of a single Segre class of a related scheme. The price to pay is an increase in dimension, and the fact that (at this time) the result only yields the push-forward of $c_{\mathrm{SM}}(X)$ to the Chow group $A_*(M)$ of the ambient nonsingular variety.

Let X be a subscheme of a nonsingular variety M. We may assume that X is the zero scheme of a section of a vector bundle E on M; in fact, we may choose $E = \mathrm{Spec}(\mathrm{Sym}\mathscr{E})$, where $\mathscr{E}$ is any locally free sheaf surjecting onto the ideal sheaf $\mathscr{I}_{X,M}$ of X in M. Note that we can assume that the rank of E is as high as we please: for example, we can replace $\mathscr{E}$ with $\mathscr{E} \oplus \mathscr{O}_M^{\oplus a}$ for any $a \geq 0$. The surjection $\mathscr{E} \twoheadrightarrow \mathscr{I}_{X,M}$ induces a morphism $\phi : \mathscr{E}|_X \to \Omega_M|_X$ whose cokernel is the sheaf of differentials Ω_X. We view this as a morphism of vector bundles over X, $\phi : E^\vee|_X \to T^\vee M|_X$. The kernel of ϕ determines a subscheme $J_E(X)$ of the projectivization $\mathbf{P}(E^\vee|_X) \xrightarrow{\pi} X$.

Definition 6.4.28 With notation as above, we will denote by $J_E(X)$ the subscheme of $\mathbf{P}(E^\vee|_X)$ defined by the vanishing of the composition of the pull-back of ϕ with the tautological inclusion $\mathscr{O}_{E^\vee}(-1) \to \pi^* E^\vee|_X$. ⌟

It may be helpful to describe $J_E(X)$ in analytic coordinates $(x_1, \dots, x_n)$ for M, over an open set U where Ω_M and E are trivial. If X is defined by $f_0(\underline{x}) = \cdots = f_r(\underline{x}) = 0$ (so $\operatorname{rk} E = r + 1$), $\phi : \mathscr{E}|_X \to \Omega_U|_X$ has matrix

$$\begin{pmatrix} \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}$$

and $J_E(X)$ is defined by the ideal

$$(y_0 df_0 + \cdots + y_r df_r) = \left(\sum_{i=0}^{r} y_i \frac{\partial f_i}{\partial x_j} \right)_{j=1,\dots,n}$$

in $\mathbf{P}(E^\vee|_{X\cap U}) = \mathbb{P}^r \times (X \cap U)$. In other words, $J_E(X)$ records linear relations between the differentials of the generators of the ideal of X. These may be due to relations between the generators themselves (note that nothing prevents us from choosing e.g., $f_0 = f_1$), or to singularities of X.

Example 6.4.29 If X is a hypersurface, defined by the vanishing of a section s of $E = \mathscr{O}(X)$, then $J_E(X)$ is the subscheme of $X \cong \mathbb{P}^0 \times X$ defined by the vanishing of ds. That is, $J_E(X) = JX$ in this case: in this sense, the definition of $J_E(X)$ generalizes the notion of 'singularity subscheme' of a hypersurface. ⌟

We view $J_E(X)$ as a subscheme of the nonsingular variety $\mathbf{P}(E^\vee)$, and denote by $\iota : J_E(X) \hookrightarrow \mathbf{P}(E^\vee)$ the inclusion and $\pi : \mathbf{P}(E^\vee) \twoheadrightarrow M$ the projection. The claim is now that the Segre class of $J_E(X)$ in $\mathbf{P}(E^\vee)$ determines the Segre term for the Chern-Schwartz-MacPherson class of X, at least after push-forward to M.

Theorem 6.4.30 ([13]) *Let $i : X \hookrightarrow M$ be a closed embedding of a scheme X in a nonsingular variety M, defined by a section of a vector bundle E of rank $>$* dim M. *Then with notation as above, $i_* c_{SM}(X)$ equals*

$$c(TM) \cap \pi_* \left(\frac{c(\pi^* E^\vee \otimes \mathscr{O}_{E^\vee}(1))}{c(\mathscr{O}_{E^\vee}(1))} \cap \left(s(J_E(X), \mathbf{P}(E^\vee))^\vee \otimes_{\mathbf{P}(E^\vee)} \mathscr{O}_{E^\vee}(1) \right) \right) . \tag{6.58}$$

Despite its rather complicated shape, (6.58) is straightforward to implement in a system capable of computing Segre classes; for example, Macaulay2 enhanced with the package `SegreClasses` [49] for computations in products of projective space. Concrete examples may be found in [13, §1].

Theorem 6.4.30 is proved by realizing $J_E(X)$ as the singularity subscheme of a hypersurface in $\mathbf{P}(E^\vee)$, applying Theorem 6.4.4, and computing the push-forward by using standard intersection-theoretic calculus and the naturality of c_*. The result is that if X is given by a section of a vector bundle E, then (6.58) computes

$$i_* c_{\mathrm{SM}}(X) - \frac{c(TM)}{c(E)} c_{\mathrm{top}}(E) \cap [M] \tag{6.59}$$

([13, Theorem 2.5]). If the rank of E exceeds the dimension of M (as required in Theorem 6.4.30), then the second term vanishes, and the theorem follows. We will come back to the more general case in Sect. 6.4.6. To our knowledge, the auxiliary hypersurface used in this argument was first introduced by Callejas-Bedregal, Morgado, and Seade in [26], in the case of local complete intersections. The construction was also considered independently by Ohmoto [75] and Xia Liao [65].

The class (6.58) may be interpreted unambiguously as a class in $A_*(X)$, and it is likely that it simply equals $c_{\mathrm{SM}}(X)$, but the argument we just sketched only shows the equality in $A_*(M)$.

The reader will certainly notice similarities between the statement of Theorem 6.4.30 and the case of hypersurfaces treated in Sect. 6.4.4. The new statement does recover Theorem 6.4.18 (after push-forward to M) in the hypersurface case, as we see in the example that follows.

Example 6.4.31 Let X be the hypersurface defined by a section s of a line bundle $\mathscr{L} \cong \mathscr{O}(X)$ on a nonsingular variety M. We may view X as the zero scheme of the section $(s, s, \dots, s)$ of $E = \mathscr{O}(X)^{\oplus r+1}$, for any $r \geq 0$. Then

$$\mathbf{P}(E^\vee) = \mathbf{P}(\mathscr{O}(-X)^{\oplus r+1}) \cong \mathbb{P}^r \times M ;$$

via this identification, $\mathscr{O}_{E^\vee}(1) \cong \mathscr{O}_{\mathbb{P}^r \times M}(1) \otimes \pi^* \mathscr{O}(X)$. Therefore

$$\frac{c(\pi^* E^\vee \otimes \mathscr{O}_{E^\vee}(1))}{c(\mathscr{O}_{E^\vee}(1))} = \frac{(1+h)^{r+1}}{1+h+\pi^* X} ,$$

where h is the hyperplane class in $\mathbb{P}^r \times M$. The scheme $J_E(X)$ is locally defined by the ideal

$$((y_0 + \cdots + y_r) ds, s)$$

in $\mathbb{P}^r \times M$, where y_i are homogeneous coordinates in $\mathbb{P}^r$. Note that (ds, s) is the ideal of the singularity subscheme JX. A generalization of the residual formula for Segre classes (6.22) shows that

$$\begin{aligned} & s(J_E(X), \mathbf{P}(E^\vee))^\vee \otimes_{\mathbf{P}(E^\vee)} \mathscr{O}_{E^\vee}(1) \\ = & \frac{h}{(1+h)(1+\pi^* X)} \cap \pi^*[X] + \frac{1+h+\pi^* X}{(1+h)(1+\pi^* X)} \cap \pi^* \left(s(JX, M)^\vee \otimes_M \mathscr{O}(X) \right) . \end{aligned}$$

Therefore, the term to push forward in (6.58) evaluates to

$$\frac{(1+h)^r \cdot h}{(1+h+\pi^*X)(1+\pi^*X)} \cap \pi^*[X] + \frac{(1+h)^r}{1+\pi^*X} \cap \pi^* \left(s(JX, M)^\vee \otimes_M \mathscr{O}(X) \right) .$$

The push-forward is carried out by the projection formula and reading off the coefficient of h^r. The second summand pushes forward to

$$\frac{1}{1+X} \cap \left(s(JX, M)^\vee \otimes_M \mathscr{O}(X) \right) = (c(\mathscr{O}(X) \cap s(JX, M))^\vee \otimes_M \mathscr{O}(X) .$$

The first summand pushes forward to

$$\left(\text{coefficient of } h^r \text{ in } \frac{(1+h)^r \cdot h}{1+h+\pi^*X} \right) \cap \frac{[X]}{1+X}$$

and elementary manipulations evaluate the coefficient, giving

$$\left(1 - \frac{X^r}{(1+X)^r} \right) \cap \frac{[X]}{1+X} .$$

In conclusion, (6.58) equals

$$c(TM) \cap \left(\frac{[X]}{1+X} + (c(\mathscr{O}(X) \cap s(JX, M))^\vee \otimes_M \mathscr{O}(X) - \frac{X^r}{(1+X)^{r+1}} \cap [X] \right) .$$

Theorem 6.4.30 asserts that for $r+1 > \dim M$, this expression equals $i_* c_{\text{SM}}(X)$. And indeed, if $r+1 > \dim M$, the last term vanishes and we recover the expression in Theorem 6.4.18. ⌟

As with Proposition 6.4.26, one intriguing feature of Theorem 6.4.30 is the vast degree of freedom in the choice of the data needed to apply it—here, the vector bundle E and the section of E whose zero-scheme defines X. The fact that different choices of bundles or of defining sections lead to the same result reflects sophisticated identities involving the relevant Segre classes, for which we do not know a more direct proof.

6.4.6 Milnor Classes

We have seen that Parusiński's generalization of the Milnor number to complex hypersurfaces with arbitrary singularities satisfies (6.35):

$$\mu(X) = (-1)^{\dim X} (\chi(X_{\text{gen}}) - \chi(X)) ,$$

where X_{gen} is a nonsingular hypersurface linearly equivalent to X. Also, we have seen that $\chi(X_{\text{gen}}) = \int c_{\text{vir}}(X)$ (Proposition 6.4.4) and $\chi(X) = \int c_{\text{SM}}(X)$ (6.50). Therefore,

$$\mu(X) = (-1)^{\dim X} \int c_{\text{vir}}(X) - c_{\text{SM}}(X) .$$

This equality motivates the following definition, which makes sense over any algebraically closed field of characteristic 0.

Definition 6.4.32 Let X be a local complete intersection. The *Milnor class* of X is the class

$$\mathscr{M}(X) := (-1)^{\dim X} (c_{\text{vir}}(X) - c_{\text{SM}}(X))$$

where $c_{\text{vir}}(X)$ is the class of the virtual tangent bundle of X. ⌟

(Recall that being a local complete intersection in a nonsingular variety is an intrinsic notion, cf. [51, Remark II.8.22.2, p.185], and that the virtual tangent bundle of a local complete intersection is well-defined as a class in the Grothendieck group of vector bundles on X.)

Definition 6.4.32 would place the class in $A_*(X)$. The class is clearly supported on the singular locus X^{sing} of X, and in the case of a hypersurface X we will produce below a well-defined class in $A_*(JX)$ whose image in $A_*(X)$ is the class of Definition 6.4.32. Formulas explicitly localizing the class to the singular locus are also given in the local complete intersection case in [22] (over $\mathbb{C}$, and in homology).

One could extend the definition of the Milnor class to more general schemes X, as measuring the difference between $c_{\text{SM}}(X)$ and $c_{\text{F}}(X)$ or $c_{\text{FJ}}(X)$ (cf. (6.38)). However, recall that in general $c_{\text{F}}(X) \neq c_{\text{FJ}}(X)$ for schemes that are not local complete intersections, so this would require a choice that seems arbitrary. For this reason, we prefer to only consider the Milnor class for local complete intersections.

The geometry associated to Milnor classes of hypersurfaces and more generally local complete intersections has been studied very thoroughly. We mention [22, 26, 71, 80] among many others, as well as [97, 98], where (to our knowledge) the notion was first introduced and studied. The contribution [29] to this Handbook includes a thorough survey of Milnor classes. Here we focus specifically on the relation between Milnor classes and Segre classes, and on consequences of this relation.

First, we note that the Milnor class of a hypersurface X of a nonsingular variety M admits an expression in terms of a Segre operator (6.41):

$$\mathscr{M}(X) = c(T_{\text{vir}}X) \cap \text{Segre}_{T^\vee M}([\mathscr{E}])_\vee , \tag{6.60}$$

where $[\mathscr{E}]$ is the class of the exceptional divisor of the blow-up $\pi : B\ell_{JX}M \to M$; as pointed out in Sect. 6.4.4, $\mathscr{E}$ may be viewed as a cycle in $\mathbf{P}(T^\vee M)$, so $\text{Segre}_{T^\vee M}([\mathscr{E}])$ is defined. To verify (6.60), let $\mathscr{X} = \pi^{-1}(X)$; then $s(X, M) =$

$\pi_* s(\mathscr{X}, B\ell_{JX}M)$, by the birational invariance of Segre classes, and this implies the expression

$$c_{\mathrm{vir}}(X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathscr{X}]}{1+\mathscr{X}} \right)$$

for the virtual Chern class of X. Also, note that $\mathscr{O}_{T^\vee M}(1)|_{\mathscr{X}} \cong \mathscr{O}(\mathscr{X} - \mathscr{E})|_{\mathscr{X}}$ (this follows from (6.46)); by Lemma 6.4.17, (6.42) implies

$$c_{\mathrm{SM}}(X) = c_*(1\!\!1_X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathscr{X}] - [\mathscr{E}]}{1+\mathscr{X}-\mathscr{E}} \right) .$$

Therefore

$$\begin{aligned}
(-1)^{\dim X}(c_{\mathrm{vir}}(X) - c_{\mathrm{SM}}(X)) &= (-1)^{\dim X} c(TM|_X) \cap \pi_* \left(\frac{[\mathscr{X}]}{1+\mathscr{X}} - \frac{[\mathscr{X}] - [\mathscr{E}]}{1+\mathscr{X}-\mathscr{E}} \right) \\
&= (-1)^{\dim X} c(TM|_X) \cap \pi_* \left(\frac{1}{1+\mathscr{X}} \cdot \frac{[\mathscr{E}]}{1+\mathscr{X}-\mathscr{E}} \right) \\
&= \frac{c(TM|_X)}{1+X} \cap \pi_* \left(\frac{[\mathscr{E}]}{1-\mathscr{X}+\mathscr{E}} \right)_\vee \\
&= \frac{c(TM|_X)}{1+X} \cap \pi_* \left(c(\mathscr{O}_{T^\vee M}(-1))^{-1} \cap [\mathscr{E}] \right)_\vee \\
&= c(T_{\mathrm{vir}} X) \cap \mathrm{Segre}_{T^\vee M}([\mathscr{E}])_\vee
\end{aligned}$$

as claimed. By Theorem 6.4.8, identity (6.60) may be written

$$\mathscr{M}(X) = c(\mathscr{O}(X))^{-1} \cap c_*(\nu_{JX})$$

for the constructible function ν_{JX} whose characteristic cycle is the exceptional divisor $\mathscr{E}$. As a Lagrangian cycle, $[\mathscr{E}]$ is a linear combination of cycles of conormal spaces of subvarieties of JX: $[\mathscr{E}] = \sum_W n_W [N_W^\vee M]$; then, as prescribed by Definition 6.4.7:

$$\nu_{JX} = \sum_W (-1)^{\dim W} n_W 1\!\!1_W .$$

Over $\mathbb{C}$, and if X is reduced, Parusiński and Pragacz [80, Corollary 2.4] prove that

$$\nu_{JX} = (-1)^{\dim X} (\chi_X - 1\!\!1_X) ,$$

where for $p \in X$, $\chi_X(p)$ denotes the Euler characteristic of the Milnor fiber of X at p. (In [80], ν_{JX} is denoted μ.)

In general, note that $\mathscr{E}$ is the projectivized normal cone of JX. If Y is any subscheme of M, then we can associate to Y a constructible function ν_Y by letting $\nu_Y = \sum_W (-1)^{\dim W} n_W 1\!\!1_W$, where the subvarieties W are the supports of the components of the normal cone $C_Y M$ and n_W is the multiplicity of the component supported on W. Then the class $c_*(\nu_Y)$ generalizes the class $c_*(\nu_{JX}) = c(\mathscr{O}(X)) \cap \mathscr{M}(X)$. Kai Behrend [19, Proposition 4.16] proves that if Y is endowed with a *symmetric obstruction theory* (the singularity subscheme of a hypersurface gives an example), then the 0-dimensional component of $c_*(\nu_Y)$ equals the corresponding 'virtual fundamental class'; its degree is a Donaldson-Thomas type invariant.

Expression (6.60) for the Milnor class may be recast in terms of the Segre class $s(JX, M)$.

Proposition 6.4.33 *Let X be a hypersurface in a nonsingular variety M. Then*

$$\mathscr{M}(X) = (-1)^{\dim M} c(TM|_{JX}) \cap \left((c(\mathscr{O}(X)) \cap s(JX, M))^\vee \otimes_M \mathscr{O}(X) \right) .$$

This is an immediate consequence of Theorem 6.4.18. Indeed,

$$\begin{aligned} c_{\mathrm{vir}}(X) &= c(T_{\mathrm{vir}}X) \cap [X] = c(TM|_X)c(N_X M)^{-1} \cap [X] = c(TM|_X)c(\mathscr{O}(X))^{-1} \cap [X] \\ &= c(TM|_X) \cap ([X] \otimes_M \mathscr{O}(X)) . \end{aligned}$$

Note that we have written the right-hand side in Proposition 6.4.33 as a class in $A_*(JX)$. The statement means that this class pushes forward to the difference defining the Milnor class in Definition 6.4.32. The formula also implies that every connected component of JX has a well-defined contribution to the Milnor class of X. Of course if a component is supported on an isolated point p, and $\hat{p}$ denotes the part of JX supported on p, then the contribution of p to the Milnor class is

$$(-1)^{\dim M} c(TM|_{JX}) \cap \left((c(\mathscr{O}(X)) \cap s(\hat{p}, M)^\vee) \otimes_M \mathscr{O}(X) \right) = s(\hat{p}, M) ,$$

a class whose degree equals (in the complex setting) the ordinary Milnor number, cf. Sect. 6.3.3.

Proposition 6.4.33 may be formulated in terms of the 'μ-class' of [3], already mentioned in Sect. 6.3.3:

$$\mu_{\mathscr{O}(X)}(JX) := c(T^\vee M \otimes \mathscr{O}(X)) \cap s(JX, M) .$$

Indeed, simple manipulations using (6.20) and (6.21) show that

$$\mathscr{M}(X) = (-1)^{\dim M} c(\mathscr{O}(X))^{\dim X} \left(\mu_{\mathscr{O}(X)}(JX)^\vee \otimes_M \mathscr{O}(X) \right) ,$$

or, equivalently,

$$\mu_{\mathscr{O}(X)}(JX) = (-1)^{\dim M} c(\mathscr{O}(X))^{\dim X} \left(\mathscr{M}(X)^\vee \otimes_M \mathscr{O}(X) \right) .$$

It is somewhat remarkable that $\mathscr{M}(X)$ and $\mu_{\mathscr{O}(X)}(JX)$ are exchanged by the 'same' operation. Such involutions are not uncommon in the theory, see [27, 37].

The μ-class has applications to e.g., duality, and such applications can be formulated in terms of the Milnor class. We give one explicit example.

Example 6.4.34 Let M be a nonsingular projective variety, and let H be a hyperplane tangent to M, that is, a point of the *dual variety* $M^\vee$ of M; so $X = M \cap H$ is a singular hypersurface of M. Rewriting [3, Proposition 2.2] in terms of the Milnor class, we obtain that *the codimension of* $M^\vee$ *in the dual projective space is the smallest integer* $r \geq 1$ *such that the component of dimension* $r-1$ *in the class*

$$(1+X)^{\dim M} \left(\mathscr{M}(X)^\vee \otimes_M \mathscr{O}(X)\right)$$

does not vanish. Further, the projective degree of this component (viewed as a class in the dual projective space) equals the multiplicity of $M^\vee$ at H, up to sign. (This result generalizes (6.30).) We do not know a 'Segre class-free' proof of these facts.

For a concrete example, consider $M = \mathbb{P}^2 \times \mathbb{P}^1$, embedded in $\mathbb{P}^5$ by the Segre embedding. Using coordinates $(x_0 : x_1 : x_2)$ for the first factor, and $(y_0 : y_1)$ for the second factor, let X be the hypersurface with equation $x_0 y_1 = 0$: Thus, X is a hyperplane section via the Segre embedding, and X is the transversal union of two surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, resp. , $\mathbb{P}^2$, meeting along a $\mathbb{P}^1$. If h_1, resp., h_2 denote the pull-back of the hyperplane class from the first, resp. second factor, then the reader can verify that

$$c_{\text{vir}}(X) = \Big((h_1+h_2) + (2h_1^2 + 3h_1h_2) + 4h_1^2h_2\Big) \cap [\mathbb{P}^2 \times \mathbb{P}^1]\,,$$

$$c_{\text{SM}}(X) = \Big((h_1+h_1) + (2h_1^2 + 4h_1h_2) + 5h_1^2h_2\Big) \cap [\mathbb{P}^2 \times \mathbb{P}^1]\,.$$

It is easy to obtain these expressions 'by hand'; in any case, the following application of [54] will confirm the second assertion.

```
i1 : load("CharacteristicClasses.m2")

i2 : R=MultiProjCoordRing {2,1}

i3 : CSM ideal(R_0*R_4)

        2        2
o3 = 5h h  + 2h  + 4h h  + h  + h
       1 2      1     1 2    1    2
```

Therefore

$$\mathscr{M}(X) = (-h_1h_2 - h_1^2h_2) \cap [\mathbb{P}^2 \times \mathbb{P}^1]\,,$$

$$(1+X)^{\dim M} \left(\mathscr{M}(X)^\vee \otimes \mathscr{O}(X)\right) = -h_1h_2 \cap [\mathbb{P}^2 \times \mathbb{P}^1]\,.$$

In fact, it is easy to verify (by hand!) that for the corresponding hypersurface in $M = \mathbb{P}^n \times \mathbb{P}^1$, we have

$$\mathscr{M}(X) = (-1)^{n+1}(1+h_1)^{n-1}h_1h_2 \cap [\mathbb{P}^n \times \mathbb{P}^1]\,,$$
$$(1+X)^{\dim M}\left(\mathscr{M}(X)^\vee \otimes \mathscr{O}(X)\right) = (-1)^{n+1}h_1h_2 \cap [\mathbb{P}^n \times \mathbb{P}^1]\,.$$

The conclusion is that $M^\vee$ has codimension n in the dual $\mathbb{P}^{2n+1}$, and is nonsingular at the point corresponding to this hyperplane section. (In fact, it is well known that the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^1$ in $\mathbb{P}^{2n+1}$ is isomorphic to its dual variety for all $n \geq 1$ [89, Example 9.1].) ⌟

It is natural to ask about extensions of Proposition 6.4.33 to more general local complete intersections. For us, $X \subseteq M$ is a *local complete intersection* if X is the zero-scheme of a regular section of a vector bundle E defined on some neighborhood of X. For notational convenience, we will restrict M if necessary and assume that E is defined over the whole of M. Recall that the bundle E and the section defining X determine a closed subscheme $J_E(X)$ of $\mathbf{P}(E^\vee|_X)$ (Definition 6.4.28). We view $J_E(X)$ as a subscheme of $\mathbf{P}(E^\vee)$, and denote by $\pi : \mathbf{P}(E^\vee) \to M$ the projection.

Theorem 6.4.35 *Let $i : X \hookrightarrow M$ be a local complete intersection in a nonsingular variety M, obtained as the zero-scheme of a regular section of a vector bundle E of rank* $\operatorname{codim}_X M$. *Then $(-1)^{\dim X+1} i_*\mathscr{M}(X)$ equals*

$$c(TM) \cap \pi_*\left(\frac{c(\pi^*E^\vee \otimes \mathscr{O}_{E^\vee}(1))}{c(\mathscr{O}_{E^\vee}(1))} \cap \left(s(J_E(X), \mathbf{P}(E^\vee))^\vee \otimes_{\mathbf{P}(E^\vee)} \mathscr{O}_{E^\vee}(1)\right)\right) \tag{6.61}$$

in $A_*(M)$.

This statement may seem puzzling at first, since (6.58) and (6.61) are *the same formula,* yet the first is stated to equal $i_*c_{\mathrm{SM}}(X)$ (for arbitrary X) and the second equals $i_*\mathscr{M}(X)$ (for local complete intersections). The difference is in the ranks of the bundle E: in Theorem 6.4.30 the rank is required to exceed the dimension of the ambient variety M, while in Theorem 6.4.35 the rank is *equal to the codimension of* X. Both statements are consequences of the more general result (6.59): the formula evaluates the CSM class up to a correction term, which is 0 if $\operatorname{rk} E \gg 0$, and it is precisely $i_*(c_{\mathrm{vir}}(X))$ if X is a local complete intersection and $\operatorname{rk} E = \operatorname{codim}_X M$.

Example 6.4.36 Let $X \subseteq M$ be a hypersurface defined by a section s of $\mathscr{O}(X)$. In Example 6.4.31 we viewed X as the zero scheme of the section $(s, \ldots, s)$ of $\mathscr{O}(X)^{\oplus r+1}$, and showed that (6.58) evaluates to

$$c(TM) \cap \left(\frac{[X]}{1+X} + (c(\mathscr{O}(X) \cap s(JX, M))^\vee \otimes_M \mathscr{O}(X) - \frac{X^r}{(1+X)^{r+1}} \cap [X]\right)\,.$$

The case considered in Theorem 6.4.35 corresponds to $r = 0$, for which the formula gives

$$c(TM) \cap \left((c(\mathscr{O}(X) \cap s(JX, M))^\vee \otimes_M \mathscr{O}(X)\right) ,$$

agreeing with $(-1)^{\dim X+1} i_* \mathscr{M}(X)$ by Proposition 6.4.33. In this sense, Theorem 6.4.35 generalizes Proposition 6.4.33. ⌟

Expression (6.61) shows that, as in the case of the 'characteristic' classes reviewed in this section, the Milnor class of a local complete intersection is determined by a Segre class, $s(J_E(X), \mathbf{P}(E^\vee))$ in this case. If $M = \mathbb{P}^n$, this class can be computed using e.g., the Macaulay2 package [49]; the other ingredients in (6.61) are straightforward. For explicit formulas and examples, see [13].

6.5 Lê Cycles

6.5.1 Stückrad-Vogel Intersection Theory and van Gastel's Result

An 'excess intersection' situation occurs when loci intersect in higher than expected dimension. For example, r hypersurfaces in a nonsingular variety M are expected to intersect in a codimension-r subscheme; if they intersect along a subscheme of higher dimension, 'excess' intersection occurs.

The ability to deal with excess intersection is one the successes of Fulton-MacPherson's intersection theory. If $X_1, \dots, X_r$ are hypersurfaces, and Z is a connected component of $X_1 \cap \cdots \cap X_r$, then the contribution of Z to the intersection product of the classes of the hypersurfaces may be written as

$$\left\{ \prod_{i=1}^{r} (1 + X_i) \cap s(Z, M) \right\}_{\dim M - r} . \tag{6.62}$$

For this, view $X_1 \cdots X_r$ as $(X_1 \times \cdots \times X_r) \cdot \Delta$, where Δ is the diagonal in $M \times \cdots \times M$: we have $(X_1 \times \cdots \times X_r) \cap \Delta \cong X_1 \cap \cdots \cap X_r$, $\Delta \cong M$, and we consider the fiber diagram

$$\begin{array}{ccc} X_1 \cap \cdots \cap X_r & \longrightarrow & \Delta \cong M \\ \downarrow & & \downarrow \\ X_1 \times \cdots \times X_r & \longrightarrow & M \times \cdots \times M . \end{array}$$

We can view Z as a connected component of $(X_1 \times \cdots \times X_r) \cap \Delta$. The restriction of the normal bundle $N_{X_1 \times \cdots \times X_r}(M \times \cdots \times M)$ to Z is then isomorphic to $\oplus_i \mathscr{O}(X_i)|_Z$, so that its Chern class is (the restriction of) $\prod_{i=1}^r (1 + X_i)$. Then (6.62) follows from (6.14). The fact that Z may be of dimension higher than $\dim M - r$ is precisely accounted for by the Segre class of Z in M.

An alternative approach to intersection theory in projective space, dealing differently with excess intersection, was developed by Jürgen Stückrad and Wolfgang Vogel ([88], and see [35] for a comprehensive account). In excess intersection situations, this approach produces a *cycle* after a transcendental extension of the base field; the intersection product can be computed from this cycle, and agrees with the Fulton-MacPherson intersection product.

We review the construction of the Stückrad-Vogel 'v-cycle', essentially following the 'geometric' account given in [91], where it is also extended to the setting of more general schemes. However, we only present the construction in the somewhat limited scope needed for our application, and we make a substantial simplification, at the price of only obtaining a cycle depending on general choices. (The Stückrad-Vogel construction produces a well-defined cycle independent of such choices, after a transcendental extension of the base field.)

Let V be a variety, $\mathscr{L}$ a line bundle on V, $s_1, \ldots, s_r$ nonzero sections of $\mathscr{L}$, and $\mathscr{D}$ the collection of the corresponding Cartier divisors $D_1, \ldots, D_r$. The sections $s_1, \ldots, s_r$ span a subspace of $H^0(V, \mathscr{L})$; by a '$\mathscr{D}$-divisor' we will mean a divisor defined by a section of this subspace. Let $Z = D_1 \cap \cdots \cap D_r$.

The following inductive procedure constructs a cycle on Z, depending on general choices of $\mathscr{D}$-divisors. The procedure only involves proper intersections with Cartier divisors, which is defined at the level of cycles: if W is a variety, and a Cartier divisor D intersects it properly, i.e., it does not contain it, then $D \cap W$ is a Cartier divisor in W (or empty), and we denote by $D * W$ the corresponding cycle (or 0). The class of this cycle is the intersection product of $[W]$ by D in the Chow group. By linearity, this operation is extended to cycles ρ such that D does not contain any component of ρ: then $D * \rho$ denotes the corresponding 'proper intersection' product.

The algorithm may be described as follows.

- Let $\alpha^0 = 0$, $\rho^0 = V$;
- For $j > 0$: if $\rho^{j-1} \neq 0$, then a general $\mathscr{D}$ divisor D'_j intersects ρ^{j-1} properly; let $D'_j * \rho^{j-1} = \alpha^j + \rho^j$, where α^j collects the components of the intersection product that are contained in $Z = D_1 \cap \cdots \cap D_r$;
- This procedure stops when $\rho^j = 0$.

It is easy to see that a general D'_j does intersect ρ^{j-1} properly, so it is always possible to make the choice needed in the second point. Also, let s'_j be the section defining D'_j. The construction implies that if $\rho^{j-1} \neq 0$, then s'_j is not in the span of $s'_1, \ldots, s'_{j-1}$. In particular, the procedure must stop at some $j \leq r$. We set $\alpha^i = \rho^i = 0$ for $j < i \leq r$.

Definition 6.5.1 We denote by $\mathscr{D} \cap V$ the sum $\sum_{i=0}^{r} \alpha^i$. This is a cycle on $Z = D_1 \cap \cdots \cap D_r$. ⌟

Remark 6.5.2 We chose the notation $\mathscr{D} \cap V$ to align with the notation used by van Gastel (in a more general context). This is the 'v-cycle' determined by $\mathscr{D}$. The definition presented above only depends on the linear system spanned by the sections defining the divisors D_i in the collection $\mathscr{D}$. ⌟

According to our definition, the cycle $\mathscr{D} \cap V$ depends on the choice of the divisors D'_j. One of the advantages of the more sophisticated Stückrad-Vogel construction is that it yields a well-defined cycle independent of any choice, albeit after extending the ground field. However, we are only interested in the rational equivalence class of $\mathscr{D} \cap V$, and this is independent of the choices. In fact, the following holds.

Theorem 6.5.3 *With notation as above,*

$$[\mathscr{D} \cap V] = s(Z, V) \otimes_V \mathscr{L}^\vee$$

in $A_*(Z)$.

In the context of Stückrad-Vogel intersection theory, this is [91, Corollary 3.6]. Theorem 6.5.3 can also be proved by interpreting $\mathscr{D} \cap V$ in terms of the blow-up of V along Z; this naturally identifies its rational equivalence class as a 'tensored Segre class' in the sense of [12], up to a product by $c(\mathscr{L})$.

By (6.21), Theorem 6.5.3 is equivalent to

$$s(Z, V) = [\mathscr{D} \cap V] \otimes_V \mathscr{L}\,. \tag{6.63}$$

Using (6.62), we see that

$$D_1 \cdots D_r \cap [V] = \left\{ c(\mathscr{L})^r \cap ([\mathscr{D} \cap V] \otimes_V \mathscr{L}) \right\}_{\dim V - r}$$

in $A_{\dim V - r} Z$. This is equivalent to the formula

$$D_1 \cdots D_r \cap [V] = \sum_{j=0}^{r} c_1(\mathscr{L})^{r-j} \cap \alpha^j\,,$$

cf. [91, Proposition 1.2 (c)].

In conclusion, the Stückrad-Vogel construction offers an alternative to the treatment of excess intersection of linearly equivalent divisors. By (6.63), the relevant Segre class may be computed in terms of the v-cycle. Among other pleasant features, this approach leads to 'positivity' statements for Segre classes: by construction, the v-cycle is effective; by (6.63), the non-effective parts of the Segre class of the intersection of sections of a line bundle $\mathscr{L}$ are due to the 'tensor' operation $\_ \otimes_V \mathscr{L}$. (Cf. [12, Corollary 1.3].)

6.5.2 Lê Cycles and Numbers

Broadly speaking, one can view singularities as arising because of an excess intersection. For example, if X is a hypersurface of $\mathbb{P}^n$, with equation $F(x_0, \dots, x_n) = 0$, the singular locus of X is the intersection of the $n+1$ hypersurfaces with equations $\partial F/\partial x_i = 0$, $i = 0, \dots, n$. Then X is singular precisely when these hypersurfaces meet with excess intersection. The scheme they define is the singularity subscheme JX of Definition 6.2.14; and the Segre class that is relevant to the Fulton-MacPherson approach is precisely, and not surprisingly, the class $s(JX, M)$ that appears in most results concerning hypersurfaces reviewed in Sect. 6.3 and 6.4. Taking the point of view of Sect. 6.5.1, we could express these results in terms of the v-cycle corresponding to the linear system spanned by the partials.

A closely related construction was provided (independently from Stückrad and Vogel) by Massey in 1986, leading to his definition of *Lê cycles* [68–70]. The theory and applications of Lê cycles are surveyed in [67]. Massey's definition may be given for analytic functions defined for a nonempty open subset of $\mathbb{C}^{n+1}$. We are going to consider the case of a homogeneous polynomial, and view it as the generator of the ideal of a hypersurface in $\mathbb{P}^n$. We will follow [67, §7.7] for the resulting *projective* Lê cycles. The considerations that follow would hold over any algebraically closed field of characteristic 0.

Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial, defining a projective hypersurface $X \subseteq \mathbb{P}^n$. Massey's definition can be phrased in terms very close to the inductive definition given in Sect. 6.5.1, applied to the linear system spanned by the derivative $\partial F/\partial x_i$ of F. We give the affine definition of the cycles first.

- Let $\Gamma^{n+1} = \mathbb{C}^{n+1}$, $\Lambda^{n+1} = 0$;
- For $1 \le k \le n+1$, define Γ^{k-1} and Λ^{k-1} by downward induction by

  $$\Gamma^k * V\left(\frac{\partial F}{\partial x_{k-1}}\right) = \Lambda^{k-1} + \Gamma^{k-1},$$

  where the (cycle-theoretic) intersection is assumed to be proper, and Λ^{k-1} consists of the components contained in JX, Γ^{k-1} of the other components.

Following [67, §7.7]:

Definition 6.5.4 The *projective Lê cycles* of X are the cycles $\mathbb{A}^k_X := \mathbb{P}(\Lambda^{k+1})$. ⌟

The projectivization of the cycles Γ^j are the *projective relative polar cycles* of X.

The Lê cycles of X evidently depend on the chosen coordinates, and may not be defined for certain choices as the cycles appearing in the definition may fail to meet properly. Massey proves that a general choice of coordinates guarantees that the intersections are proper, so that the corresponding Lê cycles exist. In the following, the Lê cycles we consider will be assumed to be obtained from a general choice of coordinates.

Comparing Massey's definition with Definition 6.5.1, we recognize that the sum $\sum_{k=0}^{n} \mathbb{A}^k$ of Lê cycles may be viewed as an instance of the v-cycle $\mathscr{D} \cap \mathbb{P}^n$, where $\mathscr{D}$ is the collection of partial derivatives of F. The dependence on the choices (e.g., the choice of coordinates in Massey's definition, or the choice of D'_j in Definition 6.5.1) is eliminated once one passes to rational equivalence, so that

$$[\mathscr{D} \cap \mathbb{P}^n] = \sum_k [\mathbb{A}_X^k]$$

in $A_*(JX)$ if all choices are general. (Note however that the indexing conventions differ, so that with notation as in Sect. 6.5.1, $[\mathbb{A}_X^k] = [\alpha^{n-k}]$.)

With this understood, the next result follows immediately from Theorem 6.5.3.

Proposition 6.5.5 *Let X be a degree-d hypersurface in $\mathbb{P}^n$, with projective Lê cycles $\mathbb{A}_X^k$. Then*

$$\sum_k [\mathbb{A}_X^k] = s(JX, \mathbb{P}^n) \otimes_{\mathbb{P}^n} \mathscr{O}(-(d-1)) \tag{6.64}$$

in $A_*(JX)$.

Remark 6.5.6 For $M = \mathbb{C}^n$, Gaffney and Gassler [43] propose a generalization of classes of Lê cycles based on more general ideals, which in the case of the Jacobian ideal of a polynomial defining a hypersurface X is closely related with the Segre class of JX (cf. the definition of the *Segre cycle* $\Lambda_k^{\mathbf{g}}(I, Y)$ in [43, (2.1)]). Partly motivated by this work, Callejas-Bedregal, Morgado, and Seade gave a definition of Lê cycles for a hypersurface X of a compact complex manifold M, which amounts essentially to a cycle representing the Segre class $s(JX, M)$ [27, Definition 3.2]. This definition is *not* compatible with Massey's Lê cycles for $M = \mathbb{P}^n$, as the authors opted to omit the extra tensor appearing in (6.64). Since the 'hyperplane' defined in [43] differs from the tautological class used in [27], this causes a discrepancy amounting to a twist of the line bundle of the hypersurface. This twist is accounted for in Proposition 6.5.5, which is compatible with the construction in [43].

See [28] and [37] for further discussions of [27, Definition 3.2]. In particular, Callejas-Bedregal, Morgado, and Seade propose an alternative 'geometric' definition in [28] (Definition 1.3), which *does* agree with Massey's for $M = \mathbb{P}^n$. Also see [29, §4] (particularly Definition 4.4) for a comprehensive account. We will come back to this definition in Sect. 6.5.3. ⌟

The fact that the Lê cycles are *cycles* is important for geometric applications. Proposition 6.5.5 only computes their *classes* up to rational equivalence, in the Chow group $A_*(JX)$ of the singularity subscheme of the hypersurface. These classes still carry useful information, even after a push-forward by the inclusion $\iota : JX \to \mathbb{P}^n$. We consider the class

$$\iota_*([\mathbb{A}_X^k]) = \lambda_X^k [\mathbb{P}^k],$$

where the integers λ_X^k are (still following Massey) called the *Lê numbers* of the hypersurface. (Massey's Lê numbers also depend on the choice of coordinates; again, we will assume that the choice of coordinates is sufficiently general.) Proposition 6.5.5 implies as an immediate corollary a formula for the Lê numbers in terms of the degrees of the components of the Segre class (and conversely).

Corollary 6.5.7 *Let $X \subseteq \mathbb{P}^n$ be a hypersurface, and denote by s_i the degree of the i-th dimensional component of the Segre class $s(JX, \mathbb{P}^n)$. Then for $k = 0, \dots, n$:*

$$\lambda_X^k = \sum_{j=k}^{n} \binom{n-k-1}{j-k} (d-1)^{j-k} s_j \tag{6.65}$$

$$s_k = \sum_{j=k}^{n} \binom{n-k-1}{j-k} (-(d-1))^{j-k} \lambda_X^j \,. \tag{6.66}$$

Proof Denote the hyperplane class by H. By Proposition 6.5.5 and the definition of $\otimes_{\mathbb{P}^n}$ (6.19):

$$\begin{aligned}(\lambda_X^n + \lambda_X^{n-1} H + \cdots + \lambda_X^0 H^n) \cap [\mathbb{P}^n]& \\ &= ((s_n + s_{n-1}H + \cdots + s_0 H^n) \cap [\mathbb{P}^n]) \otimes_{\mathbb{P}^n} \mathcal{O}(-(d-1)) \\ &= \left(s_n + \frac{s_{n-1}H}{(1-(d-1)H)} + \cdots + \frac{s_0 H^n}{(1-(d-1)H)^n} \right) \cap [\mathbb{P}^n]\end{aligned}$$

and the first formula follows by matching terms of equal degrees in the two expressions. 'Solving for $s(JX, \mathbb{P}^n)$' in Proposition 6.5.5 gives

$$s(JX, \mathbb{P}^n) = \sum_k [\Lambda_X^k] \otimes_{\mathbb{P}^n} \mathcal{O}(d-1)$$

(apply (6.21)), and the second formula follows by the same token. □

Remark 6.5.8 Formula (6.65) in Corollary 6.5.7:

$$\lambda_X^k = s_k + (n-k-1)(d-1)s_{k+1} + \binom{n-k-1}{2}(d-1)^2 s_{k+2} + \cdots \,.$$

can be viewed as the degree of the ordinary Segre class, 'corrected' by a term determined by the degree d of the hypersurface.

In the introduction to [43], Gaffney and Gassler state: *"... In fact, the Segre numbers* (of the Jacobian ideal) *are just the Lê numbers of David Massey."* Corollary 6.5.7 is compatible with this assertion: it is easy to verify that the 'Segre numbers' of [43] agree with the right-hand side of (6.65). ⌟

Example 6.5.9 Consider the hypersurface X of $\mathbb{P}^5$ defined by the polynomial

$$F = x_0^7 - x_1^7 - (x_2^3 + x_3^3 + x_4^3 + x_5^3)\, x_0^4 \,.$$

The singularity subscheme JX is a non-reduced 3-dimensional subscheme of $\mathbb{P}^5$ supported on the linear subspace $x_0 = x_1 = 0$. We can use the package [49] to compute its Segre class:

```
i1 : load("SegreClasses.m2")

i2 : R=ZZ/32749[x0,x1,x2,x3,x4,x5]

i3 : X=ideal(x1^7- x0^7 - (x2^3+x3^3+x4^3+x5^3)*x0^4)

i4 : JX=ideal jacobian X

i5 : segre(JX,ideal(0_R))

                 5        4        3       2
o5 = - 3168H  + 792H  - 144H  + 18H
            1        1        1       1
```

(Working over a finite field of large characteristic does not affect the result, and often leads to faster computations.) Thus,

$$\iota_* s(JX, \mathbb{P}^5) = 18[\mathbb{P}^3] - 144[\mathbb{P}^2] + 792[\mathbb{P}^1] - 3168[\mathbb{P}^0]\,,$$

and Corollary 6.5.7 yields

$$\begin{cases} \lambda_X^4 &= \mathbf{0} \\ \lambda_X^3 &= \mathbf{18} \\ \lambda_X^2 &= -144 + 2\cdot 6\cdot 18 = \mathbf{72} \\ \lambda_X^1 &= 792 + 3\cdot 6\cdot(-144) + \binom{3}{2}\cdot 36\cdot 18 = \mathbf{144} \\ \lambda_X^0 &= -3168 + 4\cdot 6\cdot 792 + \binom{4}{2}\cdot 36\cdot(-144) + \binom{4}{3}\cdot 216\cdot 18 = \mathbf{288}\,. \end{cases}$$

These Lê numbers agree with those obtained by applying Massey's inductive definition with coordinates $(x_0, \dots, x_5)$; the Lê cycles are complete intersections in this case, and computing their degrees is straightforward. (Using $(x_5, \dots, x_0)$ leads to a different list; this latter choice is not sufficiently general.) ⌟

We can also projectivize the cycles Γ^k appearing in Massey's definition (corresponding to the ρ-cycles in the Stückrad-Vogel algorithm). Again (loosely) following Massey, we call $\mathbb{\Gamma}_X^k := \mathbb{P}(\Gamma^{k+1})$ the 'projective polar cycles' of X, and their degrees γ_X^k the 'polar numbers' of X. We assume these are computed for a general choice of coordinates.

At the level of rational equivalence classes, Massey's algorithm implies easily the relation

$$\sum_k [\mathbb{A}_X^k] = [\mathbb{P}^n] - (1-(d-1)H)\sum_k [\Gamma_X^k]$$

from which $\lambda_X^n = 0$ and

$$\lambda_X^k = (d-1)\gamma_X^{k+1} - \gamma_X^k$$

for $0 \le k < n$. Equivalently,

$$\gamma_X^k = (d-1)^{n-k} - \sum_{j=k}^{n-1} (d-1)^{j-k}\lambda_X^j$$

for $0 \le k \le n$. (Also see [67, Corollary 7.7.3].)

Corollary 6.5.10 *With notation as in Corollary 6.5.7, and for $k = 0, \ldots, n$:*

$$\gamma_X^k = (d-1)^{n-k} - \sum_{j=k}^{n} \binom{n-k}{j-k}(d-1)^{j-k}s_j$$

$$s_k = \delta_k^n - \sum_{j=k}^{n} \binom{n-k}{j-k}(-(d-1))^{j-k}\gamma_X^j$$

where $\delta_k^n = 1$ if $k = n$, 0 otherwise.

Proof The first formula is obtained by reading off the coefficient of $[\mathbb{P}^k]$ in the identity

$$\sum_k [\Gamma_X^k] = (1-(d-1)H)^{-1} \cap \big([\mathbb{P}^n] - s(JX,\mathbb{P}^n) \otimes_{\mathbb{P}^n} \mathscr{O}(-(d-1))\big) , \qquad (6.67)$$

which follows from the above discussion and Proposition 6.5.5. Solving for $s(JX,\mathbb{P}^n)$ in (6.67) gives

$$s(JX,\mathbb{P}^n) = [\mathbb{P}^n] - (1+(d-1)H)^{-1} \cap \sum_k ([\Gamma_X^k] \otimes_{\mathbb{P}^n} \mathscr{O}(d-1))$$

(use (6.20) and (6.21)) with the stated implication on degrees. □

Example 6.5.11 For the hypersurface in Example 6.5.9, the computation of the polar numbers runs as follows.

$$\begin{cases} \gamma_X^5 & = \mathbf{1} \\ \gamma_X^4 & = \mathbf{6} \\ \gamma_X^3 & = 36 - 18 = \mathbf{18} \\ \gamma_X^2 & = 216 - (-144) - \binom{3}{1} \cdot 6 \cdot 18 = \mathbf{36} \\ \gamma_X^1 & = 1296 - 792 - \binom{4}{1} \cdot 6 \cdot (-144) - \binom{4}{2} \cdot 36 \cdot 18 = \mathbf{72} \\ \gamma_X^0 & = 7776 - (-3168) - \binom{5}{1} \cdot 6 \cdot 792 - \binom{5}{2} \cdot 36 \cdot (-144) - \binom{5}{3} \cdot 216 \cdot 18 = \mathbf{144}\,. \end{cases}$$

Again, it is straightforward to verify that these agree with the result of Massey's algorithm, applied with coordinates $(x_0, \ldots, x_5)$. ⌟

Remark 6.5.12 We already mentioned (Remark 6.4.12) Piene's seminal 1978 paper [81], including formulas for polar classes of hypersurfaces in terms of Segre classes. The reader is warned that these two uses of the term 'polar' differ: Piene's polar classes of a hypersurface X are classes in $A_*(X)$, while Massey's polar cycles are not supported on X. Therefore, the degrees of Piene's polar classes are not the polar numbers γ^k computed above. However, we note that the formula in Corollary 6.5.10 is very similar to the formula in [81, Theorem 2.3]; the main difference is in the use of $s(JX, M)$ rather than $s(JX, X)$. ⌟

6.5.3 Lê, Milnor, Segre

One moral to be drawn from the preceding considerations is that the information carried by the Lê classes of a hypersurface X of projective space, its Milnor class, and the Segre class of its singularity subscheme JX, is essentially the same. The relation between Segre classes and Milnor classes goes back to [2], while the relation between Milnor classes and Lê classes was first studied in [27, 28]. As far as hypersurfaces of projective space are concerned, many of the results covered in this review could be written in terms of any of these notions. Note however that extending Lê cycles/classes to the setting of a hypersurface of a more general nonsingular variety is nontrivial (this is one of the main goals of [27]; and see below); localizing Milnor classes to the components of the singular locus also requires nontrivial considerations (see e.g., [22]); while the Segre class of the singularity subscheme JX is naturally defined as a class in the Chow group of JX, does not require a projective embedding, and may be considered over arbitrary fields. For these reasons, it would seem that the language of Segre classes is preferable over these alternatives.

For the convenience of the reader, we collect here the formulas translating these notions into one another. For notational economy we will let

$$\mathbb{A} := \sum_k [\mathbb{A}^k_X] \quad , \quad \mathscr{M} := \mathscr{M}(X) \quad , \quad S := s(JX, \mathbb{P}^n)$$

for a degree-d hypersurface X of $\mathbb{P}^n$, and omit evident push-forwards. Then, denoting by H the hyperplane class:

$$\begin{cases} \mathbb{A} = S \otimes_{\mathbb{P}^n} \mathscr{O}(-(d-1)H) \\ S = \mathbb{A} \otimes_{\mathbb{P}^n} \mathscr{O}((d-1)H) \end{cases} \tag{6.68}$$

$$\begin{cases} \mathscr{M} = (-1)^n \dfrac{(1+H)^{n+1}}{1+dH} \cap \left(S^\vee \otimes_{\mathbb{P}^n} \mathscr{O}(dH)\right) \\ S = (-1)^n \dfrac{(1+dH)^n}{(1+(d-1)H)^{n+1}} \cap \left(\mathscr{M}^\vee \otimes_{\mathbb{P}^n} \mathscr{O}(dH)\right) \end{cases} \tag{6.69}$$

$$\begin{cases} \mathbb{A} = (-1)^n (1+H)^n (1-(d-1)H) \cap (\mathscr{M}^\vee \otimes_{\mathbb{P}^n} \mathscr{O}(H)) \\ \mathscr{M} = (-1)^n \dfrac{(1+H)^{n+1}}{1+dH} \cap (\mathbb{A}^\vee \otimes_{\mathbb{P}^n} \mathscr{O}(H)) \end{cases} \tag{6.70}$$

Indeed, (6.68) follows from Proposition 6.5.5; (6.69) from Proposition 6.4.33; and (6.70) is then an immediate consequence, using (6.20) and (6.21).

This dictionary suggests possible extensions of the notion of Lê classes to hypersurfaces of more general varieties. Let M be a nonsingular compact complex variety endowed with a very ample line bundle $\mathscr{O}(H)$. For a hypersurface X of M, Callejas-Bedregal, Morgado, and Seade have constructed *global Lê cycles,* determined by the choice of linear subspaces of $\mathbb{P}^n$, generalizing the case $M = \mathbb{P}^n$; see [28, Definition 1.3] and [29, §4.3]. Denoting the corresponding class $\mathbb{A}_{\text{CBMS}}(X)$, and letting $\mathscr{L} = \mathscr{O}(X)$, they prove the following result (which we state using our notation).

Theorem 6.5.13 ([29, Theorem 4.6])

$$\mathbb{A}_{CBMS}(X) = (-1)^{\dim M} c(\mathscr{O}(H))^{\dim M} c(\mathscr{O}(H) \otimes \mathscr{L}^\vee) \cap (\mathscr{M}(X)^\vee \otimes_M \mathscr{O}(H))$$
$$\mathscr{M}(X) = (-1)^{\dim M} c(\mathscr{O}(H))^{\dim M+1} c(\mathscr{L})^{-1} \cap (\mathbb{A}_{CBMS}(X)^\vee \otimes_M \mathscr{O}(H)) .$$

That is, the natural generalization of (6.70) holds for this class; the class $\mathbb{A}_{\text{CBMS}}$ agrees with the class of Massey's Lê cycle for $M = \mathbb{P}^n$.

It is straightforward (using Proposition 6.4.33 and (6.20) and (6.21)) to write $\mathbb{A}_{\text{CBMS}}(X)$ in terms of a Segre class:

$$\mathbb{A}_{\text{CBMS}}(X) = c(\mathscr{O}(H))\, c(T^\vee M \otimes \mathscr{O}(H)) \cap \left(s(JX, M) \otimes_M (\mathscr{O}(H) \otimes \mathscr{L}^\vee)\right) .$$

This expression reduces to (6.68) for $M = \mathbb{P}^n$, and it could be used to extend the definition of $\mathbb{A}_{\mathrm{CBMS}}(X)$ to arbitrary fields and possibly noncomplete varieties.

There are other possible extensions of Massey's Lê class to more general projective varieties; (6.68) suggests alternative generalizations. Exploring such alternatives is the subject of current research.

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Chapter 7
Milnor Number and Chern Classes for Singular Varieties: An Introduction

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In memory of our dearest Roberto
whose laughter, joy and love for life
will remain with us forever.

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Abstract We survey how the Milnor number of complex map-germs with an isolated critical point relates to various indices of vector fields on singular varieties, and the way how this number extends via the theory of Chern classes of singular varieties, to the concept of Milnor classes of varieties with arbitrary singular set in complex manifolds. This relates several deep theories in mathematics and gives rise to a necklace of beautiful jewels.

7.1 Introduction

The Milnor Fibration of holomorphic maps introduced in [94] is a fundamental object for the study of the local topology of complex hypersurfaces. When the map-germ has an isolated critical point one has the associated Milnor number, which is the most important numerical invariant associated to an isolated complex hypersurface singularity. The literature about the Milnor number is vast and we refer for instance to [73] for a recent account of the subject. It is a topological invariant, easily computable and it determines the homeomorphism type of the Milnor fiber. This invariant was extended by Hamm [64] to isolated complete intersection singularities, and it is well-known (see for instance [30]) that this number can be expressed as the difference of two indices of vector fields that extend to singular varieties the classical local index of Poincaré-Hopf, namely the GSV and radial indices.

When considering non-isolated complex hypersurface singularities there are two important viewpoints extending the Milnor number: one is local and due to work by Lê D. T., B. Teissier and D. Masssey, who introduced in [77, 78] the notions of Lê cycles and Lê numbers. These spring from the theory of polar varieties developed by Lê and Teissier, with roots in ideas by René Thom. There is a Lê cycle (and number) in each complex dimension from 0 to that of the singular set; these encode deep information about the singularity germ and they determine the homeomorphism type of the local Milnor fiber (see for instance [78, 80]). The other viewpoint is global and is due to A. Parusińsky who introduced in [97] the notion of a generalized Milnor number: this is an integer associated to each connected component of the singular set of a complex hypersurface in a compact complex manifold. There are several interpretations of that invariant by Parusińsky-Pragacz and by other authors. In particular, this notion was extended in [26, 30] to singular complete intersections in complex manifolds by expressing it (as for isolated singularities) as the difference of two indices of vector fields, the radial index and the virtual index, an extension of the GSV index. See [30] and [51] for expositions about indices of vector fields and 1-forms on singular varieties and their relations with other invariants of singularities.

On the other hand, the theory of Chern classes for singular varieties, initiated by M. H. Schwartz [116], D. P. Sullivan [121], R. MacPherson [81], and continued by J. P. Brasselet, W. Fulton and others, keeps growing fastly and it is now a rich theory that can be regarded from several points of view and has deep connections with several areas of mathematics. There are various notions extending to the singular case the classical Chern classes of complex manifolds, having each its own properties and interest. The classes introduced by M. H. Schwartz are a natural extension for stratified singular varieties of the usual Chern classes regarded as obstructions for constructing linearly independent sections of vector bundles. The classes introduced by MacPherson proved affirmatively a conjecture stated by Deligne with ideas by Grothendieck, somehow motivated by Sullivan's work for the Stiefel-Whitney classes. MacPherson's construction actually assigns a "theory of homology Chern classes" to each constructible function on a compact complex variety X. Then Brasselet and Schwartz showed in [24] that Schwartz' and MacPherson's construction for the constructible function $\mathbb{1}_X$ actually coincide up to Alexander duality; hence the name Schwartz-MacPherson classes, that we denote c_*^{SM}. On the other hand the Fulton classes c_*^{Fu} are defined using the classical Segre classes in algebraic geometry. All of these are regarded in the singular homology or in the Chow ring in the algebraic case. (See Allufi's excellent survey [7] on Segre classes in this volume.)

In the 1990s P. Aluffi, studying which schemes can arise as singular schemes of hypersurfaces in complex manifolds, realized that it was important to compare the Schwartz-MacPherson and the Fulton classes. This same issue, comparing the c_*^{SM} and c_*^{Fu} classes, also arose at almost the same time and by different reasons in the work of Parusiński-Pragacz, Yokura and Brasselet-Lehmann-Seade-Suwa. In [98–101] this appears in relation with the generalized Milnor number and the topology of degeneracy loci of sections of vector bundles. In Yokura's work this appeared in [132, 133] while looking at Chern classes in bivariant theory (cf. Brasselet's work [21]), searching for a Verdier-Riemann-Roch type theorem for the MacPherson classes of singular varieties. On the other hand this comparison of the c_*^{SM} and c_*^{Fu} classes was a natural continuation of Brasselet-Schwartz' theorem showing that the MacPherson and the Schwartz classes coincide (up to Alexander duality). Seade and Suwa proved [118, 122] that in the case of local complete intersections with only isolated singularities, the difference $c_*^{SM} - c_*^{Fu}$ is the sum of the local Mlnor numbers, up to sign. This was a clue for coining the name "Milnor classes" of compact varieties with arbitrary singular set for the difference (up to sign) between

Schwartz-MacPherson and Fulton classes. Milnor classes were studied in [25, 26] by localizing them using indices of vector fields, and we refer to [30] for a thorough account on the subject. (See Brasselet's [23] and Suwa's [123] works in this volume for thorough accounts on the subject.)

The remarkable work by Baum-Fulton-MacPherson [16] and Verdier [130] generalizing to singular varieties Hirzebruch's and Grothendieck's generalizations of Riemann-Roch, by Cappell and Shaneson for the Hirzebruch L-class that appears in the signature theorem, and by Kontsevich introducing the deep theory of motives, was notably continued by Schürmann, Yokura, Aluffi, Maxim, Brasselet and others, then leading to the concept of motivic Hirzebruch and Hirzebruch-Milnor classes. This important subject is discussed below and also in Brasselet's paper in this volume. A more detailed study will be given in Yokura's paper [137].

On the other hand, in [33] the authors of this article used work by Schürmann and Tibăr for affine varieties [113], to show that Massey's concept of Lê cycles can be globalized to projective manifolds and, surprisingly, the information encoded in those classes is equivalent to the information encoded in the Milnor classes, since the global Lê classes determine the Milnor classes and conversely.

In this work we introduce and review all these concepts and contributions. We also give a topological interpretation of the virtual index of vector fields and a short proof of the theorem in [118] that for singular varieties with only isolated complete intersection singularities, the 0-dimensional Milnor class is the sum of the local Milnor numbers: we thank José Luis Cisneros for a lemma that we use in that proof. It would be interesting to know what this invariant is for varieties with isolated singularities which are not local complete intersections.

We are grateful to the referee for many helpful comments. Our dear friend and co-author Roberto Callejas-Bedregal passed away while we were writing this work, so we dedicate it to him.

7.2 Milnor Number and Indices of Vector Fields

In this section we briefly review the various notions of indices of vector fields on singular varieties and their relation with the Milnor number of isolated complete intersection complex singularities.

7.2.1 Definition and Basic Properties

The literature about the Milnor number is vast. We refer to [94] for background material about it, or see for instance [73] for a recent account on the subject.

Consider a holomorphic function

$$f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$$

with a critical point at $\underline{0}$. Let $V = f^{-1}(0)$ and let $\mathbb{S}_\varepsilon$ be a sphere in $\mathbb{C}^{n+1}$ centered at $\underline{0}$ of radius $\varepsilon > 0$ sufficiently small so that $\mathbb{S}_\varepsilon$ is a Milnor sphere for f, cf. [73]. Let $K = V \cap \mathbb{S}_\varepsilon$ be the link of $\underline{0}$ in V. Milnor's fibration theorem in [94] says that,

$$\phi := \frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus K \longrightarrow \mathbb{S}^1 \,,$$

is a C^∞ locally trivial fibration. There is another way of looking at this. Given $\varepsilon > 0$ as above, choose $0 < \delta \ll \varepsilon$ and set $N(\varepsilon, \delta) = f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$, where $\partial\mathbb{D}_\delta$ is the boundary of the disc in $\mathbb{C}$ of radius $\delta > 0$ and centered at 0. Then,

$$f : N(\varepsilon, \delta) \longrightarrow \partial\mathbb{D}_\delta \cong \mathbb{S}^1$$

is a locally trivial fibration, equivalent to the previous one if we take the ball $\mathbb{B}_\varepsilon$ to be open. The manifold $N(\varepsilon, \delta)$ is called *a Milnor tube* for f and the fiber $F_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon$ with $t \in \partial\mathbb{D}_\delta$ is called *the Milnor fiber of* f. We denote this fiber by F_f.

One knows from [94], and it follows easily from [10], that F_f has the homotopy type of a CW-complex of middle dimension n. Furthermore, if f has an isolated critical point at $\underline{0}$ then F_f actually has the homotopy type of a bouquet of spheres of middle dimension n, $F_f \simeq \bigvee_\mu S^n$. One has:

Definition 7.2.1 If the map f has an isolated critical point, say at $\underline{0}$, then the number μ above is *the Milnor number* of f at $\underline{0}$.

It is proved in [94] that the number μ equals the local Poincaré-Hopf index at $\underline{0}$ of the gradient vector field of $\nabla f = (\partial f/\partial z_0, \dots, \partial f/\partial z_n)$, where $\{(z_0, \dots, z_n)\}$ are the coordinates in $\mathbb{C}^{n+1}$.

Recall that a vector field v on an open set $U \subset \mathbb{R}^m$ with coordinates $\{(x_1, \dots, x_m)\}$ can be written as $v = \sum_{i=1}^m f_i \;\; \partial/\partial x_i$ or, as above, simply as $v = (f_1, \dots, f_m)$. This is a section of the tangent bundle of U. The vector field is said to be continuous, smooth, analytic, etc., according as its components $(f_1, \dots, f_m)$ are continuous, smooth, analytic, etc., respectively. If m is even and the f_i are all holomorphic functions in $\mathbb{C}^{m/2}$, then we say that v is holomorphic.

A *singularity* a of v is a point where all of its components vanish, i.e., $f_i(a) = 0$ for all $i = 1, \dots, m$. The singularity is *isolated* if at every point x near a there is at least one component of v which is not zero.

The Poincaré-Hopf index of a vector field at an isolated singularity is its most basic invariant and it has many interesting properties. To define it, let v be a continuous vector field on U with an isolated singularity at a, and let $\mathbb{S}_\varepsilon$ be a small sphere in U around a. Then the (local) Poincaré-Hopf index of v at a, here denoted $\mathrm{Ind}_{\mathrm{PH}}(v, a)$, is the degree of the Gauss map $v/\|v\|$ from $\mathbb{S}_\varepsilon$ into the unit sphere in $\mathbb{R}^m$.

If the vector field v is holomorphic in $\mathbb{C}^m$ and $(v_0, \ldots, v_m)$ are its components, one has (see for instance [95, §7]) that its local index at an isolated singular point, say $\underline{0}$, can be computed as an intersection number:

$$\mathrm{Ind}_{\mathrm{PH}}(v, \underline{0}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{m,\underline{0}}}{(v_0, \ldots, v_m)},$$

where $\mathcal{O}_{m,\underline{0}}$ is the local ring of holomorphic functions in $\mathbb{C}^m$ at $\underline{0}$ and $(v_0, \ldots, v_m)$ is the ideal generated by the v_i. We thus arrive to Milnor's theorem [94, Theorem 7.2]: given a holomorphic map with an isolated critical point at $\underline{0}$, $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$, its Milnor number is always positive and it is given by:

$$\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,\underline{0}}}{(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})}. \tag{7.1}$$

We remark that much of the above discussion generalizes to isolated complete intersection singularity germs, by [64] (see also [75]). One has Hamm's theorem:

Theorem 7.2.2 *Let $f : (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, 0)$ be an isolated complex isolated complete intersection germ; set $V = f^{-1}(0)$ and let $\mathbb{B}_\varepsilon \subset \mathbb{C}^{n+k}$ be a Milnor ball for f. Let Δ be the image in $\mathbb{C}^k$ of the set of critical points of f in $\mathbb{B}_\varepsilon$. Then for every $\delta > 0$ sufficiently small with respect to ε, we have a locally trivial fiber bundle:*

$$f : \Big((\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta) \setminus f^{-1}(\Delta)\Big) \to \mathbb{D}_\delta \setminus \Delta,$$

where $\mathbb{D}_\delta$ is the disc in $\mathbb{C}$ of radius $\delta > 0$ and centered at 0. Moreover, the fiber F of this fibration has the homotopy type of a bouquet of spheres of middle dimension: $F \simeq \bigvee_\mu S^n$.

The number μ of spheres in this bouquet is called the Milnor number of f at $\underline{0}$. We refer to [64, 73, 75] for more on this topic.

7.2.2 The Radial and GSV Indices

When working with singular analytic varieties there is no obvious definition of the Poincaré-Hopf local index of vector fields at the singularities of the vector field. There are instead several possible definitions, each having its own properties and interest. We refer to [30] for a thorough account on the subject.

The radial index springs from the work [116] of M.-H. Schwartz in relation with the Chern classes for singular varieties. This index was defined in general by H. King and D. Trotman in [68] (unpublished for a long time) and later, independently, by Ebeling and Gusein-Zade [48] and by Aguilar-Seade-Verjovsky [1].

We shall use the Poincaré-Hopf theorem for manifolds with boundary:

Theorem 7.2.3 *Let M be a compact differentiable m-manifold with boundary ∂M, and let v be a continuous vector field in a neighborhood of ∂M in M, with no singularities. Then:*

- *v can be extended to all of M with finitely many singularities and the total index $Ind_{PH}(v, M)$ of the extension depends only on v and not on the way we extend it.*
- *Furthermore, if we extend v to a vector field $\tilde{v}$ in M with non-isolated singularities, then a generic perturbation turns $\tilde{v}$ into a vector field with isolated singularities and the total index is $Ind_{PH}(v, M)$, independently of the perturbation.*
- *If v is transversal to the boundary ∂M at all points, pointing outward, then $Ind_{PH}(v, M)$ equals $\chi(M)$, the Euler characteristic.*

Now let $Z \subset \mathbb{C}^m$ be a complex analytic variety of complex dimension $n \geq 1$ with an isolated singularity at $\underline{0} \in \mathbb{C}^m$. Let U be an open ball around $\underline{0}$, small enough so that every sphere in U centered at $\underline{0}$ is a Milnor sphere for Z, so it meets Z transversally (see [73, 94]). For simplicity we restrict the discussion to U. By a continuous vector field on Z with an isolated singularity at $\underline{0}$ we mean a continuous section v of $T\mathbb{C}^m|_Z$ which is tangent to $Z^* = Z \setminus \{\underline{0}\}$ and non-singular on Z^*.

Let v_{rad} be a continuous vector field on Z with an isolated singularity at $\underline{0}$, which is transverse (outward-pointing) to all spheres $\mathbb{S}_\varepsilon \subset U$ around $\underline{0}$. We call v_{rad} a *radial vector field* at $\underline{0} \in Z$. By definition *the radial index* of v_{rad} is 1, which is the Euler characteristic of a point. Now let v be another continuous vector field on Z with an isolated singularity at $\underline{0}$. To define the difference between v and v_{rad} at $\underline{0}$ consider small spheres $\mathbb{S}_\varepsilon, \mathbb{S}_{\varepsilon'}$; $\varepsilon > \varepsilon' > 0$, and let w be a vector field on the cylinder $V_{\varepsilon\varepsilon'}$ in Z bounded by the links $K_\varepsilon = \mathbb{S}_\varepsilon \cap Z$ and $K_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap Z$, such that w has finitely many singularities in the interior of Z, it restricts to v on K_ε and to v_{rad} on $K_{\varepsilon'}$. The *difference* of v and v_{rad} is defined as the total Poincaré-Hopf index of w on Z:

$$d(v, v_{\text{rad}}) = \text{Ind}_{\text{PH}}(w, Z)\,.$$

Definition 7.2.4 (cf. [1, 48, 68, 118]) *The radial (or Schwartz) index* of v at $\underline{0} \in Z$ is:

$$\text{Ind}_{\text{rad}}(v, \underline{0}; Z) = 1 + d(v, v_{\text{rad}})\,.$$

The GSV-index of vector fields was introduced in [58, 117] and then generalized to various settings by different authors; the name was coined in [74]. This index is related with the extensions of the vector field to the nearby Milnor fibers of the function defining a hypersurface germ, and one of its basic properties is its stability under small perturbations of both, the function and the vector field in question (Fig. 7.1).

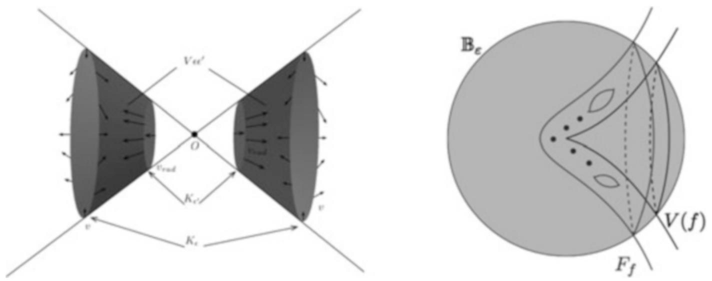

Fig. 7.1 The radial and GSV indices: the first is given by the difference with the radial vector field; the second is given by the zeros in a Milnor fiber

Let us denote by $(V, \underline{0})$ a complex analytic hypersurface in $\mathbb{C}^{n+1}$ defined by a holomorphic function

$$f : (\mathbb{C}^{n+1}, \underline{0}) \longrightarrow (\mathbb{C}, 0),$$

which may be defined only on a small ball $\mathbb{B}_\varepsilon$ around $\underline{0}$, where it has a unique critical point. Let v be a tangent vector field on V. If $n = 1$ we further assume for simplicity that V is irreducible (cf. [30, Remark 3.2.2]). Since 0 is an isolated singularity of V, it follows that the (complex conjugate) gradient vector field

$$\overline{\nabla} f := \Big(\overline{\frac{\partial f}{\partial z_0}}, \dots, \overline{\frac{\partial f}{\partial z_n}}\Big),$$

is normal to V^* for the usual Hermitian metric in $\mathbb{C}^{n+1}$. The set $\{v(x), \overline{\nabla} f(x)\}$ is a 2-frame at each point in $V^* := V \setminus \{\underline{0}\}$, and up to homotopy it can be assumed to be orthonormal. Hence this 2-frame defines a continuous function $\Phi : V^* \to W_2(n+1)$ into the Stiefel manifold of complex orthonormal 2-frames in $\mathbb{C}^{n+1}$.

Let $K = V \cap \mathbb{S}_\varepsilon$ be the link of $\underline{0}$ in V. It is an oriented, real manifold of dimension $(2n-1)$. The restriction of the above map Φ to K defines a continuous map

$$\phi_v = (v, \overline{\nabla} f) : K \longrightarrow W_2(n+1).$$

The Stiefel manifold $W_2(n+1)$ is $(2n-2)$-connected and its homology in dimension $(2n-1)$ is isomorphic to $\mathbb{Z}$. Hence ϕ_v has a degree $\deg(\phi_v) \in \mathbb{Z}$, defined by means of the induced homomorphism $H_{2n-1}(K) \to H_{2n-1}(W_2(n+1))$ in the usual way.

Definition 7.2.5 The GSV index of v at $0 \in V$, $\mathrm{Ind}_{\mathrm{GSV}}(v, 0)$, is the degree of map ϕ_v.

The definition of this index extends easily to vector fields on complex isolated complete intersection singularities (ICIS for short). We refer to [30] for a whole account on this topic. Notice that since V is a closed subspace of $\mathbb{C}^{n+1}$ and a vector field v tangent to V can be regarded as a continuous section of the tangent bundle $T\mathbb{C}^{n+1}|_V$, then v can always be extended (in infinitely many ways) to a vector field in a ball in $\mathbb{C}^{n+1}$ around $\underline{0}$, with an isolated singularity at $\underline{0}$. Furthermore, one can show that v can also be extended to the ambient space being tangent to all Milnor fibers, but for this we may need to create more singularities. In fact one has:

Proposition 7.2.6 *Let $\tilde{v}$ be a continuous extension of v to a Milnor fiber F_f. Then the GSV-index of v equals the total Poincaré-Hopf index of $\tilde{v}$ in F_f.*

One gets:

Theorem 7.2.7 *Let $f\colon (\mathbb{C}^{n+1}, \underline{0}) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic germ with an isolated singularity at $\underline{0} \in V = f^{-1}(0)$ and μ the Milnor number of f at $\underline{0}$. Let v be a tangent vector field on V, singular only at $\underline{0}$. Then:*

$$\mu = (-1)^{n+1}\big(Ind_{rad}(v, 0) - Ind_{GSV}(v, 0)\big)\,,$$

independently of the choice of the vector field.

The proof is easy. The first step consists in observing that by the definition of the radial index, together with Theorem 7.2.3 and Proposition 7.2.6, one has that the difference $\mathrm{Ind}_{\mathrm{rad}}(v, 0) - \mathrm{Ind}_{\mathrm{GSV}}(v, 0)$ is independent of the vector field. So to prove Theorem 7.2.7 it is enough to show that the formula holds for one vector field. We choose $v = v_{rad}$, a radial vector field. Then $\mathrm{Ind}_{\mathrm{rad}}(v, 0) = 1$ by definition. On the other hand Theorem 7.2.3 and Proposition 7.2.6 imply that $\mathrm{Ind}_{\mathrm{GSV}}(v, 0)$ equals the Euler characteristic $\chi(F_f)$ of the Milnor fiber. From [95] one has $\chi(F_f) = 1 + (-1)^n \mu$ and the result follows.

Notice that Theorem 7.2.7 and its proof extend easily to ICIS (cf. for instance [30]).

Remark 7.2.8 (The Virtual Index) For vector fields on a compact variety V defined by a regular section of a holomorphic vector bundle N over a complex manifold M, one also has the virtual index of vector fields. This was introduced in [74] for holomorphic vector fields and extended in [118] to continuous vector fields. The virtual index is discussed later in 7.4.10 and it is defined for vector fields on compact varieties with arbitrary singular set. At an isolated singularity of both the variety V and the vector field, the virtual and the GSV indices coincide.

7.2.3 The Homological Index

The homological index was introduced by Gómez-Mont [57]. This reminisces the classical interpretation of the Poincaré-Hopf index of holomorphic vector fields as the Euler-characteristic of a Koszul complex [62]. The homological index is defined for holomorphic vector fields on complex analytic normal isolated singularity germs, and when the germ is a complete intersection, this equals the GSV-index, by [31, 57].

Let $(V, \underline{0}) \subset (\mathbb{C}^m, \underline{0})$ be a germ of a normal complex analytic variety of pure dimension n, which is regular on $V \setminus \{\underline{0}\}$. A holomorphic vector field v on $(V, \underline{0})$ is the restriction to V of a holomorphic vector field $\widehat{v}$ in the ambient space which is tangent to $V \setminus \{\underline{0}\}$. So we may write $v = (a_1, \ldots, a_m)$ where the a_i are restrictions to V of holomorphic functions on a neighborhood of $\underline{0}$ in $(\mathbb{C}^m, \underline{0})$.

A (germ of a) holomorphic j-form on V at $\underline{0}$ means the restriction to V of a holomorphic j-form on a neighborhood of $\underline{0}$ in $\mathbb{C}^m$; two such forms in $\mathbb{C}^m$ are equivalent if their restrictions to V coincide on a neighborhood of $\underline{0} \in V$. We denote by $\Omega^j_{V,\underline{0}}$ the space of germs of all such forms; these are the Kähler differential forms on V at $\underline{0}$. So $\Omega^0_{V,\underline{0}}$ is the local structure ring $\mathcal{O}_{(V,\underline{0})}$ of holomorphic functions on V at $\underline{0}$ and each $\Omega^j_{V,\underline{0}}$ is an $\Omega^0_{V,\underline{0}}$-module. Notice that if the germ of V at $\underline{0}$ is determined by $(f_1, \ldots, f_k)$ then one has:

$$\Omega^j_{V,\underline{0}} := \frac{\Omega^j_{\mathbb{C}^m,\underline{0}}}{f_1\Omega^j_{\mathbb{C}^m,\underline{0}} + df_1 \wedge \Omega^{j-1}_{\mathbb{C}^m,\underline{0}}, \ldots, f_k\Omega^j_{\mathbb{C}^m,\underline{0}} + df_k \wedge \Omega^{j-1}_{\mathbb{C}^m,\underline{0}}}, \tag{7.2}$$

where d is the exterior derivative.

Now, given a holomorphic vector field $\widehat{v}$ at $\underline{0} \in \mathbb{C}^m$ with an isolated singularity at the origin, and a differential form $\omega \in \Omega^j_{\mathbb{C}^m,\underline{0}}$, we can always contract ω by v in the usual way. We get a differential form $i_v(\omega) \in \Omega^{j-1}_{\mathbb{C}^m,\underline{0}}$. Notice that contraction is well defined at the level of forms on V and one gets a complex $(\Omega^\bullet_{V,\underline{0}}, v)$:

$$0 \longrightarrow \Omega^n_{V,\underline{0}} \longrightarrow \Omega^{n-1}_{V,\underline{0}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{V,\underline{0}} \longrightarrow 0, \tag{7.3}$$

where the arrows are contraction by v and n is the dimension of V. We consider the homology groups of this complex:

$$H_j(\Omega^\bullet_{V,\underline{0}}, v) = \mathrm{Ker}\,(\Omega^j_{V,\underline{0}} \to \Omega^{j-1}_{V,\underline{0}})/\mathrm{Im}\,(\Omega^{j+1}_{V,\underline{0}} \to \Omega^j_{V,\underline{0}}).$$

The first observation in [57] is that if V is regular at $\underline{0}$, so that its germ at $\underline{0}$ is that of $\mathbb{C}^n$ at the origin, and if $v = (a_1, \ldots, a_n)$ has an isolated singularity at $\underline{0}$, then this is the usual Koszul complex. In that case, its homology groups vanish for $j > 0$, while

$$H_0(\Omega^{\bullet}_{V,\underline{0}}, v) \cong \mathcal{O}_{\mathbb{C}^n,\underline{0}} \big/ (a_1, \ldots, a_n)\,,$$

so its dimension is the Poincaré-Hopf local index. The above complex is exact if $v(\underline{0}) \neq 0$. Since the contraction maps are $\mathcal{O}_{V,\underline{0}}$-module maps, this implies that if V has an isolated singularity at the origin, then the homology groups of this complex are concentrated at $\underline{0}$, and they are finite dimensional because the sheaves of Kähler differentials on V are coherent. Hence it makes sense to define:

Definition 7.2.9 The *homological index* $\mathrm{Ind}_{\mathrm{hom}}(v, \underline{0}; V)$ of the holomorphic vector field v on $(V, \underline{0})$ is the Euler characteristic of the above complex:

$$\mathrm{Ind}_{\mathrm{hom}}(v, \underline{0}; V) = \sum_{i=0}^{n} (-1)^i h_i(\Omega^{\bullet}_{V,\underline{0}}, v)\,,$$

where $h_i(\Omega^{\bullet}_{V,\underline{0}}, v)$ is the dimension of the corresponding vector space over $\mathbb{C}$.

One has the following theorem from [31, 57]:

Theorem 7.2.10 *If the germ at* $\underline{0}$ *of the singular variety* V *is a hypersurface germ (or, more generally, an ICIS), then the homological and the GSV indices coincide for holomorphic vector fields.*

The proof uses that the homological index satisfies a law of conservation of the number under small perturbations. It is easy to see that the radial index also has that property and therefore the difference

$$\mathrm{Ind}_{\mathrm{rad}}(v, \underline{0}; V) \;-\; \mathrm{Ind}_{\mathrm{Hom}}(v, \underline{0}; V) \;:=\; \nu(V, \underline{0})\,,$$

is a constant that does not depend on the choice of vector field. If the germ $(V, \underline{0})$ is a hypersurface germ or an ICIS, then Theorem 7.2.7 implies that this difference $\nu(V, \underline{0})$ is the Milnor number up to sign. One has:

Question What is the constant $\nu(V, \underline{0})$ when the germ $(V, \underline{0})$ is not an ICIS?

In some sense this constant plays the role of a Milnor number for singularities which are not complete intersections (cf. [52]).

Now suppose Z is a compact analytic variety of dimension n with isolated singularities $q_1, \ldots, q_r$, and let v_i, $i = 1, \ldots, r$, be a holomorphic vector field on an open neighborhood $\widetilde{U}_i$ of each q_i, singular only at q_i. For each i, let U_i be an open neighborhood of q_i such that its closure is contained in the interior of $\widetilde{U}_i$. Then $Z^* := Z \setminus \bigcup_{i=1}^{r} U_i$ is a compact smooth manifold with boundary the union of the links of the q_i, and we have vector fields v_i defined on a neighborhood of the boundary. By Theorem 7.2.3 this extends to a $\mathbb{C}^\infty$ vector field v on all of Z

with isolated singularities at the q_i and perhaps at some points p_j in the regular part of Z. Then one has a well-defined total homological index $\mathrm{Ind}_{\mathrm{hom}}(v, Z)$: this is the sum of the homological indices of v at the q_i plus its Poincaré-Hopf index at the singularities of v in the regular part of Z. Similarly one has a total radial index $\mathrm{Ind}_{\mathrm{rad}}(v, Z)$ and a total GSV-index. It is an exercise to check that these numbers depend only on Z and not on the choice of the vector fields v_i nor their extensions to Z, so we may denote these simply by $\mathrm{Ind}_{\mathrm{rad}}(Z)$, $\mathrm{Ind}_{\mathrm{hom}}(Z)$. The following result is well-known (see [1, 48, 118]).

Theorem 7.2.11 *Let Z be a compact complex analytic variety with isolated singularities $q_1, \ldots, q_r$ in a complex manifold M, and let v be a continuous vector field on Z, singular at the q_i and possibly at some other isolated points in Z. Then:*

- *The total radial index is the Euler characteristic of Z: $Ind_{rad}(Z) = \chi(Z)$. Furthermore, this equals the 0-degree Chern-Schwartz-McPherson class of Z.*
- *If the singularities of Z are ICIS, then the total GSV index is the Euler characteristic of a smoothing $Z^\#$ of Z: $Ind_{GSV}(Z) = \chi(Z^\#)$, where $Z^\#$ is the compact differentiable manifold obtained from $Z^* := Z \setminus \bigcup_{i=1}^r U_i$ by attaching to it a Milnor fiber of each q_i. Furthermore, this equals the 0-degree Fulton class of Z.*

In the sequel we shall explain what the Chern-Schwartz-McPherson, Fulton and Milnor classes are.

This rises natural questions:

(i) What is the total homological index $\mathrm{Ind}_{\mathrm{hom}}(Z)$?, or perhaps one can ask: is the total homological index $\mathrm{Ind}_{\mathrm{hom}}(Z)$ the 0-degree Fulton class? (this is so when the singularities of Z are all ICIS). Or equivalently,
(ii) What is the difference $\mathrm{Ind}_{\mathrm{rad}}(Z) - \mathrm{Ind}_{\mathrm{hom}}(Z)$?; is this the 0-degree Milnor class?

7.2.4 The Local Euler Obstruction

The local Euler obstruction was introduced by R. MacPherson in [81] as an essential ingredient for the construction of characteristic classes of singular complex algebraic varieties; that will be explained later. An interpretation of the local Euler obstruction was given in [24] by J.-P. Brasselet and M.-H. Schwartz using vector fields. This was later extended in [28] to an obstruction for functions and vector fields on singular varieties, thus bringing the local Euler obstruction into the framework of "indices of vector fields on singular varieties"; the classical Euler obstruction corresponds to the square of the function distance to the singular point. An important analytic interpretation of the classical Euler obstruction was given in [59] by G. Gonzalez-Springer. In [105] C. Sabbah introduces a local Euler obstruction $\check{\mathrm{E}}\mathrm{u}_V(0)$ using the dual Nash bundle (see also [107]). Then Ebeling and Gusein-Zade defined in [49] the Euler obstruction for 1-forms in general; they also extended the concept to collections of 1-forms (see [50]). Also note that the Euler obstruction for functions defined in [28] was extended in [63] to maps with values in $\mathbb{C}^k$, $k > 1$.

We now recall the definition in [24] of the local Euler obstruction. Let $(V, 0)$ be a reduced, pure-dimensional complex analytic singularity germ of dimension n in an open set $U \subset \mathbb{C}^m$. Let $G(n, m)$ denote the Grassmannian of complex n-planes in $\mathbb{C}^m$. On the regular part V_{reg} of V there is a map $\sigma : V_{\text{reg}} \to U \times G(n, m)$ defined by $\sigma(x) = (x, T_x(V_{\text{reg}}))$. The *Nash transformation* $\widetilde{V}$ of V is the closure of $\text{Im}(\sigma)$ in $U \times G(n, m)$. It is a complex analytic space endowed with an analytic projection map

$$\nu : \widetilde{V} \longrightarrow V$$

which is a biholomorphism away from $\nu^{-1}(\text{Sing}(V))$. Now consider the tautological bundle over $G(n, m)$ and denote by $\mathcal{T}$ the corresponding product extension bundle over $U \times G(n, m)$. We denote by π the projection map of this bundle and let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{V}$, with projection map π.

In this section we only look at germs of analytic sets; yet, in the sequel we shall consider compact complex analytic varieties. We notice that given such a variety X, its Nash transform $\widetilde{X}$ is defined in the obvious way, that springs from the local definition. Similarly one has a bundle $\widetilde{T}$ over $\widetilde{X}$ defined as above.

Definition 7.2.12 The bundle $\widetilde{T}$ over the Nash transform $\widetilde{X}$ of X is called the Nash bundle of X (both, in the local and global cases).

Given V as before, an element of $\widetilde{T}$ is written (x, P, v) where $x \in U$, P is an n-plane in $\mathbb{C}^m$ based at x and v is a vector in P. So we have maps:

$$\widetilde{T} \xrightarrow{\pi} \widetilde{V} \xrightarrow{\nu} V\,.$$

Let us consider a complex analytic Whitney stratification (V_α) of V (see for instance [60, 128]). Adding the stratum $U \setminus V$ we obtain a Whitney stratification of U. Let us denote by $TU|_V$ the restriction to V of the tangent bundle of U. A *stratified vector field* v on V means a continuous section of $TU|_V$ such that if $x \in V_\alpha \cap V$ then $v(x) \in T_x(V_\alpha)$. By Whitney condition (a) one has the following lemma in [24]:

Lemma 7.2.13 *Every stratified vector field v on a subset $A \subset V$ has a canonical lifting to a section $\widetilde{v}$ of the Nash bundle $\widetilde{T}$ over $\nu^{-1}(A) \subset \widetilde{V}$.*

The following definition from [28, 30] extends the interpretation of the Euler obstruction given in [24] as we explain below.

Definition 7.2.14 Consider a stratified vector field $v(x)$ in a neighborhood of $\{0\}$ in V. Let $\mathbb{S}_\varepsilon$ be a small sphere centered at 0 and let $\widetilde{v}$ be the lifting of v on $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$ to a section of the Nash bundle given by Lemma 7.2.13. The *Euler obstruction of* v *at* 0, denoted $\mathrm{Eu}_V(v, 0)$, is defined to be the obstruction to extending $\widetilde{v}$ as a nowhere zero section of $\widetilde{T}$ over $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$.

That is: let $O(\widetilde{v}) \in H^{2d}\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon)\big)$ be the obstruction cocycle to extending $\widetilde{v}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$. The local Euler obstruction $\mathrm{Eu}_V(0)$ is the integer defined as the evaluation of the cocycle $O(\widetilde{v})$ on the fundamental class of the pair $\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon)\big)$.

Now recall that the stratified vector field $v(x)$ is radial in a neighborhood of $\{0\}$ in V if there is ε_0 such that for every $0 < \varepsilon \leq \varepsilon_0$, $v(x)$ is pointing outwards the ball $\mathbb{B}_\varepsilon$ over the boundary $\mathbb{S}_\varepsilon := \partial \mathbb{B}_\varepsilon$.

Definition 7.2.15 Assume the stratified radial vector field v is radial at 0. Then the *Euler obstruction* of the space V at 0, denoted $\mathrm{Eu}_V(0)$, is the Euler obstruction of v:

$$\mathrm{Eu}_V(0) := \mathrm{Eu}_V(v, 0)\,.$$

We know from [81] (see also [72]) that $\mathrm{Eu}_V(x)$ is a constructible function on V which is constant along the strata of every complex analytic Whitney stratification.

In [46] and [67] one finds the idea of studying the Euler obstruction *à la* Lefschetz, using hyperplane sections. This is also done, differently, by Lê and Teissier in [72] and in [27, 28]. We now say a a few words about this.

We start with the following lemma, which is a special case of well-known results about Lefschetz pencils. Let us denote by $\mathcal{L}$ the space of complex linear forms on $\mathbb{C}^m$. Fix a Whitney stratification of V. There are a finite number of strata of this Whitney stratification which contain 0 in their closure, and we assume that the representative of $(V, 0)$ is chosen small enough so that these are the only strata of V.

Lemma 7.2.16 ([27]) *There exists a non-empty Zariski open set Ω in $\mathcal{L}$ such that for every $l \in \Omega$, there exists a representative V of $(V, 0)$ so that:*

(1) for each $x \in V$, the hyperplane $l^{-1}(0)$ is transverse in $\mathbb{C}^m$ to every limit of tangent spaces in TV_{reg} of points in V_{reg} converging to x,
(2) for each y in the closure $\overline{V}_\alpha$ in V of each strata V_α, the hyperplane $l^{-1}(0)$ is transverse in $\mathbb{C}^m$ to every limit of tangent spaces in TV_α of points converging to y.

Then we can state the following Theorem:

Theorem 7.2.17 ([27]) *Let $(V, 0)$ be a germ of an equidimensional complex analytic space in $\mathbb{C}^m$. Let V_α, $\alpha = 1, \ldots, \ell$, be the (connected) strata of a Whitney* *stratification of a small representative V of $(V,0)$ such that 0 is in the closure of every stratum. Then for each $l \in \Omega$ as in 7.2.16 there is ε_0 such that for any ε, $\varepsilon_0 > \varepsilon > 0$ and $t_0 \neq 0$ sufficiently small, we have the following formula for the Euler obstruction of $(V,0)$:*

$$Eu_V(0) = \sum_{\alpha=1}^{\ell} \chi(V_\alpha \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)) \cdot Eu_V(V_\alpha),$$

where χ denotes the Euler-Poincaré characteristic and $Eu_V(V_\alpha)$ is the value of the Euler obstruction of V at any point of V_α, $\alpha = 1, \dots, \ell$.

The following corollary of Theorem 7.2.17 is due to Dubson [46] and provides an important relation between the Euler obstruction of an ICIS germ and the Milnor number of a general hyperplane section:

Corollary 7.2.18 *Assume that the germ $(V,0)$ is an ICIS of dimension n and let $\mu(V \cap H)$ be the Milnor number of a general hyperplane section. Then*

$$Eu_V(0) = (-1)^n \mu(V \cap H)\,.$$

Remark 7.2.19 In [28] the authors replace the general linear form l in theorem 7.2.17 by an arbitrary function on V. For this they use the invariant of vector fields in Definition 7.2.14 to introduce the Euler obstruction of a function f with an isolated singularity on a germ $V \subset \mathbb{C}^m$ by projecting to the Whitney strata of V its gradient vector field. In [28] there is also an alternative viewpoint to define the Euler obstruction of functions using derived categories; this has the advantage of working equally well for functions with non-isolated singularities. The difference between the Euler obstruction of the space and that of the function is called the Brasselet invariant in [47] and there is a vast literature on that topic. In [119] it is shown that the Euler obstruction of a function is a natural generalization of the Milnor number to the case of functions on singular spaces. The Euler obstruction of functions was elegantly extended in [49] to an invariant for 1-forms; this viewpoint is more natural than the original definition for functions and it allowed itself to a generalization for collections of 1-forms (see [50]). In [47] there is an extension of the Euler obstruction of functions to maps. We refer to [30, 51] for thorough discussions on indices of vector fields and 1-forms.

7.3 Chern Classes for Singular Varieties

Chern classes of vector bundles play a central role in geometry and topology. In the case of (almost) complex manifolds, by definition their Chern classes are those of its tangent bundle. When looking at singular varieties, the point is what plays the role of the tangent bundle at the singular set. There are several candidates, as for instance (in the sequel we say more about each of these):

- One may consider a singular variety X embedded in a complex manifold M equipped with a Whitney stratification adapted to X and consider stratified vector fields. This leads to the Chern-Schwartz classes.
- One has the Nash bundle $\widetilde{T}$ that somehow extends over the singular set the tangent bundle of the regular part of X. This leads to the Mather classes. And considering the Mather classes with "appropriate weights" given by the local Euler obstruction one arrives to the MacPherson classes. These satisfy the important functoriality properties predicted by a conjecture of Deligne and Grothendieck.
- If X is defined by a regular section of a holomorphic bundle E over M, then one has its virtual tangent bundle of X, $TX := TM|_X - E|_X$ and its total Chern class is determined by the Chern classes of $TM|_X$ and $E|_X$. This leads to the Fulton and the Fulton-Johnson classes of X. These classes actually are defined for every complex analytic variety in a complex manifold by means of the Segre class.

So there are different notions of Chern classes extending to singular varieties the classical notion for complex manifolds; as noticed in [30] each of these is also related with one of the indices of vector fields mentioned previously.

In this section we recall the classical Chern classes of vector bundles using the viewpoint of algebraic topology. Then we look at singular varieties.

7.3.1 Chern Classes of Vector Bundles

There are several alternative ways to define the Chern classes; see for instance [30, 55, 95, 124] for accounts on the subject. As a motivation we define first the Euler class of a manifold using the Poincaré-Hopf index: the paradigm to follow. Consider a real m-dimensional compact smooth oriented manifold M and a vector field v on M regarded as a section of its tangent bundle TM. Let $x_1, \dots, x_r$ be the singularities of v. We use this information to construct from it a canonical cohomology class Eu(M) $\in H^m(M; \mathbb{Z})$, called the *Euler class* of M, whose Poincaré dual is the cycle represented by the points $x_1, \dots, x_r$ weighted by their local Poincaré-Hopf index; so we get the Euler characteristic $\chi(M)$, by Poincaré-Hopf's theorem. We remark that the cohomology class Eu(M) is independent of v, but the cochain we construct to represent it does depend on the choice of v.

Let (K) be a triangulation of M such that the singularities of v are vertices (i.e., they are in the 0-skeleton). Now take the barycentric sub-division of (K), denote it $(\widehat{K})$. We use this to construct the dual cell decomposition of (K) that we denote (D_K): to each simplex σ in (K) we associate a cell $d(\sigma)$ which is the union of all simplexes in $(\widehat{K})$ whose closure meets σ exactly at its barycenter $\hat{\sigma}$. For a vertex $x_i \in K^{(0)}$ its dual cell has dimension m and it is the union of all simplexes in $(\widehat{K})$ that have x_i in its closure. Now define an m-cochain as follows: to each m-cell in

(D_K) which is dual to a singularity of v we associate its local Poincaré-Hopf index; to all other m-cells we associate 0, and we extend this to m-chains by linearity. We get a cochain with integer coefficients, which actually is a cocycle [124]. By definition this is the Euler class of M. Clearly this class is the Poincaré dual of the Euler characteristic regarded as an element in $H_0(M;\mathbb{Z}) \cong \mathbb{Z}$.

Now we define the Chern classes of a smooth complex vector bundle E over a compact simplicial complex K of real dimension n. We remark that everything we say in this context works similarly if we replace K by a CW-complex with a cell decomposition. We assume the complex dimension of the fibers of E is $k \leq m/2$ (this condition is not necessary; we refer to [124] for the general case).

Definition 7.3.1 An *r-field* for E, $r \leq k$, on a subcomplex L of K is a set $v^{(r)} = \{v_1, \dots, v_r\}$ of r continuous sections of E defined at all points in L. A singular point of $v^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A non-singular r-field is also called an *r-frame*.

The Chern class $c^q(E) \in H^{2q}(K)$, where $q = k - r + 1$, is the first possibly non-zero obstruction for constructing an r-frame of E. Let us explain this. Let $W_{r,k}$ be the Stiefel manifold of complex unitary r-frames in $\mathbb{C}^k$. Notice that we will use r-frames which are not necessarily unitary, but this does not change the results, because every frame is homotopic to a unitary one. We know (see [124]) that $W_{r,k}$ is $(2k-2r)$-connected and its first non-zero homotopy group is $\pi_{2k-2r+1}(W_{r,k}) \cong \mathbb{Z}$. The bundle of r-frames on E, denoted by $W_r(E)$, is the bundle associated with E whose fiber over $x \in K$ is the set of all r-frames in the fiber E_x over x (it is diffeomorphic to $W_{r,k}$). In the following, we fix the notation $q = k - r + 1$.

We use the standard stepwise process in obstruction theory to construct this class, similarly to the way we constructed the Euler class of a manifold. Recall that a map $X \to Y$ between topological spaces extends to the cone of X if and only if it is nulhomotopic; and a p-cell σ is homeomorphic to the cone over $\partial\sigma$.

Let σ be a p-cell in K. If the section $v^{(r)}$ of $W_r(E)$ is already defined over its boundary $\partial\sigma$, it defines a map :

$$\partial\sigma \simeq \mathbb{S}^{p-1} \xrightarrow{v^{(r)}} W_r(E)|_U \simeq U \times W_{r,k} \xrightarrow{pr_2} W_{r,k},$$

thus an element of $\pi_{p-1}(W_{r,k})$. If $p \leq 2k - 2r + 1$, this homotopy group is zero and therefore the section $v^{(r)}$ can be extended to σ without singularity. This means that we can always construct a section $v^{(r)}$ of $W_r(E)$ over the $(2q-1)$-skeleton $K^{(2q-1)}$.

If $p = 2(k - r + 1) = 2q$, we meet a possible obstruction. The r-frame on the boundary of each $2q$-cell σ defines an element, denoted by $\mathrm{ind}(v^{(r)}, \sigma)$, in the homotopy group $\pi_{2q-1}(W_{r,m}) \cong \mathbb{Z}$. The integer $\mathrm{ind}(v^{(r)}, \sigma)$ is *the (Poincaré-Hopf) index* of the r-frame $v^{(r)}$ on the cell σ. Similarly to the above case of the Euler class, this defines a cochain

$$\gamma \in C^{2q}(K; \pi_{2q-1}(W_{r,m}))\,,$$

by setting $\gamma(\sigma) = \text{ind}(v^{(r)}, \sigma)$ for each $2q$-cell σ and then by extending it linearly. This cochain is actually a cocycle [124]:

Definition 7.3.2 The cohomology class so obtained is the q-th *Chern class of the bundle* E, $c^q(E) \in H^{2q}(K; \mathbb{Z})$.

The class one gets in this way is independent of the various choices involved in its definition. Note that if K is a complex manifold M and E is its tangent bundle TM, then the top Chern class $c^m(M)$ coincides with the Euler class of the underlying real tangent bundle $T_{\mathbb{R}}M$, so Chern classes are a natural generalization of the Euler class. That is, $c^m(M)$ is the primary obstruction to construct a never-zero tangent vector field on M, where primary means the first possibly non-zero obstruction. Then $c^{m-1}(M)$ is the primary obstruction to constructing two linearly independent tangent vector fields on M and so on.

Assume now that we are given an r-frame $v^{(r)}$ on the $2q$-skeleton of a subcomplex L of K, denoted by $L^{(2q)}$. The same arguments as before say that we can always extend $v^{(r)}$ without singularity to $L^{(2q)} \cup K^{(2q-1)}$. If we wish to extend this frame to the $2q$-skeleton of K we meet an obstruction for each corresponding cell which is not in L. This gives rise to a cochain which vanishes on L and is a cocycle in $H^{2q}(K, L)$.

Definition 7.3.3 The *relative Chern class*

$$c^q(E; v^{(r)}) \in H^{2q}(K, L)\,,$$

is the class represented by the previous cocycle.

The image of $c^q(E; v^{(r)})$ by the natural map into $H^{2q}(K)$ is the usual Chern class, but as a relative class it does depend on the choice of the frame $v^{(r)}$ on L.

Example 7.3.4 Consider a continuous vector field v in $\mathbb{C}^m$ with an isolated singularity at a point x and let B be a compact $2m$-ball in $\mathbb{C}^m$ of positive radius, centered at x, containing no other singularities of v. Let TB be the restriction to B of the tangent bundle $T\mathbb{C}^m$. Since B is contractible, all the Chern classes of TB vanish. Yet, we have the top Chern class of TB relative v:

$$c^q(B, \partial B; v) \in H^{2m}(B, \partial B)\,.$$

We have the Lefschetz duality isomorphism $H^{2m}(B, \partial B) \cong H_0(B) \cong \mathbb{Z}$, and the integer we get, the dual of $c^q(M, L; v)$, is exactly the local Poincaré-Hopf index of v.

The relative Chern classes are useful in various settings including the study of Chern classes of singular varieties.

7.3.2 Schwartz Classes

The first generalization of Chern classes to singular varieties is due to M.-H. Schwartz [116]. These classes are the primary obstructions for constructing stratified frames on a singular variety V in a complex manifold (cf. [30]). Let us recall this.

We consider a compact complex analytic n-dimensional variety V embedded in a complex m-manifold M endowed with a complex analytic Whitney stratification $\{V_\alpha\}$ adapted to V. As in Sect. 7.3.1, we use a cellular decomposition (D) dual to a triangulation of M compatible with the stratification. The cells σ of (D) are transverse to the strata V_α.

Definition 7.3.5 Let L be a subspace of M which is a union of strata. A *stratified* r-field (or frame) $v^{(r)} = \{v_1, \dots, v_r\}$ on L is an r-field (or frame) on M, defined at the points in L, consisting of stratified vector fields.

A basic ingredient in the work of M. H. Schwartz is what she called "radial extension". The idea is simple though there are technical difficulties that we shall omit. See [30] for a more detailed exposition of this construction. First we describe the local process, then we say a few words about the global process.

Let v_α be a vector field in a neighborhood of a point $x \in V_\alpha$ with possibly a singularity at x. By the local topological triviality of Whitney stratifications (see [60, 126]), there is a product neighborhood $W \cong \varDelta \times U_\alpha$ of x in the ambient space, where U_α is a neighborhood of x in V_α, $\varDelta$ is a small disc in the ambient manifold, transversal to V_α at x and $V \cap W$ is a product $(\varDelta \cap V) \times U_\alpha$. We may assume that x is the only one possible singularity of v_α in U_α. Denoting by $p_1 : W \to \varDelta$ and $p_2 : W \to U_\alpha$ the projections on the two factors of the product, we have a decomposition

$$TW = p_1^* T\varDelta \oplus p_2^* TU_\alpha.$$

On the one hand, the pull-back $p_2^* v_\alpha$ is a continuous vector field on W, which is "parallel" to v_α. It is stratified, since it is tangent to the fibers of p_1. On the other hand, let $\varDelta$ be equipped with the induced stratification and let $v_\varDelta$ be a stratified vector field on $\varDelta$, which is singular at x and it is radial in the usual sense (i.e., pointing outwards in all directions). Then $p_1^* v_\varDelta$ is a stratified vector field on W since it is tangent to the fibers of p_2 and $v_\varDelta$ is stratified. It is thus radial in each slice $\varDelta \times \{q\}$ for q in U_α. The local radial extension of v_α in W is the following:

Definition 7.3.6 The local radial extension of v_α, denoted by v, is the stratified vector field defined on the neighborhood W as the sum:

$$v = p_1^* v_\varDelta + p_2^* v_\alpha.$$

A fundamental property of the local radial extension is that v has no singularity along the boundary of W, it is pointing outward W along its boundary, and if v_α has a singularity at x with index $\mathrm{Ind}_{\mathrm{PH}}(v_\alpha, x; V_\alpha)$, then the local radial extension v of v_α admits x as unique singular point in W, and one has

$$\mathrm{Ind}_{\mathrm{PH}}(v, x; W) = \mathrm{Ind}_{\mathrm{PH}}(v_\alpha, x; V_\alpha). \tag{7.4}$$

The local radial extension allows to define the global radial extension. For this we filter V by the dimension of the strata as follows:

$$V = \overline{V}_{\mathrm{reg}} = \overline{V}_n \supset \overline{V}_{n-2} \supset \cdots \supset \overline{V}_{\alpha_j} \supset \cdots \supset \overline{V}_{\alpha_2} \supset \overline{V}_{\alpha_1} \supset V_{\alpha_0}$$

where V_{α_j} are the (not necessarily connected) strata and V_{α_0} is the lowest dimensional stratum. The radial extension is defined by induction on the dimension of the strata, starting with V_{α_0}. In the first step one considers a vector field v_{α_0} with isolated singularities on V_{α_0}, which is compact. One performs the local radial extension around V_{α_0} in a tube $\mathcal{T}(V_{\alpha_0})$, union of neighborhoods W as above (see [20] for the construction of these tubes). The vector field v is pointing outward $\mathcal{T}(V_{\alpha_0})$ along its boundary and the singularities of v in $\mathcal{T}(V_{\alpha_0})$ are exactly those of v_{α_0} in V_{α_0}. The vector field v extends to the next element in the above filtration since the V_α are complex manifolds. We iterate this process and we arrive to the following theorem of M. H. Schwartz (see [30] for details):

Theorem 7.3.7 *Let V be a complex analytic variety in a complex manifold M, and let $(V_\alpha)_{\alpha\in A}$ be a complex analytic Whitney stratification of M adapted to V. Then there exist stratified vector fields on a neighborhood of V in M constructed by radial extension as above, and every such vector field v satisfies:*

(1) *Given any stratum (V_α), the total Poincaré-Hopf index of v on $\mathcal{T}(\overline{V}_\alpha)$ is $\chi(\overline{V}_\alpha)$.*
(2) *v is transverse, outwards pointing, to the boundary of every small regular neighborhood of V in M.*
(3) *The Poincaré-Hopf index of v at each singularity x is the same if we regard v as a vector field on the stratum that contains x or as a vector field in a neighborhood of x in M. Hence the total Schwartz (or radial) index of v on V is $\chi(V)$, as stated in Theorem 7.2.11.*

Now we are ready to define the Schwartz classes of the compact complex analytic singular variety V in a complex manifold M. Let n, m be the dimensions of V and M, respectively. We endow M with a Whitney stratification adapted to V and consider a triangulation (K) of M compatible with the stratification. We denote by (D) a cellular decomposition of M dual to (K). Recall that if a $2q$-cell d_α of (D) meets V, then it intersects V transversally. To define the Schwartz classes one considers particular stratified r-frames v^r. A key-step is:

Theorem 7.3.8 *Let n, m be, respectively, the complex dimensions of V and M, and we equip M with a complex analytic Whitney stratification adapted to V, a triangulation K for which every stratum is a union of simplexes, and its dual cell decomposition (D). Then, for every $r = 1, \ldots, n$, there exist stratified r-fields $v^{(r)}$ on the skeleton $(D)^{(q)}$, $q = m - r + 1$, such that:*

- *Every vector field in it is constructed by radial extension and* $v^{(r)}$ *has only isolated singularities on* $(D)^{(2m-2r+2)}$*;*
- *If we write* $v^{(r)} = (v^{(r-1)}, v_r)$*, where* $v^{(r-1)}$ *denotes the* $(r-1)$*-field consisting of the first* $(r-1)$ *vector fields in* $v^{(r)}$*, then* $v^{(r-1)}$ *is non-singular on* $(D)^{(2q)}$ *and the singularities of* $v^{(r)}$ *are the singularities of the last vector field* v_r*.*
- *The r-field* $v^{(r)}$ *extends to a neighborhood* U *of* $(D)^{(2q)}$ *in* M*, with no other singularities, and being everywhere transversal to the boundary of* U*.*

Now let U be a regular neighborhood of X in M. Notice that U, being an open set, is itself a complex manifold, so it has its own Chern classes. Recall that by definition, $c^q(U) \in H^{2q}(U)$ is the class of the obstruction cocyle obtained by trying to extend to the $2q$-skeleton of U an r-frame $v^{(r)}$ defined on the $(2q-1)$-skeleton (for some cell decomposition); $r = 2m - 2q + 1$. Assume $v^{(r)}$ can be extended to a non-singular frame on all $2q$-cells in $U \setminus X$. Then the corresponding obstruction cocycle vanishes on $U \setminus X$ and actually represents a class in $H^{2q}(U, U \setminus X)$. This is the relative Chern class $c^q(U, U \setminus X; v^{(r)})$ in Definition 7.3.3. While the usual Chern classes are independent of all choices, the relative ones do depend on the choice of the frame $v^{(r)}$ away from X, as noticed in Example 7.3.4.

In general, we have the exact sequence:

$$\cdots \longrightarrow H^{2q-1}(U \setminus V) \longrightarrow H^{2q}(U, U \setminus V) \longrightarrow H^{2q}(U) \longrightarrow \cdots \tag{7.5}$$

The natural morphism carries $c^q(U, U \setminus V; v^{(r)})$ into the usual Chern class $c^q(U)$. That is, the relative Chern class $c^q(U, U \setminus V; v^{(r)})$ is the specific lifting of $c^q(U)$ from $H^{2q}(U)$ to $H^{2q}(U, U \setminus V)$ determined by the choice of the frame $v^{(r)}$.

The various possible choices of such liftings correspond to the image of $H^{2q-1}(U \setminus V)$ in $H^{2q}(U, U \setminus V)$ in the above exact sequence (7.5).

Definition 7.3.9 The *Chern-Schwartz class*, or simply the *Schwartz class*, $c_q^{Sc}(V)$ of V is the relative Chern class of U in $H^{2q}(U, U \setminus V) \cong H^{2q}(M, M \setminus V)$ determined by an r-frame as in Theorem 7.3.8: defined on the $(2n-2r+1)$-skeleton of V in a cell decomposition of U which is dual to a triangulation of U adapted to V, and then extended radially to U, with singularities only on V. *The total Schwartz class* is $c_*^{Sc}(V) = 1 + c_1^{Sc}(V) + \ldots + c_n^{Sc}(V)$.

It is known that the classes so obtained depend only on V and not on the choice of the manifold M nor on the embedding of V in M.

Notice that usual Chern classes are defined using arbitrary frames and here we are using stratified frames obtained by radial extension, which is the original way of defining these classes. Yet, we know from [30] that one can use arbitrary stratified frames, and that will be essential for what follows. The key point is defining an appropriate index, a way of counting the contribution of each singularity of an arbitrary stratified field. Let us recall this.

We have the following definitions 10.1.3 and 10.1.4 from [30]:

Definition 7.3.10 We say that $v^{(r)}$ is *normally radial* at a_σ if for each stratum V_β having a_σ in its closure and for each sufficiently small tube $\mathcal{T}_\varepsilon(V_\alpha)$ around V_α in M, one has that each component $v_1, \dots, v_r$ of $v^{(r)}$ is transverse (pointing outwards) to the intersection $\overline{V}_\beta \cap \mathcal{T}_\varepsilon(V_\alpha)$. We say that $v^{(r)}$ is actually *radial* at a_σ if it is normally radial and it is also radial in its stratum.

So the framings constructed by radial extension are normally radial but they may not be radial.

We need to define the local Schwartz index for arbitrary (stratified) frames; this is similar to the definition of the radial index in (7.2.4). Let $v^{(r)}$ be an r-frame defined on the boundary of a (D)-cell σ of dimension $2m - 2r + 2$, whose barycenter is a point $a_\sigma \in V_\alpha \subset V$. We extend $v^{(r)}$ to a stratified frame on all of $\sigma \setminus \{a_\sigma\}$. Recall that, by construction, the cell σ meets transversally all the Whitney strata V_β containing V_α in their closure. Let $v_{\text{rad}}^{(r)}$ be a stratified radial frame around a_σ. We define the difference between $v^{(r)}$ and $v_{\text{rad}}^{(r)}$ at a_σ as follows. Consider sufficiently small spheres $\mathbb{S}_\varepsilon, \mathbb{S}_{\varepsilon'}$ in M, $\varepsilon > \varepsilon' > 0$, centered at a_σ, and consider the frame $v^{(r)}$ on $\mathbb{S}_\varepsilon \cap \sigma \cap V$ and $v_{\text{rad}}^{(r)}$ on $\mathbb{S}_{\varepsilon'} \cap \sigma \cap V$. We use again the Schwartz's technique of radial extension to get a stratified r-frame $w^{(r)}$ on the intersection of σ with the cylinder

$$X = [(V \cap \mathbb{B}_\varepsilon) \setminus (V \cap \mathring{\mathbb{B}}_{\varepsilon'})]$$

in V bounded by $K_\varepsilon = \mathbb{S}_\varepsilon \cap V$ and $K_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap V$, having finitely many singularities in the interior of X. At each of these singular points its index in the stratum, $\mathrm{Ind}_{\mathrm{PH}}(w^{(r)}, X \cap \sigma)$, equals its index in the ambient space $\mathbb{C}^m$. The *difference* of $v^{(r)}$ and $v_{\text{rad}}^{(r)}$ is defined as:

$$d(v^{(r)}, v_{\text{rad}}^{(r)}) = \sum \mathrm{Ind}_{\mathrm{PH}}(w^{(r)}, X \cap \sigma)\,,$$

where the sum on the right runs over the singular points of $w^{(r)}$ in X and each singularity is being counted with the local index of $w^{(r)}$ in the corresponding stratum. As in the work of M.-H. Schwartz, we can check that this integer does not depend on the choice of $w^{(r)}$.

Definition 7.3.11 *The Schwartz* (radial) *index* of a stratified r-field $v^{(r)}$ at $a_\sigma \in V$ is:

$$\mathrm{Ind}_{\mathrm{Sch}}\big(v^{(r)}, a_\sigma; V\big) \;=\; 1 + d(v^{(r)}, v_{\text{rad}}^{(r)})\ .$$

As before, a stratified r-frame $v^{(r)}$, $r \geq 1$, which is non-singular on $(D)^{(2m-2r+1)}$ and has isolated singularities on $(D)^{2m-2r+2}$, defines a cochain in the obvious way, and this cochain is actually a cocycle. One obtains a relative class

$$c^q(U, \partial U; v^{(r)}) \in H^{2q}(U, U_V \setminus V) \cong H^{2q}(M, M \setminus V)\,, \tag{7.6}$$

where U is a regular neighborhood of V in M. One has [30, Theorem 2.14]:

Theorem 7.3.12 *Given $V \subset M$ as before, equipped with a Whitney stratification adapted to V and a compatible triangulation (K), let (D) be its dual cellular decomposition and denote $(D)^j$ the union of all cells of dimension j. If $v^{(r)}$ is a stratified r-frame, $r \geq 1$, which is non-singular on $(D)^{(2m-2r+1)}$ and has isolated singularities on $(D)^{2m-2r+2}$, then the Schwartz indices of $v^{(r)}$, defined as in 7.3.11, determine a cocycle $c^q(V; v^{(r)}) \in H^{2q}(M, M \setminus V)$, $q = 2m - 2r + 2$, and this cocycle represents the corresponding Schwartz class of V, independently of the choice of the frame $v^{(r)}$.*

The proof is immediate from the definitions and properties of Schwartz index.

Remark 7.3.13 In short, this theorem is telling us that the Schwartz class $c^q(V)$ of a singular variety V of dimension n in a complex manifold M is the primary obstruction for constructing a stratified n-frame of $TM|_V$. Unlike the classical case, now the cell decomposition must be dual to a triangulation of V compatible with a Whitney stratification adapted to V.

Remark 7.3.14 (Localization of Schwartz Classes) The Poincaré-Hopf index theorem tells us that a choice of a vector field v on a complex manifold M localizes the top Chern class at the singularities of v. Similarly, following [30, 10.5], we have that an appropriate frame $v^{(r)}$ on a singular variety V, localizes the corresponding Schwartz class at the connected components of the singular set V_{sing} of V and at the possible singularities of the frame $v^{(r)}$ contained in the regular part $V_{\text{reg}} := V \setminus V_{\text{sing}}$. In fact, let $S_1, \ldots, S_\ell$ be the connected components of V_{sing} and U_i a regular neighborhood of each S_i. Let $v^{(r)}$ be an r-frame on the $2m - 2r + 1$ skeleton of the dual decomposition (D) with no singularities, and with isolated singularities in $(D)^{2m-2r+2}$ that are all contained either in V_{sing} or in $V^* := V \setminus (U_1 \cup \ldots \cup U_\ell)$. For instance the frames constructed by radial extension satisfy this condition. Let $\mathbb{L}_i$ be an open regular neighborhood of the boundary of each U_i. Then the above construction yields a cocycle in $H^{2m-2r+2}\big(M, M \setminus \big(V \setminus (\mathbb{L}_1 \cup \ldots \cup \mathbb{L}_\ell)\big)\big) \cong H_{2r-2}\big(V \setminus (\mathbb{L}_1 \cup \ldots \cup \mathbb{L}_\ell\big)$. This decomposes the Schwartz class into a part contained in V^*, which is the usual Chern class relative to the frame, and another part localized at the connected components of V_{sing}. If $v^{(r)}$ is obtained by radial extension, the contribution at each U_i corresponds to the Schwartz class of S_i (see [30, Theorem 10.5.2] for details).

7.3.3 MacPherson's Theory

In his paper [121] in the famous 1969 Liverpool singularities symposium, D. P. Sullivan discusses the existence of homology Stiefel classes for real analytic varieties. In the last page he explains that Deligne outlined a general conjectural theory of Chern classes for singular varieties based on ideas of Grothendieck and Hironaka's theorem about resolution of singularities. Nowadays this is known as the Deligne-Grothendieck conjecture, and it was proved by MacPherson in [81] by a different way. Let us say a few words about this.

A *constructible set* in a complex analytic variety X is a set obtained from its subvarieties by finitely many of the usual set-theoretic operations: unions, intersections and differences. A $\mathbb{Z}$-valued constructible function on X is a function $\phi : X \to \mathbb{Z}$ for which X has a finite partition into constructible sets so that ϕ is constant on each set. Or equivalently, there exists a complex analytic Whitney stratification of X such that ϕ is constant on each stratum. One has [81, Proposition 1]:

Proposition 7.3.15 *There is a unique covariant functor* **F** *from the category* $\mathcal{V}$ *of compact complex algebraic varieties to the category of abelian groups* $\mathcal{A}b$*, whose value on a variety* X *is the group of constructible functions on* X *and whose value* f_* *on a map* f *satisfies:*

$$f_*(\mathbb{1}_W)(p) := \chi(f^{-1}(p) \cap W)\,,$$

where $\mathbb{1}_W$ *is the characteristic function of* W*, defined by* $\mathbb{1}_W(x) = 1$ *for* $x \in W$ *and* $\mathbb{1}_W(x) = 0$ *for* $x \notin W$*, and* χ *denotes the usual (topological) Euler characteristic.*

MacPherson then proves the Deligne-Grothendieck conjecture:

Theorem 7.3.16 *There exists a natural transformation from the functor* **F** *to homology, which for manifolds assigns to the constant function* $\mathbb{1}$ *the Poincaré dual of the total Chern class. Explicitly, to any constructible function* α *on a compact complex algebraic variety* X *we can assign an element* $c_*(\alpha)$ *in* $H_*(X)$ *satisfying:*

1. $f_*c_*(\alpha) = c_*f_*(\alpha)$;
2. $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$;
3. $c_*(1) =$ *dual of* $c(X)$ *if* X *is non-singular, where* $c(X)$ *is the total Chern class.*

Definition 7.3.17 *The total Chern-MacPherson class* $c_*^{MP}(X)$ of any compact variety X is c_* applied to the constant function $\mathbb{1}$ on X. More generally, for a constructible function α on X, the homology class $c_*(\alpha)$ is *the total Chern-MacPherson class of the constructible function*. For simplicity, we shall often call this the MacPherson class of the constructible function α; and $c_*(\mathbb{1}_X)$ is the MacPherson class of X.

MacPherson's proof of Theorem 7.3.16 uses three important ingredients; one of these is the local Euler obstruction Eu_X of an algebraic variety, already defined in Sect. 7.2.4; another are the Mather classes that we now introduce; the third is the so-called graph construction in the algebraic context. We remark that the analyticity of the graph construction was proved by M. Kwieciński [70] and therefore MacPherson's theorem and proof work in the complex analytic category.

Let X be a complex analytic variety of dimension n, let $\widetilde{X} \xrightarrow{\nu} X$ be its Nash transformation, defined locally as in Sect. 7.2.4, and let $\widetilde{T} \to \widetilde{X}$ be the Nash bundle. Then one has the usual Chern classes of $\widetilde{T}$ defined as above, $c_i(\widetilde{T}) \in H^*(\widetilde{X})$. The variety $\widetilde{X}$ is singular in general, but since it is complex analytic, it is automatically a pseudomanifold (see for instance [20]) and therefore one has an Alexander homomorphism $H^*(\widetilde{X}) \to H_{2n-*}(\widetilde{X})$. Composing this with the homomorphism in homology induced by the projection ν, we get classes in the homology of X: these are the Mather classes, introduced in [81] (MacPherson said that these were explained to him by Mather):

Definition 7.3.18 *The Mather classes* of X, $c_i^{Ma}(X)$, are the Chern classes of the Nash bundle of X, carried to the homology of the Nash transform $\widetilde{X}$ by the Alexander homomorphism, and then pushed forward to the homology of X by the homomorphism induced by the projection. The total Mather class is $c^{Ma}(X) = \nu_*(\text{dual } c(\widetilde{T}))$. More generally, to any algebraic cycle $\sum n_i V_i$ in X, where the n_i are integers and the V_i are irreducible subvarieties of X, we can associate its *Mather class*:

$$c^{Ma}\Big(\sum n_i V_i\Big) = \sum n_i \iota_{i*} c^{Ma}(V_i)\ ,$$

where ι_i is the inclusion of V_i in X.

MacPherson's next step is writing a formula that expresses $c_*(\alpha)$ as the Mather class of an associated algebraic cycle. For this he proves [81, Lemma 2]:

Lemma 7.3.19 *There exists an isomorphism T from the group of algebraic cycles in X to the group of constructible functions on X defined by:*

$$T\Big(\sum n_i V_i\Big)(p) = \sum n_i Eu_p V_i\ ,$$

where Eu is the local Euler obstruction.

Then MacPherson proves [81, Theorem 2]:

Theorem 7.3.20 $c_* := c^{Ma} T^{-1}$ *satisfies the requirements for c_* in Theorem 7.3.16.*

Then $c^{Ma}T^{-1}(\mathbb{1}_X)$ is the (total) MacPherson class of X that we denote by $c_*^{MP}(X)$. Notice that one actually has a total MacPherson class $c^{MP}(\alpha)$ for every constructible function on X, and we know from [81] that one has:

$$c^{Ma}(X) = c^{MP}(Eu_X)\,, \tag{7.7}$$

where Eu_X is the local Euler obstruction, which is constructible.

Recall that in the previous section we defined the Schwartz classes of a singular analytic variety X of dimension n embedded in a complex manifold M of dimension m. Brasselet and Schwartz proved in [24] that these classes coincide with MacPherson's classes. In fact the theorem in [24] makes this statement precise and gives an explicit cycle representing the MacPherson class. Let us recall this.

We endow M with a Whitney stratification adapted to X and consider a triangulation (K) of M compatible with the stratification. We denote by (D) a cellular decomposition of M dual to (K). Recall that if a $2q$-cell d_α of (D) meets X, then it is dual to a $2(m-q)$-simplex σ_α of (K) contained in X. We recall too that to define the Schwartz classes one considers particular stratified r-frames v^r. These have no singularity on the $(2q-1)$-skeleton of (D), where $q = m - r + 1$, and (at most) isolated singularities on the $2q$-cells d_α. At each such cell, the frame v^r has a Poincaré-Hopf type index at the corresponding singularity $\widehat{\sigma}_\alpha$, that we may denote $I(v^r, \widehat{\sigma}_\alpha)$; of course this index is 0 if there is no singularity of v^r in that simplex. Then we have the following theorem of Brasselet and Schwartz:

Theorem 7.3.21 *The Alexander duality isomorphism $H^*(M, M \setminus X) \to H_*(X)$ carries the Schwartz class $c_*^{Sc}(X) \in H^*(M, M \setminus X)$ to the Chern-MacPherson class $c_*^{MP}(X) \in H_*(X)$. In fact, the MacPherson class $c_{r-1}^{MP}(X)$ is represented in $H_{2(r-1)}(X)$ by the cycle:*

$$\sum_{\sigma_\alpha \subset X} I(v^r, \widehat{\sigma}_\alpha) \cdot \sigma_\alpha \ ,$$

where the sum runs over all the simplexes σ_α of dimension $2(r-1)$ which are contained in X, and $I(v^r, \widehat{\sigma}_\alpha)$ is the (Poincaré-Hopf) index in the dual cell of each such simplex σ_α of a stratified vector field v^r constructed by radial extension.

Hence, from now on we denote the classes so obtained in homology by $c_*^{SM}(X)$ and call them the *Chern-Schwartz-MacPherson classes of X*, or simply *Schwartz-MacPherson classes*.

Remark 7.3.22 Brylinski, Dubson and Kashiwara [32] showed that the MacPherson classes of a singular variety can be studied by means D-modules. In fact the micro-local viewpoint, through the theory of Lagrangian cycles, has proved to be very important and fruitful to study these characteristic classes (see for instance [105, 107]).

7.3.4 Segre and Fulton Classes

When we consider holomorphic vector bundles E over algebraic varieties X, the total Segre class is inverse to the total Chern class, and thus provides equivalent information. The Segre classes were introduced in the non-singular case by B. Segre

[120]; these are defined as operators in the Chow ring, $A_kX \to A_{k-1}X$, that satisfy certain properties and they have the advantage that generalize to settings (specifically to cones) where Chern classes are not defined. We refer to the literature for more on the subject, particularly to Fulton's book [55] and various papers by P. Aluffi.

We need to recall first several basic concepts from algebraic geometry.

Let X be an algebraic variety (over $\mathbb{C}$) of dimension n; X and its subvarieties here are always assumed to be reduced and irreducible. Let $R(X)$ be its field of rational functions and $R(X)^*$ the multiplicative subgroup of its non-zero elements. Denote by $A = \mathcal{O}_{V,X}$ the local ring of X along a subvariety V.

For a $(k+1)$-dimensional subvariety W and a function $r \in R(W)^*$, the divisor of r is the k-cycle $[\mathrm{div}(r)]$ on X defined by:

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V]\,,$$

where the sum runs over all codimension one subvarieties of W and ord_V is the order of vanishing of r. If we write $r = a/b$ with $a, b \in A$, then $\mathrm{ord}_V(r) = \mathrm{ord}_V(a) - \mathrm{ord}_V(b)$.

A *k-cycle* on X is a finite formal sum $\sum n_i[V_i]$ where the n_i are integers and the V_i are k-dimensional subvarieties of X. The group of k cycles in X, Z_kX, is the free abelian group generated by the k-dimensional subvarieties of X; to a subvariety V of X corresponds $[V] \in Z_kX$. A *Weil divisor* on X is an $(n-1)$-cycle on X; these form the group $Z_{n-1}X$.

A *k-cycle* α is rationally equivalent to zero, written $\alpha \sim 0$, if it is the divisor of a rational function. That is, if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X, and $r_i \in R(W_i)^*$ such that:

$$\alpha = \sum [\mathrm{div}(r_i)]\,.$$

The cycles rationally equivalent to zero form a subgroup Rat_kX of Z_kX. *The Chow group* A_kX is the group of k-cycles in X modulo rational equivalence:

$$A_kX \;=\; Z_kX/\mathrm{Rat}_kX\,.$$

Consider now a line bundle L over an algebraic variety X. For any k-dimensional subvariety V of X, the restriction of L to V, $L|_V$, is isomorphic to $\mathcal{O}_V(C)$ for some (Cartier) divisor C on V, determined up to linear equivalence [55, §2.2]. The divisor $[C]$ determines an element in the Chow group $A_{k-1}(V)$, which we denote by $c_1(L)\cap[V]$. That is:

$$c_1(L) \cap [V] = [C]\,.$$

This is extended by linearity to algebraic cycles by $\alpha \mapsto c_1(L) \cap \alpha$, and defines a homomorphism:

$$c_1 \cap -\colon Z_k(X) \to A_{k-1}(X)\,.$$

In fact one has (see [55, 2.5.(a)]) that if α is rationally equivalent to zero on X, then $c_1(L) \cap \alpha = 0$. Hence one has a well-defined homomorphism:

$$c_1 \cap -\colon A_k(X) \to A_{k-1}(X)\,.$$

This defines the *Chern class of the line bundle* L. In fact if V is non-singular, c_1 is the usual Chern class, defined before by other means, regarded in homology via cap product with the fundamental cycle.

Remark 7.3.23 The Chern class so defined satisfies various important properties (see [55, Proposition 2.5]), in particular:

1. (Commutativity) If L, L' are line bundles on X, and α is in $Z_k(X)$, then, one has in $A_{k-2}(X)$:

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha).$$

2. (Additivity) If L, L' are line bundles on X, and α is in $Z_k(X)$, then, in $A_{k-1}(X)$ one has:

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha\,,$$

and

$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha\,,$$

where $L^\vee$ is the dual bundle.

It follows that if $L_1, \dots, L_n$ are line bundles on X, then arbitrary polynomials in their Chern classes act on A_*X. If P is a homogeneous polynomial of degree d in n variables, then

$$P\big(c_1(L_1), \dots, c_1(L_n)\big) \cap \alpha$$

is defined inductively in $A_{k-d}(X)$. In particular, and this will be used in the sequel, for a line bundle L on X and $\alpha \in A_k(X)$, $c_1(L)^d \cap \alpha$ is an element in $A_{k-d}(X)$ defined inductively by $c_1(L)^d \cap \alpha = c_1(L) \cap \big(c_1(L)^{d-1} \cap \alpha\big)$.

Now recall that if $f : X \to Y$ is a proper morphism of algebraic varieties, then for any subvariety V of X, its image $f(V)$ is a closed subvariety of Y, and one has an induced embedding of the field of rational functions $R(f(V))$ into $R(V)$. As noticed in [55, Appendix B.2.2], this is a finite field extension if V and $f(V)$ have the same dimension; in this case we denote by $[R(V) : R(f(V)]$ the degree of that field extension. Set:

$$\deg(V/f(V)) = \begin{cases} [R(V) : R(f(V)] & \text{if } \dim V = \dim f(V) \\ 0 & \text{if } \dim V > \dim f(V) \end{cases} .$$

We then define the push-forward of V by f as:

$$f_*[V] = \deg(V/f(V))[f(V)] .$$

This extends linearly to the *push-forward* homomorphism of cycles (see for instance [55, 1.4]):

$$f_* = Z_k X \to Z_k Y .$$

Now recall that a homomorphism $A \to B$ of rings is *flat* if every exact sequence of A-modules remains exact after tensoring over A with B. And a morphism $f : X \to Y$ between algebraic varieties is *flat* if for every $p \in X$ the induced map in the local rings

$$f_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p} ,$$

is flat. Flatness is an open generic condition, and its failure occurs where the map exhibits a type of "discontinuity". For instance, performing a blow up at a point exhibits a fiber where the dimension "jumps" and we have no flatness there. A flat morphism $f : X \to Y$ always has a relative fiber dimension, say n. In fact if Y is non-singular and X is Cohen-Macaulay, then flatness is equivalent to saying that the fibers have constant dimension.

Given any subvariety V of Y, set:

$$f^*[V] = [f^{-1}(V)] .$$

Notice that $f^{-1}(V)$ is a subvariety of X of pure dimension $\dim Y + n$. This extends by linearity to the *pull-back homomorphism* of cycles:

$$f^* : Z_k Y \to Z_{k+n} X .$$

We now have all the ingredients we need to define the Segre classes, and therefore Chern classes, which are their inverses.

Let $E \stackrel{p}{\to} X$ be a holomorphic bundle of rank (fiber dimension) r over an algebraic variety X. Let $P = P(E)$ be the projective bundle of lines in E, and let $\mathcal{O}(1) = \mathcal{O}_E(1)$ be the canonical line bundle on $P(E)$. For each $i = 1, \dots, r$, define a homomorphism in the Chow ring of X by:

$$s_i(E) \cap - : A_k X \to A_{k-i} X , k \geq i ,$$

by the formula

$$s_i(E) \cap \alpha = p_*\big(c_1(\mathcal{O}(1)^{r+i}) \cap p^*\alpha\big)\,,$$

where p^* is the flat pull back from $A_k X$ to $A_{k+r} P$, $(c_1(\mathcal{O}(1)^{r+i}) \cap p^*\alpha)$ is the iterated first Chern class homomorphism fro $A_{k+r} P$ to $A_{k-i} P$, and p_* is the push-forward from $A_{k-i} P$ to $A_{k-i} X$.

These are, by definition, *the Segre classes* of E. The $s_i(E)$ are endomorphisms of the Chow ring $A_* X$, with products being compositions that commute, so there is no ambiguity.

If we write the total Segre class of the bundle E over X as:

$$s(E) = 1 + s_1(E) + s_2(E) + \ldots + s_r(E)\,,$$

then the total Chern class

$$c(E) = 1 + c_1(E) + c_2(E) + \ldots + c_r(E)$$

is its inverse in the Chow ring. One has:

$$c_1(E) = -s_1(E), \quad c_2(E) = s_1(E)^2 - s_2(E), \; \ldots \;,$$

$$c_r(E) = -s_1(E)c_{r-1}(E) - s_2(E)c_{r-2}(E) - \ldots - s_r(E)\,.$$

Remark 7.3.24 (Mather and MacPherson's Classes) In the previous section we defined the Mather and MacPherson's classes of singular varieties X as elements in the homology of X. We remark however that the above construction of Chern classes of vector bundles as the inverse of the Segre classes, shows that if X is algebraic, then the Mather and the MacPherson classes actually live in the Chow ring of X.

As mentioned earlier, Segre classes extend to the more general setting of (algebraic) cones over an algebraic variety (or scheme). This includes several familiar examples, including all vector bundles. And it also includes many other important families. One of these is the normal cone $C = C_X Y$ of a closed subvariety X in a variety Y. Let us say a few words about this.

As a motivation, recall first that in algebraic geometry one studies algebraic sets, i.e., subsets of K^n, where K is an algebraically closed field, that here we take to be the complex numbers $K = \mathbb{C}$. The algebraic sets are by definition the common zeros of a set of polynomials in n variables. If X is such an algebraic set, one considers the commutative ring R of all polynomial functions $X \to \mathbb{C}$. Since $K = \mathbb{C}$ is algebraically closed, the maximal ideals of R correspond to the points of X, and the prime ideals of R correspond to the irreducible subvarieties of X.

Let us now forget this information for a moment and consider an arbitrary commutative ring R, and define its spectrum, denoted $\mathrm{Spec}(R)$, to be the set of all prime ideals. For any ideal I of R, define V_I to be the set of all prime ideals that contain I, and we equip Spec(R) with the Zariski topology by defining the closed sets to be

$$\{V_I \mid I \text{ is an ideal of } R\}.$$

Coming back to the previous example where R is the ring of polynomial functions $X \to \mathbb{C}$, the spectrum of R consists of the points of X together with elements corresponding to all subvarieties of X. The points of X are closed in the spectrum, while the elements corresponding to subvarieties of positive dimension have a closure consisting of all their points and subvarieties.

Therefore the topological space Spec(R) somehow is a refinement of the algebraic space X with its Zariski topology. By studying spectra of rings instead of algebraic sets, one can generalize concepts of algebraic geometry to non-algebraically closed fields and beyond, eventually arriving to the concept of schemes, due to A. Grothendieck.

There is a relative version of this concept (actually a functor) called the *relative or global spectrum*. If X is an algebraic variety and we are given a quasi-coherent sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras, there is a scheme $\mathrm{Spec}_X(\mathcal{A})$ and a morphism $f : \mathrm{Spec}_X(\mathcal{A}) \to X$ satisfying certain important properties. This allows us, among other things, to define key concepts for this presentation: *The normal cone and the Segre class of a subvariety X in a variety Y*. The normal cone to X in Y, $C = C_X Y$ is defined by:

$$C = \mathrm{Spec}\Big(\sum_{n=0}^{\infty} \mathfrak{I}^n/\mathfrak{I}^{n+1}\Big)$$

where $\mathfrak{I}$ is the ideal sheaf defining X in Y.

When X and Y are non-singular, this corresponds to the usual normal bundle. More generally, if the embedding of X in Y is regular, the normal cone is the vector bundle on X corresponding to the dual of the sheaf $\mathfrak{I}/\mathfrak{I}^2$, and it is also called the normal bundle of X.

Then one has:

Definition 7.3.25 Let X be a proper subvariety of Y. The *(total) Segre class* of X in Y, denoted $s(X, Y)$, is the Segre class of the normal cone $C_X Y$ in the Chow ring of X:

$$s(X, Y) = s(C_X Y) \in A_* X .$$

In case X is regularly embedded in Y, then the normal cone is a vector bundle and [55, Proposition 4.1] implies that the Segre class $s(X, Y)$ is the cap product of the total inverse Chern class of the normal bundle with $[X]$. That is:

$$s(X, Y) = c(N_X Y)^{-1} \cap [X] . \tag{7.8}$$

The following result [55, Corollary 4.2.2] gives a beautiful and useful characterization of the Segre class. This could be taken as a definition of the Segre class of X in Y with no need of introducing the previous concepts:

Proposition 7.3.26 *Let X be a subvariety of a compact variety Y, and let $\widetilde{Y}$ be the blow-up of Y along X. Let $\widetilde{X} \subset \widetilde{Y}$ be the exceptional divisor and $\eta : \widetilde{X} \to X$ the projection. Then the total Segre class of X in Y is:*

$$s(X, Y) = \sum_{i \geq 0} \eta_* \big(c_1(O(1))^i \cap [\widetilde{X}] \big) \,.$$

We remark that all terms in this formula make sense in the complex analytic category, so we can take this as the definition of the Segre class in that setting.

Observe that if X is a complex submanifold (i.e., non-singular) of a complex manifold M, then one has a C^∞ splitting of the tangent bundle of M restricted to X:

$$TM|_X = TX \oplus N_X M$$

where the latter is the normal bundle. By general properties of Chern classes (see for instance [95]) this implies:

$$c_*(TM|_X) = c_*(TX) \cdot c_*(N_X M)$$

regarded in the cohomology of X. Notice too that $TM|_X, TX$ and NX are all complex vector bundles and in the Grothendieck group $K(X)$ of vector bundles on X we have:

$$[TX] = TM|_X - N_X M \,.$$

Now following Fulton [55, 4.2.6], let X be an algebraic variety embedded in a compact algebraic manifold M, and consider the class:

$$c_*^{Fu}(X) := c(TM|_X) \cap s(X, M) \in A_*(X) \,.$$

This class is independent of the choice of embedding, and if X is a local complete intersection in M, then one has the *virtual tangent bundle* of X:

$$TX := TM|_X - N_X M$$

a well-defined element in the corresponding Grothendieck group $K(X)$, and one has:

$$c_*^{Fu}(X) = c(TM|_X)\, c(N_X M)^{-1} \cap [X] = c(TX) \cap [X] \in A_*(X) \,.$$

Definition 7.3.27 Let X be an n-dimensional algebraic variety embedded in a compact algebraic manifold M. Then the class:

$$c_*^{Fu}(X) := c(TM|_X) \cap s(X, M) \in A_*(X),$$

is called the total *Fulton class* of X.

By the above comments, if X is a local complete intersection in M, this is the cap product of the Chern class of the virtual tangent bundle with $[X]$. By definition:

$$c_*^{Fu}(X) = 1 + c_1^{Fu}(X) + \ldots + c_n^{Fu}(X),$$

with $c_i^{Fu}(X) \in A_i(X)$. The various $c_i^{Fu}(X)$ are called *the Fulton classes* of X.

If X and M are complex analytic, not algebraic, the above definitions hold in the homology of X.

Remark 7.3.28 (Fulton-Johnson) Recall that the Segre class $s(X, M)$ by definition is the Segre class of $C_X M$ and the Fulton class is $c_*^{Fu}(X) := c(TM|_X) \cap s(X, M) \in A_*(X)$. If we let $\mathfrak{I}$ be as before, the ideal sheaf defining X, one has the conormal sheaf of X in M, denoted by $\mathcal{N}_X M = \mathfrak{I}/\mathfrak{I}^2$, then one has the *Fulton-Johnson class* of X, which by definition is:

$$c_*^{FJ}(X) := c(TM|_X) \cap \mathcal{N}_X M \in A_*(X).$$

In case X is regularly embedded in M the Fulton and the Fulton-Johnson classes coincide with the total Chern class of the virtual tangent bundle $TX = TM|_X - N_X M$ capped with the fundamental class $[X]$.

7.3.5 *Topological Interpretation of the Fulton Classes*

J.-L. Verdier in [131] proved that Chern classes behave well under "specialization". In [99] this is used to give a nice interpretation of the Fulton classes under certain conditions. For this we restrict the discussion to the case where X is defined by a regular section of a very ample rank k vector bundle E over a compact (say connected) manifold M of dimension $n > k$. So the Fulton class of X is the Chern class of the virtual bundle $TM|_X - E|_X$ capped with the fundamental cycle $[X]$.

Since the bundle E is very ample, we can approximate the section s that defines X by a family of sections $\{s_t\}$, with t in some open disc Δ in $\mathbb{C}$, centered at 0, such that $s_0 = s$ and s_t is everywhere transversal to the 0-section of E for all $t \neq 0$. Hence the zero sets $\{X_t\}$ of these sections define a flat family of local complete intersections in M, which are non-singular for $t \neq 0$ and degenerate to $X_0 = X$.

Consider a regular neighborhood of X in M, actually a tube $\mathcal{T}(X)$ as in Sect. 7.3.2. Then there is a deformation retract from $\mathcal{T}(X)$ to X and a retraction map inducing an isomorphism $r_* : H_*(\mathcal{T}(X)) \to H_*(X)$. Now choose t_o sufficiently close to $0 \in \mathbb{C}$ so that the manifold X_{t_o} is contained in $\mathcal{T}(X)$; this is possible by the compactness of the X_t. Then there is a homomorphism $\iota_* : H_*(X_{t_o}) \to H_*(\mathcal{T}(X))$ induced by the inclusion and we have a special case of Verdier's specialization morphism [131]:

$$r_* \circ \iota_* : H_*(X_{t_o}) \longrightarrow H_*(X)\,.$$

We have from [131] that the Chern classes of $TM|_{X_t}$ and $E|_{X_t}$, regarded in homology, specialize to those of $TM|_X$ and $E|_X$. That is, setting $\mathcal{S} =: r_* \circ \iota_*$ we have

$$\mathcal{S}\big(c_*(TM|_{X_t})[X_t]\big) = c_*(TM|_X)[X] \quad \text{and} \quad \mathcal{S}\big(c_*(E|_{X_t})[X_t]\big) = c_*(E|_X)[X]\,. \tag{7.9}$$

Hence:

$$\mathcal{S}\Big(c_*(TM|_{X_t})[X_t]\big)\big(c_*(E|_{X_t})[X_t]\big)^{-1}\Big) \;=\; c_*^{Fu}(X)\,.$$

Since each X_t, $t \neq 0$, is non-singular, we have:

$$\Big(c_*(TM|_{X_t})[X_t]\big)\big(c_*(E|_{X_t})[X_t]\big)^{-1}\Big) \;=\; c_*(X_t)\,,$$

and we arrive to the following immediate consequence of Verdier's work (see [101]):

Theorem 7.3.29 *Under the above hypothesis, the total Chern class of the complex manifolds X_t, $t \neq 0$, regarded in homology, specializes to the Fulton class of X.*

7.4 Milnor Classes: The Foundations

So far we have discussed the Schwartz-MacPherson and the Fulton classes of singular varieties. It is natural to ask how these are related, and that is the topic we explore in this section.

Definition 7.4.1 The total Milnor class of X is, up to sign, the difference between the total Schwartz-MacPherson and Fulton classes:

$$\mathcal{M}(X) := (-1)^{\dim X}\left(c^{Fu}(X) - c^{SM}(X)\right). \tag{7.10}$$

This is the sum of the corresponding *Milnor classes* $\mathcal{M}_r(X)$ in all (even) dimensions. Milnor classes are defined globally on X, yet one has (see [5, 26, 101, 122]) that these classes have support in the singular set X_{sing} and therefore they vanish in dimensions higher than that of X_{sing}.

Milnor classes appeared first implicitly in [3, 4] and [99]. The actual name of Milnor classes was coined by various authors at about the same time (see [25, 26, 101, 132]). The genesis of the name comes from the following theorem [118, Theorem 2.4] and its corollary below:

Theorem 7.4.2 *Let X be the zero locus of a regular section s of a holomorphic bundle E of rank $k \geq 1$ over a compact complex manifold M of dimension $n+k$; assume the singular set of X consists of isolated points, say $x_1, \ldots, x_r$. Then the Fulton and the Schwartz-MacPherson classes in $H_0(X)$ differ by the sum of the local Milnor numbers:*

$$c_0^{Fu}(X) \;=\; c_0^{SM}(X) + (-1)^{n-1} \sum_{i=1}^{r} \mu_i \,.$$

The proof of Theorem 7.4.2 is via Chern-Weil theory, using the virtual index of vector fields, which is a localization of the top Fulton class (cf [123]). In Sect. 7.4.4 we give a topological interpretation of this index and a short proof of Theorem 7.4.2. It was first proved by Suwa in [122] that the Milnor classes are localized at the singular set, and therefore the theorem above yields:

Corollary 7.4.3 *With the hypotheses of Theorem 7.4.2, the total Milnor class of X is the sum of the local Milnor numbers:*

$$\mathcal{M}(X) = \sum_{i=1}^{r} \mu_i \,.$$

So Milnor classes are a generalization of the classical Milnor number to compact varieties X with arbitrary singular set. In the sequel we say more about these classes, and we relate them with the Lê cycles, introduced by D. Massey, which provide a generalization of the Milnor number to germs of functions with arbitrary critical set.

7.4.1 First Steps

Aluffi's article [3] began the study of the difference between the Schwartz-MacPherson class $c^{SM}(X)$ and the Fulton class $c^{Fu}(X)$ of a hypersurface X in a complex manifold M. A key ingredient for this is a certain μ-class defined in [2], that springs from a different (related) setting:

Definition 7.4.4 Let Y be the singular scheme of a section of a line bundle $\mathcal{L}$ on a smooth complex algebraic variety M. Let $s(Y, M)$ be the Segre class of Y in M (defined in 7.3.25). Then the μ-class of Y with respect to $\mathcal{L}$ is

$$\mu_{\mathcal{L}}(Y) := c(T^*M \otimes \mathcal{L}) \cap s(Y, M)\,,$$

in the Chow group A_*Y.

This definition depends *a priori* on the choice of M, yet Corollary 1.7 in [2] says that it is actually intrinsic to Y and $\mathcal{L}|_Y$. The name μ-class comes from the fact that, by [2, Proposition 2.1], the degree of its zero-dimensional component equals Parusiński's generalized Milnor number that we explain below, in Sect. 7.4.2.

Aluffi's Theorem I.5 in [4] says:

Theorem 7.4.5

$$c^{SM}(X) \;=\; c(TM) \cap s(X, M) + c(\mathcal{L})^{dim\, M} \cap (\mu_{\mathcal{L}}(Y)^{\vee} \otimes_M \mathcal{L}) \;.$$

This result is reformulated in [4] as follows. Recall that the singular subscheme Y of the hypersurface X in M is locally defined by the partial derivatives of an equation of X. In [4], for every integer $k \geq 0$, Aluffi defines the k-th thickening $X^{(k)}$ of X along Y. To explain this, if $\mathcal{I}_Y$ denotes the ideal of Y and $\mathcal{J}$ is the locally principal ideal of X, then $X^{(k)}$ is the subscheme of M defined by the ideal $\mathcal{J} \cdot \mathcal{I}_Y^k$. One may then consider the class $c^{Fu}(X^{(k)})$ in the Chow group $A_*(X)$. It was observed in [3] that this class is a polynomial in k with coefficients in $A_*(X)$, so it can be formally evaluated on arbitrary k's. It is also clear from the definition that $c^{Fu}(X) = c^{Fu}(X^{(0)})$. Then Theorem 7.4.5 can be stated as [4, Theorem I.2]:

Theorem 7.4.6

$$c^{SM}(X) \;=\; c^{Fu}(X^{(-1)}) \;.$$

Aluffi actually gives in [4] two other important equivalent formulations of these theorems relating Fulton and Schwartz-MacPherson classes.

7.4.2 *The Generalized Milnor Number and Milnor Classes of Hypersurfaces*

It is well-known that for a compact complex hypersurface (or complete intersection) X with only isolated singularities, the sum of the Milnor numbers at the singular points equals (up to a sign) the difference between the topological Euler characteristic of X and that of a smoothing of it, as for instance a manifold given by a non-singular hypersurface linearly equivalent to X (provided such a hypersurface exists). This is just an immediate application of Milnor's work [95] and Poincaré-Hopf's theorem for manifolds with boundary (see for instance [30, Proposition3.4.1]). This led Parusiński to extend the notion of Milnor number to non-isolated hypersurface singularities. We refer to [97] for details on the original definition and to Sect. 7.4.4 below for a generalization of this number for local complete intersections.

We now recall another interesting way to view this invariant, given in [98]. We first call to mind the classical Gauss-Bonnet theorem. This says that if M

is a compact m-dimensional complex manifold with tangent bundle TM, then its topological Euler characteristic can be expressed as:

$$\chi(M) = \int_M \Omega$$

where Ω is an $m - form$ representing the top cohomology Chern class.

As pointed out in [98, Section 5], if $\mathcal{L}$ is a holomorphic line vector bundle over M and s is a section transverse to the zero section, so its zero set Z is a non-singular hypersurface in M and its normal bundle is isomorphic to $\mathcal{L}|_Z$, then the Gauss-Bonnet theorem yields:

$$\chi(Z) = \int_M c_1(\mathcal{L}) \cdot c(M) \cdot c(\mathcal{L})^{-1} ,$$

where $c(\,)$ denotes the total cohomology Chern class. If we now drop the hypothesis of s being transversal to the zero section, then its divisor Z is a hypersurface in M with singular set the points of non-transversality with the zero section. In this setting, Parusiński's generalized Milnor number can be regarded as being (up to sign) the correction term coming from the singular set in the above formula:

$$\mu(Z) := (-1)^n \Big(\chi(Z) - \int_M c_1(\mathcal{L}) \cdot c(M) \cdot c(\mathcal{L})^{-1}\Big) .$$

It is easy to see that if Z has only isolated singularities then the formula above implies that $\mu(Z)$ is the sum of the usual Milnor numbers at the singularities of Z.

In [99] Parusiński and Pragacz study, more generally, the Euler characteristic of degeneracy loci associated with various bundle homomorphisms. Recall that given a holomorphic morphism $\phi : F \to E$ of vector bundles over a (possibly singular) analytic variety X, the r-th degeneracy locus is the set

$$D_r(\phi) = \{x \in X \mid \operatorname{rank} \phi(x) \leq r\} .$$

Several authors have worked out formulas for the Euler characteristic of degeneracy loci in terms of cohomological and numerical invariants under certain assumptions. Recalling that for a singular variety its Euler characteristic is the 0-degree Chern-Schwartz-MacPherson class, in [100, Theorem 2.1] the authors use the theory of Chern classes on singular varieties to compute the image of the whole Chern-Schwartz-MacPherson class of $D_r(\phi)$ in the homology of X of a general morphism.

The functorial approach of MacPherson is especially useful for the purposes of [99]. Then, in [100] the authors ask whether that formula can be extended to a broader family of morphisms, and explore the "simplest case" where the morphism is a nontrivial section s of a line bundle $\mathcal{L}$ and the degeneracy locus is the zero set Z. As explained above, the difference between the Euler characteristic of Z and that of the zero locus of a general section is, up to sign, the generalized Milnor number $\mu(Z)$.

In [100] the authors give a formula for the invariant $\mu(Z)$ in the vein of [99], describing the generalized Milnor number in terms of a local invariants of the singularities of Z and Chern-Schwartz-Macpherson's classes. For this, consider an analytic Whitney stratification $\mathcal{S} = \{S\}$ of Z with connected strata, such that Z_{sing} is union of strata; let γ_S be the function defined on each stratum S as follows. For each $x \in S$, let F_x be a *local Milnor fibre*, and let $\chi(F_x)$ be its Euler characteristic. Then

$$\mu(x; Z) := (-1)^n \, (\chi(F_x) - 1)\,,$$

is the *local Milnor number* of Z at x. This number is constant on each Whitney stratum, so we denote it by μ_S. Then γ_S is defined inductively by:

$$\gamma_S = \mu_S - \sum_{S' \neq S,\ \overline{S'} \supset S} \gamma_{S'}.$$

Then [100, Theorem 4] says:

Theorem 7.4.7

$$\mu(Z) = \sum_{S \in \mathcal{S}} \gamma_S \int_{\overline{S}} \left(c(\mathcal{L}_{|\overline{S}})^{-1} \cap c^{SM}(\overline{S}) \right).$$

Outline of the Proof First consider $\mathcal{L}$ to be very ample. By Bertini's theorem in the version of Verdier (see [129] and [62, p. 137]) there exists a holomorfic section s' of $\mathcal{L}$ whose zero set Z' is transverse to a Whitney stratification $\mathcal{S}$ of Z. Approximate Z by the zero sets Z_t of the sections $s_t = s - ts'$, $t \in \mathbb{C}$.

Let $Y = M \setminus \{x \in M \ ; ||s'(x)|| < \varepsilon\}$. For a sufficiently small t, we can find $\varepsilon > 0$ such that the stratification $\mathcal{S}$ is transverse to ∂Y (see [100, Step 1, p.8]). By [100, Step 2, p. 10] we see that it is possible to construct a system of tubular neighbourhoods $\varGamma_S$ of $S \cap Y$ in Y, for each $S \in \mathcal{S}$ such that:

(a) $G_S = \varGamma_S \setminus \bigcup_{S' \subset \overline{S} \setminus S} \operatorname{int}(\varGamma_{S'})$ is a manifold with corners which (as a stratified set) is transverse to $\mathcal{S}$;
(b) G_S is a locally trivial topological fibration over $\tilde{S} := S \cap G_S$; let π_S be the corresponding projection map;
(c) $\tilde{S}$ is a manifold with corners with the same homotopy type as $S \cap Y$.

Then for $t \neq 0$ small enough, Z_t is transverse to the fibres of π_S. Hence $\pi_S|_{Z_t \cap G_S} : Z_t \cap G_S \to \tilde{S}$ is a locally trivial fibration and its fibre $\tilde{F}_x$ at $x \in \tilde{S}$

is homotopically equivalent to the Milnor fibre F_x by Thom's First Isotopy Lemma. Therefore,

$$\begin{aligned}\mu(Z) &= (-1)^n[\chi(Z) - \chi(Z_t)] = (-1)^n[\chi(Z \cap Y) - \chi(Z_t \cap Y)] \\ &= (-1)^n\left[\sum_{S\in\mathcal{S}}(\chi(\tilde{S}) - \chi(Z_t \cap G_S))\right] = (-1)^n\left[\sum_{S\in\mathcal{S}}(\chi(\tilde{S}) - \chi(\tilde{S})\chi(\tilde{F}_x))\right] \\ &= \sum_{S\in\mathcal{S}} \mu_S \chi(\tilde{S}) = \sum_{S\in\mathcal{S}} \mu_S \chi(S \setminus Z')\ .\end{aligned} \tag{7.11}$$

Using that $\gamma_S = \mu_S - \sum_{S'\neq S,\ \overline{S'}\supset S} \gamma_{S'}$ and, since $s'|_{\overline{S}}$ is a general section of $\mathcal{L}|_{\overline{S}}$,

$$\chi(\overline{S} \cap Z') = \int_{\overline{S}} c(\mathcal{L}|_{\overline{S}})^{-1} c_1(\mathcal{L}|_{\overline{S}}) \cap c^{SM}(\overline{S})$$

we have (see [99] or [100, Lemma 8]),

$$\begin{aligned}\mu(Z) &= \sum_{S\in\mathcal{S}} \gamma_S \chi(\overline{S} \setminus Z') = \sum_{S\in\mathcal{S}} \gamma_S[\chi(\overline{S}) - \chi(\overline{S} \cap Z')] \\ &= \sum_{S\in\mathcal{S}} \gamma_S \left[\int_{\overline{S}} c^{SM}(\overline{S}) - \int_{\overline{S}} c(\mathcal{L}|_{\overline{S}})^{-1} c_1(\mathcal{L}|_{\overline{S}}) \cap c^{SM}(\overline{S})\right] \\ &= \sum_{S\in\mathcal{S}} \gamma_S \int_{\overline{S}} c(\mathcal{L}|_{\overline{S}})^{-1} \cap c^{SM}(\overline{S}).\end{aligned}$$

This proves the theorem when $\mathcal{L}$ is very ample. Otherwise let E be a very ample line bundle on M such that $\mathcal{L} \otimes E$ is also very ample (such a bundle exists since M is projective). Let H be the zero set of a section of E such that H is nonsingular and transverse to $\mathcal{S}$. Then the family $S \setminus H$ (for $S \in \mathcal{S}$), $S \cap H$ (for $S \in \mathcal{S}$) and $H \setminus Z$ defines a Whitney stratification of $Z \cup H$. Let T be the zero set of a general section of $\mathcal{L} \otimes M$ such that T is nonsingular and transverse to the above stratification of $Z \cup H$.

As in obtaining Eq. (7.11), using a construction of a system of tubular neighbourhoods for above stratification,

$$\mu(Z \cup H) = \sum_{S\in\mathcal{S}} \mu_{S\setminus H}\ \chi(S \setminus H \setminus T) + \mu_{H\setminus Z}\ \chi(H \setminus Z) + \sum_{S\in\mathcal{S}} \mu_{S\cap H}\ \chi(S \cap H \setminus T).$$

Using some computations and properties (see [100, Step 2, p.11]), we obtain

$$\begin{aligned}\mu(Z \cup H) = &\sum_{S\in\mathcal{S}} \gamma_S \left[\chi(\overline{S}) - \chi(\overline{S}|E) - \chi(\overline{S}|(\mathcal{L} \otimes E)) - \chi(\overline{S}|E \oplus (\mathcal{L} \otimes E))\right] \\ &- \mu(Z \cap H) - \mu(Z \cap H \cap T) + (-1)^n(\chi(M|\mathcal{L} \oplus E) - \chi(M|\mathcal{L} \oplus E \oplus (\mathcal{L} \otimes E)).\end{aligned}$$

On the other hand, by the definition of $\mu(*)$ and the additivity of Euler characteristic (see [100, Step 3, p.12]), we have

$$\mu(Z) = \mu(Z \cup H) - \mu(Z \cap H) - (-1)^n \left[\chi(M|\mathcal{L}) + \chi(M|E) - \chi(M|\mathcal{L} \oplus E) - \chi(M|\mathcal{L} \otimes E)\right].$$

Then some straightforward computations (see [100, Step 4 and 5, p.13]) yield:

$$\mu(Z) = \sum_{S \in \mathcal{S}} \gamma_S [\chi(\overline{S}) - \chi(\overline{S}|\mathcal{L})], \tag{7.12}$$

proving Theorem 7.4.7. □

Remark 7.4.8 For another proof of formula (7.12), in terms of Deligne's vanishing cycle complexes, see e.g., [83, Section 10.4]. This formula, as well as the result of Theorem 7.4.9, has been generalized to the case of global complete intersections in [85], also in the context of Hodge-theoretic Hirzebruch characteristic classes of [29]. For an important recent application of formula (7.12) to the triangulation problem in computer vision, see [84].

We know, for instance by 7.4.2 and 7.4.15, that the generalized Milnor number can be identified with the 0-degree Milnor class. Yokura conjectured (unpublished; cf. [132]) that Theorem 7.4.7 could be extended to a theorem concerning all Milnor classes. This was proved by Parusiński and Pragacz in [101]:

Theorem 7.4.9 *If M is an n-dimensional compact complex manifold, and Z is a hypersurface in M, then its total Milnor class can be expressed as:*

$$\mathcal{M}(Z) := \sum_{S \in \mathcal{S}} \gamma_S \left(c(\mathcal{L}_{|Z})^{-1} \cap (i_{\overline{S},Z})_* c^{SM}(\overline{S}) \right),$$

where $i_{\overline{S},Z} : \overline{S} \hookrightarrow Z$ is the inclusion.

Outline of the Proof Let $T^\vee M$ be the cotangent bundle of M. For V an (irreducible) subvariety of M, let

$$T^\vee_V M \;=\; \text{closure}\,\{(x,\xi) \in T^\vee M \mid x \in V_{reg}, \xi|_{T_x V_{reg}} \equiv 0\},$$

be the conormal space to V in M. Let $L(M)$ be the free abelian group of all cycles generated by the conormal spaces $T^\vee_V M$, where V varies over all subvarieties of M. Given a constructible function ξ on M with respect to a Whitney stratification $\mathcal{S} = \{S\}$ define:

$$Ch(\xi) := \sum_{S \in \mathcal{S}} (-1)^{\dim S} \eta(S,\xi) \cdot T^\vee_{\overline{S}} M,$$

which is a element in $L(M)$, $\eta(S,\xi)$ is the normal Morse index of S with respect to ξ (cf. [60] or Sect. 7.5.1 below). Hence if $F(M)$ is the free abelian group of constructible functions on M, we have the isomorphism $Ch : F(M) \to L(M)$, by [81]. Denoting by $\pi : Supp\ Ch(\mathbb{1}_V) \to V$ the restriction of the projection $\mathbb{P}(T^\vee M) \to M$, C. Sabbah in [105] described the Schwartz-MacPherson classes as

$$c^{SM}(V) = (-1)^{n-1} c\left(TM|_V\right) \cap \pi_*\left(c(\mathcal{O}(1))^{-1} \cap [\mathbb{P}(Ch(\mathbb{1}_V))]\right), \tag{7.13}$$

where $\mathcal{O}(1)$ denotes the tautological line bundle on the projectivization $\mathbb{P}(T^\vee M) \to M$.

Let $\mathcal{L}$, Z be as above. Let $B = Bl_Y X \to X$ be the blow-up of X along the singular subscheme Y of Z. Let $\mathcal{Z}$ e $\mathcal{Y}$ denote the total transform of Z and the exceptional divisor in B, respectively.

By Aluffi technique (see [4]), that uses the bundle $\mathcal{P}^1_X L$ of principal parts of L over X, B may be treated as a subvariety of $\mathbb{P}(\mathcal{P}^1_X L)$, $\mathcal{Z}$ equals $B \cap \mathbb{P}(T^\vee X \otimes L)$ and the canonical line bundle $\mathcal{O}_B(-1)$ on B is the restriction of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(\mathcal{P}^1_X L)$. Then $\mathcal{O}_B(-1)|_{\mathcal{Z}}$ is the restriction of the tautological line buncle $\mathcal{O}_{\tilde{\mathbb{P}}}(-1)$ on $\tilde{\mathbb{P}} = \mathbb{P}(T^\vee X \otimes L)$. Using the natural identification $\mathbb{P}(T^\vee X \otimes L) \cong \mathbb{P}(T^\vee X)$, Eq. (7.13) can be written as

$$c^{SM}(Z) = (-1)^{n-1} c\left(TX|_Z\right) \cap \pi_*\left(c(\mathcal{O}_B(1) \otimes \pi^* L|_Z)^{-1} \cap [\mathbb{P}(Ch(\mathbb{1}_V))]\right), \tag{7.14}$$

where $\pi : \mathcal{Z} \to Z$ is the restriction of the blow-up to $\mathcal{Z}$.

Since $\mathcal{Z} = c_1(\pi^*(L|_Z))$, $\mathcal{Y} = c_1(\mathcal{O}_B(-1))$ and $[\mathbb{P}(Ch(\mathbb{1}_V))] = (-1)^{n-1}([\mathcal{Z}] - [\mathcal{Y}])$ (see [101, Corollary 2.4]), by Eq. (7.14),

$$c^{SM}(Z) = c\left(TX|_Z\right) \cap \pi_*\left(\frac{[\mathcal{Z}] - [\mathcal{Y}]}{1 + \mathcal{Z} - \mathcal{Y}}\right)$$

and by definition of the Fulton class,

$$c^{Fu}(Z) = c\left(TX|_Z\right) \cap \pi_*\left(\frac{[\mathcal{Z}]}{1 + \mathcal{Z}}\right).$$

Then,

$$\mathcal{M}(Z) = (-1)^{n-1}\ c\left(TX|_Z\right) \cap \pi_*\left(\frac{[\mathcal{Y}]}{(1 + \mathcal{Z})(1 + \mathcal{Z} - \mathcal{Y})}\right).$$

Since $[\mathbb{P}(Ch(\mu))] = ([\mathcal{Y}])$ (see [101, Corollary 2.4]), $\mu = \sum_{S \in \mathcal{S}} \gamma_S \mathbb{1}_{\overline{S}}$ (see [101, Lemma 4.1]) and by Sabbah's result for each $\overline{S}$, we get the desired result. □

7.4.3 The Milnor Number for Compact Complete Intersections with Non-isolated Singularities

This section is largely based on [26] (see also [30]). We give an interpretation of Parusiński's generalized Milnor number as the 0-degree Milnor class, extending Parusiński's invariant to local complete intersections. This is proved in [26, 118] using the virtual index, an invariant of vector fields that generalizes the GSV index. Here we give an elementary topological definition of the virtual index, we prove Theorem (7.4.2 above) and discuss the general case.

The virtual index was introduced in [74] for holomorphic vector fields and extended in [118] to continuous vector fields. This is a localization of the Fulton class $c_n^{Fu}(X)$ at the singular set of an n-dimensional variety X. Localization is a key concept underlying this and the following sections. This can be done via topology (obstruction theory) as we do it here, and also via differential geometry (Chern-Weil theory). We refer to [15, 26, 27] for thorough discussions on that subject.

A paradigm of localization is given by Poincaré-Hopf's local index as in Example (7.3.4). More generally, given a regular neighborhood U_S of some simplicial subcomplex S in an m-manifold M (for some triangulation or cell decomposition), and a vector field v with no singularity on U_S, this determines a lifting of the Chern class $c_m(TU_S)$ from $H^{2m}(U_S)$ to $H^{2m}(U_S, U_S \setminus S)$. The relative Chern class $c_m(U_S, \partial U_S)$ that we obtain, evaluated on the orientation cycle of U_S relative to its boundary, gives the element $\mathrm{Ind}_{\mathrm{PH}}(v; M; S)$ in $H_0(S) \cong \mathbb{Z}$. This is the idea we develop. The subtle point is that the Fulton class is defined by the Chern classes of a virtual bundle.

We look first at the isolated singularity case. Unlike [26, 30], here we do it topologically. Then we discuss the general case from the topological viewpoint.

7.4.3.1 The Virtual Index for Isolated Singularities

We consider X of pure dimension n in a compact complex manifold M of dimension $n + k$, defined by a regular section s of a rank k holomorphic bundle E on M, $k < n$. By definition we have the virtual tangent bundle $TX := TM|_X - E|_X$. Its total Fulton class $c^{Fu}(X)$ is the cap product with $[X]$ of the polynomial $c(TM|_X) \cdot c(E|_X)^{-1}$, i.e.,

$$(1 + c_1(TM|_X) + \ldots + c_n(TM|_X) \cdot (1 + c_1(E|_X) + \ldots + c_k(E|_X))^{-1}. \qquad (7.15)$$

We consider the degree n term in this polynomial, so it is something of the form:

$$c_n(TM|_X) + c_{n-1}(TM|_X)d_1(E|_X) + \ldots + c_{n-k}(TM|_X)d_k(E|_X) + \ldots ; \qquad (7.16)$$

where $(1 + d_1 + \ldots + d_n)$ is $c(E|_X)^{-1} \in H^*(X)$. This lives in $H^{2n}(X)$, so evaluated on $[X]$ it is an integer.

We know from [30] that whenever we have a C^∞ tangent vector field v with isolated singularities on X_{reg}, the regular part of X, this splits $c_n^{Fu}(X)$ in two parts: one, denoted $Ind_{\text{Vir}}(v, S_i)$, is localized at each connected component S_i of the singular set of X; this is the virtual index of v at S_i, that we want to understand; the other is the total Poincaré-Hopf index of v in X_{reg}.

We look first at the case where X has only isolated singularities; and we prove Theorem 7.4.2. Let $x_1, \dots, x_\ell$ be the singular set of X. Choosing appropriate coordinate charts U_i at each x_i, $i = 1, \dots, \ell$, we let $s_j^i : (\mathbb{C}^{n+k}, x_i) \to (\mathbb{C}^k, 0)$, $j = 1, \dots, k$ be local components of the section that defines X; these define the ICIS germ (X, x_i).

Now consider a C^∞ vector field v on X which is singular at the singularities of X and possibly at some other regular points of X, say $y_1, \dots, y_p$. We want to define a localization of $c_n^{Fu}(X)$ at each of these points.

For each x_i, choose a Milnor ball $\mathbb{B}_{\varepsilon_i}$, let $\mathbb{S}_{\varepsilon_i}$ be its boundary sphere and $L_i := X \cap \mathbb{S}_{\varepsilon_i}$ its link. We remark that, by [94], $X_i := X \cap \mathbb{B}_{\varepsilon_i}$ is homeomorphic to the cone over L_i and therefore it has the homotopy of a point. Hence the bundles $TM|_{X_i}$ and $E|_{X_i}$ are trivial restricted to each X_i.

We have a canonical (up to a change of coordinates) trivialization τ_i of $E|_{L_i}$ determined by the complex conjugate gradient vector fields $\overline{\nabla} s_1^i, \dots, \overline{\nabla} s_k^i$. These define cycles in the relative cohomology $H^*(X, L_1 \cup \dots \cup L_q)$ representing relative Chern classes of $E|_X$, $\widetilde{c}_i(E|_X; \tau)$, whose images in $H^*(X)$ are the usual Chern classes.

By definition, the Chern class $c_n(TM|_X)$ is the primary obstruction to constructing a $(k+1)$-frame of this bundle, i.e., $(k+1)$ linearly independent sections, because TM has dimension $n + k$. We notice that each link L_i is a submanifold of the regular part X_{reg}, and the bundle E is isomorphic to the normal bundle of X_{reg} in M. Hence if we have a continuous vector field v on X with isolated singularities at the x_i (and possibly also at some regular points of X), then $(v, \overline{\nabla} s_1^i, \dots, \overline{\nabla} s_k^i)$ determine a $(k+1)$-frame of $TM|_{L_1 \cup \dots \cup L_q}$ and give rise to a cocycle representing a class $\widetilde{c}_n(TM|_X)$ in $H^{2n}(X, L_1 \cup \dots \cup L_q)$ whose image in $H^{2n}(X)$ is $c_n(TM|_X)$.

Consider the polynomial appearing in Eq. (7.16) but now with respect to the relative classes $\widetilde{c}_i(E|_X; \tau)$ and $\widetilde{c}_n(TM|_X)$. The element we get is a lifting of $c_n(TM|_X - E|_X) \in H^{2n}(X)$ to a class

$$\widetilde{c}_n(TM|_X - E|_X) \in H^{2n}(X, L_1 \cup \dots \cup L_q) .$$

One has:

Definition 7.4.10 At each singular point x_i of X, the *virtual index* of v at $x_i \in X$, $Ind_{\text{Vir}}(v; X, x_i) \in \mathbb{Z}$, is the contribution to the Chern number $\widetilde{c}_n(TM|_X - E|_X)[X]$ localized at $X_i := X \cap \mathbb{B}_{\varepsilon_i}$. That is:

$$Ind_{\text{Vir}}(v, X_i) := \widetilde{c}_n(TM|_{X_i} - E|_{X_i})[X_i] .$$

Lemma 7.4.11 *For each $i = 1, \dots, q$, let $\mathcal{F}_i$ be the $(k+1)$-frame on the link L_i defined by the tangent vector field v and the local gradient vector fields $(\overline{\nabla}_1, \dots, \overline{\nabla}_k)$. Let $\widetilde{c}_n(TM|_{X_i}) \in H^{2n}(X_i, L_i; \mathcal{F}_i)$ be the n-th Chern class of $TM|_{X_i}$ relative to $\mathcal{F}_i$. Then one has:*

$$\widetilde{c}_n(TM|_{X_i} - E|_{X_i}) = \widetilde{c}_n(TM|_{X_i}) .$$

Therefore the virtual index as defined above, in 7.4.10, equals the GSV-index (definition 7.2.5) and it coincides with the classical virtual index for isolated singularities, as defined in [74, 118].

Proof Notice that each monomial appearing in Eq. (7.16), other than $c_n(TM|_X)$, is decomposable: it is either a product of some (absolute) Chern class of $TM|_X$ by a class in $c(E|_X)^{-1}$ or a product of classes in $c(E|_X)^{-1}$. In the first case, the Chern classes of $TM|_{X_i}$, vanish since X_i is contractible, except c_n which has been lifted to a relative class. In the second case, the Chern classes of $c(E|_X)$ have all been lifted to relative classes, but a general fact in algebraic topology is that the cup product of two relative classes does not change if we push one of them to the absolute cohomology. And then we use that restricted to each X_i the bundle E is trivial since X_i is contractible. This proves the first statement: $\widetilde{c}_n(TM|_{X_i} - E|_{X_i}) = \widetilde{c}_n(TM|_{X_i})$. Then, by definition of the relative Chern classes (see for instance [124]), the integer $c_n(TM|_{X_i})[X_i]$ equals the degree of the map

$$(v, \overline{\nabla}_1, \dots, \overline{\nabla}_k) : L_i \to W_{k+1,n+k}$$

into the Stiefel manifold of (orthonormal up to homotopy) $(k+1)$-frames in $\mathbb{C}^{n+k}$, and this is by Definition (7.2.5) the GSV-index of v. □

The proof of Theorem 7.4.2 is now an exercise using Theorem 7.2.3, Proposition 7.2.6 and Theorem 7.2.7. □

7.4.3.2 The Virtual Index at Non-isolated Singularities

We now introduce localization at higher dimensional sets. Consider first a compact complex m-manifold M and a continuous vector field v on M with compact singular set Σ which is a subcomplex for some triangulation of M. Let S be a connected component of Σ; we assume there is a regular neighborhood U_S of S so that v is non-singular on $U_S \setminus S$. We can always perturb v slightly and get a vector field $\tilde{v}$ on U_S with finitely many singularities $x_1, \dots, x_s$. Then we define the Poincaré Hopf index of v at S, $\mathrm{Ind}_{\mathrm{PH}}(v; M; S)$, to be the sum of the local indices at the x_i: This number is independent of the way we perturbe v and if we do this over all the connected components of Σ we get the Euler characteristic of M (cf. [30, Chapter 1]).

We now let the space X in M be as in Sect. 7.3.5: X is defined by a regular section of a very ample vector bundle E and it may have arbitrary singular locus

X_{sing}. Let $S_1, \ldots, S_\ell$ be the connected components of X_{sing}. Let v be a continuous vector field on the regular part $X_{\mathrm{reg}} := X \setminus X_{\mathrm{sing}}$ and assume U_i, $i = 1, \ldots, \ell$, are regular neighborhoods of the S_i such that v is non singular on $U_i \setminus S_i$. We want to define the virtual index of v at each S_i, a localization of the 0-degree Fulton class.

We consider a family of sections $\{s_t\}$ with t in some open disc Δ in $\mathbb{C}$, centered at 0, such that $s_0 = s$ and s_t is everywhere transversal to the 0-section of E for all $t \neq 0$. The zero sets $\{X_t\}$ of these sections define a flat family of local complete intersections in M, which are non-singular for $t \neq 0$ and degenerate to $X_0 = X$. And we have Verdier's specialization morphism

$$r_* \circ \iota_* : H_*(X_{t_o}) \longrightarrow H_*(X)\,.$$

We need to refine this construction slightly. For this we assume further (with no loss of generality) that each U_i has smooth boundary ∂U_i. Let η be small enough so that the open disc Δ_η in $\mathbb{C}$ of radius η and center at 0 is contained in Δ and for each $t \in \Delta_\eta$, X_t intersects ∂U_i transversally, for all $i = 1, \ldots, \ell$. For each $t \in \Delta_\eta$, set

$$\widehat{X}_t = X_t \setminus (U_1 \cup \ldots \cup U_\ell)\,.$$

This is a compact smooth manifold with boundary the intersection of X_t with $\partial U_1 \cup \ldots \cup \partial U_\ell$, and its interior is a complex manifold which for $t = 0$ is a deformation retract of X_{reg}. We claim that the manifolds $\widehat{X}_t$ are all isotopic in $M \setminus (U_1 \cup \ldots \cup U_\ell)$. This follows from the next lemma proved for us by J. L. Cisneros-Molina.

Lemma 7.4.12 *Let $p\colon E \to M$ be a smooth vector bundle. Let $S\colon M \times I \to E$ be a homotopy between sections of E such that for all $t \in I$ the section $S_t\colon M \to E$ defined by $S_t(x) = S(x,t)$ is transversal to the zero section Z of E. Set $N_t = s_t^{-1}(Z)$ with $t \in I$. Then all the N_t are isotopic as submanifolds of M.*

Proof The sections $S_t\colon M \to E$ are all transversal to the zero section Z of E, so the map $S\colon M \times I \to E$ also is transversal to Z. Set $N = S^{-1}(Z)$, then N is a submanifold with boundary of the manifold with boundary $M \times I$. Consider the projection:

$$\pi : M \times I \to M\,.$$

Define $\pi_t\colon M \cong M \times \{t\} \to M$ as the restriction of π to $M \times \{t\}$. Then $\pi_t\colon M \to M$ is a diffeomorphism of M into itself and π is an isotopy between N_0 and N_1. □

Now let v be as above. Given $\widehat{X}_t$ with $t \in \Delta_\eta$ and $t \neq 0$, we know from the above lemma that $\widehat{X}_0$ is isotopic to $\widehat{X}_t$. Hence v can be regarded as a vector field on $\widehat{X}_t$ with no singularities on its boundary. This defines a splitting of the Chern class $c_n(\widehat{X}_t)$ in two parts, one contained in the complement of the U_i and another with support in the S_i. Now recall that Theorem 7.3.29 says that the Chern classes of the $\widehat{X}_t$, $t \neq 0$, regarded in homology, specialize to the Fulton class of X. Hence the above splitting of the Chern classes of $\widehat{X}_t$ determines the localization of the Fulton class $c_n^{Fu}(X)$:

Definition 7.4.13 The integer so obtained, by localizing the Fulton class $c_n^{Fu}(X)$ at S_i with the vector field v, is the *virtual index* of v at S_i, $Ind_{vir}(v; X, S)$.

It is clear that the virtual indices of v at the connected components of the singular set of M, together with its Poincaré-Hopf indices at the singular points v may have on the regular part of X, add up to the Fulton class $c_n^{Fu}(X)$.

Notice that this index is defined in [26, 30] in more generality than we do it here.

7.4.3.3 The Generalized Milnor Number

In (7.3.14) we remarked how Schwartz classes can be localized at connected components of the singular set by using appropriate frames. In particular, given a local complete intersection X in a manifold M, a connected component S of X_{sing} and a vector field v on a neighborhood U of S in X, non-singular on $U \setminus S$, we can localize the 0-degree Schwartz-MacPherson class $c_n^{SM}(X) \in H_0(X; \mathbb{Z})$ and get *the Schwartz index of* v *on* X *at* S, $Ind_{Sch}(v; X, S)$. It is an exercise to see that the difference

$$Ind_{vir}(v; X, S) - Ind_{Sch}(v; X, S)$$

does not depend on the choice of v. We can think of it as being up to sign the localization at S of the 0-degree Milnor class of X. Define (following [26, §6]):

Definition 7.4.14 The generalized Milnor number of X at S is:

$$\mu(X, S) := (-1)^n \big(Ind_{vir}(v; X, S) - Ind_{Sch}(v; X, S)\big) .$$

By Theorem 7.4.2 this is the usual Milnor number when S is a point; and by [26, Theorem 6.2] it is Parusiński's generalized Milnor number if X is a hypersurface.

The previous discussion yields the following theorem:

Theorem 7.4.15 *Let X be the zero set of a regular section of a rank k holomorphic bundle E over a compact complex manifold M of dimension $n + k$. Let $S_1, \ldots, S_\ell$ be the connected components of the singular set of X. Then the sum of the Milnor numbers of X at the S_i adds up to the $0 - degree$ Milnor class of X:*

$$\mathcal{M}_0(X) = \sum_{i=1}^{\ell} \mu(X, S_i) .$$

7.4.4 Milnor Classes for Complete Intersections via Localization

The previously described localization of the 0-degree Milnor class was extended in [25, 26] to define Milnor classes in general by localizing the difference between Schwartz-MacPherson and Fulton classes at the connected components of the singular set of X. Here we say a few words about this from a topological viewpoint.

The bulk of [25, 26] can be condensed in the following theorem, which actually holds in greater generality than we state here (see also [30]). Let X be an irreducible and reduced compact complex analytic n-variety in a complex m-manifold M.

Theorem 7.4.16 *Let X be defined by a regular section of a holomorphic vector bundle E over M of rank k, $1 \leq k < m$. Let $S_1, \ldots, S_\ell$ be the connected components of the singular set X_{sing}. Equip X with a Whitney stratification so that $S_1, \ldots, S_\ell$ are union of strata, and a triangulation (T) compatible with the stratification. Let (D) be the cell decomposition dual of the barycentric decomposition of (T) and for each $i = 1, \ldots, \ell$ let U_i be the regular neighborhood of S_i consisting of all cells in (D) that meet S_i. Let $(D)^j$ be the j^{th} skeleton of the cell decomposition. Then:*

1. *For every r, $1 \leq r \leq n$, there exist continuous stratified r-frames $\mathcal{F}_r^i$ on each U_i, $i = 1, \ldots, \ell$, which are non-singular on $(D)^{2n-2r+2} \cap (U_i \setminus S_i)$.*
2. *Every such frame splits the corresponding Schwartz-MacPherson and the Fulton classes of X in two pieces, one contained in the regular part of X and another localized at the components S_i of the singular set.*
3. *The difference between the localizations of the Schwartz-MacPherson and the Fulton classes of X at each S_i is independent of the choice of frame: this is, by definition, the Milnor class of X at S_i: $\mathcal{M}_r(X; S_i)$*
4. *The contributions to the Schwartz-MacPherson and the Fulton classes of X contained in the regular part are equal to the Chern classes of $X \setminus \bigcup U_i$ relative to the r-frames $\mathcal{F}_r^i$ on the boundary of each U_i.*
5. *Therefore the sum of the Milnor classes $\mathcal{M}_r(X; S_i)$ of X at the connected components of the singular set is the global Milnor class $\mathcal{M}_r(X)$ defined in 7.4.1.*

Idea of the Proof The first statement follows from the work of M. H. Schwartz constructing frames by radial extension; we refer to [30, Chapter 2] for details. Also, localization of the Schwartz-McPherson classes is straightforward (see Remark 7.3.14 and see Chapter 10 in [30] for a thorough discussion). Hence, in order to localize the Milnor classes the subtle point is localizing the Fulton classes compatibly. The degree 0 case was already discussed above.

Now, for every $r = 1, \ldots, n$, let $\mathcal{F}_r^i$ be a stratified r-frames on each U_i, $i = 1, \ldots, \ell$, as in Theorem 7.4.16. These are non-singular on $(D)^{2n-2r+2} \cap (U_i \setminus S_i)$. Given an $\widehat{X}_t$ as in Sect. 7.4.3.2 with $t \in \Delta_\eta$ and $t \neq 0$, we know from Lemma 7.4.12 that $\widehat{X}_0$ is isotopic to $\widehat{X}_t$. Hence the $\mathcal{F}_r^i$ can be regarded as an r-frame on $\widehat{X}_t$ with no singularities on its boundary. This defines a splitting of the Chern class $c_{n-r+1}(\widehat{X}_t)$ in two parts, one contained in the complement of the U_i and another with support in the S_i. Now recall that Theorem 7.3.29 says that the Chern classes of the $\widehat{X}_t$, $t \neq 0$, regarded in homology, specialize to the Fulton class of X. Hence the above splitting of the Chern classes of $\widehat{X}_t$ determines the localization of the Fulton classes stated in Theorem 7.4.16. □

As an example one has:

Example 7.4.17 [26, Corollary 5.13] Let S be a non-singular connected component of X_{sing} such that X is Whitney regular along S. Let s be the complex dimension of S and let H be a local slice transversal to S of codimension s in the ambient manifold M, passing through a point x in S, so that $X \cap H$ is a complete intersection in H with an isolated singularity at x. Let $\mu(X \cap H, x)$ be the corresponding Milnor number. Then the top Milnor class of X at S is up to sign, the transversal Milnor number $\mu(X \cap H, x)$ times the fundamental cycle $[S]$:

$$\mathcal{M}_s(X, S) = (-1)^s \mu(X \cap H, x) \cdot [S] \, .$$

Remark 7.4.18 Finally, let us note that Milnor classes can also be seen (via Verdier specialization) as Chern-Schwartz-MacPherson classes of the complex of vanishing cycles of Deligne, see, e.g., [40, 85], etc.

7.4.5 Riemann-Roch, Bivariant Theory and Milnor Classes

Riemann-Roch is one of the deepest theorems in mathematics; it is a landmark that relates the complex analysis of a compact Riemann surface S with the surface's genus g, which is a topological invariant. Its origin is Riemann's theorem that on every such surface S there are g linearly independent holomorphic 1-forms. There have been remarkable generalizations of Riemann-Roch for non-singular varieties by various people, perhaps most notably by Hirzebruch, Grothendieck and Atiyah-Singer (through the general index theorem). And then by Baum, Fulton, MacPherson, Verdier and others for singular varieties. Searching for a Verdier-Riemann-Roch type theorem for the MacPherson classes, S. Yokura realized that Milnor classes naturally come into the picture. This is much related to Chern classes in bivariant theory and Brasselet's paper [21]. In this section we glance at this material and we refer to [114, 134] for more complete discussions on the subject.

After G. Roch (1839–1866), several people tried to generalize Riemann-Roch to higher dimensions. A first step was to propose a good candidate to play the role of the genus of a Riemann surface. The arithmetic genus $\rho_a(M)$ was such a candidate. This was defined by F. Severi, and Hirzebruch points out in the introduction to [65] that it can also be defined using the Hilbert characteristic function. Hirzebruch modified its definition slightly so that it has convenient multiplicative properties. This is defined in terms of the analytic Euler characteristic:

$$\rho_a(M) := \chi(M, O_M) := \sum_{i=0}^{\dim M} H^i(M; O) .$$

In the 1930s J. A. Todd gave a slightly different definition of the arithmetic genus and showed that this can be represented in terms of certain (Eger-Todd) canonical classes, which are by definition equivalence classes of algebraic cycles; however the proof was incomplete. This was before Chern classes were defined, and it was proved later that the Eger-Todd classes essentially coincide with Chern classes.

Then J. P. Serre, in a letter to Kodaira and Spencer, conjectured that in fact for any holomorphic vector bundle E on M, the holomorphic Euler characteristic of E,

$$\chi(M, E) := \sum_{i=0}^{\dim_{\mathbb{C}} M} (-1)^i \dim_{\mathbb{C}} H^i(M, E) ,$$

can be determined by the Chern classes of M and E; this was known for curves (proved by A. Weil), extending Rieman-Roch.

In 1953 Hirzebruch proved Serre's conjecture. For this he realized how to generalize Todd's constructions and introduced what he called the Todd polynomials. These are polynomials in the Chern classes of the manifold, corresponding to a certain multiplicative sequence as we explain below.

The Hirzebruch-Riemann-Roch theorem, says the analytic Euler characteristic of a compact complex manifold M equals its Todd genus. And more generally:

$$\chi(M, E) := \mathrm{ch}(E)\mathrm{Td}(M)[M] ,$$

where ch is the Chern character and Td is the Todd class. One side is analytic and the other is topological. We now recall these concepts, for the reader's convenience, and we refer to [65] and to the excellent survey paper [114] for more on the topic.

To define this multiplicative sequence of polynomials, one considers the formal power series

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots$$

where B_i is the i-th Bernoulli number. Then one considers the coefficient of x^j in the product

$$\prod_{i=1}^{m} Q(\beta_i x) ,$$

for any $m > j$. This is symmetric in the β_i and homogeneous of weight j, so it can be expressed as a polynomial $\mathrm{Td}_j(\sigma_1, \ldots, \sigma_j)$ in the elementary functions of the β_i. This is the Todd sequence of polynomials. The first Todd polynomials are:

$$\mathrm{Td}_1 = \frac{1}{2}c_1 \quad , \quad \mathrm{Td}_2 = \frac{1}{12}(c_1^2 + c_2) \quad , \quad \mathrm{Td}_3 = \frac{1}{24}(c_1 \cdot c_2) \quad ,$$

$$\mathrm{Td}_4 = \frac{1}{24}(-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4) \; , \; \ldots$$

The Todd class of M is the formal power series in its Chern classes defined by all these polynomials:

$$\mathrm{Td}(M) := 1 + \mathrm{Td}_1(M) + \mathrm{Td}_2(M) + \ldots$$

Notice that $\chi(M, E)$ is an integer by definition, while $\mathrm{ch}(E)\mathrm{Td}(M)[M]$ is a polynomial with rational coefficients in the Chern classes of M and E; by definition this is a rational number. A highly non-trivial and yet obvious consequence of Hirzebruch-Riemann-Roch is that this number actually is an integer. In particular the Todd genus of a compact complex manifold is an integer. See for instance [19, p. 11] where Hirzebruch tells Bourguignon: "I admired already the fact that for an algebraic surface $c_1^2 + c_2$ is always divisible by 12".

Of course one may ask if this integrality property of the Todd genus is best possible. More precisely, when is the Todd genus of a compact complex manifold an even integer? For complex surfaces the answer is given by the classical Rochlin's signature theorem for spin^c 4-manifolds: the parity of the Todd genus is given by the mod(2)-index of the Dirac operator associated to the canonical spin^c-structure (cf. [11]). This statement is generalized in [53] to complex dimensions of the form $4k + 2$ using a theory of *characteristic divisors*.

The Chern character is uniquely characterized by being a ring homomorphism:

$$\mathrm{ch} : K(M) \longrightarrow H^*(M; \mathbb{Q}) \; ,$$

such that for one dimensional bundles one has

$$\mathrm{ch}(L) = e^{c_1(L)} := \sum_{i=0}^{\infty} \frac{c_1(L)^i}{i!} \; .$$

This is extended to higher dimensional bundles using the splitting principle.

Grothendieck's version of Riemann-Roch can be expressed by saying that the mapping $\xi \mapsto \mathrm{ch}(\xi) \cap \mathrm{Td}(M)$ from the Grothendieck group K^0M of algebraic vector bundles on M to a suitable cohomology theory $H^\bullet M$, is a natural transformation of covariant functors, where ch is the Chern character and Td is the Todd class; K^0M and $H^\bullet M$ are contravariant but for non-singular varieties they can be made covariant, as explained by Borel and Serre in [18]. That is, let $K_0(M)$

be the Grothendieck group of coherent algebraic sheaves on M; for a morphism $f : X \to Y$ one has the pushforward $f_! : K_0(X) \to K_0(Y)$ defined by means of the higher direct image sheaves:

$$f_!(\mathcal{G}) := \sum_{i \geq 0} (-1)^i R^i f_* \mathcal{G} \, .$$

The canonical map $K^0(M) \to K_0(M)$ taking a bundle to its sheaf of sections is an isomorphism and turns K^0 into a covariant functor. Then one has:

Theorem 7.4.19 (Grothendieck-Riemann-Roch) *For X, Y non-singular compact algebraic varieties over $\mathbb{C}$, the map $\tau_*(\mathcal{G}) = Td(X)ch(\mathcal{G}) \cup [X]$ is natural. That is, for any morphism $f : X \to Y$ the following diagram is commutative:*

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\tau_*} & H_{2*}(X;\mathbb{Q}) \\ f_! \downarrow & & \downarrow f_* \\ K_0(Y) & \xrightarrow{\tau_*} & H_{2*}(Y;\mathbb{Q}) \end{array}$$

Taking Y to be a point and $\mathcal{G}$ the sheaf of local sections of a bundle we recover Hirzebruch's theorem.

Grothendieck-Riemann-Roch was extended for singular varieties in various ways by Baum, Fulton and MacPherson in [16]. In particular, consider a possibly singular projective variety X and let K_0 be as before, the Grothendieck group of coherent sheaves on X. We consider singular cohomology with rational coefficients. Then [16] there is a unique natural transformation $\tau : K_0 \to H_*$ such that the diagram

$$\begin{array}{ccc} K^0(X) \otimes K_0(X) & \xrightarrow{\otimes} & K_0(X) \\ \mathrm{ch} \otimes \tau \downarrow & & \tau \downarrow \\ H^*(X;\mathbb{Q}) \otimes H_*(X;\mathbb{Q}) & \xrightarrow{\cap} & H_*(X;\mathbb{Q}) \end{array}$$

is commutative and if X is non-singular and $\mathcal{O}_X$ is its structure sheaf, then $\tau(\mathcal{O}_X) = \mathrm{Td}(X) \cap [X]$. The naturality of τ means, as usual, that if $f : X \to Y$ is a morphism, then the following diagram commutes:

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\tau} & H_*(X;\mathbb{Q}) \\ f_* \downarrow & & \downarrow f_* \\ K_0(Y) & \xrightarrow{\tau} & H_*(Y;\mathbb{Q}) \end{array}$$

This τ is called *the Todd class* of Baum-Fulton-MacPherson. The following Theorem was conjectured in [16] and affirmatively proved by Verdier in [130] (see [55, Theorem 18.2(3)]). We recall [130, 1.4, p. 190] that a morphism $f : X \to Y$ is a local complete intersection if at each point it factorizes as the composition of a regular embedding ι into some variety Z, followed by a smooth morphism $Z \to Y$, i.e. a flat morphism such that the sheaf $\Omega_{X/Y}$ of relative differentials is locally free.

Theorem 7.4.20 (Verdier-Riemann-Roch) *Let $f : X \to Y$ be a local complete intersection morphism and let T_f be the virtual relative tangent bundle. Then the following diagram commutes:*

$$\begin{array}{ccc} K_0(Y) & \xrightarrow{\tau} & H_*(Y;\mathbb{Q}) \\ f^* \downarrow & & \downarrow Td(T_f)\cap f^* \\ K_0(X) & \xrightarrow{\tau} & H_*(X;\mathbb{Q}) \end{array}$$

Here the homology functor can be replaced by A_*, the Chow homology covariant functor. Then Yokura in [132] asked whether a similar diagram holds for the Chern-Schwartz-MacPherson transformation $C_* : \mathcal{F} \to A_*$, where $\mathcal{F}$ is the group of constructible function. In [133] he proved that indeed such a Verdier-type Riemann-Roch formula holds in the case of smooth morphisms: given a smooth morphism $f : X \to Y$, the following diagram commutes [133, Theorem 2.2]:

$$\begin{array}{ccc} \mathcal{F}(Y) & \xrightarrow{C_*} & A_*(Y) \\ f^* \downarrow & & \downarrow c(T_f)\cap f^* \\ \mathcal{F}(X) & \xrightarrow{C_*} & A_*(X) \end{array}$$

Yokura then noticed that unlike the case of Baum-Fulton-MacPherson's Riemann-Roch, one cannot expect a Verdier-type Riemann-Roch theorem for the Chern-Schwarz-MacPherson transformation in the case of local complete intersection morphisms in general. Even in the simplest case where Y is a point, the lack of commutativity for the corresponding diagram is given by the Milnor classes of X.

Therefore one is naturally led towards considering Milnor classes in bivariant theory (cf. Brasselet's paper [21]). We refer to Yokura's quoted papers and also to Schürmann's article [106] as well as [22, 82, 114, 132]. The concept of bivariant theory was introduced by W. Fulton and R. MacPherson [56]. This topic is vast and here we say only a few words. Bivariant theory is concerned with maps, not with spaces. It consists of two functors, one covariant and another contravariant, that fit together in a nice way. More precisely, a bivariant theory $\mathbb{B}$ on a category C with values in an abelian category comes with a distinguished class of maps called *confined (or proper) morphisms* and a class of commutative squares called *independent squares*, which satisfy certain axioms (see for instance [56, Section 2.1]). Then $\mathbb{B}$ assigns to each such morphism $X \xrightarrow{f} Y$ in the category C an abelian group $\mathbb{B}(X \xrightarrow{f} Y)$, and there are three basic operations:

(i) Given morphisms $f : X \to Y$ and $g : Y \to Z$ we have a *product operation*:

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \longrightarrow \mathbb{B}(X \xrightarrow{gf} Z)\ .$$

(ii) For f, g as above with f proper, there is a *pushforward operation*:

$$f_* : \mathbb{B}(X \stackrel{gf}{\to} Z) \longrightarrow \mathbb{B}(Y \stackrel{g}{\to} Z)\,.$$

(iii) For each independent square

$$\begin{array}{ccc} X' & \stackrel{g'}{\longrightarrow} & X \\ f' \downarrow & & \downarrow f \\ Y' & \stackrel{g}{\longrightarrow} & Y \end{array} ,$$

one has a *pullback operation*:

$$g^* : \mathbb{B}(X \stackrel{f}{\to} Y) \longrightarrow \mathbb{B}(X' \stackrel{f'}{\to} Y')\,.$$

These three operations are required to satisfy certain (seven) compatibility axioms (see [56, Section 2.2] or [114, Section 9] for details).

To finish this section we remark that there is another interesting approach initiated by Yokura, to study Milnor classes in bivariant theory. One can motivate this with a natural question posed by himself in [135]: given a holomorphic bundle E over a compact complex analytic space X with compact fiber Y, what can we say about the Milnor classes (or more generally, the various types of Chern classes for singular varieties) of E out from those of X and Y?

The above question is highly non-trivial even in the simplest case where $E = X \times Y$. In fact, that setting was studied by Ohmoto and Yokura in [96]. They proved a general product formula for the Milnor class of a finite Cartesian product of local complete intersections X_i. In the particular case of a product $X \times Y$ of two local complete intersections of dimensions n, m, respectively, this formula is particularly nice:

$$\mathcal{M}(X \times Y) \;=\; \mathcal{M}(X) \times \mathcal{M}(Y) + (-1)^m \mathcal{M}(X) \times c_*^{SM}(Y) + (-1)^n c_*^{SM}(X) \times \mathcal{M}(Y)\,. \tag{7.17}$$

Since Milnor classes have support in the singular set, if Y is non-singular this implies:

$$\mathcal{M}(X \times Y) \;=\; (-1)^m \mathcal{M}(X) \times c_*^{SM}(Y)\,. \tag{7.18}$$

This last equation is nicely extended in [135] as follows. Let $f : \widetilde{X} \to X$ be a morphism of compact complex analytic varieties. Say that f is smooth of relative dimension d if it is flat, for all subvarieties V of X and all irreducible components V' of $f^{-1}(V)$ one has $\dim V' = \dim V + d$, and the sheaf of relative differentials $\Omega^1_{\widetilde{X}/X}$ is locally free. Set $\mathcal{M}_0 = \big(c^{Fu}(X) - c^{SM}(X)\big)$; so this is the total Milnor class up to sign. Then one has [135, Theorem 2.2]

Theorem 7.4.21 *Let M be an $(n+k)$-dimensional complex analytic manifold, and let E be a rank k holomorphic vector bundle over M. Let s be a regular holomorphic section of E, and let X be the zero set of s. Let $\pi : \widetilde{M} \to M$ be a smooth morphism. Set $\widetilde{E} = \pi^* E$, $\tilde{s} = \pi^* s$ and $\widetilde{X} = \pi^{-1}(X)$. Let $f = \pi|_{\widetilde{X}}$ be the restriction of π to $\widetilde{X}$. Then:*

$$\mathcal{M}_0 = c(T_f) \cap f^* \mathcal{M}_0(X) \,,$$

where T_f is the relative tangent bundle of the smooth morphism.

The aforementioned product formulas for the Milnor class of Ohmoto and Yokura in [96] inspired the recent article [34], which reminisces work by Aluffi-Faber [8] and Schürmann [110]. The main theorem in [34] considers an n-dimensional compact complex analytic manifold M and holomorphic vector bundles $\{E_1, \cdot \ldots \cdot, E_r\}$, $r \geq 1$, over M of ranks $d_i \geq 1$. For each $i = 1, \cdot \ldots \cdot r$, let X_i be the $(n - d_i)$-dimensional local complete intersection in M defined by the zeroes of a regular section s_i of E_i. Assume further that the X_i are equipped with Whitney stratifications $\mathcal{S}_i$ such that all the intersections amongst strata in the various X_i are transversal. Set $X = X_1 \cap \cdot \ldots \cdot \cap X_r$, a local complete intesection of dimension $n - d_1 - \cdot \ldots \cdot - d_r$. Then:

(i) $c^{SM}(X) = c\left((TM|_X)^{\oplus r-1}\right)^{-1} \cap \left(c^{SM}(X_1) \cdot \ldots \cdot c^{SM}(X_r)\right)$;

(ii) $c^{FJ}(X) = c\left((TM|_X)^{\oplus r-1}\right)^{-1} \cap \left(c^{FJ}(X_1) \cdot \ldots \cdot c^{FJ}(X_r)\right)$; and therefore

(iii) $\mathcal{M}(X) = (-1)^{\dim X} c\left((TM|_X)^{\oplus r-1}\right)^{-1} \cap \left(c^{FJ}(X_1) \cdot \ldots \cdot c^{FJ}(X_r) - c^{SM}(X_1) \cdot \ldots \cdot c^{SM}(X_r)\right)$.

7.5 Milnor Classes and Lê Cycles

7.5.1 Local Lê Cycles

Lê cycles are analytic cycles encoding deep information about singularity germs $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ and allow describing the topology and diffeomorphism type of the local Milnor fibres. These were introduced by D. Massey and we refer to [77, 78] for thorough discussions on this subject (see also [73], §6 and 9). Lê cycles spring from the theory of polar varieties developed by B. Teissier and Lê D. T. in the 1970s; particularly by Lê's attaching handles theorem [73, Theorem 1.6.8].

Recall (see for instance [71, 126]) that given f as above and a general linear form $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$, one has the first relative polar curve $\Gamma^1_{f,\ell}$ of f with respect to a linear form ℓ. As a set this is the union of those components in the critical set of (f, ℓ) which are not critical points of f. In other words, assume we have coordinates $(z_0, \ldots, z_n)$ so that the linear function $\ell = z_0$ is "sufficiently general". Then the critical locus of (f, ℓ) is $V(\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$, the set of points where $\partial f/\partial z_i = 0$ for all $i = 1, \ldots, n$. Now write the cycle represented by $V(\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ as a formal sum over the irreducible components:

$$\left[V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)\right] = \sum n_i [V_i] \,.$$

Then $\Gamma^1_{f,\ell}$, as a cycle, is defined by: $\Gamma^1_{f,\ell} = \sum_{V_i \not\subseteq \Sigma f} n_i [V_i]$.

More generally we may consider a generic linear functional $\mathbb{C}^{n+1} \to \mathbb{C}^r$. This gives rise to a polar variety relative to f determined by the points of non-transversality of the fibers of ℓ and f, denoted $\Gamma^r_{f,\ell}$. Let U be an open subset of $\mathbb{C}^{n+1}$ containing the origin, $z = (z_0, \ldots, z_n)$ a choice of linear coordinates in $\mathbb{C}^{n+1}$ and $\Sigma(f) = V\left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right)$ the critical set of f. For each k with $0 < k < n$, the polar variety $\Gamma^k_{f,z}$ is the analytic space $V\left(\frac{\partial f}{\partial z_k}, \ldots, \frac{\partial f}{\partial z_n}\right) / \ \Sigma(f)$, where X/Y means the analytic closure of $X - Y$. At the level of ideals $\Gamma^k_{f,z}$ consists of those components of $V\left(\frac{\partial f}{\partial z_k}, \ldots, \frac{\partial f}{\partial z_n}\right)$ which are not contained in the set $\Sigma(f)$. Massey denotes by $[\Gamma^k_{f,z}]$ the cycle associated with the space $\Gamma^k_{f,z}$. Then, for each $0 < k < n$, Massey defines the k-th Lê cycle $\Lambda^k_{f,z}$ of f with respect to the coordinate system z as the cycle:

$$\Lambda^k_{f,z} := \left[\ \Gamma^{k+1}_{f,z} \cap V\left(\frac{\partial f}{\partial z_k}\right)\ \right] - [\ \Gamma^k_{f,z}\ ].$$

If a point $p = (p_0, \ldots, p_n) \in U$ is an isolated point of the intersection of $\Lambda^k_{f,z}$ with the cycle of $V(z_0 - p_0, \ldots, z_{k-1} - p_{k-1})$, then the Lê number $\lambda^k_{f,z}(p)$ is the intersection number at p:

$$\lambda^k_{f,z}(p) := (\Lambda^k_{f,z} \cdot V(z_0 - p_0, \ldots, z_{k-1} - p_{k-1})\ )_p \ .$$

For a generic choice of coordinates all Lê numbers of f at p are defined and they are independent of the choice of coordinates. These are the (*generic*) *Lê numbers* of f at p. If the singularity of f is isolated, then there is only one generic Lê number and it coincides with the Milnor number. One has [78, Theorems 3.3 and 10.3]:

Theorem 7.5.1 *Let* $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ *be a holomorphic map-germ and let* F_f *be its Milnor fiber.*

- *If the complex dimension* s *of its critical set is* $s \leq n-2$*, then* F_f *is obtained up to diffeomorphism, from a* $2n$*-ball by successively attaching* $\lambda^{n-k}_{f,\ell}(\underline{0})$ k*-handles, where* $n - s \leq k \leq n$ *and* $\lambda^{n-k}_{f,\ell}(\underline{0})$ *is the* $(n-k)^{th}$ *Lê number.*

- *If $s = n-1$, then F_f is obtained up to diffeomorphism, from a real $2n$-manifold with the homotopy type of a bouquet of $\lambda^{n-1}_{f,\ell}(\underline{0})$ circles, by successively attaching $\lambda^{n-k}_{f,\ell}(\underline{0})$ k-handles, where $2 \leq k \leq n$.*
- *The reduced Euler characteristic of the Milnor fiber of f at $\underline{0}$ is:*

$$\tilde{\chi}(F_{f,\underline{0}}) = \sum_{i=0}^{n} (-1)^{n-i} \lambda^{i}_{f,z}(\underline{0}) \ .$$

We refer to [80] for a self-contained sketched proof of this theorem.

Massey also gave an alternative characterization of the Lê cycles of a hypersurface singularity which leads to a generalization that can be applied to any constructible complex of sheaves. To explain this, let us equip $X = f^{-1}(0)$ with a Whitney stratification $\{S_\alpha\}$. Recall that for every constructible function β on X one has the *normal Morse index* $\eta(S_\alpha, \beta)$ defined by Goresky and MacPherson in [60, Chapter 3]. To recall the definition of $\eta(S_\alpha, \beta)$, given X denote by $\mathcal{D}^b_c(X)$ the derived category of bounded constructible complexes of sheaves of $\mathbb{C}$-vector spaces on X. Denote the objects of $\mathcal{D}^b_c(X)$ by something of the form $F^\bullet$. For $F^\bullet \in \mathcal{D}^b_c(X)$ and $p \in X$ we denote by $\mathcal{H}^*(F^\bullet)_p$ the stalk cohomology of $F^\bullet$ at p and by $\chi(F^\bullet)_p$ its Euler characteristic (we refer to [66], but see also [43, 83] and [92] for background material on these topics). That is,

$$\chi(F^\bullet)_p = \sum_k (-1)^k \dim_{\mathbb{C}} \mathcal{H}^k(F^\bullet)_p \, .$$

We also denote by $\chi(X, F^\bullet)$ the Euler characteristic of X with coefficients in $F^\bullet$, i.e.,

$$\chi(X, F^\bullet) = \sum_k (-1)^k \dim_{\mathbb{C}} \mathbb{H}^k(X, F^\bullet),$$

where $\mathbb{H}^*(X, F^\bullet)$ denotes the hypercohomology groups of X with coefficients in $F^\bullet$.

Now let N be a germ of a closed complex submanifold of M which is transversal to S_α, with $N \cap S_\alpha = \{x\}$. Define the *complex link* l_{S_α} of S_α by [60]:

$$l_{S_\alpha} := X \cap N \cap B_\delta(x) \cap \{g = c\} \quad \text{for } 0 < |c| \ll \delta \ll 1,$$

where $B_\delta(x)$ is a closed ball of radius δ in some local coordinates, $g : (M, x) \to (\mathbb{C}, 0)$ is a germ of holomorphic function such that $d_x g \in T^*_{S_\alpha} M$ and $d_x g \notin T^*_{\overline{S'}} M$, for all stratum $S' \neq S_\alpha$. The *normal Morse datum* of S_α is defined by:

$$NMD(S_\alpha) := (X \cap N \cap B_\delta(x), l_{S_\alpha}),$$

and the *normal Morse index* $\eta(S_\alpha, F^\bullet)$ of the stratum is:

$$\eta(S_\alpha, F^\bullet) := \chi(NMD(S_\alpha), F^\bullet),$$

where the right-hand-side is the Euler characteristic of the relative hypercohomology.

Everything we have defined so far for a constructible complex of sheaves works equally well for constructible functions and the two constructions are somehow equivalent (cf. [109, 112]). The normal Morse index, say with respect to a constructible function β, may also be defined by

$$\eta(S_\alpha, \beta) = \chi(X \cap N \cap B_\delta(x), \beta) - \chi(\ell_{S_\alpha}, \beta)\ .$$

In general, let X be an analytic germ of an s-dimensional space which is embedded in some affine space, $M := \mathbb{C}^{n+1}$, so that the origin is a point of X. If β is a constructible function on X, for a generic linear choice of coordinates $z = (z_0, \dots, z_n)$ for $\mathbb{C}^{n+1}$, Massey in [79, Proposition 0.1] proves that there exist analytic cycles $\Lambda^i_{\beta,z}$ in X, which are purely i-dimensional, such that $\Lambda^i_{\beta,z}$ and $V(z_0 - p_0, \dots, z_{i-1} - p_{i-1})$ intersect properly at each point $p = (p_0, \dots, p_n)$ of X near the origin, and such that

$$\beta(p) = \sum_{i=0}^{s} (-1)^{s-i} (\Lambda^i_{\beta,z} \cdot V(z_0 - p_0, \dots, z_{i-1} - p_{i-1}))_p\ .$$

In [78, Corollary 10.15] is proved that, for a generic linear choice of coordinates $z = (z_0, \dots, z_n)$, if we let L^i be the i-dimensional linear subspace $V(z_0, \dots, z_i)$ then,

$$\Lambda^i_{\beta,z} = \sum_\alpha (-1)^{s-d_\alpha}\, \eta(S_\alpha, \beta)\, P_i(\overline{S_\alpha})\ ,$$

where $P_i(\overline{S_\alpha})$ is the absolute affine i-dimensional polar variety with respect to the flag given by the L^i above, as defined by Lê and Teissier in [72].

If $X = V(f)$ and the constructible function on X is defined by $w(p) = \chi(F_{f,p}) - 1$ with $F_{f,p}$ being the local Milnor fiber at $p \in X$, the Lê cycles are:

$$\Lambda^i_{f,z} = \sum_\alpha (-1)^{s-d_\alpha}\, \eta(S_\alpha, w)\, P_i(\overline{S_\alpha})\ . \tag{7.19}$$

This is the formulation that will be used in the sequel to extend the construction of local Lê cycles to the affine and projective settings.

7.5.2 Affine and Global Lê Cycles

In the affine context, Schürmann and Tibăr in [113] describe the Schwartz-MacPherson classes of a complex algebraic proper subset $X \subset \mathbb{C}^N$ using algebraic cycles that they called MacPherson cycles. In this construction a key role is played by the affine polar varieties, which we now define. For each $0 \leq i \leq N$, let L_i be a linear subvariety of $\mathbb{C}^N$ of codimension i. If X is of pure dimension $d < N$, the k-th global affine polar variety of X, with $0 \leq k \leq d$, is the algebraic set:

$$P_k(X, L_{k+1}) = \overline{\{x \in X_{\text{reg}} \mid \dim(T_x X_{\text{reg}} \cap L_{k+1}) \geq d - k\}}\,.$$

For L_{k+1} general enough $P_k(X, L_{k+1})$ has pure dimension k. We have $P_d(X, L_{k+1}) = X$ and we set $P_k(X, L_{k+1}) = \emptyset$ for $k > d$. We fix an algebraic Whitney stratification $\{S_\alpha\}$ of X with connected strata. Let β be a constructible function on X with respect to the stratification.

Definition 7.5.2 The k-th affine Lê cycle of β is:

$$\Lambda_k^{\mathbb{A}}(\beta, L_{k+1}) := \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta)\, P_k(\overline{S_\alpha}, L_{k+1})\,,$$

where d_α denotes the dimension of S_α and $\eta(S_\alpha, \beta)$ is the normal Morse index.

These polar varieties are used in [113] to define the MacPherson cycles,

$$MP_k(\beta, L_{k+1}) := \sum_\alpha (-1)^{d_\alpha} \eta(S_\alpha, \beta)\, P_k(\overline{S_\alpha}, L_{k+1})\,.$$

These are our Lê cycles up to sign and one has by [113] that $(-1)^{k+d} \Lambda_k^{\mathbb{A}}(\beta, L_{k+1})$ represents the Schwartz-MacPherson class $c_k^{SM}(\beta)$.

An interesting feature of these affine Lê cycles of X is that they are a global extension of the above local Lê cycles defined by Massey:

Proposition 7.5.3 *The restriction of the affine Lê cycles to each point $x \in X$ are the local Lê cycles at x. To be precise, let X be a closed affine algebraic subvariety of $\mathbb{C}^N$ and β a constructible function on X with respect to a Whitney stratification $\{S_\alpha\}$ of X; $x \in X$ and $U \subset \mathbb{C}^N$ is an open neighborhood of x. Consider a generic flag of linear subvarieties of $\mathbb{C}^N$, $\{x\} = L_N \subset L_{N-1} \subset \cdots \subset L_1 \subset L_0 = \mathbb{C}^N$, with L_i being of codimension i and such that $L_i \cap U = Z(z_0, \ldots, z_{i-1})$ where $z = (z_0, \ldots, z_{N-1})$ are generic linear coordinates around x. Let $\iota : U \to \mathbb{C}^N$ be the inclusion. Then, the flat pull-back of the affine Lê cycles satisfy:*

$$\iota^* \Lambda_k^{\mathbb{A}}(\beta, L_{k+1}) = \Lambda^k_{\beta \circ \iota, z}\,.$$

The proof of this proposition is easy and we refer to [35] for details.

We now define the global, or projective, Lê cycles. These were defined in [33] as follows. Let X be a d-dimensional subvariety of $\mathbb{P}^N$. Its k-th polar variety is [33, Definition 4.2]:

$$\mathbb{P}_k(X, L_{k+2}) = \overline{\{x \in X_{reg} \mid \dim\big(T_x X_{reg} \cap L_{k+2}\big) \geq d - k - 1\}},$$

where L_{k+2} is a plane of codimension $k+2$ in $\mathbb{P}^N$ and $T_x X_{reg}$ is the projective tangent space of X at a regular point x. The classes in the Chow and homology groups represented by these varieties do not depend on the choice of the linear space provided this is general enough (see [103, Prop. 1.2]). We denote these classes by $[\mathbb{P}_k(X)]$. For any given constructible function β on $X \subseteq \mathbb{P}^N$ with respect to a Whitney stratification $\mathcal{S} = \{S_\alpha\}$ of X, motivated by [113], we defined in [33, Equation (12)] the MacPherson cycles as :

$$MP_k^{\mathbb{P}}(\beta, L_{k+2}) := \sum_\alpha (-1)^{d_\alpha} \eta(S_\alpha, \beta) \mathbb{P}_k(\overline{S}_\alpha, L_{k+2}),$$

where d_α denotes the dimension of S_α. Then we defined in [33]:

Definition 7.5.4 Let M be a smooth complex submanifold of $\mathbb{P}^N$ of dimension $n+1$, let Z be the hypersurface in M defined by the set of zeroes of a reduced holomorphic section s of a line bundle L on M. Endow Z with a Whitney stratification. The k^{th} *global Lê cycle of Z with respect to a general linear subspace* L_{k+2} (of codimension $k+2$) of $\mathbb{P}^N$ is:

$$\Lambda_k(Z, L_{k+2}) = (-1)^n MP_k^{\mathbb{P}}(\omega, L_{k+2}), \tag{7.20}$$

where $\omega(x) = \chi(F_{f,x}) - 1$ with $F_{f,x}$ a local Milnor fiber of f at x of the function f corresponding to s in some local trivialization of L around x. The classes associated to these cycles will be denoted by $\Lambda_k(Z)$.

For any subvariety Z of $\mathbb{P}^N$ we denote by $Cone(Z)$ the cone in $\mathbb{C}^{N+1}$ induced by Z. Analogously, for any conical subvariety V through the origin of $\mathbb{C}^{N+1}$ we denote by $\mathbb{P}(V)$ the induced projective variety in $\mathbb{P}^N$. Let X be a subvariety of $\mathbb{P}^N$ and let L_{k+2} be a linear subvariety of $\mathbb{P}^N$ of codimension $k+2$. In this case, $Cone(L_{k+2})$ is a linear subspace of codimension $k+2$ in C^{N+1} and $P_{k+1}(Cone(X), Cone(L_{k+2}))$ is a conical subvariety of $\mathbb{C}^{N+1}$ of dimension $k+1$. One has:

The relationship between the projective and the affine polar varieties is given by

$$\mathbb{P}_k(X, L_{k+2}) = \mathbb{P}\big(P_{k+1}(Cone(X), Cone(L_{k+2}))\big)\,.$$

For any given Whitney stratification $\mathcal{S} = \{S_\alpha\}$ of X and a constructible function β on X define the k^{th} *projective Lê cycle with respect to* L_{k+2} by:

$$\Lambda_k^{\mathbb{P}}(\beta, L_{k+2}) := \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta) \mathbb{P}_k(\overline{S}_\alpha, L_{k+2})\ .$$

The next result relates the projective (or global) and affine Lê cycles:

Proposition 7.5.5

$$\Lambda_k^{\mathbb{P}}(\beta, L_{k+2}) = \mathbb{P}\big(\Lambda_{k+1}^{\mathbb{A}}(\tilde{\beta}, Cone(L_{k+2}))\big)\,,$$

where $\tilde{\beta}$ is the constructible function on $Cone(X)$ induced by β with respect the Whitney stratification $\{\pi^{-1}(S_\alpha)\} \cup \{\{\underline{0}\}\}$ and $\pi : \mathbb{C}^{N+1} \setminus \{\underline{0}\} \to \mathbb{P}^N$ is the natural projection.

Summarizing we get that the affine Lê cycles restricted to every point in X give the local Lê cycles, and the projective Lê cycles are the projectivization of the affine Lê cycles of the affine cone defined by a projective variety.

7.5.3 Lê Classes and Milnor Classes

Lê cycles are originally associated to map-germs $\mathbb{C}^{n+1} \to \mathbb{C}$ and determine the diffeomorphism type of the Milnor fiber. These were extended above to invariants of projective manifolds. On the other hand Milnor classes are by definition the difference between two extensions of the classical Chern classes to the case of singular varieties. It was proved in [33] that these two concepts are remarkably linked together in a deep way. In fact the main result in [33] says that the information encoded in the Milnor classes is essentially equivalent to the information encoded in the Lê cycles. One has:

Theorem 7.5.6 *Let M be a smooth complex submanifold of $\mathbb{P}^N$ of dimension $n+1$, let Z be the hypersurface in M defined by the set of zeroes of a reduced holomorphic section s of a line bundle L on M. Set $h := c_1(\mathcal{O}_{\mathbb{P}^N}(1)|_Z)$ and denote by $\mathcal{M}_k(Z)$ the k-th Milnor class of Z. Then, for each $k = 0, \ldots, r = \dim(Z_{sing})$, there are cycles, obtained with respect to the choice of a linear subspace of $\mathbb{P}^N$, which give rise to well defined classes $\Lambda_k(Z)$ of Z in the Chow group and integral homology group of Z, that we call the global Lê classes of Z, and these are related to the Milnor classes $\mathcal{M}_k(Z)$ by the formulas:*

$$\mathcal{M}_k(Z) = \sum_{j\geq 0}\sum_{i\geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(L|_Z)^j h^{i-k-j} \cap \Lambda_i(Z)$$

and conversely:

$$\Lambda_k(Z) = \sum_{j\geq 0} (-1)^{k+j} \binom{k+j+1}{k+1} h^j \cap \big(\mathcal{M}_{k+j}(Z) + c_1(L|_Z)\mathcal{M}_{k+j+1}(Z)\big)\,.$$

One gets the corollary below, which extends and strengthens [26, Corollary 5.13] in the hypersurface case:

Corollary *Assume M, L and Z are as above and equip M with a Whitney stratification $\{Z_\beta\}$ adapted to Z. Let d be the dimension of the singular set Z_{sing}. Then we have the following equalities of cycles in the Chow group of Z:*

$$\mathcal{M}_d(Z) \;=\; \sum_{S_\beta \subset Z_{sing}} \mu^{\perp}(S_\beta)\,[\overline{S}_\beta] \;=\; \sum_{S_\beta \subset Z_{sing}} \lambda^d_{S_\beta}\,[\overline{S}_\beta] \;=\; (-1)^d \Lambda_d(Z)\,,$$

where the sums run over the strata of dimension d which are contained in Z_{sing}, $\mu^{\perp}(S_\beta)$ is the transversal Milnor number of S_β and $\lambda^d_{S_\beta}$ is the d-th Lê number of S_β.

The trail for getting to Theorem 7.5.6 can be roughly described as follows. The first step is recalling the main theorem of A. Parusinski and P. Pragacz in [101], Theorem 7.4.9 above. This expresses the total Milnor class as a function of the Schwartz-MacPherson classes of the closure of the strata of a Whitney stratification:

$$\mathcal{M}(Z) := \sum_{S_\alpha \in \mathcal{S}} \gamma_{S_\alpha} \left(c(L_{|Z})^{-1} \cap (i_{S_\alpha,Z})_* c^{SM}(\overline{S}_\alpha) \right). \tag{7.21}$$

Then one has the aforementioned MacPherson cycles [113], associated to any constructible function on a complex algebraic proper subset $X \subset \mathbb{C}^N$ that represent the (dual) Schwartz-MacPherson classes in the Borel-Moore homology group, and also in the Chow group. We already described above the analogous result in the projective case. In this construction a key role is played by the projective polar varieties.

Next one uses R. Piegne characterization in [102] of the Mather classes via polar varieties to give a formula for the Schwartz-MacPherson classes in terms of polar varieties and the normal Morse indices. Finally we use the above described characterization of the global Lê cycles for constructible sheaves via polar varieties. This also answers a question raised by J.-P. Brasselet.

7.6 Motivic and Hirzebruch-Milnor Classes

The theory of Chern classes for singular varieties keeps growing fastly and the literature is vast. The work we speak about in this section mostly concerns work done by P. Aluffi, J.-P. Brasselet, L. Maxim, J. Schürmann and S. Yokura, and we refer to the literature for more on the subject, particularly to [6, 29, 41, 54, 82, 85, 114, 136]. Here we only glance at some of the main topics.

For other important developments, the reader may also consult the following papers: [9, 17, 36–38, 41, 42, 44, 45, 86–91, 93, 111, 137].

7.6.1 Motivic Classes

A resolution $\pi : \widetilde{X} \to X$ of a normal singular variety is *crepant* [104] if the pullback of the canonical divisor class on X is numerically the canonical divisor class on $\widetilde{X}$, i.e., the (*discrepancy*) divisor $K_{\widetilde{X}} - \pi^* K_X$ is numerically equivalent to zero. Suppose X is such that its canonical divisor K_X is $\mathbf{Q}$-Cartier (i.e., mK_X is Cartier for some integer m) and consider a resolution $\pi : \widetilde{X} \to X$. The canonical divisor of $\widetilde{X}$ is:

$$K_{\widetilde{X}} = \pi^* K_X + \sum_i a_i E_i ,$$

where the sum is over the irreducible exceptional divisors and the a_i are rational numbers, called the discrepancies (so in a crepant resolution there are no discrepancies, hence the name). The singularities of X are canonical if $a_i \geq 0$ for all i, and they are terminal if $a_i > 0$ for all i. These concepts were introduced by M. Reid in 1980 as part of the canonical and minimal models of a projective variety.

The variety X is *Gorenstein* if at each point its local ring is Cohen-Macauley and the dualizing sheaf is locally free. In [13] V. V. Batyrev gives the following definition:

Definition 7.6.1 A normal projective algebraic variety X is a *(weak) Calabi-Yau* variety if X has at worst Gorenstein canonical singularities and the canonical line bundle on X is trivial.

The notion of "Calabi-Yau" usually requires certain vanishing conditions on the cohomology of the structure sheaf. However, these are not needed for the following remarkable theorem of V. V. Batyrev [12, 13], which somehow gave rise to the birth of motivic integration, due to Kontsevich:

Theorem 7.6.2 *If X is a complex projective (weakly) Calabi-Yau variety with at worst canonical Gorenstein singularities, and $\pi_i : \widetilde{X}_i \to X$, $i = 1, 2$, are two crepant resolutions, then $\widetilde{X}_1$ and $\widetilde{X}_2$ have the same Betti numbers $h_i = \dim H^i(\ , \mathbb{C})$.*

Kontsevich, in his (December 7th) 1995 talk at Orsay [69], explained a direct approach for proving Batyrev's theorem avoiding p-adic integration and the Weil conjectures, but involving arc spaces: this led to motivic integration and proved:

Theorem 7.6.3 (Kontsevich) *Birationally equivalent smooth Calabi-Yau varieties have the same Hodge numbers*

We do not discuss here motivic integration; we refer for this to the literature, as for instance [76]. Motivic integration has led to an avalanche of applications. These include new so-called stringy invariants of singularities. In the vein of Batyrev's theorem, one has the following theorem by to P. Aluffi in [6] concerning the total Chern class of the tangent bundle, regarded as a class in the Chow group:

Theorem 7.6.4 *Let $\phi : V \to W$ be a birational morphism of nonsingular algebraic varieties over an algebraically closed field of characteristic 0. Assume that there is a resolution of indeterminacies of ϕ, Z:*

$$\begin{array}{ccccc} & & Z & & \\ & \stackrel{v}{\swarrow} & & \stackrel{w}{\searrow} & \\ V & & \xrightarrow[\phi]{\qquad} & & W \end{array}$$

such that v and w are proper and birational, and the Jacobian ideals of v and w coincide. Then there exists a class $C \in (A_*Z)_{\mathbb{Q}}$ such that:

$$c(TV) \cap [V] = v_*(C) \qquad \textit{and} \qquad c(TW) \cap [W] = w_*(C)\ ,$$

in $(A_*V)_{\mathbb{Q}}$ and $(A_*W)_{\mathbb{Q}}$ respectively.

One thus gets a type of Chern class for Y with a motivic flavor [6, Corollary 1.2]:

Corollary 7.6.5 *Let $\alpha : Y \to X$ be a crepant resolution. Then the class*

$$\alpha_*(c(TY) \cap [Y])$$

in $(A_*X)_{\mathbb{Q}}$ is independent of Y.

In fact, one has the notion of K-equivalence of varieties, where K refers to the canonical divisor, and Aluffi's work shows that after passing to rational coefficients, the Chern classes of two K-equivalent smooth varieties are the push-forward of the same class on a resolution of the indeterminacies. K-equivalence has its origin in the minimal model program where, starting with a smooth variety X (of non-negative Kodaira dimension), one tries to produce a minimal model of X. Roughly speaking this is a variety Y with mild singularities, birational to X, but whose canonical class K_Y is in some sense smallest among all such varieties birational to X. Minimal models are not necessarily unique; this and other considerations led to look for ways of putting some sort of "ordering" on the elements of the birational equivalence class of X by means of comparing canonical bundles. This led to the following:

Definition 7.6.6 Let X and Y be smooth projective varieties over the complex numbers. They are *K-equivalent* if there exists a smooth projective variety Z and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $f^*\omega_X$ is equivalent to $g^*\omega_Y$, where $\omega(\ )$ is the canonical bundle.

In the same vein as [6] one has [54] where the authors introduce a theory of Chern classes for singular varieties, called *stringy Chern classes*, with strong birational properties and an interesting orbifold interpretation when the variety is a Gorenstein quotient of a manifold by a finite group. For this, they combine techniques of motivic integration with the transformation defined by MacPherson and discussed previously. The main theorem in [54] is:

Theorem 7.6.7 *The stringy Chern class satisfies:*

- *It coincides with the usual homology Chern class when X is non-singular; and*
- *If X and X' are K-equivalent and*

$$\begin{array}{ccc} & Y & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ X & \xrightarrow[\phi]{} & X' \end{array}$$

is a common resolution, then $c_{str}(X) = f_*(C)$ and $c_{str}(X') = f'_*(C)$ for some $C \in A_*(Y)_{\mathbb{Q}}$.

A simplification of the stringy Chern class was later given in [14], where the authors give applications to mirror symmetry for Calabi-Yau complete intersections in toric varieties. We refer also to [29, 41, 41, 85, 87, 91, 108, 114, 115] and the following section for alternative viewpoints and further discussions on motivic Chern classes for singular varieties.

7.6.2 The Motivic Hirzebruch-Milnor Classes

To prove the generalization in [65] of Riemann-Roch, Hirzebruch introduced the χ_y-genus of a compact complex manifold, that specializes to the Euler characteristic, the arithmetic genus and the signature at $y = -1, 0, 1$, respectively. This led to what today are known as Hirzebruch classes of complex manifolds. We already spoke in Sect. 7.4.5 about the Todd class, Riemann-Roch and its generalizations to singular varieties. The case $y = 1$ is related to the signature and leads to what is known as the L-class. Let us say a few words about Hirzebruch classes in general. Our main references for this, besides [65], are works by Brasselet, Maxim, Schürmann, Yokura, Cappell, Shaneson and Saito, see for instance [29, 41, 82, 85, 108, 114].

We start with some basic notions from [65]. Given a commutative ring B with identity element 1, let $p_0 = 1$ and let $p_1, p_2, \ldots$ be the indeterminates. Consider the ring of polynomials $\mathcal{B} = B[p_1, p_2, \ldots]$. This can be graded by setting that the product $p_{j_1} p_{j_2} \ldots p_{j_r}$ has weight $j_1 + \ldots + j_r$. Then we can write:

$$\mathcal{B} = \sum_{k=0}^{\infty} \mathcal{B}_k ,$$

where $\mathcal{B}_k$ is the additive group of polynomials which contain only elements of weight k and $\mathcal{B}_0 = B$. Clearly $\mathcal{B}_r \cdot \mathcal{B}_s \subset \mathcal{B}_{r+s}$. A sequence $\{K_j\}$ of polynomials in the p_i with $K_0 = 1$ and $K_j \in \mathcal{B}_j$ is *multiplicative* if every identity of the form

$$1 + p_1 z + p_2 z^2 + \ldots = (1 + p_1' z + p_2' z^2 + \ldots)(1 + p_1'' z + p_2'' z^2 + \ldots)$$

implies an identity of the form:

$$\sum_{j=0}^{\infty} K_j(p_1, p_2, \ldots, p_i) z^j = \sum_{j=0}^{\infty} K_j(p_1', p_2', \ldots, p_i') z^j \sum_{j=0}^{\infty} K_j(p_1'', p_2'', \ldots, p_i'') z^j \ .$$

The *characteristic power sequence* of the multiplicative sequence is by definition:

$$K(1+z) = \sum_{j=0}^{\infty} b_j z^j \quad \text{with } b_0 = 1 \text{ and } b_j = K_j(1, 0, \ldots, 0) \in B \ .$$

We now introduce indeterminates $\beta_1, \beta_2, \ldots$ and regard the p_i as being the elementary symmetric functions in these variables:

$$1 + p_1 z + \ldots + p_m z^m \ = \prod_{i=1}^{m} (1 + \beta_i z) \ .$$

One has [65, Lemmas 1.2.1 and 1.2.2]:

Proposition 7.6.8 *The multiplicative sequence* $\{K_j\}$ *is completely determined by its characteristic power series* $Q(z) = K(1+z)$*, and to every formal power series* $Q(z) = \sum_{j=0}^{\infty} b_i z^i$ ($b_0 = 1$, $b_i \in B$) *there is an associated multiplicative sequence* $\{K_j\}$ *with* $K(1+z) = Q(z)$.

The Todd sequence of polynomials described in Sect. 7.4.5 corresponds to the formal power series $Q(z) = x/(1 - e^{-x})$. In that case the indeterminates are the Chern classes, and in the case of complex manifolds of complex dimension n the valued of the polynomial $\mathrm{Td}(c_1, \ldots, c_n)$ in the orientation cycle gives the Todd genus. Hirzebruch-Riemann-Roch's theorem says that this equals the arithmetic genus.

Also, the multiplicative sequence in indeterminates c_i with $1 + x^2$ as characteristic power series is $1, 0, p_1, 0, p_2, \ldots$. where

$$p_1 = -2c_2 + c_1^2, \ \ p_2 = 2c_4 - 2c_3 c_1 + c_2^2, \ \ p_3 = -2c_6 + 2c_5 c_1 - 2c_4 c_2 + c_3^2, \ \ldots$$

which are precisely the equations relating the Chern and Pontrjagin classes.

Recall that every closed oriented $4k$-manifold has associated a bilinear form in $H^{2k}(M; \mathbb{R})$ given by the cup product, and *its signature is by definition, the number* *of positive eigenvalues minus the number of negative eigenvalues*. The signature of this bilinear form is called the signature of M.

R. Thom had observed in [127] that for closed oriented (real) manifolds of dimensions 4 and 8, the signature is given respectively by $\frac{1}{3}(p_1(M))[M]$ and $\frac{1}{45}(7p_2(M) - p_1^2(M))[M]$, where the p_i are the Pontrjagin classes. This was a clue for Hirzebruch that introduced in the *L-sequence of polynomials*, another important multiplicative sequence; this corresponds to the formal power series

$$Q(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}} = 1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k z^k\,,$$

where the coefficients B_k are the Bernoulli numbers. The first two L-polynomials are those found by Thom: $L_1 = 1/3(p_1),\ L_2 = 1/45(7p_2 - p_1^2)$. Hirzebruch's theorem says that the signature of a $4k$ dimensional closed manifold M equals its L-genus.

Hirzebruch's proof of the generalized Riemann-Roch theorem in higher dimensions with coefficients in arbitrary vector bundles provided a unified view of the above genera by introducing what he called the χ_y-characteristic. This corresponds to considering the formal power series:

$$Q(y;x) := \frac{x(y+1)}{1 - e^{-x(y+1)}} - yx\,.$$

Hirzebruch points out [65, §1, p. 16] that:

- For $y = 0$ this is the series $x/(1 - e^{-x})$ associated to the Todd sequence;
- For $y = -1$ we have $Q(-1;x) = 1 + x$ that gives rise to the top Chern class c_n;
- For $y = 1$ we have $Q(1;x) = x/(\tanh x)$ that essentially gives the L-sequence.

So each multiplicative sequence of polynomials determines a type of cohomology classes of complex manifolds (the Todd class, the L-class, etc.); these are known as *Hirzebruch (cohomology) classes* for complex manifolds. One can also speak of Hirzebruch's homology classes. In the case of singular varieties we have already discussed how Chern classes extend. We also have the Todd class in the singular Riemann-Roch theorem of Baum-Fulton- MacPherson (Sect. 7.4.5); and one has the L-class transformations of Goreski-MacPherson[61] and Cappell-Shaneson [39].

In [29] Brasselet, Schürmann and Yokura introduced a motivic Chern Class transformation and a certain natural transformation T_{y*} that generalizes Hirzebruch's construction to the singular setting. T_{y*} is a homology class version of the motivic measure corresponding to a suitable specialization of the Hodge polynomial. This transformation remarkably unifies MacPherson's Chern class transformation (for $y = -1$), the Todd class transformation in the Baum-Fulton-MacPherson generalization of Riemann-Roch for singular varieties (for $y = 0$) and the L-class transformation of Cappell-Shaneson (for $y = 1$).

Hirzebruch also introduced in [65, §17] the notion of virtual χ_y characteristics for collections $(E_1, \ldots, E_r)$ of holomorphic line bundles and a holomorphic bundle W over a complex n-manifold M. The r-tuple $(E_1, \ldots, E_r)$ is called a *virtual submanifold* of M of complex dimension $n - r$. He then introduced an invariant in $\mathbb{Z}$ that he called the virtual χ_y-genus of the virtual submanifold. The virtual Hirzebruch classes can be defined likewise and in [85] Maxim, Schürmann and Saito show that the difference between the Hirzebruch class and the virtual one is given by what they call the Hirzebruch-Milnor class. This has support on the singular locus of X, and they prove an inductive formula to calculate it explicitly in the case of global complete intersections with arbitrary singularities. This generalizes the formula for the Chern-Milnor classes in the hypersurface case that was conjectured by S. Yokura (unpublished) and was proved by A. Parusinski and P. Pragacz [101]. It also generalizes a formula of J. Seade and T. Suwa [118] for the Chern-Milnor classes of complete intersections with isolated singularities.

Finally, a characteristic class version of the Steenbrink spectrum [125], termed the spectral Hirzebruch class, was introduced in [87] by taking into account the monodromy of the vanishing cycles of mixed Hodge modules. For hypersurfaces defined by global functions on smooth varieties, a Thom-Sebastiani type theorem for the spectral classes was obtained in [87], by using a corresponding Thom-Sebastiani theorem for the underlying filtered D-modules of vanishing cycles proved in [88]. Interestingly, these spectral characteristic classes can detect jumping coefficients of multiplier ideals, Du Bois singularities, and rational singularities for any globally defined hypersurface in a complex manifold.

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Chapter 8
Residues and Hyperfunctions

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Contents

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Abstract We discuss relative Čech-de Rham and relative Čech-Dolbeault cohomologies and their applications. In the de Rham case, we are mainly concerned with the residues that arise from the localization of characteristic classes via the Alexander duality. The relative Čech-de Rham theorem allows us to deal with the problem from both the topological and differential geometric viewpoints and the comparison of the two yields various interesting expressions of the residues and applications. In the Dolbeault case, the relative Čech-Dolbeault cohomology turns out to be canonically isomorphic with the relative cohomology of the sheaf of holomorphic forms. As an application, we give explicit expressions of Sato hyperfunctions and related operations including the embedding of the space of real analytic functions into that of hyperfunctions, where as well the Thom class plays an important role.

8.1 Introduction

Čech-de Rham cohomology, particularly its relative version, combined with the Chern-Weil theory has been extensively used in the localization problem of characteristic classes. It started with the study of residues of singular holomorphic foliations (cf. [55] and references therein) and the theory was then transferred to the index theory of holomorphic self-maps (e.g., [2]). The philosophy behind is rather simple. Namely, once we have some kind of vanishing theorem on the non-singular part of a geometric object such as a foliation, certain characteristic classes are localized at the set of singular points and the localization gives rise to residues and the residue theorem via the Alexander duality. Combined with the combinatorial description of the Alexander duality, we have various interesting expressions of the residues and applications.

The idea and the techniques turned out to be effective in many other problems including characteristic classes of singular varieties, localized intersection theory, Thom class, localized Riemann-Roch theorem for embeddings and so forth (cf. [11, 12, 54, 56, 59]).

A similar theory may be developed for the Dolbeault complex, the relevant characteristic classes in this case being the Atiyah classes (cf. [1, 60]). In particular, the relative Čech-Dolbeault cohomology turns out to be canonically isomorphic with the local (relative) cohomology of A. Grothendieck and M. Sato with coefficients in the sheaf of holomorphic forms. This gives a handy way of expressing the latter and would possibly lead to many applications. One of them is already apparent, i.e., we have a simple way of expressing the Sato hyperfunctions and some of the fundamental operations on them (cf [30, 63]).

This chapter reviews residues and hyperfunctions along the line described as above. It is organized as follows.

In Sect. 8.2, we recall the dualities on manifold s from the combinatorial viewpoint. In particular, the Alexander duality in this context (Theorem 8.2.2) is the fundamental instrument to describe residues and the residue theorem from the topological side. We discuss, in Sect. 8.3, Čech-de Rham cohomology and the integration theory on this cohomology. We then introduce its relative version and state a theorem asserting that the relative Čech-de Rham cohomology is canonically isomorphic with the relative singular or simplicial cohomology with $\mathbb{C}$-coefficients (Theorem 8.3.13). We then describe the Alexander duality in terms of relative Čech-de Rham cohomology. This constitutes a bridge between the localization theory from the topological side and that from the differential geometric side. In Sect. 8.4, we recall the obstruction theoretical definition of the Chern classes of complex vector bundles. The Euler class of real oriented vector bundles is also discussed. We then consider the localized classes in Sect. 8.5. We introduce the notion of topological residue for Chern classes, state the residue theorem (Theorem 8.5.4) and give an explicit expression of the topological residue (Theorem 8.5.7). We do similar considerations for Euler classes and recall the Poincaré-Hopf theorem as a prototype of the residue theorem. In Sect. 8.6, we review the Chern-Weil theory of characteristic classes and its modification to be adapted to Čech-de Rham cohomology. This is essential in expressing the localizations from the differential geometric side.

In Sect. 8.7, we explain general idea and techniques of localizing characteristic classes and state a general residue theorem (Theorem 8.7.3). We also give various ways of localizing characteristic classes. The localization of Chern classes by frames from the differential geometric viewpoint is discussed in Sect. 8.8. We recall residues and the residue theorem in this case and state that the topological and differential geometric localizations are essentially the same in the cases the both make sense (Theorem 8.8.8). The essential ingredient here is a differential geometric expression of the index of a family of sections (Theorem 8.8.6). We then apply these to the case of residue defined by a family of holomorphic sections of a holomorphic vector bundle and have a general expression of the residue (Theorem 8.8.10), which indicates that, in order to find the residue in the "proper case", it suffices to know the residue at an isolated singularity. We give topological, analytic and algebraic expressions of the residue at an isolated singularity in the basic case. From these, we may rephrase Theorem 8.8.10 as asserting that, for a family of holomorphic sections of a holomorphic vector bundle, the localization of the Chern class by the sections corresponds to the complex space defined by the sections via the Alexander duality (Theorem 8.8.14). In Sect. 8.9, we discuss the Thom isomorphism and the Thom class from topological and differential geometric viewpoints and make comparisons of the two. The Thom class of an oriented real vector bundle is interpreted as a localized Euler class and that of a complex vector bundle a localized top Chern class (Theorems 8.9.9 and 8.9.20). In fact the Thom class is a universal localization in each of the above cases (Remark 8.9.10. 2 and Theorem 8.9.22). We emphasize that the expression of the Thom class in relative Čech-de Rham cohomology (Theorem 8.9.18 and Remark 8.9.21) is effectively used in a number of problems, including the fixed point and coincidence point formulas of Lefschetz type as well as the hyperfunction theory (Remark 8.9.23. 3).

In Sect. 8.10, we review Dolbeault, Čech-Dolbeault as well as relative Čech-Dolbeault cohomologies almost in parallel with the de Rham case. There is the relative Čech-Dolbeault theorem asserting that the relative Čech-Dolbeault cohomology is canonically isomorphic with the relative cohomology with coefficients in the sheaf of holomorphic forms (Theorem 8.10.9). This is the key to represent hyperfunctions in terms of Čech-Dolbeault cocycles and to bring in the tools from complex geometry to the theory of hyperfunctions. We then discuss two cases where there is a direct relation between the de Rham and the Dolbeault complexes. The first one is used to define integration on the Čech-Dolbeault cohomology and the second one to have an explicit embedding morphism of the real analytic functions into the hyperfunctions. We also make consideration of local duality in parallel with the Alexander duality. Unlike the topological case, we do not have a duality isomorphism in general. One of the cases we do is given in Theorem 8.11.5 in Sect. 8.11, where we also discuss the local residue paring. These are closely related to the hyperfunction theory. We discuss in Sect. 8.12, the aforementioned application of relative Čech-Dolbeault cohomology to the Sato hyperfunction theory. We give explicit expression of hyperfunctions and fundamental operations on them. Particularly noteworthy here is that the integral of hyperfunctions are expressed as a usual Stokes type integral (8.50). We also introduce the δ-function and the δ-form in our framework. The latter has essentially the same expression as the Thom class of a trivial bundle (Remark 8.12.6. 1). Finally we explain how to regard a real analytic function as a hyperfunction. The Thom class in relative Čech-de Rham cohomology plays an essential role in this scene as well, namely the embedding morphism is constructed using its image in the relative Čech-Dolbeault cohomology by the canonical morphism mentioned above (cf. (8.52)).

This is an expository article and the original literatures and general references are cited in each place to be referred to for details. Besides these, Sects. 8.2–8.9 are based on [58, 59, 64] and Sects. 8.10–8.12 on [30, 63].

Notation and Conventions

1. $\mathbb{Z}$: the ring of integers.
2. $\mathbb{R}^m = \{(x_1, \dots, x_m)\}$: the real m-space. As a C^∞ manifold, it is oriented so that $(x_1, \dots, x_m)$ is a positive coordinate system.
3. $\mathbb{C}^n = \{(z_1, \dots, z_n)\}$: the complex n-space. As a C^∞ manifold, it is orientable. In Sects. 8.2–8.9, it is oriented the "usual" way, i.e., so that $(x_1, y_1, \dots, x_n, y_n)$ is a positive coordinate system when we write $z_i = x_i + \sqrt{-1}y_i$, $i = 1, \dots, n$. While in Sects. 8.10–8.12, it is oriented, however the orientation may not be the usual one.
4. The orientation convention as in 3 above applies also to complex manifolds.

8.2 Poincaré and Alexander Dualities

In this section, we let M denote a C^∞ manifold of dimension m. Also we take $\mathbb{Z}$ as the coefficient of homology and cohomology.

8.2.1 Algebraic Topology on Manifolds

In order to describe duality theorems for manifolds, we recall some basic notions in algebraic topology, particularly from the combinatorial viewpoint. We list [22, 27, 51] as general references for algebraic topology. See [45] for C^∞ triangulations.

C^∞ Triangulations Let X be a topological space. A *triangulation* of X is a pair (K, h) of a simplicial complex K and a homeomorphism $h : |K| \to X$, where $|K|$ is the polyhedron of K. Let Y be a subspace of X. We say that the triangulation is compatible with Y if there is a subcomplex L of K such that the restriction of h to $|L|$ is a triangulation of Y.

A triangulation $h : |K| \to M$ of a C^∞ manifold M is C^∞ if, for every simplex $\mathbf{s}$ of K, $h|_{\mathbf{s}}$ is C^∞ and its rank at each point of $\mathbf{s}$ is equal to $\dim \mathbf{s}$. Here we think of $\mathbf{s}$ as being in the affine space spanned by $\mathbf{s}$ and $h|_{\mathbf{s}}$ being C^∞ means that it admits a C^∞ extension near each point in $\mathbf{s}$.

The following is known (cf. [45, 66]):

1. Every C^∞ manifold M admits a C^∞ triangulation. In fact, if R is a closed C^∞ submanifold of M possibly with boundary, there is a C^∞ triangulation of M compatible with R and ∂R.
2. If K_1 and K_2 are C^∞ triangulations of M, there exist subdivisions of K_1 and K_2 that are simplicially isomorphic.

We take a triangulation (K_0, h) of M and let K denote the barycentric subdivision of K_0. We further let K' be the barycentric subdivision of K, i.e., the second barycentric subdivision of K_0. We take the second barycentric subdivision so that the star of a K_0-subcomplex L of K_0 with respect to K' has the same homotopy type as the polyhedron $|L|$ of L. In the sequel, a simplex $\mathbf{s}$ of K is identified with $h(\mathbf{s})$ and $|K|$ is identified with M.

Dual Cellular Decomposition For a p-simplex $\mathbf{s}$ of K, we denote by $\mathbf{s}^*$ the union of $(m-p)$-simplices of K' intersecting with $\mathbf{s}$ at its barycenter $b_{\mathbf{s}}$. It is a regular closed $(m-p)$-cell in $|K|$, called the cell dual to $\mathbf{s}$. The intersection of $\mathbf{s}$ and $\mathbf{s}^*$ consists of the one point $b_{\mathbf{s}}$. The cells dual to simplices in K form a cellular decomposition of $|K| = M$, which will be denoted by K^*.

Orientations of Simplices and Cells In order to describe the homology and cohomology of M via triangulation or dual cellular decomposition, we fix orientations of simplices of K and cells of K^*. As to the orientations of simplices of K', we impose the following conditions. Thus let $\mathbf{t}$ be a p-simplex of K'.

(1) If $\mathbf{t} \subset \mathbf{s}$, a p-simplex of K, the orientation of $\mathbf{t}$ is the same as that of $\mathbf{s}$.
(2) If $\mathbf{t} \subset (\mathbf{s}')^*$, a p-cell of K^*, the orientation of $\mathbf{t}$ is the same as that of $(\mathbf{s}')^*$.

Note that for $\mathbf{t}$ not satisfying either of the above assumptions, there is still freedom of choice of the orientation.

Homology and Cohomology We denote by $H_p(M)$ the p-th singular homology of M. An important feature in the case of a manifold is that it can be computed using either the triangulation K or the cellular decomposition K^* in the following sense. Thus let $(C_\bullet^K(M), \partial)$ be the chain complex with $C_p^K(M)$ the free Abelian group generated by the oriented p-simplices in K and $\partial : C_p^K(M) \to C_{p-1}^K(M)$ the boundary operator defined by

$$\partial(v_0, \dots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \widehat{v_i}, \dots, v_p),$$

for an oriented simplex $\mathbf{s} = (v_0, \dots, v_p)$ with vertices $v_0, \dots, v_p$, and extended linearly. We denote by $H_p^K(M)$ the p-th homology of $C_\bullet^K(M)$. Denoting by $S_p(M)$ the group of singular p-chains of M, there is a natural chain morphism $C_\bullet^K(M) \to S_\bullet(M)$, which induces an isomorphism on the homology level:

$$H_p^K(M) \xrightarrow{\sim} H_p(M).$$

Also if we denote by $(C_\bullet^{K^*}(M), \partial)$ the chain complex with $C_p^{K^*}(M)$ the free Abelian group generated by the oriented p-cells in K^*, we have a natural isomorphism:

$$H_p^{K^*}(M) \xrightarrow{\sim} H_p(M), \tag{8.1}$$

where $H_p^{K^*}(M)$ is the p-th homology of $C_\bullet^{K^*}(M)$. In our case we have injective chain morphisms

$$C_\bullet^{K^*}(M) \xrightarrow{\alpha} C_\bullet^{K'}(M) \longrightarrow S_\bullet(M), \tag{8.2}$$

where α is the morphism that regards a K^*-chain as a K'-chain, and the composition induces the isomorphism (8.1).

Also the singular cohomology $H^p(M)$ of M can be computed either from the cochain complex $(C_K^\bullet(M), \delta)$ with $C_K^p(M) = \mathrm{Hom}(C_p^K(M), \mathbb{Z})$ or from the cochain complex $(C_{K^*}^\bullet(M), \delta)$ with $C_{K^*}^p(M) = \mathrm{Hom}(C_p^{K^*}(M), \mathbb{Z})$. That is to say that the transposes of the chain morphisms above induce isomorphisms

$$H^p(M) \xrightarrow{\sim} H_K^p(M) \quad \text{and} \quad H^p(M) \xrightarrow{\sim} H_{K^*}^p(M).$$

Denoting by $\breve{C}_\bullet^K(M)$ and $\breve{S}_\bullet(M)$ the chain complexes of locally finite chains of K and of locally finite singular chains of M, respectively, we have a chain morphism $\breve{C}_\bullet^K(M) \to \breve{S}_\bullet(M)$ as before. It induces an isomorphism on the homology level:

$$\breve{H}_p^K(M) \xrightarrow{\sim} \breve{H}_p(M).$$

Likewise, considering the complex $\breve{C}_\bullet^{K^*}(M)$ of locally finite chains of K^*, we have a canonical isomorphism $\breve{H}_p^{K^*}(M) \xrightarrow{\sim} \breve{H}_p(M)$.

In the sequel, $\langle\ ,\ \rangle$ denotes the paring of chains and cochains, i.e., the Kronecker product.

8.2.2 *Poincaré and Alexander Dualities*

We discuss the dualities from the combinatorial viewpoint, following the descriptions as given in [10], except for the orientation convention.

Let M, K_0, K, K' and K^* be as in the previous subsection. In this subsection we assume that M is oriented and take orientations of the simplices and cells so that they satisfy the conditions (1) and (2) in Sect. 8.2.1 and that they are furthermore compatible with that of M in the following sense:

(3) The orientation of each m-simplex is the same as that of M.
(4) For every p-simplex $\mathbf{s}$ of K, $0 < p < m$, the orientation of $\mathbf{s}^*$ followed by the orientation of $\mathbf{s}$ gives the orientation of M.

Poincaré Duality We define a morphism

$$P : C_{K^*}^p(M) \longrightarrow \breve{C}_{m-p}^K(M) \qquad \text{by } u \mapsto \sum_{\mathbf{s}} \langle \mathbf{s}^*, u\rangle\, \mathbf{s}, \tag{8.3}$$

for a p-cochain u of K^*, where the sum is taken over all $(m-p)$-simplices $\mathbf{s}$ in M. The morphism P is in fact an isomorphism on the chain and cochain level. We may actually prove that it is compatible with boundary and coboundary operators so that it induces an isomorphism between the corresponding cohomology and homology. To be a little more precise, let $M_{K'}$ denote the fundamental cycle of M in K', i.e., the sum of all m-simplices of K'. There is a natural injective morphism $\alpha : C_p^{K^*}(M) \to C_p^{K'}(M)$ (cf. (8.2)), whose transpose is denoted by α^*. Likewise there is a natural injective morphism $\check{C}_{m-p}^K(M) \to \check{C}_{m-p}^{K'}(M)$, which is denoted by β. The key point in the proof is to show that the following diagram is commutative:

$$\begin{array}{ccc} C^p_{K^*}(M) & \xrightarrow{P} & \check{C}^K_{m-p}(M) \\ {\scriptstyle\alpha^*}\uparrow & & \downarrow{\scriptstyle\beta} \\ C^p_{K'}(M) & \xrightarrow{M_{K'}\frown} & \check{C}^{K'}_{m-p}(M), \end{array} \tag{8.4}$$

where the morphism in the bottom is the (left) cap product with $M_{K'}$. As a consequence, using properties of cap product, we have

$$P(\delta u) = (-1)^{p+1}\partial P(u) \qquad \text{for} \quad u \in C^p_{K^*}(M) \tag{8.5}$$

so that we have:

Theorem 8.2.1 (Poincaré Duality) *For an oriented C^∞ manifold M of dimension m, the isomorphism P of* (8.3) *induces an isomorphism*

$$P_M : H^p(M) \xrightarrow{\sim} \check{H}_{m-p}(M).$$

We denote by $[M]$ the class in $\check{H}_m(M)$ corresponding to $[M_{K'}]$ by the isomorphism $\check{H}_m^{K'}(M) \simeq \check{H}_m(M)$ and call it the *fundamental class of* M. Note that it does not depend on the choice of the triangulation by the property 2 of C^∞ triangulations (cf. Sect. 8.2.1). By the commutativity of (8.4), we may write

$$P_M(u) = [M] \frown u \qquad \text{for} \ \ u \in H^p(M).$$

Thus the Poincaré duality is a topological invariant.

Alexander Duality Let S be a closed set in M. Suppose that there is a triangulation K_0 of M such that S is a K_0-subcomplex of M. Recall that the *star* $S_{K'}(S)$ of S in K' is the union of simplices of K' intersecting with S. Let $O_{K'}(S) = S_{K'}(S) \smallsetminus \partial S_{K'}(S)$ denote the *open star*. Note that there is a proper deformation retraction $S_{K'}(S) \to S$ and a deformation retraction $O_{K'}(S) \to S$.

We note that, for a simplex $\mathbf{s}$ of K, the following three conditions are equivalent:

$$(1)\ \mathbf{s} \subset S, \qquad (2)\ \mathbf{s}^* \cap S \neq \emptyset, \qquad (3)\ \mathbf{s}^* \cap O_{K'}(S) \neq \emptyset.$$

Noting that $M \smallsetminus O_{K'}(S)$ is a K^*-subcomplex of M, we may write

$$C^p_{K^*}(M, M \smallsetminus O_{K'}(S)) = \{\, u \in C^p_{K^*}(M) \mid \langle \mathbf{s}^*, u\rangle = 0 \ \text{ for } \mathbf{s} \not\subset S \,\}.$$

They form a subcomplex of $C^{\bullet}_{K^*}(M)$. Denoting by $H^p_{K^*}(M, M \smallsetminus O_{K'}(S))$ its cohomology, we have a natural isomorphism:

$$H^p(M, M \smallsetminus O_{K'}(S)) \xrightarrow{\ \sim\ } H^p_{K^*}(M, M \smallsetminus O_{K'}(S)).$$

Note that there is a natural isomorphism $H^p(M, M \smallsetminus O_{K'}(S)) \simeq H^p(M, M \smallsetminus S)$.

Now in the sum in (8.3), if u is in $C^p_{K^*}(M, M \smallsetminus O_{K'}(S))$, only $(m-p)$-simplices in S appear. Thus P in (8.3) induces an isomorphism

$$A : C^p_{K^*}(M, M \smallsetminus O_{K'}(S)) \longrightarrow \check{C}^K_{m-p}(S). \tag{8.6}$$

Since $C^{\bullet}_{K^*}(M, M \smallsetminus O_{K'}(S))$ is a subcomplex of $C^{\bullet}_{K^*}(M)$ and $\check{C}^K_{\bullet}(S)$ is a subcomplex of $\check{C}^K_{\bullet}(M)$, by (8.5), we see that A is also compatible with boundary and coboundary operators. Thus we have:

Theorem 8.2.2 (Alexander Duality) *For a closed set S in M as above, the isomorphism* (8.6) *induces an isomorphism*

$$A_{M,S} : H^p(M, M \smallsetminus S) \xrightarrow{\ \sim\ } \check{H}_{m-p}(S).$$

From the construction, we have the following commutative diagram:

$$\begin{array}{ccc} H^p(M, M \smallsetminus S) & \xrightarrow{\ j^*\ } & H^p(M) \\ {\scriptstyle A}\downarrow{\wr} & & {\wr}\downarrow{\scriptstyle P} \\ \check{H}_{m-p}(S) & \xrightarrow{\ i_*\ } & \check{H}_{m-p}(M), \end{array} \tag{8.7}$$

where j^* denotes the canonical morphism and $i : S \hookrightarrow M$ the inclusion.

Example 8.2.3 For $\mathbb{R}^m$, we have:

$$H^p(\mathbb{R}^m; \mathbb{Z}) \xrightarrow[P]{\ \sim\ } \check{H}_{m-p}(\mathbb{R}^m; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H^p(\mathbb{R}^m, \mathbb{R}^m \smallsetminus \{0\}; \mathbb{Z}) \xrightarrow[A]{\ \sim\ } \check{H}_{m-p}(\{0\}; \mathbb{Z}) = H_{m-p}(\{0\}; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & p = m, \\ 0 & \text{otherwise.} \end{cases}$$

In the first isomorphism for $p = 0$, $P([1]) = [\mathbb{R}^m] \leftrightarrow 1$, where $[1]$ denotes the class of the cocycle that assigns 1 to each 0-cell and $[\mathbb{R}^m]$ the fundamental class. In the second for $p = m$, $A([u_m]) = [\{0\}] \leftrightarrow 1$, where u_m is the cocycle that assigns 1 to the dual cell of the vertex 0 and 0 to other m-cells.

Remark 8.2.4 See [24] as to the topological invariance of the Alexander duality.

8.2.3 Pseudo-Manifolds

As a generalization of manifold, we introduce the following:

Definition 8.2.5 A *pseudo-manifold* X of dimension d in M is a subcomplex of M, with respect to some triangulation of M, satisfying the following conditions:

(1) Every simplex in X is a face of some d-simplex in X.
(2) Every $(d-1)$-simplex is the face of exactly two d-simplices.
(3) The d-simplices in X can be oriented so that, if $\mathbf{s}$ is a $(d-1)$-simplex in X and if $\mathbf{s}_1$ and $\mathbf{s}_2$ are the two simplices that contain $\mathbf{s}$ in their boundary, then the prescribed orientations of $\mathbf{s}_1$ and $\mathbf{s}_2$ induce opposite orientations of $\mathbf{s}$.

A pseudo-manifold X is said to be *oriented*, once orientations of d-simplices in X satisfying (3) above are fixed. We say that X is *irreducible* if $X \smallsetminus X^{d-2}$ is connected, where X^{d-2} denotes the $(d-2)$-skeleton of X. A *general point* of X is a point in $X \smallsetminus X^{d-2}$. Then we have a decomposition into irreducible components:

$$X = \bigcup_i X_i.$$

If X is oriented, each X_i carries a fundamental cycle, the union of d-simplices in X_i and it defines a class in $H_d(X)$. In fact $H_d(X)$ is generated by these classes.

For an oriented pseudo-manifold, we may define the Poincaré morphism.

Example 8.2.6 A closed submanifold of a C^∞ manifold M is a non-singular pseudo-manifold.

An analytic variety V in a complex manifold is a pseudo-manifold. The irreducible decomposition of V as a variety gives that as a pseudo-manifold. A general point of V is nothing but a non-singular point of V.

8.3 de Rham and Relative Čech-de Rham Theorems

In this section, we let M denote a C^∞ manifold of dimension m. Also, for an open set U in M, let $\mathscr{E}^{(p)}(U)$ denote the $\mathbb{C}$-vector space of complex valued C^∞ p-forms on U.

8.3.1 de Rham Cohomology

The exterior derivative defines a complex of $\mathbb{C}$-vector spaces:

$$0 \longrightarrow \mathscr{E}^{(0)}(M) \xrightarrow{d^0} \mathscr{E}^{(1)}(M) \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} \mathscr{E}^{(m)}(M) \longrightarrow 0,$$

which is called the *de Rham complex* of M and is denoted by $(\mathscr{E}^{(\bullet)}(M), d)$.

Definition 8.3.1 The p-th *de Rham cohomology* of M is the p-th cohomology of $(\mathscr{E}^{(\bullet)}(M), d)$:

$$H_d^p(M) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

For a closed p-form ω, its class in $H_d^p(M)$ is denoted by $[\omega]$. Since $\operatorname{Ker} d^0 = \mathbb{C}(M)$, the complex valued locally constant functions on M, we have:

$$H_d^0(M) = \mathbb{C}(M).$$

In the case M is connected, $H_d^0(M) = \mathbb{C}$.

In the below we discuss the de Rham theorem which states that $H_d^p(M)$ is canonically isomorphic with the singular, simplicial or cellular cohomology $H^p(M; \mathbb{C})$ of M (cf. Theorem 8.3.2 below). One of the essential ingredients for this is the Poincaré lemma, which asserts that the de Rham complex of $\mathbb{R}^m$ is acyclic, i.e.,

$$H_d^p(\mathbb{R}^m) = 0 \qquad \text{for} \quad p > 0.$$

Compare this with the topological counterpart in Example 8.2.3.

de Rham Theorem We have a natural morphism

$$\mathscr{E}^{(p)}(M) \longrightarrow C_K^p(M; \mathbb{C}), \tag{8.8}$$

which assigns to each p-form ω the cochain $\mathbf{s} \mapsto \int_{\mathbf{s}} \omega$. By Stokes' formula, this morphism is compatible with d and δ and induces a morphism on the cohomology level. In fact we have:

Theorem 8.3.2 (de Rham) *The above induces an isomorphism*

$$H_d^p(M) \simeq H^p(M; \mathbb{C}).$$

Suppose M is oriented. In (8.8), we could use the cochain group $C_{K^*}^p(M; \mathbb{C})$ as well and may describe the Poincaré duality with $\mathbb{C}$-coefficient

$$H_d^p(M) \xrightarrow{\sim} H^p(M; \mathbb{C}) \xrightarrow[P]{\sim} \breve{H}_{m-p}(M; \mathbb{C})$$

as being induced from $\omega \mapsto \sum \int_{\mathbf{s}^*} \omega \cdot \mathbf{s}$.

Cup Product The exterior product of forms induces a product in $H_d^*(M)$, which corresponds to the cup product in $H^*(M;\mathbb{C})$ via the isomorphism of Theorem 8.3.2.

If M is compact and oriented, we also have the integration $\int_M : H_d^m(M) \to \mathbb{C}$ and we have a bilinear pairing as the composition:

$$H_d^p(M) \times H_d^{m-p}(M) \xrightarrow{\wedge} H_d^m(M) \xrightarrow{\int_M} \mathbb{C}.$$

The Poincaré duality in this case says that the pairing is non-degenerate and induces an isomorphism

$$H_d^p(M) \xrightarrow{\sim} H_d^{m-p}(M)^*.$$

8.3.2 Čech-de Rham Cohomology

The Čech-de Rham cohomology may be defined for an arbitrary covering of a manifold. Here we only consider coverings consisting of two open sets and refer to [9, 55] for details and the general case.

Let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of M. We set $U_{01} = U_0 \cap U_1$ and define the vector space $\mathscr{E}^{(p)}(\mathcal{U})$ as

$$\mathscr{E}^{(p)}(\mathcal{U}) = \mathscr{E}^{(p)}(U_0) \oplus \mathscr{E}^{(p)}(U_1) \oplus \mathscr{E}^{(p-1)}(U_{01}).$$

Thus an element σ in $\mathscr{E}^{(p)}(\mathcal{U})$ is given by a triple $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ with σ_0 a p-form on U_0, σ_1 a p-form on U_1 and σ_{01} a $(p-1)$-form on U_{01}. We define an operator $D = D^p : \mathscr{E}^{(p)}(\mathcal{U}) \to \mathscr{E}^{(p+1)}(\mathcal{U})$ by

$$D\sigma = (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01}).$$

Then it is not difficult to see that $D \circ D = 0$. Thus we have a complex $(\mathscr{E}^{(\bullet)}(\mathcal{U}), D)$, which is called the *Čech-de Rham complex* of $\mathcal{U}$.

Definition 8.3.3 The p-th *Čech-de Rham cohomology* of $\mathcal{U}$ is the p-th cohomology of $(\mathscr{E}^{(\bullet)}(\mathcal{U}), D)$:

$$H_D^p(\mathcal{U}) = \operatorname{Ker} D^p / \operatorname{Im} D^{p-1}.$$

We have the following:

Theorem 8.3.4 *The morphism* $\mathscr{E}^{(p)}(M) \to \mathscr{E}^{(p)}(\mathcal{U})$ *given by* $\omega \mapsto (\omega|_{U_0}, \omega|_{U_1}, 0)$ *induces an isomorphism*

$$H_d^p(M) \xrightarrow{\sim} H_D^p(\mathcal{U}).$$

Note that the inverse of the above isomorphism is given by assigning to the class of a cocycle $(\sigma_0, \sigma_1, \sigma_{01})$, the class of a global closed form $\rho_0\sigma_0 + \rho_1\sigma_1 - d\rho_0 \wedge \sigma_{01}$, where $\{\rho_0, \rho_1\}$ is a C^∞ partition of unity subordinate to $\mathcal{U}$.

Integration We recall the integration on Čech-de Rham cohomology (cf. [39, 55]). Let M and $\mathcal{U} = \{U_0, U_1\}$ be as above.

Definition 8.3.5 A *honeycomb system* adapted to $\mathcal{U}$ is a pair $\{R_0, R_1\}$ of m-dimensional piecewise C^∞ manifolds with boundary in M such that

(1) $R_i \subset U_i$, for $i = 0,\ 1$,
(2) $\operatorname{Int} R_0 \cap \operatorname{Int} R_1 = \emptyset$,
(3) $R_0 \cup R_1 = M$,

where "Int" means the interior.

Suppose M is oriented. We endow R_i, $i = 0, 1$, with the orientation same as that of M. Let $R_{01} = R_0 \cap R_1$ and give R_{01} the orientation as the boundary of R_0; $R_{01} = \partial R_0$, equivalently give R_{01} the orientation opposite to that of the boundary of R_1; $R_{01} = -\partial R_1$.

Suppose moreover that M is compact and let $\{R_0, R_1\}$ be as above. Then each R_i is compact, for $i = 0,\ 1$, and we may define the integration

$$\int_M : \mathscr{E}^{(m)}(\mathcal{U}) \longrightarrow \mathbb{C} \quad \text{by} \quad \int_M \sigma = \int_{R_0} \sigma_0 + \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}. \tag{8.9}$$

Then by Stokes' formula, we see that it induces the integration

$$\int_M : H^m_D(\mathcal{U}) \longrightarrow \mathbb{C},$$

which is compatible with the integration on the de Rham cohomology via the isomorphism of Theorem 8.3.4.

Čech-de Rham Theorem The Čech-de Rham theorem is described using integration on general Čech-de Rham cochains, which may be defined using C^∞ singular chains transverse to a honeycomb system, a C^∞ triangulation or the dual cellular decomposition. Here we explain the last way, which is appropriate later to express residues.

Let K_0, K, K' and K^* be as before. Let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of M and $\{R_0, R_1\}$ a honeycomb system adapted to $\mathcal{U}$.

Definition 8.3.6 We say that $\{R_0, R_1\}$ is adapted to K', if for each p-simplex $\mathbf{t}$ of K', $\mathbf{t} \cap R_i$ is an p-chain, $i = 0, 1$, and $\mathbf{t} \cap R_{01}$ is an $(p-1)$-chain with respect to some subdivision of K'.

We assume that there exists a honeycomb system $\{R_0, R_1\}$ adapted to $\mathcal{U}$ and K'. Then, for each p-cell $\mathbf{s}^*$, $\mathbf{s}^* \cap R_i$, $i = 0, 1$, are p-chains and $\mathbf{s}^* \cap R_{01}$ is a $(p-1)$-chain with respect to some subdivision of K'. We consider the morphism

$$\mathscr{E}^{(p)}(\mathcal{U}) \longrightarrow C^p_{K^*}(M; \mathbb{C}) \tag{8.10}$$

that assigns to $\sigma = (\sigma_0, \sigma_1, \sigma_{01}) \in \mathscr{E}^p(\mathcal{U})$ the cochain

$$\mathbf{s}^* \mapsto \int_{\mathbf{s}^* \cap R_0} \sigma_0 + \int_{\mathbf{s}^* \cap R_1} \sigma_1 + \int_{\mathbf{s}^* \cap R_{01}} \sigma_{01}.$$

In the above, the orientations of $\mathbf{s}^* \cap R_i$ are naturally determined from that of $\mathbf{s}^*$ and we endow $\mathbf{s}^* \cap R_{01}$ with the orientation as a part of the boundary of $\mathbf{s}^* \cap R_0$. Then, by Stokes' formula the morphism (8.10) induces a morphism on the cohomology level; $H^p_D(M) \to H^p_{K^*}(M; \mathbb{C}) \simeq H^p(M; \mathbb{C})$, which is in fact an isomorphism:

Theorem 8.3.7 (Čech-de Rham Theorem) *The above morphism is an isomorphism*

$$H^p_D(M) \simeq H^p(M; \mathbb{C}).$$

The above isomorphism is compatible with that in Theorem 8.3.2 via the isomorphism of Theorem 8.3.4.

Cup Product Let M and $\mathcal{U} = \{U_0, U_1\}$ be as above. We define the cup product

$$\mathscr{E}^{(p)}(\mathcal{U}) \times \mathscr{E}^{(p')}(\mathcal{U}) \overset{\smile}{\longrightarrow} \mathscr{E}^{(p+p')}(\mathcal{U}) \tag{8.11}$$

by assigning to (σ, τ) in $\mathscr{E}^{(p)}(\mathcal{U}) \times \mathscr{E}^{(p')}(\mathcal{U})$ the cochain $\sigma \smile \tau$ in $\mathscr{E}^{(p+p')}(\mathcal{U})$ given by

$$(\sigma \smile \tau)_i = \sigma_i \wedge \tau_i, \quad i = 0, 1, \quad \text{and} \quad (\sigma \smile \tau)_{01} = (-1)^p \sigma_0 \wedge \tau_{01} + \sigma_{01} \wedge \tau_1.$$

Then it is bilinear in (σ, τ) and we have $D(\sigma \smile \tau) = D\sigma \smile \tau + (-1)^p \sigma \smile D\tau$. Thus it induces the cup product

$$H^p_D(\mathcal{U}) \times H^{p'}_D(\mathcal{U}) \overset{\smile}{\longrightarrow} H^{p+p'}_D(\mathcal{U}),$$

which is compatible with the product in the de Rham cohomology via the isomorphism of Theorem 8.3.4.

If M is compact and oriented, we may describe the Poincaré duality as in the case of de Rham cohomology.

8.3.3 Relative Čech-de Rham Cohomology

Let S be a closed set in M. Letting $U_0 = M \smallsetminus S$ and U_1 an open neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We set

$$\mathscr{E}^{(p)}(\mathcal{U}, U_0) = \{ (\sigma_0, \sigma_1, \sigma_{01}) \in \mathscr{E}^{(p)}(\mathcal{U}) \mid \sigma_0 = 0 \}.$$

Then $(\mathscr{E}^{(\bullet)}(\mathcal{U}, U_0), D)$ is a subcomplex of $(\mathscr{E}^{(\bullet)}(\mathcal{U}), D)$. Note that we may write

$$\mathscr{E}^{(p)}(\mathcal{U}, U_0) = \mathscr{E}^{(p)}(U_1) \oplus \mathscr{E}^{(p-1)}(U_{01})$$

and $D : \mathscr{E}^{(p)}(\mathcal{U}, U_0) \to \mathscr{E}^{(p+1)}(\mathcal{U}, U_0)$ is given by $D(\sigma_1, \sigma_{01}) = (d\sigma_1, \sigma_1 - d\sigma_{01})$.

Definition 8.3.8 The *p-th relative Čech-de Rham cohomology* $H_D^p(\mathcal{U}, U_0)$ of the pair $(\mathcal{U}, U_0)$ is the p-th cohomology of $(\mathscr{E}^{(\bullet)}(\mathcal{U}, U_0), D)$.

If S is a subcomplex of M with respect to some triangulation of M, $H_D^p(\mathcal{U}, U_0)$ is canonically isomorphic with the relative singular cohomology $H^p(M, M \smallsetminus S; \mathbb{C})$ (cf. Theorem 8.3.13 below).

There is an exact sequence of complexes

$$0 \longrightarrow \mathscr{E}^{(\bullet)}(\mathcal{U}, U_0) \xrightarrow{j^\bullet} \mathscr{E}^{(\bullet)}(\mathcal{U}) \xrightarrow{i^\bullet} \mathscr{E}^{(\bullet)}(U_0) \longrightarrow 0,$$

where $j^p(\sigma_1, \sigma_{01}) = (0, \sigma_1, \sigma_{01})$ and $i^p(\sigma_0, \sigma_1, \sigma_{01}) = \sigma_0$. This gives rise to a long exact sequence

$$\cdots \longrightarrow H_d^{p-1}(U_0) \xrightarrow{\delta} H_D^p(\mathcal{U}, U_0) \xrightarrow{j^*} H_D^p(\mathcal{U}) \xrightarrow{i^*} H_d^p(U_0) \longrightarrow \cdots.$$

Note that δ assigns to the class of a closed $(p-1)$-form θ on U_0 the class of $(0, -\theta)$.

The cohomology $H_D^p(\mathcal{U}, U_0)$ is determined by the local structure of M near S in the following sense. We consider a special covering $\mathcal{U}^\star = \{U_0, U_1^\star\}$ of M given by $U_0 = M \smallsetminus S$ and $U_1^\star = M$.

Definition 8.3.9 The cohomology $H_D^p(\mathcal{U}^\star, U_0)$ is called the p-th *relative de Rham cohomology* of the pair $(M, M \smallsetminus S)$ and is denoted by $H_D^p(M, M \smallsetminus S)$.

It is easily checked that, for any covering $\mathcal{U} = \{U_0, U_1\}$ as above, the morphism $\mathscr{E}^{(p)}(\mathcal{U}^\star, U_0) \to \mathscr{E}^{(p)}(\mathcal{U}, U_0)$ given by restriction of forms induces an isomorphism

$$H_D^p(M, M \smallsetminus S) \xrightarrow{\sim} H_D^p(\mathcal{U}, U_0).$$

Thus we have:

Proposition 8.3.10 *The cohomology $H^p_D(\mathcal{U}, U_0)$ is determined uniquely modulo canonical isomorphisms, independently of the choice of U_1.*

Proposition 8.3.11 (Excision) *Let S be a closed set in M. Then, for every open set U in M containing S, there is a canonical isomorphism*

$$H^p_D(M, M \smallsetminus S) \xrightarrow{\sim} H^p_D(U, U \smallsetminus S).$$

Integration We assume that M is oriented and S is a compact set in M. Letting $U_0 = M \smallsetminus S$ and U_1 a neighborhood of S in M, we consider the covering $\mathcal{U} = \{U_0, U_1\}$. Let R_1 be a compact m-dimensional manifold with boundary in M containing S in its interior and set $R_0 = M \smallsetminus \mathrm{Int}\, R_1$. Then $\{R_0, R_1\}$ is a honeycomb system adapted to $\mathcal{U}$. The integration (8.9), which is not defined in general for $\mathscr{E}^{(m)}(\mathcal{U})$ unless M is compact, makes sense on $\mathscr{E}^{(m)}(\mathcal{U}, U_0)$:

$$\int_M : \mathscr{E}^{(m)}(\mathcal{U}, U_0) \longrightarrow \mathbb{C}, \qquad \int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}.$$

It induces the integration

$$\int_M : H^m_D(\mathcal{U}, U_0) \longrightarrow \mathbb{C}. \tag{8.12}$$

Relative Čech-de Rham Theorem Suppose now that S is a subcomplex of M with respect to a C^∞ triangulation K_0 of M and let K, K' and K^* be as in Sect. 8.2.1. We may assume without loss of generality that U_1 contains the star $S_{K'}(S)$ of S with respect to K'.

Definition 8.3.12 A honeycomb system $\{R_0, R_1\}$ is adapted to $\mathcal{U}$ and S if it is adapted to $\mathcal{U}$ and if $R_1 \subset O_{K'}(S)$.

For example, if we set $R_1 = S_{K''}(S)$, the star of S with respect to the barycentric subdivision K'' of K', and $R_0 = M \smallsetminus \mathrm{Int}\, R_1$, then $\{R_0, R_1\}$ is adapted to $\mathcal{U}$, K' and S.

If we choose a honeycomb system $\{R_0, R_1\}$ so that it is adapted to $\mathcal{U}$, K' and S, we see that the morphism (8.10) induces a morphism

$$\mathscr{E}^{(p)}(\mathcal{U}, U_0) \longrightarrow C^p_{K^*}(M, M \smallsetminus O_{K'}(S); \mathbb{C}),$$

which assigns to $\sigma = (\sigma_1, \sigma_{01}) \in \mathscr{E}^{(p)}(\mathcal{U}, U_0)$ the cochain

$$\mathbf{s}^* \mapsto \int_{\mathbf{s}^* \cap R_1} \sigma_1 + \int_{\mathbf{s}^* \cap R_{01}} \sigma_{01} \tag{8.13}$$

for an p-cell $\mathbf{s}^*$. Then it induces an isomorphism

$$H^p_D(\mathcal{U}, U_0) \xrightarrow{\sim} H^p_{K^*}(M, M \smallsetminus O_{K'}(S); \mathbb{C}).$$

Combined with $H^p_{K^*}(M, M \smallsetminus O_{K'}(S); \mathbb{C}) \simeq H^p(M, M \smallsetminus S; \mathbb{C})$, we have:

Theorem 8.3.13 (Relative Čech-de Rham Theorem) *Let S be a closed set in M. If S is a subcomplex of M with respect to some triangulation of M, there is a canonical isomorphism:*

$$H^p_D(\mathcal{U}, U_0) \simeq H^p(M, M \smallsetminus S; \mathbb{C}).$$

Remark 8.3.14 In Theorem 8.3.13, considering the case of the covering $\mathcal{U}^\star$, we see that there is a canonical isomorphism $H^p_D(M, M \smallsetminus S) \xrightarrow{\sim} H^p(M, M \smallsetminus S; \mathbb{C})$. The excision in Proposition 8.3.11 is compatible with the one in relative (singular) cohomology via this isomorphism.

Alexander Duality We now assume that M is oriented. From the Alexander duality (Theorem 8.2.2) and the relative Čech-de Rham theorem (Theorem 8.3.13), we have:

Theorem 8.3.15 *Let S be a closed set in M. If S is a subcomplex with respect to a triangulation K_0 of M, there is a canonical isomorphism:*

$$A : H^p_D(\mathcal{U}, U_0) \xrightarrow{\sim} \check{H}_{m-p}(S; \mathbb{C}).$$

Let K, K' and K^* be as before and $\{R_0, R_1\}$ a honeycomb system adapted to $\mathcal{U}$, K' and S (cf. Definitions 8.3.6 and 8.3.12). Then the isomorphism is induced from the composition

$$\mathscr{E}^{(p)}(\mathcal{U}, U_0) \longrightarrow C^p_{K^*}(M, M \smallsetminus O_{K'}(S); \mathbb{C}) \longrightarrow \check{C}^K_{m-p}(S; \mathbb{C}),$$

where the first one is defined as in (8.13) and the second one is (8.6) with $\mathbb{C}$-coefficients. It assigns to (σ_1, σ_{01}) in $\mathscr{E}^{(p)}(\mathcal{U}, U_0)$ the chain

$$\sum_{\mathbf{s}} c_{\mathbf{s}} \mathbf{s}, \qquad c_{\mathbf{s}} = \int_{\mathbf{s}^* \cap R_1} \sigma_1 + \int_{\mathbf{s}^* \cap R_{01}} \sigma_{01}, \tag{8.14}$$

where $\mathbf{s}$ runs through all the oriented $(m - p)$-simplices of K in S.

Cup Product Let S be a closed set in M and $\mathcal{U} = \{U_0, U_1\}$ a covering of M with $U_0 = M \smallsetminus S$ and U_1 a neighborhood of S in M, as before.

In the product (8.11), if σ is in $\mathscr{E}^{(p)}(\mathcal{U}, U_0)$, i.e., if $\sigma_0 = 0$, then

$$\sigma \smile \tau = (0, \sigma_1 \wedge \tau_1, \sigma_{01} \wedge \tau_1)$$

and we have a product $\mathscr{E}^{(p)}(\mathcal{U}, U_0) \times \mathscr{E}^{(p')}(U_1) \xrightarrow{\smile} \mathscr{E}^{(p+p')}(\mathcal{U}, U_0)$, which induces a product

$$H^p_D(\mathcal{U}, U_0) \times H^{p'}_d(U_1) \xrightarrow{\smile} H^{p+p'}_D(\mathcal{U}, U_0).$$

Suppose now that S is a subcomplex of M with respect to a triangulation K_0 of M. Then there is a canonical isomorphism $H^p_D(\mathcal{U}, U_0) \xrightarrow{\sim} H^p(M, M \setminus S; \mathbb{C})$ (Theorem 8.3.13). There is also the de Rham isomorphism $H^{p'}_d(U_1) \xrightarrow{\sim} H^{p'}(U_1; \mathbb{C})$ (Theorem 8.3.2). The above cup product is compatible with the topological one in the sense that the following diagram is commutative:

$$\begin{array}{ccc} H^p_D(\mathcal{U}, U_0) \times H^{p'}_d(U_1) & \xrightarrow{\smile} & H^{p+p'}_D(\mathcal{U}, U_0) \\ \wr\downarrow & & \wr\downarrow \\ H^p(M, M \setminus S; \mathbb{C}) \times H^{p'}(U_1; \mathbb{C}) & \xrightarrow{\smile} & H^{p+p'}(M, M \setminus S; \mathbb{C}). \end{array}$$

Note that there is the excision $H^p(M, M \setminus S; \mathbb{C}) \simeq H^p(U_1, U_1 \setminus S; \mathbb{C})$.

If S is compact and if U_1 has the same homotopy type as S (for example, the open star $O_K(S)$ of S in K), we have

$$\breve{H}_{m-p}(S; \mathbb{C}) = H_{m-p}(S; \mathbb{C}) \simeq H_{m-p}(U_1; \mathbb{C}) \simeq H^{m-p}_d(U_1)^*.$$

Thus we see that, in this case, the pairing

$$H^p_D(\mathcal{U}, U_0) \times H^{m-p}_d(U_1) \xrightarrow{\smile} H^m_D(\mathcal{U}, U_0) \xrightarrow{\int_M} \mathbb{C}$$

is non-degenerate and induces the Alexander duality.

8.4 Chern Classes via Obstruction Theory

Obstruction theory for characteristic classes of fiber bundles is thoroughly done in [52], to which we refer for details of the materials in this section. The descriptions given here are based on those in [11].

In the sequel, we denote by $\mathbb{B}^m$ a closed m-ball around 0 in $\mathbb{R}^m$ and $\mathbb{S}^{m-1} = \partial\mathbb{B}^m$ an $(m-1)$-sphere. We endow $\mathbb{B}^m$ with the orientation same as that of $\mathbb{R}^m$ and

$\mathbb{S}^{m-1}$ with the orientation as the boundary of $\mathbb{B}^m$. In this section the homology and cohomology are with $\mathbb{Z}$-coefficients.

8.4.1 Index of a Family of Sections

Let X be a topological space and E a complex vector bundle of rank l on X.

Definition 8.4.1 An *r-section* on a subset A of X is an ordered family $s^{(r)} = (s_1, \dots, s_r)$ of r sections of E on A. A *singular point* of $s^{(r)}$ is a point where the s_i's fail to be linearly independent over $\mathbb{C}$. An *r-frame* is an r-section without singularities.

A 1-section is nothing but a section. In this case the singular points are the zeros of the section. An l-frame is simply called a frame.

In the below, we define the q-th Chern class $c^q(E)$ of E to be the primary obstruction to constructing an r-frame of E, $r = l - q + 1$. For this purpose, we introduce the notion of the *index* of an r-section at a point where it is singular or is not defined.

Stiefel Manifold An r-frame in $\mathbb{C}^l$ is an ordered family of linearly independent r vectors in $\mathbb{C}^l$. The set of r-frames in $\mathbb{C}^l$ has a natural structure of complex manifold of dimension lr, which is called the Stiefel manifold of r-frames in $\mathbb{C}^l$ and is denoted by $W(l, r)$. Recall that it is $(2q - 2)$-connected and $\pi_{2q-1}(W(l, r)) \simeq \mathbb{Z}$, $q = l - r + 1$. Thus, by Hurewicz' theorem,

$$H_{2q-1}(W(l, r)) \simeq \mathbb{Z}. \tag{8.15}$$

Note that $\pi_{2q-1}(W(l, r))$ and $H_{2q-1}(W(l, r))$ have a canonical generator. In particular, in the case $r = 1$, $W(l, 1) = \mathbb{C}^l \smallsetminus \{0\}$ and $q = l$. There is a deformation retraction $r : \mathbb{C}^l \smallsetminus \{0\} \to \mathbb{S}^{2l-1}$. The homotopy class of the inclusion $\mathbb{S}^{2l-1} \hookrightarrow \mathbb{C}^l \smallsetminus \{0\}$ is the canonical generator of $\pi_{2l-1}(\mathbb{C}^l \smallsetminus \{0\})$. There exist canonical isomorphisms $H^{2l-1}(\mathbb{C}^l \smallsetminus \{0\}) \simeq H^{2l-1}(\mathbb{S}^{2l-1}) \simeq \mathbb{Z}$, which determine the canonical generator of $H^{2l-1}(\mathbb{C}^l \smallsetminus \{0\})$.

Mapping Degree If $\varphi : \mathbb{S}^{2q-1} \to W(l, r)$ is a continuous map, it induces a morphism

$$\varphi_* : H_{2q-1}(\mathbb{S}^{2q-1}) \longrightarrow H_{2q-1}(W(l, r))$$

and we may write

$$\varphi_* \nu_{2q-1} = d \cdot w_{2q-1}$$

with d an integer uniquely determined by the homotopy class of φ. In the above, v_{2q-1} and w_{2q-1} denote the canonical generators of $H_{2q-1}(\mathbb{S}^{2q-1})$ and $H_{2q-1}(W(l,r))$, respectively.

Definition 8.4.2 The above integer d is called the *mapping degree* of φ and is denoted by $\deg\varphi$.

Remark 8.4.3

1. Note that $\pi_{2l-1}(\mathbb{C}^l\smallsetminus\{0\})\simeq\mathbb{Z}$, which has the canonical generator $[\iota]$, $\iota:\mathbb{S}^{2l-1}\hookrightarrow\mathbb{C}^l\smallsetminus\{0\}$ being as above. A map φ as above defines an element in $\pi_{2l-1}(\mathbb{C}^l\smallsetminus\{0\})$, which is of the form $d\cdot[\iota]$ with $d=\deg\varphi$. Then $\deg\varphi=0$ if and only if φ can be extended to a map $\mathbb{B}^{2l}\to\mathbb{C}^l\smallsetminus\{0\}$ and we may think of $\deg\varphi$ as the *obstruction* to extending φ to $\mathbb{B}^{2l}$.
2. See Proposition 8.8.4 below for a differential geometric interpretation of the degree.

Index of an r-Frame We consider the bundle $W(E,r)$ of r-frames of E on X. This is a bundle associated with E whose fiber $W(E,r)_x$ at each point x of X is diffeomorphic with $W(l,r)$. An r-frame of E is nothing but a section of $W(E,r)$.

Suppose we have an oriented $2q$-cell $\mathbf{e}$ in X. We assume that $\mathbf{e}$ is regular and take a characteristic map $\chi:\mathbb{B}^{2q}\to\bar{\mathbf{e}}$, the closure of $\mathbf{e}$, so that it is a homeomorphism determining the orientation of $\mathbf{e}$. We may assume that E is trivial on $\bar{\mathbf{e}}$. Suppose we are given an r-frame $s^{(r)}$ of E on $\bar{\mathbf{e}}\smallsetminus\mathbf{e}$. Although it is not necessary, in order to fix the idea, we extend $s^{(r)}$ to an r-frame on $\bar{\mathbf{e}}\smallsetminus\{a\}$, where a is a point in in $\mathbf{e}$. It is always possible as $\bar{\mathbf{e}}\smallsetminus\{a\}$ deformation retracts to $\bar{\mathbf{e}}\smallsetminus\mathbf{e}$. We have a map

$$\varphi_{s^{(r)}}:\mathbb{S}^{2q-1}\longrightarrow W(l,r) \tag{8.16}$$

as the composition of the restriction of χ to $\mathbb{S}^{2q-1}$, $s^{(r)}$ and the projection onto the fiber $W(l,r)$ of $W(E,r)$.

Definition 8.4.4 The *index* $I(s^{(r)},a)$ of $s^{(r)}$ at a is defined by

$$I(s^{(r)},a)=\deg\varphi_{s^{(r)}}.$$

Note that the definition does not depend on the order of the members of $s^{(r)}$, the choice of a in $\mathbf{e}$ or the trivialization of E.

Example 8.4.5 Let U be a neighborhood of 0 in $\mathbb{C}=\{z\}$ and $E=\mathbb{C}\times U$ the product bundle on U. Also let $\mathbb{B}^2$ be a closed 2-ball in U with center 0. For an integer d, let s be the frame of E on $\mathbb{S}^1=\partial\mathbb{B}^2$ given by $s(z)=(z^d,z)$. Then, noting that $W(1,1)=\mathbb{C}\smallsetminus\{0\}$ and the map in (8.16) is given by $z\mapsto z^d$, we see that $I(s,0)=d$.

More generally, if the frame s on $\mathbb{S}^1$ is given by $s(z) = (f(z), z)$ with f a non-vanishing C^∞ function, we have (cf. Example 8.8.3 and Remark 8.4.3. 2):

$$I(s, 0) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{S}^1} \frac{df}{f}.$$

Remark 8.4.6 The index above coincides with the index of a section of some quotient bundle of E of rank $q = l - r + 1$.

8.4.2 Chern Classes

We briefly review the obstruction theoretic construction of Chern classes of complex vector bundles on cell complexes.

Let X be a locally finite cell complex and suppose that every cell is regular. Recall that the singular homology of X can be computed from the chain complex $C_\bullet(X)$ of oriented p-cells in X. Also the singular cohomology can be computed from the dual cochain complex $C^\bullet(X)$.

Let E be a complex vector bundle of rank l on X and r an integer with $1 \leq r \leq l$. We set $q = l - r + 1$ as before. We try to construct an r-frame, i.e., a section of $W(E, r)$, on each skeleton X^p of X starting from the 0-skeleton and then extending it inductively to larger skeletons. On the way we get a $2q$-cochain γ as the obstruction to the construction. In fact γ is a cocycle and defines a class in $H^{2q}(X)$, which will be the q-th Chern class $c^q(E)$ of E.

To be a little more precise, it is always possible to construct a section $s^{(r)}$ of $W(E, r)$ on X^0. Let $\mathbf{e}$ be a p-cell. If a section $s^{(r)}$ of $W(E, r)$ is given on $\bar{\mathbf{e}} \smallsetminus \mathbf{e}$, it defines a map as (8.16), replacing $\mathbb{S}^{2q-1}$ with $\mathbb{S}^{p-1} \simeq \bar{\mathbf{e}} \smallsetminus \mathbf{e}$. Thus $s^{(r)}$ defines an element in $\pi_{p-1}(W(l, r))$. If $p \leq 2l - 2r + 1 = 2q - 1$, the section $s^{(r)}$ can be extended to $\bar{\mathbf{e}}$, since the homotopy group vanishes. This way we may construct a section $s^{(r)}$ of $W(E, r)$ on X^{2q-1}. Then we reach to an "obstruction" when $p = 2q$. Namely, for each $2q$-cell $\mathbf{e}$, the r-frame $s^{(r)}$ on $\bar{\mathbf{e}} \smallsetminus \mathbf{e}$ is extended to an r-frame on $\bar{\mathbf{e}}$ possibly except for a point a in $\mathbf{e}$ and the obstruction is given by the index $I(s^{(r)}, a)$. We define a cochain γ by

$$\langle \mathbf{e}, \gamma \rangle = I(s^{(r)}, a) \tag{8.17}$$

for each $2q$-cell $\mathbf{e}$ and then extending it linearly. Then it is shown that the cochain γ is in fact a cocycle in $C^{2q}(X)$ and that different choices of extensions in each step leads to cocycles that are cohomologous.

Definition 8.4.7 The *q-th topological Chern class* $c^q_{\mathrm{top}}(E)$ of E is the class of γ in $H^{2q}(X)$.

Throughout this section, we denote $c^q_{\mathrm{top}}(E)$ simply by $c^q(E)$ and call it the q-th Chern class of E. The *total Chern class of* E is defined by

$$c^*(E) = 1 + c^1(E) + \cdots + c^l(E),$$

which is an element in the cohomology ring $H^*(X)$ and is invertible.

Remark 8.4.8

1. If E admits an r-frame on X, then clearly $c^q(E) = 0$. Conversely if $c^q(E) = 0$, then it is possible to construct an r-frame of E on X^{2q}, but not on X in general. Thus $c^q(E)$ is referred to as the *primary obstruction* to constructing an r-frame of E.
2. In Sect. 8.6 below, we define differential geometric Chern classes for C^∞ complex vector bundles via the Chern-Weil theory and discuss, in Sect. 8.8.3, the relation between the classes defined from two different approaches.

8.4.3 Euler Class of a Real Oriented Vector Bundle

Let E be an oriented real vector bundle of rank l' on a regular cell complex X. If $l' = 1$, the orientability of E implies that E is a trivial bundle. Thus we assume that $l' > 1$ in the sequel. We denote by $V(l', 1)$ the real Stiefel manifold of 1-frames in $\mathbb{R}^{l'}$, which is in fact $\mathbb{R}^{l'} \smallsetminus \{0\}$ so that it has the homotopy type of $\mathbb{S}^{l'-1}$. We denote by $V(E, 1)$ the bundle of 1-frames of E on X.

We may define the index of a section at a point where it is singular or it is not defined, as before. Thus let $\mathbf{e}$ be an l'-cell in X, on which E is trivial. Suppose we have a 1-frame s of E on $\bar{\mathbf{e}} \smallsetminus \{a\}$, $a \in \mathbf{e}$. We have a composition $\varphi_s : \mathbb{S}^{l'-1} \to V(l', 1)$ as in (8.16). The *index* of s at a, denoted by $I(s, a)$, is defined by

$$I(s, a) = \deg \varphi_s. \tag{8.18}$$

We then perform a similar construction as in the case of Chern classes to obtain the Euler class $e(E)$ of E as the obstruction to constructing a section of $V(E, 1)$, i.e., a non-vanishing section of E, on $X^{l'}$, which is in $H^{l'}(X)$.

Remark 8.4.9

1. As in the case of Chern classes, $e(E) = 0$ if and only if E admits a non-vanishing section on $X^{l'}$ (cf. Remark 8.4.8. 1).
2. If E is a complex vector bundle of rank l, we may think of it as a naturally oriented real bundle of rank $2l$ and $W(l, 1)$ may be identified with $V(2l, 1)$. In this case, we have $e(E) = c^l(E)$, the top Chern class of E.

8.5 Localization and Topological Residues

Let M be a C^∞ manifold of dimension m and let K_0, K, K' and K^* be as in Sect. 8.2.1.

8.5.1 Duals of Characteristic Classes

In the construction described in Sects. 8.4.2 and 8.4.3, we use the dual cellular complex K^*.

Chern Classes Let E be a complex vector bundle of rank l on M. Denoting by $s^{(r)}$ an r-frame already constructed on the $(2q-1)$-skeleton $(K^*)^{2q-1}$ of K^*, the Poincaré dual of $c^q(E)$ is represented in $\breve{H}_{m-2q}(M)$ by the cycle

$$C = \sum_{\mathbf{s}} I(s^{(r)}, b_{\mathbf{s}})\,\mathbf{s}, \tag{8.19}$$

where the sum is taken over all the $(m-2q)$-simplices $\mathbf{s}$ of K and $I(s^{(r)}, b_{\mathbf{s}})$ denotes the index of $s^{(r)}$ on $\mathbf{s}^* \cap (K^*)^{2q-1}$ at the barycenter $b_{\mathbf{s}}$ (cf. Definition 8.4.4, we may take $b_{\mathbf{s}}$ as a there).

Euler Class Let E be a real oriented vector bundle of rank l' on M. Denoting by s a non-vanishing section already constructed on $(K^*)^{l'-1}$, the Poincaré dual of $e(E)$ is represented in $\breve{H}_{m-l'}(M)$ by the cycle

$$C = \sum_{\mathbf{s}} I(s, b_{\mathbf{s}})\,\mathbf{s}, \tag{8.20}$$

where the sum is taken over all the $(m-l')$-simplices $\mathbf{s}$ of K.

8.5.2 Localized Chern Classes

Let E be a complex vector bundle of rank l on M and S a K_0-subcomplex of M. Suppose we are already given an r-frame $s^{(r)}$ of E on $(M \smallsetminus S) \cap (K^*)^{2q}$, where $(K^*)^{2q}$ denotes the $2q$-skeleton of K^*. We follow the procedure described in Sect. 8.4.2, starting with this frame to obtain an r-frame $\tilde{s}^{(r)}$ on $(K^*)^{2q} \smallsetminus \{\text{isolated points}\}$. To be more precise, recall that, for an $(m-2q)$-simplex $\mathbf{s}$ of K, $\mathbf{s}^* \cap S = \emptyset$ if and only if $\mathbf{s} \not\subset S$. The above frame $\tilde{s}^{(r)}$ has the properties:

1. if $\mathbf{s} \not\subset S$, then $\tilde{s}^{(r)}$ is defined and equals $s^{(r)}$ on $\mathbf{s}^*$,
2. if $\mathbf{s} \subset S$, then $\tilde{s}^{(r)}$ is defined on $\mathbf{s}^* \smallsetminus \{a\}$, where a is a point in $\mathbf{s}^*$.

Note that the point a may be assumed to be the barycenter $b_{\mathbf{s}}$ of $\mathbf{s}$. For every $2q$-cell $\mathbf{s}^*$ not intersecting with S, we have $I(\tilde{s}^{(r)}, b_{\mathbf{s}}) = I(s^{(r)}, b_{\mathbf{s}}) = 0$. Thus the cocycle γ defined by (8.17) for $\tilde{s}^{(r)}$ is in $C^{2q}_{K^*}(M, M \smallsetminus O_{K'}(S))$. It represents a class in $H^{2q}(M, M \smallsetminus S)$, which is denoted by $c^q_{S,\mathrm{top}}(E, s^{(r)})$ and is called the *topological localization* of $c^q(E)$ by $s^{(r)}$ at S. It will also be denoted by $c^q_S(E, s^{(r)})$, $c^q_{\mathrm{top}}(E, s^{(r)})$ or $c^q(E, s^{(r)})$. The class depends on $s^{(r)}$, but not on the choice of the extension $\tilde{s}^{(r)}$ of $s^{(r)}$. Its image by the canonical morphism $H^{2q}(M, M \smallsetminus S) \to H^{2q}(M)$ is the Chern class $c^q(E)$.

Suppose M is oriented so that we have the Alexander isomorphism (Theorem 8.2.2):

$$A : H^{2q}(M, M \smallsetminus S) \xrightarrow{\sim} \check{H}_{m-2q}(S).$$

Definition 8.5.1 The *topological residue* $\mathrm{TRes}_{c^q}(s^{(r)}, E; S)$ *of* $s^{(r)}$ *for* $c^q(E)$ *at* S is the image of $c^q_S(E, s^{(r)})$ by A.

Remark 8.5.2 In order to have the above localization and residue, it suffices to have $s^{(r)}$ on $\partial S_{K'}(S) \cap (K^*)^{2q}$.

Suppose that S has only a finite number of connected components $(S_\lambda)_\lambda$. Then we have a decomposition

$$\check{H}_{m-2q}(S) = \bigoplus\nolimits_\lambda \check{H}_{m-2q}(S_\lambda)$$

and accordingly we have the residue $\mathrm{TRes}_{c^q}(s^{(r)}, E; S_\lambda)$ in $\check{H}_{m-2q}(S_\lambda)$ for each λ.

We have (cf. (8.19)):

Proposition 8.5.3 *In the above situation, the residue* $\mathrm{TRes}_{c^q}(s^{(r)}, E; S_\lambda)$ *is represented by the cycle*

$$C_\lambda = \sum_{\mathbf{s}} I(\tilde{s}^{(r)}, b_{\mathbf{s}})\, \mathbf{s}, \tag{8.21}$$

where the sum is taken over the $(m-2q)$*-simplices of* K *in* S_λ.

In particular, if $2q = m$ and if S_λ is compact, $\check{H}_0(S_\lambda) = H_0(S_\lambda) = \mathbb{Z}$ and $\mathrm{TRes}_{c^q}(s^{(r)}, E; S_\lambda)$ is an integer given by

$$\mathrm{TRes}_{c^q}(s^{(r)}, E; S_\lambda) = \sum_{\mathbf{s}} I(\tilde{s}^{(r)}, b_{\mathbf{s}}),$$

where the sum is taken over all the 0-simplices $\mathbf{s}$ of K in S_λ, in fact $b_{\mathbf{s}} = \mathbf{s}$.

By the commutativity of (8.7), we have the following "residue theorem":

Theorem 8.5.4 *In the above situation, it holds:*

1. *For each* λ*, we have the residue* $\mathrm{TRes}_{c^q}(s^{(r)}, E; S_\lambda)$ *in* $\check{H}_{m-2q}(S_\lambda)$*, which is represented by the cycle* (8.21).

2. *We have*

$$\sum_{\lambda}(i_\lambda)_* \operatorname{TRes}_{c^q}(s^{(r)}, E; S_\lambda) = [M] \frown c^q(E) \qquad \textit{in}\ \ \check{H}_{m-2q}(M),$$

where $i_\lambda : S_\lambda \hookrightarrow M$ *denotes the inclusion.*

Transverse Residues Let M, K_0, K, K^* and S be as above. Suppose the maximum dimension of the simplices of K in S is $m-2q$ and let S' be an oriented submanifold of M of dimension $m-2q$ which is contained in S. We may assume that the orientations of simplices in K are compatible with that of S'. Let x be a point in S' and $\mathbb{D}$ a slice of S' in M at x, i.e., a $2q$-dimensional submanifold of M containing x, transverse to S' at x and diffemorphic with an open $2q$-ball. We may assume that x is the barycenter $b_{\mathbf{s}}$ of some $(m-2q)$-simplex $\mathbf{s}$ of K in S' and that $\mathbf{s}^*$ is in $\mathbb{D}$. We may also extend the triangulation K' on $\mathbf{s}^*$ throughout $\mathbb{D}$. Let $s^{(r)}$ be an r-frame of E on $(M \smallsetminus S) \cap (K^*)^{2q}$, as before. Restricting E and $s^{(r)}$ to $\mathbb{D}$, we have the localization $c_x^q(E|_{\mathbb{D}}, s^{(r)}|_{\mathbb{D}})$ and the residue $\operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$, which correspond to each other by the Alexander isomorphism

$$H^{2q}(\mathbb{D}, \mathbb{D} \smallsetminus \{x\}) \simeq H_0(\{x\}).$$

As $H_0(\{x\}) \simeq \mathbb{Z}$, we may think of $\operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$ as an integer, which is referred to as the *transverse residue* at x. In fact it is given by

$$\operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbb{D}}, E|_{\mathbb{D}}; x) = I(s^{(r)}, b_{\mathbf{s}}).$$

Remark 8.5.5 By Remark 8.5.2, in the above notation, $\operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbf{s}^*}, E|_{\mathbf{s}^*}; b_{\mathbf{s}})$ makes sense and is equal to $\operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$.

As a function of x, $\operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$ is locally constant. Thus, if S' is connected, it is constant. From Proposition 8.5.3, we have:

Proposition 8.5.6 *If* S_λ *is an* $(m-2q)$*-dimensional submanifold of* M,

$$\operatorname{TRes}_{c^q}(s^{(r)}, E; S_\lambda) = \operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbb{D}}, E|_{\mathbb{D}}; x) \cdot [S_\lambda] \qquad \textit{in}\ \ \check{H}_{m-2q}(S_\lambda),$$

where x *is a point in* S_λ *and* $\mathbb{D}$ *a slice of* S_λ *at* x.

The above expression of the residue is generalized to the case S_λ is a pseudo-manifold. Thus let S_λ be a connected component of S as above and suppose it is an oriented pseudo-manifold of dimension $m-2q$ (cf. Definition 8.2.5). If $\mathbf{s}$ is an $(m-2q)$-simplex of K in S_λ, $\mathbf{s}^*$ is a $2q$-cell such that $\mathbf{s}^* \cap S_\lambda = \{b_{\mathbf{s}}\}$. Thus we have the transverse residue $\operatorname{TRes}_{c^q}(s^{(r)}|_{\mathbf{s}^*}, E|_{\mathbf{s}^*}; b_{\mathbf{s}})$, which is equal to $I(s^{(r)}, b_{\mathbf{s}})$ (cf. Remark 8.5.5). From Proposition 8.5.3, we have:

Theorem 8.5.7 *Suppose S_λ is an oriented pseudo-manifold of dimension $m-2q$. Then the residue* $\mathrm{TRes}_{c^q}(s^{(r)}, E; S_\lambda)$ *in* $\check{H}_{m-2q}(S_\lambda)$ *is represented by the cycle*

$$\sum_{\mathbf{s}} \mathrm{TRes}_{c^q}(s^{(r)}|_{\mathbf{s}^*}, E|_{\mathbf{s}^*}; b_{\mathbf{s}}) \cdot \mathbf{s},$$

where $\mathbf{s}$ *runs through all the $(m-2q)$-simplices of K in S_λ.*

Corollary 8.5.8 *In the above situation, if $S_\lambda = \bigcup_i S_{\lambda,i}$ is the irreducible decomposition,*

$$\mathrm{TRes}_{c^q}(s^{(r)}, E; S_\lambda) = \sum_i \mathrm{TRes}_{c^q}(s^{(r)}|_{\mathbb{D}}, E|_{\mathbb{D}}; x_{\lambda,i}) \cdot [S_{\lambda,i}] \qquad \text{in } \check{H}_{m-2q}(S_\lambda),$$

where $x_{\lambda,i}$ is a general point of $S_{\lambda,i}$ and $\mathbb{D}$ a slice of $S_{\lambda,i}$ at $x_{\lambda,i}$.

In the above, $[S_{\lambda,i}]$ denotes the class of $S_{\lambda,i}$ in $\check{H}_{m-2q}(S_\lambda)$. In fact $\check{H}_{m-2q}(S_\lambda)$ is a free Abelian group generated by these classes.

In the case S_λ is a submanifold, the above reduces to Proposition 8.5.6.

Remark 8.5.9 In Sect. 8.7 below we discuss localization problems in various settings mainly from the differential geometric viewpoint and give a general residue theorem (Theorem 8.7.3). The localization of Chern classes by frames in this context is treated in Sect. 8.8, where the differential geometric counterpart of Theorem 8.5.4 is discussed. We then prove that these two are essentially the same (cf. Theorem 8.8.8).

8.5.3 Localized Euler Class

Let E be a real oriented vector bundle of rank l' on M and S a K_0-subcomplex of M. Suppose we are already given a non-vanishing section s of E on $(M \smallsetminus S) \cap (K^*)^{l'}$. We follow the procedure described in Sect. 8.5.2 for the Euler class starting with this section to obtain a class in $H^{l'}(M, M \smallsetminus S)$, which we denote by $e(E, s)$ and call the *localization of $e(E)$ by s*.

Suppose M is oriented so that we have the Alexander isomorphism

$$A : H^{l'}(M, M \smallsetminus S) \xrightarrow{\sim} \check{H}_{m-l'}(S).$$

Definition 8.5.10 The *topological residue* $\mathrm{TRes}_e(s, E; S)$ *of s for $e(E)$ at S* is the image of $e(E, s)$ by A.

If S has a finite number of connected components $(S_\lambda)_\lambda$ we have the residue $\mathrm{TRes}_e(s, E; S_\lambda)$ in $\check{H}_{m-l'}(S_\lambda)$ for each λ. It is represented by the cycle (cf. (8.20))

$$C_\lambda = \sum_{\mathbf{s}} I(\tilde{s}, b_{\mathbf{s}})\, \mathbf{s},$$

where the sum is taken over the $(m - l')$-simplices of K in S_λ.

In particular, if $l' = m$ and if S_λ is compact, $\check{H}_0(S_\lambda) = H_0(S_\lambda) = \mathbb{Z}$ and $\mathrm{TRes}_e(s, E; S_\lambda)$ is an integer given by

$$\mathrm{TRes}_e(s, E; S_\lambda) = \sum_{\mathbf{s}} I(\tilde{s}, b_{\mathbf{s}}), \tag{8.22}$$

where the sum is taken over all the 0-simplices $\mathbf{s}$ of K in S_λ, in fact $b_{\mathbf{s}} = \mathbf{s}$.

We have the residue theorem as Theorem 8.5.4 for the Euler class, replacing c^q, $s^{(r)}$ and $2q$ with e, s and l', respectively.

Transverse Residues As in the case of Chern classes, we may consider the transverse residue, replacing c^q, $s^{(r)}$ and $2q$ with e, s and l', respectively. Thus we have the transverse residue $\mathrm{TRes}_e(s|_{\mathbb{D}}, E|_{\mathbb{D}}, x)$, which is an integer, and expressions similar to the ones in Proposition 8.5.6, Theorem 8.5.7 and Corollary 8.5.8.

Euler Class of the Tangent Bundle Let M be a C^∞ manifold of dimension m. If $E = T_{\mathbb{R}}M$ is the tangent bundle of M, then $l' = m$ so that $(K^*)^{l'} = M$. The sections of $T_{\mathbb{R}}M$ are vector fields.

Definition 8.5.11 Let a be a point in M and U a neighborhood of a. For a non-vanishing vector field v on $U \smallsetminus \{a\}$, its *Poincaré-Hopf index* $\mathrm{PH}(v, a)$ at a is the index $I(v, a)$ as defined in (8.18).

Now suppose that M is oriented so that $T_{\mathbb{R}}M$ is also oriented. The *Euler class* $e(M)$ *of* M is then by definition the Euler class of $T_{\mathbb{R}}M$. In the above, we may think of a as a vertex of K and may write (cf. (8.22))

$$\mathrm{PH}(v, a) = \mathrm{TRes}_e(v, T_{\mathbb{R}}M; a).$$

In particular, if M is compact and connected, for a vector field v defined and non-vanishing on M, except for a finite number of points $a_1, \dots, a_r$,

$$\sum_{i=1}^{r} \mathrm{PH}(v, a_i) = [M] \frown e(M). \tag{8.23}$$

On the other hand, there exists a vector field v_0 having a singularity of index 1 at the barycenter of each even dimensional simplex and a singularity of index -1 at the barycenter of each odd dimensional simplex, and for v_0 we have $\sum_{\mathbf{s}} \mathrm{PH}(v_0, b_{\mathbf{s}}) = \chi(M)$, where $\mathbf{s}$ runs through all the simplices in K. From (8.23), we have

$$\chi(M) = [M] \frown e(M). \tag{8.24}$$

Thus we have:

Theorem 8.5.12 (Poincaré-Hopf Theorem) *Let M be a compact, connected and oriented C^∞ manifold. For a vector field v defined and non-vanishing on M, except for a finite number of points $a_1, \dots, a_r$,*

$$\sum_{i=1}^{r} PH(v, a_i) = \chi(M).$$

Case of Complex Vector Bundles From the construction we have the following (cf. Remark 8.4.9. 2):

Proposition 8.5.13 *If E is a complex vector bundle of rank l, we may think of it as a real oriented vector bundle of rank $2l$ and we have*

$$e(E, s) = c^l(E, s) \qquad \text{in } H^{2l}(M, M \smallsetminus S).$$

If M is oriented, we also have

$$\mathrm{TRes}_e(s, E; S_\lambda) = \mathrm{TRes}_{c^l}(s, E; S_\lambda) \qquad \text{in } \breve{H}_{m-2l}(S_\lambda).$$

Let X be a complex manifold of dimension n. Then the q-th *Chern class* $c^q(X)$ *of* X is defined to be the q-th Chern class $c^q(TX)$ of the holomorphic tangent bundle TX. Recall that TX can be naturally identified with the real tangent bundle $T_{\mathbb{R}}X$. Thus a section of TX can be considered as either a complex vector field or a real vector field.

As a special case of Proposition 8.5.13, we have:

Proposition 8.5.14 *Let v be a section of TX defined and non-vanishing on a neighborhood of a, possibly except for at a. Then its Poincaré-Hopf index* $\mathrm{PH}(v, a)$ *as defined in Definition 8.5.11 coincides with the index as defined in Definition 8.4.4 (with $r = 1$).*

We also see that, since the top Chern class $c^n(X)$ is the primary obstruction to constructing a (non-vanishing) vector field, it coincides with the Euler class $e(X)$ of $T_{\mathbb{R}}X$:

$$c^n(X) = e(X).$$

In particular, if X is compact and connected, (8.24) reads

$$\chi(X) = [X] \frown c^n(X). \tag{8.25}$$

Later in Sect. 8.6, we represent the Chern classes of a complex vector bundle by differential forms using connections for the bundle. If X is compact, (8.25) may be written

$$\chi(X) = \int_X c^n(X),$$

which is referred to as the "Gauss-Bonnet formula".

8.6 Chern-Weil Theory Adapted to Čech-de Rham Cohomology

In this section we review the Chern-Weil theory of characteristic classes of complex vector bundles and modify it to have the classes in Čech-de Rham cohomology. As general references for the Chern-Weil theory we list [5, 7, 8, 44]. As to its adaptation to Čech-de Rham cohomology, see [38, 55].

Throughout this section, we let M denote a C^∞ manifold of dimension m. For an open set U in M, we denote by $\mathscr{E}^{(0)}(U)$ the $\mathbb{C}$-algebra of C^∞ functions on U as before. Also, for a C^∞ complex vector bundle E on M, we set $\mathscr{E}^{(p)}(U; E) = C^\infty(U; \bigwedge^p (T^c_{\mathbb{R}} M)^* \otimes E)$, the $\mathscr{E}^{(0)}(U)$-module of C^∞ p-forms with coefficients in E.

Connections Let E be a C^∞ complex vector bundle of rank l on M.

Definition 8.6.1 A *connection* for E is a $\mathbb{C}$-linear map

$$\nabla : \mathscr{E}^{(0)}(M; E) \longrightarrow \mathscr{E}^{(1)}(M; E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla(s) \qquad \text{for } f \in \mathscr{E}^{(0)}(M) \text{ and } s \in \mathscr{E}^{(0)}(M; E).$$

For example, the exterior derivative $d : \mathscr{E}^{(0)}(M) \to \mathscr{E}^{(1)}(M)$ is a connection for the product bundle $\mathbb{C} \times M$. A connection ∇ is a local operator, i.e., if a section s is identically 0 on an open set U, so is $\nabla(s)$. Thus the restriction of ∇ to an open set U makes sense and it is a connection for $E|_U$. Using a partition of unity, we see that every vector bundle admits a connection.

If ∇ is a connection for E, it induces a $\mathbb{C}$-linear map

$$\nabla : \mathscr{E}^{(1)}(M; E) \longrightarrow \mathscr{E}^{(2)}(M; E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s) \qquad \text{for } \omega \in \mathscr{E}^{(1)}(M) \text{ and } s \in \mathscr{E}^{(0)}(M; E).$$

The composition

$$K = \nabla \circ \nabla : \mathscr{E}^{(0)}(M; E) \longrightarrow \mathscr{E}^{(2)}(M; E)$$

is called the *curvature* of ∇. It is easily checked that

$$K(fs) = fK(s) \qquad \text{for } f \in \mathscr{E}^{(0)}(M) \text{ and } s \in \mathscr{E}^{(0)}(M; E).$$

From this we see that the curvature K may be thought of as an element in $\mathscr{E}^{(2)}(M; \operatorname{Hom}(E, E))$ so that it is locally represented by an $l \times l$ matrix whose entries are differential 2-forms. On the other hand the connection ∇ itself may not be thought of as an element in $\mathscr{E}^{(1)}(M; \operatorname{Hom}(E, E))$. However, the fact that it is a local operator allows us to represent it locally by a matrix whose entries are 1-forms.

Thus suppose that ∇ is a connection for E and that E is trivial on U. If $e^{(l)} = (e_1, \dots, e_l)$ is a frame of E on U, we may write, for $i = 1, \dots, l$,

$$\nabla(e_i) = \sum_{j=1}^{l} \theta_{ji} \otimes e_j, \qquad \theta_{ij} \in \mathscr{E}^{(1)}(U).$$

We call $\theta = (\theta_{ij})$ the *connection matrix* of ∇ with respect to $e^{(l)}$. For the curvature K of ∇, we may write

$$K(e_i) = \sum_{j=1}^{l} \kappa_{ji} \otimes e_j, \qquad \kappa_{ij} \in \mathscr{E}^{(2)}(U).$$

We call $\kappa = (\kappa_{ij})$ the *curvature matrix* of ∇ with respect to $e^{(l)}$. From the definition we compute $\kappa_{ij} = d\theta_{ij} + \sum_{k=1}^{l} \theta_{ik} \wedge \theta_{kj}$, which we write as

$$\kappa = d\theta + \theta \wedge \theta.$$

If $e'^{(l)} = (e'_1 \dots, e'_l)$ is another frame of E on U', we have $e'_i = \sum_{j=1}^{l} p_{ji} e_j$ for some C^∞ function s p_{ij} on $U \cap U'$. The matrix $P = (p_{ij})$ is non-singular at each point of $U \cap U'$. If we denote by θ' and κ' the connection and curvature matrices of ∇ with respect to $e'^{(l)}$,

$$\theta' = P^{-1} \cdot dP + P^{-1}\theta P \quad \text{and} \quad \kappa' = P^{-1}\kappa P \quad \text{in} \quad U \cap U'. \tag{8.26}$$

The second relation signifies the fact that K is an element in $\mathscr{E}^{(2)}(M, \operatorname{Hom}(E, E))$.

8.6.1 Characteristic Classes of Complex Vector Bundles

Invariant Polynomials Let $M(l,\mathbb{C})$ denote the vector space of $l \times l$ complex matrices. A *polynomial* φ on $M(l,\mathbb{C})$ is a function $\varphi(A)$ of $A \in M(l,\mathbb{C})$ which is a polynomial in the entries of A. It is said to be *invariant* if

$$\varphi(P^{-1}AP) = \varphi(A) \qquad \text{for all } A \in M(l,\mathbb{C}) \text{ and } P \in GL(l,\mathbb{C}).$$

If we define a function σ_q, for $q = 1, 2, \dots, l$, by $\sigma_q(A) = \operatorname{tr}\left(\bigwedge^q A\right)$, it is an invariant polynomial, homogeneous of degree q. In particular, $\sigma_1(A) = \operatorname{tr}(A)$ and $\sigma_l(A) = \det(A)$. We may also write

$$\det(I_l + A) = 1 + \sigma_1(A) + \cdots + \sigma_l(A),$$

where I_l denotes the identity matrix of rank l. We call σ_q the q-th *elementary invariant polynomial*

It is known that every invariant polynomial is a polynomial in the elementary invariant polynomials.

Characteristic Forms Let E be a C^∞ complex vector bundle of rank l on M, ∇ a connection for E and K its curvature. Since K is in $\mathscr{E}^{(2)}(M; \operatorname{Hom}(E, E))$, for an invariant polynomial φ, we have a differential form $\varphi(K)$. It is shown that the form is closed and its class in the de Rham cohomology depends only on E and not on the choice of the connection.

Proposition 8.6.2 *For every invariant polynomial φ, the form $\varphi(K)$ is closed.*

By a slight abuse of notation, we introduce the following:

Definition 8.6.3 For a connection ∇ and an invariant polynomial φ homogeneous of degree k, the *characteristic form* is defined by

$$\varphi(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^k \varphi(K).$$

Difference Form We have the following:

Proposition 8.6.4 *Let ∇ and ∇' be connections for E. For an invariant polynomial φ homogeneous of degree k, there exists a $(2k-1)$-form $\varphi(\nabla, \nabla')$ such that $\varphi(\nabla', \nabla) = -\varphi(\nabla, \nabla')$ and that*

$$d\varphi(\nabla, \nabla') = \varphi(\nabla') - \varphi(\nabla).$$

Here we recall the construction of the form $\varphi(\nabla, \nabla')$. Consider the product $\mathbb{R} \times M$ with projection $\rho : \mathbb{R} \times M \to M$. Denoting by t a coordinate on $\mathbb{R}$, let $\tilde{\nabla}$ be the connection for $\rho^* E$ given by

$$\tilde{\nabla} = (1 - t)\rho^* \nabla + t\rho^* \nabla'.$$

Letting $\rho' : [0, 1] \times M \to M$ be the restriction of ρ, we have the integration along the fibers $\rho'_* : \mathscr{E}^{2k}([0, 1] \times M) \to \mathscr{E}^{2k-1}(M)$. Then we set

$$\varphi(\nabla, \nabla') = \rho'_* \varphi(\tilde{\nabla}).$$

The form $\varphi(\nabla, \nabla')$ as above is called a *difference form*. From this we see that the class $[\varphi(\nabla)]$ of the closed form $\varphi(\nabla)$ in the de Rham cohomology $H_d^{2k}(M)$ depends only on E and not on the choice of the connection ∇.

Definition 8.6.5 We denote this class by $\varphi(E)$ and call it the *characteristic class* of E for the polynomial φ.

Definition 8.6.6 The q-th *Chern form* of ∇ is defined by

$$c^q(\nabla) = \Big(\frac{\sqrt{-1}}{2\pi}\Big)^q \sigma_q(K).$$

Its class in $H_d^{2q}(M)$ is denoted by $c^q_{\mathrm{diff}}(E)$ and is called the q-th *differential geometric Chern class* of E.

If κ is the curvature matrix of ∇ with respect to some frame of E on U, the total Chern form is given by

$$c^*(\nabla) = \det\Big(I + \frac{\sqrt{-1}}{2\pi}\kappa\Big).$$

We call

$$c^*_{\mathrm{diff}}(E) = 1 + c^1_{\mathrm{diff}}(E) + \cdots + c^l_{\mathrm{diff}}(E)$$

the total Chern class of E, which is considered as an element in the cohomology ring $H_d^*(M)$. Note that the class $c^*_{\mathrm{diff}}(E)$ is invertible in $H_d^*(M)$.

Remark 8.6.7

1. As noted above, for an invariant polynomial φ, there is a polynomial P such that $\varphi = P(\sigma_1, \sigma_2, \dots)$. We have $\varphi(\nabla) = P(c^1(\nabla), c^2(\nabla), \dots)$ and $\varphi(E) = P(c^1(E), c^2(E), \dots)$.
2. For a finite number of connections, we may define the difference form.
3. As is shown below (cf. Theorem 8.8.7), the Chern class $c^q_{\mathrm{diff}}(E)$ defined above is the image of the Chern class $c^q_{\mathrm{top}}(E)$ defined by obstruction theory by the canonical morphism $H^{2q}(M; \mathbb{Z}) \to H^{2q}(M; \mathbb{C})$.

8.6.2 Characteristic Classes in Čech-de Rham Cohomology

Let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of M. Also, let $\pi : E \to M$ be a C^∞ complex vector bundle of rank l and φ an invariant polynomial homogeneous of degree k. For each $i = 0, 1$, we choose a connection ∇_i for E on U_i, and for the collection $\nabla_* = (\nabla_0, \nabla_1)$, we define the element $\varphi(\nabla_*)$ in $\mathscr{E}^{(2k)}(\mathcal{U})$ by

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)).$$

Then we have $D\varphi(\nabla_*) = 0$. Hence the form $\varphi(\nabla_*)$ defines a class in $H_D^{2k}(\mathcal{U})$. It is shown that the class does not depend on the choice of the collection of connections ∇_*. Comparing with the class defined by a global connection, we see that the class $[\varphi(\nabla_*)]$ in $H_D^{2k}(\mathcal{U})$ corresponds to the class $\varphi(E)$ in $H_d^{2k}(M)$ under the isomorphism of Theorem 8.3.4.

This way of representing characteristic classes is particularly relevant in dealing with the “localization problem”, which we discuss in the next section.

8.7 Localization and Associated Residues

In this section, we explain general philosophy and procedure of localizing characteristic classes and of obtaining the associated residues.

8.7.1 General Philosophy

Let M be a C^∞ manifold of dimension m and S a closed set in M. For a complex vector bundle E on M and an invariant polynomial φ, it sometimes happens that we have the vanishing $\varphi(E) = 0$ on $M \smallsetminus S$. Then, in view of the exact sequence

$$\cdots \longrightarrow H^*(M, M \smallsetminus S) \xrightarrow{j^*} H^*(M) \longrightarrow H^*(M \smallsetminus S) \longrightarrow \cdots,$$

there is a class $\varphi_S(E)$ in $H^*(M, M \smallsetminus S)$ which is sent to $\varphi(E)$ by j^*. We call $\varphi_S(E)$ a *localization of* $\varphi(E)$ *at* S. Note that, since j^* is not injective in general, $\varphi_S(E)$ is not uniquely determined. However, in the cases we consider below, the vanishing of $\varphi(E)$ occurs on the cocycle level of and, using this fact, we may define a natural localization. If M is oriented and if S is a subcomplex of M with respect to some triangulation of M, the localization defines the “residue” in the homology of each connected component of S through the Alexander duality $H^*(M, M \smallsetminus S) \simeq \check{H}_*(S)$ and we have the residue theorem (cf. Theorem 8.7.3 below).

We have already seen this kind of phenomenon as localization by frames from topological viewpoint (cf. Sect. 8.5). We explain the procedure in the framework of

Chern-Weil theory adapted to Čech-de Rham cohomology, which is applicable in other settings as well.

8.7.2 Residue Theorem

Let M and S be as above. Letting $U_0 = M \setminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Let E be a C^∞ complex vector bundle on M. For an invariant polynomial φ homogeneous of degree k, the characteristic class $\varphi(E)$ is represented by the cocycle $\varphi(\nabla_*)$ in $\mathscr{E}^{(2k)}(\mathcal{U})$ given by

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)), \tag{8.27}$$

where ∇_0 and ∇_1 are connections for E on U_0 and U_1 respectively (cf. Sect. 8.6.2). Sometimes it happens that we have a "vanishing theorem" on U_0 for some polynomials φ. To be a little more precise, there is some "geometric object" γ on U_0, with which is associated a class C of connections for E on U_0 such that, for a certain polynomial φ, we have

$$\varphi(\nabla_0) = 0 \qquad \text{if } \nabla_0 \text{ belongs to } C.$$

We call a connection belonging to C *special* for γ and a polynomial φ as above *adapted* to γ. If ∇_0 is special and if φ is adapted to γ, the cocycle $\varphi(\nabla_*)$ of (8.27) is in $\mathscr{E}^{2k}(\mathcal{U}, U_0)$ and defines a class in $H_D^{2k}(\mathcal{U}, U_0)$, which is denoted by $\varphi_S(E, \gamma)$. Usually we have the vanishing of the difference form for every family of finite number of special connections and, using this fact, it is shown that the class $\varphi_S(E, \gamma)$ does not depend on the choice of the special connection ∇_0 or the connection ∇_1. We call $\varphi_S(E, \gamma)$ the *localization of* $\varphi(E)$ *at* S *by* γ Sometimes we denote it simply by $\varphi(E, \gamma)$.

From now on we assume that M is oriented and that S is a subcomplex of M with respect to some triangulation K_0 of M. Then there is a canonical isomorphism $H_D^{2k}(\mathcal{U}, U_0) \simeq H^{2k}(M, M \setminus S; \mathbb{C})$ (cf. Theorem 8.3.13) and the class $\varphi_S(E, \gamma)$ is sent to $\varphi(E)$ by the canonical morphism $j^* : H^{2k}(M, M \setminus S; \mathbb{C}) \to H^{2k}(M; \mathbb{C})$. We have the Alexander duality (Theorems 8.2.2 and 8.3.15):

$$A : H^{2k}(M, M \setminus S; \mathbb{C}) \xrightarrow{\sim} \check{H}_{m-2k}(S; \mathbb{C}).$$

Definition 8.7.1 The *residue* $\mathrm{Res}_\varphi(\gamma, E; S)$ *of* γ *at* S *for* $\varphi(E)$ is the image of $\varphi_S(E, \gamma)$ by A.

Suppose that S has only a finite number of connected components $(S_\lambda)_\lambda$. Then we have a decomposition

$$\check{H}_{m-2k}(S; \mathbb{C}) = \bigoplus_\lambda \check{H}_{m-2k}(S_\lambda; \mathbb{C})$$

and accordingly we have the residue $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ in $\check{H}_{m-2k}(S_\lambda; \mathbb{C})$ for each λ. For each λ, we take an open neighborhood U_λ of S_λ in U_1 so that $U_\lambda \cap U_\mu = \emptyset$, if $\lambda \neq \mu$. Let K, K' and K^* be as in Sect. 8.2.1. We may assume that $O_{K'}(S_\lambda) \subset U_\lambda$. Also let $\{R_0, R_1\}$ be a honeycomb system adapted to $\mathcal{U}$, K' and S (cf. Definitions 8.3.6 and 8.3.12) and set $R_\lambda = R_1 \cap U_\lambda$ and $R_{0\lambda} = -\partial R_\lambda$. Then we have (cf. (8.14)):

Proposition 8.7.2 *In the above situation, the residue* $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ *is represented by the cycle*

$$C_\lambda = \sum_{\mathbf{s}} c_{\mathbf{s}}\mathbf{s}, \qquad c_{\mathbf{s}} = \int_{\mathbf{s}^* \cap R_\lambda} \varphi(\nabla_1) + \int_{\mathbf{s}^* \cap R_{0\lambda}} \varphi(\nabla_0, \nabla_1), \tag{8.28}$$

where $\mathbf{s}$ *runs through the* $(m-2k)$*-simplices of* K *in* S_λ*.*

In particular, if $2k = m$ and if S_λ is compact, $\check{H}_0(S_\lambda; \mathbb{C}) = H_0(S_\lambda; \mathbb{C}) = \mathbb{C}$ and we have:

$$\mathrm{Res}_\varphi(\gamma, E; S_\lambda) = \int_{R_\lambda} \varphi(\nabla_1) + \int_{R_{0\lambda}} \varphi(\nabla_0, \nabla_1). \tag{8.29}$$

From the commutativity of the diagram (8.7) with $\mathbb{C}$-coefficients, we have:

Theorem 8.7.3 (Residue Theorem) *In the above situation, it holds:*

1. *For each* λ*, we have the residue* $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ *in* $\check{H}_{m-2k}(S_\lambda; \mathbb{C})$*, which is represented by the cycle* (8.28)*.*
2. *We have*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_\varphi(\gamma, E; S_\lambda) = [M] \frown \varphi(E) \qquad \text{in } \check{H}_{m-2k}(M; \mathbb{C}),$$

where $i_\lambda : S_\lambda \hookrightarrow M$ *denotes the inclusion.*

If $2k = m$ and if M is compact, we may write the right hand side of the identity in 2 as $\int_M \varphi(E)$.

Remark 8.7.4

1. The above arguments also work, if we replace E with a virtual bundle and M with a possibly singular variety, with some modifications.
2. The above theorem becomes especially meaningful every time we have an explicit description of the residues.

Transverse Residues Let M, K_0, K, K^* and S be as above. Suppose the maximum dimension of the simplices of K in S is $m-2k$ and let S' be an oriented submanifold of M of dimension $m-2k$ which is contained in S. Let x be a point in S' and $\mathbb{D}$ a slice of S' in M at x (cf. the paragraph right after Theorem 8.5.4).

Suppose

(*) the restriction $\gamma|_{\mathbb{D}}$ of γ to $\mathbb{D}$ makes sense and the restriction of any connection special for γ is special for $\gamma|_{\mathbb{D}}$.

Then, restricting the bundle E and the connections ∇_0 and ∇_1 to $\mathbb{D}$, we have the localization $\varphi_x(E|_{\mathbb{D}}, \gamma|_{\mathbb{D}})$ and the residue $\mathrm{Res}_\varphi(\gamma|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$, which correspond to each other by the Alexander isomorphism

$$H^{2k}(\mathbb{D}, \mathbb{D} \smallsetminus \{x\}; \mathbb{C}) \simeq H_0(\{x\}; \mathbb{C}).$$

As $H_0(\{x\}; \mathbb{C}) \simeq \mathbb{C}$, we may think of $\mathrm{Res}_\varphi(\gamma|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$ as a number, which is called the *transverse residue* at x. If we take a closed $2k$-ball $\mathbb{B}_1$ with center x in $\mathbb{D}$ and set $\mathbb{B}_{01} = -\partial\mathbb{B}_1$, which is a $(2k-1)$-sphere with the orientation opposite to the natural one, it is given by (cf. (8.29))

$$\mathrm{Res}_\varphi(\gamma|_{\mathbb{D}}, E|_{\mathbb{D}}; x) = \int_{\mathbb{B}_1} \varphi(\nabla_1) + \int_{\mathbb{B}_{01}} \varphi(\nabla_0, \nabla_1). \tag{8.30}$$

With these, we have:

Proposition 8.7.5 *As a function of x, $\mathrm{Res}_\varphi(\gamma|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$ is locally constant. Thus, if S' is connected, it is constant.*

Remark 8.7.6

1. We may assume that x is the barycenter $b_{\mathbf{s}}$ of some $(m-2k)$-simplex $\mathbf{s}$ of K in S' and $B_1 = \mathbf{s}^* \cap R_1$. The transverse residue is then expressed as (cf. Proposition 8.7.2)

$$\mathrm{Res}_\varphi(\gamma|_{\mathbb{D}}, E|_{\mathbb{D}}; x) = \int_{\mathbf{s}^*\cap R_1} \varphi(\nabla_1) + \int_{\mathbf{s}^*\cap R_{01}} \varphi(\nabla_0, \nabla_1).$$

2. In the cases we consider below, the assumption (*) is always satisfied.
3. If $2k = m$ and if S is compact, we do not have to assume that S is a subcomplex of M. Simply take arbitrary mutually disjoint open neighborhoods as the U_λ's and define $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ by (8.29) with R_λ as above, then Theorem 8.7.3 is still valid.

We give a formula for the residue at S_λ when it is a compact oriented submanifold of M of dimension $m-2k$ with orientation compatible with that of M. In this case we have the transverse residue $\mathrm{Res}_\varphi(\gamma|_{\mathbb{D}}, E|_{\mathbb{D}}; x)$ at each point x in S_λ and it is in fact constant (cf. Proposition 8.7.5).

Theorem 8.7.7 *Suppose that S_λ is a compact oriented submanifold of M of dimension $m-2k$. Then we have:*

$$\mathrm{Res}_\varphi(\gamma, E; S_\lambda) = \mathrm{Res}_\varphi(\gamma|_{\mathbb{D}}, E|_{\mathbb{D}}; x) \cdot [S_\lambda] \qquad \textit{in } H_{m-2k}(S_\lambda; \mathbb{C}).$$

We give an explicit expression of $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ in the case S_λ is a pseudo-manifold (cf. Definition 8.2.5).

Let S_λ be a connected component of S as above and suppose it is an oriented pseudo-manifold of dimension $m-2k$. If $\mathbf{s}$ is an $(m-2k)$-simplex of K in S_λ, $\mathbf{s}^*$ is a $2k$-cell such that $\mathbf{s}^* \cap S_\lambda = \{b_\mathbf{s}\}$. Thus we have the residue $\mathrm{Res}_\varphi(\gamma, E|_{\mathbf{s}^*}; b_\mathbf{s})$, which is given by (cf. Remark 8.7.6. 1)

$$\mathrm{Res}_\varphi(\gamma, E|_{\mathbf{s}^*}; b_\mathbf{s}) = \int_{\mathbf{s}^* \cap R_1} \varphi(\nabla_1) + \int_{\mathbf{s}^* \cap R_{01}} \varphi(\nabla_0, \nabla_1).$$

From Proposition 8.7.2, we have:

Theorem 8.7.8 *Suppose S_λ is an oriented pseudo-manifold of dimension $m-2k$. Then the residue $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ in $\check{H}_{m-2k}(S_\lambda; \mathbb{C})$ is represented by the cycle*

$$\sum_{\mathbf{s}} \mathrm{Res}_\varphi(\gamma, E|_{\mathbf{s}^*}; b_\mathbf{s}) \cdot \mathbf{s},$$

where $\mathbf{s}$ runs through all the $(m-2k)$-simplices of K in S_λ.

Corollary 8.7.9 *In the above situation, if $S_\lambda = \bigcup_i S_{\lambda,i}$ is the irreducible decomposition,*

$$\mathrm{Res}_\varphi(\gamma, E; S_\lambda) = \sum_i \mathrm{Res}_\varphi(\gamma|_\mathbb{D}, E|_\mathbb{D}; x_{\lambda,i}) \cdot [S_{\lambda,i}] \qquad \text{in } \check{H}_{m-2k}(S_\lambda; \mathbb{C}),$$

where $x_{\lambda,i}$ is a general point of $S_{\lambda,i}$ and $\mathbb{D}$ a slice of $S_{\lambda,i}$ at $x_{\lambda,i}$.

In the case S_λ is a compact submanifold, the above reduces to Theorem 8.7.7. Note that we do not need the compactness of M or of S_λ in Corollary 8.7.9.

8.7.3 Grothendieck Residues

In the complex analytic setting, the transverse residue is usually expressed in term of Grothendieck residues, which we briefly review (cf. [21]). We denote by $\mathscr{O}_n$ the ring of germs of holomorphic functions at 0 in $\mathbb{C}^n$.

Let U be a neighborhood of the origin 0 in $\mathbb{C}^n$ and $f_1, \dots, f_n$ holomorphic functions on U such that their common set of zeros $V(f_1, \dots, f_n)$ consists only of 0. For small positive numbers ε_i, $i = 1, \dots, n$, we set

$$\Gamma = \{ z \in U \mid |f_i(z)| = \varepsilon_i, \ i = 1, \dots, n \},$$

which is an n-cycle in U. It is oriented so that the form $d\theta_1 \wedge \cdots \wedge d\theta_n$ is positive, $\theta_i = \arg f_i$. For a holomorphic n-form ω on U, we set

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix} = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n}$$

and call it the *Grothendieck residue* of $\omega/f_1 \cdots f_n$ at 0. In the case $n = 1$, the above residue is the usual Cauchy residue at 0 of the meromorphic 1-form ω/f_1.

Note that this residue is alternating in $(f_1, \ldots, f_n)$. In general, this is computed as follows. From the condition $V(f_1, \ldots, f_n) = \{0\}$, we see that, for each i, z_i is in the radical $\sqrt{(f_1, \ldots, f_n)}$ of the ideal $(f_1, \ldots, f_n)$ in $\mathscr{O}_n$ generated by (the germs at 0 of) the f_i's (Nullstellensatz). Hence there is a positive integer k_i such that $z_i^{k_i}$ is in $(f_1, \ldots, f_n)$ and we may write $z_i^{k_i} = \sum_{j=1}^n g_{ij} f_j$ with $g_{ij} \in \mathscr{O}_n$. Then

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix} = \mathrm{Res}_0 \begin{bmatrix} \det(g_{ij})\, \omega \\ z_1^{k_1}, \ldots, z_n^{k_n} \end{bmatrix}.$$

If we write $\omega = f\, dz_1 \wedge \cdots \wedge dz_n$ with f in $\mathscr{O}_n$, the right hand side of the above is, by the Cauchy integral formula, the coefficient of $z_1^{k_1-1} \cdots z_n^{k_n-1}$ in the power series expansion of $f \det(g_{ij})$.

In particular, if $f = (f_1, \ldots, f_n)$ is non-degenerate, i.e., if the Jacobian $J_f = \det\big(\partial f_i/\partial z_j\big)$ is non-zero at 0, then we have

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix} = \frac{f(0)}{J_f(0)}.$$

Example 8.7.10 If $\omega = df_1 \wedge \cdots \wedge df_n$, then

$$\mathrm{Res}_0 \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \ldots, f_n \end{bmatrix}$$

is a positive integer which is simultaneously equal to (cf. Sect. 8.8.5 below):

1. The mapping degree of $f = (f_1, \ldots, f_n)$, thus the Poincaré-Hopf index at 0 in $\mathbb{C}^n$ of the vector field $v = \sum_{i=1}^n f_i \cdot \partial/\partial z_i$.
2. $\dim_{\mathbb{C}} \mathscr{O}_n/(f_1, \ldots, f_n)$.

Example 8.7.11 In particular, if $f_i = \partial f/\partial z_i$ for some holomorphic function f on U, then it is the *Milnor number* $\mu(V, 0)$ of the hypersurface V defined by f at 0 (cf. [43, 47]):

$$\mathrm{Res}_0 \begin{bmatrix} d\big(\frac{\partial f}{\partial z_1}\big) \wedge \cdots \wedge d\big(\frac{\partial f}{\partial z_n}\big) \\ \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \end{bmatrix} = \mu(V, 0).$$

8.7.4 Various Types of Localizations

In the sequel, we discuss some vanishing theorems and the corresponding localizations. In the case (I) below, both the obstruction theory and the Chern-Weil theory are effective and the comparison of the two approaches yields interesting results. In the case (II), it seems difficult to apply the obstruction theory directly. We use either the relative K-theory or the Chern-Weil theory. In the case (III), we have continuous invariants and the topological method seems to be non-applicable in general. The method using connections is effective.

(I) Localization by a Frame
Let M be a C^∞ manifold and E a C^∞ complex vector bundle of rank l on M. Suppose E admits an r-frame $s^{(r)}$ on $M \smallsetminus S$. If we use the obstruction theory, this means the vanishing of the cocycle representing the Chern class $c^q(E)$, $q = l - r + 1$, on $M \smallsetminus S$ and we have naturally the topological localization $c^q_{\mathrm{top}}(E, s^{(r)})$ (cf. Sect. 8.5). In the differential geometric framework as in the above, the geometric object γ is $s^{(r)}$ and the connections special for γ are the $s^{(r)}$-trivial connections. Here a connection ∇ is said to be $s^{(r)}$-trivial if $\nabla(s_i) = 0$ for every member s_i of $s^{(r)}$. The Chern polynomial c^q is adapted to γ (in fact c^i, $q \le i \le l$, are adapted, however the case $i = q$ is of particular interest). The corresponding localization of $c^q(E)$ is denoted by $c^q_S(E, s^{(r)})$. This will be discussed in detail in Sect. 8.8 below.

The relevant vanishing theorem is the following:

Proposition 8.7.12 *Let $s^{(r)}$ be an r-frame of E on an open set U in M and let ∇ be an $s^{(r)}$-trivial connection for E on U, then on U,*

$$c^i(\nabla) = 0 \qquad \textit{for} \quad i \ge l - r + 1.$$

(II) Localization by Exactness
Suppose we have a complex of complex vector bundles on M

$$0 \longrightarrow E_q \longrightarrow \cdots \longrightarrow E_0 \longrightarrow 0,$$

which is exact on $M \smallsetminus S$. In this situation we are to consider the characteristic classes of the "virtual bundle" $\xi = \sum_{\nu=0}^{q} (-1)^j E_\nu$. The geometric object γ is the exact sequence and the connections special for γ are the families of connections compatible with the sequence. Any polynomial φ without constant term is adapted to γ and we have the localization $\varphi_S(\xi)$ in $H^*(M, M \smallsetminus S)$.

For example, let X be a complex manifold and $\mathscr{O}_X$ the sheaf of holomorphic function s on X. Then, for a coherent $\mathscr{O}_X$-module $\mathscr{S}$, we may define the localized class $\varphi_S(\mathscr{S})$ with S the support of $\mathscr{S}$, taking a resolution of $\mathscr{S}$ by real analytic vector bundles (cf. [4, 56])

(III) Localization by Bott Type Vanishing Theorems
This type of vanishing occurs on some characteristic forms of vector bundles or virtual bundles that admit an action of a subbundle of the holomorphic tangent bundle. Thus the geometric object γ is such an action and the connections special for γ are the ones compatible with the action. It leads to a general residue theory for singular holomorphic foliations or distributions. Here we discuss only a basic case and refer to [55] and references therein for the general case.

Let X be a complex manifold of dimension n and TX the holomorphic tangent bundle of X, which is naturally identified with the real tangent bundle $T_{\mathbb{R}}X$ of X as a C^∞ manifold. Let v be a non-vanishing holomorphic vector field on X. Thus in this case we are to consider the subbundle F of TX spanned by v and the action $\mathscr{E}^{(0)}(X, F) \times \mathscr{E}^{(0)}(X, TX) \to \mathscr{E}^{(0)}(X, TX)$ of F on TX given by the Lie bracket $[\ ,\ ]$.

Definition 8.7.13 A connection ∇ for TX is a v-*connection* if

1. $\nabla w(v) = [v, w]$ for every w in $\mathscr{E}^{(0)}(X, TX)$,
2. ∇ is of type $(1, 0)$, i.e., the entries in the connection matrix with respect to a holomorphic frame are of type $(1, 0)$.

With these we have the following:

Theorem 8.7.14 *Let v be a non-vanishing holomorphic vector field on X, ∇ a v-connection for TX and φ an invariant polynomial homogeneous of degree n. Then*

$$\varphi(\nabla) = 0.$$

Note that there exists a v-connection which is also v-trivial.

Let v be a holomorphic vector field on X and $S = \mathrm{zero}(v)$ the set of zeros of v. Letting $W_0 = X \smallsetminus S$ and W_1 a neighborhood of S, we consider the covering $\mathcal{W} = \{W_0, W_1\}$ of X, as in Sect. 8.7.2. Then, by the above theorem, for an invariant polynomial φ homogeneous of degree n, we have the localization $\varphi_S(v, TX)$ in $H_D^{2n}(\mathcal{W}, W_0)$ and the residue $\mathrm{Res}_\varphi(v, TX; S)$ in $\check{H}_0(S; \mathbb{C})$. They correspond each other by the Alexander isomorphism $H_D^{2n}(\mathcal{W}, W_0) \simeq \check{H}_0(S; \mathbb{C})$ and we have a residue theorem as Theorem 8.7.3. If S_λ is a compact connected component of S, $\check{H}_0(S_\lambda; \mathbb{C}) = H_0(S_\lambda; \mathbb{C}) \simeq \mathbb{C}$ and we may think of $\mathrm{Res}_\varphi(v, TX; S_\lambda)$ as a complex number.

We give an explicit expression of the residue in the case S_λ consists of an isolated point. Thus let v be a holomorphic vector field in a neighborhood W of 0 in $\mathbb{C}^n = \{(z_1, \dots, z_n)\}$. We write

$$v = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}$$

with a_i holomorphic functions on W and assume that the set of their common zeros is $\{0\}$. Let A be the matrix whose (i, j)-entry is $\frac{\partial a_i}{\partial z_j}$ and define $\sigma_q(A)$ as in Sect. 8.6.1. For an invariant polynomial φ homogeneous of degree 0, we write $\varphi = P(\sigma_1, \sigma_2, \dots)$ and set $\varphi(A) = P(\sigma_1(A), \sigma_2(A), \dots)$ (cf. Remark 8.6.7. 1). With these we have an expression of the residue as a Grothendieck residue:

$$\mathrm{Res}_\varphi(v, TW; 0) = \mathrm{Res}_0 \begin{bmatrix} \varphi(A) dz_1 \wedge \cdots \wedge dz_n \\ a_1, \dots, a_n \end{bmatrix}.$$

In particular, if $\varphi = \sigma_n$, the right hand side is

$$\mathrm{Res}_0 \begin{bmatrix} da_1 \wedge \cdots \wedge da_n \\ a_1, \dots, a_n \end{bmatrix},$$

which is the Poincaré-Hopf index $\mathrm{PH}(v, 0)$ of v at 0. Thus the residue theorem in this case generalizes the Poincaré-Hopf theorem (Theorem 8.5.12).

Example 8.7.15 Letting λ_1 and λ_2 be non-zero complex numbers, consider the vector field

$$v = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2}$$

on $\mathbb{C}^2 = \{(z_1, z_2)\}$. Then $S = \{0\}$ and

$$\mathrm{Res}_{\sigma_1^2}(v, 0) = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} \quad \text{and} \quad \mathrm{Res}_{\sigma_2}(v, 0) = 1.$$

In general, a holomorphic subbundle F of TX is called a (non-singular) distribution. It is a (non-singular) foliation, if it is involutive, i.e., $[F, F] \subset F$. Let p be the rank of F. If E is a holomorphic vector bundle admitting an action of F, for an invariant polynomial φ homogeneous of degree greater than $n - p$ and a connection ∇ for E compatible with the action, we have the vanishing $\varphi(\nabla) = 0$.

We give some example of holomorphic actions.

(A) Action on the Normal Bundle of the Foliation Suppose F is involutive, i.e., a foliation. We call $N_F = TX/F$ the normal bundle of the foliation. We denote by $\eta : TX \to N_F$ the canonical surjection and define

$$\alpha : \mathscr{E}^{(0)}(X, F) \times \mathscr{E}^{(0)}(X, N_F) \longrightarrow \mathscr{E}^{(0)}(X, N_F) \qquad \text{by } \alpha(v, \eta(w)) = \eta([v, w]).$$

Then it is a well-defined action (cf. [5]). The case we discussed above may be considered as a special case of this where F is trivial.

(B) Action on the Normal Bundle of an Invariant Submanifold Let V be a complex submanifold of X. Let N_V be the normal bundle of V in X so that we have the exact sequence

$$0 \longrightarrow TV \longrightarrow TM|_V \xrightarrow{\varpi} N_V \longrightarrow 0.$$

Let F be a distribution on X. We say that F leaves V invariant if $F|_V \subset TV$. In this case we set $F_V = F|_V$, which is a distribution on V. In this situation, there is a natural holomorphic action of F_V on N_V, which is defined as follows. Let u and ν be C^∞ sections of F_V and N_V, respectively. Take sections $\tilde{u}$ of F and $\tilde{v}$ of TX so that $\tilde{u}|_V = u$ and $\varpi(\tilde{v}|_V) = \nu$, where $|_V$ means the restriction as sections. Define

$$\alpha : \mathscr{E}^{(0)}(V, F_V) \times \mathscr{E}^{(0)}(V, N_V) \longrightarrow \mathscr{E}^{(0)}(V, N_V) \qquad \text{by } \alpha(u, \nu) = \varpi([\tilde{u}, \tilde{v}]|_V).$$

Then it is a well-defined action (cf. [15, 40, 53]).

(C) Action on the Normal Bundle of the Ambient Foliation Let V be as in (B) and let F be a foliation on X leaving V in variant. Then the action in (A) induces an action of F_V on $N_F|_V$ (cf. [33, 41]).

Remark 8.7.16 For applications of the above residue theories, see for example [14]. A general index theory for complex discrete dynamical systems is developed in parallel with the above in a series of papers including [2].

8.8 Localization of Chern Classes by Frames

8.8.1 Differential Geometric Localization by Frames

Let M be a C^∞ manifold of dimension m and E a C^∞ complex vector bundle of rank l on M. Also let S be a closed set in M and suppose we have a C^∞ r-frame $s^{(r)}$ of E on $M \setminus S$. Then we will see that there is a natural localization $c_S^q(E, s^{(r)})$ in $H^{2q}(M, M \setminus S; \mathbb{C})$ of the Chern class $c^q(E)$, $q = l - r + 1$.

Letting $U_0 = M \setminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Recall the class $c^q(E)$ is represented by the cocycle $c^q(\nabla_*)$ in $\mathscr{E}^{(2q)}(\mathcal{U})$ given by

$$c^q(\nabla_*) = (c^q(\nabla_0), c^q(\nabla_1), c^q(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an $s^{(r)}$-trivial connection, then $c^q(\nabla_0) = 0$ and the cocycle is in $\mathscr{E}^{(2q)}(\mathcal{U}, U_0)$ (cf. Proposition 8.7.12). Moreover, the class of $c^q(\nabla_*)$ in $H^{2q}(M, M \setminus S; \mathbb{C})$ does not depend on the choice of the $s^{(r)}$-trivial connection ∇_0 or the connection ∇_1. Thus the class is well-defined in $H^{2q}(M, M \setminus S; \mathbb{C})$, which we denote by $c^q_{S,\mathrm{diff}}(E, s^{(r)})$ and call the *differential geometric localization* of $c^q(E)$ by $s^{(r)}$ at S. Sometimes it will simply be denoted by $c^q_S(E, s^{(r)})$, $c^q_{\text{diff}}(E, s^{(r)})$ or $c^q(E, s^{(r)})$. Its image by the canonical morphism $H^{2q}(M, M \smallsetminus S; \mathbb{C}) \to H^{2q}(M; \mathbb{C})$ is the class $c^q(E)$.

Suppose M is oriented and S is a subcomplex of M with respect to some triangulation of M. Then we have the Alexander duality (Theorems 8.2.2 and 8.3.15):

$$A : H^{2q}(M, M \smallsetminus S; \mathbb{C}) \xrightarrow{\sim} \check{H}_{m-2q}(S; \mathbb{C}).$$

Definition 8.8.1 The *differential geometric residue* $\operatorname{Res}_{c^q}(s^{(r)}, E; S)$ *of* $s^{(r)}$ *for* $c^q(E)$ *at* S is the image of $c^q_S(E, s^{(r)})$ by A.

Suppose that S has a finite number of connected components $(S_\lambda)_\lambda$. Then we have the residue $\operatorname{Res}_{c^q}(s^{(r)}, E; S_\lambda)$ in $\check{H}_{m-2q}(S_\lambda; \mathbb{C})$ for each λ. It is represented by a cycle as in (8.28), or given by a number as in (8.29), with k, $\varphi(\nabla_1)$ and $\varphi(\nabla_0, \nabla_1)$ replaced by q, $c^q(\nabla_1)$ and $c^q(\nabla_0, \nabla_1)$, respectively.

Also we have the differential geometric counterpart of Theorem 8.5.4 by letting $k = q$, $\gamma = s^{(r)}$ and $\varphi = c^q$ in Theorem 8.7.3.

Moreover we have the transverse residues and the statements corresponding to Theorems 8.7.7 and 8.7.8 and Corollary 8.7.9.

Remark 8.8.2

1. As noted in Remark 8.7.6.3, if $2q = m$ and if S is compact, it is not necessary to assume that S is a subcomplex of M to have the residue theorem.
2. In Sect. 8.5, we defined the topological localization $c^q_{S,\text{top}}(E, s^{(r)})$ and the topological residue $\operatorname{TRes}_{c^q}(s^{(r)}, E; S_\lambda)$. In Sect. 8.8.3 below, we see that they are essentially the same as the ones defined using connections.

Before we proceed further we give some basic examples to illustrate the procedure explained above.

Example 8.8.3 Let U be a neighborhood of 0 in $\mathbb{R}^2$ and $E = \mathbb{C} \times U$ the product bundle on U. Suppose we have a non-vanishing C^∞ section s of E on $U_0 = U \smallsetminus \{0\}$. Set $U_1 = U$ and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of U. Then we have the localization $c^1(E, s)$ in $H^2(\mathcal{U}, U_0) \simeq H^2(U, U \smallsetminus \{0\}; \mathbb{C})$ and the residue $\operatorname{Res}_{c^1}(s, E; 0)$ in $H_0(\{0\}; \mathbb{C}) = \mathbb{C}$, which we try to find.

Denoting by e the frame of E given by $e(z) = (1, z)$, we write $s = fe$ with f a non-vanishing C^∞ function on U_0. Let $\mathbb{B}^2$ be a closed 2-ball around 0 in U. Its boundary $\partial\mathbb{B}^2$ is a circle $\mathbb{S}^1$ oriented counterclockwise. In the expression of the residue corresponding to (8.29), we take as ∇_1 the e-trivial connection on U, thus $c^1(\nabla_1) = 0$ and

$$\operatorname{Res}_{c^1}(s, E; 0) = -\int_{\mathbb{S}^1} c^1(\nabla_0, \nabla_1)$$

with ∇_0 the s-trivial connection on U_0. Now we recall how the difference form $c^1(\nabla_0, \nabla_1)$ is defined (cf. the paragraph right after Proposition 8.6.4). Let θ_i be the connection matrix, in fact a form, of ∇_i with respect to the frame e, $i = 0, 1$. Then

$\theta_1 = 0$. Noting that the connection form of ∇_0 with respect to s is zero, we get from (8.26),

$$\theta_0 = -\frac{df}{f}.$$

Hence $\tilde{\theta} = (1-t)\theta_0 = (t-1)\frac{df}{f}$ and the curvature form $\tilde{\kappa}$ is given by

$$\tilde{\kappa} = d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta} = dt \wedge \frac{df}{f}.$$

Thus

$$c^1(\nabla_0, \nabla_1) = \rho'_* c^1(\tilde{\nabla}) = \frac{\sqrt{-1}}{2\pi}\rho'_*(dt \wedge \frac{df}{f}) = -\frac{1}{2\pi\sqrt{-1}}\frac{df}{f}$$

and

$$\mathrm{Res}_{c^1}(s, E; 0) = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{S}^1}\frac{df}{f}.$$

If we denote by $p : E \to \mathbb{C}$ the projection in the fiber direction and by ζ a coordinate on $\mathbb{C}$, we may write $\frac{df}{f} = f^*\frac{d\zeta}{\zeta}(= s^*p^*\frac{d\zeta}{\zeta})$. Thus $\mathrm{Res}_{c^1}(s, E, 0)$ is equal to $\deg f$ (cf. Proposition 8.8.4 below).

8.8.2 Angular Form and Bochner-Martinelli Form

Recall that $H_d^{m-1}(\mathbb{R}^m \setminus \{0\}) \simeq H^{m-1}(\mathbb{S}^{m-1}; \mathbb{C}) \simeq \mathbb{C}$. We give an explicit closed $(m-1)$-form generating $H_d^{m-1}(\mathbb{R}^m \setminus \{0\})$. As we will see, it is in the core of the Thom class of a real oriented vector bundle (cf. Theorem 8.9.18). In the complex case, we have the Bochner-Martinelli form on $\mathbb{C}^n \setminus \{0\}$, which is a closed $(2n-1)$-form generating the cohomology $H_d^{2n-1}(\mathbb{C}^n \setminus \{0\}) \simeq H^{2n-1}(\mathbb{S}^{2n-1}; \mathbb{C}) \simeq \mathbb{C}$. In fact it is a $\bar{\partial}$-closed $(n, n-1)$-form. As we see later, it is in the core of the Thom class of a complex vector bundle (cf. Remark 8.9.21).

Consider the forms on $\mathbb{R}^m = \{(x_1, \dots, x_m)\}$ given by

$$\Phi(x) = dx_1 \wedge \cdots \wedge dx_m, \quad \Phi_i(x) = (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m$$

so that $d\,\Phi_i(x) = \Phi(x)$. Also let C'_m be the constant given by

$$C'_m = \begin{cases} \frac{(k-1)!}{2\pi^k} & m = 2k, \\ \frac{(2k)!}{2^{2k+1}\pi^k k!} & m = 2k+1. \end{cases}$$

Then the form

$$\psi_m = C'_m \frac{\sum_{i=1}^m \Phi_i(x)}{\|x\|^m}$$

is a closed $(m-1)$-form on $\mathbb{R}^m \smallsetminus \{0\}$ such that

$$\int_{\mathbb{S}^{m-1}} \psi_m = 1, \tag{8.31}$$

where $\mathbb{S}^{m-1}$ is the unit sphere, in fact it may be a sphere of arbitrary radius. It is called the *angular form* on $\mathbb{R}^m$.

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, then $\psi_{2n} = (\beta_n + \overline{\beta_n})/2$, where

$$\beta_n = C_n \frac{\sum_{i=1}^n \overline{\Phi_i(z)} \wedge \Phi(z)}{\|z\|^{2n}}, \qquad C_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n}.$$

We call β_n the *Bochner-Martinelli form* on $\mathbb{C}^n$. Note that

$$\beta_1 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z},$$

the *Cauchy form* on $\mathbb{C}$. The form β_n is a closed form of type $(n, n-1)$ on $\mathbb{C}^n \smallsetminus \{0\}$, real on $\mathbb{S}^{2n-1}$ and

$$\int_{\mathbb{S}^{2n-1}} \beta_n = 1.$$

8.8.3 Coincidence of Topological and Differential Geometric Localizations

First we give an expression of the mapping degree in terms of differential forms.

Let $W(l, r)$ be the Stiefel manifold of r-frames in $\mathbb{C}^l$ and set $q = l - r + 1$ as before. We have $H^{2q-1}((W(l, r); \mathbb{C}) \simeq \mathbb{C}$ with a canonical generator. By the de Rham theorem, the generator is represented by a closed $(2q-1)$-form ω_{2q-1} on $W(l, r)$. By definition of the mapping degree, we have:

Proposition 8.8.4 *For a C^∞ map $\varphi : \mathbb{S}^{2q-1} \to W(l, r)$,*

$$\deg \varphi = \int_{\mathbb{S}^{2q-1}} \varphi^* \omega_{2q-1}.$$

In particular, if $r = 1$, then $q = l$ and $W(l, 1) = \mathbb{C}^l \smallsetminus \{0\}$. In this case we may take as ω_{2q-1} the Bochner-Martinelli form β_l (cf. Sect. 8.8.2).

We first state a generalization of Example 8.8.3 to the higher dimensional case, which will be of fundamental importance in the subsequent discussions.

Let U be a neighborhood of 0 in $\mathbb{R}^{2n}$ and $E = \mathbb{C}^n \times U$ the product bundle on U. Suppose we have a non-vanishing C^∞ section s of E on $U_0 = U \smallsetminus \{0\}$. Set $U_1 = U$ and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of U. We have the localization $c^n(s, E)$ in $H^{2n}(\mathcal{U}, U_0) \simeq H^{2n}(U, U \smallsetminus \{0\}; \mathbb{C})$ and the associated residue $\operatorname{Res}_{c^n}(s, E, 0)$ in $H_0(\{0\}; \mathbb{C}) = \mathbb{C}$. If we denote by $e^{(n)} = (e_1, \dots, e_n)$ the frame of E given by $e_i(x) = ({}^t(0, \dots, \overset{i}{1}, \dots, 0), x)$, we may write $s = \sum_{i=1}^n f_i e_i$ with f_i C^∞ functions on U_0 such that, at every point of U_0, at least one of them is non-zero. Let $\mathbb{B}^{2n}$ be a closed $2n$-ball around 0 in U and $\mathbb{S}^{2n-1} = \partial \mathbb{B}^{2n}$. In the expression (8.29) (with $\varphi = c^n$ and $\gamma = s$) of the residue, we take as ∇_1 the $e^{(n)}$-trivial connection on U so that $c^n(\nabla_1) = 0$ and

$$\operatorname{Res}_{c^n}(s, E; 0) = -\int_{\mathbb{S}^{2n-1}} c^n(\nabla_0, \nabla_1)$$

with ∇_0 an s-trivial connection on U_0. Let β_n denote the Bochner-Martinelli form on $\mathbb{C}^n$.

Theorem 8.8.5 (Fundamental Theorem for Residues) *In the above situation, we have $c^n(\nabla_0, \nabla_1) = -f^*\beta_n$ for a suitable choice of the s-trivial connection ∇_0. Thus*

$$\operatorname{Res}_{c^n}(s, E; 0) = \int_{\mathbb{S}^{2n-1}} f^*\beta_n.$$

From the above we see that $\operatorname{Res}_{c^n}(s, E; 0)$ is equal to $\deg f$ (cf. Proposition 8.8.4). Moreover, if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and if s, thus f is defined and holomorphic on U, this may be expressed as either $\dim \mathcal{O}_n/(f_1, \dots, f_n)$ or $\operatorname{Res}_0 \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix}$ (cf. Sect. 8.8.5 below).

More generally, let $\mathbb{B}^{2q}$ be a closed ball of dimension $2q$ in $\mathbb{R}^{2q}$ and E a complex vector bundle of rank l on a neighborhood U of $\mathbb{B}^{2q}$. Suppose we have an r-frame $s^{(r)}$, $r = l - q + 1$, on a neighborhood of $\mathbb{S}^{2q-1} = \partial \mathbb{B}^{2q}$. We may extend $s^{(r)}$ to an r-frame on $U \smallsetminus \{x\}$, where x is a point in the interior of $\mathbb{B}^{2q}$.

Theorem 8.8.6 *In the above situation, we have*

$$\operatorname{Res}_{c^q}(s^{(r)}, E; x) = I(s^{(r)}, x).$$

Thus $\operatorname{Res}_{c^q}(s^{(r)}, E; x)$ *is an integer.*

Using the above, we may prove that the Chern class defined by obstruction theory and the one defined by Chern-Weil theory are essentially the same.

Let E be a C^∞ complex vector bundle of rank l on M. We have the q-th Chern class $c^q_{\mathrm{diff}}(E)$ in $H^{2q}(M;\mathbb{C})$ defined via Chern-Weil theory and the q-th Chern class $c^q_{\mathrm{top}}(E)$ in $H^{2q}(M;\mathbb{Z})$ defined via obstruction theory.

Theorem 8.8.7 *In the above situation, $c^q_{diff}(E)$ is the image of $c^q_{top}(E)$ by the canonical morphism $H^{2q}(M;\mathbb{Z}) \to H^{2q}(M;\mathbb{C})$.*

We have a similar theorem for localized classes. Thus let S be a K_0-subcomplex of M and $s^{(r)}$ an r-frame of E on $M \smallsetminus S$, $r = l - q + 1$. Then we have the localized classes $c^q_{\mathrm{diff}}(E, s^{(r)})$ in $H^{2q}(M, M \smallsetminus S;\mathbb{C})$ and $c^q_{\mathrm{top}}(E, s^{(r)})$ in $H^{2q}(M, M \smallsetminus S;\mathbb{Z})$.

Theorem 8.8.8 *In the above situation, if the codimension of S is greater than or equal to two, $c^q_{diff}(E, s^{(r)})$ is the image of $c^q_{top}(E, s^{(r)})$ by the canonical morphism $H^{2q}(M, M \smallsetminus S;\mathbb{Z}) \to H^{2q}(M, M \smallsetminus S;\mathbb{C})$.*

Corollary 8.8.9 *Suppose M is oriented. Then* $\mathrm{Res}_{c^q}(s^{(r)}, E; S)$ *is the image of* $\mathrm{TRes}_{c^q}(s^{(r)}, E; S)$ *by the canonical morphism $\breve{H}_{m-2q}(S;\mathbb{Z}) \to \breve{H}_{m-2q}(S;\mathbb{C})$.*

In particular, if $m = 2q$ and if S is compact and connected, the both are the identical integers.

8.8.4 Complex Spaces Defined by Families of Holomorphic Sections

Let X be a complex manifold of dimension n and E a holomorphic vector bundle of rank l on X. Let $s^{(r)} = (s_1, \dots, s_r)$ be a holomorphic r-section of E and $\mathbf{S}$ the complex space in X defined by $s^{(r)}$, i.e., the complex space defined by the $(r \times r)$-minors of the matrix of local components of the s_i's. Thus the support S of $\mathbf{S}$ is the singular set $S(s^{(r)})$ of $s^{(r)}$. It is an analytic variety in X and $\dim S \geq n - q$, $q = l - r + 1$. It is known that X admits a C^1 triangulation compatible with S and that two such triangulations have a common refinement (cf. [46]).

In this situation, we have the topological localization $c^q_{\mathrm{top}}(E, s^{(r)})$ in the relative cohomology $H^{2q}(X, X \smallsetminus S;\mathbb{Z})$ (cf. Sect. 8.5.2) and the differential geometric localization $c^q_{\mathrm{diff}}(E, s^{(r)})$ in $H^{2q}(X, X \smallsetminus S;\mathbb{C})$ (cf. Sect. 8.8.1). We also have the associated residues $\mathrm{TRes}_{c^q}(s^{(r)}, E; S)$ in $\breve{H}_{2(n-q)}(S;\mathbb{Z})$ and $\mathrm{Res}_{c^q}(s^{(r)}, E; S)$ in $\breve{H}_{2(n-q)}(S;\mathbb{C})$. We have seen (cf. Corollary 8.8.9) that $\mathrm{Res}_{c^q}(s^{(r)}, E; S)$ is the image of $\mathrm{TRes}_{c^q}(s^{(r)}, E; S)$ by the canonical morphism

$$\breve{H}_{2n-2q}(S;\mathbb{Z}) \longrightarrow \breve{H}_{2n-2q}(S;\mathbb{C}). \tag{8.32}$$

Now we consider the case where S is of pure dimension $n - q$. In the sequel, we refer this situation as a *proper case*. Suppose S has a finite number of irreducible components S_i, $i = 1, \dots, \rho$. Each S_i defines a class $[S_i]$ in $\breve{H}_{2n-2q}(S;\mathbb{Z})$ and it is the free Abelian group generated by the $[S_i]$'s. Thus the morphism (8.32) is injective so that we may identify the two residues:

$$\mathrm{Res}_{c^q}(s^{(r)}, E; S) = \mathrm{TRes}_{c^q}(s^{(r)}, E; S).$$

Let p_i be a non-singular point of $S_i \smallsetminus \bigcup_{j \neq i} S_j$ and $\mathbb{D}_i$ a complex slice of S_i in X at p_i, i.e., a locally closed complex sudmanifold of dimension q in X through p_i and transverse to S_i at p_i. The r-section $s_i^{(r)} = s^{(r)}|_{\mathbb{D}_i}$ of $E_i = E|_{\mathbb{D}_i}$ has an isolated singularity at p_i so that we have the residue $\mathrm{Res}_{c^q}(s_i^{(r)}, E_i; p_i)$, which is an integer (cf. Theorem 8.8.6). We may assume that p_i is the barycenter $b_{\mathbf{s}}$ of a $2(n-q)$-simplex $\mathbf{s}$ in the non-singular part of S_i and that the $2q$-cell $\mathbf{s}^*$ dual to $\mathbf{s}$ is in $\mathbb{D}_i$. Then by Corollary 8.8.9,

$$\mathrm{Res}_{c^q}(s_i^{(r)}, E_i; p_i) = \mathrm{TRes}_{c^q}(s_i^{(r)}, E_i; p_i), \quad p_i = b_{\mathbf{s}}.$$

Note that this number does not depend on the choice of p_i on the non-singular part of S_i, since the residue is locally constant in p_i and the non-singular part of S_i is connected.

We restate Corollary 8.5.8 (see also Corollary 8.7.9) in the above situation:

Theorem 8.8.10

$$\mathrm{TRes}_{c^q}(s^{(r)}, E; S) = \sum_{i=1}^{\rho} \mathrm{TRes}_{c^q}(s_i^{(r)}, E_i; p_i) \cdot [S_i] \qquad \textit{in} \quad \check{H}_{2(n-q)}(S; \mathbb{Z}).$$

Thus we see that, in order to find the residue in the proper case, it suffices to know the residue at an isolated singularity.

8.8.5 Residues at an Isolated Singularity

Let W be a neighborhood of 0 in $\mathbb{C}^n$ and $E = \mathbb{C}^l \times W$ the product bundle of rank l on W, $l \geq n$. Let $r = l - n + 1$ and suppose we have a holomorphic r-section $s^{(r)}$ of E on W with an isolated singularity at 0. Thus this is a proper case. In this situation, we have $\mathrm{Res}_{c^n}(s^{(r)}, E; 0)$, which is an integer, in fact we will see that it is positive in the holomorphic case. Here we only review the case $r = 1$, thus $l = n$, and refer to [57] and the references therein for details and the general case. Letting $W_0 = W \smallsetminus \{0\}$ and $W_1 = W$, we consider the covering $\{W_0, W_1\}$ of W. We take an s-trivial connection ∇_0 for E on W_0 and a connection ∇_1 for E on W_1 trivial with respect to some holomorphic frame $e^{(n)} = (e_1, \dots, e_n)$ of E. Thus $c^n(\nabla_0) = 0$ and $c^n(\nabla_1) = 0$. Let R be a compact real $2n$-dimensional manifold with C^∞ boundary in W containing p in its interior. Then we have (cf. (8.29) with $\varphi = c^n$, $\gamma = s$ and $S_\lambda = \{0\}$)

$$\mathrm{Res}_{c^n}(s, E; 0) = -\int_{\partial R} c^n(\nabla_0, \nabla_1).$$

We give various expressions of this number.

Topological Expression We have already seen that $\mathrm{Res}_{c^n}(s, E; p)$ may be expressed as a mapping degree, even if E and s are not holomorphic, in fact s may not be defined at p (cf. Definition 8.4.4 and Theorem 8.8.6). Let $\mathbb{S}^{2n-1}$ denote a small $(2n-1)$-sphere in W with center p. Then we have the mapping as given in (8.16):

$$\varphi : \mathbb{S}^{2n-1} \longrightarrow W(n, 1).$$

In the above situation, we have :

$$\mathrm{Res}_{c^n}(s, E; p) = \deg \varphi.$$

Remark 8.8.11 If E and s are holomorphic, then $\mathrm{Res}_{c^n}(s, E; p)$ is a positive integer, by our orientation convention.

Analytic Expression Let E, W, $e^{(n)}$ and s be as above. We write $s_i = \sum_{i=1}^n f_i\, e_i$, with f_i holomorphic functions on W. Then $\mathbf{S}$ is the complex space in W defined by $f_1, \dots, f_n$ and its support $S = S(s)$ is the set of common zeros of the f_i's.

Theorem 8.8.12 *In the above situation,*

$$\mathrm{Res}_{c^n}(s, E; p) = \mathrm{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix}.$$

This is proved by considering Čech-de Rham cohomology for a covering consisting of n open sets. In the case $n = 1$, the proof can be done by directly computing the difference form $c^1(\nabla_0, \nabla_1)$ (cf. Example 8.8.3).

Algebraic Expression Let E, W, $e^{(n)}$ and s be as above. Also let f_i be as before. We denote by $\mathscr{O}_n$ the ring of germs of holomorphic functions at 0 in $\mathbb{C}^n$ and by $(f_1, \dots, f_n)$ the ideal generated by the germs of $f_1, \dots, f_n$ in $\mathscr{O}_n$.

Theorem 8.8.13 *In the above situation,*

$$\mathrm{Res}_{c^n}(s, E; p) = \dim_{\mathbb{C}} \mathscr{O}_n/(f_1, \dots, f_n).$$

The proof involves the theory of Cohen-Macaulay rings.

8.8.6 Duals of Complex Subspaces

We again consider the proper case in the situation of Sect. 8.8.4 and let $\mathbf{S}$, S_i, p_i, $\mathbb{D}_i$, E_i and $s_i^{(r)}$ be as before. We define the *multiplicity* m_i of S_i in $\mathbf{S}$ by

$$m_i = \mathrm{Res}_{c^q}(s_i^{(r)}, E_i; p_i).$$

Note that this definition is justified by Theorem 8.8.13. We then define the class of $\mathbf{S}$ by $[\mathbf{S}] = \sum m_i\,[S_i]$ in $\check{H}_{2(n-q)}(X)$ or in $\check{H}_{2(n-q)}(S)$.

From Theorem 8.8.10 we have:

Theorem 8.8.14 *The class $c^q(E, s^{(r)})$ corresponds to* $[\mathbf{S}]$ *via the Alexander duality* $H^{2q}(X, X \smallsetminus S; \mathbb{Z}) \xrightarrow{\sim} \check{H}_{2(n-q)}(S; \mathbb{Z})$.

The above is a precise form of the fact that the class $c^q(E)$ corresponds to $[\mathbf{S}]$ via the Poincaré duality $H^{2q}(X; \mathbb{Z}) \xrightarrow{\sim} \check{H}_{2(n-q)}(X; \mathbb{Z})$.

Here is an example.

Divisors Let X be a complex manifold of dimension n. A *divisor* D on X is represented by a system $(\{W_\alpha\}, \{\varphi^\alpha\})$, where $\{W_\alpha\}$ is a covering of X, φ^α is a meromorphic function on W_α, for each α, and $f^{\alpha\beta} = \varphi^\alpha/\varphi^\beta$ is a non-vanishing holomorphic function on $W_\alpha \cap W_\beta$, for each pair (α, β). We may assume that each W_α is small enough so that we may write $\varphi^\alpha = f^\alpha/g^\alpha$ with f^α and g^α holomorphic functions on W_α. The factorization of f^α and g^α leads to an expression of D as a locally finite sum $D = \sum n_i V_i$ with n_i integers and V_i irreducible hypersurfaces in X. We have the class $[D] = \sum_i n_i[V_i]$ of D in $\check{H}_{2(n-1)}(X; \mathbb{Z})$ or in $\check{H}_{2(n-1)}(|D|; \mathbb{Z})$, where $|D| = \bigcup_i V_i$ is the support of D.

On the other hand, the system $\{f^{\alpha\beta}\}$ defines a line bundle L_D, the bundle associated with D. It has a natural meromorphic section, i.e., the section s_D that is represented by φ^α on each W_α.

Suppose for the moment that D is positive, i.e., $n_i > 0$ for all i. Then each φ^α is holomorphic and we may think of D as a complex space in X defined by the holomorphic section s_D of L_D. In this case, n_i is the multiplicity of V_i in D. In this situation, we have the localized class $c^1(L_D, s_D)$ in $H^2(X, X \smallsetminus |D|; \mathbb{Z})$ that correspond to $[D]$ in $\check{H}_{2(n-1)}(|D|; \mathbb{Z})$ via the Alexander duality $H^2(X, X \smallsetminus |D|; \mathbb{Z}) \xrightarrow{\sim} \check{H}_{2(n-1)}(|D|; \mathbb{Z})$ (cf. Theorem 8.8.14). In particular, the first Chern class $c^1(L_D)$ is the Poincaré dual of $[D]$.

Considering the localization by meromorphic sections, we have similar statements in the case of not necessarily positive divisors.

8.9 Thom Isomorphism and Thom Class

We discuss the Thom isomorphism and the Thom class first from the combinatorial viewpoint, following [10]. We then consider the differential geometric counterparts in Čech-de Rham cohomology (cf. [55]). In particular, it gives explicit expressions of the Thom class.

Throughout this section we let M denote a C^∞ manifold of dimension m. In Sects. 8.9.1–8.9.4 the homology and cohomology are with $\mathbb{Z}$-coefficients.

8.9.1 Thom Class of a Submanifold

Let V be a closed submanifold of dimension d of M. We set $k = m - d$. We take a triangulation K_0 of M compatible with V and let K and K' be as in Sect. 8.2.1. We denote by K_V the set of simplices of K that are in V and by K'_V its barycentric subdivision. Then K'_V is the set of simplices in K' that are in V. We denote by K^* and K^*_V the cellular decompositions of M and V dual to K and K_V, respectively. Note that, for a p-simplex $\mathbf{s}$ of K in V, its dual $\mathbf{s}^*$ in K^* is an $(m-p)$-cell and that its dual $\mathbf{s}^*_V$ in K^*_V is a $(d-p)$-cell. They are related by $\mathbf{s}^*_V = \mathbf{s}^* \cap V$ as sets.

The simplices and cells of K, K' and K^* are oriented so that the conditions (1) and (2) in Sect. 8.2.1 are satisfied. The simplices of K_V and K'_V are oriented as simplices of K and K', respectively. In order to describe the homology and cohomology of V, we impose similar conditions for simplices and cells of K_V, K'_V and K^*_V. The condition corresponding to (1) is automatically satisfied. Thus we impose:

$(2)_V$ Let $\mathbf{t}$ be a p-simplex of K'_V. If $\mathbf{t} \subset (\mathbf{s}')^*_V$, a p-cell of K^*_V, the orientation of $\mathbf{t}$ is the same as that of $(\mathbf{s}')^*_V$.

We have an isomorphism

$$T : C^p_{K^*_V}(V) \longrightarrow C^{p+k}_{K^*}(M, M \smallsetminus O_{K'}(V)), \qquad u_V \mapsto u,$$

where u is given by, for each $(d-p)$-simplex $\mathbf{s}$ in K,

$$\langle \mathbf{s}^*, u \rangle = \begin{cases} \langle \mathbf{s}^*_V, u_V \rangle & \text{if } \mathbf{s} \subset V, \\ 0 & \text{if } \mathbf{s} \not\subset V. \end{cases}$$

Now we wish to have T compatible with coboundary operators so that it induces an isomorphism on cohomologies. For this we further impose the following condition. Let $\mathbf{s}$ be a p-simplex of K_V. We take a d-simplex $\mathbf{s}_0$ of K_V so that $\mathbf{s} \prec \mathbf{s}_0$. Then there exist an $(m-p)$-simplex $\mathbf{t}$ of K' in $\mathbf{s}^*$, a k-simplex $\mathbf{t}_0$ of K' in $\mathbf{s}^*_0$ and a $(d-p)$-simplex $\mathbf{t}_V$ of K'_V in $\mathbf{s}^*_V$ such that $\mathbf{t}_0$ and $\mathbf{t}_V$ span $\mathbf{t}$. Note that the simplices

$\mathbf{t}$, $\mathbf{t}_0$ and $\mathbf{t}_V$ have the same orientations as $\mathbf{s}^*$, $\mathbf{s}_0^*$ and $\mathbf{s}_V^*$, respectively (cf. (2) and $(2)_V$).

(5) The simplices and cells are oriented so that the orientation of $\mathbf{s}_0^*$ followed by the orientation of $\mathbf{s}_V^*$ gives the orientation of $\mathbf{s}^*$.

Note that, by the tubular neighborhood theorem, the above can be done consistently with other conditions, in particular independently of the choice of $\mathbf{s}_0$, if the normal bundle $N_{\mathbb{R},V}$ of V in M is orientable. With the assumption that the bundle $N_{\mathbb{R},V}$ is oriented as the condition (5), with $\mathbf{s}_0^*$ being thought of as in the fiber direction,

$$\delta \circ T = (-1)^k T \circ \delta$$

and thus the following:

Theorem 8.9.1 *If the normal bundle of V in M is oriented, the above T induces an isomorphism*

$$T : H^p(V) \xrightarrow{\sim} H^{p+k}(M, M \smallsetminus V),$$

called the Thom isomorphism.

Definition 8.9.2 The *Thom class* of V in M, denoted by $\Psi_{M,V}$ or simply by Ψ_V, is the image of the unity [1] in $H^0(V)$ by T:

$$\Psi_V = T([1]) \in H^k(M, M \smallsetminus V).$$

From the definition we have:

Proposition 8.9.3 *The Thom class Ψ_V is represented by a cocycle that assigns* 1 *or* 0 *to each oriented k-cell* $\mathbf{s}^*$ *according as* $\mathbf{s}^*$ *intersects with V or not.*

The Thom isomorphism is also described as follows. There is a deformation retraction $r : O_{K'}(V) \to V$ inducing an isomorphism $r^* : H^p(V) \xrightarrow{\sim} H^p(O_{K'}(V))$. We also have $H^{p+k}(M, M \smallsetminus V) \simeq H^{p+k}(O_{K'}(V), O_{K'}(V) \smallsetminus V)$ and, for a class a in $H^p(V)$, we have

$$T(a) = \Psi_V \smile r^* a. \tag{8.33}$$

Remark 8.9.4 In the above, we do not assume that M or V to be orientable. In the case they are, we have the dualities on M and V and we have the commutative diagram (8.35) below.

8.9.2 Thom Class of an Oriented Real Vector Bundle

Let $\pi : E \to M$ be a C^∞ real oriented vector bundle of rank l'. We denote by Σ the image of the zero section $s_0 : M \to E$. Note that s_0 is a diffeomorphism of M onto Σ and that $s_0^* N_{\mathbb{R},\Sigma} = E$. We apply the above considerations by letting M, V and k be E, Σ and l', respectively. Thus K_0 is a triangulation of E compatible with Σ. Then we have the Thom class $\Psi_{E,\Sigma}$, which we simply denote by Ψ_E and call the *Thom class* of E.

Rephrasing Proposition 8.9.3, we have:

Proposition 8.9.5 *The Thom class $\Psi_E \in H^{l'}(E, E \smallsetminus \Sigma)$ is represented by a cocycle that assigns* 1 *or* 0 *to the dual l'-cell* $\mathbf{s}^*$ *in E of an m-simplex* $\mathbf{s}$ *the value* 1 *or* 0 *according as* $\mathbf{s}$ *is in Σ or not.*

Example 8.9.6 If M is a point, then $E = \mathbb{R}^{l'}$ and we have

$$H^{l'}(\mathbb{R}^{l'}, \mathbb{R}^{l'} \smallsetminus 0) \simeq H_0(\{0\}) \simeq \mathbb{Z}.$$

The Thom class $\Psi_{\mathbb{R}^{l'}}$ is the canonical generator of $H^{l'}(\mathbb{R}^{l'}, \mathbb{R}^{l'} \smallsetminus 0)$ (cf. Example 8.2.3).

The relation between the Thom class of E and that of each fiber is given as follows. For each point x in M, let $i_x : (E_x, E_x \smallsetminus 0) \hookrightarrow (E, E \smallsetminus \Sigma)$ be the inclusion. We have the commutative diagram:

$$\begin{array}{ccccccc}
 & H^{l'-1}(E_x \smallsetminus 0) & \xrightarrow{\delta} & H^{l'}(E_x, E_x \smallsetminus 0) & & & \\
 & \uparrow{\scriptstyle i_x^*} & & \uparrow{\scriptstyle i_x^*} & & & \\
\longrightarrow & H^{l'-1}(E \smallsetminus \Sigma) & \xrightarrow{\delta} & H^{l'}(E, E \smallsetminus \Sigma) & \xrightarrow{j^*} & H^{l'}(E) & \longrightarrow \\
 & & & & & \wr\uparrow{\scriptstyle \pi^*} & \\
 & & & & & H^{l'}(M). &
\end{array} \tag{8.34}$$

Note that δ in the first row is an epimorphism for $l' = 1$ and an isomorphism for $l' > 1$. From the above description, we have:

Proposition 8.9.7 *A class Ψ in $H^{l'}(E, E \smallsetminus \Sigma)$ coincides with Ψ_E if and only if $i_x^* \Psi = \Psi_{E_x}$ for all x in M.*

We see below (cf. (8.37)) that $(\pi^*)^{-1} j^* \Psi_E = e(E)$, the Euler class of E.

Remark 8.9.8

1. Let V be a closed submanifold of M. We denote the normal bundle $N_{\mathbb{R},V}$ simply by N. By the tubular neighborhood theorem, there is a neighborhood U of V in M, a neighborhood W of the zero section, identified with V, in N and a homeomorphism $\tau : (U, V) \to (W, V)$, which is the identity on V. It induces an isomorphism

$$H^p(N, N \smallsetminus V) = H^p(W, W \smallsetminus V) \xrightarrow{\sim} H^p(U, U \smallsetminus V) = H^p(M, M \smallsetminus V).$$

If N is orientable, the Thom class Ψ_N corresponds to Ψ_V.

2. We discuss the Thom isomorphism and the Thom class in terms of differential forms in Sect. 8.9.5 below, where they are treated in cohomology with $\mathbb{C}$-coefficients.

8.9.3 Poincaré, Alexander and Thom Isomorphisms

Let V be a closed submanifold of dimension d of M, as in Sect. 8.9.1. We assume that M and V are oriented. In order to describe the dualities for M we impose the conditions (1), (2) in Sect. 8.2.1 and (3), (4) in Sect. 8.2.2. We also impose the corresponding conditions for the simplices and cells of K_V, K'_V and K^*_V. Note that the one corresponding to (1) is already satisfied. Besides $(2)_V$ we further impose:

$(3)_V$ The orientation of each d-simplex of K_V is the same as that of V.

$(4)_V$ For every p-simplex $\mathbf{s}$, $0 < p < d$, of K_V, the orientation of $\mathbf{s}^*_V$ followed by the orientation of $\mathbf{s}$ gives the orientation of V.

We have the exact sequence of real vector bundles:

$$0 \longrightarrow T_{\mathbb{R}}V \longrightarrow T_{\mathbb{R}}M|_V \xrightarrow{\varpi} N_{\mathbb{R},V} \longrightarrow 0.$$

Since we assumed that M and V to be oriented, the normal bundle $N_{\mathbb{R},V}$ is orientable. We orient the bundle sno that, if $(x_1, \dots, x_m)$ and $(x_{k+1}, \dots, x_m)$ are positive coordinate systems on M and V, then the frame $(\varpi(\frac{\partial}{\partial x_1}), \dots, \varpi(\frac{\partial}{\partial x_k}))$ is positive. The total space of $N_{\mathbb{R},V}$ is then oriented so that the orientation of the fiber followed by that of V gives the orientation. By identifying neighborhoods of V in M and $N_{\mathbb{R},V}$ by the tubular neighborhood theorem, we may rephrase the above conventions as:

Convention We orient the bundle $N_{\mathbb{R},V}$ so that the orientation of the fiber of $N_{\mathbb{R},V}$ followed by that of V gives the orientation of M.

We also impose the condition (5) in Sect. 8.9.1, which is consistent with the above convention.

With these, we have the following commutative diagram:

$$\begin{array}{ccc} H^p(V) & \xrightarrow[T]{\sim} & H^{p+k}(M, M \smallsetminus V) \\ {\scriptstyle P}\downarrow\wr & \swarrow{\scriptstyle A}\,\sim & \\ \check{H}_{d-p}(V), & & \end{array} \tag{8.35}$$

from which we see that

$$A(\Psi_{M,V}) = [V], \tag{8.36}$$

the fundamental class of V.

8.9.4 Thom Class as a Localized Euler Class

Let $\pi : E \to M$ be an oriented real vector bundle of rank l'. We apply the considerations in Sect. 8.5.3 to the "diagonal section" of the pull-back bundle π^*E. Recall that it is a vector bundle on E given by

$$\pi^*E = \{\, (\xi_1, \xi_2) \in E \times E \mid \pi(\xi_1) = \pi(\xi_2) \,\}.$$

We think of it as a vector bundle $\varpi : \pi^*E \to E$ on the second factor with ϖ the restriction of the projection. We denote by Σ the image of the zero section of $\pi : E \to M$, which is naturally diffeomorphic with M. The bundle π^*E admits the diagonal section s_Δ defined by $s_\Delta(\xi) = (\xi, \xi)$ for ξ in E, whose zero set is Σ. Thus we have the localization $e(\pi^*E, s_\Delta)$ in $H^{l'}(E, E \smallsetminus \Sigma)$ of $e(\pi^*E)$ by s_Δ.

Suppose M is oriented so that Σ is also oriented. We orient the total space E so that the orientation of the fiber followed by that of Σ gives the orientation of E (cf. Convention in the previous subsection). Then we have the corresponding residue $\mathrm{TRes}_e(s_\Delta, \pi^*E; \Sigma)$ in $\check{H}_m(\Sigma)$ (note that E is $m + l'$ dimensional).

Recall that we have the Thom class Ψ_E of E in $H^{l'}(E, E \smallsetminus \Sigma)$.

Theorem 8.9.9 *In the above situation,*

1. $e(\pi^*E, s_\Delta) = \Psi_E \quad \text{in} \ \ H^{l'}(E, E \smallsetminus \Sigma)$.
2. *If M is oriented,*

$$\mathrm{TRes}_e(s_\Delta, \pi^*E; \Sigma) = \Sigma \qquad in \ \ \check{H}_m(\Sigma).$$

Remark 8.9.10

1. By the functoriality of the obstruction cocycles and Theorem 8.9.9, we have $\pi^* e(E) = e(\pi^* E) = j^* \Psi_E$. As the map $\pi : E \to M$ is a deformation retraction, it induces an isomorphism $\pi^* : H^{l'}(M) \xrightarrow{\sim} H^{l'}(E)$ and we have

$$e(E) = (\pi^*)^{-1} j^* \Psi_E. \tag{8.37}$$

2. The Thom class is a universal localization of the euler class in the following sense. Given a section $s : M \to E$ of E with singular set S. We have the induced morphism

$$s^* : H^{l'}(E, E \smallsetminus \Sigma) \longrightarrow H^{l'}(M, M \smallsetminus S).$$

By the functoriality of relative obstruction cocycles, we have

$$e(E, s) = s^* e(\pi^* E, s_\Delta) = s^* \Psi_E.$$

If E is a complex vector bundle of rank l, we have the topological localization $c^l_{\text{top}}(\pi^* E, s_\Delta)$ in $H^{2l}(E, E \smallsetminus M)$, which coincides with $e(\pi^* E, s_\Delta)$ (cf. Proposition 8.5.13) so that we have:

Corollary 8.9.11 *For a complex vector bundle E of rank l,*

1. $c^l_{top}(\pi^* E, s_\Delta) = \Psi_E$ *in* $H^{2l}(E, E \smallsetminus \Sigma)$.
2. *If M is oriented,*

$$\mathrm{TRes}_{c^l}(s_\Delta, \pi^* E; \Sigma) = \Sigma \qquad in \;\; \check{H}_m(\Sigma).$$

Remark 8.9.12 Remark 8.9.10. 2 applies with e and l' replaced by c^l and $2l$, i.e., the Thom class of a complex vector bundle is a universal localization of the top Chern class.

We come back to this point and review this from differential geometric viewpoint in the subsequent subsections.

8.9.5 Thom Class in Relative Čech-de Rham Cohomology

The Thom isomorphism and the Thom class are introduced in the previous subsections from topological viewpoint in cohomology with $\mathbb{Z}$-coefficients. In this subsection we express them in terms of differential forms. Thus the cohomologies involved are with $\mathbb{C}$-coefficients.

Case of Submanifolds Let M be a C^∞ manifold of dimension m and V a closed submanifold of dimension d. We set $k = m - d$. Recall that, assuming the normal bundle of V in M is oriented, we have the Thom isomorphism

$$T : H^p(V;\mathbb{Z}) \xrightarrow{\sim} H^{p+k}(M, M \smallsetminus V;\mathbb{Z})$$

and the Thom class $\Psi_V = T([1])$ (cf. Sect. 8.9.1). Its image by the canonical morphism $H^k(M, M \smallsetminus V;\mathbb{Z}) \to H^k(M, M \smallsetminus V;\mathbb{C})$ is still called the Thom class of V and is denoted by Ψ_V.

Let K_0, K, K', K^* and K_V^* be as before. Letting $U_0 = M \smallsetminus V$ and U_1 a neighborhood of V, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Then we have the isomorphism (cf. Theorem 8.3.13):

$$H_D^k(\mathcal{U}, U_0) \xrightarrow{\sim} H^k(M, M \smallsetminus V;\mathbb{C}).$$

If we choose a honeycomb system $\{R_0, R_1\}$ so that it is adapted to $\mathcal{U}$, K' and V (cf. Definitions 8.3.6 and 8.3.12), by Proposition 8.9.3, the Thom class Ψ_V is represented by a cocycle (ψ_1, ψ_{01}) in $\mathscr{E}^{(k)}(\mathcal{U}, U_0)$ such that, for each oriented k-cell $\mathbf{s}^*$ forming a basis of $C_k^{K^*}(M)$,

$$\int_{\mathbf{s}^*\cap R_1} \psi_1 + \int_{\mathbf{s}^*\cap R_{01}} \psi_{01} = \begin{cases} 1 & \text{if } \mathbf{s}^* \cap V \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

(cf. (8.13)). In fact the above characterizes the Thom class.

If we choose U_1 to be a tubular neighborhood of V with a C^∞ deformation retraction $r : U_1 \to V$, then $r^* : H^p(V;\mathbb{C}) \xrightarrow{\sim} H^p(U_1;\mathbb{C})$ and, by (8.33), the Thom isomorphism T assigns to the class of θ the class of $(\psi_1 \wedge r^*\theta, \psi_{01} \wedge r^*\theta)$.

Case of Oriented Vector Bundles Let $\pi : E \to M$ be a C^∞ real oriented vector bundle of rank l'. We denote by Σ the image of the zero section of E and set $W_0 = E \smallsetminus \Sigma$. Letting $W_1 = E$ (in fact we may take arbitrary neighborhood of Σ in E as W_1 by Corollary 8.3.10), we consider the covering $\mathcal{W} = \{W_0, W_1\}$ of (the total space of) E. Let T_1 be a closed ball bundle in W_1 and let $T_0 = E \smallsetminus \operatorname{Int} T_1$. Then $\{T_0, T_1\}$ is a honeycomb system adapted to $\mathcal{W}$. We denote by π_1 and π_{01}, respectively, the restrictions of π to T_1 and T_{01}. Thus $\pi_1 : T_1 \to M$ is a closed l'-ball bundle and $\pi_{01} : T_{01} \to M$ an $(l'-1)$-sphere bundle. Note that the orientation of T_{01} is opposite to that of the boundary of T_1. Let $(\pi_1)_*$ and $(\pi_{01})_*$ denote the integrations along the fibers of π_1 and π_{01}, respectively.

Definition 8.9.13 The *fiber integration* on Čech-de Rham cochains

$$\pi_* : \mathscr{E}^{(p)}(\mathcal{W}, W_0) \longrightarrow \mathscr{E}^{(p-l')}(M)$$

is defined by

$$\pi_*\xi = (\pi_1)_*\xi_1 + (\pi_{01})_*\xi_{01} \qquad \text{for } \xi = (\xi_1, \xi_{01}).$$

Then we have the following:

Proposition 8.9.14 (Projection Formula) *In the above situation,*

1. *For* ξ *in* $\mathscr{E}^{(p)}(\mathcal{W}, W_0)$ *and* θ *in* $\mathscr{E}^{(q)}(M)$*,*

$$\pi_*(\xi \cdot \pi^*\theta) = \pi_*\xi \wedge \theta,$$

where $\pi^*\theta$ *is considered as an element in* $\mathscr{E}^{(q)}(W_1)$.

2. *If* M *is compact and oriented, for* ξ *in* $\mathscr{E}^{(p)}(\mathcal{W}, W_0)$ *and* θ *in* $\mathscr{E}^{(m+r-p)}(M)$*,*

$$\int_E \xi \cdot \pi^*\theta = \int_M \pi_*\xi \wedge \theta.$$

Also noting that $(\partial\pi_1)_* = -(\pi_{01})_*$, we have $\pi_* \circ D + (-1)^{l'+1} d \circ \pi_* = 0$. From this we see that the fiber integration induces a morphism

$$\pi_* : H^p(E, E \smallsetminus \Sigma; \mathbb{C}) \longrightarrow H^{p-l'}(M; \mathbb{C}). \tag{8.38}$$

Theorem 8.9.15 *The morphism* (8.38) *is an isomorphism. In fact, it is the inverse of the Thom isomorphism with coefficients in* $\mathbb{C}$.

From Proposition 8.9.14.1, we recover the expression (8.33) in terms of differential forms:

$$T_E(a) = \Psi_E \smile \pi^*a \qquad \text{for} \quad a \in H^p(M; \mathbb{C}).$$

Thus, if Ψ_E is represented by a cocycle $(\psi_1, \psi_{01}) \in \mathscr{E}^{(l')}(\mathcal{W}, W_0)$, T_E is induced in cohomology by the map $\theta \mapsto (\psi_1 \wedge \pi^*\theta, \psi_{01} \wedge \pi^*\theta)$ of $\mathscr{E}^{(p)}(M)$ to $\mathscr{E}^{(p+l')}(\mathcal{W}, W_0)$.

We now describe the Thom class in terms of Čech-de Rham cohomology. First note that, from Theorem 8.9.15, we have:

Corollary 8.9.16 *A class* Ψ *in* $H^{l'}(\mathcal{W}, W_0)$ *coincides with the Thom class* Ψ_E *if and only if*

$$\pi_*\Psi = 1.$$

Proposition 8.9.17 *The Thom class* Ψ_E *in* $H^{l'}(\mathcal{W}, W_0)$ *is represented by a cocycle in* $\mathscr{E}^{(l')}(\mathcal{W}, W_0)$ *of the form*

$$(\pi^*\epsilon, -\psi),$$

where ϵ *is a closed* l'*-form on* M *and* ψ *is an* $(l'-1)$*-form on* W_{01} *such that* $d\psi = -\pi^*\epsilon$ *in* W_{01} *and* $-(\pi_{01})_*\psi = 1$.

The form ψ above is called a *global angular form*. In particular, if M is a point, then $E = \mathbb{R}^{l'}$ and we have

$$H_D^{l'}(\mathcal{W}, W_0) = H^{l'}(\mathbb{R}^{l'}, \mathbb{R}^{l'} \smallsetminus \{0\}; \mathbb{C}) \simeq \mathbb{C}.$$

The Thom class Ψ_E is then represented by a cocycle $(0, -\psi)$ with

$$\int_{\mathbb{S}^{l'-1}} \psi = 1,$$

i.e., we may take as ψ the angular form $\psi_{l'}$ on $\mathbb{R}^{l'}$ (cf. Sect. 8.8.2).

In the diagram (8.34) with $\mathbb{C}$-coefficients, we see that the global angular form ψ restricts to an angular form on each fiber E_x and we recover Proposition 8.9.7 using differential forms.

Thus we have:

Theorem 8.9.18 *Let $E = \mathbb{R}^{l'} \times M$ be the product bundle and $\rho : E \to \mathbb{R}^{l'}$ the projection onto the fiber direction. Then the Thom class Ψ_E is represented by a cocycle in $\mathscr{E}^{l'}(\mathcal{W}, W_0)$ of the form*

$$(0, -\rho^* \psi_{l'}),$$

where $\psi_{l'}$ is the angular form on $\mathbb{R}^{l'}$.

If $E = \mathbb{C}^l \times M$, Ψ_E is represented by a cocycle $(0, -\rho^* \beta_l)$ with β_l the Bochner-Martinelli form on $\mathbb{C}^l$ (cf. Sect. 8.8.2 and the next subsection).

The Euler class in terms of differential forms naturally arises in this context:

Proposition 8.9.19 *The form ϵ in Proposition 8.9.17 represents the Euler class $e(E)$ in $H_d^{l'}(M) \simeq H^{l'}(M; \mathbb{C})$.*

Thus the Euler class $e(E)$ vanishes if and only if there is a closed $(l-1)$-form ψ' on W_{01} such that Ψ_E is represented by $(0, -\psi')$.

8.9.6 Thom Class of a Complex Vector Bundle

Thom Class as a Localized Top Chern Class Let $\pi : E \to M$ be a complex C^∞ vector bundle of rank l. We apply the considerations in Sect. 8.8.1 to the diagonal section s_Δ of the pull-back bundle $\pi^* E$ (cf. Sect. 8.9.4). The zero set of s_Δ is the image Σ of the zero section of $E \to M$, which is naturally diffeomorphic with M. In this situation, on the one hand there is the differential geometric localization $c^l_{\mathrm{diff}}(\pi^* E, s_\Delta)$ in $H^{2l}(E, E \smallsetminus \Sigma; \mathbb{C})$ of $c^l(\pi^* E)$ by s_Δ. On the other hand, as a real vector bundle of rank $2l$, there is the Thom class Ψ_E of E in $H^{2l}(E, E \smallsetminus \Sigma; \mathbb{Z})$ or $H^{2l}(E, E \smallsetminus \Sigma; \mathbb{C})$ (cf. Sects. 8.9.2 and 8.9.5).

If M is oriented, we have the residue $\mathrm{Res}_{c^l}(s_\Delta, \pi^*E;\ \Sigma)$ as the image of the class $c^l_{\mathrm{diff}}(\pi^*E, s_\Delta)$ by the Alexander isomorphism

$$A : H^{2l}(E, E \smallsetminus \Sigma; \mathbb{C}) \xrightarrow{\sim} \check{H}_m(\Sigma; \mathbb{C}).$$

In this case, Ψ_E corresponds to Σ by A.

Theorem 8.9.20 *In the above situation,*

1. $c^l_{diff}(\pi^*E, s_\Delta) = \Psi_E \qquad in\ \ H^{2l}(E, E \smallsetminus \Sigma; \mathbb{C}).$
2. *If M is oriented,*

$$\mathrm{Res}_{c^l}(s_\Delta, \pi^*E;\ \Sigma) = \Sigma \qquad in\ \ \check{H}_m(\Sigma; \mathbb{C}).$$

Remark 8.9.21 The above follows from Corollary 8.9.11 and Theorem 8.8.8. It can also be proved directly along the following line. Thus take a covering $\mathcal{W} = \{W_0, W_1\}$ of E as in Sect. 8.9.5. Suppose E is trivial on an open set U in M; $E|_U \simeq \mathbb{C}^l \times U$. Then we may choose an s_Δ-trivial connection D_0 for π^*E on W_0 and a connection D_1 for π^*E on W_1 so that

$$(c^l(D_1), c^l(D_0, D_1))|_{\pi^{-1}(U)} = (0, -p^*\beta_l),$$

where $p : E|_U \to \mathbb{C}^l$ is the projection onto the fiber and β_l is the Bochner-Martinelli form on $\mathbb{C}^l$ (cf. Sect. 8.8.2). Then by Corollary 8.9.16, we have the theorem.

The above choice of D_0 and D_1 gives explicit local expressions of the forms in Proposition 8.9.17 for a complex vector bundle.

Universality of the Thom Class Let $\pi : E \to M$ be as above. Suppose we have a section s of $E \to M$ with zero set S. Then it induces a morphism

$$s^* : H^{2l}(E, E \smallsetminus \Sigma; \mathbb{C}) \longrightarrow H^{2l}(M, M \smallsetminus S; \mathbb{C}).$$

Theorem 8.9.22 *In the above situation,*

$$c^l_S(E, s) = s^*\Psi_E.$$

Remark 8.9.23

1. The above follows from the fact that the Thom class is a universal localization of the Euler class (cf. Remark 8.9.12). It can also be proved directly taking suitable connections.
2. The residues defined in Sects. 8.5 and 8.8 correspond, in the case $r = 1$, to what is called the "localized top Chern class" defined in [18] in the algebraic category. In particular, see Example 19.2.6 loc.cit. as to Theorem 8.9.22.

3. For applications of the expression as in Theorem 8.9.18 (cf. also Remark 8.9.21), see [6, 13] and the embedding of real analytic forms into hyperforms in Sect. 8.12 below.
4. An equivariant version of the above theory is developed in [17].

8.10 Dolbeault and Relative Čech-Dolbeault Theorems

For generalities on sheaves and sheaf cohomology, we refer to [19, 28]. See also [61] and the references therein. As references for complex manifolds and the theory of analytic functions of several complex variables, we list [21, 23, 36].

Throughout this section, we let X denote a complex manifold of dimension n. We also denote by $\mathscr{O}_X^{(p)}$, $\mathscr{E}_X^{(p)}$ and $\mathscr{E}_X^{(p,q)}$ the sheaves of holomorphic p-forms, C^∞ p-forms and C^∞ forms of type (p,q), respectively, on X. We omit the suffix X, if there is no fear of confusion. For a sheaf $\mathscr{S}$ on X and an open set W in X, we denote by $\mathscr{S}(W)$ the set of sections on W. This is consistent with the notation $\mathscr{E}^{(p)}(W)$ in the previous sections.

8.10.1 Dolbeault Cohomology

For each p, the $\bar\partial$-operator defines a complex of $\mathbb{C}$-vector spaces:

$$0 \longrightarrow \mathscr{E}^{(p,0)}(X) \xrightarrow{\bar\partial^{p,0}} \mathscr{E}^{(p,1)}(X) \xrightarrow{\bar\partial^{p,1}} \cdots \xrightarrow{\bar\partial^{p,n-1}} \mathscr{E}^{(p,n)}(X) \longrightarrow 0,$$

which is called the p-th *Dolbeault complex* of X and is denoted by $(\mathscr{E}^{(p,\bullet)}(X), \bar\partial)$.

Definition 8.10.1 The *Dolbeault cohomology* of type (p,q) of X is the q-th cohomology of $(\mathscr{E}^{(p,\bullet)}(X), \bar\partial)$:

$$H_{\bar\partial}^{p,q}(X) = \operatorname{Ker} \bar\partial^{p,q} / \operatorname{Im} \bar\partial^{p,q-1}.$$

For a $\bar\partial$-closed (p,q)-form ω, its class in $H_{\bar\partial}^{p,q}(X)$ is denoted by $[\omega]$.

On the other hand, for the sheaf $\mathscr{O}^{(p)}$, we may define the cohomology $H^q(X;\mathscr{O}^{(p)})$ taking, for example, a flabby resolution of $\mathscr{O}^{(p)}$. Also, if $\mathcal{W}$ is an open covering of X, we have the Čech cohomology $H^q(\mathcal{W};\mathscr{O}^{(p)})$. If $\mathcal{W}$ is a Stein covering, there is a canonical isomorphism $H^q(\mathcal{W};\mathscr{O}^{(p)}) \simeq H^q(X;\mathscr{O}^{(p)})$.

We have:

Theorem 8.10.2 (Canonical Dolbeault Theorem) *There is a canonical isomorphism*

$$H^{p,q}_{\bar\partial}(X) \simeq H^q(X; \mathscr{O}^{(p)}).$$

Remark 8.10.3 The above isomorphism is given via the Čech-Dolbeault cohomology, regarding both Dolbeault and Čech cocycles as being Čech-Dolbeault cocycles (cf. [61, 63]).

The use of the resolution of $\mathscr{O}^{(p)}$ by the complex $\mathscr{E}^{(p,\bullet)}$ leads to an isomorphism $H^{p,q}_{\bar\partial}(X) \simeq H^q(X; \mathscr{O}^{(p)})$, which differs from the above by a sign of $(-1)^{\frac{q(q+1)}{2}}$. For example, in Theorem 8.11.1 below, the sign $(-1)^{\frac{n(n-1)}{2}}$ does not appear this way.

8.10.2 Čech-Dolbeault Cohomology

The Čech-Dolbeault cohomology may be defined for an arbitrary covering of a manifold. Here we only consider coverings consisting of two open sets and refer to [60, 63] for details and the general case.

Let $\mathcal{W} = \{W_0, W_1\}$ be an open covering of X. We set $W_{01} = W_0 \cap W_1$ and define the vector space $\mathscr{E}^{(p,q)}(\mathcal{W})$ as

$$\mathscr{E}^{(p,q)}(\mathcal{W}) = \mathscr{E}^{(p,q)}(W_0) \oplus \mathscr{E}^{(p,q)}(W_1) \oplus \mathscr{E}^{(p,q-1)}(W_{01}).$$

Thus an element ξ in $\mathscr{E}^{(p,q)}(\mathcal{W})$ is given by a triple $\xi = (\xi_0, \xi_1, \xi_{01})$ with ξ_0 a (p, q)-form on W_0, ξ_1 a (p, q)-form on W_1 and ξ_{01} a $(p, q-1)$-form on W_{01}. We define an operator $\bar\vartheta = \bar\vartheta^p : \mathscr{E}^{(p,q)}(\mathcal{W}) \to \mathscr{E}^{(p,q+1)}(\mathcal{W})$ by

$$\bar\vartheta\xi = (\bar\partial\xi_0, \bar\partial\xi_1, \xi_1 - \xi_0 - \bar\partial\xi_{01}).$$

Then it is not difficult to see that $\bar\vartheta \circ \bar\vartheta = 0$. Thus we have a complex $(\mathscr{E}^{(p,\bullet)}(\mathcal{W}), \bar\vartheta)$, the p-th *Čech-Dolbeault complex* of $\mathcal{W}$.

Definition 8.10.4 The *Čech-Dolbeault cohomology* of type (p, q) of $\mathcal{W}$ is the q-th cohomology of $(\mathscr{E}^{(p,\bullet)}(\mathcal{W}), \bar\vartheta)$:

$$H^{p,q}_{\bar\vartheta}(\mathcal{W}) = \operatorname{Ker} \bar\vartheta^{p,q} / \operatorname{Im} \bar\vartheta^{p,q-1}.$$

We denote the image of ξ by the canonical surjection $\operatorname{Ker} \bar\vartheta^{p,q} \to H^{p,q}_{\bar\vartheta}(\mathcal{W})$ by $[\xi]$.

Theorem 8.10.5 *The morphism $\mathscr{E}^{(p,q)}(X) \to \mathscr{E}^{(p,q)}(\mathcal{W})$ given by $\omega \mapsto (\omega, \omega, 0)$ induces an isomorphism*

$$H^{p,q}_{\bar\partial}(X) \xrightarrow{\sim} H^{p,q}_{\bar\vartheta}(\mathcal{W}).$$

Note that the inverse of the above isomorphism is given by assigning to the class of a cocycle (ξ_0, ξ_1, ξ_{01}), the class of a global closed form $\rho_0\xi_0 + \rho_1\xi_1 - \bar\partial\rho_0 \wedge \xi_{01}$, where $\{\rho_0, \rho_1\}$ is a C^∞ partition of unity subordinate to $\mathcal{W}$.

8.10.3 Relative Čech-Dolbeault Cohomology

Let S be a closed set in X. Letting $W_0 = X \smallsetminus S$ and W_1 an open neighborhood of S, we consider the covering $\mathcal{W} = \{W_0, W_1\}$ of X. We have the cohomology $H^{p,q}_{\bar\vartheta}(\mathcal{W}, W_0)$ as the cohomology of the complex $(\mathscr{E}^{(p,\bullet)}(\mathcal{W}, W_0), \bar\vartheta)$, where

$$\mathscr{E}^{(p,q)}(\mathcal{W}, W_0) = \mathscr{E}^{(p,q)}(W_1) \oplus \mathscr{E}^{(p,q-1)}(W_{01}), \qquad W_{01} = W_0 \cap W_1,$$

and $\bar\vartheta : \mathscr{E}^{(p,q)}(\mathcal{W}, W_0) \to \mathscr{E}^{(p,q+1)}(\mathcal{W}, W_0)$ is given by

$$\bar\vartheta(\xi_1, \xi_{01}) = (\bar\partial\xi_1, \xi_1 - \bar\partial\xi_{01}).$$

We have the exact sequence

$$0 \longrightarrow \mathscr{E}^{(p,\bullet)}(\mathcal{W}, W_0) \xrightarrow{j^*} \mathscr{E}^{(p,\bullet)}(\mathcal{W}) \xrightarrow{i^*} \mathscr{E}^{(p,\bullet)}(W_0) \longrightarrow 0,$$

where $j^*(\xi_1, \xi_{01}) = (0, \xi_1, \xi_{01})$ and $i^*(\xi_0, \xi_1, \xi_{01}) = \xi_0$. This gives rise to the exact sequence

$$\cdots \longrightarrow H^{p,q-1}_{\bar\partial}(W_0) \xrightarrow{\delta} H^{p,q}_{\bar\vartheta}(\mathcal{W}, W_0) \xrightarrow{j^*} H^{p,q}_{\bar\vartheta}(\mathcal{W}) \xrightarrow{i^*} H^{p,q}_{\bar\partial}(W_0) \longrightarrow \cdots, \tag{8.39}$$

where δ assigns to the class of θ the class of $(0, -\theta)$.

Now we consider the special case where $W_1 = X$. Thus, letting $W_0 = X \smallsetminus S$ and $W_1^\star = X$, we consider the covering $\mathcal{W}^\star = \{W_0, W_1^\star\}$ of X.

Definition 8.10.6 We denote $H^{p,q}_{\bar\vartheta}(\mathcal{W}^\star, W_0)$ by $H^{p,q}_{\bar\vartheta}(X, X \smallsetminus S)$ and call it the *relative Dolbeault cohomology* of (X, S).

Let $\mathcal{W} = \{W_0, W_1\}$ be as in the beginning of this subsection, with W_1 an arbitrary open set containing S. Then we see that the restriction $\mathscr{E}^{(p,\bullet)}(\mathcal{W}^\star, W_0) \to \mathscr{E}^{(p,\bullet)}(\mathcal{W}, W_0)$ induces an isomorphism

$$H^{p,q}_{\bar\vartheta}(X, X \smallsetminus S) \xrightarrow{\sim} H^{p,q}_{\bar\vartheta}(\mathcal{W}, W_0).$$

Thus we have:

Proposition 8.10.7 *The cohomology* $H^{p,q}_{\bar\vartheta}(\mathcal{W}, W_0)$ *is uniquely determined modulo canonical isomorphisms, independently of the choice of* W_1.

Proposition 8.10.8 (Excision) *Let* S *be a closed set in* X. *Then, for every open set* W *in* X *containing* S, *there is a canonical isomorphism*

$$H^{p,q}_{\bar\vartheta}(X, X \smallsetminus S) \xrightarrow{\sim} H^{p,q}_{\bar\vartheta}(W, W \smallsetminus S).$$

Denoting by $H^q(X, X \smallsetminus S; \mathscr{O}^{(p)})$ the relative cohomology of $\mathscr{O}^{(p)}$ for the pair $(X, X \smallsetminus S)$, we have:

Theorem 8.10.9 (Relative Čech-Dolbeault Theorem) *There is a canonical isomorphism:*

$$H^{p,q}_{\bar\vartheta}(\mathcal{W}, W_0) \simeq H^q(X, X \smallsetminus S; \mathscr{O}^{(p)}).$$

The excision of Proposition 8.10.8 is compatible with that of the relative cohomology, via the isomorphism of Theorem 8.10.9.

We finish this subsection by presenting the following topic:

Differential Let X, S and $\mathcal{W} = \{W_0, W_1\}$ be as above. We set $X' = X \smallsetminus S$. If we define an operator

$$\mathfrak{d} : \mathscr{E}^{(p,q)}(\mathcal{W}, W_0) \longrightarrow \mathscr{E}^{p+1,q}(\mathcal{W}, W_0) \qquad \text{by} \ \ (\xi_1, \xi_{01}) \mapsto (-1)^q\,(\partial\xi_1, -\partial\xi_{01}),$$

then it is compatible with $\bar\vartheta$ and induces

$$\mathfrak{d} : H^{p,q}_{\bar\vartheta}(\mathcal{W}, W_0) \longrightarrow H^{p+1,q}_{\bar\vartheta}(\mathcal{W}, W_0).$$

Also, $d : \mathscr{O}^{(p)} \to \mathscr{O}^{(p+1)}$ induces $d : H^q(X, X'; \mathscr{O}^{(p)}) \to H^q(X, X'; \mathscr{O}^{(p+1)})$.

The operators $\mathfrak{d}$ and d are compatible with the isomorphism in Theorem 8.10.9:

Proposition 8.10.10 *The following diagram is commutative:*

$$\begin{array}{ccc} H^{p,q}_{\bar\vartheta}(\mathcal{W}, W_0) & \xrightarrow{\ \mathfrak{d}\ } & H^{p+1,q}_{\bar\vartheta}(\mathcal{W}, W_0) \\ \wr\| & & \wr\| \\ H^q(X, X'; \mathscr{O}^{(p)}) & \xrightarrow{\ d\ } & H^q(X, X'; \mathscr{O}^{(p+1)}). \end{array}$$

Remark 8.10.11 Relative Dolbeault cohomology has already been considered in some form or another (cf. [31, 32]).

8.10.4 Relative de Rham and Relative Dolbeault Cohomologies

We consider the following two cases where there is a natural relation between the two cohomology theories.

(I) Note that, for every (n, q)-form ω, $\bar\partial\omega = d\omega$. Thus the inclusion $\mathscr{E}^{(n,q)}(W) \hookrightarrow \mathscr{E}^{(n+q)}(W)$ is compatible with $\bar\partial$ and d for every open set W in X and induces morphisms

$$H^{n,q}_{\bar\partial}(X) \longrightarrow H^{n+q}_{d}(X) \quad \text{and} \quad H^{n,q}_{\bar\vartheta}(\mathcal{W}) \longrightarrow H^{n+q}_{D}(\mathcal{W}), \tag{8.40}$$

where $\mathcal{W} = \{W_0, W_1\}$ is a covering of X. The two morphisms correspond to each other via the isomorphisms of Theorems 8.3.4 and 8.10.5. Let S be a closed set in X. Letting $W_0 = X \smallsetminus S$ and W_1 a neighborhood of S, consider the covering $\mathcal{W} = \{W_0, W_1\}$. Then we also have a natural morphism

$$H^{n,q}_{\bar\vartheta}(\mathcal{W}, W_0) \longrightarrow H^{n+q}_{D}(\mathcal{W}, W_0). \tag{8.41}$$

In particular, this is used later to define the integration on Čech-Dolbeault cohomology.

(II) We define $\rho^q : \mathscr{E}^{(q)} \to \mathscr{E}^{(0,q)}$ by assigning to a q-form ω its $(0, q)$-component $\omega^{(0,q)}$. Then $\rho^{q+1}(d\omega) = \bar\partial(\rho^q\omega)$ and we have:

Proposition 8.10.12 *There is a natural morphism of complexes*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathscr{E}^{(0)} & \xrightarrow{d} & \mathscr{E}^{(1)} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathscr{E}^{(q)} & \xrightarrow{d} & \cdots \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \rho^0} & & \downarrow{\scriptstyle \rho^1} & & & & \downarrow{\scriptstyle \rho^q} & & \\
0 & \longrightarrow & \mathscr{O} & \longrightarrow & \mathscr{E}^{(0,0)} & \xrightarrow{\bar\partial} & \mathscr{E}^{(0,1)} & \xrightarrow{\bar\partial} & \cdots & \xrightarrow{\bar\partial} & \mathscr{E}^{(0,q)} & \xrightarrow{\bar\partial} & \cdots .
\end{array}$$

Let S and $\mathcal{W}$ be as above.

Corollary 8.10.13 *There is a natural morphism* $\rho^q : H^q_D(\mathcal{W}, W_0) \to H^{0,q}_{\bar\vartheta}(\mathcal{W}, W_0)$ *that makes the following diagram commutative:*

$$\begin{array}{ccc}
H^q_D(\mathcal{W}, W_0) & \xrightarrow{\rho^q} & H^{0,q}_{\bar\vartheta}(\mathcal{W}, W_0) \\
\wr\vert & & \wr\vert \\
H^q(X, X \smallsetminus S; \mathbb{C}) & \xrightarrow{\iota} & H^q(X, X \smallsetminus S; \mathscr{O}).
\end{array}$$

Recall that we have the analytic de Rham complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathscr{O} \xrightarrow{d} \mathscr{O}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{O}^{(n)} \longrightarrow 0,$$

which yields a complex

$$0 \longrightarrow H^q(X, X'; \mathbb{C}) \xrightarrow{\iota} H^q(X, X'; \mathscr{O}) \xrightarrow{d} \cdots \xrightarrow{d} H^q(X, X'; \mathscr{O}^{(n)}) \longrightarrow 0,$$

where $X' = X \setminus S (= W_0)$.

The following is proved by a spectral sequence argument:

Proposition 8.10.14 *If $H^q(X, X'; \mathbb{C}) = 0$ and $H^q(X, X'; \mathscr{O}^{(p)}) = 0$ for $p \geq 0$ and $q \neq q_0$, then the above sequence is exact for $q = q_0$.*

As an application, we have the de Rham complex for hyperforms (cf. (8.49) below).

8.10.5 Cup Product and Integration

Cup Product We again consider the case of coverings by two open sets. Thus let $\mathcal{W} = \{W_0, W_1\}$ be an open covering of X and let $\mathscr{E}^{(p,q)}(\mathcal{W})$ be as in Sect. 8.10.2. We define the cup product (cf. (8.11))

$$\mathscr{E}^{(p,q)}(\mathcal{W}) \times \mathscr{E}^{(p',q')}(\mathcal{W}) \xrightarrow{\smile} \mathscr{E}^{(p+p',q+q')}(\mathcal{W}) \tag{8.42}$$

by assigning to ξ in $\mathscr{E}^{(p,q)}(\mathcal{W})$ and η in $\mathscr{E}^{(p',q')}(\mathcal{W})$ the cochain $\xi \smile \eta$ in $\mathscr{E}^{(p+p',q+q')}(\mathcal{W})$ given by

$$(\xi \smile \eta)_i = \xi_i \wedge \eta_i, \quad i = 0, 1, \quad \text{and} \quad (\xi \smile \eta)_{01} = (-1)^{p+q} \xi_0 \wedge \eta_{01} + \xi_{01} \wedge \eta_1.$$

Then it is bilinear in (ξ, η) and we have $\bar{\vartheta}(\xi \smile \eta) = \bar{\vartheta}\xi \smile \eta + (-1)^{p+q} \xi \smile \bar{\vartheta}\eta$. Thus it induces the cup product

$$H^{p,q}_{\bar{\vartheta}}(\mathcal{W}) \times H^{p',q'}_{\bar{\vartheta}}(\mathcal{W}) \longrightarrow H^{p+p',q+q'}_{\bar{\vartheta}}(\mathcal{W})$$

compatible, via the isomorphism of Theorem 8.10.5, with the product in the Dolbeault cohomology induced by the exterior product of forms.

Let S be a closed set in X. Let $W_0 = X \setminus S$ and W_1 a neighborhood of S and consider the covering $\mathcal{W} = \{W_0, W_1\}$. Then we see that (8.42) induces a product

$$\mathscr{E}^{(p,q)}(\mathcal{W}, W_0) \times \mathscr{E}^{(p',q')}(W_1) \xrightarrow{\smile} \mathscr{E}^{(p+p',q+q')}(\mathcal{W}, W_0) \tag{8.43}$$

assigning to (ξ_1, ξ_{01}) and η_1 the cochain $(\xi_1 \wedge \eta_1, \xi_{01} \wedge \eta_1)$. It induces the cup product

$$H^{p,q}_{\bar{\vartheta}}(\mathcal{W}, W_0) \times H^{p',q'}_{\bar{\partial}}(W_1) \xrightarrow{\smile} H^{p+p',q+q'}_{\bar{\vartheta}}(\mathcal{W}, W_0). \tag{8.44}$$

Integration Let X be a complex manifold of dimension n. As a C^∞ manifold it is orientable. In the sequel we suppose that X is oriented, however the orientation may not be the usual one (cf. Glossary at the end of Sect. 8.1).

Using the natural morphism $H^{n,n}_{\bar\partial}(X) \to H^{2n}_d(X)$ (cf. (8.40)), if X is compact, we may define the integration on $H^{n,n}_{\bar\partial}(X)$ as the composition

$$H^{n,n}_{\bar\partial}(X) \longrightarrow H^{2n}_d(X) \xrightarrow{\int_X} \mathbb{C}. \tag{8.45}$$

This may as well expressed in terms of Čech-Dolbeault and Čech-de Rham cohomologies.

Let K be a compact set in X (X may not be compact). Letting $W_0 = X \smallsetminus K$ and W_1 a neighborhood of K, we consider the covering $\mathcal{W}_K = \{W_0, W_1\}$. Let $\{R_0, R_1\}$ be a honeycomb system adapted to $\mathcal{W}_K$. In this case we may take as R_1 a compact $2n$-dimensional manifold with C^∞ boundary in W_1 containing K in its interior and set $R_0 = X \smallsetminus \mathrm{Int}\, R_1$. Then $R_{01} = -\partial R_1$ (cf. Sect. 8.3.2) and we have the integration on $\mathscr{E}^{(n,n)}(\mathcal{W}, W_0)$ given by, for $\xi = (\xi_1, \xi_{01})$,

$$\int_X \xi = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}.$$

This again induces the integration on the cohomology

$$\int_X : H^{n,n}_{\bar\vartheta}(\mathcal{W}_K, W_0) \longrightarrow \mathbb{C}, \tag{8.46}$$

which is the composition of (8.41) and (8.12).

Local Duality Morphism First, if X is compact, the bilinear pairing

$$H^{p,q}_{\bar\partial}(X) \times H^{n-p,n-q}_{\bar\partial}(X) \xrightarrow{\wedge} H^{n,n}_{\bar\partial}(X) \xrightarrow{\int_X} \mathbb{C}$$

given as the composition of the wedge product and the integration induces the Kodaira-Serre duality

$$KS_X : H^{p,q}_{\bar\partial}(X) \xrightarrow{\sim} H^{n-p,n-q}_{\bar\partial}(X)^*,$$

where * denotes the algebraic dual. We may also express the above using Čech-Dolbeault cohomology.

Now we consider the case where X may not be compact. Let K be a compact set in X and $W_K = \{W_0, W_1\}$ a covering of X as considered in the previous paragraph. The cup product (8.44) followed by the integration (8.46) gives a bilinear pairing

$$H^{p,q}_{\bar\vartheta}(\mathcal{W}_K, W_0) \times H^{n-p,n-q}_{\bar\partial}(W_1) \xrightarrow{\smile} H^{p,q}_{\bar\vartheta}(\mathcal{W}_K, W_0) \xrightarrow{\int_X} \mathbb{C}.$$

Setting

$$H_{\bar\partial}^{n-p,n-q}[K] = \varinjlim_{W_1 \supset K} H_{\bar\partial}^{n-p,n-q}(W_1),$$

where W_1 runs through open neighborhoods of K, this induces a morphism

$$\bar{A}_{X,K} : H_{\bar\vartheta}^{p,q}(\mathcal{W}_K, W_0) \longrightarrow H_{\bar\partial}^{n-p,n-q}[K]^*$$

which we call the *complex analytic Alexander morphism*, or the $\bar\partial$-*Alexander morphism* for short. If X is compact, we have the following commutative diagram:

$$\begin{array}{ccc} H_{\bar\vartheta}^{p,q}(\mathcal{W}_K, W_0) & \xrightarrow{j^*} & H_{\bar\partial}^{p,q}(X) \\ \bar{A}\downarrow & & \wr\downarrow KS \\ H_{\bar\partial}^{n-p,n-q}[K]^* & \xrightarrow{i_*} & H_{\bar\partial}^{n-p,n-q}(X)^*. \end{array}$$

Compare this with (8.7).

An interesting problem would be to see when $\bar{A}$ is an isomorphism. For this, we need to consider topological duals instead of algebraic duals. See Theorem 8.11.5 below for an example.

8.11 Examples, Applications and Related Topics

8.11.1 A Canonical Dolbeault-Čech Correspondence

We consider the covering $\mathcal{W}' = \{W_i\}_{i=1}^n$ of $\mathbb{C}^n \smallsetminus \{0\}$ given by $W_i = \{z_i \neq 0\}$. Here we put $'$ as we later consider the covering $\mathcal{W} = \{W_i\}_{i=1}^{n+1}$ of $\mathbb{C}^n$ with $W_{n+1} = \mathbb{C}^n$ (cf. Remark 8.12.6. 2 below). In the sequel we denote $\mathbb{C}^n \smallsetminus \{0\}$ by $\mathbb{C}^n \smallsetminus 0$.

On the one hand we have the Bochner-Martinelli form β_n, which is a $\bar\partial$-closed $(n, n-1)$-form on $\mathbb{C}^n \smallsetminus 0$ (cf. Sect. 8.8.2). On the other hand we have the Cauchy form in n-variables

$$\kappa_n = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \frac{\Phi(z)}{z_1 \cdots z_n},$$

which may be regarded as a cocycle c in $C^{n-1}(\mathcal{W}'; \mathscr{O}^{(n)})$, the $(n-1)$-st group of Čech cochains on $\mathcal{W}'$ with coefficients in $\mathscr{O}^{(n)}$, given by $c_{1\dots n} = \kappa_n$. Note that, since $\mathcal{W}'$ is a Stein covering, there is a canonical isomorphism $H^{n-1}(\mathcal{W}'; \mathscr{O}^{(n)}) \simeq H^{n-1}(\mathbb{C}^n \smallsetminus 0; \mathscr{O}^{(n)})$.

Theorem 8.11.1 *Under the isomorphism*

$$H_{\bar\partial}^{n,n-1}(\mathbb{C}^n \smallsetminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)})$$

of Theorem 8.10.2, the class of β_n corresponds to the class of $(-1)^{\frac{n(n-1)}{2}}\kappa_n$.

Remark 8.11.2 If we set

$$\beta_n^0 = C_n \frac{\sum_{i=1}^n \overline{\Phi_i(z)}}{\|z\|^2}, \qquad \kappa_n^0 = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \frac{1}{z_1\cdots z_n},$$

under the isomorphism

$$H_{\bar\partial}^{0,n-1}(\mathbb{C}^n \smallsetminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O}),$$

the class of β_n^0 corresponds to the class of $(-1)^{\frac{n(n-1)}{2}}\kappa_n^0$.

In the sequel we endow $\mathbb{C}^n$ with the usual orientation. In the above situation set

$$R_1 = \{\, z \in \mathbb{C}^n \mid \|z\|^2 \le n\varepsilon^2 \,\}.$$

The boundary ∂R_1 is a usually oriented $(2n-1)$-sphere $\mathbb{S}^{2n-1}$. We also set

$$\Gamma = \{\, z \in \mathbb{C}^n \mid |z_i| = \varepsilon,\ i = 1, \dots, n \,\},$$

which is an n-cycle oriented so that $\arg z_1 \wedge \cdots \wedge \arg z_n$ is positive.

Theorem 8.11.3 *Let θ be a $\bar\partial$-closed $(n, n-1)$-form on $\mathbb{C}^n \smallsetminus 0$ and γ a cocycle in $C^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)})$. If the class of θ corresponds to the class of γ by the canonical isomorphism*

$$H_{\bar\partial}^{n,n-1}(\mathbb{C}^n \smallsetminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)}),$$

then

$$\int_{\mathbb{S}^{2n-1}} \theta = (-1)^{\frac{n(n-1)}{2}} \int_\Gamma \gamma.$$

Note that the above is consistent with Theorem 8.11.1:

$$\int_{\mathbb{S}^{2n-1}} \beta_n = 1 = \int_\Gamma \kappa_n.$$

Remark 8.11.4 The above correspondence is studied in [26], with a different sign convention.

8.11.2 *Local Duality*

A Theorem of Martineau The following theorem of A. Martineau [42] (see also [25, 37]) may naturally be interpreted in our framework as one of the cases where the $\bar{\partial}$-Alexander morphism is an isomorphism with topological duals so that the duality pairing is given by the cup product followed by integration as described in Sect. 8.10.5. In the below we assume that $\mathbb{C}^n$ is oriented, however the orientation may not be the usual one.

Theorem 8.11.5 *Let K be a compact set in $\mathbb{C}^n$ such that $H_{\bar{\partial}}^{p,q}[K] = 0$ for $q \geq 1$. Then for every open set $W \supset K$, $H_{\bar{\vartheta}}^{p,q}(W, W \smallsetminus K)$ and $H_{\bar{\partial}}^{n-p,n-q}[K]$ admits natural structures of Fréchet-Schwartz and dual Fréchet-Schwartz spaces, respectively, and we have :*

$$\bar{A} : H_{\bar{\vartheta}}^{p,q}(W, W \smallsetminus K) \xrightarrow{\sim} H_{\bar{\partial}}^{n-p,n-q}[K]' = \begin{cases} 0 & q \neq n \\ \mathscr{O}^{(n-p)}[K]' & q = n, \end{cases}$$

where $'$ denotes the strong dual.

The theorem is originally stated for $p = 0$ in terms of local cohomology. The hypothesis $H_{\bar{\partial}}^{p,q}[K] = 0$, for $q \geq 1$, is satisfied if K is a subset of $\mathbb{R}^n \subset \mathbb{C}^n$ by the following theorem (cf. [20]):

Theorem 8.11.6 (Grauert) *Any subset of $\mathbb{R}^n$ admits a fundamental system of neighborhoods consisting of Stein open sets in $\mathbb{C}^n$.*

In our framework, the duality is described as follows (cf. Sect. 8.10.5). Let $W_0 = W \smallsetminus K$ and W_1 a neighborhood of K in W and consider the covering $\mathcal{W}_K = \{W_0, W_1\}$ of W. The duality pairing is given, for a cocycle (ξ_1, ξ_{01}) in $\mathscr{E}^{(p,n)}(\mathcal{W}_K, W_0)$ and a holomorphic $(n-p)$-form η near K, by

$$\int_{R_1} \xi_1 \wedge \eta + \int_{R_{01}} \xi_{01} \wedge \eta, \tag{8.47}$$

where R_1 is a compact real $2n$-dimensional manifold with C^∞ boundary in W_1 containing K in its interior and $R_{01} = -\partial R_1$. We may always choose a cocycle with $\xi_1 = 0$, if W is Stein.

Local Residue Pairing Now we consider Theorem 8.11.5 in the case $K = \{0\}$ and $(p, q) = (n, n)$. We also let $W = \mathbb{C}^n$. In this paragraph we consider the usual orientation on $\mathbb{C}^n$. We have the exact sequence

$$\cdots \longrightarrow H_{\bar{\partial}}^{n,n-1}(\mathbb{C}^n \smallsetminus 0) \xrightarrow{\delta} H_{\bar{\vartheta}}^{n,n}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus 0) \longrightarrow 0.$$

Thus every element in $H^{n,n}_{\bar\vartheta}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus 0)$ is represented by a cocycle of the form $(0, -\theta)$. Since $\mathscr{O}[K] = \mathscr{O}_{\mathbb{C}^n,0} = \mathscr{O}_n$ in this case, the duality in Theorem 8.11.5 is induced by the pairing

$$H^{n,n}_{\bar\vartheta}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus 0) \times \mathscr{O}_n \xrightarrow{\int} \mathbb{C}$$

given by

$$((0, -\theta), h) \mapsto -\int_{R_{01}} h\theta = \int_{\mathbb{S}^{2n-1}} h\theta.$$

In the above, h is a holomorphic function in a neighborhood W of 0 in $\mathbb{C}^n$. We may take as R_1 a $2n$-ball around 0 in W so that $R_{01} = -\partial R_1 = -\mathbb{S}^{2n-1}$ with $\mathbb{S}^{2n-1}$ a usually oriented $(2n-1)$-sphere. Thus if θ corresponds to γ, the above integral is equal to

$$(-1)^{\frac{n(n-1)}{2}} \int_{\Gamma} h\gamma$$

(cf. Theorem 8.11.3). In particular, if $\theta = \beta_n$ the pairing is given by

$$\int_{\mathbb{S}^{2n-1}} h\beta_n = \int_{\Gamma} h\kappa_n = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_{\Gamma} \frac{h dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n} = h(0). \tag{8.48}$$

Likewise in the case $(p, q) = (0, n)$, the duality in Theorem 8.11.5 is induced by the pairing

$$H^{0,n}_{\bar\vartheta}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus 0) \times \mathscr{O}^{(n)}_{\mathbb{C}^n,0} \xrightarrow{\int} \mathbb{C}$$

given by

$$((0, -\theta), \eta) \mapsto -\int_{R_{01}} \theta \wedge \eta = \int_{\mathbb{S}^{2n-1}} \theta \wedge \eta.$$

The above subjects are closely related to the theory of hyperfunctions, which we discuss in the next section.

8.11.3 Some Others

We may develop the theory of Atiyah classes in the context of Čech-Dolbeault cohomology, which is conveniently used to define their localizations in the relative

Dolbeault cohomology. In particular we have the $\bar\partial$-Thom class of a holomorphic vector bundle, see [1, 60] for details.

The Bott-Chern cohomology introduced in [8] refines both de Rham and Dolbeault cohomologies. The above idea and techniques may further be pushed forward to develop the theory of Čech-Bott-Chern cohomology. In particular the relative Bott-Chern cohomology which arise naturally in this context is used for the localization of Bott-Chern classes of vector bundles admitting a Hermitian connection compatible with an action of a distribution. For details and the relation with the relative Dolbeault cohomology theory, we refer to [16].

We refer to [3] for another application of the relative Dolbeault cohomology, namely to the study of Hodge structures under blowing-up. We may equally use the complex of currents, instead of that of differential forms, to define the relative Dolbeault cohomology. One of the advantages of this is that the push-forward morphism is available, see [65] for details and applications in the context of [3].

8.12 Sato Hyperfunctions

Sato hyperfunctions are defined in terms of relative cohomology with coefficients in the sheaf of holomorphic functions and the theory is developed in the language of derived functors (cf. [35, 48–50]). The use of relative Dolbeault cohomology via the relative Dolbeault theorem (Theorem 8.10.9) provides us with another way of treating the theory. This approach gives simple and explicit expressions of hyperfunctions and some fundamental operations on them and leads to a number of new results. These are discussed in detail in [30], see also [29, 62]. Here we pick up some of the essentials of the contents therein. In general the theory of hyperfunctions may be developed on an arbitrary real analytic manifold and it involves various orientation sheaves. However for simplicity, here we consider hyperfunctions on open sets in $\mathbb{R}^n$ fixing various orientations.

Hyperfunctions and Hyperforms We consider the standard inclusion $\mathbb{R}^n \subset \mathbb{C}^n$, i.e., if $(z_1, \dots, z_n)$, $z_i = x_i + \sqrt{-1}y_i$, is a coordinate system on $\mathbb{C}^n$, then $\mathbb{R}^n$ is given by $y_i = 0$, $i = 1, \dots, n$. We orient $\mathbb{R}^n$ and $\mathbb{C}^n$ so that $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n, x_1, \dots, x_n)$ are positive coordinate systems. Thus $(y_1, \dots, y_n)$ is a positive coordinate system in the normal direction. This is consistent with Convention in Sect. 8.9.3. Note that the difference between this orientation of $\mathbb{C}^n$ and the usual one, where $(x_1, y_1, \dots, x_n, y_n)$ is positive, is a sign of $(-1)^{\frac{n(n+1)}{2}}$.

With these, for an open set U in $\mathbb{R}^n$, the space of hyperfunctions on U is given by

$$\mathscr{B}(U) = H^n(W, W \smallsetminus U; \mathscr{O}),$$

where W is an open set in $\mathbb{C}^n$ containing U as a closed set and $\mathscr{O}$ the sheaf of holomorphic functions on $\mathbb{C}^n$. We call such a W a *complex neighborhood* of U. Note that, by excision, the definition does not depend on the choice of the complex neighborhood W. By the relative Dolbeault theorem (cf. Theorem 8.10.9), there is a canonical isomorphism:

$$\mathscr{B}(U) \simeq H^{0,n}_{\bar{\vartheta}}(W, W \setminus U).$$

More generally we introduce the following:

Definition 8.12.1 The space of *p-hyperforms* on U is defined by

$$\mathscr{B}^{(p)}(U) = H^{p,n}_{\bar{\vartheta}}(W, W \setminus U).$$

Note that the definition does not depend on the choice of W by excision (cf. Proposition 8.10.8). Denoting by $\mathscr{O}^{(p)}$ the sheaf of holomorphic p-forms on $\mathbb{C}^n$, we have a canonical isomorphism (cf. Theorem 8.10.9):

$$H^{p,n}_{\bar{\partial}}(W, W \setminus U) \simeq H^n(W, W \setminus U; \mathscr{O}^{(p)})$$

so that $\mathscr{B}^{(0)}(U)$ is canonically isomorphic with $\mathscr{B}(U)$. Hyperforms are essentially the same with what have conventionally been referred to as differential forms with coefficients in hyperfunctions.

Remark 8.12.2 In the above, we implicitly used the fact that $\mathbb{R}^n$ is "purely n-codimensional" in $\mathbb{C}^n$ with respect to $\mathscr{O}^{(p)}$ and $\mathbb{Z}$ (cf. [34]). This means that $H^q(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{R}^n; \mathscr{O}^{(p)}) = 0$ and $H^q(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{R}^n; \mathbb{Z}) = 0$ for $q \neq n$. For the latter, it can be seen from the Alexander isomorphism

$$H^q(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{R}^n; \mathbb{Z}) \xrightarrow{\sim} \check{H}_{2n-q}(\mathbb{R}^n; \mathbb{Z})$$

(cf. Theorem 8.2.2 and Example 8.2.3).

Expression of Hyperforms Let U and W be as above. Letting $W_0 = W \setminus U$ and W_1 a neighborhood of U in W, we consider the open covering $\mathcal{W} = \{W_0, W_1\}$ of W. Then $\mathscr{B}^{(p)}(U) = H^{p,n}_{\bar{\vartheta}}(W, W \setminus U) = H^{p,n}_{\bar{\vartheta}}(\mathcal{W}, W_0)$ and a p-hyperform is represented by a pair (ξ_1, ξ_{01}) with ξ_1 a (p, n)-form on W_1, which is automatically $\bar{\partial}$-closed, and ξ_{01} a $(p, n-1)$-form on W_{01} such that $\xi_1 = \bar{\partial}\xi_{01}$ on W_{01}. We have the exact sequence (cf. (8.39)):

$$H^{p,n-1}_{\bar{\partial}}(W) \longrightarrow H^{p,n-1}_{\bar{\partial}}(W \setminus U) \xrightarrow{\delta} \mathscr{B}^{(p)}(U) \xrightarrow{j^*} H^{p,n}_{\bar{\partial}}(W).$$

By Theorem 8.11.6, we may take as W a Stein open set and, if we do this, we have $H^{p,n}_{\bar{\partial}}(W) \simeq H^n(W; \mathscr{O}^{(p)}) = 0$. Thus δ is surjective and every p-hyperform is represented by a cocycle of the form $(0, -\theta)$ with θ a $\bar\partial$-closed $(p, n-1)$-form on $W \smallsetminus U$.

In the case $n > 1$, $H^{p,n-1}_{\bar\partial}(W) \simeq H^{n-1}(W; \mathscr{O}^{(p)}) = 0$ and δ is an isomorphism:

$$H^{p,n-1}_{\bar\partial}(W \smallsetminus U) \simeq \mathscr{B}^{(p)}(U), \qquad [\theta] \leftrightarrow [(0, -\theta)].$$

In the case $n = 1$, as $H^{p,0}_{\bar\partial}(W \smallsetminus U) = H^0(W \smallsetminus U; \mathscr{O}^{(p)})$ and $H^{p,0}_{\bar\partial}(W) = H^0(W; \mathscr{O}^{(p)})$, $p = 0, 1$, we have the isomorphism

$$H^0(W \smallsetminus U; \mathscr{O}^{(p)})/H^0(W; \mathscr{O}^{(p)}) \simeq \mathscr{B}^{(p)}(U), \qquad [\omega] \leftrightarrow [(0, -\omega)].$$

In particular, for $p = 0$, the left hand side is the original expression of hyperfunctions by Sato in the one-dimensional case and the right hand side is the expression in terms of relative Dolbeault cohomology.

Remark 8.12.3 Although a hyperform may be represented by a single differential form in most of the cases, it is important to keep in mind that it is represented by a pair (ξ_1, ξ_{01}) in general.

Now we describe some of the operations on hyperforms.

Multiplication by Real Analytic Functions Let $\mathscr{A}(U)$ denote the space of real analytic functions on U. We define the multiplication

$$\mathscr{A}(U) \times \mathscr{B}^{(p)}(U) \longrightarrow \mathscr{B}^{(p)}(U)$$

by assigning to $(f, [\xi])$ the class of $(\tilde f \xi_1, \tilde f \xi_{01})$ with $\tilde f$ a holomorphic extension of f.

Partial Derivatives We define the partial derivative

$$\frac{\partial}{\partial x_i} : \mathscr{B}(U) \longrightarrow \mathscr{B}(U)$$

as follows. Let (ξ_1, ξ_{01}) represent a hyperfunction on U. Write $\xi_1 = f\, d\bar z_1 \wedge \cdots \wedge d\bar z_n$ and $\xi_{01} = \sum_{j=1}^n g_j\, d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_j} \wedge \cdots \wedge d\bar z_n$. Then $\frac{\partial}{\partial x_i}[\xi]$ is represented by the cocycle

$$\Big(\frac{\partial f}{\partial z_i}\, d\bar z_1 \wedge \cdots \wedge d\bar z_n, \sum_{j=1}^n \frac{\partial g_j}{\partial z_i}\, d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_j} \wedge \cdots \wedge d\bar z_n\Big).$$

Thus for a differential operator $P(x, D)$, $P(x, D) : \mathscr{B}(U) \to \mathscr{B}(U)$ is well-defined.

Differential We define the differential

$$d : \mathscr{B}^{(p)}(U) \longrightarrow \mathscr{B}^{(p+1)}(U).$$

by assigning to the class of (ξ_1, ξ_{01}) the class of $(-1)^n(\partial\xi_1, -\partial\xi_{01})$ (cf. Proposition 8.10.10). From Proposition 8.10.14, we have the exact sequence (de Rham complex for hyperforms, cf. Remark 8.12.2):

$$0 \longrightarrow \mathbb{C}(U) \longrightarrow \mathscr{B}(U) \xrightarrow{d} \mathscr{B}^{(1)}(U) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{B}^{(n)}(U) \longrightarrow 0. \tag{8.49}$$

We come back to the first part below.

Integration Let K be a compact set in U. We define the space $\mathscr{B}_K^{(p)}(U)$ of p-hyperforms on U with support in K as the kernel of the restriction $\mathscr{B}^{(p)}(U) \to \mathscr{B}^{(p)}(U \smallsetminus K)$. Then, for any complex neighborhood W of U, there is a canonical isomorphism

$$\mathscr{B}_K^{(p)}(U) \simeq H_{\bar{\vartheta}}^{p,n}(W, W \smallsetminus K).$$

Thus we may define the integration on $\mathscr{B}_K^{(n)}(U)$ by directly applying (8.46), which we recall for the sake of completeness. Let W be a complex neighborhood of U and consider the covering $\mathcal{W}_K = \{W_0, W_1\}$ with $W_0 = W \smallsetminus K$ and W_1 a neighborhood of K in W. Then we have a canonical identification $\mathscr{B}_K^{(p)}(U) = H_{\bar{\vartheta}}^{p,n}(\mathcal{W}_K, W_0)$. Let R_1 be a compact real $2n$-dimensional manifold with C^∞ boundary in W_1 containing K in its interior and set $R_{01} = -\partial R_1$. Then the integration

$$\int_U : \mathscr{B}_K^{(n)}(U) \longrightarrow \mathbb{C}$$

is given as follows. Noting that $u \in \mathscr{B}_K^{(n)}(U) = H_{\bar{\vartheta}}(\mathcal{W}_K, W_0)$ is represented by

$$\xi = (\xi_1, \xi_{01}) \in \mathscr{E}^{(n,n)}(\mathcal{W}_K, W_0) = \mathscr{E}^{(n,n)}(W_1) \oplus \mathscr{E}^{(n,n-1)}(W_{01}),$$

we have

$$\int_U u = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}. \tag{8.50}$$

Duality By Theorem 8.11.5 we have

$$\mathscr{B}_K^{(p)}(U) = H_{\bar{\partial}}^{p,n}(W, W \smallsetminus K) \simeq \mathscr{O}^{(n-p)}[K]' = \mathscr{A}^{(n-p)}[K]', \tag{8.51}$$

where $\mathscr{A}^{(n-p)}$ denotes the sheaf of germs of real analytic $(n-p)$-forms on $\mathbb{R}^n$ and

$$\mathscr{A}^{(n-p)}[K] = \varinjlim \mathscr{A}^{(n-p)}(U_1),$$

the direct limit over the set of neighborhoods U_1 of K in U. Recall that the pairing is induced by (8.47).

δ-Function and δ-Form We consider the case $K = \{0\} \subset \mathbb{R}^n$.

Definition 8.12.4 The δ-*function* is the element in $\mathscr{B}_{\{0\}}(\mathbb{R}^n) = H^{0,n}_{\bar{\vartheta}}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus \{0\})$ which is represented by

$$(0, -(-1)^{\frac{n(n+1)}{2}} \beta_n^0),$$

where β_n^0 is as defined in Remark 8.11.2.

The isomorphism (8.51) reads in this case:

$$\mathscr{B}_0(\mathbb{R}^n) \simeq (\mathscr{A}_0^{(n)})',$$

where $\mathscr{A}_0^{(n)}$ denotes the stalk of $\mathscr{A}^{(n)}$ at 0. For a representative $\omega = h(x)\Phi(x)$ of an element in $\mathscr{A}_0^{(n)}$, $h(z)\Phi(z)$ is its complex representative. Let R_1 be a small $2n$-ball around 0 in $\mathbb{C}^n$ so that $R_{01} = -\partial R_1 = -(-1)^{\frac{n(n+1)}{2}} \mathbb{S}^{2n-1}$ with $\mathbb{S}^{2n-1}$ a usually oriented $(2n-1)$-sphere. Then the δ-function is the hyperfunction that assigns to a representative $\omega = h(x)\Phi(x)$ the value (cf. (8.48))

$$-(-1)^{\frac{n(n+1)}{2}} \int_{R_{01}} h(z)\beta_n = \int_{\mathbb{S}^{2n-1}} h(z)\beta_n = h(0).$$

Definition 8.12.5 The δ-*form* is the element in $\mathscr{B}^{(n)}_{\{0\}}(\mathbb{R}^n) = H^{n,n}_{\bar{\vartheta}}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus \{0\})$ which is represented by

$$(0, -(-1)^{\frac{n(n+1)}{2}} \beta_n).$$

Recall the isomorphism (8.51), which reads in this case:

$$\mathscr{B}_0^{(n)}(\mathbb{R}^n) \simeq (\mathscr{A}_0)'.$$

For a representative $h(x)$ of an element in $\mathscr{A}_0$, $h(z)$ is its complex representative. Let R_1 be as above. Then the δ-form is the hyperform that assigns to a representative $h(x)$ the value

$$-(-1)^{\frac{n(n+1)}{2}} \int_{R_{01}} h(z)\beta_n = \int_{S^{2n-1}} h(z)\beta_n = h(0).$$

Remark 8.12.6

1. If we orient $\mathbb{C}^n$ the usual way so that the coordinate system $(x_1, y_1, \dots, x_n, y_n)$ is positive, the delta function $\delta(x)$ is represented by $(0, -\beta_n^0)$. Also, the delta form is represented by $(0, -\beta_n)$. Incidentally, it has the same expression as the Thom class of the trivial complex vector bundle of rank n (cf. Remark 8.9.21).
2. Set $W_i = \{z_i \neq 0\}$, $i = 1, \dots, n$, and $W_{n+1} = \mathbb{C}^n$ and consider the coverings $\mathcal{W} = \{W_i\}_{i=1}^{n+1}$ and $\mathcal{W}' = \{W_i\}_{i=1}^{n}$ of $\mathbb{C}^n$ and $\mathbb{C}^n \setminus 0$. We have the natural isomorphisms

$$\mathscr{B}_{\{0\}}(\mathbb{R}^n) \simeq H_{\bar\partial}^{0,n-1}(\mathbb{C}^n \setminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathscr{O}) \simeq H^n(\mathcal{W}, \mathcal{W}'; \mathscr{O}).$$

   As noted in Remark 8.11.2, under the middle isomorphism above, the class of β_n^0 corresponds to the class of $(-1)^{\frac{n(n-1)}{2}} \kappa_n^0$. If we choose the usual orientation on $\mathbb{C}^n$, the class corresponding to $[\kappa_n^0]$ in $H^n(\mathcal{W}, \mathcal{W}'; \mathscr{O})$ is the traditional representation of the δ-function (cf. (8.48)).

1 **as a Hyperfunction** We examine the map $\mathbb{C}(U) \to \mathscr{B}(U)$ in (8.49). Let $\mathcal{W}$ be as before. Then it is given by $\rho^n : H_D^n(\mathcal{W}, W_0) \to H_{\bar\vartheta}^{0,n}(\mathcal{W}, W_0)$, which is induced by $(\omega_1, \omega_{01}) \mapsto (\omega_1^{(0,n)}, \omega_{01}^{(0,n-1)})$ (cf. Corollary 8.10.13). For simplicity we assume that U is connected.

Then we have the commutative diagram:

$$\begin{array}{ccccc} \mathbb{C} = H^0(U;\mathbb{C}) \xrightarrow[T]{\sim} & H^n(W, W \setminus U; \mathbb{C}) & \xrightarrow{\iota} & H^n(W, W \setminus U; \mathscr{O}) = \mathscr{B}(U) \\ & \wr\,\big| & & \wr\,\big| \\ & H_D^n(\mathcal{W}, W_0) & \xrightarrow{\rho^n} & H_{\bar\vartheta}^{0,n}(\mathcal{W}, W_0), \end{array}$$

where T denotes the Thom isomorphism, which sends $1 \in \mathbb{C}$ to the Thom class $\Psi_U \in H^n(W, W \setminus U; \mathbb{C})$ of U (cf. Sect. 8.9.5). If Ψ_U is represented by (ψ_1, ψ_{01}) in $H_D^n(\mathcal{W}, W_0)$, as a hyperfunction, 1 is represented by $\rho^n(\psi_1, \psi_{01}) = (\psi_1^{(0,n)}, \psi_{01}^{(0,n-1)})$. In particular, we may set $(\psi_1, \psi_{01}) = (0, -\psi_n(y))$, where $\psi_n(y)$ is the angular form on $\mathbb{R}_y^n$ (cf. Theorem 8.9.18). Thus as a hyperfunction, 1 is represented by $(0, -\psi_n^{(0,n-1)}(y))$. Noting that $y_i = 1/(2\sqrt{-1})(z_i - \bar z_i)$, we compute

$$\psi_n^{(0,n-1)}(y) = (\sqrt{-1})^n C_n \frac{\sum_{i=1}^n (-1)^i (z_i - \bar z_i) d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_i} \wedge \cdots \wedge d\bar z_n}{\|z - \bar z\|^n}.$$

In particular, if $n = 1$,

$$\psi_1^{(0,0)}(y) = \frac{1}{2} \frac{y}{|y|}.$$

Embedding of Real Analytic Forms Let U and W be as above. Using the above expression of 1, we may define an embedding

$$\mathscr{A}^{(p)}(U) \hookrightarrow \mathscr{B}^{(p)}(U) = H_{\bar{\vartheta}}^{p,n}(W, W \smallsetminus U) \text{ by } \omega \mapsto [(\psi_1^{(0,n)} \wedge \tilde{\omega}, \psi_{01}^{(0,n-1)} \wedge \tilde{\omega})], \tag{8.52}$$

where (ψ_1, ψ_{01}) is a representative of the Thom class as above and $\tilde{\omega}$ the complexification of ω. Note that $(\psi_1^{(0,n)} \wedge \tilde{\omega}, \psi_{01}^{(0,n-1)} \wedge \tilde{\omega})$ is a cocycle as $\tilde{\omega}$ is holomorphic. It is compatible with the differentials d of $\mathscr{A}^{(\bullet)}(U)$ and $\mathscr{B}^{(\bullet)}(U)$.

In particular, if $p = 0$, we have the embedding $\mathscr{A}(U) \hookrightarrow \mathscr{B}(U)$, which is given by $f \mapsto [(\tilde{f}\psi_1^{(0,n)}, \tilde{f}\psi_{01}^{(0,n-1)})]$ with $\tilde{f}$ the complexification of f.

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Chapter 9
Mixed Hodge Structures Applied to Singularities

Joseph Steenbrink

Contents

Abstract We give a survey of applications of mixed Hodge theory to the study of isolated singularities. A summary of mixed Hodge theory is followed by some examples. The formalism of vanishing cycles is realized on the Hodge level by a sheaf complex with three filtrations, making the application to the cohomology of the Milnor fibre possible. The approaches by algebraic analysis and by motivic integration are discussed, and the spectrum with its properties is considered. The paper ends with a treatment of Du Bois singularities.

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9.1 Introduction

In this paper we focus on the applications of mixed Hodge theory to the study of singularities. Hodge theory deals with the cohomology of smooth complex projective varieties, or more generally, compact Kähler manifolds. By de Rham's theorem, cohomology classes of compact oriented differentiable manifolds can be considered as classes of closed differential forms modulo exact ones. The choice of a Riemannian metric enables one to define the Laplace operator Δ on differential forms. Each de Rham cohomology class contains exactly one closed form ω with $\Delta\omega = 0$, the *harmonic* representative, see [31]. For compact complex manifolds there exists the Hodge decomposition of complex-valued diffential k-forms $\omega = \sum_{p+q=k} \omega^{p,q}$. Here $\omega^{p,q}$ is a form of type (p,q). In terms of local holomorphic coordinates $(z_1,\ldots,z_n)$, the representation of such a form involves p factors dz_i and q factors $d\bar{z}_j$. On a compact Kähler manifold, the (p,q)-components of a harmonic form are again harmonic. As a consequence, the Hodge decomposition of forms induces a Hodge decomposition of cohomology classes via their harmonic representatives:

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the subspace of $H^k(X,\mathbb{C})$ consisting of classes of forms containing harmonic forms of type (p,q). Complex conjugation with respect to $H^k(X,\mathbb{R})$ interchanges $H^{p,q}(X)$ and $H^{q,p}(X)$: this equips $H^k(X)$ with a *Hodge structure of weight* k.

Using Leray's theory of sheaves and resolution of singularities, Grothendieck [24] defined the de Rham cohomology of complex algebraic varieties in purely algebraic terms. For smooth complete varieties, the analogue of the Hodge decomposition is the degeneration at the E_1-term of the spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Rightarrow H^{p+q}_{\mathrm{DR}}(X/\mathbb{C}) := \mathbb{H}^{p+q}(X, \Omega^{\cdot}_{X/\mathbb{C}})$$

where $\mathbb{H}$ stand for hypercohomology of a sheaf complex. Though there is no algebraic description of the Hodge *decomposition*, this formula does give an algebraic description of a decreasing filtration $F^{\cdot}$ on the de Rham cohomology, the *Hodge filtration* which has the property that $F^p/F^{p+1} \simeq H^q(X, \Omega^p_{X/\mathbb{C}})$. By GAGA, this space is isomorphic to $H^{p,q}(X)$.

On the level of the sheaf complex $\Omega^{\cdot}_{X/\mathbb{C}}$ the Hodge filtration is given by

$$F^p\Omega^k_{X/\mathbb{C}} = 0 \text{ if } k < p \text{ and } = \Omega^k_{X/\mathbb{C}} \text{ if } k \geq p.$$

A generalization of Hodge theory to arbitrary complex algebraic varieties was developed by Deligne [17, 18]. He showed that the cohomology of a complex algebraic variety (not necessarily complete or nonsingular) carries a slightly more general structure, which presents $H^k(X, \mathbb{C})$ as a successive extension of Hodge structures of decreasing weights, contained between 0 and $2k$, whose Hodge numbers $h^{pq} = \dim H^{pq}$ are zero unless $0 \leq p, q \leq k$. This generalization is called a *mixed Hodge structure*. Morphisms between varieties give rise to morphisms of mixed Hodge structures in a functorial way. In the smooth case, the main tool is the logarithmic de Rham complex with its Hodge and weight filtrations.

Griffiths [22], see also [16], studied the asymptotic behaviour of the Hodge structure on the cohomology of compact Kähler manifolds in a one parameter degeneration. Deligne conjectured in [23, Conjecture 9.17] that the limit object would be a mixed Hodge structure, whose weight filtration is determined by the logarithm of the unipotent part of the Picard-Lefschetz monodromy. This conjecture was proved by Schmid [52].

In the mean time, Deligne advised the author to study the relative logarithmic de Rham complex $\Omega^{\cdot}_{X/S}(\log E)$ to control the behaviour of intermediate Jacobians in a one-parameter degeneration. Using unpublished notes of Katz, a sheaf double complex $A^{\cdot,\cdot}$ was constructed in [55] which carries three filtrations: a Hodge filtration F, a weight filtration W and a monodromy filtration M. The main properties are (under the hypothesis that the special fibre E is reduced):

1. There is a bifiltered quasi-isomorphism $(\Omega^{\cdot}_{X/S}(\log E) \otimes \mathcal{O}_E, W, F) \rightarrow (A^{\cdot,\cdot}, W, F)$;
2. $(A^{\cdot,\cdot}, M, F)$ is a mixed Hodge complex of sheaves (its hypercohomology groups are mixed Hodge structures);
3. the logarithm N of the monodromy (which is unipotent in this case) has a lift to $A^{\cdot,\cdot}$ which maps M_k to M_{k-2}.

It was claimed in [55] that M induces on the hypercohomology groups of $A^{\cdot,\cdot}$ the weight filtration of N, but F. El Zein remarked that the proof is incomplete. A correct proof was given in [27].

Deligne observed that the existence of the double complex $A^{\cdot,\cdot}$ enables one to localize the study of one-parameter degenerations. This meant the entrance of mixed Hodge theory in singularity theory, and gave rise to [56], where a mixed Hodge structure on the cohomology of the Milnor fibre of an isolated hypersurface singularity is constructed, and its relation with monodromy, intersection form and local cohomology is established.

In this paper, we describe the construction of this mixed Hodge structure and give a survey of several applications. We start with some general background. Then we describe the formalism of nearby and vanishing cycles and show how the complex $A^{\cdot,\cdot}$ fits into this pattern. We describe the relation between monodromy and weight filtration for arbitrary smoothings of isolated singularities. Then we pass to two alternative approaches to this theme: algebraic analysis and motivic integration. We return to isolated hypersurface singularities and consider the spectrum and its properties. Finally, we discuss the notion of du Bois singularities, which plays an important role in the minimal model program.

More background on mixed Hodge theory can be found in [42]; for further reading about mixed Hodge structures and singularities we refer to Kulikov's monograph [36]. The referee deserves credit for the remarks on recent developments at the end of this paper.

9.2 Background

9.2.1 Mixed Hodge Structures

Let $H_{\mathbb{Z}}$ be a finitely generated abelian group. Consider the finite-dimensional vector spaces $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$ and $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$. A mixed Hodge structure on $H_{\mathbb{Z}}$ is a pair (W, F) where

- W is a finite increasing filtration of $H_{\mathbb{Q}}$, the *weight filtration*, and
- F is a finite decreasing filtration of $H_{\mathbb{C}}$, the *Hodge filtration*

(here "finite" means that there exist integers $a < b$ with $W_a = (0),\ W_b = H_{\mathbb{Q}}$ resp. $F^a = H_{\mathbb{C}},\ F^b = (0)$). These data must satisfy the conditions, that for all integers p,

$$F^{p+1} \cap (W_k \otimes \mathbb{C}) + \overline{F^{k-p}} \cap (W_k \otimes \mathbb{C}) + W_{k-1} \otimes \mathbb{C} = W_k \otimes \mathbb{C};$$

$$\left(F^{p+1} \cap W_k \otimes \mathbb{C} + W_{k-1} \otimes \mathbb{C}\right) \cap \left(\overline{F^{k-p}} \cap W_k \otimes \mathbb{C} + W_{k-1} \otimes \mathbb{C}\right) = W_{k-1} \otimes \mathbb{C}.$$

The filtration F induces a filtration F_k on the subquotient $\mathrm{Gr}_k^W H_{\mathbb{C}} := W_k \otimes \mathbb{C}/W_{k-1} \otimes \mathbb{C}$ of $H_{\mathbb{C}}$ by

$$F_k^p := \left(F^p(W_k \otimes \mathbb{C}) + W_{k-1} \otimes \mathbb{C}\right) / W_{k-1} \otimes \mathbb{C}.$$

The conditions above imply that

$$\mathrm{Gr}_k^W H_{\mathbb{C}} = F_k^r \oplus \overline{F_k^s} \text{ if } r + s = k + 1.$$

We let

$$H^{p,q} := F_{p+q}^p \cap \overline{F_{p+q}^q} \subset \mathrm{Gr}_{p+q}^W H_{\mathbb{C}}$$

where the bar refers to complex conjugation with respect to $\mathrm{Gr}_k^W H_{\mathbb{Q}}$. One has the *Hodge decomposition*

$$\mathrm{Gr}_k^W H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q} \text{ with } H^{q,p} = \overline{H^{p,q}}.$$

The *Hodge numbers* are the numbers $h^{p,q} = \dim_{\mathbb{C}} H^{pq}$. The mixed Hodge structure $(H_{\mathbb{Z}}, W, F)$ is called *pure of weight k* if $\mathrm{Gr}_m^W H_{\mathbb{Q}} = 0$ for all $m \neq k$. An important example is the Tate Hodge structure $\mathbb{Z}(m)$ for $m \in \mathbb{Z}$, given by $(2\pi i)^m \mathbb{Z} \subset \mathbb{C}$. It is pure of weight $-2m$, with $h^{p,q} = 0$ unless $p = q = -m$, and $h^{-m,-m} = 1$.

A *morphism of mixed Hodge structures* $(H_{\mathbb{Z}}, W, F) \to (H'_{\mathbb{Z}}, W, F)$ is a group homomorphism $\phi : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ such that for all $k, p \in \mathbb{Z}$ one has

$$(\phi \otimes \mathbb{Q})(W_k H_{\mathbb{Q}}) \subset W_k H'_{\mathbb{Q}} \text{ and } (\phi \otimes \mathbb{C})(F^p H_{\mathbb{C}}) \subset F^p H'_{\mathbb{C}}.$$

For R a noetherian subring of $\mathbb{R}$ such that $R \otimes \mathbb{Q}$ is a field, one defines the notion of an R-mixed Hodge structure as above, replacing $\mathbb{Z}$ by R. The category of R-mixed Hodge structures is abelian.

Recall that a morphism of filtered vector spaces

$$f : (V, F) \to (V', F)$$

is a linear map $f : V \to V'$ with the property that $f(F^p V) \subset F^p V'$ for all p. It is called *strict* if $f(V) \cap F^p V' = f(F^p V)$ for all p. Every morphism of mixed Hodge structures is strict with respect to the Hodge and weight filtrations (cf. [42, Cor. 3.6]). As a consequence, the functors Gr_k^W, Gr_F^p and $\mathrm{Gr}_F^p \mathrm{Gr}_k^W$ are exact (cf. [42, Cor. 3.8]).

Let H and H' be two mixed Hodge structures. Then $\mathrm{Hom}(H, H')$ becomes a mixed Hodge structure with the following definitions:

$$\mathrm{Hom}(H, H')_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$$

$$W_k \mathrm{Hom}(H_{\mathbb{Q}}, H'_{\mathbb{Q}}) = \{f \mid f(W_m H_{\mathbb{Q}}) \subset W_{m+k} H'_{\mathbb{Q}} \text{ for all } m\}$$

$$F^p \mathrm{Hom}(H_{\mathbb{C}}, H'_{\mathbb{C}}) = \{f \mid f(F^\ell H_{\mathbb{C}}) \subset F^{p+\ell} H'_{\mathbb{C}} \text{ for all } \ell\}$$

The *tensor product* of these mixed Hodge structures is defined by

$$(H \otimes H')_{\mathbb{Z}} = H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}},$$

$$W_k (H \otimes H')_{\mathbb{Q}} = \sum_{m+m'=k} W_m H_{\mathbb{Q}} \otimes W_{m'} H'_{\mathbb{Q}},$$

$$F^p (H \otimes H')_{\mathbb{C}} = \sum_{r+s=p} F^r H_{\mathbb{C}} \otimes F^s H'_{\mathbb{C}}.$$

Let $m \in \mathbb{Z}$. The m-th *Tate twist* of the mixed Hodge structure H is given by

$$H(m) := H \otimes \mathbb{Z}(m).$$

A *polarization* of an R-Hodge structure H of weight k is a (non-degenerate) R-valued bilinear form $Q : H_R \otimes H_R \to R$ which is symmetric if k is even, anti-symmetric if k is odd, and satifies

1. $i^{p-q} Q(u, \bar{u}) > 0$ for $0 \neq u \in H^{p,q}$;
2. the Q-orthogonal complement of F^m is F^{k-m-1} for all m.

If R is a field, the category of polarized R-Hodge structures of weight k is semisimple.

9.2.2 Compact Kähler Manifolds

On a complex manifold X of dimension n with tangent bundle T_X, the sheaf of differentiable sections $\mathcal{E}^1_X$ of $T^*_X \otimes \mathbb{C}$ has the decomposition

$$\mathcal{E}^1_X = \mathcal{E}^{1,0}_X \oplus \mathcal{E}^{0,1}_X,$$

where a section of $\mathcal{E}^{1,0}_X$ (resp. $\mathcal{E}^{0,1}_X$) is locally of the form $\sum f_i \alpha_i$ for differentiable functions f_i and holomorphic (resp. anti-holomorphic) one-forms ω_i. This induces a canonical decomposition (the *Hodge decomposition*) of each complex-valued differential m-form

$$\omega = \sum_{p+q=m} \omega^{p,q}$$

where $\omega^{p,q}$ is a section of $\bigwedge^p \mathcal{E}^{1,0}_X \otimes \bigwedge^q \mathcal{E}^{0,1}_X \subset \bigwedge^m \mathcal{E}^1_X$. Under complex conjugation, the Hodge type changes from (p, q) to (q, p).

A *Kähler metric* on X is a hermitian metric on its tangent bundle whose imaginary part is a closed 2-form. A *Kähler manifold* is a complex manifold equipped with a Kähler metric.

For a complex manifold X we consider its de Rham cohomology:

$$H^m(X, \mathbb{C}) = \frac{\{\text{closed complex valued } m\text{-forms}\}}{\{\text{exact complex valued } m\text{-forms}\}}.$$

We define $H^{p,q} \subset H^{p+q}(X, \mathbb{C})$ as the subspace of de Rham cohomology classes which contain a form of Hodge-type (p, q). Then $\overline{H^{p,q}} = H^{q,p}$.

Theorem 9.2.1 *Suppose that X is a compact Kähler manifold of dimension n. Then for all $m = 0, 1, \ldots, 2n$ one has*

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}.$$

Corollary 9.2.2 *If the compact complex manifold X admits a Kähler metric, then $H^m(X)$ carries a Hodge structure which is pure of weight m. This Hodge structure does not depend on the choice of a Kähler metric.*

In order to obtain polarized Hodge structures, we need to consider *primitive cohomology*. In contrast with the Hodge decomposition, this depends on the choice of a Kähler metric in general.

Let (X, h) be a compact Kähler manifold of dimension n. The form $\omega = \Im h$ is real, closed and of type $(1, 1)$. It defines an endomorphism L of the de Rham cohomology algebra: $L([\eta]) = [\omega \wedge \eta]$. Then $L(H^{p,q}) \subset H^{p+1,q+1}$.

Theorem 9.2.3 (Hard Lefschetz Theorem) *Cup product with the Kähler class $[\omega]$ induces isomorphisms*

$$L^{n-k} : H^k(X, \mathbb{C}) \to H^{2n-k}(X, \mathbb{C}),\ k \leq n$$

$$L^{n-p-q} : H^{p,q} \to H^{n-q,n-p},\ p+q \leq n.$$

For each $k \leq n$ one defines the primitive cohomology groups

$$H^k_{\text{prim}}(X) = \ker(L^{n-k+1}) \subset H^k(X, \mathbb{C})$$

and considers the Hodge-Riemann form Q_k on $H^k(X, \mathbb{C})$ defined by

$$Q_k(\alpha, \beta) = (-1)^{k(k-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{n-k}.$$

Theorem 9.2.4 *For each $k \leq n$*

1. *the group $H^k_{\text{prim}}(X)$ is a $\mathbb{R}$-Hodge substructure of $H^k(X, \mathbb{R})$;*
2. *the form Q_k defines a polarization on $H^k_{\text{prim}}(X)$.*

If K is a subfield of $\mathbb{R}$ such that the Kähler class $[\omega]$ lies in $H^2(X, K) \subset H^2(X, \mathbb{R})$, then the map L is defined over K, so the primitive cohomology groups are K-Hodge structures.

Theorem 9.2.5 (Kodaira's Embedding Theorem) *The complex Kähler manifold is projective algebraic if and only if $[\omega] \in H^2(X, \mathbb{Q})$.*

It follows that for projective algebraic complex manifolds, the primitive cohomology groups are polarized $\mathbb{Q}$-Hodge structures.

Theorem 9.2.6 (Hodge Index Theorem) *Let X be a compact Kähler manifold of even dimension m. Then the symmetric intersection form on $H_m(X, \mathbb{R})$ has index*

$$\sigma(X) = \sum_q (-1)^q h^{p,q}.$$

Following Griffiths, the Hodge filtration on $H^k(X,\mathbb{C})$ can be described in the following way. For each $p\in\mathbb{Z}$ define

$$\sigma^{\geq p}\Omega^{\cdot}_X=\{0\to\cdots\to\Omega^p_X\to\Omega^{p+1}_X\to\cdots\to\Omega^n_X\}$$

where $n = \dim X$. (We call this filtration of a complex the "obvious" filtration, from the French "filtration bête"). Then $\sigma^{\geq p}\Omega^{\cdot}_X$ is a subcomplex of $\Omega^{\cdot}_X$ and for a compact Kähler manifold the resulting maps

$$\mathbb{H}^k(X,\sigma^{\geq p}\Omega^{\cdot}_X)\to\mathbb{H}^k(X,\Omega^{\cdot}_X)$$

are all injective (this results from the Dolbeault isomorphism $H^q(X,\Omega^p_X)\simeq H^{p,q}$). Then $F^pH^k(X,\mathbb{C})=\mathbb{H}^k(X,\sigma^{\geq p}\Omega^{\cdot}_X)$.

Until now there has been no reason to consider the Hodge filtration instead of the Hodge decomposition. This has changed drastically through Griffiths' work on variation of Hodge structure.

Consider a one parameter family of compact Kähler manifolds, i.e. a Kähler manifold $\mathcal{X}$ provided with a proper and smooth holomorphic map $\pi:\mathcal{X}\to S$, where S is a Riemann surface. Then for $m\in\mathbb{N}$ one has the local system $R^m\pi_*\mathbb{C}_{\mathcal{X}}$ on S with fibre $H^m(X_s,\mathbb{C})$ where $X_s=\pi^{-1}(s)$. The holomorphic vector bundle $R^m\pi_*\mathbb{C}_{\mathcal{X}}\otimes\mathcal{O}_S$ contains the C^∞-subbundles $\mathcal{H}^{p,m-p}$ with fibres $H^{p,m-p}(X_s)$ and the bundles $\mathcal{F}^p=\bigoplus_{r\geq p}\mathcal{H}^{r,m-r}$. In contrast with the $\mathcal{H}^{p,m-p}$ the bundles $\mathcal{F}^p$ are holomorphic subbundles of $R^m\pi_*\mathbb{C}_{\mathcal{X}}\otimes\mathcal{O}_S$. This means that the Hodge filtration varies holomorphically with s, whereas the Hodge decomposition does not.

This can be clarified using relative hypercohomology of the relative de Rham complex. One defines

$$\Omega^p_{\mathcal{X}/S}:=\Omega^p_{\mathcal{X}}/\pi^*\Omega^1_S\wedge\Omega^{p-1}_{\mathcal{X}}.$$

The complex $\Omega^{\cdot}_{\mathcal{X}/S}$ carries the "obvious" filtration $\sigma^{\geq p}$ and one has isomorphisms

$$\mathcal{F}^p\simeq\mathbb{R}^m\pi_*\sigma^{\geq p}\Omega^{\cdot}_{\mathcal{X}/S}.$$

9.2.3 *Smooth Complex Algebraic Varieties*

The de Rham cohomology groups of a complex manifold X can be described by the holomorphic de Rham complex $\Omega^{\cdot}_X$ in the following way. First, one has the embedding $\Omega^{\cdot}_X\hookrightarrow\mathcal{E}^{\cdot}_X$, which is a quasi-isomorphism (i.e. induces an isomorphism of cohomology sheaves, because of the Poincaré Lemma for differentiable and holomorphic forms), and hence induces an isomorphism on hypercohomology

$$\mathbb{H}^k(X,\Omega^{\cdot}_X)\simeq\mathbb{H}^k(X,\mathcal{E}^{\cdot}_X)\simeq H^k(X,\mathbb{C}).$$

The Hodge filtration on $H^k(X, \mathbb{C})$ can be described in the following way. For each $p \in \mathbb{Z}$ consider the obvious filtration

$$\sigma^{\geq p}\Omega^{\cdot}_X = \{0 \to \cdots \to \Omega^p_X \to \Omega^{p+1}_X \to \cdots \to \Omega^n_X\}$$

where $n = \dim X$. Then $\sigma^{\geq p}\Omega^{\cdot}_X$ is a subcomplex of $\Omega^{\cdot}_X$ and for a compact Kähler manifold the resulting maps

$$\mathbb{H}^k(X, \sigma^{\geq p}\Omega^{\cdot}_X) \to \mathbb{H}^k(X, \Omega^{\cdot}_X)$$

are all injective (this results from the Dolbeault isomorphism $H^q(X, \Omega^p_X) \simeq H^{pq}$). Then $F^p H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \sigma^{\geq p}\Omega^{\cdot}_X)$.

This description has two important consequences.

First, for a smooth complex projective variety X_a, with associated complex manifold X, the sheaf complex of holomorphic differentials can be replaced by its subcomplex of algebraic differentials. By GAGA, this gives an isomorphism $H^q(X_a, \Omega^p_{X_a}) \simeq H^q(X, \Omega^p_X)$ for all p, q. Therefore one has isomorphisms $\mathbb{H}^k(X_a, \sigma^{\geq p}\Omega^{\cdot}_{X_a}) \simeq \mathbb{H}^k(X, \sigma^{\geq p}\Omega^{\cdot}_X)$. So the Hodge filtration admits an algebraic description. In particular, if X_a is defined over a subfield K of $\mathbb{C}$, then $H^k(X, \mathbb{C})$ and its Hodge filtration are defined over K.

Second, it enables us to describe the cohomology of smooth quasiprojective varieties in an analogous fashion. Let U be a smooth complex quasiprojective variety. Then by resolution of singularities, there exists a smooth projective variety X containing U as a Zariski-dense open subset, such that $D = X \setminus U$ is a divisor with simple normal crossings on X. This means that each irreducible component of D is a smooth subvariety of X of codimension one, and that at each $x \in D$, the tangent spaces of the irreducible components of D that contain x are in general position. If $n = \dim X$, this means that there exists a system of holomorphic local coordinates $(z_1, \dots, z_n)$ at x such that near x, the divisor D is given by the equation $\prod_{i=1}^r z_i = 0$.

A holomorphic differential α on an open subset of U is said to have *logarithmic poles* along D if both α and $d\alpha$ have at most a pole of order one along D. Letting $j : U \hookrightarrow X$ be the inclusion map, the differentials with logarithmic poles along D form a subcomplex of sheaves $\Omega^{\cdot}_X(\log D) \subset j_*\Omega^{\cdot}_U$: the *logarithmic de Rham complex along D*.

Theorem 9.2.7 *With these notations, the following hold.*

1. *For all $k \in \mathbb{N}$ one has a canonical isomorphism*

$$H^k(U, \mathbb{C}) \simeq \mathbb{H}^k(X, \Omega^{\cdot}_X(\log D));$$

2. *The increasing filtration W on $\Omega^{\cdot}_X(\log D)$ defined by*

$$W_m\Omega^p_X(\log D) = \begin{cases} 0 & \text{for } m < 0 \\ \Omega^p_X(\log D) & \text{for } m \geq p \\ \Omega^{p-m}_X \wedge \Omega^m_X(\log D) & \text{if } 0 \leq m \leq p. \end{cases}$$

induces in cohomology

$$W_m H^k(U;\mathbb{C}) = \textit{Image of } \left(\mathbb{H}^k(X, W_{m-k}\Omega^{\bullet}_X(\log D)) \to H^k(U;\mathbb{C})\right),$$

a filtration which can be defined over the rationals.

3. *The filtration $\sigma^{\geq \cdot}$ on the complex $\Omega^{\cdot}_X(\log D)$ in cohomology gives a filtration*

$$F^p H^k(U,\mathbb{C}) = \textit{Image of } \left(\mathbb{H}^k(X, \sigma^{\geq p}\Omega^{\cdot}_X(\log D)) \hookrightarrow H^k(U;\mathbb{C})\right)$$

4. *The filtrations W and F define a mixed Hodge structure on $H^k(U)$.*

The inclusion $\Omega^{\cdot}_X = W_0\Omega^{\cdot}_X(\log D) \hookrightarrow \Omega^{\cdot}_X(\log D)$ induces the restriction map

$$H^k(X,\mathbb{C}) = \mathbb{H}^k(X,\Omega^{\cdot}_X) \to \mathbb{H}^k(X,\Omega^{\cdot}_X(\log D)) = H^k(U,\mathbb{C})$$

with image $W_k H^k(U,\mathbb{C})$.

9.2.4 Varieties with Simple Normal Crossings

A *variety with normal crossings* is a complex projective variety which locally analytically is isomorphic to a divisor with normal crossings on a complex manifold. If all of its irreducible components are smooth, it is called a *variety with simple normal crossings*.

Let X be a variety with simple normal crossings and irreducible components $X_1,\ldots,X_N$. Then for each i, the intersection D_i of X_i with $\bigcup_{j\neq i} X_j$ is a divisor with simple normal crossings on X_i. A mixed Hodge structure on $H^k(X)$ is constructed by defining a suitable de Rham complex for it, equipped with filtrations W and F.

Let $X^{(p)}$ denote the union inside X of all intersections $X_{i_1}\cap\ldots\cap X_{i_p}$ for $1 \leq i_1 < \ldots < i_p \leq N$ and let $\tilde{X}^{(p)}$ denote their disjoint union. One has a natural finite morphism $a_p : \tilde{X}^{(p)} \to X$. Define $\omega^{\cdot}_X := \Omega^{\cdot}_X/\text{torsion}$. It is a resolution of the constant sheaf $\mathbb{C}_X$ hence

$$H^k(X,\mathbb{C}) \simeq \mathbb{H}^k(X,\omega^{\cdot}_X).$$

Moreover one has exact sequences for all q

$$0 \to \omega_X^q \to (a_1)_* \Omega^q_{\tilde{X}^{(1)}} \to (a_2)_* \Omega^q_{\tilde{X}^{(2)}} \to \dots \tag{9.1}$$

of Mayer-Vietoris type. One defines a double complex

$$\omega_X^{\cdot,\cdot} = \bigoplus_{p,q \geq 0} (a_{q+1})_* \Omega^p_{\tilde{X}^{(q+1)}}$$

with differentials d' and d'', where d' is just the derivative of holomorphic differentials and d'' is a combination of pull-back maps under inclusions of components of $\tilde{X}^{(q+2)}$ in components of $\tilde{X}^{(q+1)}$. This double complex is equipped with filtrations W and F as follows. One defines

$$F^p \omega_X^{\cdot,\cdot} = \bigoplus_{r \geq p, q \geq 0} (a_{q+1})_* \Omega^r_{\tilde{X}^{(q+1)}}.$$

Then from the exact sequence (9.1) one sees that the natural map $\omega_X^{\cdot} \to (a_1)_* \Omega^{\cdot}_{\tilde{X}^{(1)}}$ induces a *filtered quasi-isomorphism*

$$\left(\omega_X^{\cdot}, \sigma^{\geq}\right) \to \left(\omega_X^{\cdot,\cdot}, F\right)$$

The filtration W on $\omega_X^{\cdot,\cdot}$ is defined by

$$W_m \omega_X^{\cdot,\cdot} = \bigoplus_{p \geq 0, q \geq -m} (a_{q+1})_* \Omega^p_{\tilde{X}^{(q+1)}}$$

so $\mathrm{Gr}_m^W \omega_X^{\cdot,\cdot} = (a_{1-m})_* \Omega^{\cdot}_{\tilde{X}^{(1-m)}}$. The mixed Hodge structure on $H^k(X)$ is obtained from the filtrations induced on the hypercohomology of the sheaf complex $\omega_X^{\cdot,\cdot}$ in the same way as in Theorem 9.2.7.

9.2.5 Weighted Homogeneous Isolated Hypersurface Singularities

Consider a polynomial $f = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C}[z_0, \dots, z_n]$ with $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$. The *support* of f is the finite set $\{\alpha \mid c_\alpha \neq 0\}$. Given positive rational numbers $(w_0, \dots, w_n)$ one says that f is *weighted homogeneous with weights* $(w_0, \dots, w_n)$ if its support is contained in the hyperplane with equation $w_0\alpha_0 + \dots + w_n\alpha_n = 1$. Let d be the smallest common denominator of $w_0, \dots w_n$ and let $v_i = dw_i$. Then f is weighted homogeneous with weights $(w_0, \dots, w_n)$ if and only if

$$f(\lambda^{v_0} z_0, \dots, \lambda^{v_n} z_n) = \lambda^d f(z_0, \dots, z_n) \text{ for all } \lambda \in \mathbb{C}. \tag{9.2}$$

Suppose that f is weighted homogeneous with weights $(w_0, \ldots, w_n)$ and that 0 is the only singular point of the affine variety $V(f)$. Then $X := V(f-1)$ is a smooth affine variety, homotopy equivalent with the Milnor fibre of f at 0. This Milnor fibre has the homotopy type of a wedge of μ n-spheres, where $\mu = \dim \mathbb{C}[z_0, \ldots, z_n]/\mathrm{Jac}(f)$. Here $\mathrm{Jac}(f)$ is the ideal generated by the partial derivatives of f.

This fact can be made explicit using algebraic de Rham cohomology. Let $\Omega^k(X)$ denote the space of regular n-forms on X. As X is affine and nonsingular, its algebraic de Rham cohomology is just the cohomology of the complex $\Omega^{\cdot}(X)$ of regular differential forms. See [24].

Let $U = \mathbb{C}^{n+1} \setminus X$. The Poincaré residue map defines an isomorphism res : $H^{n+1}(U, \mathbb{C}) \to \tilde{H}^n(X, \mathbb{C})$. Let $A \subset \mathbb{N}^{n+1}$ be a finite subset with the property that the monomials z^α for $\alpha \in A$ form a $\mathbb{C}$-basis for $\mathbb{C}[z_0, \ldots, z_n]/\mathrm{Jac}(f)$. For $\alpha \in A$ we set $\ell(\alpha) := \sum_{i=0}^n w_i(\alpha_i + 1)$ and

$$\omega_\alpha := \frac{z^\alpha dz_0 \wedge \cdots \wedge dz_n}{(f-1)^{\lceil \ell(\alpha \rceil}}.$$

Theorem 9.2.8 *A basis for $\tilde{H}^n(X, \mathbb{C})$ is given by the classes η_α of the forms* $\mathrm{res}(\omega_\alpha)$ *for $\alpha \in A$.*

For $t \in [0,1]$ the map $(z_0, \ldots, z_n) \mapsto (\exp(2\pi i t w_0) z_0, \ldots, \exp(2\pi i t w_n) z_n)$ defines an isomorphism $h_t : X \to f^{-1}(\exp(2\pi i t))$. Hence the geometric monodromy of f is given by the automorphism h_1 of X. The cohomological monodromy is then $T = (h_1^*)^{-1}$. So

$$T(\eta_\alpha) = \exp(-\ell(\alpha))\eta_\alpha.$$

Let $\tilde{H}^n(X, \mathbb{C})_1$ (resp. $\tilde{H}^n(X, \mathbb{C})_{\neq 1}$) denote the eigenspace of T for the eigenvalue 1 (resp. the sum of eigenspaces of T for eigenvalues different from 1). Then one has the decomposition $\tilde{H}^n(X, \mathbb{C}) = \tilde{H}^n(X, \mathbb{C})_1 \oplus \tilde{H}^n(X, \mathbb{C})_{\neq 1}$ which is defined over $\mathbb{Q}$.

The weight filtration on $\tilde{H}^n(X, \mathbb{C})$ is defined by

$$W_k = 0 \text{ for } k < n, \;\; W_n = \tilde{H}^n(X, \mathbb{C})_{\neq 1} \text{ and } W_k = \tilde{H}^n(X, \mathbb{C}) \text{ for } k > n.$$

Then $\mathrm{Gr}^W_{n+1} \simeq \tilde{H}^n(X, \mathbb{C})_1$. The classes in W_n are represented by regular forms which are square integrable on X.

The Hodge filtration on $\tilde{H}^n(X, \mathbb{C})$ is given by

$$F^p = \text{ the subspace generated by all } \eta_\alpha \text{ with } \ell(\alpha) \leq n - p + 1.$$

The pure Hodge structure W_n is polarized by the form $(\alpha, \beta) \mapsto (-1)^{n(n-1)/2} \int_X \alpha \wedge \beta$. In the case that f is homogeneous, the projective closure $\bar{X}$ of X in projective space is smooth, and the restriction map $H^n(\bar{X}) \to H^n(X)$ identifies $W_n H^n(X)$ with the primitive cohomology $H^n_{\mathrm{prim}}(\bar{X})$. If f is merely weighted homogeneous, one considers instead the closure of X in some weighted projective space. Then $\bar{X}$ has quotient singularities. In [57] the Hodge theory of varieties with quotient singularities is developed with a suitable de Rham complex, and the situation is similar.

The Hodge numbers of the mixed Hodge structure on $\tilde{H}^n(X)$ are therefore the following:

- $h^{n+1-q,q} = \sharp\{\alpha \in A \mid \ell(\alpha) = q\}$ for $q = 1, \ldots, n$;
- $h^{n-q,q} = \sharp\{\alpha \in A \mid q - 1 < \ell(\alpha) < q\}$ for $q = 0, \ldots, n$.

If n is even, the intersection form on $H_n(X, \mathbb{R})$ is symmetric. By the Hodge Index Theorem its index is equal to $\sum_q (-1)^q h^{n-q,q}$. This has been conjectured by V.I. Arnol'd (private communication by Varchenko on the boat trip during the Arbeitstagung in 1975). It was known in the case of Pham-Brieskorn polynomials, where $w_i^{-1} \in \mathbb{N}$, see [30, §14].

9.2.6 Varieties with Isolated Singularities

Let X be a projective variety such that Σ, the singular set of X, has dimension zero. Choose a good resolution $\pi : \tilde{X} \to X$ and let $E = \pi^{-1}(\Sigma)$, a divisor with simple normal crossings on $\tilde{X}$. The resolution

$$0 \to \mathbb{Q}_X \to \mathbb{R}\pi_* \mathbb{Q}_{\tilde{X}} \oplus \mathbb{Q}_\Sigma \to \mathbb{R}\pi_* \mathbb{Q}_E \to 0$$

gives on the complex level the resolution of $\mathbb{C}_X$ by the sheaf complex $\underline{\Omega}^{\cdot}_X$ which is the mapping cone of the morphism

$$\mathbb{R}\pi_* \Omega^{\cdot}_{\tilde{X}} \oplus \mathbb{C}_\Sigma \to \mathbb{R}\pi_* \omega^{\cdot}_E \; .$$

It is equipped with filtrations W and F which induce the mixed Hodge structure on the cohomology of X. Moreover there is a natural morphism of filtered complexes $\lambda : (\Omega^{\cdot}_X, F) \to (\underline{\Omega}^{\cdot}_X)$.

The filtered complex $(\underline{\Omega}^{\cdot}_X, F)$ is unique in a suitable derived category by Du Bois [6]. See Sect. 9.9.

9.3 Nearby and Vanishing Cycles Formalism

Let X be a complex manifold of dimension $n+1$ and let $f : X \to \mathbb{C}$ be a holomorphic function. For $t \in \mathbb{C}$ we set $X_t = f^{-1}(t)$ and $i : X_0 \hookrightarrow X$. We choose $\epsilon > 0$ so small that 0 is the only critical value of f in the disc $|t| < \epsilon$ and put

$$X_\infty := \{(x,u) \in X \times \mathbb{C} \mid f(x) = \exp(2\pi\sqrt{-1}u), |f(x)| < \epsilon\}$$

and let $k : X_\infty \to X$ be the projection on the first factor.

Let $\mathcal{K}^\cdot$ be a sheaf complex of abelian groups on X which is bounded from below. Choose an injective resolution $k^*\mathcal{K}^\cdot \to \mathcal{I}^\cdot$, i.e. a quasi-isomorphism of sheaf complexes such that $\mathcal{I}^\cdot$ is bounded from below and $\mathcal{I}^p$ is an injective sheaf for each p. Then $\psi_f(\mathcal{K}^\cdot) := i^*k_*\mathcal{I}^\cdot$. This defines a functor on the derived category of sheaf complexes of abelian groups which are bounded from below: the *nearby cycles functor*

$$\psi_f : D^b(X) \to D^b(X_0).$$

Let $x \in X_0$ and choose $\epsilon, \eta > 0$ with $\eta \ll \epsilon \ll 1$. Then the restriction of f to $\{z \in X \mid |z - x| < \epsilon, 0 < |f(z)| < \eta\}$ is a C^∞ fibre bundle (the *Milnor fibration*). Let $X_{f,x}$ denote a typical fibre of this fibration. Then

$$H^k(\psi_f(\mathcal{K})_x) \simeq \mathbb{H}^k(X_{f,x}, \mathcal{K}^\cdot_{|X_{f,x}}).$$

The morphism $k^*\mathcal{K}^\cdot \to \mathcal{I}^\cdot$ induces the *specialization morphism*

$$\mathrm{sp} : i^*\mathcal{K}^\cdot \to \psi_f(\mathcal{K}^\cdot).$$

We let $\phi_f(\mathcal{K}^\cdot)$ be the mapping cone over this morphism sp. This defines the *vanishing cycle functor*

$$\phi_f : D^b(X) \to D^b(X_0).$$

Observe that $\phi_f(\mathcal{K}^\cdot)_x$ is acyclic (i.e. a complex with zero cohomology) if and only if $\mathrm{sp}_x : \mathcal{K}^\cdot_x \to \psi_f(\mathcal{K}^\cdot)_x$ is a quasi-isomorphism. In that sense the vanishing cycles functor measures the cohomological difference between the special fibre X_0 and the general fibre, which is homotopy equivalent to X_∞. The inclusion of the second factor of the mapping cone induces the *canonical map*

$$\mathrm{can} : \psi_f\mathcal{K}^\cdot \to \phi_f\mathcal{K}^\cdot$$

and one has the distinguished triangle in $D^b(X_0)$

$$i^*\mathcal{K}^\cdot \xrightarrow{\text{sp}} \psi_f\mathcal{K}^\cdot \xrightarrow{\text{can}} \phi_f\mathcal{K}^\cdot \xrightarrow{+1} \cdots \tag{9.3}$$

The map $(x, u) \mapsto (x, u + 1)$ is a covering transformation h of k and hence determines an automorphism h^* of $\psi_f\mathcal{K}^\cdot$. The *monodromy transformation* for f is the map $T = (h^*)^{-1} : \psi_f\mathcal{K}^\cdot \to \psi_f\mathcal{K}^\cdot$. Because $(T - I) \circ \text{sp} = 0$ we obtain the *variation morphism* $\text{var} : \phi_f\mathcal{K}^\cdot \to \psi_f\mathcal{K}^\cdot$ with the property that $\text{var} \circ \text{can} = T - I$ on $\psi_f\mathcal{K}^\cdot$.

Consider the special case where $\mathcal{K}^\cdot$ is the constant sheaf $\mathbb{C}_X$ and X is smooth. Then for each $x \in X_0$ the germ $f : (X, x) \to (\mathbb{C}, 0)$ is a hypersurface singularity, and $H^j(\phi_f(\mathcal{K}^\cdot)_x) \simeq \tilde{H}^j(X_{f,x}, \mathbb{C})$ is the reduced cohomology of the Milnor fibre. Moreover T is the cohomological monodromy.

9.4 Mixed Hodge Structure on the Nearby and Vanishing Cohomology

In this section we will describe a mixed Hodge structure on the nearby and vanishing cohomology of a proper regular function $f : X \to \mathbb{C}$ where X is a quasi-projective complex algebraic variety whose singular locus has finite image. We choose a good embedded resolution $\pi : \tilde{X} \to X$ of $X_0 = f^{-1}(0)$, i.e.

- $\tilde{X}$ is smooth and π is a projective birational morphism;
- $E := \pi^{-1}(X_0)$ is a divisor with simple normal crossings on $\tilde{X}$.

Let I denote the set of irreducible components of E, so $E = \bigcup_{i\in I} E_i$. For $i \in I$ we let e_i denote the multiplicity of f along E_i. As a divisor we have $E = \sum_{i\in I} e_i E_i$. We choose a common multiple e of the multiplicities e_i and let $\tilde{Y}$ denote the normalization of the fibre product

$$Y = \{(z, t) \in \tilde{X} \times \mathbb{C} \mid f(\pi(z)) = t^e\}.$$

Let us write $\tilde{f}$ and ρ for the natural projections from $\tilde{Y}$ to $\mathbb{C}$ and $\tilde{X}$ respectively. The Semistable Reduction Theorem [37] tells us, that for a suitable choice of π and e, the variety $\tilde{Y}$ is smooth, the fibre $D := \tilde{f}^{-1}(0)$ is reduced and ρ is a cyclic covering of degree e branched only over E. Note that in this case $\psi_{f\pi} = \rho_*\psi_{\tilde{f}}$.

The mixed Hodge structure we look for has been constructed by realizing the cohomology of the nearby fibre as the hypercohomology of a suitable complex of differential forms, equipped with filtrations F and W. Following an idea of P. Deligne, I studied in my thesis [55] the relative logarithmic de Rham complex.

Choose a disk S in $\mathbb{C}$ such that 0 is the only critical value of f in S. Let $\tilde{S} = \{t \in \mathbb{C} \mid t^e \in S\}$. We may assume that $X = f^{-1}(S)$. A first step is

Theorem 9.4.1 *Let* $\Omega^p_{\tilde{Y}/\tilde{S}}(\log D) = \Omega^p_{\tilde{Y}}(\log D)/\frac{d}{dt} \wedge \Omega^{p-1}_{\tilde{Y}}(\log D)$ *for* $p \in \mathbb{Z}$. *Then*

$$\psi_{\tilde{f}}(\mathbb{C}_{\tilde{Y}}) \simeq \Omega^{\cdot}_{\tilde{Y}/\tilde{S}}(\log D) \otimes \mathcal{O}_D$$

in $D^b(\tilde{Y}_0)$.

A Hodge filtration F on $\Omega^{\cdot}_{\tilde{Y}/\tilde{S}}(\log D) \otimes \mathcal{O}_D$ is defined by $F^p = \sigma^{\geq p}$ as usual. However, a suitable weight filtration cannot be defined on this complex. We need a further (double) complex to make this filtration visible. Define

$$A^{p,q} = \Omega^{p+q+1}_{\tilde{Y}}(\log D)/W_p\Omega^{p+q+1}_{\tilde{Y}}(\log D) \text{ for } p, q \geq 0$$

with differentials

$$d' : A^{p,q} \to A^{p+1,q}, \text{ and } d'' : A^{p,q} \to A^{p,q+1}$$

defined by

$$d'(\omega) = \frac{dt}{t} \wedge \omega, \text{ and } d''(\omega) = d\omega.$$

For each $q \geq 0$ the map $\mu : \Omega^q_{\tilde{Y}/\tilde{S}}(\log D) \otimes \mathcal{O}_D \to A^{0,q}$ defined by $\mu(\omega) = (-1)^q \frac{dt}{t} \wedge \omega$ extends to an exact sequence

$$0 \to \Omega^q_{\tilde{Y}/\tilde{S}}(\log D) \otimes \mathcal{O}_D \to A^{0,q} \to A^{1,q} \to A^{2,q} \to \cdots$$

and $\mu d = d'\mu$. Hence, if we equip the total single complex $s(A^{\cdot,\cdot})$ with the filtration F defined by

$$F^r s(A^{\cdot,\cdot}) = \bigoplus_p \bigoplus_{q \geq r} A^{p,q},$$

then μ defines a filtered quasi-isomorphism

$$(\Omega^{\cdot}_{\tilde{Y}/\tilde{S}}(\log D) \otimes \mathcal{O}_D, F) \to (s(A^{\cdot,\cdot}), F).$$

The *monodromy filtration* M on $s(A^{\cdot,\cdot})$ is defined by

$$M_r A^{p,q} = \text{ the image of } W_{r+2p+1}\Omega^{p+q+1}_{\tilde{Y}}(\log D) \text{ in } A^{p,q}.$$

Observe that $d'M_r \subset M_{r-1}$ so the differential on M_r/M_{r-1} is just the one induced by d''. Hence we obtain the direct sum splitting

$$\mathrm{Gr}_r^M s(A^{\cdot\cdot}) =\simeq \bigoplus_{k\geq 0,-r} \mathrm{Gr}^W_{r+2k+1}\Omega^{\cdot}_{\tilde{Y}}(\log D)[1] \simeq \bigoplus_{k\geq 0,-r} (a_{r+2k+1})_*\Omega^{\cdot}_{\tilde{D}^{r+2k+1}}[-r-2k].$$

A careful study of the underlying rational structure (see [42, Sect. 11.2.6]) leads to a Tate twist $\otimes\mathbb{Q}(-r-k)$ for the summand $(a_{r+2k+1})_*\Omega^{\cdot}_{\tilde{D}^{r+2k+1}}[-r-2k]$. Thus $\mathbb{H}^m(D, \mathrm{Gr}_r^M s(A^{\cdot\cdot}))$ is a Hodge structure of weight $m+r$. The general yoga of mixed Hodge theory tells us that F and M induce a mixed Hodge structure on $H^m(\tilde{Y}_\infty, \mathbb{Q})$ and that the weight spectral sequence

$$_ME_1^{-r,q+r} = \bigoplus_k H^{q-r-2k}(\tilde{D}^{r+2k+1})(-r-k) \Rightarrow H^q(\tilde{Y}_\infty, \mathbb{Q})$$

degenerates at the term E_2.

The terminology *monodromy weight filtration* needs an explanation. First some linear algebra. Let V be finite dimensional vector space, k an integer and N a nilpotent endomorphism of V. Then there exists a unique increasing filtration $W(N,k)$ of V with the following properties:

1. $N(W(N,k)_j) \subset W(N,k)_{j-2}$ for all $j \in \mathbb{Z}$;
2. For all $j \in \mathbb{N}$ the induced linear map

$$N^j : W(N,k)_{k+j}/W(N,k)_{k+j-1} \to W(N,k)_{k-j}/W(N,k)_{k-j-1}$$

is an isomorphism.

We call $W(N,k)$ the *weight filtration of N centered at k*.

The transformation T of $\psi_f\mathbb{Q}_X$ induces the *monodromy automorphism* of $H^m(\tilde{Y}_\infty, \mathbb{Q})$. By the Monodromy Theorem this transformation is *quasi-unipotent*. The Jordan decomposition $T = T_sT_u = T_uT_s$ with T_s semisimple and T_u unipotent enables us to define $N = \log T_u$, a nilpotent endomorphism of $H^m(\tilde{Y}_\infty, \mathbb{Q})$.

Theorem 9.4.2

- *The filtration M of $s(A^{\cdot\cdot})$ induces on $H^m(\tilde{Y}_\infty, \mathbb{C})$ the filtration $W(N,m)$.*
- *The semisimple part T_s of the monodromy is an automorphism of the mixed Hodge structure.*

See [56, Theorem 2.13]. In other words, the weight filtration of the mixed Hodge structure on the nearby cohomology of the map f is equal to the weight filtration of the logarithm of the unipotent monodromy.

Let $Z \subset X_0$ be a closed subvariety and let $i_Z : Z \hookrightarrow X_0$ and $j_Z : X_0 \setminus Z \hookrightarrow X_0$ be the inclusion mappings. Then we have the distinguished triangle

$$j_{Z!}j_Z^*\psi_f\mathbb{Q}_X \to \psi_f\mathbb{Q}_X \to (i_Z)_*i_Z^*\psi_f\mathbb{Q}_X \stackrel{+1}{\to} \cdots \tag{9.4}$$

The cohomology of $j_{Z*}j_Z^*\psi_f\mathbb{Q}_X$ can be interpreted as the cohomology of the part $Z_\infty \subset X_\infty$ of the nearby fibre which is "near to Z". (To understand this, take a sufficiently small open neighborhood of Z inside X and take the closure of its intersection with the fibre X_t for t sufficiently small but nonzero.) If Z is a point x, this is the (closed) Milnor fibre $X_{f,x}$. The long exact cohomology sequence for the triangle (9.4) is then

$$\cdots H_c^k(X_\infty \setminus Z_\infty) \to H^k(X_\infty) \to H^k(Z_\infty) \to H_c^{k+1}(X_\infty \setminus Z_\infty) \to \cdots \quad (9.5)$$

We perform the same geometric construction as above but with the extra condition that $\pi^{-1}(Z)$ is a subdivisor of E. That means that there is a subset I' of I such that

$$\pi^{-1}(Z) = \bigcup_{i\in I'} E_i =: E'.$$

In order to get a mixed Hodge structure on the cohomology of Z_∞ we restrict the complex $\psi_{\tilde{f}}$ to the subdivisor $D' := \rho^{-1}E'$ of D. The restriction map $H^q(X_\infty) \to H^q(Z_\infty)$ corresponds to the quotient map $\Omega^{\cdot}_{\tilde{Y}/\tilde{S}}(\log D)\otimes\mathcal{O}_D \to \Omega^{\cdot}_{\tilde{Y}/\tilde{S}}(\log D)\otimes\mathcal{O}_{D'}$ on the level of sheaf complexes. We let D'' denote the divisor $D - D'$ and define

$$W'_k\Omega^{\ell}_{\tilde{Y}}(\log D) = \Omega^{\ell-k}_{\tilde{Y}}(\log D'') \wedge \Omega^k_{\tilde{Y}}(\log D) \text{ for } k \le \ell;$$

$$A_Z^{p,q} = \Omega^{p+q+1}_{\tilde{Y}}(\log D)/W'_q\Omega^{p+q+1}_{\tilde{Y}}(\log D) \text{ for } p, q \ge 0.$$

We obtain a quotient $A_Z^{\cdot,\cdot}$ of $A^{\cdot,\cdot}$ which we equip with the induced (quotient) filtrations M and F. We have a commutative diagram

$$\begin{array}{ccc} \Omega^{\cdot}_{\tilde{Y}/\tilde{S}}(\log D) \otimes \mathcal{O}_D & \to & \Omega^{\cdot}_{\tilde{Y}/\tilde{S}}(\log D) \otimes \mathcal{O}_{D'} \\ \downarrow & & \downarrow \\ A^{\cdot,\cdot} & \to & A_Z^{\cdot,\cdot} \end{array}$$

where the vertical arrows are filtered quasi-isomorphisms with respect to F and the second horizontal arrow is compatible with the filtrations M and F. This defines a mixed Hodge structure on $H^q(Z_\infty) \simeq \mathbb{H}^q(D', s(A_Z^{\cdot,\cdot}))$ such that the restriction map $H^q(Y_\infty) \to H^q(Z_\infty)$ is a morphism of mixed Hodge structures.

By defining $A_{Z,c}^{\cdot,\cdot}$ as the kernel of $A^{\cdot,\cdot} \to A_Z^{\cdot,\cdot}$ with the induced filtrations F and M we get a mixed Hodge structure on $H_c^k(X_\infty \setminus Z_\infty)$ in such a way that (9.5) becomes an exact sequence of mixed Hodge structures.

9.5 Smoothings of Isolated Singularities

We consider an isolated singularity (X, x) of pure dimension $n + 1$ and a holomorphic function germ $f : (X, x) \to (\mathbb{C}, 0)$. By Steenbrink [62, Theorem 1] a representative $f : X \to \mathbb{C}$ for this germ can be found such that $X \setminus \{x\}$ is smooth, the map f is flat, projective and with x as its only critical point in X_0. Moreover the restriction mapping $H^n(X_\infty, \mathbb{C}) \to H^n(X_{f,x}, \mathbb{C})$ is surjective.

In this section we deal with the relation between monodromy and weight filtration on $H^n(X_{f,x}, \mathbb{C})$. Let us first consider the hypersurface case (i.e. x is a regular point of X) as in [56]. Then one has the exact sequence of mixed Hodge structures

$$0 \to H^n(X_0) \to H^n(X_\infty) \to \tilde{H}^n(X_{f,x}) \to 0,$$

from the distinguished triangle (9.3) and the fact that $H^n(\phi_f \mathbb{C}_X) \simeq \tilde{H}^n(X_{f,x}, \mathbb{C})$. The monodromy T acts on this sequence, and by the Invariant Cycle Theorem [42, Theorem 11.43] we have $H^n(X_0) = \ker(T - I; H^n(X_\infty)$. If the subscript 1 (resp. $\neq 1$) refers to the generalized eigenspace of T for the eigenvalue 1 (resp. for the eigenvalues $\neq 1$), then one has $H^n(X_\infty)_{\neq 1} \simeq H^n(X_{f,x})_{\neq 1}$ and $H^n(X_\infty)_1 / \ker(T - I) \simeq H^n(X_{f,x})_1$. This means that

$$W = W(N, n) \text{ on } H^n(X_{f,x})_{\neq 1}. \tag{9.6}$$

This is true for any globalized smoothing of an isolated singularity, for the isomorphism $H^n(X_\infty)_{\neq 1} \simeq H^n(X_{f,x})_{\neq 1}$ is always valid.

For the eigenvalue 1 we use that $\ker(T - I) \cap H^n(X_\infty)_1 = \ker(N) \cap H^n(X_\infty)_1$. Moreover, the filtration $W(N, k)$ of a vector space V induces on the quotient $V/\ker(N)$ the filtration $W(\bar{N}, k + 1)$ where $\bar{N} \in \operatorname{End}(V/\ker(N))$ is induced by N. So

$$W = W(N, n + 1) \text{ on } H^n(X_{f,x})_1. \tag{9.7}$$

Here the hypersurface case is special in the sense that the variation mapping in rational cohomology $\operatorname{var} : H^n(X_{f,x}) \to H^n_c(X_{f,x})$ is an isomorphism. This is true if and only if X is a rational homology manifold.

In the general case we have a canonical decomposition $\operatorname{Gr}^W H^n(X_{f,x})_1 = A \oplus B$ such that $W = W(N, n)$ on B and $W = W(N, n + 1)$ on A. See [62, Remark 9], where graded polarizations of A and B are given. Unlike the hypersurface case, this decomposition of $\operatorname{Gr}^W H^n(X_{f,x})_1$ does not lift in general to a direct sum decomposition of $H^n(X_{f,x})_1$ as a mixed Hodge structure. To see this we consider the case $n = 1$.

Suppose (X, x) is a normal surface singularity. The local cohomology group $H^2_x(X)$ has a mixed Hodge structure with weights 0 and 1. If $(\tilde{X}, E) \to (X, x)$ is a good resolution, then we have an isomorphism $H^1(E) \to H^2_x(X)$, and

$W_0 H^2_x(X) = \mathbb{Q}^{h_X}$ where h_X is the first Betti number of the dual graph of the curve E. Moreover $\mathrm{Gr}^W_1 H^2_x(X) \simeq H^1(\tilde{E})$. Suppose $f : (X, x) \to (\mathbb{C}, 0)$ defines a curve singularity X_0 with r irreducible components. Then the decomposition $\mathrm{Gr}^W H^1(X_{f,x})_1 = A \oplus B$ has $B = B_0 \oplus B_1 \oplus B_2$ with $B_1 = H^1(\tilde{E})$, $B_0 = \mathbb{Q}^{h_X}$ and $B_2 = \mathbb{Q}(-1)^{h_X}$. Moreover $N : B_2 \to B_0$ is an isomorphism. Finally $A = A_2 = \mathbb{Q}(-1)^{r-1} = \ker(N : \mathrm{Gr}^W_2 H^1(X_{f,x}) \to \mathrm{Gr}^W_0 H^1(X_{f,x})$.

Suppose the decomposition would lift to a decomposition of $H^1(X_{f,x})$. Then A_2 would be a direct summand of $H^1(X_{f,x})$. Consider the Hodge substructure $\ker(N)$ of $H^1(X_{f,x})$. We have $\mathrm{Gr}^W_2 \ker(N) \simeq A_2$ and A_2 would be a direct summand. So $\ker(N) \simeq W_1 \ker(N) \oplus A_2$. However, $\ker(N)$ is the cohomology of the open curve E' which is $\tilde{E}$ with the r points of its intersection with the strict transfom of X_0 removed. We have the extension of mixed Hodge structures

$$0 \to H^1(E) \to H^1(E') \to A_2 \to 0$$

The position of these points on E is reflected in the extension data of this mixed Hodge structure, as described by Carlson [13]. So in general A_2 will not be a direct summand.

The following formula for the signature of the cup product form $h : (\omega.\eta) \mapsto \int \omega \wedge \eta$ on $H^n_c(X_{f,x}, \mathbb{R})$ holds:

$$\sigma(h) = \sum_{p+q=n} (-1)^p \left(h^{pq} + 2 \sum_{i \geq 1} (-1) h^{p+i,q+i} \right).$$

See [62, Theorem 11]. For a detailed exposition of several bilinear forms associated to this situation see [3].

9.6 Hodge Structure via $\mathcal{D}$-Modules

The use of algebraic analysis in the study of isolated singularities began with Brieskorn's proof of the monodromy theorem [11], followed by Malgrange [38] and Pham [44]. The first reference I found which deals with the description of the Hodge filtration using D-modules is [12]. Our treatment follows this text closely. Brylinski's aim was to show that the intersection homology groups of a complete complex algebraic variety carry a pure Hodge structure.

Varchenko [65] developed a method to describe the mixed Hodge structure on the cohomology of the Milnor fibre in the case of isolated hypersurface singularities which does not use resolution of singularities. With the assistance of F. Pham and M. Saito this method was reformulated into the language of D-modules in [51].

For each complex manifold X we have the coherent sheaf of rings $\mathcal{D}_X$ of germs of holomorphic differential operators on X. It is equipped with the increasing filtration F by the order of differential operators, such that $\mathrm{Gr}_F\mathcal{D}_X \simeq \mathrm{Sym}(\mathcal{T}_X)$ where $\mathcal{T}_X$ is the holomorphic tangent bundle of X.

There exists an algebraic definition of $\mathcal{D}_X$ for smooth algebraic varieties X over a field K of characteristic zero as follows. One defines $F_k\mathcal{D}_X$ as the subsheaf of $End_K\mathcal{O}_X$ for $k \in \mathbb{N}$ recursively: $F_0\mathcal{D}_X = \mathcal{O}_X$ and a local section P of $End_K\mathcal{O}_X$ is in F_{k+1} if and only its commutator $Pf - fP$ is a section of $F_{k-1}\mathcal{D}_X$ for $k \geq 1$. Finally $\mathcal{D}_X = \bigcup_{k\geq 0} F_k\mathcal{D}_X$.

Let Y be a complex submanifold of codimension one in the complex manifold X. We define $\mathcal{O}_X(*Y)$ as the sheaf of germs of meromorphic functions on X which are holomorphic on $X \setminus Y$. It is a (left) $\mathcal{D}_X$-module. Moreover, if for each $k \in \mathbb{N}$ we let $\mathcal{O}_X((k+1)Y) =: F_k\mathcal{O}_X(*Y)$ be the subsheaf whose sections have a pole of order at most $k+1$ along Y, then $\mathcal{O}_X(*Y)$ becomes a filtered $\mathcal{D}_X$-module: for local sections P of $F_k\mathcal{D}_X$ and g of $F_\ell\mathcal{O}_X(*Y)$ the section $P(f)$ is in $F_{k+\ell}\mathcal{O}_X(*Y)$.

Consider the de Rham complex $\Omega^{\cdot}_X(*Y)$ of X with poles along Y. The filtration $F_{\cdot}$ on $\mathcal{O}_X(*Y)$ extends to this complex by $F_k\Omega^p_X(*Y) = \Omega^p_X((k+p+1)Y)$ if $k+p \geq 0$ and $F_k\Omega^p_X(*Y) = 0$ else. Then the cohomology sheaves of the quotient complex $F_{-p}\Omega^p_X(*Y)/F_{-p+1}$ are given by $\mathcal{H}^q = 0$ for $q \neq p$ and $\mathcal{H}^p \simeq \Omega^p_X(\log Y)$. Putting F^p for F_{-p} we find that the inclusion map

$$(\Omega^{\cdot}_X(\log Y), F^{\cdot}) \to (\Omega^{\cdot}_X(*Y), F^{\cdot})$$

is a *filtered quasi-isomorphism*. See [15, Prop. II.3.13]. In the case that X is smooth projective and Y a closed codimension one subvariety this means that the Hodge filtration of the mixed Hodge structure on $H^*(X \setminus Y, \mathbb{C})$ can be obtained by consideration of the order of pole of differentials along Y. This was used by Griffiths [22] to describe the cohomology of (complements of) smooth hypersurfaces in projective space. It is the main inspiring example of the de Rham complexes of holonomic D-modules. Note that working with filtered D-modules, the natural filtrations are increasing, and this is why we use subscripts rather than superscripts for them.

Consider a coherent $\mathcal{D}_X$-module $\mathcal{M}$. A *good filtration* on $\mathcal{M}$ is an increasing filtration $\{F_k\mathcal{M}\}_{k\in\mathbb{Z}}$ by $\mathcal{O}_X$-submodules such that

1. $F_m\mathcal{D}_X \cdot F_k\mathcal{M} \subset F_{m+k}\mathcal{M}$ for all m, k;
2. locally there exists $K \in \mathbb{Z}$ such that $F_m\mathcal{D}_X \cdot F_k\mathcal{M} = F_{m+k}\mathcal{M}$ for $m \geq 0,\ k \geq K$.

A good filtration always exists locally on X. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ with a good filtration F, the associated graded module $\bigoplus_{k\in\mathbb{Z}} F_k\mathcal{M}/F_{k-1}\mathcal{M}$ has an associated coherent sheaf on the holomorphic cotangent bundle T^*X. The support of this module, the *characteristic variety* $Ch(\mathcal{M})$, is a homogeneous analytic subvariety of dimension at least $\dim(X)$. It does not depend on the choice of a good filtration, hence is globally defined. The module $\mathcal{M}$ is called *holonomic* if $\dim Ch(\mathcal{M}) = \dim X$, and *regular holonomic* if there exist a global good filtration for which the characteristic variety is reduced. In that case, Kashiwara and Kawai [33] showed the existence of a canonical good filtration.

The de Rham complex of a $\mathcal{D}_X$-module $\mathcal{M}$ is the complex $DR(\mathcal{M}) := \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ with differential

$$d(\omega \otimes m) = (d\omega) \otimes m + \sum_{i=1}^{\dim X} (dz_i \wedge \omega) \otimes (\frac{\partial}{\partial z_i} \cdot m)$$

where $(z_1, \ldots, z_n)$ is a local holomorphic coordinate system on X. For $\mathcal{M}$ holonomic, the cohomology sheaves of $DR(\mathcal{M})$ are analytically constructible by Kashiwara [32]. The de Rham complex is functorial. The Riemann-Hilbert correspondence [33, 41], tells us that the de Rham functor defines an equivalence between the derived category $D^b_{rh}(\mathcal{D}_X)$ of bounded complexes of coherent $\mathcal{D}_X$-modules with regular holonomic cohomology sheaves and the derived category $D^b_c(\mathbb{C}_X)$ of bounded complexes of sheaves of $\mathbb{C}$-vector spaces with analytically constructible cohomology sheaves. The de Rham complexes of single regular holonomic $\mathcal{D}_X$-modules are exactly the *perverse sheaves* of $\mathbb{C}_X$-modules for the middle perversity. Perverse sheaves K are sheaf complexes characterized by the conditions that the dimensions of the support of the cohomology sheaves $\mathcal{H}^i(K)$ and $\mathcal{H}^i(\mathbf{D}K)$ are at most $-i$. Here $\mathbf{D}$ stands for Verdier duality.

The origin of the study of isolated hypersurface singularities using D-modules is Brieskorn's proof of the monodromy theorem [11]: the eigenvalues of the monodromy are roots of unity. He "construct(s) by algebraic methods a regular singular ordinary linear differential operator, such that the monodromy of this singular operator coincides with the Picard-Lefschetz monodromy".

The *Brieskorn lattice* $\mathcal{H}^{(0)}$ of an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is defined by

$$\mathcal{H}^{(0)} := \Omega^{n+1}_{\mathbb{C}^{n+1},0} / df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}.$$

It is a free $\mathbb{C}\{t\}$ module of rank equal to the Milnor number

$$\mu(f) = \dim_{\mathbb{C}} \Omega^{n+1}_{\mathbb{C}^{n+1},0} / df \wedge \Omega^{n}_{\mathbb{C}^{n+1},0}.$$

Here t acts by multiplication with f. The submodule

$$\mathcal{H}' := df \wedge \Omega^{n}_{\mathbb{C}^{n+1},0} / df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}.$$

is a free $\mathbb{C}\{t\}$ module of rank $\mu(f)$ as well and the operator $\partial_t : [df \wedge \omega] \mapsto d\omega$ is a bijection from $\mathcal{H}'$ to $\mathcal{H}^{(0)}$. If we let

$$\mathbb{C}\{\{\partial_t^{-1}\}\} := \left\{ \sum_{i \geq 0} a_i \partial_t^{-i} \mid \sum_{i \geq 0} a_i z^i / i! \in \mathbb{C}\{z\} \right\}$$

then $\mathcal{H}^{(0)}$is equipped with the operator ∂_t^{-1} and becomes a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank $\mu(f)$. The D-module we arrive at is the localization $\mathcal{H} := \mathcal{H}^{(0)}[\partial_t]$. It is a vector space of dimension $\mu(f)$ over the field $\mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]$. Note that $\partial_t \mathcal{H}' = \mathcal{H}^{(0)}$.

For each $a \in \mathbb{C}$ one defines $C_a = \bigcup_{r>0} \ker(t\partial_t - a)^r \subset \mathcal{H}$. This is a finite dimensional vector space and $C_a = 0$ unless $\exp(-2\pi i a)$ is a monodromy eigenvalue. By the monodromy theorem, $C_a \neq 0$ implies that a is a rational number. For $a \in \mathbb{Q}$ we define $V_a\mathcal{H}$ (resp. $V_{>a}\mathcal{H}$) to be the $\mathbb{C}\{t\}$-submodule of $\mathcal{H}$ generated by all subspaces C_b with $b \geq a$ (resp. $b > a$). Then for each $a \in \mathbb{Q}$ we have $V_a\mathcal{H} = C_a \oplus V_{>a}\mathcal{H}$, and the cohomology of the Milnor fibre is identified in [51] with $\bigoplus_{-1<a\leq 0} C_a$. The Hodge filtration on C_a is given by

$$F^p C_a = \text{ the image of } \partial_t^{n-p}\mathcal{H}^{(0)} \cap V_a \text{ in } C_a.$$

For any $\omega = g(z)dz_0 \wedge \ldots \wedge dz_n \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$ we may define a family of holomorphic n-forms $\eta_t = \mathrm{Res}_{X_t}\left(\frac{\omega}{f-t}\right)$ on the Milnor fibres X_t of the singularity f. A multivalued family of n-cycles γ_t on X_t may be obtained by choosing one n-cycle on a fibre and then moving this continuously to the other fibres, using monodromy. Then the multivalued function $I(t) = \int_{\gamma_t} \eta_t$ may be written as

$$I(t) = \sum_{\alpha,q} c_{\alpha,q} t^{\alpha} (\log t)^q / q!$$

for certain constants $c_{\alpha,q}$ with $\alpha \in \mathbb{Q}$ and $q \in \mathbb{N}$. By a result of Malgrange [38] we know that $c_{\alpha,q} \neq 0$ implies that

1. $0 \leq q \leq n$ (this gives the bound $n+1$ for the Jordan blocks of the monodromy);
2. $\exp(-2\pi i\alpha)$ is an eigenvalue of the monodromy T;
3. $\alpha > -1$.

This result has been used by Varchenko [65] to define the *order* $\alpha(\omega)$ of ω as the smallest α such that there exists (γ_t) and q with $c_{\alpha,q} \neq 0$. It appears, due to Stokes' theorem, that $\alpha(\omega) = \infty$ if and only if $\omega \in df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}$. This order function corresponds to the filtration V above.

We let

$$Q^f = \Omega^{n+1}_{\mathbb{C}^{n+1},0} / df \wedge \Omega^{n}_{\mathbb{C}^{n+1},0} = \mathcal{H}^{(0)}/\mathcal{H}'$$

and define $V_\alpha Q^f$ as the image of $V^\alpha \mathcal{H}^{(0)}$ in Q^f. Multiplication by f maps $V_\alpha Q^f$ to $V_{\alpha+1} Q^f$ and hence defines an endomorphism $\{f\}$ of degree one of $\mathrm{Gr}^V Q^f$.

Theorem 9.6.1 *The maps* $\{f\}$ *and* $N := \log T_u \in \mathrm{End} H^n(X_{f,x}, \mathbb{C})$ *have the same Jordan normal form.*

See [66] and [51, Theorem 7.1].

Mixed Hodge Modules
The description of Hodge structures in terms of D-modules found its culmination in the fundamental work of M. Saito on mixed Hodge modules [48]. Mixed Hodge modules on a smooth complex manifold X form an abelian category MHM(X) whose objects are tuples consisting of

1. a perverse sheaf K of $\mathbb{Q}$-vector spaces, equipped with an increasing filtration W;
2. a regular holonomic D_X-module M with an increasing filtration W by D_X-submodules and a good filtration F;
3. a filtered quasi-isomorphism between $(K \otimes \mathbb{C}, W)$ and $(\mathrm{DR}(M), W)$

which satisfy certain conditions. The formulation of these conditions involves nearby cycle functors and induction on the dimension. See [46] for a short summary. When X is a point, one obtains the category of graded-polarizable mixed Hodge structures. In particular, this formalism puts mixed Hodge structures on the cohomology of *any* hypersurface singularity, and shows functoriality in many cases.

9.7 Motivic Milnor Fibre

The notions of motivic nearby fibre and motivic Milnor fibre were introduced by Denef and Loeser [19] using arc spaces. They are defined for non-constant regular functions $f : X \to k$ on a smooth connected quasi-projective variety over a field k of characteristic zero, and take their value in the group $K_0(\mathrm{Var}_k^{\hat{\mu}})$, the Grothendieck group of varieties over k equipped with an action of a finite cyclic group.

Let us recall the Grothendieck group $K_0(\mathrm{Var}_k)$ of varieties over k. We will use Bittner's description from [4]. Generators are isomorphism classes $[X]$ of smooth projective varieties over k. Relations are generated by the following. Let X be a smooth projective variety and $Y \subset X$ a smooth closed subvariety. Let $\pi : X' \to X$ be the blowing-up with center Y and let $Y' = \pi^{-1}(Y)$. Then

$$[X'] - [Y'] = [X] - [Y] \text{ in } K_0(\mathrm{Var}_k) .$$

For an arbitrary projective k-variety X of dimension n, one defines $[X] \in K_0(\mathrm{Var}_k)$ by induction on n. Choose a resolution of singularities $\pi : \tilde{X} \to X$, let Σ be the singular locus of X and $E = \pi^{-1}(\Sigma)$. Then E and Σ have smaller dimension than X so we may assume $[E]$ and $[\Sigma]$ are well-defined. Then we put $[X] := [\tilde{X}] - [E] + [\Sigma]$. For X quasi-projective there exists projective T such that $\bar{X} = X \cup T$ is projective and $[X] := [\bar{X}] - [T]$.

The group $K_0(\mathrm{Var}_k^{\hat{\mu}})$ is an equivariant version of this construction. For $n \in \mathbb{N}$ we let μ_n denote the group of n-th roots of unity in k. By mapping μ_{nd} to μ_n by $x \mapsto x^d$ we obtain a projective system and we let $\hat{\mu} = \varprojlim \mu_n$. A *good* $\hat{\mu}$*-action* on a k-variety X is given by an action of the group μ_n of n-th roots of unity on X for some n.

The group $K_0(\mathrm{Var}_k^{\hat{\mu}})$ has generators $[X, \hat{\mu}]$ where X is a k-variety with good $\hat{\mu}$-action and relations $[X, \hat{\mu}] = [Y, \hat{\mu}] + [X \setminus Y, \hat{\mu}]$ and $[X \times V, \hat{\mu}] = [X, \hat{\mu}] \cdot \mathbb{L}^m$ when V is an m-dimensional affine space with any good $\hat{\mu}$-action. Finally $\mathcal{M}_k^{\hat{\mu}} = K_0(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{L}^{-1}]$, where $\mathbb{L} = [\mathbb{A}_k^1]$.

For a degeneration $f : X \to \mathbb{C}$ with special fibre E a divisor with normal crossings the motivic nearby fibre ψ_f is defined by Bittner in [5] as

$$\psi_f := \sum_{m \geq 1} (-1)^{m-1} [\tilde{D}^m \times \mathbb{P}^{m-1}] \in K_0^{\hat{\mu}}(\mathrm{Var_k})$$

with notations as in Sect. 9.3. For $\zeta \in \mu_e$ the covering transformation $z \mapsto \zeta z$ of τ extends to an automorphism $\gamma(\zeta)$ of order e of $\tilde{X}$ and in this way we obtain a good $\hat{\mu}$-action on $\tilde{X}$ and D.

A similar formula holds under the hypothesis that $X \setminus X_0 \hookrightarrow X$ is a toroidal embedding without self-intersection, see [64, Theorem 3].

9.8 Spectrum and Spectral Pairs

9.8.1 Definitions

We let HS denote the category of mixed $\mathbb{Q}$-Hodge structures. Its Grothendieck ring $K_0(\mathrm{HS})$ is the target of a ring homomorphism

$$\chi_{\mathrm{Hdg}} : K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{HS})$$

defined for a generator $[X]$ with X smooth projective by

$$\chi_{\mathrm{Hdg}}([X]) = \sum_k (-1)^k [H^k(X)].$$

For a blowing-up $(X', Y') \to (X, Y)$ with $Y \subset X$ smooth projective one has exact sequences $0 \to H^k(X) \to H^k(X') \oplus H^k(Y) \to H^k(Y') \to 0$, so χ_{Hdg} respects the relation $[X'] - [Y'] = [X] - [Y]$. For arbitrary X one has $\chi_{\mathrm{Hdg}}([X]) = \sum_k (-1)^k [H_c^k(X)]$. See [43] for more details.

On the other hand, one has a ring homomorphism

$$P_{\mathrm{hn}} : K_0(\mathrm{HS}) \to \mathbb{Z}[u, v, u^{-1}, v^{-1}]$$

defined by $P_{\mathrm{hn}}(V) = \sum_{p,q \in \mathbb{Z}} h^{p,q}(V) u^p v^q$, the *Hodge number polynomial.*

Both χ_{Hdg} and P_{hn} have their equivariant companions. To describe this we consider the category $\mathrm{HS}^{\hat{\mu}}$ of $\mathbb{Q}$-mixed Hodge structures with a finite order automorphism γ. The functor χ_{Hdg} extends in a natural way to a functor

$$\chi_{\mathrm{Hdg}}^{\hat{\mu}} : K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}) \to K_0(\mathrm{HS}^{\hat{\mu}})$$

The equivariant Hodge number polynomial $P_{\mathrm{hn}}^{\hat{\mu}}$ takes its values in the ring of Laurent-Puiseux polynomials $R := \bigcup_{n\in\mathbb{N}} \mathbb{Z}[u^{\frac{1}{n}}, v^{\frac{1}{n}}, u^{-1}, v^{-1}]$. Let (V, γ) be a Hodge structure of weight k with an automorphism of finite order. We have decompositions

$$V_{\mathbb{C}} = \bigoplus_{0\le a<1} V_a \text{ and } V_{\mathbb{C}} = \bigoplus_{p} V^{p,k-p}$$

where $V_a = \ker(\gamma - \exp(2\pi i a))$. This leads to a double decomposition

$$V_{\mathbb{C}} = \bigoplus_{0\le a<1, p\in\mathbb{Z}} V_a \cap V^{p,k-p}.$$

For $0 < a < 1$ and $p \in \mathbb{Z}$ we define $b = a+p$ and $\tilde{h}^{b,k+1-b} = \dim V_a \cap V^{p,k-p}$ and we let $\tilde{h}^{p,k-p} = \dim V^{p,k-p} \cap V_0$. Then the equivariant Hodge number polynomial is defined by

$$P_{\mathrm{hn}}^{\hat{\mu}}(V, \gamma) = \sum_{b\in\mathbb{Q}} \tilde{h}^{b,k-b}(V) u^b v^{k-b}.$$

The *singularity spectrum* (in Varchenko's sense, see [68]) of the isolated hypersurface singularity $f : (X, x) \to (\mathbb{C}, 0)$ is defined by

$$\mathrm{Sp}_V(f, x) := t^{-1} P_{\mathrm{hn}}^{\hat{\mu}}(\phi_{f,x})(t, 1) = t^{-1}\mathrm{Sp}_{\mathrm{Sa}}(f, x)$$

where $\mathrm{Sp}_{\mathrm{Sa}}(f, x)$ is the generating function for the exponents of f in Saito's sense [45], which is also equal to $\sum_\alpha \dim \mathrm{Gr}_\alpha^V(Q^f)t^\alpha$ by Sect. 9.5. The characteristic pairs from [56] contain the same information as $P_{\mathrm{hn}}^{\hat{\mu}}(\phi_{f,x})$.

9.8.2 Examples

1. Let f be a weighted homogeneous isolated singularitiy with weights $w_0, \ldots, w_n$. Then by Steenbrink [56, Example 5.11]

$$\mathrm{Sp}_{\mathrm{Sa}}(f, x) = \prod_{i=0}^{n} \frac{t - t^{w_i}}{t^{w_i} - 1}.$$

If $A \subset \mathbb{N}^{n+1}$ is a finite subset with the property that the monomials z^α for $\alpha \in A$ form a $\mathbb{C}$-basis for $\mathbb{C}[z_0, \ldots, z_n]/\mathrm{Jac}(f)$, then

$$\mathrm{Sp}_{\mathrm{Sa}}(f, x) = \sum_{\alpha \in A} t^{\ell(\alpha)}$$

where $\ell(\alpha) = \sum_{i=0}^{n} w_i(\alpha_i + 1)$.

2. A formula for the spectrum of an irreducible plane curve singularity was found by M. Saito [50]. It was reproved by Guibert using motivic integration in [25].
3. In [56] a conjectural formula occurs for $P^{\hat{\mu}}_{\mathrm{hn}}(\phi_{f,x})$ in the case of isolated hypersurface singularities which are nondegenerate with respect to their Newton diagram. It was proven there for functions of two variables. This conjecture was corrected by Arnol'd [1]. V.I. Danilov in [14] described an algorithm calculating $P^{\hat{\mu}}_{\mathrm{hn}}(\phi_{f,x})$ for Newton nondegenerate functions. The corresponding formula for the spectrum was proved by M. Saito, who showed in [47] that for these the V-filtration on Q^f coincides with the Newton filtration. A more elementary proof was given in [71].
4. The behaviour of the spectrum in a Yomdin series of isolated singularities is described in [61] as the spectrum of a non-isolated singularity plus a correction term. This was proved for series arising from homogeneous functions and conjectured in general. Proofs of the formula in the general case were given by M. Saito [49] (using mixed Hodge modules) and Guibert, Loeser and Merle [26] (using motivic integration).

9.8.3 Some Properties of the Spectrum

Range: Consider an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with $\mathrm{Sp}_{\mathrm{Sa}}(f, x) = \sum_{\alpha \in \mathbb{Q}} m_\alpha t^\alpha$. If $m_\alpha \neq 0$ then $0 < \alpha < n + 1$.

Symmetry: Moreover $m_\alpha = m_{n+1-\alpha}$.

Thom-Sebastiani: Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ be isolated singuarities. Then the germ $f \oplus g : (\mathbb{C}^{n+m+2}, 0) \to (\mathbb{C}, 0)$ defined by $(f \oplus g)(x, y) = f(x) + g(y)$ is also an isolated singularity and

$$\mathrm{Sp}_{\mathrm{Sa}}(f \oplus g, (x, y)) = \mathrm{Sp}_{\mathrm{Sa}}(f, x) \cdot \mathrm{Sp}_{\mathrm{Sa}}(g, y)$$

(product in the ring of Laurent polynomials). This was conjectured in [56, Conjecture 5.4] and proved in [67] and [51, Theorem 8.11]. See also [70, Sect. 7].

Semicontinuity: It was observed by Arnol'd [2] that the spectral numbers in certain deformations display semicontinuity behaviour. Work of Varchenko on this problem led to the following formulation. Let $F : X \times S \to T \times S$ be a good representative of a one-parameter unfolding of the isolated singularity

$f = F_0$, where S is a small disc in $\mathbb{C}$ containing 0. Let $s \in S$ and $t \in T$ such that the fibre $F^{-1}(t, s)$ has the singular points $x_1, \ldots, x_r$. Then the spectrum of f at its critical point x is bigger than the sum of the spectra of f_s at its critical points x_i in the following sense. For any subset I of $\mathbb{R}$ and Laurent-Puiseux polynomial $h = \sum_\alpha m_\alpha t^\alpha$ one defines $\deg_I(h) := \sum_{\alpha \in I} m_\alpha$. Then I is called a *semicontinuity domain* if for any unfolding as above,

$$\sum_{i=1}^{r} \deg_I \mathrm{Sp}_V(f_s, x_i) \leq \deg_I \mathrm{Sp}_V(f, x).$$

Varchenko [69] showed that for deformations of negative weight of weighted homogeneous isolated singularities, each open interval of length one is a semicontinuity domain. This leads to a rather sharp bound on the number of isolated singular points that can occur on a projective hypersurface of given dimension and degree. The author [59] proved that for arbitrary deformations of isolated singularities, each half-open interval $(\alpha, \alpha + 1]$ with $\alpha \in \mathbb{Q}$ is a semicontinuity domain. This result gives conditions on which multigerms can appear as small deformations of a given isolated singularity. A recent application is the classification of projective hypersurfaces with small polar degree [54]. The proof uses Varchenko's formula for the spectrum of $f + w^q$ (special case of the Thom-Sebastiani formula) and a semicontinuity result for Hodge numbers of Milnor fibres of isolated complete intersection singularities. This again relies on the existence of a mixed Hodge structure on the Milnor fibre of deformations which are not necessarily smoothings [7].

Geometric genus: It was shown by M. Saito [45] that the geometric genus of an isolated hypersurface singularity is equal to the number of spectral numbers in the interval $(-1, 0]$, i.e. to $\deg_{(-1,0]} \mathrm{Sp}_V(f, x)$.

Spectrum for isolated complete intersection singularities: The construction of a mixed Hodge structure on the Milnor fibre in Sect. 9.3 is not restricted to the case of isolated hypersurface singularities. In [21] the case of isolated complete intersection singularities is considered. Unlike in the hypersurface case, there does not exist a privileged one parameter smoothing for such a singularity. The topology of the Milnor fibre is independent of the choice of smoothing, but the monodromy does depend on it, and a fortiori the mixed Hodge structure on its cohomology. Moreover, the resulting spectrum will not be symmetric in general. This leads to the introduction of the notion of a *smoothing pair* of an isolated complete intersection singularity.

A two-parameter deformation $F = (f, g) : (\mathcal{X}, x) \to (\mathbb{C}^2, 0)$ of an n-dimensional isolated complete intersection singularity (X, x) is called a smoothing pair if x is an isolated singularity of $\mathcal{X}$ and of $X' := g^{-1}(0)$. Then $g : \mathcal{X} \to \mathbb{C}$ and $f : X' \to \mathbb{C}$ are one-parameter smoothings of these isolated singularities, with Milnor fibres $\mathcal{X}_t$ and X'_t respectively. The mixed Hodge structure involved is on the relative cohomology group $H^{n+1}(\mathcal{X}_t, X'_t)$.

To define this mixed Hodge structure, we apply Saito's theory of mixed Hodge modules [48]. It enables one to iterate the functors of nearby and vanishing cycles and stay within the category of mixed Hodge modules. Restriction to a point gives a mixed Hodge module with support on this point, which is nothing but a mixed Hodge structure. The group $H^{n+1}(\mathcal{X}_t, X'_t)$ can be identified with $\phi_f \psi_g \mathbb{Q}^H_{\mathcal{X}}$ and is equipped with a mixed Hodge structure in this way. The relevant monodromy action is T_f.

There exists an obvious notion of a deformation of smoothing pairs, and the symmetry and semicontinuity of the spectrum also hold in this context.

Distribution of the spectral numbers: Let us write $\mathrm{Sp}_V(f, x) = \sum_{i=1}^{\mu} t^{\alpha_i}$ with $\alpha_1 \leq \cdots \leq \alpha_\mu$. By the symmetry of the spectrum, $\alpha_i + \alpha_j = n - 1$ when $i + j = \mu + 1$. The spectrum can therefore be considered as a probability measure with support in the interval $(-1, n)$ and mean $(n-1)/2$. Its variance is then given by

$$V_2 := \frac{1}{\mu} \sum_{i=1}^{\mu} (\frac{n-1}{2} - \alpha_i)^2.$$

Hertling [29] showed that $V_2 = (\alpha_\mu - \alpha_1)/12$ for quasi-homogeneous isolated hypersurface singularities. His proof uses the theory of Frobenius manifolds. An elementary proof was given by Dimca [20]. Moreover, Hertling conjectured that $(\alpha_\mu - \alpha_1)/12$ is an upper bound for V_2 for all isolated hypersurface singularities. This conjecture was proved for irreducible curve singularities by M. Saito [50] and by Brélivet [9] for curves in general. Moreover, he showed that for curves the equality $V_2 = (\alpha_\mu - \alpha_1)/12$ holds only for semi-quasihomogeneous functions (which are μ-constant deformations of quasi-homogeneous isolated curve singularities, and therefore have the same spectrum). In [10], results and conjectures concerning higher moments of the spectral distribution are discussed.

9.9 The Filtered de Rham Complex and Applications

Let X be a complex algebraic variety of dimension n. Its filtered de Rham complex is a sheaf complex $\underline{\Omega}^{\cdot}_X$ equipped with a filtration F by subcomplexes with the following properties:

1. Its analytic partner $\underline{\Omega}^{\cdot}_X \otimes \mathcal{O}_{X^{\mathrm{an}}}$ is a resolution of the constant sheaf $\mathbb{C}_X$;
2. The differentials in $\underline{\Omega}^{\cdot}_X$ are differential operators of order at most one, and the induced differentials on the quotient complexes $\underline{\Omega}^p_X := \mathrm{Gr}^p_F \underline{\Omega}^{\cdot}_X[p]$ are $\mathcal{O}_X$-linear.

3. If X is complete, then the filtration F induces on $H^k(X,\mathbb{C}) = \mathbb{H}^k(X, \underline{\Omega}^{\cdot}_X)$ the Hodge filtration of its mixed Hodge structure, and the spectral sequence
$$E_1^{p,q} = \mathbb{H}^q(X, \underline{\Omega}^p_X) \Rightarrow H^{p+q}(X,\mathbb{C})$$
degenerates at the E_1-term.
4. There exists a natural morphism of filtered sheaf complexes $(\Omega^{\cdot}_X, F) \to (\underline{\Omega}^{\cdot}_X, F)$, where $(\Omega^{\cdot}_X, F)$ is the complex of Kähler differentials with its usual filtration.

This filtered de Rham complex was defined by Du Bois [6]; it is constructed using hyperresolutions of X and is unique in a suitable filtered derived category. There also exists a filtered de Rham complex for morphisms of varieties $f : Y \to X$, the cone over a morphism $f^* : \underline{\Omega}^{\cdot}_X \to Rf_*\underline{\Omega}^{\cdot}_Y$, which we denote by $\underline{\Omega}^{\cdot}_{X,Y}$.

Example Let E be a variety with normal crossings. Then $\underline{\Omega}^{\cdot}_E \simeq \omega_E$ (see Sect. 9.2.4). The morphism $\Omega^{\cdot}_E \to \omega^{\cdot}_E$ is just dividing out the torsion. If $E \subset Y$ is the inclusion of E as a divisor with normal crossings in a nonsingular variety Y, then one has an isomorphism $\underline{\Omega}^{\cdot}_{Y,E} \simeq \Omega^{\cdot}_Y(\log E)(-E)$. See [60, Proposition 3.3].

Let (X, x) be a singularity, purely of dimension n. Its du Bois invariants are defined using the filtered de Rham complex. We follow the simpler description in [63]. Let Σ denote the singular locus of X. Choose a good resolution $\pi : (Y, E) \to (X, \Sigma)$. Then $\underline{\Omega}^p_{X,\Sigma} = R\pi_*\Omega^p_Y(\log E)(-E)$ for $0 \le p \le n$. These are complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves $R^q\pi_*\Omega^p_Y(\log E)(-E)$. These do not depend on the choice of the good resolution. Two important vanishing properties from [28], see also [60], are:

Theorem 9.9.1 *Let X be an n-dimensional complex projective variety, $\Sigma \subset X$ a subvariety such that $X \setminus \Sigma$ is nonsingular, $\mathcal{L}$ an ample line bundle on X and $\pi : Y \to X$ a proper birational morphism such that Y is nonsingular, $E := \pi^{-1}(\Sigma)$ is a divisor with normal crossings on Y and π maps $Y \setminus E$ isomorphically to $X \setminus \Sigma$. Then*

(a) $H^q(Y, \Omega^p_Y(\log E)(-E) \otimes \pi^*\mathcal{L}) = 0$ *for* $p + q > n$,
(b) $R^q\pi_*\Omega^p_Y(\log E)(-E) = 0$ *for* $p + q > n$.

For a regular point $x \in X$, the complex $\underline{\Omega}^p_{X,x}$ is a resolution of $\Omega^p_{X,x}$. Hence in the isolated singularity case, the cohomology sheaves $\mathcal{H}^q(\underline{\Omega}^p_X)$ have support in x for $q > 0$. Their lengths give rise to the du Bois invariants of (X, x):
$$b^{p,q}(X, x) := \dim_{\mathbb{C}} R^q\pi_*\Omega^p_Y(\log E)(-E)_x.$$

We list some properties of these invariants:

1. $b^{p,q}(X,x) = 0$ when $p+q > n$. This follows from Theorem 9.9.1(b).
2. For a toric isolated singularity (X,x) we have $b^{p,q}(X,x) = 0$ for all p,q.
3. If (X,x) has depth $\geq k$, then $b^{0,q}(X,x) = 0$ for $q < k-1$. See [63, Proposition 1].
4. If (X,x) is an isolated complete intersection singularity, then $b^{p,q}(X,x) = 0$ unless $p+q \in \{n-1, n\}$. See [63, Theorem 5].

As mentioned above, the filtered de Rham complex $(\underline{\Omega}^{\cdot}_X, F)$ is equipped with a morphism of filtered complexes $(\Omega^{\cdot}_X, F) \to (\underline{\Omega}^{\cdot}_X, F)$. In particular one has a morphism $\mathcal{O}_X \to \underline{\Omega}^0_X$. One says that (X,x) is a *du Bois singularity* if this morphism is a quasi-isomorphism, i.e. $\mathcal{H}^0\underline{\Omega}^0_X \simeq \mathcal{O}_X$ and $\mathcal{H}^i\underline{\Omega}^0_X = 0$ for all $i \neq 0$. The first condition is equivalent with weak normality of X. In the isolated singularity case, the second condition means that $b^{0,q}(X,x) = 0$ for all $q > 0$.

In [53] the following characterization of du Bois singularities can be found:

Theorem 9.9.2 *Let X be a reduced separated scheme of finite type over a field of characteristic zero. Embed X in a smooth scheme Y and let $\pi : \tilde{Y} \to Y$ be a log resolution of X in Y that is an isomorphism outside of X. If E is the reduced pre-image of X in $\tilde{Y}$, then X has Du Bois singularities if and only if the natural map $\mathcal{O}_X \to R\pi_*\mathcal{O}_E$ is a quasi-isomorphism.*

By definition, X has *rational singularities* if for some resolution $\pi : Y \to X$ the natural map $\mathcal{O}_X \to \mathbb{R}\pi_*\mathcal{O}_Y$ is a quasi-isomorphism. Rational singularities are du Bois. See [58] for the case of isolated singularities, and [35] for the general case.

By definition, X has *log canonical singularities* if X is normal, its canonical divisor K_X is $\mathbb{Q}$-Cartier and for some good resolution $\pi : Y \to X$ with exceptional divisor $E = \sum_i E_i$ one has $K_Y = f^*(K_X) + \sum_i a_i E_i$ with $a_i \geq -1$ for all i. Log canonical singularities are du Bois [34].

Isolated hypersurface singularities are rational (resp. du Bois) if and only if each spectral number α satsifies $\alpha > 0$ (resp. $\alpha \geq 0$).

The filtered de Rham complex $(\underline{\Omega}, F)$, which is also referred to as the Du Bois complex, has been used in [8] for defining *motivic Chern and Hirzebruch classes* of singular complex algebraic varieties, i.e. characteristic classes analogous to χ_{Hdg}. Moreover, a characteristic class version of the spectrum, termed the *spectral Hirzebruch class*, was introduced in [39] by using vanishing cycles of mixed Hodge modules. For hypersurfaces defined by global functions on smooth varieties, a Thom-Sebastiani type theorem for the spectral classes was obtained in [39], by using a corresponding Thom-Sebastiani theorem for the underlying filtered D-modules of vanishing cycles proved in [40]. Notably, these spectral characteristic classes can be used to detect jumping coefficients of multiplier ideals, Du Bois singularities, and rational singularities for any globally defined hypersurface in a complex manifold.

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Chapter 10
Constructible Sheaf Complexes in Complex Geometry and Applications

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Contents

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Abstract We present a detailed introduction of the theory of constructible sheaf complexes in the complex algebraic and analytic setting. All concepts are illustrated by many interesting examples and relevant applications, while some important results are presented with complete proofs. This paper is intended as a broadly accessible user's guide to these topics, providing the readers with a taste of the subject, reflected by concrete examples and applications that motivate the general theory. We discuss the stability of constructible sheaf complexes under the standard functors, and explain the relation of these functors to perverse sheaves and the perverse t-structure. We introduce the main results of stratified Morse theory in the framework of constructible sheaves, for proving the basic vanishing and finiteness results. Applications are given to various index theorems, the functorial calculus of characteristic cycles of constructible functions, and to weak Lefschetz and Artin-Grothendieck type theorems. We recall the construction of Deligne's nearby and vanishing cycle functors, prove that they preserve constructible complexes, and discuss their relation with the perverse t-structure. We finish this paper with a description and applications of the Kähler package for intersection cohomology of complex algebraic varieties, and the recent study of perverse sheaves on semi-abelian varieties.

10.1 Introduction

The main goal of this paper is to provide a user's guide, both for the novice and the expert, for the theory of (weakly) constructible sheaf complexes in complex geometry and their many applications. Our guiding principle for writing these notes was to provide an explicit and geometric introduction of the mathematical concepts, while also explaining some of the most important examples of the general theory. For this reason, we aim to present as many of the basic results as possible, sometimes even with complete proofs in some special important cases. Moreover, these results and definitions are then always explained and illustrated by many examples.

Constructible (complexes of) sheaves are the algebraic counterpart of the decomposition of a variety into manifolds pieces (strata), and they are, roughly speaking, obtained by gluing local systems defined along strata of a Whitney stratification $\mathscr{S}$. *Perverse sheaves* are an important class of constructible complexes, introduced in [6] as a formalization of the celebrated Riemann-Hilbert correspondence of Kashiwara [64], which relates the topology of algebraic, resp., analytic varieties (intersection homology) and the algebraic, resp., analytic theory of differential equations (holonomic D-modules).

In recent years, constructible sheaf complexes and especially perverse sheaves have become indispensable tools for studying complex algebraic and analytic varieties. They have seen spectacular applications in geometry and topology (e.g., the decomposition theorem [6] and the topology of complex algebraic maps), but also in fields such as representation theory (e.g., proof of the Kazhdan-Lusztig conjecture, proof of the geometrization of the Satake isomorphism, and proof of the fundamental lemma in the Langlands program) or combinatorics (e.g., Stanley's proof of McMullen's conjecture, or the resolution of the Dowling-Wilson top-heavy conjecture); see, e.g., [24, 89] for more recent surveys of such applications. Furthermore, perverse sheaves and the *nearby and vanishing cycle functors* of Deligne [52] are the backbone of Saito's mixed Hodge module theory [105, 106], a far-reaching generalization of Deligne's mixed Hodge theory.

However, despite their fundamental importance, perverse sheaves as special constructible complexes of sheaves remain rather mysterious objects. It is our hope that the present paper will help readers become better acquainted with various aspects of the general theory. Those looking to delve further into more specialized topics or wishing to explore problems of current research will find ample references to facilitate navigation of both classic and recent literature.

Let us next give a brief summary of the content of the paper.

In Sect. 10.2, we define the notion of (weak) constructibility, and discuss the stability of (weakly) constructible sheaf complexes under the standard functors. As far as possible, we allow also weakly constructible sheaf complexes, where one does not impose any finiteness conditions for the stalks (and which is sometimes even more natural or simpler to work with, especially as long as no duality is used). Similarly, we try to work in such a way that it applies to the complex algebraic as well as complex analytic context (sometimes only under suitable compactness assumptions in the complex analytic context). The presented calculus includes *external products, Künneth isomorphisms, Verdier duality* and the relation to the *Euler characteristic calculus of constructible functions*.

We also introduce here the perverse t-structure and perverse sheaves (with respect to middle perversity), and explain their relation with the standard functors (following their counterparts in l-adic cohomology as presented in [6, Chapter 4]). Several aspects of the general theory are worked out in detail for intersection cohomology complexes, which provide some of the main examples of perverse sheaves. Furthermore, for constructible sheaf complexes of R-modules with R a *Dedekind domain*, we also consider the *dual perverse t-structure* and its relation to the *rectified homological depth* of Grothendieck, as studied by Hamm and Lê [56] (also for the corresponding homotopical notion).

In Sect. 10.3, we explain the basic results from [109] about *stratified Morse theory* in the framework of (weakly) constructible sheaves in the *complex context*, continuing and extending the recent survey of Goresky [49] in this handbook series, as well as Massey's survey [85]. We follow here the notions of the geometric stratified Morse theory of Goresky-MacPherson [47], so that one can easily compare the results of our paper with those of *loc.cit.*. We introduce, for example, the sheaf theoretic counterparts of the *local and normal Morse data*, as well as their relations for a *stratified Morse critical point* of a C^{∞}-function on a complex algebraic (or analytic) variety. The normal Morse data of (weakly) constructible sheaf complexes are studied via the *complex link* of a stratum of a Whitney stratification, which allows to prove the basic *vanishing and finiteness theorems* by induction on the dimension of the underlying complex analytic variety. In particular, we get a description of the (dual) perverse t-structure in terms of properties of the normal Morse data. Moreover, we also explain some relations to the general *micro-local sheaf theory* of

Kashiwara-Schapira [66], e.g., like a description of the *micro-support* of a (weakly) constructible sheaf complex in terms of the normal Morse data.

We use the language of *stratified Morse theory for constructible functions and sheaves* to also give in this section an introduction to the functorial theory of *Lagrangian cycles* in the complex analytic and algebraic context. We discuss from this viewpoint the *Euler isomorphism* between constructible functions and Lagrangian cycles given by the *characteristic cycle* of a constructible function, together with some index theorems. Examples are given to *Poincaré-Hopf index theorems for singular spaces*, *effective characteristic cycles on abelian varieties* [4] and the *global Euler obstruction for affine varieties* [116], as well as the famous *local Euler obstruction* of MacPherson [80]. We also explain (using this language of *stratified Morse theory for constructible functions*) the translation into the context of Lagrangian cycles of the following operations for constructible functions and sheaves: external product, proper direct image, non-characteristic pullback and specialization (i.e., nearby cycles), together with an intersection formula for vanishing cycles.

Finally, the last part of Sect. 10.3 deals with applications of the *stratified Morse theory for constructible sheaves* to *vanishing and weak Lefschetz theorems* in the complex algebraic and analytic context. This includes different versions of the *Artin vanishing theorem* for complex algebraically (weakly) constructible complexes on an *affine* variety, and *vanishing theorems* for (weakly) constructible complexes on complex analytic *Stein and q-complete varieties*, as well as relative counterparts for morphisms given by *Artin-Grothendieck types theorems* in the complex algebraic and analytic context.

In Sect. 10.4, we recall the construction of Deligne's *nearby and vanishing cycle functors* [52], and prove that they preserve (weakly) constructible complexes. For the constructible context we also need and explain the calculation of their *(co)stalks* in terms of the *Milnor fibers* for the corresponding local Milnor fibrations [70], based on the existence of an adapted Whitney stratification satisfying the *a_f-condition of Thom* [14, 57, 73]. Then we state the relation of *nearby and vanishing cycle functors* with duality [86], and prove their relation with the (dual) perverse t-structure and perverse sheaves (from the point of view of stratified Morse theory, as developed in the previous section), i.e.,

$$\psi_f[-1] \quad \text{and} \quad \varphi_f[-1] \quad \text{are } t\text{-}exact \text{ for the (dual) perverse t-structure.}$$

We also include here a discussion on the *Thom-Sebastiani theorem* for vanishing cycles [50, 84], and give a brief description of Beilinson's and Deligne-Verdier's procedure for *gluing perverse sheaves* via vanishing cycles [7, 124]. As a final application we explain the equality $Rf_! = Rf_*$ on the level of Grothendieck groups of algebraically constructible complexes [69, 125], as well as analytically constructible functions in a compactifiable complex analytic context [109].

In Sect. 10.5, we give an overview of properties of the *intersection cohomology* groups of complex algebraic varieties, which generalize the corresponding features of the cohomology groups of smooth varieties. These properties, consisting of

Poincaré duality, (weak and hard) Lefschetz theorems and the *decomposition theorem*, are collectively termed the *Kähler package* for intersection cohomology [6, 23, 24, 105, 106].

We also mention briefly a recent combinatorial application of the Kähler package for intersection cohomology, namely a proof (for realizable matroids) by Huh-Wang [59] of the *Dowling-Wilson top-heavy conjecture* [31, 32].

Finally, in Sect. 10.6, we survey recent developments in the study of *perverse sheaves on semi-abelian varieties*. We also include several concrete applications of this theory, e.g., to the study of homotopy types of complex algebraic manifolds (formulated in terms of their cohomology jump loci), as well as new topological characterizations of semi-abelian varieties [74, 76, 77].

We assume reader's familiarity with derived categories and the derived calculus; for a quick refresher on these topics the interested reader may consult [30, 89] or [66, Chapter I–III].

We work in the complex algebraic or analytic setting with reduced Hausdorff spaces, e.g., in the complex algebraic context we are working with the complex analytic space associated to a reduced separated scheme of finite type over Spec($\mathbb{C}$). Unless otherwise specified, all dimensions are taken to be complex dimensions. In the complex analytic setting we always assume that our spaces have bounded dimension and a countable topology, e.g., the disjoint union of all $\mathbb{C}^n$'s ($n \in \mathbb{N}_0$) is not allowed.

10.2 Constructible and Perverse Sheaf Complexes

10.2.1 Constructibility

Let X be a complex algebraic (or analytic) variety (here a variety does not need to be irreducible). It is well known (see, e.g., [47, 121, 122]) that such a variety can be endowed with a *Whitney stratification*, i.e., a (locally) finite partition $\mathscr{S}$ into non-empty, connected, locally closed nonsingular subvarieties S of X (called "strata") which satisfy the following properties:

(i) *Frontier condition*: for any stratum $S \in \mathscr{S}$, the frontier $\partial S := \bar{S} \setminus S$ is a union of strata of $\mathscr{S}$, where $\bar{S}$ denotes the closure of S.
(ii) *Constructibility*: the closure $\bar{S}$ and the frontier ∂S of any stratum $S \in \mathscr{S}$ are closed complex algebraic (respectively, analytic) subspaces in X.

These conditions already imply that X gets an induced dimension filtration

$$X_\bullet : \emptyset := X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

by the closed algebraic (or analytic) subsets $X_i := \bigcup_{\dim S \le i} S$ $(0 \le i \le n = \dim X)$, with the strata S of dimension i given by the connected components of $X_i \setminus X_{i-1}$.

In addition, whenever two strata S_1 and S_2 are such that $S_2 \subseteq \partial S_1$, the pair (S_2, S_1) is required to satisfy the following Whitney b-regularity condition that guarantee that the variety X is topologically or cohomologically equisingular along each stratum (as in [109, Section 4.2]):

(iii) *Whitney b-condition*: If $x_n \in S_1$ and $y_n \in S_2$ are sequences converging to $x \in S_2$ such that the tangent planes $T_{x_n} S_1$ converge to some limiting plane τ and the secant lines $l_n = \overline{x_n, y_n}$ converge to some limiting line l (in some local coordinates), then $l \subset \tau$.

Note that the Whitney b-condition implies the following.

(iv) *Whitney a-condition*: If $x_n \in S_1$ is a sequence converging to $x \in S_2$ such that the tangent planes $T_{x_n} S_1$ converge to some limiting plane τ, then $T_x S_2 \subset \tau$.

These conditions are independent of the choice of local coordinates, and any algebraic (or analytic) partition $\mathscr{S}$ as above has a refinement to an algebraic (or analytic) Whitney stratification (see, e.g., [121, Theorem 1.2 and Proposition 2.1], [122], as well as the references given in [47, Section 1.7]). Moreover, if the Whitney b-conditions hold for all connected components S of the $X_i \setminus X_{i-1}$ and a filtration $X_\bullet$ by closed algebraic (or analytic) subsets as above, then the partition $\mathscr{S}$ with these strata S is (locally) finite and satisfies the frontier condition (see, e.g., [109, Section 4.2.1]) as well as the constructibility condition.

Example 10.2.1 (Whitney Umbrella) The singular locus of the Whitney umbrella

$$X = \{z^2 = xy^2\} \subset \mathbb{C}^3$$

is the x-axis, but the origin is "more singular" than any other point on the x-axis. A Whitney stratification of X is given by the strata

$$S_1 = X \setminus \{x - \text{axis}\}, \quad S_2 = \{(x, 0, 0) \mid x \neq 0\}, \quad S_3 = \{(0, 0, 0)\}.$$

Another example is given by X a complex manifold, with strata S given by its connected components.

In the following, let R be a Noetherian and commutative ring of finite global dimension. Let X be a complex algebraic (or analytic) variety, and denote by $D^b(X; R)$ the derived category of bounded complexes of sheaves of R-modules. By our assumptions on X and R, these bounded derived categories $D^b(-; R)$ are closed under Grothendieck's six operations: Rf_*, $Rf_!$, f^*, $f^!$, $R\mathscr{H}om^\bullet$ and $\overset{L}{\otimes}$, with f an algebraic (or analytic) morphism.

Definition 10.2.2 A sheaf $\mathscr{F}$ of R-modules on X is said to be *weakly constructible* if there is a Whitney stratification $\mathscr{S}$ of X so that the restriction $\mathscr{F}|_S$ of $\mathscr{F}$ to every stratum $S \in \mathscr{S}$ is an R-local system (i.e., a locally constant sheaf). In this case we also say that $\mathscr{F}$ is $\mathscr{S}$*-weakly constructible*. A ($\mathscr{S}$-)weakly constructible sheaf $\mathscr{F}$ on X is called ($\mathscr{S}$-)*constructible*, if all stalks $\mathscr{F}_x$ for $x \in X$ are finitely generated R-modules. A bounded complex $\mathscr{F}^\bullet \in D^b(X; R)$ is called (weakly) constructible if all its cohomology sheaves $\mathscr{H}^j(\mathscr{F}^\bullet)$ are (weakly) constructible. Similarly for $\mathscr{S}$-(weakly) constructible complexes in case one works with a fixed Whitney stratification $\mathscr{S}$.

Note that the category $Sh_{(\mathscr{S}-)wc}(X)$ of ($\mathscr{S}$-)weakly constructible sheaves on X is an abelian subcategory of the category of all sheaves of R-modules on X, which is stable under extensions (like the subcategory of R-local systems). Similarly for the category $Sh_{(\mathscr{S}-)c}(X)$ of ($\mathscr{S}$-)constructible sheaves (like the subcategory of R-local systems with finitely generated stalks), since we assume R to be Noetherian.

Example 10.2.3 On a point space $X = \{pt\}$ any sheaf is weakly constructible. For general X, the constant sheaf R_X (resp., an R-local system $\mathscr{L}$ on X) is (weakly) constructible on X with respect to any Whitney stratification. On the other hand, if $i\colon X \hookrightarrow \mathbb{C}^*$ denotes the inclusion of the closed analytic subset $X := \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{C}^*$, then the direct image sheaf i_*R_X is constructible on $\mathbb{C}^*$. But if $k\colon \mathbb{C}^* \hookrightarrow \mathbb{C}$ is the open inclusion, then the closure $\bar{X} = X \cup \{0\}$ in $\mathbb{C}$ is not analytic and the direct image sheaf $k_*i_*R_X = (k \circ i)_*R_X$ is not weakly constructible on $\mathbb{C}$.

We denote by $D^b_{(w)c}(X; R)$ the full triangulated subcategory of $D^b(X; R)$ consisting of (weakly) constructible complexes (that is, complexes which are (weakly) constructible with respect to *some* Whitney stratification). We will also use the simpler notation $D^b_{(w)c}(X)$ if the coefficient ring R is understood from the context. Similarly for $D^b_{\mathscr{S}-(w)c}(X; R)$ in case we work with (weakly) constructible complexes with respect to a *fixed* Whitney stratification $\mathscr{S}$. By viewing a sheaf as a complex concentrated in degree zero, one gets an isomorphism of Grothendieck groups (and similarly for weakly constructible sheaves)

$$K_0\left(Sh_{(\mathscr{S}-)c}(X)\right) \xrightarrow{\sim} K_0\left(D^b_{(\mathscr{S}-)c}(X; R)\right), \tag{10.1}$$

whose inverse is given by taking the alternating sum of (classes of) cohomology sheaves (see [109, Lemma 3.3.1]).

Remark 10.2.4 Even in the complex analytic setting a sheaf $\mathscr{F}$ (resp., a bounded sheaf complex $\mathscr{F}^\bullet$) on X is (weakly) constructible iff this is locally the case, in the sense that there is an open covering (U_i) of X such that $\mathscr{F}|_{U_i}$ (resp., $\mathscr{F}^\bullet|_{U_i}$) is (weakly) constructible for all i (see [30, Proposition 4.1.13] and compare also with [53, Corollary 3.4]). In fact, the complement of the largest open U of X such $\mathscr{F}|_U$ is locally constant (resp., $\mathscr{F}^\bullet|_U$ has locally constant cohomology sheaves) is then an analytic subset of X (see [30, Proposition 4.1.12]).

The derived category $D^b_{(w)c}(X; R)$ of bounded (weakly) constructible complexes is closed under Grothendieck's six operations: Rf_*, $Rf_!$, f^*, $f^!$, $R\mathscr{H}om^\bullet$ and $\overset{L}{\otimes}$. More precisely, one has the following (e.g., see [11] or the unified treatment of [109, Theorem 4.0.2 and Proposition 4.0.2]):

Theorem 10.2.5 *Let $f: X \to Y$ be a morphism of complex algebraic (or analytic) varieties, with $j: U \hookrightarrow Y$ the inclusion of the open complement of a closed algebraic (or analytic) subset.*

(a) If $\mathscr{G}^\bullet \in D^b_c(Y; R)$, then $f^*\mathscr{G}^\bullet$, $f^!\mathscr{G}^\bullet \in D^b_c(X; R)$.
(b) If $\mathscr{F}^\bullet \in D^b_c(X; R)$ and f is an algebraic map, then $Rf_*\mathscr{F}^\bullet, Rf_!\mathscr{F}^\bullet \in D^b_c(Y; R)$. If $\mathscr{F}^\bullet \in D^b_c(X; R)$ and f is an analytic map so that the restriction of f to supp$(\mathscr{F}^\bullet)$ *is proper (e.g., f is proper), then $Rf_*\mathscr{F}^\bullet \simeq Rf_!\mathscr{F}^\bullet \in D^b_c(Y; R)$.*
(c) If $\mathscr{G}^\bullet \in D^b_c(Y; R)$, then $Rj_*j^*\mathscr{G}^\bullet$, $Rj_!j^*\mathscr{G}^\bullet \in D^b_c(Y; R)$.
(d) If $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^b_c(X; R)$, then $\mathscr{F}^\bullet \overset{L}{\otimes} \mathscr{G}^\bullet$, $R\mathscr{H}om^\bullet(\mathscr{F}^\bullet, \mathscr{G}^\bullet) \in D^b_c(X; R)$.

Similarly for weakly constructible instead of constructible complexes.

In fact, the above theorem in an application of the following more precise version, where one considers a stratified morphism with respect to given Whitney stratifications (e.g., see [11] or [109, Proposition 4.0.2 and Corollary 4.2.1]):

Theorem 10.2.6 *Let $f: X \to Y$ be a stratified morphism of complex algebraic (or analytic) varieties mapping all strata $S \in \mathscr{S}$ of a Whitney stratification of X into strata $T \in \mathscr{T}$ of a Whitney stratification of Y, with $j: U \hookrightarrow Y$ the inclusion of the open complement of a closed union of strata of $\mathscr{T}$ (with its induced Whitney stratification $\mathscr{T}|_U$).*

(a) If $\mathscr{G}^\bullet \in D^b_{\mathscr{T}-c}(Y; R)$, then $f^*\mathscr{G}^\bullet$, $f^!\mathscr{G}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$.
(b) If $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$ and f is a stratified submersion (i.e., it maps all strata $S \in \mathscr{S}$ submersively to a stratum $T \in \mathscr{T}$) so that the restriction of f to supp$(\mathscr{F}^\bullet)$ *is proper (e.g., f is proper), then $Rf_*\mathscr{F}^\bullet \simeq Rf_!\mathscr{F}^\bullet \in D^b_{\mathscr{T}-c}(Y; R)$.*
(c) If $\mathscr{G}^\bullet \in D^b_{\mathscr{T}|_U-c}(U; R)$, then $Rj_*\mathscr{G}^\bullet$, $Rj_!\mathscr{G}^\bullet \in D^b_{\mathscr{T}-c}(Y; R)$.
(d) If $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$, then $\mathscr{F}^\bullet \overset{L}{\otimes} \mathscr{G}^\bullet$, $R\mathscr{H}om^\bullet(\mathscr{F}^\bullet, \mathscr{G}^\bullet) \in D^b_{\mathscr{S}-c}(X; R)$.

Similarly for weakly constructible instead of constructible complexes.

By considering the constant map $c: X \to pt$ to a point, one gets the following.

Example 10.2.7 Let X be a complex algebraic (or analytic) variety with a Whitney stratification $\mathscr{S}$, and let $i: Z \hookrightarrow X$ be the inclusion of a closed algebraic (or analytic) subset given as a union of strata of $\mathscr{S}$ with its induced Whitney stratification $\mathscr{S}|_Z$.

1. The constant sheaf $R_X = c^*R$ is constructible with respect to $\mathscr{S}$, with

$$H^k_c(X; R) = H^k(Rc_!c^*R) \quad \text{resp.,} \quad H^k(X; R) = H^k(Rc_*c^*R) \quad (k \in \mathbb{Z})$$

the corresponding cohomology (with compact support) of X.
2. The sheaf complex $i^!R_X = i^!c^*R$ is constructible with respect to $\mathscr{S}|_Z$, with

$$H^k_Z(X; R) = H^k(Rc_*i^!c^*R) = H^k(X, X\backslash Z; R) \quad (k \in \mathbb{Z})$$

the corresponding cohomology with support in Z of X (or relative cohomology of the pair $(X, X\backslash Z)$).
3. The *dualizing complex* $\mathbb{D}^\bullet_X := c^!R$ of X is constructible with respect to $\mathscr{S}$, with

$$H_k(X; R) = H^{-k}_c(X; \mathbb{D}^\bullet_X) = H^{-k}(Rc_!c^!R) \quad (k \in \mathbb{Z})$$

resp.,

$$H^{BM}_k(X; R) = H^{-k}(X; \mathbb{D}^\bullet_X) = H^{-k}(Rc_*c^!R) \quad (k \in \mathbb{Z})$$

the corresponding (Borel-Moore) homology of X. If X is smooth (or more generally an R-homology manifold) of pure dimension d, then $\mathbb{D}^\bullet_X = c^!R \simeq c^*R[2d]$, which implies the *Poincaré duality*:

$$H_{2d-k}(X; R) \simeq H^k_c(X; R) \quad \text{and} \quad H^{BM}_{2d-k}(X; R) \simeq H^k(X; R) \quad (k \in \mathbb{Z})\,.$$

Corollary 10.2.8 (External Tensor Product) *Let X_i be complex algebraic (or analytic) varieties, with $\mathscr{F}^\bullet_i \in D^b_{(w)c}(X_i; R)$ for $i = 1, 2$. Consider the external tensor product*

$$\mathscr{F}^\bullet_1 \overset{L}{\boxtimes} \mathscr{F}^\bullet_2 := p_1^*\left(\mathscr{F}^\bullet_1\right) \overset{L}{\otimes} p_2^*\left(\mathscr{F}^\bullet_2\right) \in D^b_{(w)c}(X_1 \times X_2; R)\,,$$

with $p_i\colon X_1 \times X_2 \to X_i$ the projection on the corresponding factor for $i = 1, 2$. If $\mathscr{F}^\bullet_i$ is (weakly) constructible with respect to a Whitney stratification $\mathscr{S}_i$ of X_i for $i = 1, 2$, then $\mathscr{F}^\bullet_1 \overset{L}{\boxtimes} \mathscr{F}^\bullet_2$ is (weakly) constructible with respect to the product Whitney stratification $\mathscr{S}_1 \times \mathscr{S}_2$ of $X_1 \times X_2$ (with strata $S_1 \times S_2$ for $S_1 \in \mathscr{S}_1$ and $S_2 \in \mathscr{S}_2$).

The external tensor product of (weakly) constructible complexes behaves nicely with respect to products of two morphisms, as the following result shows (see, e.g., [109, Eq. (1.16) on p.78, Proposition 2.0.1 and Corollary 2.0.4] for more general versions).

Proposition 10.2.9 (Künneth Isomorphisms) *Let $f_k\colon X_k \to Y_k$ be two morphisms of complex algebraic (or analytic) varieties, with $i_k\colon Z_k \hookrightarrow X_k$ the inclusion of a closed complex algebraic (or analytic) subset, and $j_k\colon U_k := X_k \backslash Z_k \hookrightarrow X_k$ the inclusion of the open complement ($k = 1, 2$).*

1. *For any $\mathscr{F}_k^\bullet \in D^b(Y_k; R)$ for $k = 1, 2$ one has:*

$$(f_1 \times f_2)^* \left(\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet \right) \simeq f_1^* \left(\mathscr{F}_1^\bullet \right) \overset{L}{\boxtimes} f_2^* \left(\mathscr{F}_2^\bullet \right) .$$

2. *For any $\mathscr{F}_k^\bullet \in D^b(X_k; R)$ for $k = 1, 2$ one has:*

$$R(f_1 \times f_2)_! \left(\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet \right) \simeq Rf_{1!} \left(\mathscr{F}_1^\bullet \right) \overset{L}{\boxtimes} Rf_{2!} \left(\mathscr{F}_2^\bullet \right) .$$

3. *For any $\mathscr{F}_k^\bullet \in D^b_{wc}(Y_k; R)$ for $k = 1, 2$ one has:*

$$(i_1 \times i_2)^! \left(\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet \right) \simeq i_1^! \left(\mathscr{F}_1^\bullet \right) \overset{L}{\boxtimes} i_2^! \left(\mathscr{F}_2^\bullet \right) .$$

4. *For any $\mathscr{F}_k^\bullet \in D^b_{wc}(X_k; R)$ for $k = 1, 2$ one has in the complex algebraic context:*

$$R(f_1 \times f_2)_* \left(\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet \right) \simeq Rf_{1*} \left(\mathscr{F}_1^\bullet \right) \overset{L}{\boxtimes} Rf_{2*} \left(\mathscr{F}_2^\bullet \right) .$$

5. *For any $\mathscr{F}_k^\bullet \in D^b_{wc}(X_k; R)$ for $k = 1, 2$ one has:*

$$R(j_1 \times j_2)_*(j_1 \times j_2)^* \left(\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet \right) \simeq Rj_{1*}j_1^* \left(\mathscr{F}_1^\bullet \right) \overset{L}{\boxtimes} Rj_{2*}j_2^* \left(\mathscr{F}_2^\bullet \right) .$$

By taking for both morphisms f_k the constant map $f_k = c : X_k \to pt$ to a point space ($k = 1, 2$) one gets the following.

Example 10.2.10 (Classical Künneth Formulae) For any $\mathscr{F}_k^\bullet \in D^b_{wc}(X_k; R)$ for $k = 1, 2$ one has:

$$R\Gamma_c \left(X_1 \times X_2; \mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet \right) \simeq R\Gamma_c \left(X_1; \mathscr{F}_1^\bullet \right) \overset{L}{\otimes} R\Gamma_c \left(X_2; \mathscr{F}_2^\bullet \right) ,$$

$$R\Gamma_{Z_1 \times Z_2} \left(X_1 \times X_2; \mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet \right) \simeq R\Gamma_{Z_1} \left(X_1; \mathscr{F}_1^\bullet \right) \overset{L}{\otimes} R\Gamma_{Z_2} \left(X_2; \mathscr{F}_2^\bullet \right) ,$$

and in the complex algebraic context:

$$R\Gamma\left(X_1 \times X_2; \mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet\right) \simeq R\Gamma\left(X_1; \mathscr{F}_1^\bullet\right) \overset{L}{\otimes} R\Gamma\left(X_2; \mathscr{F}_2^\bullet\right) .$$

In the special case when R is a field, one further gets

$$H_c^*\left(X_1 \times X_2; \mathscr{F}_1^\bullet \boxtimes \mathscr{F}_2^\bullet\right) \simeq H_c^*\left(X_1; \mathscr{F}_1^\bullet\right) \otimes H_c^*\left(X_2; \mathscr{F}_2^\bullet\right) ,$$

and in the complex algebraic context also

$$H^*\left(X_1 \times X_2; \mathscr{F}_1^\bullet \boxtimes \mathscr{F}_2^\bullet\right) \simeq H^*\left(X_1; \mathscr{F}_1^\bullet\right) \otimes H^*\left(X_2; \mathscr{F}_2^\bullet\right) .$$

Another important application of the general calculus of (weakly) constructible complexes deals with *Verdier duality* (see, e.g., [109, Corollary 4.2.2]):

Corollary 10.2.11 *Let X be a complex algebraic (or analytic) variety with a Whitney stratification $\mathscr{S}$, with morphisms $f: Z \to X$ and $g: X \to Y$.*

1. If $\mathscr{F}^\bullet \in D^b(X; R)$ is $\mathscr{S}$-(weakly) constructible, then its Verdier dual

$$\mathcal{D}\mathscr{F}^\bullet := R\mathscr{H}om^\bullet(\mathscr{F}^\bullet, \mathbb{D}_X^\bullet)$$

is also $\mathscr{S}$-(weakly) constructible, with

$$f^!\left(\mathcal{D}\mathscr{F}^\bullet\right) \simeq \mathcal{D}\left(f^*\mathscr{F}^\bullet\right) \quad \textit{resp.,} \quad Rg_*\left(\mathcal{D}\mathscr{F}^\bullet\right) \simeq \mathcal{D}\left(Rg_!\mathscr{F}^\bullet\right) .$$

2. If $\mathscr{F}^\bullet \in D^b(X; R)$ is constructible, then biduality holds:

$$\mathscr{F}^\bullet \simeq \mathcal{D}\mathcal{D}\mathscr{F}^\bullet \tag{10.2}$$

so that $\mathscr{F}^\bullet \in D^b(X; R)$ is ($\mathscr{S}$-)constructible iff its Verdier dual $\mathcal{D}\mathscr{F}^\bullet \in D^b(X; R)$ is ($\mathscr{S}$-)constructible. Moreover

$$f^*\left(\mathcal{D}\mathscr{F}^\bullet\right) \simeq \mathcal{D}\left(f^!\mathscr{F}^\bullet\right) \quad \textit{resp.,} \quad Rg_!\left(\mathcal{D}\mathscr{F}^\bullet\right) \simeq \mathcal{D}\left(Rg_*\mathscr{F}^\bullet\right) .$$

Here, for the last isomorphism one has to assume that $Rg_!\left(\mathcal{D}\mathscr{F}^\bullet\right)$ is constructible (e.g., as in the algebraic context).

Note that already for a point space $X = \{pt\}$ the biduality result uses our assumption that R is a Noetherian and commutative ring of finite global dimension (see also [66, Exercise I.30]). Similarly, Verdier duality commutes for constructible sheaf complexes with external tensor products (see, e.g., [109, Corollary 2.0.4] in a more general context).

Proposition 10.2.12 *Let X_i be a complex algebraic (or analytic) variety with $\mathscr{F}_i^\bullet \in D_c^b(X; R)$ for $i = 1, 2$. Then*

$$\mathcal{D}\left(\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet\right) \simeq \mathcal{D}\left(\mathscr{F}_1^\bullet\right) \overset{L}{\boxtimes} \mathcal{D}\left(\mathscr{F}_2^\bullet\right) .$$

In particular, $\mathbb{D}^\bullet_{X_1 \times X_2} \simeq \mathbb{D}^\bullet_{X_1} \overset{L}{\boxtimes} \mathbb{D}^\bullet_{X_2}$, *so that one gets the classical Künneth formula for homology, e.g., for R a field:*

$$H_*(X_1 \times X_2; R) \simeq H_*(X_1; R) \otimes H_*(X_2; R) ,$$

and in the algebraic context also for Borel Moore homology, e.g., for R a field:

$$H_*^{BM}(X_1 \times X_2; R) \simeq H_*^{BM}(X_1; R) \otimes H_*^{BM}(X_2; R) .$$

The general calculus of constructible sheaves also includes finiteness results for the cohomology (with compact support) of constructible sheaf complexes.

Corollary 10.2.13 *Assume that $\mathscr{F}^\bullet \in D_c^b(X; R)$ and that either*

(a) X is a complex algebraic variety, or
(b) X is an analytic space and $\mathrm{supp}(\mathscr{F}^\bullet)$ *is compact.*

Then the hypercohomology groups $H^i(X; \mathscr{F}^\bullet)$ and $H_c^i(X; \mathscr{F}^\bullet)$ are finite type R-modules for every $i \in \mathbb{Z}$ (which are zero for $|i|$ large enough).

(c) Assume X is a compact analytic space with a Whitney stratification $\mathscr{S}$ so that $j : U \hookrightarrow X$ is the inclusion of the open complement of a closed union of strata. If $\mathscr{G}^\bullet \in D^b_{\mathscr{S}|_U - c}(U; R)$, then also the hypercohomology groups $H^i(U; \mathscr{G}^\bullet)$ and $H_c^i(U; \mathscr{G}^\bullet)$ are finite type R-modules for every $i \in \mathbb{Z}$ (which are zero for $|i|$ large enough).

With $\mathscr{F}^\bullet \in D_c^b(X; R)$ as in the above corollary, we make the following.

Definition 10.2.14 Assume R is a field. The *(compactly supported) Euler characteristic of* $\mathscr{F}^\bullet \in D_c^b(X; R)$ is defined as:

$$\chi_{(c)}(X, \mathscr{F}^\bullet) := \chi(H^*_{(c)}(X; \mathscr{F}^\bullet)) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_R H^i_{(c)}(X; \mathscr{F}^\bullet).$$

(Here, we use the notation $\chi_{(c)}$ and $H^i_{(c)}$ to indicate that the definition applies to the compactly supported Euler characteristic χ_c by using H_c^i, as well as to the usual Euler characteristic χ by using H^i.)

As it will be explained later on (see Example 10.4.37, and also [109, Section 2.3 and Section 6.0.6]), in this complex context we have the equality:

$$\chi(X, \mathscr{F}^\bullet) = \chi_c(X, \mathscr{F}^\bullet). \tag{10.3}$$

Moreover, this Euler characteristic depends only on the associated constructible function

$$\chi_{stalk}(\mathscr{F}^\bullet) \in CF(X) \tag{10.4}$$

given by the stalkwise Euler characteristic $\chi_{stalk}(\mathscr{F}^\bullet)(x) := \chi\left(\mathscr{F}_x^\bullet\right)$ for $x \in X$, with $CF(X)$ the corresponding abelian group of constructible functions given by (locally) finite $\mathbb{Z}$-linear combinations of indicator functions 1_Z, for $Z \subset X$ a closed irreducible algebraic (or analytic) subset of X. Similarly for the abelian group $CF_{\mathscr{S}}(X)$ of $\mathscr{S}$-constructible functions given by (locally) finite $\mathbb{Z}$-linear combinations of indicator functions 1_S or $1_{\bar{S}}$ for $S \in \mathscr{S}$ a stratum, i.e., $\mathbb{Z}$-valued functions which are constant on all strata $S \in \mathscr{S}$. This induces a surjective group homomorphism

$$\chi_{stalk} \colon\ K_0\left(D^b_{(\mathscr{S}-)c}(X; R)\right) \to CF_{(\mathscr{S})}(X)\,, \tag{10.5}$$

with

$$\chi_c(X, \mathscr{F}^\bullet) = \sum_{S \in \mathscr{S}} \chi_c(S) \cdot \chi_{stalk}(\mathscr{F}^\bullet)(S) \tag{10.6}$$

for an $\mathscr{S}$-constructible complex $\mathscr{F}^\bullet$ on X in the complex algebraic context (or analytic context with X compact). Here $\chi_c(S) := \chi_c(R_S) = \chi(H^*_c(S; R))$ is the corresponding Euler characteristic of a stratum $S \in \mathscr{S}$.

Let us also mention here the following (co)stalk calculation.

Proposition 10.2.15 *Let $\mathscr{F}^\bullet \in D^b_c(X; R)$, $x \in X$, and $i_x \colon \{x\} \hookrightarrow X$ the inclusion. Then*

$$\mathscr{H}^j(\mathscr{F}^\bullet)_x \simeq H^j(i_x^*\mathscr{F}^\bullet) \simeq H^j(\mathring{B}_{\epsilon,x}; \mathscr{F}^\bullet), \tag{10.7}$$

$$H^j(i_x^!\mathscr{F}^\bullet) \simeq H^j_c(\mathring{B}_{\epsilon,x}; \mathscr{F}^\bullet) \simeq H^j(\mathring{B}_{\epsilon,x}, \mathring{B}_{\epsilon,x} \setminus x; \mathscr{F}^\bullet), \tag{10.8}$$

where $\mathring{B}_{\epsilon,x}$ is the intersection of X with an open small ϵ-ball neighborhood of x in some local embedding of X in $\mathbb{C}^N$. Here, $i_x^*\mathscr{F}^\bullet$ and $i_x^!\mathscr{F}^\bullet$ are called the stalk and, respectively, costalk of $\mathscr{F}^\bullet$ at x.

In fact, this will be an easy application of the Morse theoretical results explained later on, since the proper real analytic function r given by the squared distance to x (in these local coordinates) has no stratified critical values in an interval $]0, \epsilon[$ for $\epsilon > 0$ small enough (see Lemma 10.3.3 and, e.g., [109, Lemma 5.1.1]).

10.2.2 Perverse Sheaves

Perverse sheaves are an important class of constructible complexes, introduced in [6] as a formalization of Kashiwara's Riemann–Hilbert correspondence [64] (see also [58]), which relates the topology of complex algebraic, resp., analytic varieties (intersection homology) and the algebraic, resp., analytic theory of differential equations (holonomic D-modules). We recall their definition below.

Definition 10.2.16

(a) The *perverse t-structure* on $D^b_c(X; R)$ consists of the two strictly full subcategories ${}^pD^{\leq 0}(X; R)$ and ${}^pD^{\geq 0}(X; R)$ of $D^b_c(X; R)$ defined as:

$$ {}^pD^{\leq 0}(X; R) := \{\mathscr{F}^\bullet \in D^b_c(X; R) \mid \dim \operatorname{supp}^{-j}(\mathscr{F}^\bullet) \leq j, \forall j \in \mathbb{Z}\}, $$

$$ {}^pD^{\geq 0}(X; R) := \{\mathscr{F}^\bullet \in D^b_c(X; R) \mid \dim \operatorname{cosupp}^{j}(\mathscr{F}^\bullet) \leq j, \forall j \in \mathbb{Z}\}, $$

where, for $i_x : \{x\} \hookrightarrow X$ denoting the point inclusion, we define the *j-th support* and, respectively, the *j-th cosupport* of $\mathscr{F}^\bullet \in D^b_c(X; R)$ by:

$$ \operatorname{supp}^j(\mathscr{F}^\bullet) = \overline{\{x \in X \mid H^j(i_x^*\mathscr{F}^\bullet) \neq 0\}}, $$

$$ \operatorname{cosupp}^j(\mathscr{F}^\bullet) = \overline{\{x \in X \mid H^j(i_x^!\mathscr{F}^\bullet) \neq 0\}}. $$

(For a constructible complex $\mathscr{F}^\bullet$, the sets $\operatorname{supp}^j(\mathscr{F}^\bullet)$ and $\operatorname{cosupp}^j(\mathscr{F}^\bullet)$ are closed algebraic (or analytic) subvarieties of X, hence their dimensions are well defined.)

(b) For a given Whitney stratification $\mathscr{S}$ of X this also induces the *perverse t-structure* on $D^b_{\mathscr{S}-c}(X; R)$ with ${}^pD^{\leq 0}_{\mathscr{S}}(X; R)$, resp., ${}^pD^{\geq 0}_{\mathscr{S}}(X; R)$ defined by the same (co)support conditions.

(c) A ($\mathscr{S}$-)constructible complex $\mathscr{F}^\bullet \in D^b_{(\mathscr{S}-)c}(X; R)$ is called a *perverse sheaf* on X if

$$ \mathscr{F}^\bullet \in Perv_{(\mathscr{S})}(X; R) := {}^pD^{\leq 0}_{(\mathscr{S})}(X; R) \cap {}^pD^{\geq 0}_{(\mathscr{S})}(X; R). $$

The category of ($\mathscr{S}$-constructible) perverse sheaves is the *heart* of the perverse t-structure, hence it is an abelian category, and it is stable by extensions (see, e.g., [6, Theorem 1.3.6]).

Remark 10.2.17

(a) The same definition also defines the *perverse t-structure* for ($\mathscr{S}$-) weakly constructible complexes with heart the abelian category of ($\mathscr{S}$-)weakly constructible perverse sheaves.
(b) An algebraically (weakly) constructible complex is perverse if and only if it is so when viewed as an analytically (weakly) constructible complex.
(c) If R is a *field*, the Universal Coefficient Theorem can be used to show that the Verdier duality functor $\mathcal{D}\colon D^b_c(X;R) \to D^b_c(X;R)$ satisfies:

$$\operatorname{cosupp}^j(\mathscr{F}^\bullet) = \operatorname{supp}^{-j}(\mathcal{D}\mathscr{F}^\bullet), \tag{10.9}$$

In particular, $\mathcal{D}$ exchanges ${}^pD^{\leq 0}(X;R)$ and ${}^pD^{\geq 0}(X;R)$, so that it preserves ($\mathscr{S}$-) constructible perverse sheaves with field coefficients.

Recall here that the condition that two subcategories ${}^pD^{\leq 0}(X;R)$ and ${}^pD^{\geq 0}(X;R)$ of $D^b_{(\mathscr{S}-w)c}(X;R)$ define a *t-structure* just means (see, e.g., [6, Definition 1.3.1]):

1. $\operatorname{Hom}_{D^b(X;R)}(\mathscr{F}^\bullet, \mathscr{G}^\bullet[-1]) = 0$ for all $\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X;R)$ and $\mathscr{G}^\bullet \in {}^pD^{\geq 0}(X;R)$.
2. ${}^pD^{\leq 0}(X;R)$ is stable under $[1]$, and ${}^pD^{\geq 0}(X;R)$ is stable under $[-1]$.
3. For any $\mathscr{E}^\bullet \in D^b_{(\mathscr{S}-w)c}(X;R)$ there is a distinguished triangle

$$\mathscr{F}^\bullet \longrightarrow \mathscr{E}^\bullet \longrightarrow \mathscr{G}^\bullet[-1] \xrightarrow{[1]}$$

for some $\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X;R)$ and $\mathscr{G}^\bullet \in {}^pD^{\geq 0}(X;R)$.

Then it is enough to check these properties for a fixed Whitney stratification $\mathscr{S}$, where they can be proved by induction on $\dim X$ via the *gluing of t-structures* as in [6, Corollary 2.1.4, Proposition 2.1.14]. Here it is important to note that the conditions $\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X;R)$ and, resp., $\mathscr{F}^\bullet \in {}^pD^{\geq 0}(X;R)$ can also be described in terms of a fixed Whitney stratification of X for which $\mathscr{F}^\bullet$ is $\mathscr{S}$-(weakly) constructible. Indeed, the perverse t-structure can be characterized as follows:

Theorem 10.2.18 *Assume $\mathscr{F}^\bullet \in D^b_{(w)c}(X;R)$ is (weakly) constructible with respect to a Whitney stratification $\mathscr{S}$ of X. Then:*

(i) stalk vanishing:

$$\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X;R) \iff \forall S \in \mathscr{S},\ \forall x \in S:\ H^j(i_x^*\mathscr{F}^\bullet) = 0 \textit{ for all } j > -\dim S.$$

(ii) costalk vanishing:

$$\mathscr{F}^\bullet \in {}^pD^{\geq 0}(X;R) \iff \forall S \in \mathscr{S},\ \forall x \in S:\ H^j(i_x^!\mathscr{F}^\bullet) = 0 \ \textit{for all } j < \dim S.$$

Of course the isomorphism class of the stalk $i_x^*\mathscr{F}^\bullet$ for $x \in S$ as above does not depend on the choice of the point $x \in S$. In fact, if $i_S\colon S \hookrightarrow X$ denotes the inclusion of the stratum S, the stalk vanishing condition is equivalent to

(i') $\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X;R) \iff \forall S \in \mathscr{S}\colon \mathscr{H}^j(i_S^*\mathscr{F}^\bullet) = 0$ for all $j > -\dim S$.

Similarly for the costalk $i_x^!\mathscr{F}^\bullet$ for $x \in S$, since this costalk is also isomorphic to $k_x^! i_S^! \mathscr{F}^\bullet$ with $k_x\colon \{x\} \hookrightarrow S$ the inclusion of the point $\{x\}$ into S. But $i_S^!\mathscr{F}^\bullet$ has locally constant cohomology sheaves so that $k_x^! i_S^!\mathscr{F}^\bullet[2\dim S] \simeq k_x^* i_S^!\mathscr{F}^\bullet$ for all $x \in S$. And then the costalk condition is equivalent to

(ii') $\mathscr{F}^\bullet \in {}^pD^{\geq 0}(X;R) \iff \forall S \in \mathscr{S}\colon \mathscr{H}^j(i_S^!\mathscr{F}^\bullet) = 0$ for all $j < -\dim S$.

Example 10.2.19 Assume X is of pure complex dimension with $c\colon X \to pt$ the constant map to a point space. As usual, for an R-module M we denote by $M_X = c^*M$ the constant R-sheaf on X with stalk $M_x = M$ for all $x \in X$. Then:

(a) $M_X[\dim X] \in {}^pD^{\leq 0}(X;R)$. More generally, if $\mathscr{L}$ is a local system on X, then $\mathscr{L}[\dim X] \in {}^pD^{\leq 0}(X;R)$ (since this is a local condition).
(b) If X is smooth with the trivial stratification, and $\mathscr{L}$ is a local system on X, then $\mathscr{L}[\dim X]$ is perverse on X (since in this case $i_x^!\mathscr{L}[2\dim X] \simeq i_x^*\mathscr{L}$ for all $x \in X$).
(c) The *intersection* (IC) *complexes* on X of Goresky-MacPherson [46] (for the middle perversity) are examples of perverse sheaves. For a given Whitney stratification $\mathscr{S}$ of X these are defined by further imposing, for all strata $S \in \mathscr{S}$ with $\dim S < \dim X$, the stronger stalk/costalk vanishing conditions in Theorem 10.2.18 obtained by replacing $>$ with $\geq$ and $<$ with $\leq$. Such an IC complex on X is $\mathscr{S}$-weakly constructible and determined by its restriction to the top dimensional strata, which is (isomorphic to) a shifted local system $\mathscr{L}[\dim X]$, so we may denote it unambiguously by $IC_X(\mathscr{L})$. It is $\mathscr{S}$-constructible if $\mathscr{L}$ has finitely generated stalks. If $\mathscr{L}$ is the constant sheaf R on the top dimensional strata, we use the notation IC_X. In the end, these perverse IC-complexes do not depend on the chosen Whitney stratification $\mathscr{S}$, but only on the *generically defined* local system $\mathscr{L}$ (on the complement of a closed algebraic or analytic subset of X of dimension smaller than $\dim X$).
(d) If X is a local complete intersection then $R_X[\dim X]$ is a perverse sheaf on X (see Example 10.4.29 and, e.g., [109, Example 6.0.11]). More generally, if $\mathscr{L}$ is a local system on X, then $\mathscr{L}[\dim X]$ is perverse on X. By (a) one only has to show that $\mathscr{L}[\dim X] \in {}^pD^{\geq 0}(X;R)$. Since this is a local condition in the classical topology, we can assume $\mathscr{L} = M_X$ is a constant sheaf, with $i\colon X \hookrightarrow X'$ the inclusion of a closed analytic subset defined as the zero set of k holomorphic functions on a pure dimensional complex manifold X' with $\dim X' - k = \dim X$. Then $M_X = i^*M_{X'}$ with $M_{X'}[\dim X'] \in {}^pD^{\geq 0}(X';R)$ by (b), so that the claim follows from Proposition 10.4.28 below (see, e.g., [109, Proposition 6.0.2]).

To simplify the formulation of some results, let us recall the following.

Definition 10.2.20 For $n \in \mathbb{Z}$ one defines the following two strictly full subcategories ${}^pD^{\leq n}_{(\mathscr{S})}(X; R)$ and ${}^pD^{\geq n}_{(\mathscr{S})}(X; R)$ of $D^b_{(\mathscr{S}-)c}(X; R)$, resp., $D^b_{(\mathscr{S}-)wc}(X; R)$ by

$${}^pD^{\leq n}_{(\mathscr{S})}(X; R) := {}^pD^{\leq 0}_{(\mathscr{S})}(X; R)[-n] \quad \text{and} \quad {}^pD^{\geq n}_{(\mathscr{S})}(X; R) := {}^pD^{\geq 0}_{(\mathscr{S})}(X; R)[-n]$$

so that ${}^pD^{\leq -1}_{(\mathscr{S})}(X; R) \subset {}^pD^{\leq 0}_{(\mathscr{S})}(X; R)$ and ${}^pD^{\geq 1}_{(\mathscr{S})}(X; R) \subset {}^pD^{\geq 0}_{(\mathscr{S})}(X; R)$.

Remark 10.2.21 The two subcategories ${}^pD^{\leq n}_{(\mathscr{S})}(X; R)$ and ${}^pD^{\geq n}_{(\mathscr{S})}(X; R)$ define a *shifted perverse t-structure* on $D^b_{(\mathscr{S}-)c}(X; R)$ and $D^b_{(\mathscr{S}-)wc}(X; R)$. But for most of the results of this paper (especially those proved by stratified Morse theory later on), we only need the following obvious properties (as in [109, Chapter VI]):

1. The zero object belongs to ${}^pD^{\leq n}_{(\mathscr{S})}(X; R)$ and ${}^pD^{\geq n}_{(\mathscr{S})}(X; R)$.
2. ${}^pD^{\leq n}_{(\mathscr{S})}(X; R)$ and ${}^pD^{\geq n}_{(\mathscr{S})}(X; R)$ are stable by extensions.
3. ${}^pD^{\leq n}(X; R)$ is stable under $[1]$, and ${}^pD^{\geq n}(X; R)$ is stable under $[-1]$.

Example 10.2.22 On a point space $X = \{pt\}$ one gets:

1. ${}^pD^{\leq n}(\{pt\}; R)$ is given by the bounded complexes of R-modules, whose cohomology is concentrated in degree $\leq n$. This condition is stable under shifting a complex to the left (i.e., the shift $[1]$).
2. ${}^pD^{\geq n}(\{pt\}; R)$ is given by the bounded complexes of R-modules, whose cohomology is concentrated in degree $\geq n$. This condition is stable under shifting a complex to the right (i.e., the shift $[-1]$).

Lemma 10.2.23 *Let $f : X \to Y$ be a morphism of complex algebraic (or analytic) varieties, whose fiber dimension is bounded by $d \in \mathbb{N}_0$. Then:*

(a) $f^!$ maps ${}^pD^{\geq n}(Y; R)$ into ${}^pD^{\geq n-d}(X; R)$.
(b) f^* maps ${}^pD^{\leq n}(Y; R)$ into ${}^pD^{\leq n+d}(X; R)$.

This follows from the definitions using $i^!_x f^! \simeq i^!_{f(x)}$ and $i^*_x f^* \simeq i^*_{f(x)}$ for all $x \in X$. As an example, one can take for f a locally closed inclusion or an unramified covering map, both of which have fiber dimension $d = 0$.

Example 10.2.24 Let X be a complex algebraic (or analytic) variety of dimension $d = \dim X$, with $c : X \to pt$ a constant map. Then

$$\mathbb{D}^\bullet_X = c^! R \in {}^pD^{\geq -d}(X; R) \quad \text{and} \quad R_X = c^* R \in {}^pD^{\leq d}(X; R)\,.$$

Corollary 10.2.25 *Let $f: X \to Y$ be a smooth morphism (i.e., a submersion) of complex algebraic (or analytic) varieties, with constant relative (or fiber) dimension $d \in \mathbb{N}_0$. Then $f^! \simeq f^*[2d]$ so that*

(a) f^* maps ${}^pD^{\geq n}(Y; R)$ into ${}^pD^{\geq n+d}(X; R)$.
(b) $f^!$ maps ${}^pD^{\leq n}(Y; R)$ into ${}^pD^{\leq n-d}(X; R)$.

In particular $f^![-d] \simeq f^*[d]$ maps $Perv(Y; R)$ into $Perv(X; R)$.

Example 10.2.26 Let $f: X \to Y$ be a *smooth* morphism (i.e., a submersion) of complex algebraic (or analytic) varieties, with constant relative (or fiber) dimension d. Assume f is surjective and Y (and then also X) is pure dimensional. Then

$$f^* IC_Y(\mathscr{L})[d] \simeq IC_X(f^*\mathscr{L})$$

for a generically defined local system $\mathscr{L}$ on Y, with $f^*\mathscr{L}$ the corresponding generically defined local system on X defined by pullback.

The following result will be very important for the applications of the *stratified Morse theory for constructible sheaves* in the next sections.

Proposition 10.2.27 *Let $Y \hookrightarrow M$ be a closed complex algebraic (or analytic) subvariety of an ambient complex algebraic (or analytic) manifold M. Assume $N \hookrightarrow M$ is a closed complex algebraic (or analytic) submanifold of constant codimension $d = \dim M - \dim N$, which is transversal to a Whitney stratification $\mathscr{S}$ of Y (i.e., N is transversal to all strata $S \in \mathscr{S}$). Then $X := Y \cap N$ gets an induced Whitney stratification $\mathscr{S}'$ with strata S' the connected components of the intersections $S \cap N$ for $S \in \mathscr{S}$, with $\dim S \cap N = \dim S - d$ for all $S \in \mathscr{S}$ (and $S \cap N \neq \emptyset$). Let $i: X = Y \cap N \hookrightarrow Y$ be the (stratified) closed inclusion. Then*

(a) i^* maps ${}^pD^{\leq n}_{\mathscr{S}}(Y; R)$ into ${}^pD^{\leq n-d}_{\mathscr{S}'}(X; R)$.
(b) i^* maps ${}^pD^{\geq n}_{\mathscr{S}}(Y; R)$ into ${}^pD^{\geq n-d}_{\mathscr{S}'}(X; R)$.

In particular $i^*[-d]$ maps $Perv_{\mathscr{S}}(Y; R)$ into $Perv_{\mathscr{S}'}(X; R)$.

Proof Consider the following cartesian diagram of closed inclusions for $S \in \mathscr{S}$:

$$\begin{array}{ccc} S' := S \cap N & \xrightarrow{i_{S'}} & X = Y \cap N \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ S & \xrightarrow{i_S} & Y\,. \end{array}$$

Then (a) follows from $i^*_{S'}i^* \simeq i'^*i^*_S$ for checking the stalk vanishing condition (i'). Similarly (b) follows from the *base change isomorphism*

$$i^!_{S'}i^*\mathscr{F}^\bullet \simeq i'^*i^!_S\mathscr{F}^\bullet \quad \text{for } \mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X;R) \tag{10.10}$$

for checking the costalk vanishing condition (ii'). □

Example 10.2.28 Consider the context of Proposition 10.2.27, with Y (and therefore also $X = Y \cap N$) pure dimensional. Let $\mathscr{L}$ be a local system on the open subset U of Y given by the top dimensional stratum of $\mathscr{S}$ (which is then dense in Y). Similarly $U \cap N$ is open and dense in X. Then

$$i^*IC_Y(\mathscr{L})[-d] \simeq IC_X(i'^*\mathscr{L}) ,$$

with $i' \colon U \cap N \to U$ the induced inclusion.

In the base change isomorphism (10.10) used above, we can even assume that $S \hookrightarrow Y$ is a *closed* stratum, by restriction to the open complement of ∂S. Then it is a special case (with $Y' = S$) of the following more general result (see, e.g., [109, Proposition 4.3.1 and Remark 4.3.6]).

Theorem 10.2.29 (Base Change Isomorphisms) *Let $Y \hookrightarrow M$ be a closed complex algebraic (or analytic) subvariety of an ambient complex algebraic (or analytic) manifold M. Assume $N \hookrightarrow M$ is a closed complex algebraic (or analytic) submanifold, which is transversal to a Whitney stratification $\mathscr{S}$ of Y. Let $i \colon Y' \hookrightarrow Y$ be the inclusion of a closed union of strata of $\mathscr{S}$, with $j \colon U := Y \backslash Y' \hookrightarrow Y$ the inclusion of the open complement with its induced stratification $\mathscr{S}|_U$. Consider the cartesian diagram*

$$\begin{array}{ccccc} X' := Y' \cap N & \xrightarrow{i'} & X := Y \cap N & \xleftarrow{j'} & U' := U \cap N \\ \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k''} \\ Y' & \xrightarrow{i} & Y & \xleftarrow{j} & U\,. \end{array}$$

Then one has the following base change isomorphisms:

$$k^*Rj_*\mathscr{F}^\bullet \simeq Rj'_*k''^*\mathscr{F}^\bullet \quad \textit{for } \mathscr{F}^\bullet \in D^b_{\mathscr{S}|_U-wc}(U;R) \tag{10.11}$$

and

$$k'^*i^!\mathscr{F}^\bullet \simeq i'^!k^*\mathscr{F}^\bullet \quad \textit{for } \mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(Y;R). \tag{10.12}$$

Next we study the relation between the *perverse t-structure* and *external tensor products*.

Proposition 10.2.30 *Let X_i be complex algebraic (or analytic) varieties ($i = 1, 2$). Then*

(a) The external tensor product $\overset{L}{\boxtimes}$ induces

$$\overset{L}{\boxtimes}:\ {}^pD^{\leq n}(X_1; R) \times {}^pD^{\leq m}(X_2; R) \to {}^pD^{\leq n+m}(X_1 \times X_2; R)\,.$$

(b) Assume R is a field. Then $\overset{L}{\boxtimes}$ induces

$$\overset{L}{\boxtimes}:\ {}^pD^{\geq n}(X_1; R) \times {}^pD^{\geq m}(X_2; R) \to {}^pD^{\geq n+m}(X_1 \times X_2; R)\,.$$

In particular, if R is a field, $\overset{L}{\boxtimes}$ induces

$$\overset{L}{\boxtimes}:\ Perv(X_1; R) \times Perv(X_2; R) \to Perv(X_1 \times X_2; R)\,.$$

Property (a) follows from $i^*_{(x_1,x_2)}(- \overset{L}{\boxtimes} -) \simeq i^*_{x_1}(-) \overset{L}{\otimes} i^*_{x_2}(-)$ and the right exactness of the tensor product $\otimes$. Property (b) is a consequence of the Künneth isomorphism $i^!_{(x_1,x_2)}(- \overset{L}{\boxtimes} -) \simeq i^!_{x_1}(-) \overset{L}{\otimes} i^!_{x_2}(-)$ and the exactness of the tensor product $\otimes$ for R a field.

Example 10.2.31 Let X_i be pure dimensional complex algebraic (or analytic) varieties ($i = 1, 2$), with R a field. Then

$$IC_{X_1}(\mathscr{L}_1) \overset{L}{\boxtimes} IC_{X_2}(\mathscr{L}_2) \simeq IC_{X_1 \times X_2}(\mathscr{L}_1 \boxtimes \mathscr{L}_2)$$

for a generically defined local system $\mathscr{L}_i$ on X_i ($i = 1, 2$).

Example 10.2.32 Let X be a complex algebraic (or analytic) variety, with R a field. If $\mathscr{F}^\bullet \in Perv(X; R)$ and $\mathscr{L}$ is a locally constant sheaf on X, then $\mathscr{F}^\bullet \otimes \mathscr{L} \in Perv(X; R)$. In particular, if X is also pure dimensional, one gets:

$$IC_X(\mathscr{L}_1) \otimes \mathscr{L}_2 \simeq IC_X(\mathscr{L}_1 \otimes \mathscr{L}_2),$$

with $\mathscr{L}_1$ a generically defined local system on X, and $\mathscr{L}_2$ a local system on all of X. As a special case, one also gets:

$$IC_X \otimes \mathscr{L} \simeq IC_X(\mathscr{L})$$

if X is pure dimensional and $\mathscr{L}$ is a local system on X. These assertions can be checked locally in the analytic topology (so it suffices to assume that $\mathscr{L}_2$ is a constant sheaf), by using Proposition 10.2.30 and Example 10.2.31 in which we take X_2 to be a point space.

Corollary 10.2.33 *Let $X_i \hookrightarrow M$ be closed complex algebraic (or analytic) subvarieties of the complex algebraic (or analytic) manifold M of pure dimension d $(i = 1, 2)$. Let $\mathscr{S}_i$ be Whitney stratifications of X_i $(i = 1, 2)$ which are transversal in M, i.e., all strata $S_1 \in \mathscr{S}_1$ are transversal to all strata $S_2 \in \mathscr{S}_2$. This is equivalent to the diagonal embedding $\Delta\colon M \to M \times M$ being transversal to the product stratification $\mathscr{S}_1 \times \mathscr{S}_2$ of $X_1 \times X_2$. In particular, $X_1 \cap X_2 \simeq (X_1 \times X_2) \cap \Delta(M)$ gets an induced Whitney stratification $\mathscr{S}_1 \cap \mathscr{S}_2$ with strata the connected components of the intersections $S_1 \cap S_2$. Then one gets for the induced map $\Delta\colon X_1 \cap X_2 \to X_1 \times X_2$ the following:*

(a) The tensor product $- \overset{L}{\otimes} - \simeq \Delta^*(- \overset{L}{\boxtimes} -)$ induces

$$\overset{L}{\otimes}\colon\ {}^pD^{\leq n}_{\mathscr{S}_1}(X_1; R) \times {}^pD^{\leq m}_{\mathscr{S}_2}(X_2; R) \to {}^pD^{\leq n+m-d}_{\mathscr{S}_1 \cap \mathscr{S}_2}(X_1 \cap X_2; R)\,.$$

(b) Assume R is a field. Then the tensor product $- \overset{L}{\otimes} - \simeq \Delta^*(- \overset{L}{\boxtimes} -)$ induces

$$\overset{L}{\otimes}\colon\ {}^pD^{\geq n}_{\mathscr{S}_1}(X_1; R) \times {}^pD^{\geq m}_{\mathscr{S}_2}(X_2; R) \to {}^pD^{\geq n+m-d}_{\mathscr{S}_1 \cap \mathscr{S}_2}(X_1 \cap X_2; R)\,.$$

In particular, if R is a field, $\left(- \overset{L}{\otimes} -\right)[-d]$ induces

$$Perv_{\mathscr{S}_1}(X_1; R) \times Perv_{\mathscr{S}_2}(X_2; R) \to Perv_{\mathscr{S}_1 \cap \mathscr{S}_2}(X_1 \cap X_2; R)\,.$$

Example 10.2.34 In the context of Corollary 10.2.33, let X_i be in addition pure dimensional $(i = 1, 2)$, with R a field. Then

$$\left(IC_{X_1}(\mathscr{L}_1) \overset{L}{\otimes} IC_{X_2}(\mathscr{L}_2)\right)[-d] \simeq IC_{X_1 \cap X_2}(\mathscr{L}_1 \otimes \mathscr{L}_2)$$

for a local system $\mathscr{L}_i$ defined on the open dense subset U_i of X_i given by the top dimensional stratum $(i = 1, 2)$. Here $X_1 \cap X_2$ is also pure dimensional with $\mathscr{L}_1 \otimes \mathscr{L}_2$ defined on the open dense subset $U_1 \cap U_2$ of $X_1 \cap X_2$.

The existence of the perverse t-structure on $D^b_{(w)c}(X; R)$ implies the existence of *perverse truncation functors* ${}^p\tau_{\leq 0}$, ${}^p\tau_{\geq 0}$, which are adjoint to the inclusions

$${}^pD^{\leq 0}(X; R) \hookrightarrow D^b_{(w)c}(X; R) \hookleftarrow {}^pD^{\geq 0}(X; R)\,.$$

In particular, for every $k \in \mathbb{Z}$, there are adjunction maps $\mathscr{F}^\bullet \to {}^p\tau_{\geq k}\mathscr{F}^\bullet$ and ${}^p\tau_{\leq k}\mathscr{F}^\bullet \to \mathscr{F}^\bullet$ (see, e.g., [6, Proposition 1.3.3]). These perverse truncation functors can be used to associate to any constructible complex $\mathscr{F}^\bullet \in D^b_{(w)c}(X; R)$ its *perverse cohomology* sheaves defined as:

$${}^p\mathscr{H}^i(\mathscr{F}^\bullet) := {}^p\tau_{\leq 0}{}^p\tau_{\geq 0}(\mathscr{F}^\bullet[i]) \in Perv(X; R).$$

These are $\mathscr{S}$-(weakly) constructible for $\mathscr{F}^\bullet$ $\mathscr{S}$-(weakly) constructible. It then follows that $\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X; R)$ if and only if ${}^p\mathscr{H}^i(\mathscr{F}^\bullet) = 0$ for all $i > 0$. Similarly, $\mathscr{F}^\bullet \in {}^pD^{\geq 0}(X; R)$ if and only if ${}^p\mathscr{H}^i(\mathscr{F}^\bullet) = 0$ for all $i < 0$. In particular, $\mathscr{F}^\bullet \in Perv(X; R)$ if and only if ${}^p\mathscr{H}^i(\mathscr{F}^\bullet) = 0$ for all $i \neq 0$ and ${}^p\mathscr{H}^0(\mathscr{F}^\bullet) = \mathscr{F}^\bullet$ (see, e.g., [6, Proposition 1.3.7]). Perverse cohomology sheaves can be used to calculate the (hyper)cohomology groups of any $\mathscr{F}^\bullet \in D^b_{(w)c}(X)$ via the *perverse cohomology spectral sequence*

$$E_2^{i,j} = H^i(X; {}^p\mathscr{H}^j(\mathscr{F}^\bullet)) \Longrightarrow H^{i+j}(X; \mathscr{F}^\bullet). \tag{10.13}$$

Definition 10.2.35 A functor $F\colon D_1 \to D_2$ of triangulated categories with t-structures is *left t-exact* if $F(D_1^{\geq 0}) \subseteq D_2^{\geq 0}$, *right t-exact* if $F(D_1^{\leq 0}) \subseteq D_2^{\leq 0}$, and *t-exact* if F is both left and right t-exact.

Example 10.2.36 The inclusion of full subcategories

$$D^b_{\mathscr{S}-(w)c}(X; R) \subset D^b_{(w)c}(X; R) \quad \text{and} \quad D^b_{(\mathscr{S}-)wc}(X; R) \subset D^b_{(\mathscr{S}-)c}(X; R)$$

are t-exact with respect to the perverse t-structures.

Remark 10.2.37 If F is a t-exact functor, it restricts to a functor on the corresponding hearts. More generally, if $F\colon D_1 \to D_2$ is a functor of triangulated categories with t-structures, and we let C_1, C_2 be the corresponding hearts with $k_i\colon C_i \hookrightarrow D_i$, then

$${}^pF := {}^tH^0 \circ F \circ k_1\colon C_1 \to C_2$$

is called the *perverse functor associated to* F. (Here, if $\tau_{\leq 0}$ and $\tau_{\geq 0}$ are the truncation functors on a triangulated category D with heart C, we set ${}^tH^0 := \tau_{\geq 0}\tau_{\leq 0} = \tau_{\leq 0}\tau_{\geq 0}\colon D \to C$.) In this paper we work only with the perverse t-structure, so a t-exact functor preserves perverse sheaves.

Example 10.2.38 Let X be a complex analytic (or algebraic) variety, and let $Z \subseteq X$ be a closed subset. Fix a Whitney stratification $\mathscr{S}$ of the pair (X, Z), i.e., Z is a union of strata of $\mathscr{S}$. Then Z and $U := X \setminus Z$ inherit Whitney stratifications as well, and if we denote by $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$ the stratified inclusion maps, then the functors $j^* = j^!$, $i^!$, i^*, $i_* = i_!$, $j_!$ and Rj_* preserve (weak) constructibility with respect to the above fixed stratifications, with $j^*i_! = 0$ and $i^!Rj_* = 0$. Moreover, the functors $j_!$, i^* are right t-exact, the functors $j^! = j^*$, $i_* = i_!$ are t-exact, and Rj_*, $i^!$ are left t-exact. Similarly for the functors $j^* = j^!$, $i^!$, i^*, $i_* = i_!$, $j_!j^*$ and Rj_*j^* (as well as $j_!$ and Rj_* in the complex algebraic context), if we do not fix a Whitney stratification.

Example 10.2.39 Let $f\colon X \to Y$ be a *smooth* morphism (i.e., a submersion) of complex algebraic (or analytic) varieties, with constant relative (or fiber) dimension

d. Then $f^![-d] \simeq f^*[d]$ is t-exact. Similarly, the external tensor product $\overset{L}{\boxtimes}$ is t-exact in each variable in case R is a field.

Example 10.2.40 Let $f: X \to Y$ be a finite map (i.e., proper, with finite fibers). Then $Rf_* = f_!$ is t-exact (see Example 10.3.31). If, moreover, X is pure dimensional with f surjective and generically bijective, then $Rf_* IC_X \simeq f_* IC_X \simeq IC_Y$. The latter fact applies, in particular, to the case when X is the (algebraic) normalization of Y.

The perverse cohomology sheaf construction provides a way to get perverse sheaves out of any (weakly) constructible complex. Another important method for constructing perverse sheaves, the *intermediate extension*, will be discussed below.

Let $j: U \hookrightarrow X$ be the inclusion of an open constructible subset of the complex algebraic (or analytic) variety X, with $i: Z = X \setminus U \hookrightarrow X$ the closed inclusion. We can also work with a fixed Whitney stratification $\mathscr{S}$ of X so that U is an open union of strata of $\mathscr{S}$. A complex $\mathscr{F}^\bullet \in D^b_{(w)c}(X; R)$ is a (weakly) constructible *extension* of $\mathscr{G}^\bullet \in D^b_{(w)c}(U; R)$ if $j^*\mathscr{F}^\bullet \simeq \mathscr{G}^\bullet$. In what follows, we are interested to find perverse extensions of $\mathscr{G}^\bullet \in Perv_{(\mathscr{S}|_U)}(U; R)$.

Definition 10.2.41 The *intermediate extension* $j_{!*}\mathscr{G}^\bullet$ of the perverse sheaf $\mathscr{G}^\bullet \in Perv_{(\mathscr{S}|_U)}(U; R)$ is the image in the abelian category $Perv_{(\mathscr{S})}(X; R)$ of the morphism

$$^p j_! \mathscr{G}^\bullet := {}^p\mathscr{H}^0(j_!\mathscr{G}^\bullet) \to {}^p\mathscr{H}^0(Rj_*\mathscr{G}^\bullet) =: {}^p j_* \mathscr{G}^\bullet.$$

Here, ${}^p j_! \mathscr{G}^\bullet \to {}^p j_* \mathscr{G}^\bullet$ is obtained by applying the functor ${}^p\mathscr{H}^0$ to the natural morphism $j_!\mathscr{G}^\bullet \to Rj_*\mathscr{G}^\bullet$ in $D^b_{(w)c}(X; R)$. In the analytic context we explicitly have to assume that $j_!\mathscr{G}^\bullet$ (and then also $Rj_*\mathscr{G}^\bullet \simeq Rj_*j^*j_!\mathscr{G}^\bullet$) is (weakly) constructible, e.g., $\mathscr{G}^\bullet \in Perv(U; R)$ is (weakly) constructible with respect to $\mathscr{S}|_U$.

Example 10.2.42 If X is of pure dimension n, and $j: U \hookrightarrow X$ is the inclusion of a smooth open subset whose complement is an algebraic (or analytic) subset of dimension $< \dim X$, then for a local system $\mathscr{L}$ on U one has that $\mathscr{L}[n] \in Perv(U; R)$ and

$$IC_X(\mathscr{L}) \simeq j_{!*}(\mathscr{L}[n]) .$$

Example 10.2.43 Let X be smooth of pure dimension one and let $j: U \hookrightarrow X$ be the inclusion of a Zariski open and dense subset. If $\mathscr{L}$ is a local system on U, then:

$$IC_X(\mathscr{L}) \simeq j_{!*}(\mathscr{L}[1]) \simeq (j_*\mathscr{L})[1] .$$

The intermediate extension functor plays an important role in describing the simple objects in the abelian category of ($\mathscr{S}-$)constructible perverse sheaves on X. Indeed, we have the following.

Proposition 10.2.44 *Consider the context of Definition 10.2.41 above.*

(a) The intermediate extension $j_{!*}\mathscr{G}^\bullet$ *of* $\mathscr{G}^\bullet \in Perv_{(\mathscr{S}|_U)}(U; R)$ *has no non-trivial sub-object and no non-trivial quotient object whose supports are contained in* $Z = X \setminus U$.

(b) Moreover, if $\mathscr{G}^\bullet \in Perv_{(\mathscr{S}|_U)}(U; R)$ *is a simple* ($\mathscr{S}|_U$*-constructible) object then* $j_{!*}\mathscr{G}^\bullet \in Perv_{(\mathscr{S})}(X; R)$ *is a simple* ($\mathscr{S}$*-constructible) object.*

Moreover, the following important result holds, see [6, Theorem 4.3.1] (where the assumptions are needed for the use of biduality and the stability of perverse sheaves under duality):

Theorem 10.2.45 *Let* X *be a complex algebraic variety, assume that the coefficient ring* R *is a field, and consider (constructible) perverse sheaves* $Perv(X; R) \subset D^b_c(X; R)$.

(a) The category of perverse sheaves $Perv(X; R)$ *is Artinian and Noetherian, i.e., every perverse sheaf on* X *admits an increasing finite filtration with quotients simple perverse sheaves .*

(b) The simple R*-perverse sheaves on* X *are the twisted intersection complexes* $IC_{\overline{V}}(\mathscr{L})$ *(regarded as complexes on* X *via extension by zero), where* V *runs through the family of smooth connected constructible subvarieties of* X, $\mathscr{L}$ *is a simple (i.e., irreducible)* R*-local system of finite rank on* V, *and* $\overline{V}$ *is the closure of* V *in* X.

Remark 10.2.46 A similar result as in Theorem 10.2.45 is also true for a compact analytic variety, or if one works in the algebraic or analytic context with $\mathscr{S}$-constructible perverse sheaves $Perv_{\mathscr{S}}(X; R) \subset D^b_{\mathscr{S}-c}(X; R)$ for a fixed Whitney stratification $\mathscr{S}$ of X with only *finitely* many strata. In the latter case one needs to use in (b) for V only the strata $S \in \mathscr{S}$.

Example 10.2.47 Let R be a field of coefficients, and let $X = \mathbb{C}$, $U = \mathbb{C}^*$ with open inclusion $j: U \hookrightarrow X$, and $Z = \{0\}$ with closed inclusion $i: Z \hookrightarrow X$. Let $\mathscr{L}$ be a local system of finite rank on U with stalk V and monodromy automorphism $h: V \to V$. Then $\mathscr{L}[1] \in Perv(U; R)$ and $IC_X(\mathscr{L}) \simeq j_*\mathscr{L}[1]$. Moreover, it is an instructive exercise to show that the following assertions hold:

(a) $j_!\mathscr{L}[1]$ and $Rj_*\mathscr{L}[1]$ are perverse sheaves on X.

(b) There is a short exact sequence of perverse sheaves on X:

$$0 \longrightarrow IC_X(\mathscr{L}) \longrightarrow Rj_*\mathscr{L}[1] \longrightarrow i_!IC_Z(V_h) \longrightarrow 0\,, \tag{10.14}$$

where $V_h = \text{Coker}\,(h-1)$. In particular, if the local system $\mathscr{L}$ is simple, then the perverse sheaf $Rj_*\mathscr{L}[1]$ admits the filtration

$$IC_X(\mathscr{L}) \subset Rj_*\mathscr{L}[1]$$

with simple quotients $IC_Z(V_h)$ and $IC_X(\mathscr{L})$.

(c) There is a short exact sequence of perverse sheaves on X:

$$0 \longrightarrow i_! IC_Z(V^h) \longrightarrow j_! \mathscr{L}[1] \longrightarrow IC_X(\mathscr{L}) \longrightarrow 0\,, \tag{10.15}$$

where $V^h = \ker(h-1)$. Hence, if the local system $\mathscr{L}$ is simple, then the perverse sheaf $j_!\mathscr{L}[1]$ admits the filtration

$$IC_Z(V^h) \subset Rj_*\mathscr{L}[1]$$

with simple quotients $IC_X(\mathscr{L})$ and $IC_Z(V^h)$.

The next result describes the behavior of the intermediate extension with respect to the dualizing functor (see, e.g., [89, Proposition 8.4.15]).

Proposition 10.2.48 *Consider the context of Definition 10.2.41 with coefficient ring R a field, and let $\mathscr{G}^\bullet \in Perv_{(\mathscr{S}|_U)}(U;R)$ be ($\mathscr{S}|_U$-)constructible. Then*

$$\mathcal{D}(j_{!*}\mathscr{G}^\bullet) \simeq j_{!*}(\mathcal{D}\mathscr{G}^\bullet). \tag{10.16}$$

In particular, $\mathcal{D}IC_X \simeq IC_X$ for X pure dimensional.

10.2.3 Strongly Perverse Sheaves, Dual t-Structure and Rectified Homological Depth

In this section we only consider ($\mathscr{S}-$)constructible sheaves so that biduality is available, with a *dual perverse t-structure*

$$^{p^+}D^{\leq 0}(X;R) := \mathcal{D}\left({}^pD^{\geq 0}(X;R)\right) \quad \text{and} \quad {}^{p^+}D^{\geq 0}(X;R) := \mathcal{D}\left({}^pD^{\leq 0}(X;R)\right)$$

on $D^b_{(\mathscr{S}-)c}(X;R)$. Here $\mathcal{D}$ denotes the Verdier duality functor on X. But only for R a *Dedekind domain* one can give a more explicit description of this *dual perverse t-structure*. If R s a field, the Universal Coefficient Theorem yields that $\mathscr{F}^\bullet \in {}^pD^{\geq 0}(X;R)$ if and only if $\mathcal{D}\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X;R)$. More generally, one has the following result (see also [6, Section 3.3]):

Proposition 10.2.49 *Assume that the ring R is a Dedekind domain (e.g., a field or a principal ideal domain). If $\mathscr{F}^\bullet \in D^b_c(X;R)$ is constructible with respect to a Whitney stratification $\mathscr{S}$ of X, then $\mathcal{D}\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X;R)$, or equivalently $\mathscr{F}^\bullet \in {}^{p^+}D^{\geq 0}(X;R)$, if and only if the following two conditions are satisfied:*

(i) $\mathscr{F}^\bullet \in {}^pD^{\geq 0}(X;R)$;

(ii) for any stratum $S \in \mathscr{S}$ and any $x \in S$, the costalk cohomology $H^{\dim S}(i_x^!\mathscr{F}^\bullet)$ is torsion-free.

Proof Let $S \in \mathscr{S}$ and $x \in S$, with inclusion $i_x \colon \{x\} \hookrightarrow X$. Properties of the dualizing functor and the Universal Coefficient Theorem yield:

$$H^j(i_x^* \mathcal{D}\mathscr{F}^\bullet) \simeq H^j(\mathcal{D} i_x^! \mathscr{F}^\bullet) \simeq \mathrm{Hom}(H^{-j}(i_x^! \mathscr{F}^\bullet), R) \oplus \mathrm{Ext}(H^{-j+1}(i_x^! \mathscr{F}^\bullet), R).$$

The desired equivalence can now be checked easily. In particular ${}^pD^{\geq -1}(X; R) \subset {}^{p^+}D^{\geq 0}(X; R) \subset {}^pD^{\geq 0}(X; R)$. □

Definition 10.2.50 Assume that R is a Dedekind domain. We say that $\mathscr{F}^\bullet \in D_c^b(X; R)$ is *strongly perverse* if $\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X; R)$ and $\mathcal{D}\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X; R)$. Equivalently, $\mathscr{F}^\bullet \in {}^pD^{\leq 0}(X; R) \cap {}^{p^+}D^{\geq 0}(X; R)$.

If R is a field, the notions of perverse sheaf and strongly perverse sheaf coincide. If R is a Dedekind domain, then $\mathscr{F}^\bullet$ is strongly perverse (with respect to $\mathscr{S}$) if and only if $\mathscr{F}^\bullet$ is perverse and property (ii) above holds (i.e., costalks of $\mathscr{F}^\bullet$ in the lowest possible degree are torsion-free on each stratum).

Example 10.2.51 Let X be a pure dimensional complex algebraic or analytic variety, and let R be a Dedekind domain. Then $IC_X(\mathscr{L})$ is strongly perverse if the generically defined local system $\mathscr{L}$ has finitely generated and torsion-free stalks (since then condition (ii) only needs to be checked for the top dimensional strata S, with $H^{\dim S}(i_x^! \mathscr{L}[\dim X]) \simeq \mathscr{L}_x$ for x in such a stratum S).

Strongly perverse sheaves are related to the notion of *rectified homological depth*, which we now define.

Let X be a complex algebraic or analytic variety. Following [109, Definition 6.0.4], we make the following.

Definition 10.2.52 The *rectified homological depth* $\mathrm{rHd}(X, R)$ of X with respect to the commutative base ring R is $\geq d$ (for some $d \in \mathbb{Z}$) if

$$\mathcal{D}(R_X[d]) \in {}^pD^{\leq 0}(X; R)\,, \tag{10.17}$$

or, equivalently,

$$R_X \in {}^{p^+}D^{\geq d}(X; R)\,. \tag{10.18}$$

The *rectified homological depth* $\mathrm{rHd}(X, R)$ of $X \neq \emptyset$ with respect to the commutative base ring R is the maximum of such integers d.

As indicated in [109, p. 387], the above definition agrees with the notion of *rectified homological depth* introduced by Hamm and Lê [56] (following an earlier definition of Grothendieck, together with the corresponding homotopical notion) in more geometric terms. In the following, let $\dim_- X$ be the minimum of the dimension of the irreducible components of X. So for $X \neq \emptyset$ smooth, this is the minimum of the dimension of the connected components of X. Similarly, $\dim_- X = \dim_- X_{reg}$, for X_{reg} the open dense regular part of X.

Example 10.2.53

(a) One always has $\mathrm{rHd}(X, R) \leq \dim_{-} X$, and $\mathrm{rHd}(X, R) = \dim_{-} X$ if X is smooth and nonempty (since then $\mathbb{D}_X^\bullet \simeq R_X[2\dim X]$ with $\dim X$ viewed as a locally constant function). Moreover, $\mathrm{rHd}(X, R) = \dim X$ implies that X is pure dimensional (by looking at the regular part).
(b) If X is a pure-dimensional local complete intersection, then $\mathrm{rHd}(X, R) = \dim X$ (see Example 10.4.29 and, e.g., [109, Example 6.0.11]).

In view of Example 10.2.19(a) and Definition 10.2.50, one has the following equivalence (see also [109, (6.14)]):

Proposition 10.2.54 *Let R be a Dedekind domain. For any nonempty complex algebraic or analytic variety X one has:*

$$\mathrm{rHd}(X, R) = \dim X \iff X \text{ is pure-dimensional and } R_X[\dim X] \text{ is strongly perverse.}$$

As a consequence, Proposition 10.2.49 yields the following.

Corollary 10.2.55 *Let X be a nonempty pure-dimensional complex algebraic or analytic variety with a Whitney stratification $\mathscr{S}$.*

(a) If R is a field, then:

$$\mathrm{rHd}(X, R) = \dim X \iff R_X[\dim X] \text{ is perverse.}$$

(b) If R is a Dedekind domain (e.g., a PID), then $\mathrm{rHd}(X, R) = \dim X$ *if and only if the following two conditions are satisfied:*

(i) $R_X[\dim X]$ is perverse.
(ii) for any stratum $S \in \mathscr{S}$ and any $x \in S$ with $i_x \colon \{x\} \hookrightarrow X$, the costalk cohomology $H^{\dim S}(i_x^! R_X[\dim X])$ is torsion-free.

Let us finish this section with citing the following result from [6, Section 3.3].

Proposition 10.2.56 *Assume that the ring R is a Dedekind domain (e.g., a field or a PID). If $\mathscr{F}^\bullet \in D_c^b(X; R)$ is constructible with respect to a Whitney stratification $\mathscr{S}$ of X, then $\mathcal{D}\mathscr{F}^\bullet \in {}^pD^{\geq 0}(X; R)$, or equivalently $\mathscr{F}^\bullet \in {}^{p^+}D^{\leq 0}(X; R)$, if and only if the following two conditions are satisfied:*

(i) $\mathscr{F}^\bullet \in {}^pD^{\leq 1}(X; R)$;
(ii) for any stratum $S \in \mathscr{S}$ and any $x \in S$, the stalk cohomology $H^{-\dim S+1}(i_x^*\mathscr{F}^\bullet)$ is a torsion module.

In particular, ${}^pD^{\leq 0}(X; R) \subset {}^{p^+}D^{\leq 0}(X; R) \subset {}^pD^{\leq 1}(X; R)$.

Assume R is a Dedekind domain. Let us denote by $D^b_{(\mathscr{S}-)tc}(X; R)$ the full triangulated subcategory of ($\mathscr{S}$-)constructible sheaf complexes $\mathscr{F}^\bullet \in D^b_{(\mathscr{S}-)c}(X; R)$, with all stalk cohomology modules $H^k(i_x^*\mathscr{F}^\bullet)$ being torsion modules for all $x \in$

$X, k \in \mathbb{Z}$. These are also stable by the usual truncation functors. Similarly, in the context of Example 10.2.38, all considered functors preserve the subcategory $D^b_{\mathscr{S}-tc}(-; R)$ of these torsion $\mathscr{S}$-constructible sheaf complexes. For $i_! = i_*$ and $j^* = j^!$ this is trivial, and for $i^!$ it follows from the corresponding property for Rj_* and the standard distinguished triangle

$$i_* i^! \longrightarrow id \longrightarrow Rj_* j^* \longrightarrow \ .$$

But the stalk cohomology modules of the open push-forward Rj_* can be expressed in terms of the compact link [109, Remark 4.4.2], and by the Künneth formula they vanish after tensoring with the quotient field $Q(R)$ of R. Then the usual perverse t-structure and its truncation functors restrict to the perverse t-structure

$$\left({}^p D^{\leq 0}_{\mathscr{S}-t}(X; R), {}^p D^{\geq 0}_{\mathscr{S}-t}(X; R) \right) \quad \text{on } D^b_{\mathscr{S}-tc}(X; R), \tag{10.19}$$

and the corresponding heart of $\mathscr{S}$-constructible torsion perverse sheaves

$$Perv_{\mathscr{S}-t}(X; R) \ .$$

Example 10.2.57 Let X be a pure dimensional complex algebraic or analytic variety with a Whitney stratification $\mathscr{S}$, and let R be a Dedekind domain. Then $IC_X(\mathscr{L})$ is a $\mathscr{S}$-constructible torsion perverse sheaf if the local system $\mathscr{L}$ defined on the open top dimensional strata has only finitely generated torsion stalks.

Corollary 10.2.58 *Assume R is a Dedekind domain. Then one gets for the shifted Verdier duality functor* $(\mathcal{D}(-))[1]$*:*

(a) $(\mathcal{D}(-))[1]$ *maps* ${}^p D^{\leq 0}_{\mathscr{S}-t}(X; R)$ *into* ${}^p D^{\geq 0}_{\mathscr{S}-t}(X; R)$.
(b) $(\mathcal{D}(-))[1]$ *maps* ${}^p D^{\geq 0}_{\mathscr{S}-t}(X; R)$ *into* ${}^p D^{\leq 0}_{\mathscr{S}-t}(X; R)$.

In particular, the category $Perv_{\mathscr{S}-t}(X; R)$ of $\mathscr{S}$-constructible torsion perverse sheaves is stable under the shifted Verdier duality functor $(\mathcal{D}(-))[1]$.

Example 10.2.59 Assume R is a Dedekind domain. Then on a point space $X = \{pt\}$ the *shifted duality functor* $(\mathcal{D}(-))[1]$ is given by

$$R\mathscr{H}om^\bullet(-, R)[1] \simeq \mathscr{H}om^\bullet(-, Q(R)/R) : \ D^b_{tc}(\{pt\}; R) \to D^b_{tc}(\{pt\}; R) \ ,$$

preserving finitely generated torsion modules (viewed as complexes concentrated in degree zero). Here $Q(R)$ is the quotient field of R so that the short exact sequence

$$0 \longrightarrow R \longrightarrow Q(R) \longrightarrow Q(R)/R \longrightarrow 0$$

gives the injective resolution $[Q(R) \longrightarrow Q(R)/R]$ of R used for the calculation of $R\mathscr{H}om^\bullet(-, R)$.

In fact this example is also used on a general X for showing that the Verdier duality functor $\mathcal{D}$ preserves $D^b_{(\mathscr{S}-)tc}(X; R)$, together with $i_x^*\mathcal{D} \simeq \mathcal{D}i_x^!$ for all $x \in X$. For applications and a discussion of related results see also [20, 48, 87, 88, 115].

10.3 Stratified Morse Theory for Constructible Sheaves

In this section, we explain the basic results from [109] about *stratified Morse theory* in the framework of (weakly) constructible sheaves in the *complex context*, continuing and extending the recent survey of Goresky [49] in this handbook series (as well as the survey of Massey [85]). We follow the notions of the geometric *stratified Morse theory* of Goresky-MacPherson so that one can easily compare our results with those of [47]. Moreover, we also explain some relations to the general *micro-local sheaf theory* of Kashiwara-Schapira [66]. But unlike most of these references, we do not need a *global embedding* into an ambient complex manifold (except for the index theorems for characteristic cycles of constructible sheaves and the comparison with the notion of micro-support from [66]).

10.3.1 Morse Functions, Local and Normal Morse Data

We work with a fixed complex algebraic (or analytic) variety X with a given Whitney stratification $\mathscr{S}$.

Definition 10.3.1 A function $f: X \to \mathbb{R}$ is a C^k-differentiable function (with $2 \leq k \leq \infty$) if, for any $x \in X$, there exists a local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x)$ such that f is induced by restriction of a C^k-function germ $\hat{f}: (\mathbb{C}^n, x) \to (\mathbb{R}, f(x))$.

Of course, a complex algebraic (or analytic) function or morphism $h: X \to \mathbb{C}$ is by definition locally given by restriction of a complex algebraic (or analytic) function germ $\hat{h}: (\mathbb{C}^n, x) \to (\mathbb{C}, h(x))$.

Definition 10.3.2 A point $x \in X$ is called a *stratified critical point* of $f: X \to \mathbb{R}$, resp., $h: X \to \mathbb{C}$ (with respect to $\mathscr{S}$) if $f: S \to \mathbb{R}$, resp., $h: S \to \mathbb{C}$ is not a submersion at the point x belonging to the stratum $S \in \mathscr{S}$. Let $\mathrm{Sing}(f|_S)$, resp., $\mathrm{Sing}(h|_S)$ be the set of critical points of f, resp., h in the stratum $S \in \mathscr{S}$, with

$$\mathrm{Sing}_{\mathscr{S}}(f) := \bigcup_{S\in\mathscr{S}} \mathrm{Sing}(f|_S) \quad \text{resp.,} \quad \mathrm{Sing}_{\mathscr{S}}(h) := \bigcup_{S\in\mathscr{S}} \mathrm{Sing}(h|_S)\,.$$

Then $\mathrm{Sing}_{\mathscr{S}}(f)$ (resp., $\mathrm{Sing}_{\mathscr{S}}(h)$) is a closed (complex algebraic or analytic) subset of X by the *Whitney a-condition*.

The following result shows that for $\mathscr{S}$-weakly constructible sheaf complexes the local change of cohomology with respect to f is located at the critical points $x \in \mathrm{Sing}_{\mathscr{S}}(f)$ (see, e.g., [109, Example 5.1.1]).

Lemma 10.3.3 *Let $f\colon X \to]c, d[\subset \mathbb{R}$ be a proper differentiable function, with $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ a $\mathscr{S}$-weakly constructible complex.*

1. *If f is a stratified submersion at a point $x \in \{f = e\}$ (with $e \in]c, d[$, and here f does not need to be proper), then*

$$\left(R\Gamma_{\{f\geq e\}}(\mathscr{F}^\bullet)\right)_x \simeq 0 \simeq \left(R\Gamma_{\{f\leq e\}}(\mathscr{F}^\bullet)\right)_x .$$

2. *If f is a stratified submersion at all points of $\{f = e\}$ (for $e \in]c, d[$), then*

$$R\Gamma(\{f = e\}, R\Gamma_{\{f\geq e\}}(\mathscr{F}^\bullet)) \simeq 0 \simeq R\Gamma(\{f = e\}, R\Gamma_{\{f\leq e\}}(\mathscr{F}^\bullet)) .$$

3. *If f is a stratified submersion at all points of $\{a < f < b\}$ (with $]a, b[\subset]c, d[$), then all cohomology sheaves of $(Rf_*\mathscr{F}^\bullet)|_{]a,b[}$ are locally constant.*

These results can be proved by induction on the dimension of X *without using the "first isotopy Lemma of Thom"*. Property 1. above just means that a Whitney b-regular stratification satisfies the *"local stratified acyclicity"* property form [109, Definition 4.0.3]. In the embedded context of a global closed embedding $k\colon X \hookrightarrow M$ into a complex manifold it also implies the following estimate [109, Corollary 4.0.3] of the *micro-support* of $Rk_*\mathscr{F}^\bullet$ in the sense of Kashiwara-Schapira [66, Definition 5.1.2] (with $Rk_* = k_* = k_!$ the extension by zero):

$$\mu supp(Rk_*\mathscr{F}^\bullet) \subset \bigcup_{S\in\mathscr{S}} T^*_S M \hookrightarrow T^*M \quad \text{for any } \mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R). \tag{10.20}$$

Here, $\mu supp(\mathscr{G}^\bullet) \subset T^*M$ is by definition the complement of the largest open subset $U \subset T^*M$ such that $R\Gamma_{\{f\geq f(x)\}}(\mathscr{G}^\bullet)_x \simeq 0$ for any differentiable function germ $f\colon (M, x) \to (\mathbb{R}, f(x))$ with $df_x \in U$ (see [66, Definition 5.1.2]). We also use the notation $T^*_S M$ for the *conormal bundle* of the stratum S in M, i.e., for the kernel of the natural vector bundle epimorphism $T^*M|_S \to T^*S$ dual to the inclusion $TS \hookrightarrow TM|_S$. Let us denote by $T^*_{\mathscr{S}} M$ this union of conormal spaces of strata $S \in \mathscr{S}$. Then the Whitney a-condition for $\mathscr{S}$ is equivalent to the fact that $T^*_{\mathscr{S}} M \hookrightarrow T^*M$ is closed in T^*M. Moreover it is *conic* in the sense that it is invariant under the natural $\mathbb{C}^*$-action on the cotangent bundle. The inclusion of the micro-support in (10.20) implies that $\mu supp(Rk_*\mathscr{F}^\bullet) \subset T^*M$ is a closed complex algebraic (or analytic) conic Lagrangian subset of T^*M, and (10.20) is even equivalent to $\mathscr{S}$-weak constructibility (see, e.g., [109, Lemma 4.1]).

Lemma 10.3.4 *Let $\mathscr{G}^\bullet \in D^b(M; R)$ be given with $\mu supp(\mathscr{G}^\bullet) \subset T^*_{\mathscr{S}} M$. Then $supp(\mathscr{G}^\bullet) \subset X$ and $\mathscr{G}^\bullet|_X$ is $\mathscr{S}$-weakly constructible.*

Example 10.3.5 Assume $X = M$ with $\mathscr{G}^\bullet \in D^b(M; R)$ be given. Then $\mu supp(\mathscr{G}^\bullet) \subset T^*_M M$ the zero section of T^*M if and only if all cohomology sheaves $\mathscr{H}^i(\mathscr{G}^\bullet)$ are locally constant on M.

This Example can be proved directly with the *"non-characteristic deformation Lemma"* of Kashiwara (see, e.g., [109, Proposition 4.1.1]). Moreover, it can be used as a substitute for the *"first isotopy Lemma of Thom"*.

If one does not want to fix a complex Whitney stratification $\mathscr{S}$, then one gets the following micro-local characterization of weak constructibility (see, e.g., [109, Theorem 4.0.1] and [66, Theorem 8.5.5]).

Theorem 10.3.6 *Let M be a complex algebraic (or analytic) manifold. Then $\mathscr{G}^\bullet \in D^b(M; R)$ is a weakly constructible complex (i.e., $\mathscr{G}^\bullet \in D^b_{wc}(M; R)$) in the complex algebraic (or analytic) sense if and only if $\mu supp(\mathscr{G}^\bullet) \subset T^*M$ is a closed complex algebraic (or analytic) conic Lagrangian subset of T^*M.*

Let us now come back to the general non-embedded context with $\mathscr{S}$ a fixed Whitney stratification of X. Then Lemma 10.3.3 implies (see, e.g., [109, Corollary 5.1.1]).

Corollary 10.3.7 *Let $f: X \to]c, d[\subset \mathbb{R}$ be a proper differentiable function, with $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ a $\mathscr{S}$-weakly constructible complex. If f is a stratified submersion at all points of $\{a \le f \le b\} \cap \{f \ne e\}$ (with $[a, b] \subset]c, d[$ and $e \in]a, b[$), then one has distinguished triangles*

$$R\Gamma(\{f = e\}, R\Gamma_{\{f \ge e\}}(\mathscr{F}^\bullet)) \to R\Gamma(\{f \le b\}, \mathscr{F}^\bullet) \to R\Gamma(\{f \le a\}, \mathscr{F}^\bullet) \to , \tag{10.21}$$

and

$$R\Gamma_c(\{f < a\}, \mathscr{F}^\bullet) \to R\Gamma_c(\{f < b\}, \mathscr{F}^\bullet) \to R\Gamma(\{f = e\}, R\Gamma_{\{f \le e\}}(\mathscr{F}^\bullet)) \to . \tag{10.22}$$

Here, $R\Gamma(\{f = e\}, R\Gamma_{\{f \ge e\}}(\mathscr{F}^\bullet))$ is the cohomological counterpart of the *"coarse Morse datum"* as defined in [47, p.62, Definition 3.4]. In particular,

$$R\Gamma(\{f = e\}, R\Gamma_{\{f \le e\}}(\mathscr{F}^\bullet) \simeq R\Gamma(\{-f = -e\}, R\Gamma_{\{-f \ge -e\}}(\mathscr{F}^\bullet))$$

is the "coarse Morse datum" of $-f$. So this "coarse Morse datum" of $-f$ corresponds to the "relative cohomology with compact support" in the triangle (10.22). But the cohomology with compact support is in some sense "dual" to the cohomology with closed support. So this reflects the "duality" observed in [47, p.27, 2.7]. In the context of stratified Morse theory for constructible sheaves, this is a special form of *Poincaré-Verdier duality* (see Equation (10.25)).

The next step is to consider a differentiable function f with *isolated* stratified critical points in $\{f = e\}$, and to localize the "coarse Morse datum" at these critical points. In terms of sheaf theory, this is quite easy (since $R\Gamma_{\{f\geq e\}}(\mathscr{F}^{\bullet})|_{\{f=e\}}$ is supported on the stratified critical points).

Lemma 10.3.8 *Let $f\colon X \to]c,d[\subset \mathbb{R}$ be a proper differentiable function, with $\mathscr{F}^{\bullet} \in D^b_{\mathscr{S}-wc}(X;R)$ a $\mathscr{S}$-weakly constructible complex. Assume f is a stratified submersion at all points of $\{a \leq f \leq b\}$ except for finitely many $x_i \in \{f = e\}$ (with $[a,b] \subset]c,d[$ and $e \in]a,b[$). Then*

$$R\Gamma(\{f=e\}, R\Gamma_{\{f\geq e\}}\mathscr{F}^{\bullet})) \simeq \bigoplus_i \left(R\Gamma_{\{f\geq e\}}(\mathscr{F}^{\bullet})\right)_{x_i} . \tag{10.23}$$

The stalk complex

$$LMD(\mathscr{F}^{\bullet}, f, x) := \left(R\Gamma_{\{f\geq f(x)\}}(\mathscr{F}^{\bullet})\right)_x \tag{10.24}$$

is the sheaf theoretic counterpart of the *"local Morse datum"* of [47, p.63, Definition 3.5.2]. The "duality" between f and $-f$ discussed above is closely related to *Verdier duality for the local Morse data* in the following form of [109, Equation (5.54) on p.314]:

$$\mathcal{D}(LMD(\mathscr{F}^{\bullet}, -f, x)) \simeq LMD(\mathcal{D}(\mathscr{F}^{\bullet}), f, x) \quad \text{for } \mathscr{F}^{\bullet} \in D^b_{\mathscr{S}-wc}(X;R). \tag{10.25}$$

Let us now recall two notions from [47].

Definition 10.3.9 Fix a point x in the stratum $S \in \mathscr{S}$.

1. x is called a *stratified Morse critical point* of the differentiable function germ $f\colon (X,x) \to \mathbb{R}$ (with respect to $\mathscr{S}$), if $f|_S$ has at x a Morse critical point in the classical sense (i.e., its Hessian at x is non-degenerate), and $f = \hat{f}|_X$ is induced in some local embedding $(X,x) \hookrightarrow (\mathbb{C}^n, x)$ by a differentiable function germ $\hat{f}\colon (\mathbb{C}^n, x) \to (\mathbb{R}, f(x))$ such that the covector $d\hat{f}_x$ is *non-degenerate* in the sense of [47, p.44, Definition 1.8], i.e., it does not vanish on any generalized tangent space
$$\tau := \lim_{x_n \to x} T_{x_n} S' \quad \text{at } x,$$
with x_n a sequence in a stratum $S' \in \mathscr{S}$ with $S \subset \partial S'$.
2. A *normal slice* N at $x \in S \hookrightarrow X \hookrightarrow \mathbb{C}^n$ is a closed complex submanifold germ N of $(\mathbb{C}^n, x)$, with $N \cap S = \{x\}$ such that N intersects S *transversally* at x (so that $\operatorname{codim} N = \dim S$).

Note that a *normal slice* N intersects all strata transversally near x by *Whitney a-regularity*. Therefore $N \cap X$ gets an induced Whitney stratification $\mathscr{S}|_N$ near x with strata the connected components of intersections $S' \cap N \neq \emptyset$ ($S' \in \mathscr{S}$) and $\{x\} = S \cap N$ now a point stratum.

Similarly, a *stratified Morse critical point* of f is an isolated stratified critical point of f. Finally, if $f = \hat{f}|_X$ is induced in some local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x)$ by a differentiable function germ $\hat{f}\colon (\mathbb{C}^n, x) \to (\mathbb{R}, f(x))$ such that the covector $d\hat{f}_x$ is *non-degenerate*, then one can find such an $\hat{f}$ in any local embedding $(X, x) \hookrightarrow (\mathbb{C}^m, x)$.

Example 10.3.10 Let S be an open stratum of X. Then $x \in S$ is a stratified Morse critical point of f if and only if $f|_S$ has at x a classical Morse critical point. If $S = \{x\}$ is a point stratum, then x is by definition a stratified critical point for any differentiable function $f\colon X \to \mathbb{R}$. It is a stratified Morse critical point if and only if $f = \hat{f}|_X$ is induced in some local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x)$ by a differentiable function germ $\hat{f}\colon (\mathbb{C}^n, x) \to (\mathbb{R}, f(x))$ such that the covector $d\hat{f}_x$ is non-degenerate.

Example 10.3.11 Let $X = M$ be a complex manifold, with $M' \hookrightarrow M$ a closed complex submanifold of positive codimension and the Whitney stratification $\mathscr{S}$ given by the connected components of M' and $M \backslash M'$. Then a differentiable function $f\colon M \to \mathbb{R}$ has a stratified Morse critical point at $x \in M'$ if and only if x is not a critical point of $f\colon M \to \mathbb{R}$ (this is equivalent to df_x is nondegenrate) and $x \in M'$ is a classical Morse critical point for $f\colon M' \to \mathbb{R}$.

Let us now state the first main result from stratified Morse theory for constructible sheaves (see, e.g., [109, Theorem 5.0.1]).

Theorem 10.3.12 *Let X be a complex algebraic (or analytic) variety with a Whitney stratification $\mathscr{S}$, and let $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ be a given $\mathscr{S}$-weakly constructible complex. Let $f\colon (X, x) \to \mathbb{R}$ be a differentiable function germ with a stratified Morse critical point at $x \in S$ for some stratum $S \in \mathscr{S}$. Take a normal slice $N \subset \mathbb{C}^n$ at x in some local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x)$.*

1. *Then one has an isomorphism*

$$\left(R\Gamma_{\{f \geq f(x)\}}(\mathscr{F}^\bullet)\right)_x \simeq \left(R\Gamma_{\{f|_{N\cap X} \geq f(x)\}}(\mathscr{F}^\bullet|_{N\cap X})\right)_x[-\tau]\,,$$

   with τ the Morse index of $f|_S$ (i.e., its Hessian at x has exactly τ negative eigenvalues).
2. *The isomorphism class of*

$$\left(R\Gamma_{\{f|_{N\cap X} \geq f(x)\}}(\mathscr{F}^\bullet|_{N\cap X})\right)_x$$

   is independent of the choice of such a local embedding and normal slice N. Moreover, this isomorphism class only depends on the stratum $S \in \mathscr{S}$, but not on the point $x \in S$ or the function germ $f = \hat{f}|_X$ with $d\hat{f}_x$ non-degenerate in such a local embedding.

The isomorphism class of

$$NMD(\mathscr{F}^\bullet, S) := \left(R\Gamma_{\{f|_{N\cap X} \geq f(x)\}}(\mathscr{F}^\bullet|_{N\cap X})\right)_x \tag{10.26}$$

is the sheaf theoretic counterpart of the *"normal Morse data"* of [47, p.65, Definition 3.6.1]. In the (locally) embedded context $k: X \hookrightarrow M$, $NMD(\mathscr{F}^\bullet, S)$ is also isomorphic to the "*micro-local type of $Rk_*\mathscr{F}^\bullet$ at the non-degenerate covector $\omega = d\hat{f}_x$*" in the sense of Kashiwara-Schapira [66, Proposition 6.6.1(ii)] (as shown in [109, Equation (5.52) on p.311]). Part 1. of Theorem 10.3.12 corresponds to [47, p.8, Theorem SMT Part B; p.65, Main Theorem 3.7], and part 2. to [47, p.93, Theorem 7.5.1] (compare also with [47, p.223, Proposition 6.A.1]).

Let us consider again the embedded context with $k: X \hookrightarrow M$ a closed embedding into a complex algebraic (or analytic) manifold, and $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$. By using the estimate (10.20) and the *involutivity* of $\mu supp(Rk_*\mathscr{F}^\bullet)$ [66, Theorem 6.5.4] one can show that $\mu supp(Rk_*\mathscr{F}^\bullet)$ is a union of closures of conormal bundles $T^*_S M$ of some strata $S \in \mathscr{S}$. Then the explicit characterization of the micro-support $\mu supp(Rk_*\mathscr{F}^\bullet)$ is given as follows (see, e.g., [109, Proposition 5.0.1]).

Proposition 10.3.13 *Let $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ be given. Then*

$$\overline{T^*_S M} \subset \mu supp(Rk_*\mathscr{F}^\bullet) \iff NMD(\mathscr{F}^\bullet, S) \not\simeq 0\,.$$

For the applications of stratified Morse theory for constructible sheaves, it is important to get information about this *normal Morse datum* $NMD(\mathscr{F}^\bullet, S)$ for a stratum $S \in \mathscr{S}$. In our complex context this can be obtained in the following way. Since it is a local study, we can assume that for a given point $x \in S$ we have (locally) a closed embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x = 0)$.

We already know that the isomorphism class of the "normal Morse datum" $NMD(\mathscr{F}^\bullet, S)$ only depends on the non-degenerate covector $\omega = d\hat{f}_x \in T^*_S\mathbb{C}^n|_x$. So one can use a *holomorphic* function germ $g: (\mathbb{C}^n, x) \to (\mathbb{C}, 0)$ with $d\hat{f}_x = d(Re(g))_x$. Also choose a local *normal slice* N at $x \in S \hookrightarrow X \hookrightarrow \mathbb{C}^n$, with $\{x\} = S \cap N$ a point stratum of the induced Whitney stratification $\mathscr{S}|_N$ of $X \cap N$. Then one gets (see Remark 10.4.18 and also, e.g., [109, Proposition 5.0.3]).

Proposition 10.3.14 *Let $k_x: \{x\} \to N \cap X$ be the inclusion of this point stratum, and $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ be given. Then there exists two distinguished triangles*

$$R\Gamma(l_X, \mathscr{F}^\bullet)[-1] \longrightarrow NMD(\mathscr{F}^\bullet, S) \longrightarrow k_x^*(\mathscr{F}^\bullet|_{N\cap X}) \longrightarrow\,, \tag{10.27}$$

and

$$k_x^!(\mathscr{F}^\bullet|_{N\cap X}) \longrightarrow NMD(\mathscr{F}^\bullet, S) \longrightarrow R\Gamma(l_X, \partial l_X, \mathscr{F}^\bullet)[-1] \longrightarrow\,. \tag{10.28}$$

Moreover, the isomorphism classes of $R\Gamma(l_X, \mathscr{F}^\bullet)$ and $R\Gamma(l_X, \partial l_X, \mathscr{F}^\bullet)$ only depend on the stratum S of X (but on no other choice like the local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x = 0)$, or the choice of N or g as above).

Here we use the following notation (with $0 < |w| \ll \delta \ll 1$ and $r(z) := \sum_{1\leq i\leq n} z_i \bar{z}_i$ the square of the local distance to $x = 0 \in \mathbb{C}^n$)

$$(l_X, \partial l_X) := (X \cap N \cap \{r \leq \delta, g = w\}, X \cap N \cap \{r = \delta, g = w\}) \tag{10.29}$$

for the *complex link* of X with respect to $S \in \mathscr{S}$ (see [47, p.161, Definition.2.2]). In other words, this is the local *Milnor fiber (with its boundary)* (see, e.g., [71] or [109, Example 1.1.2]) of the holomorphic function germ $g\colon (X \cap N, x) \to (\mathbb{C}, 0)$, which has an isolated stratified critical point in x with respect to $\mathscr{S}|_N$ (since dg_x is a non-degenerate covector).

Example 10.3.15 Assume $S \in \mathscr{S}$ is an *open* stratum in X. Then for $x \in S$ and a normal slice N at x one gets $X \cap N = \{x\}$, so that the corresponding complex link is empty, $l_X = \emptyset$, with

$$NMD(\mathscr{F}^\bullet, S) \simeq k_x^*(\mathscr{F}^\bullet|_{N\cap X}) \simeq i_x^*\mathscr{F}^\bullet$$

just the stalk of $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ at x.

Example 10.3.16 We state here cohomological counterparts of some results of [47].

1. If we take $\mathscr{F}^\bullet = \mathbb{Z}_X$, then we get by the first distinguished triangle (10.27) for the cohomology of the "normal Morse datum" $NMD(\mathscr{F}^\bullet, S)$ for $x \in S$:

$$\left(R^k\Gamma_{\{Re(g)|_{N\cap X}\geq 0\}}(\mathscr{F}^\bullet|_{N\cap X})\right)_x \simeq \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and } l_X = \emptyset, \\ 0 & \text{for } k \neq 0 \text{ and } l_X = \emptyset, \\ \widetilde{H}^{k-1}(l_X; \mathbb{Z}) & \text{for } l_X \neq \emptyset. \end{cases}$$

   This corresponds to the "first part of the fundamental theorem of complex stratified Morse theory" [47, p.16, Theorem CSMT Part A, p.166 Corollary 2.4.1].

2. Consider now the case $\mathscr{F}^\bullet = Rj_*\mathbb{Z}_U$ for $j\colon U \hookrightarrow X$ the inclusion of an open subspace, which is a union of strata. Assume $x \notin U$. Then one gets by the second distinguished triangle (10.28) for the cohomology of the "normal Morse datum" $NMD(\mathscr{F}^\bullet, S)$ for $x \in S$:

$$\left(R^k\Gamma_{\{Re(g)|_{N\cap X}\geq 0\}}(\mathscr{F}^\bullet|_{N\cap X})\right)_x \simeq H^{k-1}(l_X, \partial l_X; Rj_*\mathbb{Z}_U)\,.$$

   Moreover, by the base change isomorphism (10.11) one gets

$$H^{k-1}(l_X, \partial l_X; Rj_*\mathbb{Z}_U) \simeq H^{k-1}(l_U, \partial l_U; \mathbb{Z})\,,$$

   with $(l_U, \partial l_U) := (l_X, \partial l_X) \cap U$ the *complex link of* U. This corresponds to the "second part of the fundamental theorem of complex stratified Morse theory" [47, p.18, Theorem CSMT Part B, p.169, Corollary 2.6.1].

Proposition 10.3.14 and the corresponding distinguished triangles for $dg_x \in T^*_S\mathbb{C}^n$ a non-degenerate covector are related to the theory of the *nearby and vanishing cycle functors* of Deligne, as will be explained and discussed in later sections (see Remark 10.4.18):

$$NMD(\mathscr{F}^\bullet, S) \simeq \left(\varphi_{g|_{N\cap X}}[-1](\mathscr{F}^\bullet|_{N\cap X})\right)_x \tag{10.30}$$

and

$$R\Gamma(l_X, \mathscr{F}^\bullet) \simeq \left(\psi_{g|_{N\cap X}}(\mathscr{F}^\bullet|_{N\cap X})\right)_x \ , \tag{10.31}$$

resp.,

$$R\Gamma(l_X, \partial l_X, \mathscr{F}^\bullet) \simeq k_x^! \left(\psi_{g|_{N\cap X}}(\mathscr{F}^\bullet|_{N\cap X})\right) \ . \tag{10.32}$$

One also has for a holomorphic function germ $g\colon (X,x) \to (\mathbb{C},0)$ and $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X;R)$ an isomorphism (see Corollary 10.4.14 or, e.g., [109, Corollary 1.1.1])

$$LMD(\mathscr{F}^\bullet, Re(g), x) = \left(R\Gamma_{\{Re(g)\geq 0\}}(\mathscr{F}^\bullet)\right)_x \simeq \left(\varphi_g[-1](\mathscr{F}^\bullet)\right)_x \ . \tag{10.33}$$

Example 10.3.17 (Normal Morse Data for Products) Let X_i be a complex algebraic (or analytic) variety, with $\mathscr{F}_i^\bullet$ weakly constructible with respect to a Whitney stratification $\mathscr{S}_i$ of X_i, for $i = 1, 2$. Then $\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet$ is weakly constructible with respect to the product Whitney stratification $\mathscr{S}_1 \times \mathscr{S}_2$ of $X_1 \times X_2$ (with strata $S_1 \times S_2$ for $S_1 \in \mathscr{S}_1$ and $S_2 \in \mathscr{S}_2$). Moreover

$$NMD\left(\mathscr{F}_1^\bullet \overset{L}{\boxtimes} \mathscr{F}_2^\bullet, S_1 \times S_2\right) \simeq NMD(\mathscr{F}_1^\bullet, S_1) \overset{L}{\otimes} NMD(\mathscr{F}_2^\bullet, S_2) \ .$$

Let $N_i \subset \mathbb{C}^{n_i}$ be normal slices at $x_i \in S_i$, with a corresponding non-degenerate covector $dg_{i,x_i} \in T^*_{S_i}\mathbb{C}^{n_i}$ for a holomorphic function germ $g_i\colon (\mathbb{C}^{n_i}, x_i) \to (\mathbb{C}, 0)$ as above ($i = 1, 2$). Then $N = N_1 \times N_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ is a normal slice at $(x_1, x_2) \in S_1 \times S_2$, with $dg_{(x_1,x_2)} = \left(dg_{1,x_1}, dg_{2,x_2}\right)$ a corresponding non-degenerate covector defined by $g(z_1, z_2) := g_1(z_1) + g_2(z_2)$. Example 10.3.17 is then by (10.30) a very special case of the *Thom-Sebastiani Theorem for vanishing cycles* stated in Theorem 10.4.30.

10.3.2 Perverse Sheaf Description via Normal Morse Data

The distinguished triangles of Proposition 10.3.14 can be used to get (inductive) information about the "normal Morse data". The advantage in the complex context comes from the fact that $X \cap N \cap \{g = w\}$ is a complex analytic space (of lower dimension) with an induced *Whitney stratification*. Moreover, the distance function r used for the stratified Morse theory on $X \cap N \cap \{g = w\}$ (in the induction step) is now *strongly plurisubharmonic* so that one has nice estimates for the Morse index (of an approximation by a Morse function) of r and $-r$.

Definition 10.3.18 A C^k-function $f : X \to \mathbb{R}$ $(k \geq 2)$ is called *q-convex in* $x \in X$, if one can choose $\hat{f} : (\mathbb{C}^n, x) \to (\mathbb{R}, f(x))$ in a local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x)$ with $\hat{f}|_X = f$ so that its *Levi form* $L_{\hat{f}}(x)$ at the point x has at most q non-positive eigenvalues. Here the Levi form of $\hat{f}$ at the point x is the Hermitian form given by the matrix of partial derivatives:

$$L_{\hat{f}}(x) := \left(\frac{\partial^2 \hat{f}}{\partial z_j \, \partial \bar{z}_k}(x) \right)_{1 \leq j,k \leq n} .$$

f is called *q-convex* (in a subset $Z \subset X$) if it is q-convex in all points (of Z).

For example, a *strongly plurisubharmonic* C^k-function on an open subset U of $\mathbb{C}^n$ (with $k \geq 2$) is 0-convex in all of U. In particular, the distance function

$$r : \mathbb{C}^n \to \mathbb{R}^{\geq 0} ; \; r(z) := \sum_{i=1}^{n} z_i \bar{z}_i \quad \text{is 0-convex in } \mathbb{C}^n .$$

We next state the second *main result* for stratified Morse theory of constructible sheaves in the complex context (see [109, Theorem 6.0.1] for a more general formulation). We include here a complete proof based on Theorem 10.3.12, for explaining the use of stratified Morse theory and because similar ideas can also be used later on for other important results.

Theorem 10.3.19 *Let $\mathcal{S}$ be a Whitney stratification of the complex algebraic (or analytic) variety X, and consider a C^∞-function $f : X \to \mathbb{R}$. Assume $a < b$ are stratified regular values of f with respect to $\mathcal{S}$ such that $\{a \leq f \leq b\}$ is compact. Let $\mathcal{F}^\bullet \in D^b_{\mathcal{S}-(w)c}(X; R)$ be given.*

1. *Suppose f is q-convex in $\{a \leq f \leq b\}$, and let $q' := \min\{q, \dim X\}$.*

$$\mathcal{F}^\bullet \in {}^p D^{\leq n}_{\mathcal{S}}(X; R) \Rightarrow R\Gamma(\{a \leq f \leq b\}, \{f = a\}, \mathcal{F}^\bullet) \in {}^p D^{\leq n+q'}(\{pt\}; R).$$

If R is a Dedekind domain and $\mathcal{F}^\bullet$ is $\mathcal{S}$-constructible, then also

$$\mathcal{F}^\bullet \in {}^{p^+} D^{\leq n}_{\mathcal{S}}(X; R) \;\Rightarrow\; R\Gamma(\{a \leq f \leq b\}, \{f = a\}, \mathcal{F}^\bullet) \in {}^{p^+} D^{\leq n+q'}(\{pt\}; R) .$$

2. *Suppose $-f$ is q-convex in $\{a \leq f \leq b\}$, and let $q' := \min\{q, \dim X\}$.*

$$\mathcal{F}^\bullet \in {}^p D^{\geq n}_{\mathcal{S}}(X; R) \Rightarrow R\Gamma(\{a \leq f \leq b\}, \{f = a\}, \mathcal{F}^\bullet) \in {}^p D^{\geq n-q'}(\{pt\}; R).$$

If R is a Dedekind domain and $\mathscr{F}^\bullet$ is $\mathscr{S}$-constructible, then also

$$\mathscr{F}^\bullet \in {}^{p^+}D^{\geq n}_{\mathscr{S}}(X; R) \Rightarrow R\Gamma(\{a \leq f \leq b\}, \{f = a\}, \mathscr{F}^\bullet) \in {}^{p^+}D^{\geq n-q'}(\{pt\}; R)\,.$$

Here $R\Gamma(\{a \leq f \leq b\}, \{f = a\}, \mathscr{F}^\bullet) \in D^b_c(\{pt\}; R)$ in case $\mathscr{F}^\bullet$ is $\mathscr{S}$-constructible.

Proof We prove the claim by induction over $\dim X$, where the case $\dim X = 0$ is trivial (since the conditions to be checked contain the zero object and are stable by finite direct sums).

After approximating f by a stratified Morse function (see [8]), we can assume that f has only finitely many stratified Morse critical points in $\{a \leq f \leq b\}$. If the approximation is close enough, then also this new $\pm f$ is q-convex. This implies the following important estimate for the Morse index $\tau(f|_S)$ of $f|_S$ for a stratum $S \in \mathscr{S}$ in terms of $q' := \min\{q, \dim(X)\}$ (compare with [47, p.191, Lemma 4.A.2]):

$$\begin{cases} \tau(f|_S) \leq \dim S + q', & \text{if } f \text{ is q-convex.} \\ \tau(f|_S) = 2\dim S - \tau(-f|_S) \geq \dim S - q', & \text{if } -f \text{ is q-convex.} \end{cases} \tag{10.34}$$

Then we apply Lemma 10.3.8. Since the conditions to be checked contain the zero object and are stable by extensions, we only have to show that the "local Morse datum"

$$LMD(\mathscr{F}^\bullet, f, x) = (R\Gamma_{\{f \geq f(x)\}}(\mathscr{F}^\bullet))_x$$

of f at such a stratified Morse critical point x in a stratum $S \in \mathscr{S}$ belongs to ${}^{p^{(+)}}D^{\leq n+q'}(\{pt\}; R)$ for statement 1. (resp., belongs to ${}^{p^{(+)}}D^{\geq n+q'}(\{pt\}; R)$ for statement 2.).

By Theorem 10.3.12 we get

$$LMD(\mathscr{F}^\bullet, f, x) \simeq NMD(\mathscr{F}^\bullet, S)[-\tau],$$

with $NMD(\mathscr{F}^\bullet, S)$ the "normal Morse datum" of $\mathscr{F}^\bullet$ in the stratum S and τ the Morse index of $f|_S$. Here we can use the estimate (10.34) for this Morse index. Since the conditions to be checked contain the zero object and are stable by extensions as well as shifting by [1] (resp., [−1]) for the first (resp., second) result, we only have to show that (with $s := \dim S$):

$$NMD(\mathscr{F}^\bullet, S) \in \begin{cases} {}^{p^{(+)}}D^{\leq n-s}(\{pt\}; R) & \text{in part 1. of the theorem.} \\ {}^{p^{(+)}}D^{\geq n-s}(\{pt\}; R) & \text{in part 2. of the theorem.} \end{cases} \tag{10.35}$$

By Proposition 10.3.14 we have two distinguished triangles:

$$R\Gamma(l_X, \mathscr{F}^\bullet)[-1] \longrightarrow NMD(\mathscr{F}^\bullet, S) \longrightarrow k_x^*(\mathscr{F}^\bullet|_{N\cap X}) \longrightarrow \ ,$$

and

$$k_x^!(\mathscr{F}^\bullet|_{N\cap X}) \longrightarrow NMD(\mathscr{F}^\bullet, S) \longrightarrow R\Gamma(l_X, \partial l_X, \mathscr{F}^\bullet)[-1] \longrightarrow \ ,$$

where we use, as before, the notation (with $0 < |w| \ll \delta \ll 1$)

$$(l_X, \partial l_X) := (X \cap N \cap \{r \le \delta, g = w\}, X \cap N \cap \{r = \delta, g = w\})$$

for the *complex link* of X with respect to $S \in \mathscr{S}$. Also, $k_x : \{x\} \hookrightarrow N \cap X$ is the inclusion, and r is the *0-convex distance function* $r(z) := \sum_{i=1}^n z_i \bar{z}_i$ in our fixed local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x = 0)$.

But $N \cap \{g = w\}$ is *transversal* to $\mathscr{S}$ near x (for $0 < |w| \ll 1$). So

$$L := X \cap N \cap \{g = w\}$$

gets an induced Whitney stratification $\mathscr{S}|_L$. Moreover, for $0 < |w| \ll \delta \ll 1$, δ is a regular value of r with respect to $\mathscr{S}|_L$, since dg_x is non-degenerate. Then we can use the induction hypothesis with $q = 0$ for

$$\begin{cases} \mathscr{F}^\bullet|_L,\ r & \text{and } [a, b] := [-1, \delta] \text{ in part 1. of the theorem.} \\ \mathscr{F}^\bullet|_L,\ -r & \text{and } [a, b] := [-\delta, 1] \text{ in part 2. of the theorem.} \end{cases}$$

Note that

$$\mathscr{F}^\bullet|_L[-1] \in {}^{p^{(+)}}D^{\le n-s}_{\mathscr{S}|_L}(L; R) \quad \text{or} \quad {}^{p^{(+)}}D^{\ge n-s}_{\mathscr{S}|_L}(L; R)$$

by transversality and Proposition 10.2.27 (or its proof), with $\operatorname{codim} L = s + 1$.

By the induction hypothesis we get

$$R\Gamma(l_X, \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\le n-s}(\{pt\}; R)$$

or

$$R\Gamma(l_X, \partial l_X, \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\ge n-s}(\{pt\}; R)\ .$$

But, by definition, $k_x^*(\mathscr{F}^\bullet|_{N\cap X}) \in {}^{p^{(+)}}D^{\le n-s}(\{pt\}; R)$, which implies part 1. of the theorem by the first distinguished triangle above. Similarly

$$\mathscr{F}^\bullet|_{X\cap N} \in {}^{p^{(+)}}D^{\ge n-s}_{\mathscr{S}|_N}(X \cap N; R)$$

by Proposition 10.2.27 (or its proof), so that $k_x^!(\mathscr{F}^\bullet|_{N\cap X}) \in {}^{p^{(+)}}D^{\geq n-s}(\{pt\}; R)$. This implies part 2. of the theorem by the second distinguished triangle above. The last claim of Theorem 10.3.19 follows by inspection of the proof. □

Corollary 10.3.20 *Let $\bar{X}$ be a compact complex algebraic (or analytic) variety of dimension d, with $\mathscr{F}^\bullet \in D^b_{wc}(\bar{X}; R)$ (resp., $\mathscr{F}^\bullet \in D^b_c(\bar{X}; R)$ and R is a Dedekind domain in case we consider the dual perverse t-structure). Let $j: X \hookrightarrow \bar{X}$ be the inclusion of the open dense complement of a closed algebraic (or analytic) subset $Z \subset \bar{X}$ (so that $d = \dim X = \dim \bar{X}$). Then:*

1. $j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(X; R) \Rightarrow Rj_!j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(\bar{X}; R)$

$$\Rightarrow R\Gamma_c(X, j^*\mathscr{F}^\bullet) \simeq R\Gamma(\bar{X}, Rj_!j^*\mathscr{F}^\bullet) \in {}^{p^{(+)}}D^{\leq n+d}(\{pt\}; R)\,.$$

2. $j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(X; R) \Rightarrow Rj_*j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(\bar{X}; R)$

$$\Rightarrow R\Gamma(X, j^*\mathscr{F}^\bullet) \simeq R\Gamma(\bar{X}, Rj_*j^*\mathscr{F}^\bullet) \in {}^{p^{(+)}}D^{\geq n-d}(\{pt\}; R)\,.$$

Note that by a theorem of Nagata (see, e.g., [22, 79]) such a compactification is always available in the complex algebraic context.

The (method of) proof of Theorem 10.3.19 also implies the following.

Proposition 10.3.21 *Let $\mathscr{S}$ be a Whitney stratification of the complex algebraic (or analytic) variety X, and consider a differentiable function $f: X \to \mathbb{R}$. Assume $a < b$ are stratified regular values of f with respect to $\mathscr{S}$ such that $\{a \leq f \leq b\}$ is compact. Let $T \subset D^b(\{pt\}; R)$ be a fixed "null system", i.e., a full triangulated subcategory stable by isomorphisms.*

1. *Denote by $D^b_{(\mathscr{S}-)T-stalk}(X; R)$ the induced "null system" given by all $\mathscr{F}^\bullet \in D^b_{(\mathscr{S}-)wc}(X; R)$ with stalks $i_x^*\mathscr{F}^\bullet \in T$ for all $x \in X$. Then*

$$\mathscr{F}^\bullet \in D^b_{\mathscr{S}-T-stalk}(X; R) \Rightarrow R\Gamma(\{a \leq f \leq b\}, \{f = a\}, \mathscr{F}^\bullet) \in T\,.$$

2. *Denote by $D^b_{(\mathscr{S}-)T-costalk}(X; R)$ the induced "null system" given by all $\mathscr{F}^\bullet \in D^b_{(\mathscr{S}-)wc}(X; R)$ with costalks $i_x^!\mathscr{F}^\bullet \in T$ for all $x \in X$. Then*

$$\mathscr{F}^\bullet \in D^b_{\mathscr{S}-T-costalk}(X; R) \Rightarrow R\Gamma(\{a \leq f \leq b\}, \{f = a\}, \mathscr{F}^\bullet) \in T\,.$$

Corollary 10.3.22 *Let $\bar{X}$ be a compact complex algebraic (or analytic) variety, with $\mathscr{F}^\bullet \in D^b_{wc}(\bar{X}; R)$. Let $j: X \hookrightarrow \bar{X}$ be the inclusion of the open complement of a closed algebraic (or analytic) subset $Z \subset \bar{X}$, and fix a given "null system" $T \subset D^b(\{pt\}; R)$.*

1. $j^*\mathscr{F}^\bullet \in D^b_{T-stalk}(X; R) \Rightarrow Rj_!j^*\mathscr{F}^\bullet \in D^b_{T-stalk}(\bar{X}; R)$

$$\Rightarrow R\Gamma_c(X, j^*\mathscr{F}^\bullet) \simeq R\Gamma(\bar{X}, Rj_!j^*\mathscr{F}^\bullet) \in T\,.$$

2. $j^*\mathscr{F}^\bullet \in D^b_{T-costalk}(X; R) \Rightarrow Rj_*j^*\mathscr{F}^\bullet \in D^b_{T-costalk}(\bar{X}; R)$

$$\Rightarrow R\Gamma(X, j^*\mathscr{F}^\bullet) \simeq R\Gamma(\bar{X}, Rj_*j^*\mathscr{F}^\bullet) \in T .$$

Example 10.3.23 Let us mention here some important examples of such a "null system" $T \subset D^b(\{pt\}; R)$.

1. $T = D^b_c(\{pt\}; R)$ are the complexes with finitely generated cohomology. Then $D^b_{(\mathscr{S}-)T-stalk}(X; R)$ is the category of ($\mathscr{S}$-)constructible sheaf complexes.
2. Assume R is a Dedekind domain and $T = D^b_{tc}(\{pt\}; R)$ are the complexes with finitely generated torsion cohomology. Then $D^b_{(\mathscr{S}-)T-stalk}(X; R)$ is the category of ($\mathscr{S}$-)constructible torsion sheaf complexes.
3. Assume R is a field and $T = D^b_{c,\chi=0}(\{pt\}; R)$ are the complexes with finitely generated cohomology whose Euler characteristic is zero. Then $D^b_{(\mathscr{S}-)T-stalk}(X; R)$ is the category of ($\mathscr{S}$-)constructible sheaf complexes $\mathscr{F}^\bullet \in D^b_{(\mathscr{S}-)c}(X; R)$ with associated ($\mathscr{S}$-)constructible function

$$\chi_{stalk}(\mathscr{F}^\bullet) = 0 \in CF_{(\mathscr{S})}(X) ,$$

i.e., whose stalkwise Euler characteristic $\chi_{stalk}(\mathscr{F}^\bullet)(x) := \chi\left(\mathscr{F}^\bullet_x\right)$ vanishes for all $x \in X$. Then $[\mathscr{F}^\bullet] \in K_0\left(D^b_{(\mathscr{S}-)c}(X; R)\right)$ is in the kernel of

$$\chi_{stalk} \colon K_0\left(D^b_{(\mathscr{S})-c}(X; R)\right) \to CF_{(\mathscr{S})}(X) \iff \mathscr{F}^\bullet \in D^b_{(\mathscr{S}-)T-stalk}(X; R).$$

Example 10.3.24 Let $\bar{X}$ be a compact complex algebraic (or analytic) variety, with a given Whitney stratification $\mathscr{S}$. Let $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^b_{\mathscr{S}-c}(\bar{X}; R)$ and assume R a field. Let $j \colon X \hookrightarrow \bar{X}$ be the inclusion of the open complement of a closed algebraic (or analytic) subset $Z \subset \bar{X}$, which is a union of strata $S \in \mathscr{S}$, with $\mathscr{S}|_X$ the induced Whitney stratification of X. Then $\chi_{stalk}(j^*\mathscr{F}^\bullet) = \chi_{stalk}(j^*\mathscr{G}^\bullet)$ as a constructible function implies

$$\chi_c(X, j^*\mathscr{F}^\bullet) = \chi_c(X, j^*\mathscr{G}^\bullet) .$$

So the global Euler characteristic with compact support $\chi_c(X, j^*\mathscr{F}^\bullet)$ only depends on the underlying constructible function $\chi_{stalk}(j^*\mathscr{F}^\bullet) = \alpha \in CF_{\mathscr{S}}(X)$, with

$$\chi_c(X, \mathscr{F}^\bullet) = \int_X \alpha d\chi := \sum_{S \in \mathscr{S}|_X} \chi_c(S) \cdot \alpha(S) . \tag{10.36}$$

Here $\chi_c(S) := \chi_c(S, R_S) = \chi(H^*_c(S; R))$ is the corresponding Euler characteristic of a stratum $S \in \mathscr{S}|_X$.

Similar considerations apply if X is given as a relatively compact open subset $X = \{f < b\}$ (resp., a compact subset $X = \{f \leq b\}$) for a proper differentiable function $f: \bar{X} \to [a, d[\subset \mathbb{R}$, with $b < d$ a stratified regular value of f with respect to $\mathscr{S}$. Here $\bar{X}$ does not need to be compact, with strata S of $\mathscr{S}|_{\{f<b\}}$ given by the connected components of $S' \cap \{f < b\}$ for $S' \in \mathscr{S}$. Similarly for the decomposition $\mathscr{S}|_{\{f \leq b\}}$ of $\{f \leq b\}$, but here the parts $S' \cap \{f \leq b\}$ for $S' \in \mathscr{S}$ are complex manifolds with possible non-empty boundary $S' \cap \{f = b\}$.

The proof of Theorem 10.3.19 implies directly the following characterizations (see, e.g., [109, Corollary 6.0.2]).

Corollary 10.3.25 *Let $\mathscr{S}$ be a Whitney stratification of the complex algebraic (or analytic) variety X, and consider $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ (resp., $\mathscr{F}^\bullet \in D^b_c(X; R)$ and R is a Dedekind domain in case we consider the dual perverse t-structure). Then we have:*

1. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq 0}_{\mathscr{S}}(X; R) \iff NMD(\mathscr{F}^\bullet, S)[-\dim(S)] \in {}^{p^{(+)}}D^{\leq 0}(\{pt\}; R)$ *for all strata $S \in \mathscr{S}$.*
2. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq 0}_{\mathscr{S}}(X; R) \iff NMD(\mathscr{F}^\bullet, S)[-\dim(S)] \in {}^{p^{(+)}}D^{\geq 0}(\{pt\}; R)$ *for all strata $S \in \mathscr{S}$.*
3. *Let $T \subset D^b(\{pt\}; R)$ be a fixed "null system". Then*

$$\mathscr{F}^\bullet \in D^b_{\mathscr{S}-T-stalk}(X; R) \iff NMD(\mathscr{F}^\bullet, S) \in T \quad \text{for all strata } S \in \mathscr{S}$$
$$\iff \mathscr{F}^\bullet \in D^b_{\mathscr{S}-T-costalk}(X; R)\,.$$

Proof The implications $\Rightarrow$ are already contained in (10.35) from the proof of Theorem 10.3.19. For the other implications $\Leftarrow$ in case of a point stratum $S = \{x\}$ one applies the proof of Theorem 10.3.19 for the (co)stalk description

$$i_x^*\mathscr{F}^\bullet \simeq R\Gamma(\{r \leq \delta\}, \mathscr{F}^\bullet) \simeq R\Gamma(\{-1 \leq r \leq \delta\}, \{r = -1\}, \mathscr{F}^\bullet)$$

and

$$i_x^!\mathscr{F}^\bullet \simeq R\Gamma_c(\{r < \delta\}, \mathscr{F}^\bullet) \simeq R\Gamma(\{-\delta \leq -r \leq 1\}, \{-r = -\delta\}, \mathscr{F}^\bullet)\,,$$

with $0 < \delta$ small and r the *0-convex distance function* $r(z) := \sum_{i=1}^n z_i\bar{z}_i$ in some fixed local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x = 0)$. The general case is reduced to a point stratum via intersecting with a normal slice N to $S \in \mathscr{S}$ at a given point $x \in S$. Here, the normal Morse data $NMD(\mathscr{F}^\bullet, S') = NMD(\mathscr{F}^\bullet|_N, S' \cap N)$ for $S' \in \mathscr{S}$ close to x does not change. □

Example 10.3.26 Let $\mathscr{S}$ be a Whitney stratification of the complex algebraic (or analytic) variety X, with $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$. Then $\mathscr{F}^\bullet$ is a *perverse sheaf* if and only if

$$H^i\,(NMD(\mathscr{F}^\bullet, S)[-\dim(S)]) = 0 \quad \text{for } i \neq 0 \text{ and all strata } S \in \mathscr{S}.$$

If, moreover, $\mathscr{F}^\bullet$ is constructible and R a Dedekind domain, then $\mathscr{F}^\bullet$ is a *strongly perverse sheaf* if and only if, in addition,

$$H^0\,(NMD(\mathscr{F}^\bullet, S)[-\dim(S)]) \quad \text{is finitely generated and torsion-free}$$

for all strata $S \in \mathscr{S}$.

Example 10.3.27 Let $\mathscr{S}$ be a Whitney stratification of the complex algebraic (or analytic) variety X, with $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$ and R a field. Then $\chi_{stalk}(\mathscr{F}^\bullet) = \chi_{stalk}(\mathscr{G}^\bullet)$ as an $\mathscr{S}$-constructible function if and only if

$$\chi\,(NMD(\mathscr{F}^\bullet, S)) = \chi\,(NMD(\mathscr{G}^\bullet, S)) \quad \text{for all strata } S \in \mathscr{S}.$$

So the Euler characteristic $\chi\,(NMD(\mathscr{F}^\bullet, S)) =: \chi\,(NMD(\alpha, S))$ for $S \in \mathscr{S}$ depends only on the underlying $\mathscr{S}$-constructible function $\alpha = \chi_{stalk}(\mathscr{F}^\bullet)$.

Remark 10.3.28 Corollary 10.3.25 also shows that the corresponding properties of the normal Morse data do not depend on the choice of the Whitney stratification $\mathscr{S}$ of X with $\mathscr{F}^\bullet$ $\mathscr{S}$-(weakly) constructible. The given characterization of the perverse t-structure in terms of normal Morse data is equivalent (in an embedded context) to [66, Theorem 10.3.12], if one identifies the normal Morse data with the corresponding "micro-local type" in the sense of Kashiwara-Schapira [66, Proposition 6.6.1(ii)] (as shown in [109, Equation (5.52) on p.311]). The characterization of perverse sheaves in terms of normal Morse data corresponds to the "dimension axiom" in the Morse theoretic approach to perverse sheaves as outlined in [81]. It also implies the *purity result* stated in [47, p.223, 6.A.3].

Here are the final applications of Theorem 10.3.19 for this section (see, e.g., [109, Corollary 6.0.5]), which for Y a point space reduce to Corollary 10.3.20.

Proposition 10.3.29 *Let $\bar{f}\colon \bar{X} \to Y$ be a proper holomorphic map of complex varieties, with $Z \subset \bar{X}$ a closed complex subvariety. Consider the induced holomorphic map*

$$f := \bar{f}|_X\colon\; X := \bar{X}\backslash Z \to Y\,.$$

Suppose the fiber dimension of f is bounded above by d, and consider $\mathscr{F}^\bullet \in D^b_{(w)c}(X; R)$ (resp., $\mathscr{F}^\bullet \in D^b_c(X; R)$ and R is a Dedekind domain in case we consider the dual perverse t-structure). Then:

1. $\mathscr{F}^\bullet|_X \in {}^{p^{(+)}}D^{\geq n}(X; R) \;\Rightarrow\; Rf_*(\mathscr{F}^\bullet|_X) \in {}^{p^{(+)}}D^{\geq n-d}(Y; R)$.
2. $\mathscr{F}^\bullet|_X \in {}^{p^{(+)}}D^{\leq n}(X; R) \;\Rightarrow\; Rf_!(\mathscr{F}^\bullet|_X) \in {}^{p^{(+)}}D^{\leq n+d}(Y; R)$.

Note that by a theorem of Nagata (see, e.g., [22, 79]) such a (partial) compactification $\bar{f}\colon \bar{X} \to Y$ is always available in the complex algebraic context for a morphism $f\colon X \to Y$.

Corollary 10.3.30 *Let $f: X \to Y$ be a morphism of complex algebraic varieties. Suppose the fiber dimension of f is bounded above by d, and consider $\mathscr{F}^\bullet \in D^b_{(w)c}(X; R)$ (resp., $\mathscr{F}^\bullet \in D^b_c(X; R)$ and R is a Dedekind domain in case we consider the dual perverse t-structure). Then:*

1. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(X; R) \Rightarrow Rf_*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n-d}(Y; R)$.
2. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(X; R) \Rightarrow Rf_!\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n+d}(Y; R)$.

Example 10.3.31 (Finite Morphism) Let $f: X \to Y$ be a *finite* morphism of complex algebraic (or analytic) varieties (i.e., f is proper with fiber dimension $d = 0$). Then $Rf_! = Rf_* = f_*$ is *t-exact* with respect to the perverse t-structure, and in case R a Dedekind domain also with respect to the dual t-structure.

10.3.3 Characteristic Cycles and Index Theorems

In this section, we give an introduction to the theory of Lagrangian cycles in the complex analytic and algebraic context, using the language of *stratified Morse theory for constructible functions and sheaves*, as developed in the previous sections. We explain from this viewpoint the Euler isomorphism between constructible functions and Lagrangian cycles, together with some index theorems.

The theory of Lagrangian cycles in the complex analytic context started in 1973 with Kashiwara's local index formula for holonomic D-modules, as formulated in [62] and proved in [63, Chapter 6] (compare also with [82]). Kashiwara introduced for this a local invariant of a singular complex analytic set, which around the same time was independently introduced by MacPherson as the *local Euler obstruction* in his celebrated work [80] on Chern homology classes for singular complex algebraic varieties (establishing a conjecture of Grothendieck and Deligne). It was Dubson [15, 33, 34] who observed some years later that these two invariants are the same. The definitions of Kashiwara and MacPherson are of transcendental nature. A purely algebraic definition of the "local Euler obstruction" was found by Gonzalez-Sprinberg and Verdier [44] (compare with [40, Example 4.2.9]).

In this section we work in a global embedded context, with $k: X \hookrightarrow M$ the closed embedding of a complex algebraic (or analytic) variety X into a complex algebraic (or analytic) manifold M of dimension $\dim M = m$ (with M pure-dimensional or otherwise $\dim M$ viewed as a locally constant function). Let $\mathscr{S}$ be a given Whitney stratification of X, with conormal space

$$T^*_{\mathscr{S}}M := \bigcup_{S\in\mathscr{S}} T^*_S M \hookrightarrow T^*M|_X \hookrightarrow T^*M\,.$$

Here $T^*_{\mathscr{S}}M$ is a closed subset by the *Whitney a-condition* of our stratification $\mathscr{S}$. The open subset of *non-degenerate* covectors is given by

$$\left(T^*_{\mathscr{S}}M\right)^\circ = \bigcup_{S\in\mathscr{S}} \left(T^*_S M\right)^\circ ,$$

with

$$\left(T^*_S M\right)^\circ := T^*_S M \setminus \bigcup_{S\neq S'\in\mathscr{S}} \overline{T^*_{S'}M} .$$

Definition 10.3.32 The abelian group $L(\mathscr{S}, T^*M)$ of ($\mathbb{C}^*$-conic) *Lagrangian cycles* in $T^*_{\mathscr{S}}M$ is given by

$$L(\mathscr{S}, T^*M) := H^{BM}_{2m}\left(T^*_{\mathscr{S}}M; \mathbb{Z}\right) ,$$

so that a corresponding Lagrangian cycle is uniquely given by a (locally) finite sum

$$\sum_{S\in\mathscr{S}} m(S)\cdot\left[\overline{T^*_S M}\right] \quad \text{with } m(S)\in\mathbb{Z}, \tag{10.37}$$

and $\left[\overline{T^*_S M}\right]$ the corresponding *fundamental class*. If we do not want to fix the stratification $\mathscr{S}$, then we denote by $L(T^*M|_X)$ the abelian group of ($\mathbb{C}^*$-conic) *Lagrangian cycles* in $T^*M|_X$ given by a similar (locally) finite $\mathbb{Z}$-linear combination of fundamental classes

$$\left[T^*_Z M\right] := \left[\overline{T^*_{Z_{reg}} M}\right]$$

with $Z \hookrightarrow X$ an irreducible closed algebraic (or analytic) subvariety.

Remark 10.3.33 In the complex algebraic context, the Borel-Moore homology group

$$H^{BM}_{2m}\left(T^*_{\mathscr{S}}M; \mathbb{Z}\right) \simeq A_m\left(T^*_{\mathscr{S}}M\right) \simeq Z_m\left(T^*_{\mathscr{S}}M\right)$$

is also the same as the corresponding *Chow- and cycle group* of $T^*_{\mathscr{S}}M$ as in [40], since $T^*_{\mathscr{S}}M$ is of dimension m. So many of the following results could also be stated in this language, but this does not apply to our method of proof based on *stratified Morse theory for constructible functions and sheaves*. For this reason, we work in this section only with the homological language, which at the same time also applies to the complex analytic context.

For simplicity, in this section we only consider ($\mathscr{S}$-)constructible sheaf complexes in $D^b_{(\mathscr{S}-)c}(-; R)$ with R a field and their corresponding (compactly supported) *Euler characteristic*, even though most of our results and proofs work for more general "stalk properties and corresponding additive functions" (see, e.g., [109, Section 5.0.3]).

Definition 10.3.34 The *characteristic cycle* $CC(\mathscr{F}^\bullet) \in L(\mathscr{S}, T^*M)$ of a constructible complex $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$ is defined via the following multiplicities in (10.37):

$$m(S) := (-1)^{\dim S} \cdot \chi(NMD(\mathscr{F}^\bullet, S)) \in \mathbb{Z} \quad \text{for a stratum } S \in \mathscr{S}. \tag{10.38}$$

By Example 10.3.27 the Euler characteristic

$$\chi(NMD(\mathscr{F}^\bullet, S)) = \chi(NMD(\alpha, S))$$

only depends on the associated $\mathscr{S}$-constructible function $\alpha = \chi_{stalk}(\mathscr{F}^\bullet) \in CF_{\mathscr{S}}(X)$ so that we can also define the characteristic cycle $CC(\alpha)$ of $\alpha \in CF_{\mathscr{S}}(X)$ via

$$(-1)^{\dim S} \cdot m(S) := \chi(NMD(\alpha, S)) = \alpha(x) - \int_{l_X} \alpha \, d\chi \in \mathbb{Z}\,, \tag{10.39}$$

with $x \in S$, $(l_X, \partial l_X)$ the corresponding *complex link* of X in $S \in \mathscr{S}$ and $l_X^\circ := l_X \backslash \partial l_X$. This can also be reformulated as

$$(-1)^{\dim S} \cdot m(S) = \alpha(x) - \sum_{S \subset \partial S'} c(S, S') \cdot \alpha(S') \in \mathbb{Z}\,, \tag{10.40}$$

with $c(S, S') := \int_{l_X} 1_{S'} d\chi = \int_{l_X^\circ} 1_{S'} d\chi = \chi_c(l_X^\circ \cap S', \mathbb{Q})$ for $S \subset \partial S'$ and $S, S' \in \mathscr{S}$ a topological invariant of the Whitney stratification $\mathscr{S}$ of X. Note that

$$\chi_c(l_X \cap S', \mathbb{Q}) = \chi_c(l_X^\circ \cap S', \mathbb{Q})\,, \tag{10.41}$$

since ∂l_X is a compact real analytic Whitney stratified set with *odd-dimensional* strata so that $\chi_c(\partial l_X \cap S', \mathbb{Q}) = 0$ (see, e.g., [109, Lemma 5.0.3] and [120]).

Here, the last equality in (10.39) follows from the distinguished triangle (10.27). The choice of the sign $(-1)^{\dim S}$ will become clear in a moment.

Example 10.3.35 Let $X \hookrightarrow M$ be a *closed* complex submanifold of M, with strata $S \in \mathscr{S}$ given by the connected components of X. Let $\mathscr{L}$ be a local system of rank r on X. Then

$$CC(\mathscr{L}) = CC(r \cdot 1_X) = (-1)^{\dim X} \cdot r \cdot \left[T^*_X M\right] .$$

Similarly, in the general context of the above Definition with $S \in \mathscr{S}$ an *open* stratum of X and $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$. Then by Example 10.3.15, the multiplicity $m(S)$ of the characteristic cycle $CC(\mathscr{F}^\bullet)$ is given by $m(S) = (-1)^{\dim S} \cdot \chi_{stalk}(\mathscr{F}^\bullet)(S) \in \mathbb{Z}$.

Remark 10.3.36 Let $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$ be given, with $k\colon X \hookrightarrow M$ the global closed embedding into the complex manifold M. Then for a stratum $S \in \mathscr{S}$ we get by definition

$$\overline{T^*_S M} \subset supp(CC(\mathscr{F}^\bullet)) \iff \chi(NMD(\mathscr{F}^\bullet, S)) \neq 0 .$$

And, in general, this is much weaker than the corresponding condition

$$\overline{T^*_S M} \subset \mu supp(Rk_*\mathscr{F}^\bullet) \iff NMD(\mathscr{F}^\bullet, S) \not\simeq 0$$

from Proposition 10.3.13. So in general the inclusion

$$supp(CC(\mathscr{F}^\bullet)) \subset \mu supp(Rk_*\mathscr{F}^\bullet) \quad \text{for } \mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R) \tag{10.42}$$

does not need to be an equality. But Example 10.3.26 implies

$$supp(CC(\mathscr{F}^\bullet)) = \mu supp(Rk_*\mathscr{F}^\bullet) \quad \text{for } \mathscr{F}^\bullet \in Perv_{\mathscr{S}}(X; R). \tag{10.43}$$

Moreover, Example 10.3.26 also implies that $CC(\mathscr{F}^\bullet)$ is an *effective* Lagrangian cycle for $0 \neq \mathscr{F}^\bullet \in Perv_{\mathscr{S}}(X; R)$, i.e., the multiplicity $m(S) \geq 0$ for all $S \in \mathscr{S}$, with $m(S) > 0$ for S a top-dimensional stratum in $supp(\mathscr{F}^\bullet)$.

By induction on $\dim X$ and Example 10.3.35, one easily gets that the induced group homomorphism

$$CC\colon K_0(D^b_{\mathscr{S}-c}(X; R)) \to L(\mathscr{S}, T^*M) = H^{BM}_{2m}\left(T^*_{\mathscr{S}}M; \mathbb{Z}\right)$$

is *surjective*. Moreover, CC factorizes over χ_{stalk} and both homomorphisms

$$CC\colon K_0(D^b_{\mathscr{S}-c}(X; R)) \to L(\mathscr{S}, T^*M) \quad \text{and} \quad CC\colon CF_{\mathscr{S}}(X) \to L(\mathscr{S}, T^*M)$$

have by Example 10.3.27 the *same kernel*, so that CC induces for a fixed Whitney stratification $\mathscr{S}$ of X an isomorphism of abelian groups

$$CC\colon CF_{\mathscr{S}}(X) \xrightarrow{\sim} L(\mathscr{S}, T^*M) = H^{BM}_{2m}\left(T^*_{\mathscr{S}}M; \mathbb{Z}\right) . \tag{10.44}$$

Finally, $CC(\mathscr{F}^\bullet) \in L(T^*M|_X)$ (and then also $CC(\alpha) \in L(T^*M|_X)$) does not depend on the choice of the Whitney stratification $\mathscr{S}$ with $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$. In fact, if $\mathscr{T}$ is another Whitney stratification of X which refines $\mathscr{S}$, then any stratum $T \in \mathscr{T}$ is contained in a stratum $S \in \mathscr{S}$. If $T \subset S$ is *open*, then $NMD(\mathscr{F}^\bullet, S) = NMD(\mathscr{F}^\bullet, T)$ just by the definitions. If $T \subset S$ is a proper closed subset, then $NMD(\mathscr{F}^\bullet, T) = 0$. In fact, if $x \in T$ is a stratified Morse critical point of a function germ $f\colon (M, x) \to (\mathbb{R}, f(x))$ with respect to $\mathscr{T}$, then $LMD(\mathscr{F}^\bullet, f, x) \simeq NMD(\mathscr{F}^\bullet, T)[-\tau]$ by Theorem 10.3.12. But then $x \in S$ is not a stratified critical point of $f\colon (M, x) \to (\mathbb{R}, f(x))$ with respect to $\mathscr{S}$ so that $LMD(\mathscr{F}^\bullet, f, x) = 0$ by Lemma 10.3.3. So in the limit over all such Whiney stratifications $\mathscr{S}$ of X, we get a surjective group homomorphism

$$CC\colon K_0(D_c^b(X;R)) \to L(T^*M|_X)\,,$$

which factorizes via χ_{stalk} over an isomorphism

$$CC\colon CF(X) \xrightarrow{\ \sim\ } L(T^*M|_X)\,. \tag{10.45}$$

Let us now explain the choice of the sign $(-1)^{\dim S}$ in the Definition 10.3.34 of the characteristic cycle CC.

The differentiable C^k-function germ $f\colon (M, x) \to (\mathbb{R}, f(x))$ (with $2 \le k \le \infty$) has a *stratified Morse critical point* at $x \in S$, for a stratum $S \in \mathscr{S}$, if and only if (see, e.g., [66, p.311] or [109, p.286]):

$$\begin{cases} df_x \in \left(T_S^*M\right)^\circ\,, & \text{i.e., } df_x \in T_S^*M \text{ is non-degenerate, and} \\ \text{the graph } df(M) \subset T^*M \text{ of } df \text{ intersects } T_S^*M \ \textit{transversally} \text{ at } df_x\,. \end{cases} \tag{10.46}$$

Using the *complex orientations* of T^*M and $M \simeq df(M)$ one gets in this case for the corresponding *local intersection number*:

$$(-1)^{\dim S} \cdot \sharp_{df_x}\big([T_S^*M] \cap [df(M)]\big) = (-1)^\lambda\,, \tag{10.47}$$

with λ the *Morse index* of $f|_S$ at x (i.e., its Hessian at x has exactly λ negative eigenvalues). Here $[df(M)] \in H_{2m}^{BM}(df(M);\mathbb{Z})$ is the fundamental class of the oriented manifold $df(M) \simeq M$. The local intersection number $\sharp_{df_x}$ in (10.47) is defined similarly (with $A \cap B = \{df_x\}$) to the following *global intersection number*

$$\sharp\colon\ H_{2m}^{BM}(A;\mathbb{Z}) \times H_{2m}^{BM}(B;\mathbb{Z}) \to \mathbb{Z}$$

for two *closed* subsets $A, B \subset T^*M$ with $A \cap B$ *compact*:

$$\begin{array}{ccccc} H_A^{2m}(T^*M;\mathbb{Z}) \times H_B^{2m}(T^*M;\mathbb{Z}) & \xrightarrow{\ \cup\ } & H_c^{4m}(T^*M;\mathbb{Z}) & \xrightarrow{\ tr\ } & \mathbb{Z} \\ {\scriptstyle PD}\downarrow\wr & & {\scriptstyle PD}\downarrow\wr & & \| \\ H_{2m}^{BM}(A;\mathbb{Z}) \times H_{2m}^{BM}(B;\mathbb{Z}) & \xrightarrow{\ \cap\ } & H_0(T^*M;\mathbb{Z}) & \xrightarrow{\ deg\ } & \mathbb{Z} \end{array}$$

Here, PD is *Poincaré duality* given by the cap-product with the fundamental class $[T^*M]$ of the complex manifold T^*M.

Example 10.3.37 Let $g\colon (M, x) \to (\mathbb{C}, g(x))$ be a holomorphic function germ so that the graph $dg(M) \subset T^*M$ of dg intersects T_S^*M *transversally* at dg_x. Then

$$\sharp_{dg_x}\big([T_S^*M]\cap[dg(M)]\big)=1$$

and $g|_S$ has a *complex Morse* critical point at $x\in S$. But if we identify the complex cotangent bundle T^*M with the real cotangent bundle of the underlying real (oriented) manifold, then the graph $dg(M)$ of g gets identified with the graph $df(M)$ of the real part $f:=Re(g)$ of g. So, with the complex orientations, one also has

$$\sharp_{df_x}\big([T_S^*M]\cap[df(M)]\big)=1\,,$$

but $f|_S$ has at $x\in S$ a classical Morse critical point of index $\lambda=\dim S$.

The intersection theory in the ambient cotangent bundle T^*M is used for the formulation of the following beautiful *intersection formula* (see, e.g., [109, Theorem 5.0.4] for a more general version).

Theorem 10.3.38 *Let $f\colon M\to[a,d[\subset\mathbb{R}$ be a C^∞-function ($a\le d\le\infty$), with $f|_X$ proper. Suppose that $T^*_{\mathscr{S}}M\cap df(M)$ is compact, with $T^*_{\mathscr{S}}M$ the union of conormal spaces T^*_SM to the strata $S\in\mathscr{S}$ of the Whitney stratification $\mathscr{S}$ of $X\hookrightarrow M$. Then we have for all $\mathscr{F}^\bullet\in D^b_{\mathscr{S}-c}(X;R)$:*

$$\begin{aligned}\dim_R H^*(X,\mathscr{F}^\bullet)<\infty\,,\ \textit{with}&\\ \chi(X,\mathscr{F}^\bullet)=\sharp(\,CC(\chi_{stalk}(\mathscr{F}^\bullet))\cap[df(M)]\,)\,,&\end{aligned}\tag{10.48}$$

and

$$\begin{aligned}\dim_R H_c^*(X,\mathscr{F}^\bullet)<\infty\,,\ \textit{with}&\\ \chi_c(X,\mathscr{F}^\bullet)=\sharp(\,CC(\chi_{stalk}(\mathscr{F}^\bullet))\cap[-df(M)]\,).&\end{aligned}\tag{10.49}$$

Proof The reader should compare the following proof also with the proof of Theorem 10.3.19. By approximation [8], we can assume that f has only *stratified Morse critical points* with respect to $\mathscr{S}$, since the corresponding intersection number does not change by a homotopy argument. By assumption, $T^*_{\mathscr{S}}M\cap df(M)$ is compact so that f has only finitely many stratified Morse critical points. Choose $b\in[a,d[$ with all these critical points contained in $\{a\le f<b\}$. Then one gets by Lemma 10.3.3

$$R\Gamma(X\cap\{f\le b\},\mathscr{F}^\bullet)\simeq R\Gamma(X,\mathscr{F}^\bullet)\,,$$

and

$$R\Gamma_c(X\cap\{f<b\},\mathscr{F}^\bullet)\simeq R\Gamma_c(X,\mathscr{F}^\bullet)\,.$$

And these complexes belong to $D^b_c(\{pt\};R)$ by Proposition 10.3.21. Similarly, the normal Morse data $NMD(\mathscr{F}^\bullet,S)\in D^b_c(\{pt\};R)$ by Corollary 10.3.25. Since the

Euler characteristic χ is additive, by Corollary 10.3.7 and Lemma 10.3.8 it is enough to show that for such a stratified Morse critical point $x \in S$:

$$\chi(LMD(\mathscr{F}^\bullet, \pm f, x)) = \sharp_{\pm df_x}\left(CC(\mathscr{F}^\bullet) \cap [\pm df(M)] \right).$$

But this follows from Theorem 10.3.12:

$$LMD(\mathscr{F}^\bullet, \pm f, x) \simeq NMD(\mathscr{F}^\bullet, S)[-\tau],$$

with τ the Morse index of $\pm f|_S$ at x. Finally, by the definition of $CC(\mathscr{F}^\bullet)$ and Equation (10.47) we have

$$\begin{aligned}\chi(LMD(\mathscr{F}^\bullet, \pm f, x)) &= (-1)^\tau \cdot \chi(NMD(\mathscr{F}^\bullet, S)) \\ &= \sharp_{\pm df_x}\left(CC(\mathscr{F}^\bullet) \cap [\pm df(M)] \right),\end{aligned}$$

which completes the proof. □

This intersection Theorem 10.3.38 goes back to Dubson [35], Sabbah [103] and Ginsburg [42] (partially in the context of holonomic D-modules).

Note that the condition that $T^*_{\mathscr{S}}M \cap df(M)$ is compact just means that the stratified critical locus $\mathrm{Sing}_{\mathscr{S}}(f) := \bigcup_{S\in\mathscr{S}} \mathrm{Sing}(f|_S)$ is compact. The intersection formula above can also be reformulated in the language of constructible functions.

Assume that $f : M \to [a, d[\subset \mathbb{R}$ is a C^∞-function with $f|_X$ *proper* and $\pi\left(T^*_{\mathscr{S}}M \cap df(M)\right) \subset [a, b]$ for some $b \in [a, d[$, with π the projection $T^*M \to M$. In particular, $T^*_{\mathscr{S}}M \cap df(M)$ is *compact*.

Then one also has the following counterpart of Theorem 10.3.38 for a constructible function $\alpha \in CF_{\mathscr{S}}(X)$:

$$\int_{X\cap\{f\le r\}} \alpha \, d\chi = \sharp\left(CC(\alpha) \cap [df(M)] \right) \quad \text{for all } r \in]b, d[\,, \tag{10.50}$$

and

$$\int_{X\cap\{f< r\}} \alpha \, d\chi = \sharp\left(CC(\alpha) \cap [-df(M)] \right) \quad \text{for all } r \in]b, d[\,. \tag{10.51}$$

Note that $r \in]b, d[$ is a stratified regular value of $f|_X$ so that the left Euler characteristics are defined by Example 10.3.24. Let us give some examples, where the above conditions on f are satisfied.

Example 10.3.39 (Global Index Formula and Poincaré-Hopf Theorem for Singular Spaces) Let $X \hookrightarrow M$ be a compact complex algebraic (or analytic) subvariety of M, with $\mathscr{S}$ a given Whitney stratification of X. Then we can take for f a constant function so that $df(M)$ is the zero-section. Then we get for any $\mathscr{S}$-constructible function $\alpha \in CF_{\mathscr{S}}(X)$:

$$\int_X \alpha \, d\chi = \sharp(\, CC(\alpha) \cap [T^*_M M]\,) = \sharp(\, CC(\alpha) \cap [\omega(M)]\,) \tag{10.52}$$

for any differentiable one-form ω on M, i.e., a section of $T^*M \to M$. Assume that $T^*_{\mathscr{S}}M \cap \omega(M)$ is finite, i.e., the set $\mathrm{Sing}_{\mathscr{S}}(\omega) := \bigcup_{S\in\mathscr{S}} \mathrm{Sing}(\omega|_S)$ of critical points of ω with respect to $\mathscr{S}$ is finite. If one defines for $x \in \mathrm{Sing}_{\mathscr{S}}(\omega)$ and $\alpha \in CF_{\mathscr{S}}(X)$ the *local index* $ind_x(\omega, \alpha)$ of ω with respect to α by

$$ind_x(\omega, \alpha) := \sharp_{\omega_x}\big(CC(\alpha) \cap [\omega(M)]\big)\,, \tag{10.53}$$

then one gets the Poincaré-Hopf theorem

$$\int_X \alpha \, d\chi = \sum_{x\in\mathrm{Sing}_{\mathscr{S}}(\omega)} ind_x(\omega, \alpha)\,. \tag{10.54}$$

See also [117] for an overview of different indices of vector fields and one forms in the context of singular complex varieties. The following recent application is due to [4] (and implicitly already contained in [39, 113]).

Example 10.3.40 (Effective Characteristic Cycles on an Abelian Variety) Let $M = A$ be a complex abelian variety, so that A is a projective abelian algebraic group with trivial cotangent bundle T^*A. Let $\alpha \in CF(A)$ be a constructible function with $CC(\alpha)$ an *effective* cycle in T^*A. By Kleiman's transversality theorem, a *generic algebraic one-form* ω intersects $supp(CC(\alpha))$ only in finitely many points ω_x (see, e.g., [113, Proposition 2.8]). Then the local intersection number (10.53) as well as the global Euler characteristic (10.53) are non-negative:

$$ind_x(\omega, \alpha) = \sharp_{\omega_x}\big(CC(\alpha) \cap [\omega(A)]\big) \geq 0 \quad \text{and} \quad \int_A \alpha \, d\chi \geq 0\,. \tag{10.55}$$

For example α could be given by

1. $\alpha = Eu^\vee_Z = (-1)^{\dim Z} \cdot Eu_Z$ is the dual Euler obstruction of a pure dimensional closed subvariety $Z \hookrightarrow A$ (see Eq. (10.59) below).
2. $\alpha = \chi_{stalk}(\mathscr{F}^\bullet)$ for a perverse sheaf $\mathscr{F}^\bullet \in Perv(A; R)$.
3. $\alpha = \chi_{stalk}(Rp_*\mathscr{F}^\bullet)$ for a perverse sheaf $\mathscr{F}^\bullet \in Perv(G; R)$ on a *semi-abelian variety* G, with $p\colon G \to A$ the projection onto the corresponding abelian variety A. Then

$$\chi(G; \mathscr{F}^\bullet) = \chi(A; Rp_*\mathscr{F}^\bullet) = \int_A \alpha \, d\chi \geq 0\,.$$

See [4, Proposition 8.4, Example 8.5] for a more general class of morphisms $p\colon G \to A$ to an abelian variety with this property for any perverse sheaf $\mathscr{F}^\bullet \in Perv(G; R)$ (for a different approach via *generic vanishing theorems* on semi-abelian varieties see (10.128) in Sect. 10.6).

Example 10.3.41 (Affine Varieties and Global Euler Obstruction) Let $X \hookrightarrow \mathbb{C}^n$ be an *affine* complex algebraic variety endowed with a complex algebraic Whitney stratification $\mathscr{S}$. Then the *semi-algebraic* distance function $r\colon \mathbb{C}^n \to [0, \infty[$, $r(z) := \sum_{i=1}^n z_i \cdot \bar{z}_i$ has only finitely many critical values with respect to $\mathscr{S}$. So one gets for $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$:

$$\chi(X, \mathscr{F}^\bullet) = \sharp(\, CC(\chi_{stalk}(\mathscr{F}^\bullet)) \cap [dr(\mathbb{C}^n)]\,)\,,$$

and

$$\chi_c(X, \mathscr{F}^\bullet) = \sharp(\, CC(\chi_{stalk}(\mathscr{F}^\bullet)) \cap [-dr(\mathbb{C}^n)]\,)\,.$$

Similarly, for a constructible function $\alpha \in CF_{\mathscr{S}}(X)$:

$$\int_{X\cap\{r\leq\delta\}} \alpha \, d\chi = \sharp(\, CC(\alpha) \cap [dr(\mathbb{C}^n)]\,) \quad \text{for all } \delta > 0 \text{ large enough,}$$

and

$$\int_{X\cap\{r<\delta\}} \alpha \, d\chi = \sharp(\, CC(\alpha) \cap [-dr(\mathbb{C}^n)]\,) \quad \text{for all } \delta > 0 \text{ large enough.}$$

If X is pure dimensional and $CC(\alpha) = (-1)^{\dim X} \cdot \left[\overline{T^*_{X_{reg}} M}\right]$, then the intersection number

$$Eu(X) := (-1)^{\dim X} \cdot \sharp\left(\left[\overline{T^*_{X_{reg}} M}\right] \cap [dr(\mathbb{C}^n)]\right)$$

is by [109, Equation (5.6.4)] the *global Euler obstruction* of X in the sense of Seade-Tibăr-Verjovsky [116] (and compare also with [35, Theorem 1]). The sign $(-1)^{\dim X}$ comes from different orientation conventions. As we will see in a moment, here $\alpha = Eu_X$ is the *local Euler obstruction* function of X as defined by MacPherson [80].

Example 10.3.42 (Local Index Formula) Let $f\colon M \to [0, d[\, \subset \mathbb{R}$ be a *real analytic* function, with $f|_X$ proper. Then

$$f \circ \pi(\, T^*_{\mathscr{S}} M \cap df(M)\,) \subset [0, d[$$

is *subanalytic* and, by the *curve selection lemma*, it is *discrete*. Especially,

$$\pi(\, T^*_{\mathscr{S}} M \cap df(U)\,) \subset X \cap \{f = 0\}\,,$$

if we restrict to $f\colon U := \{f < \epsilon\} \to [0, \epsilon[$, with $0 < \epsilon$ small enough. Then one gets for $\alpha \in CF_{\mathscr{S}}(X)$ and $0 < r < \epsilon$:

$$\int_{X\cap\{f=0\}} \alpha \, d\chi = \int_{X\cap\{f\leq r\}} \alpha \, d\chi = \sharp(\, CC(\alpha) \cap [df(U)]\,)\,. \tag{10.56}$$

The most important special case is the local situation $(M, x) \simeq (\mathbb{C}^n, 0)$ with $f(z) := r(z) := \sum_{i=1}^n z_i \cdot \bar{z}_i$. In this case we get for $0 < \epsilon \ll 1$:

$$\alpha(x) = \sharp_{dr_x}(\, CC(\alpha) \cap [dr(\{r < \epsilon\})]\,)\,. \tag{10.57}$$

In particular, this local intersection number is independent of the choice of the real analytic function r with $\{r = 0\} = \{x\}$. Moreover, it defines the inverse of the characteristic cycle map (see, e.g., [109, Corollary 5.0.1] and compare with [42, Theorem 11.7]).

Corollary 10.3.43 *The inverse of the characteristic cycle map*

$$CC\colon CF_{\mathscr{S}}(X) \to L(\mathscr{S}; T^*M)$$

is given by

$$Eu^\vee\colon L(\mathscr{S}; T^*M) \to CF_{\mathscr{S}}(X);$$

$$[C] \mapsto \alpha\,, \quad \textit{with } \alpha(x) := \sharp_{dr_x}(\,[C] \cap [dr(\{r < \epsilon\})]\,)\,,$$

for r, ϵ as above. In particular, for $CC(\alpha) = \sum_{S\in\mathscr{S}} m(S) \cdot \left[\overline{T_S^*M}\right]$, we get for $x \in X$:

$$\alpha(x) = \sum_{S\in\mathscr{S}} m(S) \cdot Eu^\vee\left(\left[\overline{T_S^*M}\right]\right)\,. \tag{10.58}$$

Note that this corollary is also a *refinement* of [42, Theorem 11.7], since we work with a *fixed stratification*. Assume now that $X \hookrightarrow M$ is pure dimensional, so that

$$C = \left[T_X^*M\right] := \left[\overline{T_{X_{reg}}^*M}\right] \in L(\mathscr{S}; T^*M)$$

defines a corresponding Lagrangian cycle. Then one gets by [109, Equation (5.35) and p.323–324] that

$$Eu^\vee\left(\left[T_X^*M\right]\right) = Eu_X^\vee := (-1)^{\dim X} \cdot Eu_X \in CF_{\mathscr{S}}(X) \tag{10.59}$$

is the *dual local Euler obstruction* function $Eu_X^\vee$, with Eu_X the famous *local Euler obstruction* function of X as defined by MacPherson [80] (compare also with [42, Theorem 11.7]). The sign $(-1)^{\dim X}$ comes again from different orientation conventions. In particular, we get without any calculation that the *local Euler obstruction* Eu_X is constructible with respect to *any Whitney stratification* of X. Since this is a local result, it is then also true for any pure-dimensional complex algebraic (or analytic) variety X (without any embedding into a complex manifold). This result is due to Dubson [33, Proposition 1, Theorem 3] and Brasselet-Schwartz [13, p.125, Corollary 10.2]. Then the *local index formula* (10.58) can be reformulated as

$$\alpha(x) = \sum_{S \in \mathscr{S}} m(S) \cdot (-1)^{\dim S} Eu_{\bar{S}}(x) \quad \text{for } x \in X \tag{10.60}$$

and $CC(\alpha) = \sum_{S \in \mathscr{S}} m(S) \cdot \left[\overline{T^*_S M}\right]$ for a given $\alpha \in CF_{\mathscr{S}}(X)$.

Remark 10.3.44 The *local index formula* (10.60) goes back to a corresponding *index formula* of Kashiwara [62] (compare with [63, Theorem 6.3.1]) for the *solution complex* $Sol_M(\mathscr{M}) := Rhom_{\mathcal{D}_M}(\mathscr{M}, \mathscr{O}_M)$ of a *holonomic* $\mathcal{D}_M$-module on the complex manifold M. Note that this solution complex is a complex analytically (or algebraically) constructible complex of sheaves of $\mathbb{C}$-vector spaces, with finite dimensional stalks ([66, Theorem 11.3.7]). Kashiwara's formula corresponds to (10.60) for the constructible function

$$\alpha := \chi_{stalk}(\, Rhom_{\mathcal{D}_M}(\mathscr{M}, \mathscr{O}_M)[\dim M]\,)\,.$$

Moreover, he works directly with the *characteristic cycle* of a holonomic $\mathcal{D}_M$-module. The corresponding multiplicities of characteristic cycles fit by [109, Example 5.3.4] with our conventions. Similarly, Kashiwara introduced for his index formula some topological invariants, which are nothing else but the *Euler obstructions* of the closures $\bar{S}$ for the strata $S \in \mathscr{S}$ of a Whitney stratification $\mathscr{S}$ of M so that the characteristic variety $char(\mathscr{M}) \subset T^*_{\mathscr{S}}M$. But this fact was only observed later on by Dubson (compare with [15], and also with [63, Introduction, p.xiii]).

Other references for these formulae are [35], [14, p.545] and [42, Theorem 8.2, Theorem 11.7, p.393] (but with an incorrect sign in [42, Corollary 6.19(b) and Theorem 8.2]).

So one gets in the complex analytic (or algebraic) context a commutative diagram (see also [42]):

$$\begin{array}{ccc} K_0(\text{holonomic } \mathcal{D}_M\text{-modules}) & \xrightarrow{CC} & L(T^*M) \\ {\scriptstyle DR_M}\downarrow & & \wr\uparrow{\scriptstyle CC} \\ K_0(D^b_c(M;\mathbb{C})) & \xrightarrow{\chi_{stalk}} & CF(M)\,. \end{array} \tag{10.61}$$

Here $DR_M(\mathscr{M}) \simeq Rhom_{\mathcal{D}_M}(\mathscr{O}_M, \mathscr{M})[\dim M]$ is the *De Rham complex* of the holonomic $\mathcal{D}_M$-module $\mathscr{M}$. In the algebraic context one has of course to use the De Rham or solution complex of the associated analytic $\mathcal{D}_M$-module. That $DR_M(\mathscr{M})$ and $Sol_M(\mathscr{M})[\dim M]$ for a holonomic $\mathcal{D}_M$-module $\mathscr{M}$ have the same characteristic cycle (or associated constructible function χ_{stalk}) follows, e.g., from the fact that they are exchanged by the *duality* of holonomic $\mathcal{D}_M$-modules, resp., *Verdier duality* for constructible sheaf complexes (see, e.g., [58, Proposition 4.6.4(iii), Corollary 4.6.5]). Finally, also the left vertical arrow becomes an isomorphism by the famous *Riemann-Hilbert correspondence* (see, e.g., [58, Theorem 7.2.1]), if we restrict ourselves to *regular holonomic* $\mathcal{D}_M$-modules.

10.3.4 Functorial Calculus of Characteristic Cycles

We continue to work in an embedded context and discuss the functorial calculus of characteristic cycles (see also, e.g., [42, 103] and compare with [66, Chapter IX] for counterparts in real geometry). We explain the translation into the context of Lagrangian cycles of the following operations for constructible functions and sheaves: external product, proper direct image, non-characteristic pullback and specialization (i.e., nearby cycles), together with an intersection formula for vanishing cycles.

Note that by Corollary 10.3.22 and Corollary 10.3.25, the constructible complexes

$$\mathcal{F}^\bullet \in D^b_{(\mathcal{S}-)c}(X; R) \quad \text{with} \quad \chi_{stalk}(\mathcal{F}^\bullet) = 0 \in CF_{(\mathcal{S})}(X)$$

are *preserved* by all Grothendieck functors like pullback f^*, direct image $Rf_! = Rf_*$ for a proper morphism, and the (external) tensor products $\otimes$ resp., $\boxtimes$ (which are exact in the case R a field as considered here). So they induce similar transformations of constructible functions $CF_{(\mathcal{S})}(X)$ via the surjection

$$\chi_{stalk}\colon\ K_0\left(D^b_{(\mathcal{S})-c}(X; R)\right) \to CF_{(\mathcal{S})}(X)\,.$$

Similarly for the nearby and vanishing cycles functors ψ_f, φ_f of Deligne, as studied in Sect. 10.4. So it is natural to ask for a direct description of the corresponding transformations of characteristic cycles in the embedded context, compatible with the isomorphisms

$$CC\colon CF_{\mathcal{S}}(X) \xrightarrow{\ \sim\ } L(\mathcal{S}, T^*M) \quad \text{and} \quad CC\colon CF(X) \xrightarrow{\ \sim\ } L(T^*M|_X)\,.$$

Whenever possible, we formulate the corresponding results for the refined context of fixed Whitney stratifications.

Example 10.3.45 (Characteristic Cycle of External Products) Let $X_i \hookrightarrow M_i$ be a closed embedding of the complex algebraic (or analytic) variety X_i into a complex algebraic (or analytic) manifold M_i (for $i = 1, 2$). Assume $\mathcal{S}_i$ is a Whitney stratification of X_i, with $\alpha_i \in CF_{\mathcal{S}_i}(X_i)$ (for $i = 1, 2$). Then $\alpha_1 \boxtimes \alpha_2 \in CF_{\mathcal{S}_1 \times \mathcal{S}_2}(X_1 \times X_2)$ is defined by

$$\alpha_1 \boxtimes \alpha_2(x_1, x_2) := \alpha_1(x_1) \cdot \alpha_2(x_2) \quad \text{for } (x_1, x_2) \in X_1 \times X_2,$$

with

$$CC(\alpha_1 \boxtimes \alpha_2) = CC(\alpha_1) \boxtimes CC(\alpha_2) \in L(\mathscr{S}_1 \times \mathscr{S}_2, T^*M_1 \times T^*M_2)\,. \tag{10.62}$$

Here, we use the identification $T^*(M_1 \times M_2) = T^*M_1 \times T^*M_2$, and this result follows directly from the *product formula for normal Morse data* as in Example 10.3.17. Alternatively, it can be deduced from the *multiplicativity of the local Euler obstruction functions* (as stated in [80, p. 426]):

$$Eu_{X_1 \times X_2} = Eu_{X_1} \boxtimes Eu_{X_2}$$

in case the X_i are pure-dimensional ($i = 1, 2$).

To formulate the results for suitable pullbacks f^* and direct images f_* for a morphism $f: M \to N$ of complex (algebraic) manifolds, we need to compare the corresponding cotangent bundles as in the following commutative diagram (whose right square is cartesian):

$$\begin{array}{ccccc} T^*M & \xleftarrow{\;{}^t f'\;} & f^*(T^*N) & \xrightarrow{\;f_\pi\;} & T^*N \\ \big\downarrow{\scriptstyle \pi_M} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_N} \\ M & = \!= \!= & M & \xrightarrow{\;f\;} & N\,. \end{array} \tag{10.63}$$

Here ${}^t f'$ is the dual of the differential of f, with f_π induced by base change. We first consider the case of direct images. Let $X \hookrightarrow M$, resp., $Y \hookrightarrow N$ be closed complex algebraic (or analytic) subvarieties endowed with Whitney stratifications $\mathscr{S}$ of X, resp., $\mathscr{T}$ of Y in such a way that $f: X \to Y$ is a *proper stratified submersion*. Then also $f_\pi: f^*(T^*N)|_X \to T^*N|_Y$ is *proper* by base change. Moreover

$$f_\pi\left({}^t f'^{-1}\left(T^*_{\mathscr{S}}M\right)\right) \subset T^*_{\mathscr{T}}Y\,,$$

since $f: X \to Y$ is a stratified submersion. So on the level of *Lagrangian cycles* one gets an induced group homomorphism

$$f_{\pi *}{}^t f'^*: L(\mathscr{S}, T^*M) \to L(\mathscr{T}, T^*N)\,, \tag{10.64}$$

with the pullback ${}^t f'^*$ defined via Poincaré duality:

$$\begin{array}{ccc} H^{2m}_{T^*_{\mathscr{S}}M}(T^*M) & \xrightarrow{\;{}^t f'^*\;} & H^{2m}_{{}^t f'^{-1}(T^*_{\mathscr{S}}M)}(f^*(T^*N), \mathbb{Z}) \\ {\scriptstyle PD}\big\downarrow{\scriptstyle \wr} & & {\scriptstyle PD}\big\downarrow{\scriptstyle \wr} \\ H^{BM}_{2m}(T^*_{\mathscr{S}}M, \mathbb{Z}) & \xrightarrow{\;{}^t f'^*\;} & H^{BM}_{2n}({}^t f'^{-1}(T^*_{\mathscr{S}}M), \mathbb{Z})\,, \end{array}$$

with $m = \dim M, n = \dim N$ and $T^*M, T^*N, f^*(T^*N)$ oriented as complex manifolds.

Similarly, the pushforward of constructible functions fits into a commutative diagram

$$\begin{array}{ccc} K_0\left(D^b_{\mathscr{S}-c}(X;R)\right) & \xrightarrow{f_*} & K_0\left(D^b_{\mathscr{T}-c}(Y;R)\right) \\ \chi_{stalk}\downarrow & & \downarrow\chi_{stalk} \\ CF_{\mathscr{S}}(X) & \xrightarrow{f_*} & CF_{\mathscr{T}}(Y), \end{array}$$

with

$$f_*(\alpha)(y) := \int_{X\cap\{f=y\}} \alpha d\chi \quad \text{for } y \in Y \text{ and } \alpha \in CF_{\mathscr{S}}(X). \tag{10.65}$$

And these pushforwards of Lagrangian cycles and constructible functions are compatible with the characteristic cycle map (see, e.g., [111, Section 4.6] for the following proof).

Proposition 10.3.46 *Let $\alpha \in CF_{\mathscr{S}}(X)$ be given. Then*

$$CC(f_*\alpha) = f_{\pi*}{}^t f'^* CC(\alpha) \in L(\mathscr{T}, T^*N)\ .$$

Proof Let $\beta \in CF_{\mathscr{T}}(Y)$ be defined by $CC(\beta) = f_{\pi*}{}^t f'^* CC(\alpha) \in L(\mathscr{T}, T^*N)$. Then we need to show $\beta = f_*\alpha$. For this we calculate $\beta(y)$ for $y \in Y$ as in Corollary 10.3.43 via

$$\beta(y) := \sharp_{dr_y}(\ f_{\pi*}{}^t f'^* CC(\alpha) \cap [dr(\{r < \epsilon\})]\ )\ ,$$

with $r\colon (N,y) \simeq (\mathbb{C}^n, 0) \to [0,\infty[$ given by $r(z) = \sum_{i=1}^n z_i \cdot \bar{z}_i$ the distance function to y (in local coordinates) and $0 < \epsilon$ small enough. By the *projection formula* one gets

$$\beta(y) = \sharp(\ CC(\alpha) \cap [d(r \circ f)(\{r \circ f < \epsilon\})]\ )\ .$$

And the last intersection number is by the *local index formula* (10.56) given by

$$\sharp(\ CC(\alpha) \cap [d(r \circ f)(\{r \circ f < \epsilon\})]\ ) = \int_{X\cap\{f=y\}} \alpha\, d\chi = f_*\alpha(y)\ ,$$

thus completing the proof. □

Example 10.3.47 Let $Y = N = \{pt\}$ be a point with f a constant map, so that $f^*(T^*N) = T^*_M M$ is the zero section of T^*M. Then we recover the *global index formula* (10.52):

$$\int_X \alpha \, d\chi = \sharp(\, CC(\alpha) \cap [T^*_M M] \,) \, .$$

To define similarly a pullback of Lagrangian cycles, we have to go in diagram (10.63) into the opposite direction, and need the following properness condition for ${}^t f' \colon f^*(T^*N) \to T^*M$ (see, e.g., [109, Lemma 4.3.1]).

Definition 10.3.48 Let $C \subset T^*N$ be a closed $\mathbb{C}^*$-conic subset. Then $f\colon M \to N$ is *non-characteristic* with respect to C if one of the following two equivalent conditions holds:

1. $f_\pi^{-1}(C) \cap kern({}^t f') \subset f^*(T^*_N N)$, with $kern({}^t f')$ the corresponding kernel bundle, and $f^*(T^*_N N)$ the zero-section of $f^*(T^*N)$.
2. The map ${}^t f' \colon f_\pi^{-1}(C) \to T^*M$ is *proper* and therefore finite.

Example 10.3.49 If $f\colon M \to N$ is a *submersion*, then f is *non-characteristic* with respect to any closed $\mathbb{C}^*$-conic subset $C \subset T^*N$, since then $kern({}^t f') = f^*(T^*_N N)$ is the zero-section of $f^*(T^*N)$.

More generally, *transversality* with respect to a Whitney stratification $\mathscr{T}$ of a closed complex algebraic (or analytic) subset $Y \hookrightarrow N$ can be characterized as follows (see, e.g., [109, Example 4.3.2] and [66, Definition 4.1.5]).

Example 10.3.50 The morphism $f\colon M \to N$ is *transversal* to $\mathscr{T}$, i.e., f is transversal to all strata $T \in \mathscr{T}$, if and only if f is *non-characteristic* with respect to the closed $\mathbb{C}^*$-conic subset $T^*_{\mathscr{T}} N \subset T^*N$. In this case, $X := f^{-1}(Y)$ gets an induced Whitney stratification $\mathscr{S} = f^{-1}\mathscr{T}$ with strata S given by the connected components of the locally closed complex submanifolds $f^{-1}(T) \subset M$ (for $T \in \mathscr{T}$). Here, the codimension $\operatorname{codim} f^{-1}(T) = \operatorname{codim} T$ is preserved, i.e.,

$$\dim f^{-1}(T) = \dim T + \dim M - \dim N \quad \text{for all } T \in \mathscr{T}.$$

Moreover, ${}^t f' \colon f_\pi^{-1}(T^*_T N) \to T^*M$ is injective with image $T^*_{f^{-1}(T)} M$. By Poincaré duality one gets an induced pullback map of Lagrangian cycles

$$ {}^t f'_* f_\pi^* \colon L(\mathscr{T}, T^*N) \to L(\mathscr{S}, T^*M), \tag{10.66}$$

with

$$ {}^t f'_* f_\pi^* \left(\left[\overline{T^*_T N} \right] \right) = \left[\overline{T^*_{f^{-1}(T)} M} \right] \quad \text{for all } T \in \mathscr{T}.$$

Moreover, normal slices to $f^{-1}(T) \subset M$ and $T \subset N$ get identified via f, so that the corresponding Euler characteristics of normal Morse data do not change under pullback of constructible functions given by $f^*\alpha := \alpha \circ f$ for $\alpha \in CF_{\mathcal{T}}(Y)$. In particular, the characteristic cycle map CC commutes with this pullback only up to a sign:

$$(-1)^{\dim M - \dim N} \cdot CC(f^*\alpha) = {}^t f'_* f_\pi^* CC(\alpha) \quad \text{for all } \alpha \in CF_{\mathcal{T}}(Y).$$

For example, if Y (and hence also $X = f^{-1}(Y)$) is pure-dimensional, then one gets for such a transversal map

$$f^*(Eu_Y) = Eu_X \ .$$

More generally one has the following result (see, e.g., [114, Theorem 3.1]).

Theorem 10.3.51 *Let $f : M \to N$ be a morphism of complex algebraic (or analytic) manifolds of dimension $m = \dim M, n = \dim N$, with $Y \subset N$ a closed complex algebraic (or analytic) subvariety and $X := f^{-1}(Y) \subset M$. Assume that f is non-characteristic with respect to the support $C := supp(CC(\alpha)) \subset T^*N|_Y$ of the characteristic cycle $CC(\alpha)$ of a constructible function $\alpha \in CF(Y)$. Then $C' := {}^t f'\left(f_\pi^{-1}(C)\right)$ is pure m-dimensional, with*

$$ {}^t f'_* f_\pi^*(CC(\alpha)) = (-1)^{m-n} \cdot CC(f^*(\alpha)) \ . \tag{10.67}$$

In particular, the left hand side is a Lagrangian cycle in $T^*M|_X$.

The following Example 10.3.52 has nice applications in *geometric representation theory* for the *Weyl group* $\mathbb{W}$ of a connected semisimple complex Lie group G (see, e.g., [43]). Here, $M_1 = M_2 = G/B$ is the *Flag manifold* of G (with $B \subset G$ a Borel subgroup), and the Whitney stratification $\mathcal{S}$ of $G/B \times G/B$ is given by the finitely many G-orbits S_w of the diagonal G-action (indexed by $w \in \mathbb{W}$). Then any $\alpha \in CF_{\mathcal{S}}(G/B \times G/B)$ satisfies the following assumption for

$$C := \bigcup_{w \in \mathbb{W}} T^*_{S_w}(G/B \times G/B) \subset T^*(G/B \times G/B)$$

the corresponding *Steinberg variety*.

Example 10.3.52 Let M_i be three complex algebraic (or analytic) complex manifolds of dimension $m_i = \dim M_i$ ($i = 1, 2, 3$). Assume $\alpha \in CF(M_1 \times M_2)$ satisfies one of the following two equivalent conditions for a closed $\mathbb{C}^*$-conic subset $C \subset T^*(M_1 \times M_2)$ with $supp(CC(\alpha)) \subset C$:

1. The projection $T^*(M_1 \times M_2) = T^*M_1 \times T^*M_2 \to T^*M_1 \times M_2$ restricted to C is proper and therefore finite,
2. $C \cap M_1 \times T^*M_2$ is contained in the zero section of $T^*(M_1 \times M_2)$.

Then the embedding $d\colon M_1 \times M_2 \times M_3 \to (M_1 \times M_2) \times (M_2 \times M_3)$ induced by the diagonal embedding $M_2 \to M_2 \times M_2$ is *non-characteristic* with respect to $supp(CC(\alpha \boxtimes \beta))$ for *any* $\beta \in CF(M_2 \times M_3)$, so that

$$ {}^t d'_* d^*_\pi (CC(\alpha \boxtimes \beta)) = (-1)^{m_2} \cdot CC(d^*(\alpha \boxtimes \beta)) . $$

Another application of Theorem 10.3.51 is the following *intersection formula* (see, e.g., [114, Corollary 3.1]).

Corollary 10.3.53 *Let M a complex algebraic (or analytic) manifold of dimension $m = \dim M$, with $\alpha, \beta \in CF(M)$ given constructible functions. Assume that the diagonal embedding $d\colon M \to M \times M$ is non-characteristic with respect to $supp(CC(\alpha \boxtimes \beta))$, with $supp(\alpha \cdot \beta)$ compact.*

Then also $supp(CC(\alpha) \cap CC(\beta)) \subset T^*M$ is compact, with

$$ \int_M \alpha \cdot \beta \, d\chi = (-1)^m \cdot deg(CC(\alpha) \cap CC(\beta)) . \tag{10.68} $$

Example 10.3.54 The assumption $d\colon M \to M \times M$ is *non-characteristic* with respect to $supp(CC(\alpha \boxtimes \beta))$ for $\alpha, \beta \in CF(M)$ holds in the following cases:

1. α, resp., β is constructible with respect to a Whitney stratification $\mathscr{S}$, resp., $\mathscr{T}$ of M, with $\mathscr{S}$ and $\mathscr{T}$ intersecting transversally (i.e., all strata $S \in \mathscr{S}$ and $T \in \mathscr{T}$ intersect transversally in M), so that $d\colon M \to M \times M$ is transversal to the product stratification $\mathscr{S} \times \mathscr{T}$ of $M \times M$.
2. α and β are *splayed* in the sense of [114, Definition 2.1], i.e., for any $p \in M$ there are locally analytic isomorphisms $M = V_1 \times V_2$ of analytic manifolds so that $\alpha = \pi_1^*(\alpha')$ and $\beta = \pi_2^*(\beta')$ for some $\alpha' \in CF(V_1)$ and $\beta' \in CF(V_2)$, with $\pi_i\colon V_1 \times V_2 \to V_i$ the projection ($i = 1, 2$). For $\alpha = 1_X$ and $\beta = 1_Y$, with $X, Y \subset M$ closed complex algebraic (or analytic) subvarieties, this just means that X and Y are *splayed* in the sense of Aluffi-Faber [1, 2].

We next explain the relation between *specialization* of Lagrangian cycles and *nearby cycles* for constructible functions (see, e.g., [14, 42, 103]).

Let $f\colon N \to \mathbb{C}$ be a submersion of complex algebraic (or analytic) manifolds, with $k\colon M := \{f = 0\} \hookrightarrow N$ the inclusion of a smooth hypersurface with open complement $U = \{f \neq 0\}$. Consider the exact sequence of vector bundles on N

$$ 0 \to \langle df \rangle = kern(p) \to T^*N \to T^*_f \to 0 , $$

with the projection $p\colon T^*N \to T^*_f$ dual to the inclusion $T_f \to TN$ of the subvector bundle of tangents to the fibers of f. Let $X \hookrightarrow N$ be a closed complex algebraic (or analytic) subvariety endowed with a Whitney stratification $\mathscr{S}$ so that $X \cap \{f = 0\} =: X_0$ is a union of strata, with $\mathscr{S}|_{X_0}$, resp., $\mathscr{S}|_U$ the induced Whitney stratification of X_0, resp., $X \cap U$. After shrinking to an open neighborhood, we can and will assume by the *curve selection lemma* that $f|_S$ is a submersion for all strata $S \in \mathscr{S}|_U$.

Finally, we assume that $\mathscr{S}$ satisfies the following *a_f-condition of Thom*:

$$\begin{cases} \text{If } x_n \in S', \text{ for } S' \in \mathscr{S}|_U, \text{ is a sequence converging to } x \in S \in \mathscr{S}|_{X_0}, \\ \text{such that } kern(df_{x_i}|_{T_{x_i}S'}) = T_{x_i}(S' \cap \{f = f(x_i)\}) \\ \text{converges to some limiting plane } \tau, \text{ then } T_x S \subset \tau. \end{cases} \tag{10.69}$$

By a classical theorem of Hironaka [57, Corollary 1 of Theorem 2, p.248] (and see also [73, Corollary 1.3.5.1]), one can always refine a given stratification so that it satisfies the a_f-condition. In fact, by a more recent result [14, Theorem 4.2.1], this a_f-condition is true for any Whitney stratification $\mathscr{S}$ as above (with X_0 a union of strata).

Then $T^*_{S'}N \cap kern(p)$ is contained in the zero-section $T^*_N N|_U$ of $T^*N|_U$, since $f: S' \to \mathbb{C}$ is a submersion for $S' \in \mathscr{S}|_U$. Therefore $p' = p: T^*_{S'}N \to T^*_f|_S$ is a proper injection with image $p'(T^*_{S'}N) =: T^*_{S'}f$, the *relative conormal space* of $f|_{S'}$. Here we consider the commutative diagram

$$\begin{array}{ccccccc} T^*_{S'}N & \longrightarrow & T^*N|_U & \xrightarrow{j} & T^*N & & \\ {\scriptstyle p'}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} & & \\ T^*_{S'}f & \longrightarrow & T^*_f|_U & \xrightarrow{j'} & T^*_f & \xleftarrow{i} & T^*M = T^*_f|_M . \end{array} \tag{10.70}$$

Note that by the a_f-condition the closure $\overline{T^*_{S'}f} \subset T^*_f$ for a stratum $S' \in \mathscr{S}|_U$ is contained in the closed subset

$$T^*_{\mathscr{S}}f := \bigcup_{S \in \mathscr{S}} T^*_S f \hookrightarrow T^*_f ,$$

with $T^*_S f := T^*_S M$ for $S \in \mathscr{S}|_{X_0}$ and $p': T^*_{\mathscr{S}}N|_U \to T^*_f|_U$ proper. Then the specialization of Lagrangian cycles

$$sp: L(\mathscr{S}, T^*N) \to L(\mathscr{S}|_{X_0}, T^*M)$$

is defined as the composition of the following homomorphisms (with $n = \dim N = \dim M + 1$):

$$\begin{array}{ccccc} H^{BM}_{2n}(T^*_{\mathscr{S}}N, \mathbb{Z}) & \xrightarrow{j^*} & H^{BM}_{2n}(T^*_{\mathscr{S}}N|_U, \mathbb{Z}) & \xrightarrow{p'_*} & H^{BM}_{2n}(T^*_{\mathscr{S}}f|_U, \mathbb{Z}) \\ {\scriptstyle sp}\downarrow & & & & {\scriptstyle j'^*}\uparrow\wr \\ L(\mathscr{S}|_{X_0}, T^*M) & = & H^{BM}_{2n-2}(T^*_{\mathscr{S}|_{X_0}}M, \mathbb{Z}) & \xleftarrow{i^*} & H^{BM}_{2n}(T^*_{\mathscr{S}}f, \mathbb{Z}) , \end{array} \tag{10.71}$$

with the Gysin map i^* again defined by Poincaré duality as the intersection with $[T^*M]$. So $sp([T^*_S N]) = 0$ for $S \in \mathscr{S}|_{X_0}$ and

$$sp([\overline{T^*_{S'} N}]) = \left[\overline{T^*_{S'} f}\right] \cap [T^*M] \quad \text{for } S' \in \mathscr{S}|_U .$$

The *nearby cycles* of constructible functions used in the next Theorem are induced from Deligne's nearby cycle functor ψ_f for constructible sheaf complexes as discussed later on in Sect. 10.4, fitting into a commutative diagram

$$\begin{array}{ccc} K_0(D^b_{\mathscr{S}-c}(X;R)) & \xrightarrow{\psi_f} & K_0(D^b_{\mathscr{S}|_{X_0}-c}(X_0;R)) \\ {\scriptstyle \chi_{stalk}}\downarrow & & \downarrow{\scriptstyle \chi_{stalk}} \\ CF_{\mathscr{S}}(X) & \xrightarrow{\psi_f} & CF_{\mathscr{S}|_{X_0}}(X_0) . \end{array} \tag{10.72}$$

With these notations, we can now show the following result (see, e.g., [103, Theorem 4.3] and compare with [14, 42]), which is also used in [114] in the proof of Theorem 10.3.51 above.

Theorem 10.3.55 *Let $\alpha \in CF_{\mathscr{S}}(X)$ be given. Then*

$$sp(CC(\alpha)) = CC(-\psi_f(\alpha)) \in L(\mathscr{S}|_{X_0}, T^*M) , \tag{10.73}$$

with $\psi_f(\alpha)(x) := \int_{M_{f|_X,x}} \alpha d\chi$, for $M_{f|_X,x}$ a local Milnor fiber of $f|_X$ at $x \in X_0$.

Proof Let $\beta \in CF_{\mathscr{S}|_{X_0}}(X_0)$ be defined by $CC(\beta) = sp(CC(\alpha)) \in L(\mathscr{S}|_{X_0}, T^*M)$. Then we need to show that $\beta = -\psi_f(\alpha)$. For this, we calculate $\beta(x)$ for $x \in X_0$ as in Corollary 10.3.43 via

$$\beta(x) := \sharp_{dr'_x}(sp(CC(\alpha)) \cap [dr'(\{r' < \epsilon\})]) ,$$

with $r' := r|_M$, for $r\colon (N, x) \simeq (\mathbb{C}^n, 0) \to [0, \infty[$ given by $r(z) = \sum_{i=1}^n z_i \cdot \bar{z}_i$ the distance function to 0 (in local coordinates of N), and $0 < \epsilon$ small enough. But this intersection number is locally constant in the family $T^*_f|_{\{f=w\}}$ for $|w| \ll 1$ small compared to ϵ, since $\{r = \epsilon\}$ is transversal to all $S \cap \{f = w\}$, for $S \in \mathscr{S}$ and $|w| \ll 1$ small, by the a_f-condition as well as the Whitney condition for $\mathscr{S}|_{X_0}$. So instead of specializing at $w = 0$, we can do this at a small stratified regular value $w \neq 0$ and use Example 10.3.50:

$$\beta(x) = -\sharp(CC(\alpha|_{\{f=w\}}) \cap [dr''(\{r'' < \epsilon\})]) ,$$

with $r'' := r|_{\{f=w\}}$. Note that the sign comes from $\dim N - \dim\{f = w\} = 1$, i.e., we intersect transversally with a submanifold of codimension 1. And the last intersection number is by Eq. (10.50) given by

$$\sharp(\, CC(\alpha|_{\{f=w\}}) \cap [dr''(\{r'' < \epsilon\})]\,) = \int_{M_{f|_X,x}} \alpha \, d\chi = \psi_f(\alpha)(x)\,,$$

with $M_{f|_X,x} := X \cap \{f = w\} \cap \{r \leq \epsilon'\}$, for $0 < |w| \ll \epsilon' < \epsilon$, a local Milnor fiber of $f|_X$ at $x \in X_0$. □

Note that the a_f-condition is also needed to have such a *local Milnor fibration* with Milnor fiber $M_{f,x}$ for a general holomorphic function germ $f\colon (X, x) \to (\mathbb{C}, 0)$ on a singular complex analytic variety X (see, e.g., [109, Example 1.3.3] and [70]). The definition of *nearby cycles* of constructible functions in terms of (weighted) Euler characteristics of these *local Milnor fibers* goes back to Verdier [123].

As an example, let us explain how Theorem 10.3.55 implies Theorem 10.3.51 for the inclusion $k\colon M = \{f = 0\} \hookrightarrow N$ of a global smooth hypersurface (i.e., of codimension one), with $f\colon N \to \mathbb{C}$ a submersion and the notations as before. Consider the following cartesian diagram:

$$\begin{array}{ccccc} T^*N|_M & \xrightarrow{k_\pi} & T^*N & \xleftarrow{j} & T^*N|_U \\ {}^t k' \big\downarrow & & p \big\downarrow & & \big\downarrow p' \\ T^*M & \xrightarrow{i} & T^*_f & \xleftarrow{j'} & T^*_f|_U\,. \end{array} \tag{10.74}$$

Assume that $k\colon M \hookrightarrow N$ is *non-characteristic* with respect to $supp(CC(\alpha)) \subset T^*N$ for a given $\alpha \in CF(N)$, so that ${}^t k'\colon k_\pi^{-1}(supp(CC(\alpha))) \to T^*M$ is proper. After shrinking N we can then assume that also $p\colon supp(CC(\alpha)) \to T^*_f$ is proper. Then one gets by base change

$$j'^* p_* CC(\alpha) = p'_* j^* CC(\alpha)\,,$$

and

$$\begin{aligned} {}^t k'_* k_\pi^* CC(\alpha) &= i^* p_* CC(\alpha) = sp(CC(\alpha)) \\ &= CC(-\psi_f(\alpha)) = -CC(k^*\alpha)\,. \end{aligned}$$

Here, the fact that $\psi_f(\alpha) = k^*\alpha$ follows from (10.79) below, since we have $0 = \psi_f(\alpha)(x) - k^*\alpha(x) = \psi_f(\alpha)(x) - \alpha(x)$ for all $x \in M$ by the non-characteristic assumption, which gives $df_x \notin supp(CC(\alpha))$ for all $x \in M$.

We finish this section with citing from [110, Corollary 0.3] (and compare with [103, Theorem 4.5]) a nice intersection formula related to the vanishing cycle functor φ_f introduced in Sect. 10.4. For a description of the *characteristic cycle of vanishing cycles* we refer to [83, Theorem 2.10].

Theorem 10.3.56 (Global Intersection Formula for Vanishing Cycles) *Let M be a complex algebraic (or analytic) manifold and $f\colon M \to \mathbb{C}$ an algebraic (or holomorphic) function, with $\mathscr{F}^\bullet \in D^b_c(M; R)$, resp., $\alpha \in CF(M)$ be given. Suppose that the intersection of $df(M)$ and the support of the characteristic cycle of $\mathscr{F}^\bullet$, resp., α is contained in a compact complex algebraic (or analytic) subset $I \subset T^*M$, with $K := \pi(I) \subset \{f = 0\}$ for $\pi\colon T^*M \to M$ the projection. Then one has*

$$\chi\big(R\Gamma(K, \varphi_f[-1]\,\mathscr{F}^\bullet)\big) = \sharp\big(\,[CC(\mathscr{F}^\bullet)] \cap [df(M)]\,\big)\,, \tag{10.75}$$

resp.,

$$-\int_K \varphi_f(\alpha)\, d\chi = -\chi(K, \varphi_f(\alpha)) = \sharp\big(\,[CC(\alpha)] \cap [df(M)]\,\big)\,. \tag{10.76}$$

Example 10.3.57 Let $f\colon M \to \{0\}$ be the constant zero function so that $df(M) = T^*_M M$ is the zero section of T^*M, with $-\varphi_f(\alpha) = \alpha$ for all $\alpha \in CF(M)$. Assume $M =: K$ is compact. Then we recover the *global index formula* (10.52):

$$\int_X \alpha\, d\chi = \sharp(\,CC(\alpha) \cap [T^*_M M]\,)\,.$$

In the special case $I = \{\omega\}$ given by a point $\omega \in T^*M$, we get back by Theorem 10.3.56 a formula conjectured by Deligne (with $x := \pi(\omega)$ and $K := \{x\}$):

$$\chi\big((\varphi_f[-1]\,\mathscr{F}^\bullet)_x\big) = \sharp_{df_x}\big(\,[CC(\mathscr{F}^\bullet] \cap [df(M)]\,\big) \tag{10.77}$$

and

$$-\,\varphi_f(\alpha)(x) = \alpha(x) - \int_{M_{f;x}} \alpha\, d\chi = \sharp_{df_x}\big(\,[CC(\alpha)] \cap [df(M)]\,\big)\,, \tag{10.78}$$

with $M_{f,x}$ the *local Milnor fiber* of f in x. In particular, for $df_x \notin supp(CC(\alpha))$, we get

$$-\,\varphi_f(\alpha)(x) = \alpha(x) - \psi_f(\alpha)(x) = \alpha(x) - \int_{M_{f;x}} \alpha\, d\chi = 0\,. \tag{10.79}$$

Example 10.3.58 Let $f\colon (M, x) = (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with an *isolated singularity* in $0 \in \mathbb{C}^{n+1}$. Then the local Milnor fiber $M_{f,0}$ has the homotopy type of a wedge of a finite number of n-spheres (for $n \geq 1$). The *Milnor number* $\mu(f)$ is the number of these n-spheres, so that

$$\chi(M_{f,x}, \mathbb{Q}) = 1 + (-1)^n \cdot \mu(f)\,.$$

For $n = 0$, the local Milnor fiber $M_{f,0}$ consists of $1 + \mu(f)$ points. If we apply the above formula to the perverse sheaf $\mathscr{F}^\bullet = \mathbb{Q}_M[n+1]$, resp., $\alpha = (-1)^{n+1} \cdot 1_M$, with $CC(\mathbb{Q}_M[n+1]) = CC((-1)^{n+1} \cdot 1_M) = T^*_M M$ the zero section of T^*M, we recover the formula

$$\mu(f) = \sharp_{df_x}\big([T^*_M M] \cap [df(M)] \big)$$

for the Milnor number, i.e.,

$$\mu(f) = \dim_{\mathbb{C}} \frac{\big(\mathscr{O}_{\mathbb{C}^{n+1}}\big)_0}{(\nabla f)_0}$$

for the Milnor number in terms of the Jacobian ideal $(\nabla f)_0 \subset \big(\mathscr{O}_{\mathbb{C}^{n+1}}\big)_0$.

The *local intersection formula for the vanishing cycle functor* (10.75) is due to Dubson [36, Theorem 1], Ginsburg [42, Proposition 7.7.1], Lê [72, Theorem 4.1.2] and Sabbah [103, Theorem 4.5]. For a discussion of the history of this *local intersection formula for vanishing cycles* we recommend the paper [72].

But most of these references are in the language of *holonomic D-modules* or *perverse sheaves*. So the assumption on the intersection for a constructible complex of sheaves $\mathscr{F}^\bullet$ corresponds to an assumption on the *micro-support* $\mu supp(\mathscr{F}^\bullet)$. To be able to state the result also for constructible functions, it is important to work with the weaker assumption about the *support of the characteristic cycle* $supp(CC(\mathscr{F}^\bullet)) \subset \mu supp(\mathscr{F}^\bullet)$.

Let us finally point out that MacPherson's theory [80] of *Chern classes* c_* *of singular varieties and constructible functions* in the embedded context can easily be recovered and improved from the *functorial theory of characteristic cycles* as presented above, e.g., like the following results about c_*:

1. Functoriality of c_* for proper morphism via Proposition 10.3.46.
2. Multiplicativity of c_* with respect to external products via Example 10.3.45.
3. Specialization of c_* in one parameter families via Theorem 10.3.55.
4. A Verdier-Riemann-Roch Theorem for c_* with respect to non-characteristic pullbacks (e.g., submersions or transversal pullbacks) via Theorem 10.3.51.
5. An intersection formula for c_* via Theorem 10.3.51 in the context of Example 10.68.

In fact, the most recent approach of [3] deduces $c_*(\alpha)$ for $\alpha \in CF(X)$ from $CC(\alpha)$, viewed as a $\mathbb{C}^*$-invariant cycle, via intersection with the zero-section $T^*_M M$ in $\mathbb{C}^*$-equivariant Borel-Moore homology. For further reading also on relations between *MacPherson Chern classes* c_* and *characteristic cycles* we refer to, e.g., [1–4, 19, 42, 43, 103, 111, 113, 114, 123].

10.3.5 Vanishing Results

Let us now explain other applications of the *stratified Morse theory for constructible sheaves* and of Theorem 10.3.19, to *vanishing and weak Lefschetz theorems* in the complex algebraic and analytic context. In the following we are back to the general context that the base ring R is commutative and Noetherian, of finite global dimension, and we work with (weakly) constructible complexes of sheaves of R-modules.

First we consider the complex algebraic context with an *affine* complex algebraic variety $X \hookrightarrow \mathbb{C}^n$ and the *strongly plurisubharmonic and semi-algebraic* distance function

$$r\colon \mathbb{C}^n \to \mathbb{R}^{\geq 0}\,;\; r(z) := \sum\nolimits_{i=1}^{n} z_i \bar{z}_i\ .$$

If $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X;R)$ is $\mathscr{S}$-weakly constructible with respect to a *complex algebraic* Whitney stratification $\mathscr{S}$ of X, then the proper semi-algebraic distance function r has only *finitely* many stratified critical values so that one gets as in the proof of Theorem 10.3.38 and Example 10.3.41:

$$R\Gamma(X \cap \{r \leq b\}, \mathscr{F}^\bullet) \simeq R\Gamma(X, \mathscr{F}^\bullet) \quad \text{for } b > 0 \text{ large enough,}$$

and

$$R\Gamma_c(X \cap \{r < b\}, \mathscr{F}^\bullet) \simeq R\Gamma_c(X, \mathscr{F}^\bullet) \quad \text{for } b > 0 \text{ large enough.}$$

Then Theorem 10.3.19 implies for $f := r$ (with $q = 0$, since r is strongly plurisubharmonic) directly the following important *Artin-Grothendieck type result* (see also [109, Corollary 6.0.4] for a more general version):

Theorem 10.3.59 (Artin Vanishing Theorem) *Let X be an affine complex algebraic variety, with $\mathscr{F}^\bullet \in D^b_{(w)c}(X;R)$ a complex algebraically (weakly) constructible sheaf complex (respectively, complex algebraically constructible in case of a Dedekind domain R, if we want to use the dual perverse t-structure). Then:*

1. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(X;R) \Rightarrow R\Gamma(X,\mathscr{F}^\bullet) \in {}^{p^{(+)}}D^{\leq n}(\{pt\};R) \subset D^b_{(w)c}(\{pt\};R)$,
2. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(X;R) \Rightarrow R\Gamma_c(X,\mathscr{F}^\bullet) \in {}^{p^{(+)}}D^{\geq n}(\{pt\};R) \subset D^b_{(w)c}(\{pt\};R)$.

Remark 10.3.60 The same proof as indicated above gives the *Artin vanishing Theorem* more generally for an *open semi-algebraic* subset $j\colon X \hookrightarrow X'$ of a complex algebraic variety X', if it has a proper *strongly plurisubharmonic and semi-algebraic* function $r\colon X \to \mathbb{R}^{\geq 0}$, and we consider only (weakly) constructible sheaves of the form $\mathscr{F}^\bullet = j^*\mathscr{G}^\bullet$ for some algebraically (weakly) constructible $\mathscr{G}^\bullet \in D^b_{(w)c}(X',R)$.

Compare with [6, Section 4], especially [6, Theorem 4.1.1]), for the corresponding relative counterpart for an affine morphism in the context of the *perverse t-structure* in l-adic cohomology. The relative version of Theorem 10.3.59 for an *affine complex algebraic morphism* is also true, as we will see later on. For now, let us only illustrate this Theorem 10.3.59 by the following example (see, e.g., [109, Example 6.0.4]).

Example 10.3.61 (Weak Lefschetz Theorem for Singular Spaces) Let X be a closed algebraic subvariety of the complex projective space, with H a hyperplane. Consider the open inclusion $j\colon U := X\backslash H \hookrightarrow X$, with $i\colon X\cap H \hookrightarrow X$ the corresponding closed inclusion. Assume $\mathscr{F}^\bullet \in D^b_{(w)c}(X;R)$ is *complex algebraically* (weakly) constructible (respectively, complex algebraically constructible in case of a Dedekind domain R, if we want to use the dual perverse t-structure). Then

$$j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(U;R) \Rightarrow$$

$$R\Gamma(X, Rj_*j^*\mathscr{F}^\bullet) \simeq R\Gamma(U, j^*\mathscr{F}^\bullet) \in {}^{p^{(+)}}D^{\leq n}(\{pt\};R)\,,$$

and

$$j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(U;R) \Rightarrow$$

$$R\Gamma(X, X\cap H, \mathscr{F}^\bullet) := R\Gamma(X, Rj_!j^*\mathscr{F}^\bullet) \simeq R\Gamma_c(U, j^*\mathscr{F}^\bullet) \in {}^{p^{(+)}}D^{\geq n}(\{pt\};R).$$

In particular:

1. The *relative homology*

$$H_k(X, X\cap H; R) = H^{-k}(U; j^*\mathbb{D}^\bullet_X)$$

vanishes for $k < \mathrm{rHd}(U,R)$.
2. If R is a Dedekind domain, then the we get for the *relative cohomology* that

$$H^k(X, X\cap H; R) \simeq H^k_c(U; R_U) = \begin{cases} 0 & \text{for } k < \mathrm{rHd}(U,R), \\ \text{torsion-free} & \text{for } k = \mathrm{rHd}(U,R). \end{cases}$$

3. Suppose X is purely n-dimensional, and H is a *generic* hyperplane, i.e., H is transversal to a Whitney stratification $\mathscr{S}$ of X, so that $i^*IC_X[-1] \simeq IC_{X\cap H}$ by Example 10.2.28. Then the *relative intersection cohomology*

$$IH^k(X, X\cap H; R) := H^{k-n}_c(U; j^*IC_X) = 0 \quad \text{for } k < n.$$

Moreover, $IH^n(X, X\cap H; R)$ is torsion-free in case R is a Dedekind domain.

Here, statements 1. and 2. show the role of the *rectified homological depth* for the "weak Lefschetz theorem for singular spaces" as conjectured by Grothendieck. Compare also with [56, Theorem 3.4.1] for the corresponding homotopy result.

The "weak Lefschetz theorem for intersection homology" 3. is due to [46, Theorem 7.1], and compare also with [47, Theorem 6.10]. Here we only cite the following remark of Goresky-MacPherson from the beginning of [47, Section 6.10]: "...The following Lefschetz hyperplane theorem was our original motivation for developing Morse theory on singular spaces ...".

The *weak Lefschetz theorem* of Example 10.3.61 can be generalized in many different directions. Especially, it is enough to assume that X is a *quasi-projective* algebraic subvariety $X = X'\backslash A$, with $A \subset X' \subset \mathbb{C}P^N$ closed subvarieties, if H is a *generic* hyperplane. Just take a Whitney stratification of X' such that A is a union of strata, and H is *transversal* to all strata. Then one can apply the following *base change isomorphisms* (see, e.g., [109, Lemma 6.0.5]).

Lemma 10.3.62 *Let $A \subset X'$ be closed analytic subvarieties of the complex manifold M. Fix a Whitney stratification $\mathscr{S}$ of X', with A a union of strata. Suppose H is closed complex submanifold of M, which is transversal to all strata, and consider the following cartesian diagram of inclusions:*

$$\begin{array}{ccc} X := X'\backslash A & \xleftarrow{\ j\ } & U := X\backslash H \\ \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle k} \\ X' & \xleftarrow{\ j'\ } & U' := X'\backslash H\,. \end{array}$$

Then for $\mathscr{F}^\bullet \in D^b_{\mathscr{S}|_X-wc}(X;R)$ a weakly constructible complex with respect to the induced stratification $\mathscr{S}|_X$ of X, one has natural isomorphisms:

$$j'_! Rk_* j^* \mathscr{F}^\bullet \simeq Rk'_* j_! j^* \mathscr{F}^\bullet \quad \textit{and} \quad Rj'_* k_! j^* \mathscr{F}^\bullet \simeq k'_! Rj_* j^* \mathscr{F}^\bullet\,. \tag{10.80}$$

If X' is also compact, then one gets for $\mathscr{F}^\bullet \in D^b_{\mathscr{S}|_X-wc}(X;R)$:

$$R\Gamma_c(X, Rj_* j^* \mathscr{F}^\bullet \simeq R\Gamma(X', k'_! Rj_* j^* \mathscr{F}^\bullet) \simeq R\Gamma(U', k_! j^* \mathscr{F}^\bullet)\,,$$

and

$$R\Gamma(X, X \cap H, \mathscr{F}^\bullet) \simeq R\Gamma(X', Rk'_* j_! j^* \mathscr{F}^\bullet) \simeq R\Gamma_c(U', Rk_* j^* \mathscr{F}^\bullet)\,.$$

Moreover,

$$j^* \mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(U;R) \;\Rightarrow\; k_! j^* \mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(U';R)\,,$$

and

$$j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(U; R) \;\Rightarrow\; Rk_* j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(U'; R)\,.$$

Applying the Artin vanishing Theorem 10.3.59 to these complexes on the right hand side, one gets that the *weak Lefschetz theorem* of Example 10.3.61 remains true for $X = X'\backslash A$ a quasi-projective variety, with $A \subset X' \subset \mathbb{C}P^N$ closed subvarieties, if H is a *generic* hyperplane as discussed above (with $U' \hookrightarrow \mathbb{C}P^N\backslash H = \mathbb{C}^N$ affine).

We next discuss the counterpart of the Artin vanishing Theorem 10.3.59 in the complex analytic context for *q-complete* varieties in the following sense.

Definition 10.3.63 A complex variety X is called *q-complete* ($q \in \mathbb{N}_0$), if there exists a *proper q-convex C^∞-function* $f\colon X \to \mathbb{R}^{\geq 0}$ (with q-convexity in the sense of Definition 10.3.18). In particular, the 0-*complete* complex varieties are just the complex analytic *Stein spaces* (see, e.g., [100]), and a closed complex subvariety of a q-complete variety is again q-complete.

Then we need to consider a Morse approximation of f (as in [8]) with possibly infinitely many critical points, and get by Theorem 10.3.19 the following result (see, e.g., [109, Corollary 6.1.2] and compare also with [66, Theorem 10.3.8]).

Theorem 10.3.64 (Vanishing Theorem for q-Complete Varieties) *Let X be a complex analytic variety with a Whitney stratification $\mathscr{S}$, $f\colon X \to \mathbb{R}^{\geq 0}$ a proper q-convex function, and*

$$K_n := \{f \leq r_n\},\; U_n := \{f < r_n\}, \quad \textit{for } r_n \nearrow \infty$$

a sequence of regular values of f with respect to $\mathscr{S}$ ($n \in \mathbb{N}$). Then one gets for $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X; R)$ a $\mathscr{S}$-weakly constructible complex and $q' := \min\{q, \dim X\}$:

1. *If $\mathscr{F}^\bullet \in {}^pD^{\leq m}_{\mathscr{S}}(X; R)$, then*

$$H^k(K_{n+1}, K_n; \mathscr{F}^\bullet) \simeq H^k(\{r_n \leq f \leq r_{n+1}\}, \{f = r_n\}; \mathscr{F}^\bullet) = 0 \quad \textit{for } k > m + q',$$

so that $H^k(K_{n+1}; \mathscr{F}^\bullet) \to H^k(K_n; \mathscr{F}^\bullet)$ is surjective for $k \geq m + q'$. In particular, the projective system $H^k(K_n; \mathscr{F}^\bullet)$, $n \in \mathbb{N}$, satisfies the Mittag-Leffler condition for $k \geq m + q'$ so that

$$H^k(X; \mathscr{F}^\bullet) \simeq \lim_{\leftarrow} H^k(K_n; \mathscr{F}^\bullet) = 0 \quad \textit{for } k > m + q'. \tag{10.81}$$

2. *If $\mathscr{F}^\bullet \in {}^pD^{\geq m}_{\mathscr{S}}(X; R)$, then*

$$H^k_c(X; \mathscr{F}^\bullet) \simeq \lim_{\rightarrow} H^k_c(U_n; \mathscr{F}^\bullet) = 0 \quad \textit{for } k < m - q'. \tag{10.82}$$

3. *If* $\mathscr{F}^\bullet \in {}^{p^+}D^{\geq m}_{\mathscr{S}}(X; R)$ *is constructible and* R *is a Dedekind domain, then*

$$H_c^{m-q'}(X; \mathscr{F}^\bullet) \simeq \lim_{\to} H_c^{m-q'}(U_n; \mathscr{F}^\bullet) \quad \text{is torsion-free.} \tag{10.83}$$

Note that in the general complex analytic context one cannot expect any *finiteness or torsion-properties* for the corresponding cohomology groups. If X is the Stein space given by an infinite discrete set $X = \mathbb{N} \subset \mathbb{C}$ and $\mathscr{F}$ is a sheaf on X, then

$$H^0(X; \mathscr{F}) = \prod_{x \in X} \mathscr{F}_x \quad \text{and} \quad H_c^0(X; \mathscr{F}) = \bigoplus_{x \in X} \mathscr{F}_x \,.$$

So even if all stalks $\mathscr{F}_x$ ($x \in X$) are finitely generated (torsion) R-modules, this does not need to be the case for $H^k(X; \mathscr{F})$ or $H_c^k(X; \mathscr{F})$. See [108] for similar finiteness results on *q-convex and q-concave complex varieties* (using the same method of proof as in Theorem 10.3.64).

Example 10.3.65 Let X be a q-complete variety of dimension $n = \dim X$ (e.g., a Stein space for $q = 0$), with $q' := \min\{q, n\}$. Then

1. The *homology* $H_k(X; R) = H_c^{-k}(X; \mathbb{D}_X^\bullet)$ vanishes for $k > n + q'$. Moreover, $H_{n+q'}(X; R)$ is torsion-free in case R is a Dedekind domain.
2. The *cohomology* $H^k(X; R)$ vanishes for $k > n + q'$.
3. Suppose X is purely n-dimensional. Then we get for the *intersection cohomology* that

$$IH^k(X; R) := H^{k-n}(X; IC_X) = 0 \quad \text{for } k > n + q',$$

and the *intersection homology* $IH_k(X; R) := H_c^{n-k}(X; IC_X)$ vanishes for $k > n + q'$. Moreover, $IH_{n+q'}(X; R)$ is torsion-free in case R is a Dedekind domain.

The results in 1. and 2. are due to Hamm [54, 55], whereas 3. is due to Goresky-MacPherson [47, Section 6.9] (at least for Stein spaces). See [109, Chapter 6] for more general results and further references.

Before stating the relative versions of these vanishing results, let us recall the following.

Definition 10.3.66 A morphism $f : X \to Y$ of complex algebraic varieties is called *affine*, if any point $y \in Y$ has an open affine neighborhood $U \subset Y$ such that $f^{-1}(U)$ is affine. Similarly, a morphism $f : X \to Y$ of complex analytic varieties is called *q-complete*, resp., *Stein*, if any point $y \in Y$ has an open neighborhood $U \subset Y$ such that $f^{-1}(U)$ is q-complete, resp., Stein ($q \in \mathbb{N}_0$). So, by definition, a *Stein morphism* corresponds to a *0-complete morphism*.

Example 10.3.67 A finite morphism (e.g., a closed embedding) is an affine, resp., Stein morphism. Similarly, if the closed embedding $i : X \hookrightarrow Y$ is locally given by one algebraic, resp., analytic equation $X = \{f = 0\}$, then the inclusion of the open complement $j\colon U := Y\backslash X \hookrightarrow Y$ is an affine, resp., Stein morphism. Also, an *affine algebraic* morphism is *Stein* when viewed as a morphism of the underlying analytic varieties.

Remark 10.3.68 Let us explain the crucial property of a q-complete (resp., affine) morphism $f\colon X \to Y$ used for (co)stalk calculations in the following Theorems. In the analytic context consider a local embedding $(Y, y) \hookrightarrow (\mathbb{C}^n, 0)$ and an open neighborhood $U \subset Y$ of y such that $f^{-1}(U)$ is q-complete, with $g\colon f^{-1}(U) \to \mathbb{R}^{\geq 0}$ a proper q-convex function. Consider the strongly plurisubharmonic distance function $r\colon \mathbb{C}^n \to \mathbb{R}^{\geq 0}$ with $r(z) := \sum_{i=1}^n z_i \cdot \bar{z}_i$. Then for any small $r_0 > 0$ the function

$$g' := g + \frac{1}{r - r_0} \circ f \,:\, f^{-1}(U \cap \{r < r_0\}) \to \mathbb{R}^{\geq 0} \quad \text{is also proper and q-convex} \tag{10.84}$$

(see, e.g., [109, p.429] and [119, Proposition 2.2, Proposition 2.4]). Therefore one can apply the vanishing Theorem 10.3.64 to $f^{-1}(U \cap \{r < r_0\})$.

For an affine morphism $f\colon X \to Y$, one takes a global affine embedding $(U, y) \hookrightarrow (\mathbb{C}^n, 0)$ of an open affine neighborhood $U \subset Y$ of y such that $f^{-1}(U)$ is also affine with $g\colon f^{-1}(U) \to \mathbb{R}^{\geq 0}$ a proper strongly plurisubharmonic and semialgebraic function. Then g' as before is a *proper strongly plurisubharmonic and semialgebraic function* on the *open semi-algebraic* subset $f^{-1}(U \cap \{r < r_0\}) \subset X$, so that one can use the Artin vanishing Theorem in the version of Remark 10.3.60.

Let us now state the following *Artin-Gothendieck type* result in the complex algebraic context (see, e.g., [109, Theorem 6.0.4]).

Theorem 10.3.69 (Algebraic Artin-Grothendieck Type Theorem) *Let $f\colon X \to Y$ be an affine morphism of complex algebraic varieties, with $\mathscr{F}^\bullet \in D^b_{(w)c}(X; R)$ a complex algebraically (weakly) constructible sheaf complex (respectively, complex algebraically constructible sheaf complex in case of a Dedekind domain R, if we want to use the dual perverse t-structure). Then:*

1. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(X; R) \Rightarrow Rf_*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(Y; R)$,
2. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(X; R) \Rightarrow Rf_!\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(Y; R)$.

Compare with [6, Section 4], especially [6, Theorem 4.1.1], for the corresponding relative counterpart for an affine morphism in the context of the *perverse t-structure* in l-adic cohomology. In the case $Y = \{pt\}$ a point space, Theorem 10.3.69 just reduces to Theorem 10.3.59.

In the complex analytic context one has in addition to assume that the corresponding direct image complexes are again (weakly) constructible (see, e.g., [109, Corollary 6.0.8] and [66, Proposition 10.3.17] for the case of a Stein map).

Theorem 10.3.70 (Analytic Artin-Grothendieck Type Theorem) *Let $f: X \to Y$ be a q-complete morphism of complex analytic varieties (e.g., a Stein map for $q = 0$), with $\mathscr{F}^\bullet \in D^b_{(w)c}(X; R)$ a (weakly) constructible sheaf complex. Then:*

1. $\mathscr{F}^\bullet \in {}^pD^{\leq n}(X; R)$ *and* $Rf_*\mathscr{F}^\bullet \in D^b_{(w)c}(Y; R) \Rightarrow Rf_*\mathscr{F}^\bullet \in {}^pD^{\leq n+q}(Y; R)$,
2. $\mathscr{F}^\bullet \in {}^pD^{\geq n}(X; R)$ *and* $Rf_!\mathscr{F}^\bullet \in D^b_{(w)c}(Y; R) \Rightarrow Rf_!\mathscr{F}^\bullet \in {}^pD^{\geq n-q}(Y; R)$.
3. *Assume* $\mathscr{F}^\bullet \in D^b_c(X; R)$ *is constructible with* R *a Dedekind domain. then* $\mathscr{F}^\bullet \in {}^{p^+}D^{\geq n}(X; R)$ *and* $Rf_!\mathscr{F}^\bullet \in D^b_c(Y; R) \Rightarrow Rf_!\mathscr{F}^\bullet \in {}^{p^+}D^{\geq n-q}(Y; R)$.

In the case $Y = \{pt\}$ a point space, Theorem 10.3.70 corresponds to Theorem 10.3.64.

Let us finish this section with the following:

Example 10.3.71 (Relative Weak Lefschetz Theorem for Singular Spaces) Let $V \to Y$ be a complex algebraic (or analytic) vector bundle, and $W \hookrightarrow V$ be a subbundle with $rank\ V = rank\ W + 1$. Let $i: \mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$ be the closed inclusion of the associated projective bundles, with open complement $j: U := \mathbb{P}(V)\backslash\mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$. Then the projection $\pi: U \to Y$ is an affine (resp., Stein) morphism.

Assume $\mathscr{F}^\bullet \in D^b_{(w)c}(\mathbb{P}(V); R)$ is (weakly) constructible (respectively, constructible in case of a Dedekind domain R, if we want to use the dual perverse t-structure), so that

$$R\pi_! j^*\mathscr{F}^\bullet,\ R\pi_* j^*\mathscr{F}^\bullet \in D^b_{(w)c}(Y; R)\,.$$

Then

$$j^*\mathscr{F}^\bullet \in {}^pD^{\leq n}(U; R) \Rightarrow R\pi_* j^*\mathscr{F}^\bullet \in {}^pD^{\leq n}(Y; R)\,,$$

and

$$j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(U; R) \Rightarrow R\pi_! j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(Y; R)\,.$$

10.4 Nearby and Vanishing Cycles, Applications

In this section we recall the construction of Deligne's nearby and vanishing cycle functors ([52]), and indicate their relation with perverse sheaves. We continue to assume that the base ring R is commutative and Noetherian, of finite global dimension, and we work with (weakly) constructible complexes of sheaves of R-modules in the complex algebraic (or analytic) context.

10.4.1 Construction

Let $f \colon X \to \mathbb{C}$ be a morphism from a complex algebraic (or analytic) variety X to $\mathbb{C}$. Let $X_0 = f^{-1}(0)$ be the central fiber, with inclusion map $i \colon X_0 \hookrightarrow X$. Let $X^* := X \setminus X_0$ and let $f^* \colon X^* \to \mathbb{C}^*$ be the induced morphism to the punctured affine line. Consider the following cartesian diagram:

$$\begin{array}{ccccccc} X_0 & \overset{i}{\hookrightarrow} & X & \overset{j}{\hookleftarrow} & X^* & \overset{\widehat{\pi}}{\longleftarrow} & \widetilde{X^*} \\ \downarrow & & \downarrow f & & \downarrow f^* & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{C} & \hookleftarrow & \mathbb{C}^* & \underset{\pi}{\longleftarrow} & \widetilde{\mathbb{C}^*}, \end{array} \tag{10.85}$$

where $\pi \colon \widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ is the infinite cyclic (universal) cover of $\mathbb{C}^*$ given by $z \mapsto \exp(2\pi i z)$. Then $\widehat{\pi} \colon \widetilde{X^*} \to X^*$ is an infinite cyclic cover with deck group $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$. Note that π is only a holomorphic but not an algebraic map, so that this fiber square only exists in the complex analytic category, even if we start with a morphism $f \colon X \to \mathbb{C}$ in the complex algebraic context.

Definition 10.4.1 The *nearby cycle functor of* f assigns to a bounded complex $\mathscr{F}^\bullet \in D^b(X; R)$ the complex on X_0 defined by

$$\psi_f \mathscr{F}^\bullet := i^* R(j \circ \widehat{\pi})_* (j \circ \widehat{\pi})^* \mathscr{F}^\bullet \simeq i^* R j_* R\widehat{\pi}_* \widehat{\pi}^* j^* \mathscr{F}^\bullet \in D^b(X_0; R) . \tag{10.86}$$

Remark 10.4.2 By definition, $\psi_f \mathscr{F}^\bullet$ depends only on the restriction $j^* \mathscr{F}^\bullet$ of $\mathscr{F}^\bullet$ to X^*. In the complex analytic context, it would have been enough for the definition of the nearby cycles to start with a holomorphic map $f \colon X \to D \subset \mathbb{C}$ to a small open disc D around zero in the complex plane, with $\pi \colon \widetilde{D^*} \to D^*$ the induced infinite cyclic (universal) cover of D^*.

Note that $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$ acts naturally on $R\widehat{\pi}_* \widehat{\pi}^*$, with an induced action on the nearby cycles $\psi_f \mathscr{F}^\bullet$. In our complex context we choose the generator of $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$ fitting with the complex orientation of $\mathbb{C}^*$ and call the induced action

$$h = h_f \colon \psi_f \mathscr{F}^\bullet \to \psi_f \mathscr{F}^\bullet$$

the *monodromy automorphism* of the nearby cycles. The adjunction morphism

$$id \to R(j \circ \widehat{\pi})_* (j \circ \widehat{\pi})^*$$

induces the *specialization map* $sp \colon i^* \to \psi_f$ commuting with the monodromy automorphism h acting trivially on i^* (i.e., acting as the identity on i^*). Let us now explain the use of the base change induced by the universal cover $\pi \colon \widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ in the definitions above in a simple but important example.

Example 10.4.3 (Nearby Cycle Functor for the Identity Map) Let $f = id$ be the identity map of $\mathbb{C}$, with $i\colon \{0\} \hookrightarrow \mathbb{C}$ the inclusion of the point zero. Similarly, let $i_x\colon \{x\} \hookrightarrow D_r := \{|z| < r\} \subset \mathbb{C}$, for $0 < |x| < r \leq \infty$, be the inclusion of a (nearby) point $x \neq 0$ in a (small) open disc of radius r around zero. Finally consider a complex $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(\mathbb{C}; R)$ which is weakly constructible with respect to the Whitney stratification $\mathscr{S}$ of $\mathbb{C}$ given by the two strata $S = \{0\}$ and $S' = \mathbb{C}^*$.

Then the cohomology sheaves $\mathscr{H}^k(j^*\mathscr{F}^\bullet)$ are only *locally constant* on $\mathbb{C}^*$, but their pullbacks $\mathscr{H}^k(\widehat{\pi}^* j^*\mathscr{F}^\bullet)$ to $\mathbb{C} \simeq \widetilde{\mathbb{C}^*}$ (or their restrictions to $\widetilde{D_r^*}$) are locally constant and therefore *constant* (for all $k \in \mathbb{Z}$) since $\mathbb{C} \simeq \widetilde{\mathbb{C}^*}$ (or $\widetilde{D_r^*}$) is convex and *contractible*. Then

$$\begin{array}{ccc} \psi_{id}\mathscr{F}^\bullet & \xleftarrow{\sim} & R\Gamma(D_r^*; R\widehat{\pi}_*\widehat{\pi}^* j^*\mathscr{F}^\bullet) \simeq R\Gamma(\widetilde{D_r^*}; \widehat{\pi}^* j^*\mathscr{F}^\bullet) \\ & & \downarrow \wr \\ & & i_{\tilde{x}}^*\widehat{\pi}^* j^*\mathscr{F}^\bullet \simeq i_x^*\mathscr{F}^\bullet \end{array}$$

for $i_{\tilde{x}}\colon \{\tilde{x}\} \hookrightarrow \widetilde{D_r^*}$ the inclusion of a point $\tilde{x}$ with $\pi(\tilde{x}) = x$. And the *monodromy* h on $H^k(\psi_{id}(\mathscr{F}^\bullet))$ gets identified with the *monodromy of the local system* $\mathscr{H}^k(j^*\mathscr{F}^\bullet)$ acting on $i_x^*\mathscr{H}^k(j^*\mathscr{F}^\bullet)$ (for all $k \in \mathbb{Z}$). Finally the *specialization* map is given by

$$sp\colon\ i^*\mathscr{F}^\bullet \xleftarrow{\sim} R\Gamma(D_r; \mathscr{F}^\bullet) \longrightarrow i_x^*\mathscr{F}^\bullet \simeq \psi_{id}\mathscr{F}^\bullet\,.$$

Definition 10.4.4 In the context of the diagram (10.85), the *complex of vanishing cycles* of $\mathscr{F}^\bullet \in D^b(X; R)$, denoted $\varphi_f\mathscr{F}^\bullet$, is the bounded complex on X_0 defined by taking "the" cone of the *specialization map* $sp\colon i^*\mathscr{F}^\bullet \to \psi_f\mathscr{F}^\bullet$. In particular, one gets a unique distinguished triangle

$$i^*\mathscr{F}^\bullet \xrightarrow{sp} \psi_f\mathscr{F}^\bullet \xrightarrow{can} \varphi_f\mathscr{F}^\bullet \xrightarrow{[1]} \tag{10.87}$$

in $D^b(X_0; R)$.

Remark 10.4.5 Note that cones are not functorial, but in the above construction one can for example work with (a suitable truncation of) the canonical flabby resolution to get φ_f as a functor (see [66, Chapter 8] or [109, pp. 25–26] for more details). The vanishing cycle functor also comes equipped with a monodromy automorphism, denoted also by h, so that h induces an automorphism of the triangle (10.87).

Example 10.4.6 Assume $\mathscr{F}^\bullet \in D^b(X; R)$ is supported on X_0, i.e., $j^*\mathscr{F}^\bullet \simeq 0$. Then $\psi_f\mathscr{F}^\bullet \simeq 0$ and $(\varphi_f\mathscr{F}^\bullet)[-1] \simeq i^*\mathscr{F}^\bullet$, with a trivial monodromy action h.

Example 10.4.7 (Vanishing Cycle Functor for the Identity Map) Consider the context of the Example 10.4.3. Then for any point $0 \neq x \in D_r$ in a small open disc D_r around zero:

$$(\varphi_{id}\mathscr{F}^\bullet)[-1] \simeq R\Gamma(D_r, \{x\}; \mathscr{F}^\bullet) \simeq \left(R\Gamma_{\{l\geq 0\}}(\mathscr{F}^\bullet)\right)_0 =: LMD(\mathscr{F}^\bullet, l, 0)$$

for any $\mathbb{R}$-linear map $l: \mathbb{C} \to \mathbb{R}$ with $l(x) < 0$. In particular the following properties are equivalent:

1. $sp: i^*\mathscr{F}^\bullet \to \psi_{id}\mathscr{F}^\bullet$ is an isomorphism,
2. $\varphi_{id}\mathscr{F}^\bullet \simeq 0$,
3. all cohomology sheaves $\mathscr{H}^k(\mathscr{F}^\bullet)$ are locally constant on $\mathbb{C}$ or D_r ($k \in \mathbb{Z}$),
4. the normal Morse datum $NMD(\mathscr{F}^\bullet, \{0\}) \simeq LMD(\mathscr{F}^\bullet, l, 0)$ vanishes.

Next we explain why the nearby and vanishing cycle functors preserve (weak) constructibility. Consider a complex algebraic (or analytic) morphism $f: X \to \mathbb{C}$ as in diagram (10.85). Assume X is endowed with a Whitney stratification $\mathscr{S}$ such that $X_0 = \{f = 0\}$ is a union of strata, with induced stratification $\mathscr{S}|_{X_0}$. Similarly for the induced stratification $\mathscr{S}|_{X^*}$ of the open complement $X^* = \{f \neq 0\}$. But $\widehat{\pi}: \widetilde{X^*} \to X^*$ is an infinite cyclic covering with deck group $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$, so that $\widetilde{X^*}$ gets an induced complex analytic Whitney stratification $\widetilde{\mathscr{S}}$, making $\widehat{\pi}: \widetilde{X^*} \to X^*$ a *stratified map* such that, for any stratum $S \in \mathscr{S}$, $\widehat{\pi}: \widehat{\pi}^{-1}(S) \to S$ is also an infinite cyclic covering with deck group $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$, i.e., a locally trivial fibration with fiber $\mathbb{Z}$. But this implies by induction on dim X^* the following (see, e.g. [109, Corollary 4.2.1(4)]).

Lemma 10.4.8 *$R\widehat{\pi}_*\widehat{\pi}^*$ maps $D^b_{\mathscr{S}|_{X^*}-wc}(X^*; R)$ to itself.*

Together with Example 10.2.38 this implies (also in the complex algebraic context) the following important fact.

Corollary 10.4.9 *The nearby and vanishing cycle functors ψ_f, φ_f induce*

$$\psi_f,\ \varphi_f:\ D^b_{(\mathscr{S}-)wc}(X; R) \to D^b_{(\mathscr{S}|_{X_0}-)wc}(X_0; R)\ ,$$

i.e., they preserve weak constructibility (with respect to $\mathscr{S}$ and $\mathscr{S}|_{X_0}$).

But note that $R\widehat{\pi}_*\widehat{\pi}^*$ *does not preserve* $D^b_{(\mathscr{S}|_{X^*}-)c}(X^*; R)$, i.e., constructibility. Let g be a generator of $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$, given by the complex orientation of $\mathbb{C}^*$, which we interpret now as an *automorphism* of $\widetilde{X^*}$. Let $\mathscr{G}$ be a sheaf on X^*. We claim that the following sequence of sheaves on X^* is *exact*:

$$0 \longrightarrow \mathscr{G} \xrightarrow{ad_{\widehat{\pi}}} \widehat{\pi}_*\widehat{\pi}^*\mathscr{G} \xrightarrow{g^*-id} \widehat{\pi}_*\widehat{\pi}^*\mathscr{G} \longrightarrow 0\,.$$

Take a ball B in $\mathbb{C}^*$ so that the restriction of the universal covering map of $\mathbb{C}^*$ to B is isomorphic to the projection $B \times \mathbb{Z} \to B$, with g corresponding to the translation $\mathbb{Z} \to \mathbb{Z}$, $i \mapsto i + 1$. If we take an open subset $V \subset f^{-1}(B)$, then $\widehat{\pi}: \widehat{\pi}^{-1}(V) \to V$ is isomorphic to the projection $V \times \mathbb{Z} \to V$, with g acting as before on the second factor. Then

$$\Gamma(V, \widehat{\pi}_*\widehat{\pi}^*\mathscr{G}) \simeq \prod\nolimits_{i\in\mathbb{Z}}\ \Gamma(V, \mathscr{G})$$

such that g^* acts by the *permutation* $i \mapsto i + 1$. Moreover the adjunction map corresponds to the diagonal embedding $\Gamma(V, \mathscr{G}) \to \prod_{i\in\mathbb{Z}} \Gamma(V, \mathscr{G})$. Then the sequence

$$0 \longrightarrow \Gamma(V,\mathscr{G}) \xrightarrow{diag} \prod_{i\in\mathbb{Z}} \Gamma(V,\mathscr{G}) \xrightarrow{g^*-id} \prod_{i\in\mathbb{Z}} \Gamma(V,\mathscr{G}) \longrightarrow 0$$

is exact. Since each point of $X^* = \{f \neq 0\}$ has a fundamental system of open neighborhoods V as before, this implies our claim.

By using a flabby resolution we therefore get the distinguished triangles (see, e.g., [109, (5.88) on p.369])

$$\mathscr{G} \xrightarrow{ad_{\widehat{\pi}}} R\widehat{\pi}_*\widehat{\pi}^*\mathscr{G} \xrightarrow{g^*-id} R\widehat{\pi}_*\widehat{\pi}^*\mathscr{G} \xrightarrow{[1]}$$

for any $\mathscr{G}^\bullet \in D^b(X^*; R)$, and

$$i^*Rj_*j^*\mathscr{F}^\bullet \xrightarrow{ad_{\widehat{\pi}}} \psi_f\mathscr{F}^\bullet \xrightarrow{h_f-id} \psi_f\mathscr{F}^\bullet \xrightarrow{[1]} \tag{10.88}$$

for any $\mathscr{F}^\bullet \in D^b(X; R)$.

Consider also the distinguished triangle

$$i^*\mathscr{F}^\bullet \xrightarrow{ad_j} i^*Rj_*j^*\mathscr{F}^\bullet \longrightarrow i^!\mathscr{F}^\bullet[1] \xrightarrow{[1]} . \tag{10.89}$$

Since the map $can\colon i^*\mathscr{F}^\bullet \to \psi_f\mathscr{F}^\bullet$ factorizes as

$$can\colon i^*\mathscr{F}^\bullet \xrightarrow{ad_j} i^*Rj_*j^*\mathscr{F}^\bullet \xrightarrow{ad_{\widehat{\pi}}} \psi_f\mathscr{F}^\bullet ,$$

we get from the distinguished triangles (10.89), (10.88), (10.87) and the *octahedral axiom* a distinguished triangle

$$i^!\mathscr{F}^\bullet[1] \longrightarrow \varphi_f\mathscr{F}^\bullet \xrightarrow{var} \psi_f\mathscr{F}^\bullet \xrightarrow{[1]} \tag{10.90}$$

for $\mathscr{F}^\bullet \in D^b(X; R)$. Here the variation morphism

$$var\colon \varphi_f\mathscr{F}^\bullet \to \psi_f\mathscr{F}^\bullet$$

can be defined by the cone of the pair of morphisms (applied to a flabby resolution):

$$(0, h - id)\colon [i^*\mathscr{F}^\bullet \to \psi_f\mathscr{F}^\bullet] \longrightarrow [0 \to \psi_f\mathscr{F}^\bullet] ,$$

with $h = h_f$ the monodromy automorphism, so that

$$can \circ var = h - id \quad \text{and} \quad var \circ can = h - id \,. \tag{10.91}$$

Remark 10.4.10 Note that the *variation morphism var* depends of the choice of a generator $g \in \pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$, i.e., on the choice of an orientation for $\mathbb{C}^*$. Moreover there are then two choices for *var* in the literature, using a different sign convention so that

$$can \circ var = \pm(h - id) = \mp(id - h) \quad \text{and} \quad var \circ can = \pm(h - id) = \mp(id - h) \,.$$

For example, our choice here only fits with $-var$ as used in [66, Equation (8.6.8)] and [109, Equation (5.90)].

Let us now explain an important description of the *(co)stalks of the nearby cycles* in terms of *local Milnor fibers*. Consider a complex algebraic (or analytic) morphism $f: X \to \mathbb{C}$ as in diagram (10.85). Assume X is endowed with a Whitney stratification $\mathscr{S}$ such that $X_0 = \{f = 0\}$ is a union of strata, with induced stratification $\mathscr{S}|_{X_0}$. Recall again that by a classical theorem of Hironaka [57, Corollary 1 of Theorem 2, p.248] (see also [73, Corollary 1.3.5.1]), one can always refine a given stratification so that it satisfies the a_f-condition (10.69). Moreover, by [14, Theorem 4.2.1], this a_f-condition is true for the given Whitney stratification $\mathscr{S}$, with X_0 a union of strata.

Note that the a_f-condition is needed to have a *local Milnor fibration* at a given point $x \in X_0$, with *Milnor fiber*

$$(M_{f,x}, \partial M_{f,x}) := (X \cap \{f = w\} \cap \{r \leq \epsilon\}, X \cap \{f = w\} \cap \{r = \epsilon\})$$

$$\text{and} \quad \mathring{M}_{f,x} := M_{f,x} \backslash \partial M_{f,x}$$

for $0 < |w| \ll \epsilon$ and a general holomorphic function germ $f: (X, x) \to (\mathbb{C}, 0)$ on a singular complex analytic variety X (see, e.g., [109, Example 1.3.3] and [70]). Here we consider a local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, 0)$ with $r(z) := \sum_{i=1}^n z_i \cdot \bar{z}_i$ the distance to $x = 0$. By the *curve selection lemma* one can assume that (locally near $x \in X_0$) w is a stratified regular value, i.e., $\{f = w\}$ is transversal to $\mathscr{S}$ for $0 < |w|$ small enough. Then by the a_f-condition, and for $0 < |w| \ll \epsilon$ small enough, $r = \epsilon$ is a stratified regular value of r with respect to the induced Whitney stratification of $X \cap \{f = w\}$ (see, e.g., [109, Example 1.1.3]).

Then the following local calculation is a direct consequence of the definition of the *nearby cycle functor* and the existence of such a *local Milnor fibration* (see, e.g., [109, Example 5.4.2]).

Proposition 10.4.11 *For every $x \in X_0$, with $i_x: \{x\} \hookrightarrow X_0 = \{f = 0\}$ the inclusion, there are isomorphisms:*

$$i_x^*(\psi_f \mathscr{F}^\bullet) \simeq R\Gamma(M_{f,x}, \mathscr{F}^\bullet) \simeq R\Gamma(\mathring{M}_{f,x}, \mathscr{F}^\bullet) \tag{10.92}$$

compatible with the corresponding monodromy actions, and

$$i_x^!(\psi_f \mathscr{F}^\bullet) \simeq R\Gamma(M_{f,x}, \partial M_{f,x}, \mathscr{F}^\bullet) \simeq R\Gamma_c(\mathring{M}_{f,x}, \mathscr{F}^\bullet) \tag{10.93}$$

for any $\mathscr{F}^\bullet \in D^b_{wc}(X; R)$.

By Proposition 10.3.21 and the triangle (10.87) we get the following.

Corollary 10.4.12 *Let* $T \subset D^b(\{pt\}; R)$ *be a fixed "null system", i.e., a full triangulated subcategory stable by isomorphisms. If* $\mathscr{F}^\bullet \in D^b_{(\mathscr{S}-)T-stalk}(X; R)$*, then also*

$$\psi_f \mathscr{F}^\bullet,\ \varphi_f \mathscr{F}^\bullet \in D^b_{(\mathscr{S}|_{X_0}-)T-stalk}(X; R)\,.$$

In particular, $\psi_f \mathscr{F}^\bullet$ *and* $\varphi_f \mathscr{F}^\bullet$ *are constructible for* $\mathscr{F}^\bullet$ *constructible.*

Remark 10.4.13 By taking $T = D^b_{c,\chi=0}(\{pt\}; R)$, we get the nearby and vanishing cycles for constructible functions:

$$\psi_f,\ \varphi_f\colon\ CF_{(\mathscr{S})}(X) \to CF_{(\mathscr{S}|_{X_0})}(X_0)\,,$$

with

$$\psi_f(\alpha)(x) := \int_{M_{f,x}} \alpha d\chi \quad \text{and} \quad \varphi_f(\alpha)(x) := \int_{M_{f,x}} \alpha d\chi - \alpha(x)$$

for $\alpha \in CF_{(\mathscr{S})}(X)$ and $x \in X_0 = \{f = 0\}$.

Using Proposition 10.4.11 and the distinguished triangle (10.87), one gets the following (see, e.g., [109, Lemma 5.4.1, Example 5.4.1]).

Corollary 10.4.14 *For every* $x \in X_0 = \{f = 0\}$*, with* $i_x\colon \{x\} \hookrightarrow X_0 = \{f = 0\}$ *the inclusion, there are isomorphisms:*

$$\begin{aligned} i_x^*(\varphi_f \mathscr{F}^\bullet)[-1] &\simeq R\Gamma(\mathring{B}_{\epsilon,x}, \mathring{B}_{\epsilon,x} \cap \{f = w\}, \mathscr{F}^\bullet) \\ &\simeq \left(R\Gamma_{\{Re(f)\geq 0\}}(\mathscr{F}^\bullet)\right)_x =: LMD(\mathscr{F}^\bullet, Re(f), x) \end{aligned} \tag{10.94}$$

for $\mathscr{F}^\bullet \in D^b_{wc}(X; R)$ *and* $0 < |w| \ll \epsilon$ *small enough. Here* $\mathring{B}_{\epsilon,x} = X \cap \{r < \epsilon\}$ *is the intersection of* X *with a small open* ϵ*-ball around* $x \in X$.

Example 10.4.15 As a special case of (10.94), let $\mathscr{F}^\bullet = R_X$ be the constant sheaf on X. Since $\mathring{B}_{\epsilon,x} \cap X_0$ is contractible, one gets

$$\mathscr{H}^k(\varphi_f R_X)_x \simeq \widetilde{H}^k(M_{f,x}; R)$$

is the *reduced cohomology* of the Milnor fiber $M_{f,x}$ of f at x. If, moreover, X is smooth, then Milnor fibers at smooth points of X_0 are contractible, so the above calculation yields the inclusion:

$$\mathrm{supp}(\varphi_f R_X) := \bigcup_k \mathrm{supp}\ \mathscr{H}^k(\varphi_f \mathscr{F}^\bullet) \subseteq \mathrm{Sing}(X_0).$$

A more general estimation of the support of vanishing cycles is provided by the following result (see, e.g., [83] or [109, Remark 4.2.4]).

Proposition 10.4.16 *Let X be a complex algebraic (or analytic) variety with a given Whitney stratification $\mathscr{S}$, and let $f\colon X \to \mathbb{C}$ be a morphism with $X_0 = \{f = 0\}$. For every $\mathscr{S}$-weakly constructible complex $\mathscr{F}^\bullet$ on X and every integer k, one has the inclusion*

$$\mathit{supp}\mathscr{H}^k(\varphi_f \mathscr{F}^\bullet) \subseteq X_0 \cap \mathrm{Sing}_{\mathscr{S}}(f), \tag{10.95}$$

where

$$\mathrm{Sing}_{\mathscr{S}}(f) := \bigcup_{S \in \mathscr{S}} \mathrm{Sing}(f|_S)$$

is the stratified singular set of f with respect to the stratification $\mathscr{S}$.

Example 10.4.17 (Isolated Stratified Critical Point) In the context of Proposition 10.4.16, assume that $x \in X_0 = \{f = 0\}$ is an *isolated* stratified critical point of f. Then

$$i_x^!(\varphi\mathscr{F}^\bullet)[-1] \simeq i_x^*(\varphi\mathscr{F}^\bullet)[-1] \simeq \big(R\Gamma_{\{Re(f)\geq 0\}}(\mathscr{F}^\bullet)\big)_x =: LMD(\mathscr{F}^\bullet, Re(f), x)$$

for $i_x\colon \{x\} \to X_0$ the point inclusion. Applying i_x^* to the distinguished triangle (10.87), one gets

$$i_x^*(\psi_f\mathscr{F}^\bullet)[-1] \xrightarrow{can} i_x^*(\varphi_f\mathscr{F}^\bullet)[-1] \longrightarrow i_x^*(i^*\mathscr{F}^\bullet) \xrightarrow{[1]} ,$$

with $i_x^*(\psi_f\mathscr{F}^\bullet) \simeq R\Gamma(M_{f,x}, \mathscr{F}^\bullet)$ for a local Milnor fiber $M_{f,x}$ as in (10.92).

Applying $i_x^!$ to the distinguished triangle (10.90), one gets

$$i_x^!(i^!\mathscr{F}^\bullet) \longrightarrow i_x^*(\varphi_f\mathscr{F}^\bullet)[-1] \xrightarrow{var} i_x^!(\psi_f\mathscr{F}^\bullet)[-1] \xrightarrow{[1]} ,$$

with $i_x^!(\psi_f\mathscr{F}^\bullet) \simeq R\Gamma(M_{f,x}, \partial M_{f,x}, \mathscr{F}^\bullet)$ for a local Milnor fiber $(M_{f,x}, \partial M_{f,x})$ as in (10.93).

Remark 10.4.18 Consider a local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, x)$, with N a *normal slice* to $x \in S$ for a stratum $S \in \mathscr{S}$. Assume $g\colon (\mathbb{C}^N, x) \to (\mathbb{C}, 0)$ is a holomorphic function germ such that the covector dg_x is *non-degenerate* with respect to $\mathscr{S}$. Then x is an isolated stratified critical point of $g|_N$ with respect to the induced

Whitney stratification $\mathscr{S}|_N$ of $X \cap N$, so that Example 10.4.17 gives for this case the distinguished triangles from Proposition 10.3.14.

The nearby and vanishing cycle functors have the following *base change properties*. Consider a cartesian diagram of morphism

$$\begin{array}{ccccc} Y_0 & \xrightarrow{k'} & X_0 & \longrightarrow & \{0\} \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\ Y & \xrightarrow{k} & X & \xrightarrow{f} & \mathbb{C}, \end{array} \tag{10.96}$$

with $f' := f \circ k$. Then one has the following *base change isomorphisms* (see, e.g., [109, Remark 4.3.7, Lemma 4.3.4]).

Proposition 10.4.19 (Base Change Isomorphisms for Nearby and Vanishing Cycles) *The following base change isomorphisms commute with the maps can and var:*

1. *Assume $k\colon Y \to X$ is proper. Then*

$$Rk'_*(\psi_{f'}\mathscr{F}^\bullet) \simeq \psi_f(Rk_*\mathscr{F}^\bullet) \quad \textit{and} \quad Rk'_*(\varphi_{f'}\mathscr{F}^\bullet) \simeq \varphi_f(Rk_*\mathscr{F}^\bullet)$$

   for all $\mathscr{F}^\bullet \in D^b(Y;R)$.
2. *Assume $k\colon Y \to X$ is smooth. Then*

$$k'^*(\psi_f\mathscr{F}^\bullet) \simeq \psi_{f'}(k^*\mathscr{F}^\bullet) \quad \textit{and} \quad k'^*(\varphi_f\mathscr{F}^\bullet) \simeq \varphi_{f'}(k^*\mathscr{F}^\bullet)$$

   for all $\mathscr{F}^\bullet \in D^b(X;R)$.
3. *Assume $X \hookrightarrow M$ is a closed subvariety of the complex algebraic (or analytic) manifold M, with $\mathscr{S}$ a Whitney stratification of X. Let $N \hookrightarrow M$ be a closed complex algebraic (or analytic) submanifold which is transversal to $\mathscr{S}$ (i.e., transversal to all strata $S \in \mathscr{S}$), with $k\colon Y := X \cap N \hookrightarrow X$ the induced inclusion. Then*

$$k'^*(\psi_f\mathscr{F}^\bullet) \simeq \psi_{f'}(k^*\mathscr{F}^\bullet) \quad \textit{and} \quad k'^*(\varphi_f\mathscr{F}^\bullet) \simeq \varphi_{f'}(k^*\mathscr{F}^\bullet)$$

   for all $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-wc}(X;R)$.

Then one gets by *proper base change* and the Examples 10.4.3 and 10.4.7 the following (see, e.g., [109, Example 1.1.1]).

Example 10.4.20 Let $f\colon X \to \mathbb{C}$ be a *proper* morphism, with $X_0 = \{f = 0\}$. Then

$$R\Gamma(X_0, \psi_f\mathscr{F}^\bullet) \simeq \psi_{id}(Rf_*\mathscr{F}^\bullet) \simeq i_x^*(Rf_*\mathscr{F}^\bullet) \simeq R\Gamma(\{f = x\}, \mathscr{F}^\bullet) \tag{10.97}$$

and

$$R\Gamma(X_0, \varphi_f \mathscr{F}^\bullet)[-1] \simeq \varphi_{id}(Rf_*\mathscr{F}^\bullet)[-1]$$
$$\simeq R\Gamma(D_r, \{x\}, Rf_*\mathscr{F}^\bullet) \simeq R\Gamma(\{|f| < r, \{f = x\}, \mathscr{F}^\bullet) \tag{10.98}$$

for $\mathscr{F}^\bullet \in D^b_{wc}(X; R)$ and a point $0 \neq x \in D_r$ in a small open disc D_r around zero. In particular, it follows from (10.87) and (10.97) that for any $0 \neq x \in D_r$ as above, one has for f proper the following *specialization sequence*:

$$\cdots \to H^k(X_0; \mathscr{F}^\bullet) \to H^k(\{f = x\}; \mathscr{F}^\bullet) \to H^k(X_0; \varphi_f \mathscr{F}^\bullet) \to H^{k+1}(X_0; \mathscr{F}^\bullet) \cdots \tag{10.99}$$

In the case $\mathscr{F}^\bullet = R_X$ and by analogy with the local situation, the groups $H^*(X_0; \varphi_f R_X)$ are usually referred to as the the *vanishing cohomology of* f (see, e.g., [92]).

10.4.2 Relation with Perverse Sheaves and Duality

Let $f : X \to \mathbb{C}$ be a morphism of complex algebraic (or analytic) varieties. The behavior of the nearby and vanishing cycle functors with regard to Verdier duality is described by the following result (for instance, see [86, Theorem 3.1, Corollary 3.2]).

Theorem 10.4.21 *The shifted functors $\psi_f[-1]$ and $\varphi_f[-1]$ commute with the Verdier duality functor $\mathcal{D}$ up to natural isomorphisms.*

Note that this duality result also fits with the following behavior of the nearby and vanishing cycle functors with regard to the (dual) perverse t-structure (but this is not used in its proof, see, e.g., [109, Theorem 6.0.2]).

Theorem 10.4.22 *Let $f : X \to \mathbb{C}$ be a morphism of complex algebraic (or analytic) varieties, with $X^* := \{f \neq 0\}$. Assume $\mathscr{F}^\bullet \in D^b(X; R)$ is weakly constructible (resp., $\mathscr{F}^\bullet$ is constructible with R a Dedekind domain, in case we want to use the dual perverse t-structure). Then we have:*

1. $j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(X^*; R) \Rightarrow (\psi_f \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\leq n}(X_0; R)$.
2. $j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(X^*; R) \Rightarrow (\psi_f \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\geq n}(X_0; R)$.
3. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(X; R) \Rightarrow (\varphi_f \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\leq n}(X_0; R)$.
4. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(X; R) \Rightarrow (\varphi_f \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\geq n}(X_0; R)$.

Proof Note that the result (3.) resp., (4.) for the *vanishing cycle functor* follows directly form the corresponding result (1.) resp., (2.) for the nearby cycle functor, if one uses the distinguished triangle (10.87) or (10.90).

The argument for the nearby cycles is similar to the proof of Theorem 10.3.19. But this time we use the corresponding description of Proposition 10.4.11 for the *(co)stalk of the nearby cycle functor*. Choose a complex algebraic (or analytic) Whitney stratification $\mathscr{S}$ of X with $X_0 = \{f = 0\}$ and $X^* = \{f \neq 0\}$ a union of strata, which satisfies the a_f*-condition* of Thom (10.69). Then we already know, by Corollary 10.4.9, that $\psi_f \mathscr{F}^\bullet$ is (weakly) constructible with respect to the induced Whitney stratification $\mathscr{S}|_{X_0}$ of X_0.

Consider a point $x \in S$ for a stratum $S \subset \{f = 0\}$ of dimension s. First we assume that $S = \{x\}$ is a point stratum. By Proposition 10.4.11 we get

$$i_x^*(\psi_f \mathscr{F}^\bullet) \simeq R\Gamma(X \cap \{r \leq \delta,\, f = w\}, \mathscr{F}^\bullet)\,,$$

and

$$i_x^!(\psi_f \mathscr{F}^\bullet) \simeq R\Gamma(X \cap \{r \leq \delta,\, f = w\}, X \cap \{r = \delta,\, f = w\}, \mathscr{F}^\bullet)$$

for $0 < |w| \ll \delta \ll 1$, with $i_x \colon \{x\} \to \{f = 0\}$ the inclusion and

$$r(z) := \sum\nolimits_{i=1}^{n} z_i \bar{z}_i \quad \text{in a local embedding} \quad (X, x) \hookrightarrow (\mathbb{C}^n, 0)\,.$$

Then $L := \{f = w\}$ is *transversal* to $\mathscr{S}$ near x for $0 < |w| \ll 1$ (by the curve selection lemma). So we get by Proposition 10.2.27:

$$\mathscr{F}^\bullet|_L[-1] \in {}^{p^{(+)}}D^{\leq n}(L; R) \quad \text{or} \quad \mathscr{F}^\bullet|_L[-1] \in {}^{p^{(+)}}D^{\geq n}(L; R)\,.$$

Moreover, for $0 < |w| \ll \delta \ll 1$, δ is a regular value of r with respect to $\mathscr{S}|_L$. This follows from the a_f*-condition*. If we apply Theorem 10.3.19 with $q = 0$ to r (or $-r$), then we get by the (co)stalk formulae above:

$$i_x^*(\psi_f \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\leq n}(\{pt\}; R) \quad \text{or} \quad i_x^!(\psi_f \mathscr{F}^\bullet)[-1] \in {}^{p^{(+)}}D^{\geq n}(\{pt\}; R)\,.$$

This proves our claim for a point stratum.

We reduce the general case to the first case by taking a *complex analytic normal slice* N at x (in some local embedding), with $\operatorname{codim} N = \dim S = s$. Consider the cartesian diagram

$$\begin{array}{ccccc}
\{x\} = N \cap S & \xrightarrow{\kappa_x} & N \cap \{f = 0\} & \xrightarrow{i'} & N \cap X \\
\downarrow k_x & & \downarrow k' & & \downarrow k \\
S & \xrightarrow{i_S} & \{f = 0\} & \xrightarrow{i} & X\,.
\end{array}$$

Then we have

$$k_x^* i_S^*(\psi_f \mathscr{F}^\bullet) \simeq \kappa_x^* k'^*(\psi_f \mathscr{F}^\bullet) \,.$$

Since $\psi_f \mathscr{F}^\bullet$ is weakly constructible with respect to the induced stratification of $\{f = 0\}$, we get by the *base change property* (10.10):

$$k_x^* i_S^!(\psi_f \mathscr{F}^\bullet) \simeq \kappa_x^! k'^*(\psi_f \mathscr{F}^\bullet) \,.$$

But we also have by Proposition 10.4.19 the *base change isomorphism*

$$k'^*(\psi_f \mathscr{F}^\bullet) \simeq \psi_{f'}(k^* \mathscr{F}^\bullet) \,, \quad \text{with } f' := f \circ k.$$

Then the claim follows from the first case for f', $X' := N \cap X$ and $k^* \mathscr{F}^\bullet$, since N is transversal to $\mathscr{S}$ near x, with codim $N = \dim S = s$, so that by Proposition 10.2.27:

$$k^* \mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n-s}(X'; R) \quad \text{or} \quad k^* \mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n-s}(X'; R) \,.$$

This completes the proof. □

Remark 10.4.23 We get in particular that the (shifted) *nearby and vanishing cycle functors* ${}^p\psi_f := \psi_f[-1]$ and ${}^p\varphi_f := \varphi_f[-1]$ are *t-exact* functors with respect to the *perverse t-structure*.

This is a result of Gabber in the algebraic context for l-adic cohomology (unpublished, but compare with [6, Proposition 4.4.2], [16, Theorem 1.2] and [61, Corollary 4.5, 4.6]). In the complex analytic context it is due to Goresky-MacPherson [45, Theorem 6.5], and [47, p.222, Corollary 6.13.6, p.224 6.A.5]. For another proof of this classical case see [66, Corollary 10.3.11, 10.3.13].

But it does not seem to be well known that the result also applies for R a Dedekind domain to the *dual t-structure*. In particular, the functors ${}^p\psi_f = \psi_f[-1]$ and ${}^p\varphi_f = \varphi_f[-1]$ preserve then *strongly perverse* sheaves.

Example 10.4.24 If X is a pure-dimensional complex algebraic (or analytic) variety satisfying $\mathrm{rHd}(X, R) = \dim X$ for R a Dedekind domain, then Proposition 10.2.54 and the above Remark yield that ${}^p\psi_f R_X[\dim X]$ and ${}^p\varphi_f R_X[\dim X]$ are strongly perverse sheaves on X_0. Therefore these perverse sheaves have torsion-free costalks in the lowest possible degree (cf. Corollary 10.2.55).

Example 10.4.25 Let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be a nonconstant holomorphic function germ defined on a pure $(n+1)$-dimensional complex singularity germ contained in some ambient $(\mathbb{C}^N, 0)$. Denote by $M_{f,0}$ the Milnor fiber of the singularity at the origin in $X_0 = f^{-1}(0)$. Let $\Sigma := \mathrm{Sing}_{\mathscr{S}}(f)$ be the stratified singular locus of f with respect to a fixed Whitney stratification $\mathscr{S}$ of X, and set $r := \dim_0 \Sigma$. Let R be a Dedekind domain and assume that $\mathrm{rHd}(X, R) = n + 1$. The support condition for the perverse sheaf ${}^p\varphi_f R_X[n+1]$ (which is supported on Σ) yields that the only possibly non-trivial reduced cohomology $\widetilde{H}^k(M_{f,0}; R)$ of $M_{f,0}$ is concentrated in degrees $n - r \leq k \leq n$. Moreover, as shown, e.g., in [93, Theorem 3.4(d)] or [109, Example 6.0.12], the lowest (possibly) non-trivial module $\widetilde{H}^{n-r}(M_{f,0}; R)$ is torsion-free.

For more applications of the fact that vanishing and nearby cycle functors preserve strongly perverse sheaves see [109], e.g., [109, Example 6.0.14] for applications to *local Lefschetz and vanishing theorems*, as well as [92] (for the study of vanishing cohomology of complex projective hypersurfaces) and [93] (for understanding the Milnor fiber cohomology).

Using Theorem 10.4.22 and Corollary 10.3.25, together with Example 10.4.17 and Remark 10.4.18, one gets the following characterization (see, e.g., [109, Corollary 6.0.7]).

Corollary 10.4.26 *Let X be a complex algebraic (or analytic) variety. Assume $\mathscr{F}^\bullet \in D^b(X; R)$ is weakly constructible (resp., $\mathscr{F}^\bullet$ is constructible with R a Dedekind domain, in case we want to use the dual perverse t-structure). Then we have for "? given by $\leq$" or "? given by $\geq$":*

$$\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{?n}(X; R) \quad \Leftrightarrow \quad \big((\varphi_f \mathscr{F}^\bullet)[-1]\big)_x \in {}^{p^{(+)}}D^{?n}(\{pt\}; R)$$

for all holomorphic function germs $f : (X, x) \to (\mathbb{C}, 0)$ such that x is isolated in the support of $\varphi_f(\mathscr{F}^\bullet)$.

As an application we get once more the *effectivity of characteristic cycles of perverse sheaves*, this time via *vanishing cycles*. Assume X is a complex manifold and R is a field. Let $\mathscr{F}^\bullet \in D^b_{\mathscr{S}-c}(X; R)$ be a constructible complex with respect to some Whitney stratification $\mathscr{S}$. Consider the characteristic cycle of $\mathscr{F}^\bullet$, i.e.,

$$CC(\mathscr{F}^\bullet) = \sum_{S \in \mathscr{S}} m(S) \cdot \overline{T^*_S X},$$

with multiplicities $m(S)$ given as in Definition 10.3.34 in terms of the normal Morse data of $\mathscr{F}^\bullet$. Here, we recall the calculation of $CC(\mathscr{F}^\bullet)$ in terms of vanishing cycles.

For a stratum $S \in \mathscr{S}$, let $x \in S$ be a point, and let $g : (X, x) \to (\mathbb{C}, 0)$ be a holomorphic function germ at x such that dg_x is a non-degenerate covector and x is a complex Morse critical point of $g|_S$ (i.e., $dg(X)$ intersects $T^*_S X$ transversally at dg_x). Then, as in Example 10.3.37, $Re(g)|_S$ has a classical Morse critical point at x of Morse index equal to dim S, and the multiplicities $m(S)$ of $CC(\mathscr{F}^\bullet)$ can be computed from Theorem 10.3.12 and Corollary 10.4.14:

$$m(S) = \chi(({}^p\varphi_g \mathscr{F}^\bullet)_x). \tag{10.100}$$

Then we get by Example 10.4.17 and Corollary 10.4.26:

Corollary 10.4.27 *If, in the above notations, $\mathscr{F}^\bullet$ is a perverse sheaf, then its characteristic cycle is effective, i.e., $m(S) \geq 0$ for all strata S in $\mathscr{S}$.*

Let us come back to the general context for finishing this section with the following applications of Theorem 10.4.22 (see, e.g., [109, Proposition 6.0.2]).

Proposition 10.4.28 *Let $i: Y \hookrightarrow X$ be the inclusion of a closed complex algebraic (or analytic) subset, with $j: U := X\backslash Y \hookrightarrow X$ the inclusion of the open complement. Suppose that Y can locally be described (at each point $x \in Y$) as the common zero-set of at most k algebraic (or holomorphic) functions on X ($k \geq 1$).*

Assume $\mathscr{F}^\bullet \in D^b(X; R)$ is weakly constructible (resp., $\mathscr{F}^\bullet$ is constructible with R a Dedekind domain, in case we want to use the dual perverse t-structure). Then one has:

1. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(X; R) \Rightarrow i^!\mathscr{F}^\bullet[k] \in {}^{p^{(+)}}D^{\leq n}(Y; R)$.
2. $\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(X; R) \Rightarrow i^*\mathscr{F}^\bullet[-k] \in {}^{p^{(+)}}D^{\geq n}(Y; R)$.
3. $j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\leq n}(U; R) \Rightarrow i^*Rj_*j^*\mathscr{F}^\bullet[k-1] \in {}^{p^{(+)}}D^{\leq n}(Y; R)$ *and* $Rj_*j^*\mathscr{F}^\bullet[k-1] \in {}^{p^{(+)}}D^{\leq n}(X; R)$.
4. $j^*\mathscr{F}^\bullet \in {}^{p^{(+)}}D^{\geq n}(U; R) \Rightarrow i^!Rj_!j^*\mathscr{F}^\bullet[-(k-1)] \in {}^{p^{(+)}}D^{\geq n}(Y; R)$ *and* $Rj_!j^*\mathscr{F}^\bullet[-(k-1)] \in {}^{p^{(+)}}D^{\geq n}(X; R)$.

These are local results, so that we can assume that Y is the common zero-set of k' algebraic (or holomorphic) functions (with $1 \leq k' \leq k$). Then the claim for (1.) or (2.) follows by induction from the case $k' = 1$, which is a direct application of Theorem 10.4.22 and the distinguished triangle (10.87) or (10.90). Finally (3.) or (4.) is, by $i^*Rj_*j^* \simeq i^!Rj_!j^*[1]$ (see, e.g., [6, (1.4.6.4)]), a special case of (1.) or (2.).

Example 10.4.29 (Purity) Let $i: Y \hookrightarrow X$ be the inclusion of a closed complex algebraic (or analytic) subset, which can locally be described as the common zero-set of at most k algebraic (or holomorphic) functions on X. Then $i^!\mathbb{D}_X \simeq \mathbb{D}_Y$ implies by Proposition 10.4.28 the following estimate of the *rectified homological depth*:

$$\mathrm{rHd}(X, R) \geq n \;\Rightarrow\; \mathrm{rHd}(Y, R \geq n-k\,. \tag{10.101}$$

Especially for X smooth of pure dimension n, we get (using also $i^*R_X = R_Y$)

$$\mathrm{rHd}(Y, R) \geq n-k = \dim Y \geq \mathrm{rHd}(Y, R) \quad \text{and} \quad R_Y[\dim Y] \in {}^pD^{\geq 0}(Y; R)\,,$$

if Y is locally a *set-theoretical complete intersection* of codimension k in X. In particular

$$\mathrm{rHd}(Y, R) = \dim Y \quad \text{and} \quad R_Y[\dim Y] \text{ is a perverse sheaf}$$

for Y a *pure-dimensional local complete intersection*.

Compare also with [56, Theorem 3.2.1, Corollary.3.2.2] for the corresponding homotopy results.

10.4.3 Thom-Sebastiani for Vanishing Cycles

In this section, we state a Thom-Sebastiani result for vanishing cycles, generalizing [118] to functions defined on singular ambient spaces, with arbitrary critical loci, and with arbitrary weakly constructible sheaf coefficients. For complete details, see [84] and also [109, Corollary 1.3.4].

Let $f: X \to \mathbb{C}$ and $g: Y \to \mathbb{C}$ be complex algebraic (or analytic) functions. Let pr_1 and pr_2 denote the projections of $X \times Y$ onto X and Y, respectively. Consider the function

$$f \boxtimes g := f \circ pr_1 + g \circ pr_2 : X \times Y \to \mathbb{C}.$$

The goal is to express the vanishing cycle functor $\varphi_{f\boxtimes g}$ in terms of the corresponding functors φ_f and φ_g for f and, resp., g.

We let $V(f) = \{f = 0\}$, and similarly for $V(g)$ and $V(f \boxtimes g)$. Denote by ℓ the inclusion of $V(f) \times V(g)$ into $V(f \boxtimes g)$. With these notations, one has the following result.

Theorem 10.4.30 *For $\mathscr{F}^\bullet \in D^b_{wc}(X; R)$ and $\mathscr{G}^\bullet \in D^b_{wc}(Y; R)$, there is a natural isomorphism*

$$\ell^* {}^p\varphi_{f\boxtimes g}(\mathscr{F}^\bullet \overset{L}{\boxtimes} \mathscr{G}^\bullet) \simeq {}^p\varphi_f \mathscr{F}^\bullet \overset{L}{\boxtimes} {}^p\varphi_g \mathscr{G}^\bullet \tag{10.102}$$

commuting with the corresponding monodromies.

Moreover, if $p = (x, y) \in X \times Y$ is such that $f(x) = 0$ and $g(y) = 0$, then, in an open neighborhood of p, the complex ${}^p\varphi_{f\boxtimes g}(\mathscr{F}^\bullet \overset{L}{\boxtimes} \mathscr{G}^\bullet)$ has support contained in $V(f) \times V(g)$, and, in every open set in which such a containment holds, there are natural isomorphisms

$${}^p\varphi_{f\boxtimes g}(\mathscr{F}^\bullet \overset{L}{\boxtimes} \mathscr{G}^\bullet) \simeq \ell_!({}^p\varphi_f \mathscr{F}^\bullet \overset{L}{\boxtimes} {}^p\varphi_g \mathscr{G}^\bullet) \simeq \ell_*({}^p\varphi_f \mathscr{F}^\bullet \overset{L}{\boxtimes} {}^p\varphi_g \mathscr{G}^\bullet). \tag{10.103}$$

Corollary 10.4.31 *In the notations of the above theorem and with integer coefficients, there is an isomorphism*

$$\begin{aligned} \widetilde{H}^{i-1}(M_{f\boxtimes g,p}) \cong & \bigoplus_{a+b=i} \left(\widetilde{H}^{a-1}(M_{f,pr_1(p)}) \otimes \widetilde{H}^{b-1}(M_{g,pr_2(p)}) \right) \\ & \oplus \bigoplus_{c+d=i+1} \operatorname{Tor}\left(\widetilde{H}^{c-1}(M_{f,pr_1(p)}), \widetilde{H}^{d-1}(M_{g,pr_2(p)}) \right), \end{aligned} \tag{10.104}$$

where $M_{f,x}$ denotes as usual the Milnor fiber of a function f at x, and similarly for $M_{g,y}$.

Remark 10.4.32 Note that the Thom-Sebastiani Theorem 10.4.30 implies directly the *multiplicativity of normal Morse data* in terms of vanishing cycles with respect to *external products* as mentioned in Example 10.3.17.

Example 10.4.33 (Brieskorn Singularities and Intersection Cohomology) For $i = 1, \ldots, n$, consider a $\mathbb{C}$-local system $\mathscr{L}_i$ of rank r_i on $\mathbb{C}^*$, with monodromy automorphism h_i, and denote the corresponding intersection cohomology complex on $\mathbb{C}$ by $IC_{\mathbb{C}}(\mathscr{L}_i)$. The complex $IC_{\mathbb{C}}(\mathscr{L}_i)$ agrees with $\mathscr{L}_i[1]$ on $\mathbb{C}^*$, and has stalk cohomology at the origin concentrated in degree -1, where it is isomorphic to $\ker(id - h_i)$. For positive integers a_i, consider the functions $f_i(x) = x^{a_i}$ on $\mathbb{C}$. The complex ${}^p\varphi_{f_i} IC_{\mathbb{C}}(\mathscr{L}_i)$ is a perverse sheaf supported only at 0; therefore, ${}^p\varphi_{f_i} IC_{\mathbb{C}}(\mathscr{L}_i)$ is non-zero only in degree zero, where it has dimension $a_i r_i - \dim \ker(id - h_i)$.

Next, consider the $\mathbb{C}$-local system $\mathscr{L}_1 \boxtimes \cdots \boxtimes \mathscr{L}_n$ on $(\mathbb{C}^*)^n$ with monodromy automorphism $h := \boxtimes_{i=1}^n h_i$, and note that, as in Example 10.2.31,

$$IC_{\mathbb{C}}(\mathscr{L}_1) \overset{L}{\boxtimes} \cdots \overset{L}{\boxtimes} IC_{\mathbb{C}}(\mathscr{L}_n) \simeq IC_{\mathbb{C}^n}(\mathscr{L}_1 \boxtimes \cdots \boxtimes \mathscr{L}_n).$$

The perverse sheaf

$${}^p\varphi_{x_1^{a_1} + \cdots + x_n^{a_n}} IC_{\mathbb{C}^n}(\mathscr{L}_1 \boxtimes \cdots \boxtimes \mathscr{L}_n)$$

is supported only at the origin, and hence is concentrated only in degree zero. In degree zero, it can be seen by iterating the Thom-Sebastiani isomorphism that it has dimension equal to

$$\prod_i (a_i r_i - \dim \ker(id - h_i)) \, .$$

In the special case when $r_i = 1$ and $h_i = 1$ for all i, the above calculation recovers the classical result stating that the Milnor number of the isolated singularity at the origin of $x_1^{a_1} + \cdots + x_n^{a_n} = 0$ is $\prod_i (a_i - 1)$.

10.4.4 Gluing Perverse Sheaves via Vanishing Cycles

Assume $R = \mathbb{C}$ (or, more generally, R is an algebraically closed field; but only the case $R = \mathbb{C}$ also nicely fits with the corresponding theory of (regular) holonomic D-modules). Let $f : X \to \mathbb{C}$ be a complex algebraic (or analytic) morphism with corresponding nearby and vanishing cycle functors ψ_f, φ_f. Recall that these two functors come equipped with monodromy automorphisms, both of which are denoted here by h. For $\mathscr{F}^\bullet \in D^b_c(X;\mathbb{C})$, the morphism

$$can\colon \psi_f\mathscr{F}^\bullet \longrightarrow \varphi_f\mathscr{F}^\bullet$$

of (10.87) is called the *canonical morphism*, and it is compatible with monodromy. Similarly for the *variation morphism*

$$var\colon \varphi_f\mathscr{F}^\bullet \longrightarrow \psi_f\mathscr{F}^\bullet$$

of (10.90). In the above notations we also have

$$can \circ var = h - id \quad \text{and} \quad var \circ can = h - id\,. \tag{10.105}$$

The monodromy automorphisms acting on the nearby and vanishing cycle functors have Jordan decompositions

$$h = h_u \circ h_s = h_s \circ h_u,$$

where h_s is *semi-simple* (and locally of finite order) and h_u is *unipotent*. For any $\lambda \in \mathbb{C}$ and $\mathscr{F}^\bullet \in D^b_c(X;\mathbb{C})$, denote by $\psi_{f,\lambda}\mathscr{F}^\bullet$ the generalized λ-eigenspace for h, and similarly for $\varphi_{f,\lambda}\mathscr{F}^\bullet$. By the definition of vanishing cycles, the canonical morphism can induces morphisms

$$can\colon \psi_{f,\lambda}\mathscr{F}^\bullet \longrightarrow \varphi_{f,\lambda}\mathscr{F}^\bullet,$$

which (since the monodromy acts trivially on $i^*\mathscr{F}^\bullet$) are isomorphisms for $\lambda \neq 1$, and there is a distinguished triangle

$$i^*\mathscr{F}^\bullet \xrightarrow{sp} \psi_{f,1}\mathscr{F}^\bullet \xrightarrow{can} \varphi_{f,1}\mathscr{F}^\bullet \xrightarrow{[1]}\ . \tag{10.106}$$

There are decompositions

$$\psi_f = \psi_{f,1} \oplus \psi_{f,\neq 1} \ \text{ and } \ \varphi_f = \varphi_{f,1} \oplus \varphi_{f,\neq 1} \tag{10.107}$$

so that $h_s = 1$ on $\psi_{f,1}$ and $\varphi_{f,1}$, and h_s has no 1-eigenspace on $\psi_{f,\neq 1}$ and $\varphi_{f,\neq 1}$. Moreover, $can\colon \psi_{f,\neq 1} \to \varphi_{f,\neq 1}$ and $var\colon \varphi_{f,\neq 1} \to \psi_{f,\neq 1}$ are isomorphisms.

The canonical and variation morphisms play an important role in the following *gluing* of perverse sheaves.

Let X be a complex algebraic variety, with $i\colon Z \hookrightarrow X$ the inclusion of a closed algebraic subvariety, and $j\colon U \hookrightarrow X$ the inclusion of the open complement $U := X\backslash Z$. A natural question to address is if one can "glue" the categories $Perv(Z)$ and $Perv(U)$ to recover the category $Perv(X)$ of perverse sheaves on X. We discuss here only the case when Z is a hypersurface, but see also [124] for a more general setup. The gluing procedure, due to Beilinson [7] and Deligne-Verdier [124], establishes an equivalence of categories between perverse sheaves on the algebraic variety X and a pair of perverse sheaves, one on Z, the other on U, together with a gluing datum. A similar method was used by M. Saito for constructing his mixed Hodge modules [106].

As a warm-up situation, consider $X = \mathbb{C}$ with coordinate function z, $Z = \{0\}$ and $U = \mathbb{C}^*$. Let $\mathscr{F}^\bullet$ be a $\mathbb{C}$-perverse sheaf on X. One can form the diagram

$$^p\psi_z\mathscr{F}^\bullet \underset{var}{\overset{can}{\rightleftarrows}} {}^p\varphi_z\mathscr{F}^\bullet$$

whose objects are perverse sheaves on $Z = \{0\}$, i.e., complex vector spaces. This leads to the following elementary description of the category of perverse sheaves on $\mathbb{C}$, cf. [124].

Proposition 10.4.34 *The category of perverse sheaves on $\mathbb{C}$ which are locally constant on $\mathbb{C}^*$ is equivalent to the category of quivers (that is, diagrams of vector spaces) of the form*

$$\psi \underset{v}{\overset{c}{\rightleftarrows}} \varphi$$

with ψ, φ finite dimensional vector spaces, and $id + c \circ v$, $id + v \circ c$ invertible.

Example 10.4.35 Let $\mathscr{L}$ be a local system on $\mathbb{C}^*$ with stalk V and monodromy $h \colon V \to V$. The perverse sheaf $j_*\mathscr{L}[1]$ (e.g., see Example 10.2.47) corresponds to

$$V \underset{v}{\overset{c}{\rightleftarrows}} V/\ker(h - id),$$

where c is the projection and v is induced by $h - id$. Thus a quiver

$$\psi \underset{v}{\overset{c}{\rightleftarrows}} \varphi$$

with c surjective arises from $j_*\mathscr{L}[1]$, where $\mathscr{L}_1 = \psi$ is the stalk of $\mathscr{L}$ and $h = id + v \circ c$.

More generally, let f be a regular function on a smooth algebraic variety X, with $Z = f^{-1}(0)$ and $U = X \setminus Z$. Let $Perv(U, Z)_{gl}$ be the category whose objects are $(\mathscr{A}^\bullet, \mathscr{B}^\bullet, c, v)$, with $\mathscr{A}^\bullet \in Perv(U)$, $\mathscr{B}^\bullet \in Perv(Z)$, $c \in \mathrm{Hom}({}^p\psi_{f,1}\mathscr{A}^\bullet, \mathscr{B}^\bullet)$, $v \in \mathrm{Hom}(\mathscr{B}^\bullet, {}^p\psi_{f,1}\mathscr{A}^\bullet)$, and so that $id + v \circ c$ is invertible. Then one has the following.

Theorem 10.4.36 (Beilinson [7], Deligne-Verdier [124]) *There is an equivalence of categories*

$$Perv(X) \simeq Perv(U, Z)_{gl}$$

defined by:

$$\mathscr{F}^\bullet \mapsto (\mathscr{F}^\bullet|_U, {}^p\varphi_{f,1}\mathscr{F}^\bullet, can, var).$$

See also [115, Section 3.3, 3.4] for some other examples of quiver descriptions of perverse sheaves, especially also for the corresponding description of Verdier duality on the quiver side.

10.4.5 Other Applications

There are many more applications of the nearby and vanishing cycle functors than we can mention in an expository paper, e.g., see [30, 66, 90, 109] and the references therein. Let us only mention here that vanishing cycles play an important role in the theory of characteristic classes for singular hypersurfaces (e.g., see [19, 96, 97, 112]), which have recently seen applications in birational geometry (cf. [97]), and they have also found applications to concrete optimization problems in applied algebraic geometry and algebraic statistics (e.g., see [94, 95] and the survey [91]).

We finish this section with an application to the calculus of Grothendieck groups of constructible sheaves and their relation to the theory of constructible functions (see, e.g., [109, Section 6.0.6] for a more general version).

Let $f: X \to \mathbb{C}$ be an algebraic (or analytic) morphism with $i: X_0 := \{f = 0\} \hookrightarrow X$ and $j: U := \{f \neq 0\} \hookrightarrow X$ the inclusions. Then one has the distinguished triangle (10.88)

$$i^*Rj_*j^*\mathscr{F}^\bullet \longrightarrow \psi_f\mathscr{F}^\bullet \xrightarrow{h_f - id} \psi_f\mathscr{F}^\bullet \xrightarrow{[1]}$$

for any $\mathscr{F}^\bullet \in D^b_c(X; R)$. But then also $\psi_f\mathscr{F}^\bullet \in D^b_c(X_0; R)$ by Corollary 10.4.9 and Corollary 10.4.12, so that

$$[i^*Rj_*j^*\mathscr{F}^\bullet] = 0 \in K_0(D^b_c(X_0; R)) \tag{10.108}$$

for the corresponding class in the Grothendieck group of constructible sheaf complexes on X_0.

Let more generally $X_0 := \{f_i = 0,\ i = 1, .., k+1\}$ be the common zero-set of the algebraic (or holomorphic) functions $f_i: X \to \mathbb{C}$ $(i = 1, \dots, k+1)$. Let $j: U := X\backslash X_0 \hookrightarrow X$ be again the inclusion. Then the equality (10.108) is also true. This follows by induction from a *Mayer-Vietoris triangle* (see, e.g., [66, p.114, (2.6.28)], with $R\Gamma_U(\cdot) \simeq Rj_*j^*$) associated to the open covering $\{U_1, U_2\}$ of U, with

$$U_1 := \{f_{k+1} \neq 0\} \quad \text{and} \quad U_2 := X\backslash\{f_i = 0,\ i = 1, \dots, k\}\,.$$

Note that

$$U_1 \cap U_2 = X \backslash \{f_i \cdot f_{k+1} = 0,\ i = 1, \dots, k\}$$

is the complement of the common zero-set of k algebraic (or holomorphic) functions.

Let $i\colon X_0 \hookrightarrow X$ be the inclusion. By the distinguished triangle

$$Rj_! j^* \mathscr{F}^\bullet \longrightarrow Rj_* j^* \mathscr{F}^\bullet \longrightarrow i_* i^* Rj_* j^* \mathscr{F}^\bullet \xrightarrow{[1]}$$

we get

$$Rj_! j^* = Rj_* j^* :\ K_0(D_c^b(X; R)) \to K_0(D_c^b(X; R))\ . \tag{10.109}$$

Similarly, the distinguished triangle

$$i^! \mathscr{F}^\bullet \longrightarrow i^* \mathscr{F}^\bullet \longrightarrow i^* Rj_* j^* \mathscr{F}^\bullet \xrightarrow{[1]}$$

implies

$$i^! = i^* :\ K_0(D_c^b(X; R)) \to K_0(D_c^b(X_0; R))\ . \tag{10.110}$$

Using again a *Mayer-Vietoris triangle* (see, e.g., [66, p.114, (2.6.28)], with $R\Gamma_U(\cdot) \simeq Rj_* j^*$) associated to an open affine covering, these equalities (10.109) and (10.110) are then available in the general complex algebraic context. Consider in addition a proper morphism $h\colon X \to Y$. Then $Rh_! = Rh_*$ maps $D_c^b(X; R)$ into $D_c^b(X; R)$ and we get the following equality for the induced maps on the level of Grothendieck groups:

$$R(h \circ j)_! j^* = R(h \circ j)_* j^* :\ K_0(D_c^b(X; R)) \to K_0(D_c^b(Y; R)) \tag{10.111}$$

in the complex algebraic context. Let $f\colon X \to Y$ be a morphism of complex algebraic varieties. By a theorem of Nagata (see, e.g., [22, 79]) this can be (partially) compactified to $f = \bar{f} \circ j$, with $j\colon X \hookrightarrow \bar{X}$ an open inclusion and $\bar{f}\colon \bar{X} \to Y$ a proper morphism of complex algebraic varieties. Then the functors $Rf_!$, Rf_* preserve algebraically constructible complexes and one gets the equality (compare also with [69, p.210] and [125]):

$$Rf_! = Rf_* :\ K_0(D_c^b(X; R)) \to K_0(D_c^b(Y; R))\ , \tag{10.112}$$

which also implies for the induced group homomorphisms of complex algebraically constructible functions the equality

$$f_! = f_* :\ CF(X) \to CF(X)\ . \tag{10.113}$$

Let us now consider the complex analytic context, with $CF(X)$ the abelian group of complex analytically constructible functions on X.

Example 10.4.37 Let $i\colon X_0 \hookrightarrow X$ be the inclusion of a closed complex analytic subset, with $j\colon U := X\backslash X_0 \hookrightarrow X$ the inclusion of the open complement. Fix also a proper holomorphic map $f\colon X \to Y$ of complex analytic varieties. Then we get the following equalities for the corresponding group homomorphisms on the level of constructible functions:

1. $j_! j^* = j_* j^* : CF(X) \to CF(X)$, especially $i^* j_* j^* = 0$.
2. $i^! = i^* : CF(X) \to CF(X_0)$.
3. $(f \circ j)_! j^* = (f \circ j)_* j^* : CF(X) \to CF(Y)$.

Note that (1.) implies (3.). Moreover, (1.) and (2.) are local results so that we can assume that X_0 is defined by finitely many holomorphic functions. Then they follow from the corresponding result above on the level of Grothendieck groups of constructible sheaf complexes.

A typical application is the following generalization of a classical result of Sullivan for R a field and X_0 compact so that the constant map $f\colon X_0 \to pt$ is proper. Then

$$\chi(X_0, i^* Rj_* j^* \mathscr{F}^\bullet) = f_* i^* j_* j^* (\chi_{stalk}(\mathscr{F}^\bullet)) = 0 \tag{10.114}$$

for any $\mathscr{F}^\bullet \in D^b_c(X; R)$.

Remark 10.4.38 $R\Gamma(X_0, i^* Rj_* j^* \mathscr{F}^\bullet)$ calculates the *global link cohomology* of $\mathscr{F}^\bullet$ as defined in [37]. For example, for $X_0 = \{x\}$ a point, one has (for $0 < \epsilon \ll 1$):

$$R\Gamma(X_0, i^* Rj_* j^* \mathscr{F}^\bullet) \simeq R\Gamma(X \cap \{||z|| = \epsilon\}, \mathscr{F}^\bullet) \ ,$$

with $||\cdot||$ defined by some local embedding $(X, x) \hookrightarrow (\mathbb{C}^n, 0)$. So, for $\mathscr{F}^\bullet := R_X$ the constant sheaf, we get that the *link*

$$X \cap \{||z|| = \epsilon\} \quad \text{of } X \text{ in } x$$

has a *vanishing Euler characteristic*. For a proof of this classical result of Sullivan [120], based on the *Milnor fibration theorem*, compare with [18, Proposition 4.1].

10.5 Intersection Cohomology, the Decomposition Theorem, Applications

In this section, we overview properties of the intersection cohomology groups of complex algebraic varieties, which generalize the corresponding features of the cohomology groups of smooth varieties. These properties, consisting of Poincaré duality, Lefschetz theorems and the decomposition theorem, are collectively termed the *Kähler package* for intersection cohomology. In this section we work over $R = \mathbb{Q}$.

Recall the following.

Definition 10.5.1 The (compactly supported) *intersection cohomology groups* of a pure complex n-dimensional complex algebraic variety X are defined as:

$$IH^k(X;R) := H^{k-n}(X;IC_X)\,,\;\; IH^k_c(X;R) := H^{k-n}_c(X;IC_X).$$

First note that Poincaré duality for the intersection cohomology groups of X is an immediate corollary of the self-duality of IC_X (cf. Proposition 10.2.48). Specifically, if X is a pure n-dimensional complex algebraic variety, there is an isomorphism

$$IH^{2n-k}(X;\mathbb{Q}) \simeq IH^k_c(X;\mathbb{Q})^\vee, \tag{10.115}$$

for any integer k.

Note also that if X is a pure n-dimensional complex algebraic variety, there exists a natural morphism

$$\alpha_X \colon \mathbb{Q}_X[n] \longrightarrow IC_X, \tag{10.116}$$

extending the natural quasi-isomorphism on the smooth locus of X. Moreover, α_X becomes a quasi-isomorphism if X is assumed to be a rational homology manifold (see, e.g., [89, Theorem 6.6.3]). Applying the hypercohomology functor to (10.116) yields a morphism

$$H^k(X;\mathbb{Q}) \longrightarrow IH^k(X;\mathbb{Q}),$$

which is an isomorphism if X is a rational homology manifold.

10.5.1 Lefschetz Type Results for Intersection Cohomology

Lefschetz type results for intersection cohomology can be derived from sheaf theoretic statements about perverse sheaves. We begin with the following immediate consequence of the Artin vanishing Theorem 10.3.59 for perverse sheaves:

Theorem 10.5.2 (Weak Lefschetz Theorem for Perverse Sheaves) *If X is a complex projective variety and $i \colon D \hookrightarrow X$ denotes the inclusion of a hyperplane section, then for every $\mathscr{F}^\bullet \in Perv(X;\mathbb{Q})$ the restriction map*

$$H^k(X;\mathscr{F}^\bullet) \longrightarrow H^k(D;i^*\mathscr{F}^\bullet)$$

is an isomorphism for $k < -1$ and is injective for $k = -1$.

Example 10.5.3 Let $X \subset \mathbb{C}P^{n+1}$ be a complex projective hypersurface, with $i: D \hookrightarrow X$ the inclusion of a hyperplane section. By applying Theorem 10.5.2 to the perverse sheaf $\mathscr{F}^\bullet := \mathbb{Q}_X[n]$ on X, one gets isomorphisms

$$H^k(X;\mathbb{Q}) \longrightarrow H^k(D;\mathbb{Q})$$

for all $k < n-1$ and a monomorphism for $k = n-1$.

Theorem 10.5.2 has the following important consequence (see also Example 10.3.61).

Corollary 10.5.4 (Lefschetz Hyperplane Section Theorem for Intersection Cohomology) *Let $X^n \subset \mathbb{C}P^N$ be a pure n-dimensional closed algebraic subvariety with a Whitney stratification $\mathscr{S}$. Let $H \subset \mathbb{C}P^N$ be a generic hyperplane (i.e., transversal to all strata of $\mathscr{S}$), with $i: D := X \cap H \hookrightarrow X$ the inclusion of the corresponding hyperplane section. Then the natural homomorphism*

$$IH^k(X;\mathbb{Q}) \longrightarrow IH^k(D;\mathbb{Q})$$

is an isomorphism for $0 \le k \le n-2$ and a monomorphism for $k = n-1$.

(Indeed, by transversality, one gets that $i^* IC_X \simeq IC_D[1]$, e.g., see Example 10.2.28.)

The Hard Lefschetz theorem for intersection cohomology can also be deduced from a more general sheaf-theoretic statement. Let $f: X \to Y$ be a projective morphism and let $L \in H^2(X;\mathbb{Q})$ be the first Chern class of an f-ample line bundle on X. Then L corresponds to a map of complexes $L: \mathbb{Q}_X \to \mathbb{Q}_X[2]$, which, after tensoring with IC_X, yields a map

$$L: IC_X \longrightarrow IC_X[2].$$

This induces $L: Rf_*IC_X \to Rf_*IC_X[2]$, and after applying perverse cohomology one gets a map of perverse sheaves on Y:

$$L: {}^p\mathscr{H}^i(Rf_*IC_X) \longrightarrow {}^p\mathscr{H}^{i+2}(Rf_*IC_X).$$

Iterating, one gets maps of perverse sheaves

$$L^i: {}^p\mathscr{H}^{-i}(Rf_*IC_X) \longrightarrow {}^p\mathscr{H}^{i}(Rf_*IC_X)$$

for every $i \ge 0$. Then one has the following result, proved initially by positive characteristic methods [6, Theorem 6.2.10] (see also [105, 106], or the more geometric approach of [23]):

Theorem 10.5.5 (Relative Hard Lefschetz) *Let $f: X \to Y$ be a projective morphism of complex algebraic varieties with X pure-dimensional, and let $L \in$*

$H^2(X;\mathbb{Q})$ be the first Chern class of an f-ample line bundle on X. For every $i>0$, one has isomorphisms of perverse sheaves

$$L^i : {}^p\mathscr{H}^{-i}(Rf_*IC_X) \xrightarrow{\simeq} {}^p\mathscr{H}^{i}(Rf_*IC_X).$$

By taking f in Theorem 10.5.5 to be the constant map $f: X \to point$, one obtains as a consequence the Hard Lefschetz theorem for intersection cohomology groups:

Corollary 10.5.6 (Hard Lefschetz Theorem for Intersection Cohomology) *Let X be a complex projective variety of pure complex dimension n, with $L \in H^2(X;\mathbb{Q})$ the first Chern class of an ample line bundle on X. Then there are isomorphisms*

$$L^i : IH^{n-i}(X;\mathbb{Q}) \xrightarrow{\simeq} IH^{n+i}(X;\mathbb{Q}) \tag{10.117}$$

for every integer $i>0$, induced by the cup product by L^i. In particular, the intersection cohomology Betti numbers of X are unimodal, i.e., $\dim_{\mathbb{Q}} IH^{i-2}(X;\mathbb{Q}) \leq \dim_{\mathbb{Q}} IH^i(X;\mathbb{Q})$ *for all $i \leq n/2$.*

10.5.2 The Decomposition Theorem and Immediate Applications

A great deal of information about intersection cohomology groups can be derived from the *BBD decomposition theorem* [6, Theorem 6.2.5], one of the most important results in the theory of perverse sheaves. It was conjectured by S. Gelfand and R. MacPherson, and proved soon after by Beilinson, Bernstein, Deligne and Gabber by reduction to positive characteristic. The proof given in [6] ultimately rests on Deligne's proof of the Weil conjectures. Different proofs were given later on by M. Saito (as a consequence of his theory of mixed Hodge modules [105]) and, more recently, by de Cataldo and Migliorini [23] (involving only classical Hodge theory). A more general decomposition theorem (for semi-simple perverse sheaves) has been obtained by Mochizuki [98, 99] (with substantial contributions of Sabbah [104]), in relation to a conjecture of Kashiwara [65]; see also [25]. For more topological versions of the decomposition theorem for self-dual perverse sheaves on the level of Witt and cobordism groups, see [21, 115]. In what follows, we explain the statement of the decomposition theorem, together with a few applications.

In its initial form of [6], the decomposition theorem calculates the derived pushforward of an IC-complex under a proper algebraic map. For simplicity, in this section we assume that all varieties are *irreducible*.

Recall that every algebraic map $f: X \to Y$ of complex algebraic varieties can be *stratified*, i.e., there exist algebraic Whitney stratifications $\mathscr{S}$ of X and $\mathscr{T}$ of Y such that, given any connected component T of a $\mathscr{T}$-stratum on Y one has the following properties:

(a) $f^{-1}(T)$ is a union of connected components of strata of $\mathscr{S}$, each of which is mapped submersively to T by f;
(b) For every point $y \in T$, there is an Euclidean neighborhood U of y in T and a stratum-preserving homeomorphism $h\colon U \times f^{-1}(y) \to f^{-1}(U)$ such that $f|_{f^{-1}(U)} \circ h$ is the projection to U.

Property (b) is just *Thom's isotopy lemma*: for every stratum T in Y, the restriction $f|_{f^{-1}(T)}\colon f^{-1}(T) \to T$ is a topologically locally trivial fibration.

Theorem 10.5.7 (BBD Decomposition Theorem [6]) *Let $f\colon X \to Y$ be a proper map of complex algebraic varieties. Then:*

(i) (Decomposition) There is a (non-canonical) isomorphism in $D^b_c(Y;\mathbb{Q})$:

$$Rf_*IC_X \simeq \bigoplus_i {}^p\mathscr{H}^i(Rf_*IC_X)[-i]. \tag{10.118}$$

(ii) (Semi-simplicity) Each ${}^p\mathscr{H}^i(Rf_*IC_X)$ is a semi-simple object in $Perv(Y;\mathbb{Q})$, i.e., if $\mathscr{T}$ is the set of connected components of strata of Y in a stratification of f, there is a canonical isomorphism of perverse sheaves on Y:

$${}^p\mathscr{H}^i(Rf_*IC_X) \simeq \bigoplus_{T\in\mathscr{T}} IC_{\overline{T}}(\mathscr{L}_{i,T}) \tag{10.119}$$

where the local systems $\mathscr{L}_{i,T}$ are semi-simple.

Remark 10.5.8 If f is a projective submersion of smooth complex algebraic varieties, Theorem 10.5.7 reduces to Deligne's decomposition theorem, see [27] and [28, Theorem 4.2.6].

Standard facts in algebraic geometry (e.g., resolution of singularities and Chow's lemma) reduce the proof of the BBD decomposition theorem (Theorem 10.5.7) to the case when $f\colon X \to Y$ is a projective morphism, with X a smooth variety. If the morphism f is projective, then (10.118) is a formal consequence of the relative Hard Lefschetz theorem (cf. [29]). So the heart of the BBD decomposition theorem consists of the semi-simplicity statement. After the above-mentioned reductions for f, the proof given in [23] is by induction on the pair of indices $(\dim Y, r(f))$, where $r(f)$ is the *degree of semi-smallness* of f. The problem is then reduced to the case $r(f) = 0$, i.e., that of a *semi-small* map, when $Rf_*\mathbb{Q}_X[\dim X] \simeq {}^p\mathscr{H}^0(f_*\mathbb{Q}_X[n])$ is perverse on Y. This case is handled via the non-degeneracy of a certain refined intersection pairing associated to the fibers over the most singular points of f.

One of the first consequences of the BBD decomposition theorem is that it gives a splitting of $IH^*(X;\mathbb{Q})$ in terms of twisted intersection cohomology groups of closures of strata in Y, namely:

Corollary 10.5.9 *Under the assumptions and notations of Theorem 10.5.7, there is a splitting*

$$IH^j(X;\mathbb{Q}) \simeq \bigoplus_{i\in\mathbb{Z}} \bigoplus_{T\in\mathscr{T}} IH^{j-\dim X+\dim T-i}(\overline{T};\mathscr{L}_{i,T}), \tag{10.120}$$

for every $j \in \mathbb{Z}$.

By applying Theorem 10.5.7 to the case of a resolution of singularities one gets the following.

Corollary 10.5.10 *The intersection cohomology of a complex algebraic variety is a direct summand of the cohomology of a resolution of singularities.*

More generally, one has the following nice application of the BBDG decomposition theorem (e.g., see [26, Section 4.5] or [89, Theorem 9.3.37]):

Theorem 10.5.11 *Let $f : X \to Y$ be a proper map of complex algebraic varieties, and let $Y' := f(X)$ be the image of f. Denote by $d = \dim X - \dim Y'$ the relative dimension of f. Then $IC_{Y'}[d]$ is a direct summand of Rf_*IC_X. In particular, $IH^j(Y';\mathbb{Q})$ is a direct summand of $IH^j(X;\mathbb{Q})$ for every integer j.*

An important application of the decomposition statement (10.118) for $f : X \to Y$ is the E_2-degeneration of the corresponding *perverse Leray spectral sequence*:

$$E_2^{i,j} = H^i(Y; {}^p\mathscr{H}^j(Rf_*IC_X)) \Longrightarrow H^{i+j}(Y; Rf_*IC_X) = IH^{i+j+\dim X}(X;\mathbb{Q}).$$

This can be used to prove the following singular version of the classical global invariant cycle theorem for smooth projective maps, as well as a local version of it (see [6, Corollary 6.2.8, Corollary 6.2.9]):

Theorem 10.5.12 (Global and Local Invariant Cycle Theorems) *Let $f : X \to Y$ be a proper map of complex algebraic varieties. Let $U \subseteq Y$ be a Zariski-open subset on which the sheaf $R^i f_* IC_X$ is a local system. Then the following assertions hold:*

(a) (Global) The natural restriction map

$$IH^i(X;\mathbb{Q}) \longrightarrow H^0(U; R^i f_* IC_X)$$

is surjective.

(b) (Local) Let $u \in U$ and $B_u \subset U$ be the intersection with a sufficiently small Euclidean ball (chosen with respect to a local embedding of (Y, u) into a

manifold) centered at u. Then the natural restriction/retraction map

$$H^i(f^{-1}(u); IC_X) \simeq H^i(f^{-1}(B_u); IC_X) \longrightarrow H^0(B_u; R^i f_* IC_X)$$

is surjective.

10.5.3 A Recent Application of the Kähler Package for Intersection Cohomology

In this section, we mention briefly a recent combinatorial application of the Kähler package for intersection cohomology. For more applications, the interested reader may consult [24, 89] and the references therein.

Let $E = \{v_1, \cdots, v_n\}$ be a subset of generators of a d-dimensional complex vector space V, and let $w_i(E)$ be the number of i-dimensional subspaces spanned by subsets of E. In 1974, Dowling and Wilson [31, 32] proposed the following conjecture (which is a special case of a more general conjecture for matroids; see [12] for more recent developments):

Conjecture 10.5.13 (Dowling–Wilson Top-Heavy Conjecture) For all $i < d/2$ one has:

$$w_i(E) \leq w_{d-i}(E). \tag{10.121}$$

Remark 10.5.14 If $d = 3$, de Bruijn–Erdös showed that $w_1(E) \leq w_2(E)$. More generally, Motzkin showed that $w_1(E) \leq w_{d-1}(E)$.

Another conjecture concerning the numbers $w_i(E)$ was proposed by Rota [101, 102] in 1971, and it can be formulated as follows:

Conjecture 10.5.15 (Rota's Unimodal Conjecture) There is some j so that

$$w_0(E) \leq \cdots \leq w_{j-1}(E) \leq w_j(E) \geq w_{j+1}(E) \geq \cdots \geq w_d(E). \tag{10.122}$$

Huh-Wang [59] used the Kähler package on intersection cohomology to prove the Dowling-Wilson top-heavy conjecture, and the unimodality of the "lower half" of the sequence $\{w_i(E)\}$. Specifically, they showed the following:

Theorem 10.5.16 (Huh-Wang) *For all $i < d/2$, the following properties hold:*

(a) (top heavy) $w_i(E) \leq w_{d-i}(E)$.
(b) (unimodality) $w_i(E) \leq w_{i+1}(E)$.

Proof *(Sketch of Proof)* The key step in the proof is to show that the numbers $w_i(E)$ are realized geometrically, i.e., there exists a complex d-dimensional projective variety Y such that for every $0 \le i \le d$ one has:

$$H^{2i+1}(Y;\mathbb{Q}) = 0 \text{ and } \dim_{\mathbb{Q}} H^{2i}(Y;\mathbb{Q}) = w_i(E).$$

To define the variety Y one first uses $E = \{v_1, \cdots, v_n\}$ to construct a map $i_E \colon V^\vee \to \mathbb{C}^n$ by regarding each $v_i \in E$ as a linear map on the dual vector space $V^\vee$. Precomposing i_E with the open inclusion $\mathbb{C}^n \hookrightarrow (\mathbb{C}P^1)^n$ yields a map

$$f \colon V^\vee \to (\mathbb{C}P^1)^n.$$

Set

$$Y := \overline{\mathrm{Im}\,(f)} \subset (\mathbb{C}P^1)^n.$$

Ardila-Boocher [5] showed that the variety Y has an algebraic cell decomposition (i.e., it is paved by complex affine spaces), the number of $\mathbb{C}^i$'s appearing in the decomposition of Y being exactly $w_i(E)$. However, the space Y is in this case highly singular, so its rational cohomology does not satisfy the Kähler package. Instead, one needs to use the corresponding intersection cohomology results.

Next, note that for any complex projective variety Y one has that

$$\ker\left(H^i(Y;\mathbb{Q}) \xrightarrow{\alpha} IH^i(Y;\mathbb{Q})\right) = W_{\le i-1}H^i(Y;\mathbb{Q})$$

is the subspace of $H^i(Y;\mathbb{Q})$ consisting of classes of Deligne weight $\le i-1$ (e.g., see [126, Theorem 9.2]). Since the complex projective variety Y constructed above has an algebraic cell decomposition, its cohomology $H^i(Y;\mathbb{Q})$ is pure of weight i. (This follows easily by induction using the fact that $H^i_c(\mathbb{C}^n;\mathbb{Q}) = 0$ for $i \ne 2n$ and $H^{2n}_c(\mathbb{C}^n;\mathbb{Q}) = \mathbb{C}$ is pure of weight $2n$.) Hence, the natural map

$$\alpha \colon H^*(Y;\mathbb{Q}) \to IH^*(Y;\mathbb{Q})$$

is injective in any degree i.

For $i < d/2$, consider the following commutative diagram:

$$\begin{array}{ccc} H^{2i}(Y;\mathbb{Q}) & \overset{\alpha}{\hookrightarrow} & IH^{2i}(Y;\mathbb{Q}) \\ \downarrow{\scriptstyle \smile L^{d-2i}} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \smile L^{d-2i}} \\ H^{2d-2i}(Y;\mathbb{Q}) & \overset{\alpha}{\hookrightarrow} & IH^{2d-2i}(Y;\mathbb{Q}) \end{array}$$

where the right-hand vertical arrow is the Hard Lefschetz isomorphism for the intersection cohomology groups of Y (see Corollary 10.5.6). Since the maps labelled by α are injective, it follows that

$$H^{2i}(Y;\mathbb{Q}) \xrightarrow{\smile L^{d-2i}} H^{2d-2i}(Y;\mathbb{Q})$$

is injective as well. In particular,

$$w_i(E) = \dim_{\mathbb{Q}} H^{2i}(Y;\mathbb{Q}) \leq \dim_{\mathbb{Q}} H^{2d-2i}(Y;\mathbb{Q}) = w_{d-i}(E)$$

for every $i < d/2$, thus proving part (a) of the theorem. Part (b) follows similarly, by using the unimodality of the intersection cohomology Betti numbers (cf. Corollary 10.5.6). □

10.6 Perverse Sheaves on Semi-Abelian Varieties

In this section, we survey recent developments in the study of perverse sheaves on semi-abelian varieties, with concrete applications to the study of homotopy types of complex algebraic manifolds (formulated in terms of their cohomology jump loci), as well as new topological characterizations of semi-abelian varieties. We begin by introducing and recalling the main ingredients needed to formulate our results. For complete details, the interested reader may consult [74, 76, 77].

10.6.1 Cohomology Jump Loci

Let X be a connected CW complex of finite type (e.g., a complex quasi-projective variety) with positive first Betti number, i.e., $b_1(X) > 0$. The *character variety* $\mathrm{Char}(X)$ of X is the identity component of the moduli space of rank-one $\mathbb{C}$-local systems on X, i.e.,

$$\mathrm{Char}(X) := \mathrm{Hom}(H_1(X;\mathbb{Z})/\mathrm{Torsion}, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{b_1(X)}.$$

Definition 10.6.1 The *i-th cohomology jump locus of* X is defined as:

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X; L_\rho) \neq 0\},$$

where L_ρ is the unique rank-one $\mathbb{C}$-local system on X associated to the representation $\rho \in \mathrm{Char}(X)$.

The jump loci $\mathcal{V}^i(X)$ are closed subvarieties of $\mathrm{Char}(X)$ and homotopy invariants of X. For projective varieties, they can be seen as topological counterparts of the Green-Lazarsfeld jump loci of topologically trivial line bundles [50, 51]. Cohomology jump loci emerged from work of Novikov on Morse theory for closed

1-forms on manifolds, and provide a unifying framework for the study of a host of questions concerning homotopy types of complex algebraic varieties.

10.6.2 Jump Loci via Constructible Complexes

The Albanese map construction allows one to interpret the cohomology jump loci of a smooth connected complex quasi-projective variety as cohomology jump loci of certain constructible complexes of sheaves (or even of perverse sheaves, if the Albanese map is proper) on a semi-abelian variety. This motivates the study of cohomology jump loci of such complexes, with a view towards characterizing important classes of objects (such as perverse sheaves) on semi-abelian varieties.

An *abelian variety* of dimension g is a compact complex torus $\mathbb{C}^g / \Gamma$ with $\Gamma \simeq \mathbb{Z}^{2g}$, which is also a complex projective variety. A *semi-abelian variety* G is an abelian complex algebraic group, which is an extension of an abelian variety by a complex affine torus. (Semi-)abelian varieties are naturally associated to smooth (quasi-)projective varieties via the *Albanese map* construction, see [60]. Specifically, if X is a smooth complex (quasi-)projective variety, the *Albanese map* of X is a morphism

$$\mathrm{alb}\colon X \to \mathrm{Alb}(X)$$

from X to a (semi-)abelian variety $\mathrm{Alb}(X)$ such that for any morphism $f: X \to G$ to a semi-abelian variety G, there exists a unique morphism $g: \mathrm{Alb}(X) \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}} & \mathrm{Alb}(X) \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & G \end{array}$$

Here, $\mathrm{Alb}(X)$ is called the *Albanese variety* associated to X.

The Albanese map induces an isomorphism on the torsion-free part of the first integral homology groups, i.e.,

$$H_1(X; \mathbb{Z})/\mathrm{Torsion} \xrightarrow{\simeq} H_1(\mathrm{Alb}(X); \mathbb{Z}). \tag{10.123}$$

In particular, there is an identification:

$$\mathrm{Char}(X) \simeq \mathrm{Char}(\mathrm{Alb}(X)), \tag{10.124}$$

and the equality of Betti numbers $b_1(X) = b_1(\mathrm{Alb}(X))$.

By using the projection formula, for any $\rho \in \mathrm{Char}(X) \simeq \mathrm{Char}(\mathrm{Alb}(X))$ one gets:

$$H^i(X; L_\rho) \simeq H^i(X; \mathbb{C}_X \otimes L_\rho) \simeq H^i\big(\mathrm{Alb}(X); (R\,\mathrm{alb}_* \mathbb{C}_X) \otimes L_\rho\big). \tag{10.125}$$

If, moreover, the Albanese map $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ is proper (e.g., if X is projective), then the BBD decomposition theorem [6] yields that $R\,\mathrm{alb}_* \mathbb{C}_X$ is a direct sum of (shifted) perverse sheaves. In view of (10.125), this provides a description of the cohomology jump loci of X in terms of cohomology jump loci of certain perverse sheaves on the semi-abelian variety $\mathrm{Alb}(X)$. This motivates the following.

Definition 10.6.2 Let $\mathscr{F}^\bullet \in D^b_c(G; \mathbb{C})$ be a bounded constructible complex of $\mathbb{C}$-sheaves on a semi-abelian variety G. The *degree i cohomology jump locus of $\mathscr{F}^\bullet$* is defined as:

$$\mathcal{V}^i(G, \mathscr{F}^\bullet) := \{\rho \in \mathrm{Char}(G) \mid H^i(G; \mathscr{F}^\bullet \otimes_{\mathbb{C}} L_\rho) \neq 0\}.$$

Back to the cohomology jump loci of X, we note that (10.125) yields the following identification:

$$\mathcal{V}^i(X) = \mathcal{V}^i(\mathrm{Alb}(X), R\,\mathrm{alb}_* \mathbb{C}_X). \tag{10.126}$$

10.6.3 Mellin Transformation and Applications

Let G be a semi-abelian variety defined by an extension

$$1 \to T \to G \to A \to 1,$$

where A is an abelian variety of dimension g and $T \simeq (\mathbb{C}^*)^m$ is a complex affine torus of dimension m. Set

$$\Gamma_G := \mathbb{C}[\pi_1(G)],$$

and note that Γ_G is a Laurent polynomial ring in $m + 2g$ variables. Moreover,

$$\mathrm{Char}(G) \simeq \mathrm{Spec}\,\Gamma_G.$$

Let $\mathscr{L}_G$ be the (universal) rank-one local system of Γ_G-modules on G, defined by mapping the generators of $\pi_1(G)$ to the multiplication by the corresponding variables of Γ_G.

Definition 10.6.3 ([41]) The *Mellin transformation* $\mathcal{M}_*\colon D^b_c(G;\mathbb{C}) \to D^b_{coh}(\Gamma_G)$ is given by

$$\mathcal{M}_*(\mathscr{F}^\bullet) := Ra_*(\mathscr{L}_G \otimes_{\mathbb{C}} \mathscr{F}^\bullet),$$

where $a\colon G \to \textit{point}$ is the constant map to a point space, and $D^b_{coh}(\Gamma_G)$ denotes the bounded coherent complexes of Γ_G-modules (i.e., whose cohomology modules are all finitely generated Γ_G-modules).

The Mellin transformation can be used to completely characterize perverse sheaves on complex affine tori. More precisely, one has the following result due to Gabber-Loeser [41, Theorem 3.4.1 and Theorem 3.4.7] in the l-adic context, and extended to the present form in [74, Theorem 3.2]:

Theorem 10.6.4 *A constructible complex $\mathscr{F}^\bullet \in D^b_c(T;\mathbb{C})$ on a complex affine torus T is perverse if, and only if,*

$$H^i(\mathcal{M}_*(\mathscr{F}^\bullet)) = 0 \textit{ for all } i \neq 0.$$

In the context of abelian varieties, the Mellin transformation was used in [9] for proving generic vanishing results for the cohomology of perverse sheaves. By induction on the dimension of the complex affine torus T, the result of [9] was extended to the semi-abelian context as follows.

Theorem 10.6.5 ([77, Theorem 4.3]) *For any $\mathbb{C}$-perverse sheaf $\mathscr{F}^\bullet$ on a semi-abelian variety G, one has:*

$$H^i(\mathcal{M}_*(\mathscr{F}^\bullet)) = 0 \textit{ for } i < 0,$$

and

$$H^i(D_{\Gamma_G}(\mathcal{M}_*(\mathscr{F}^\bullet))) = 0 \;\textit{ for } i < 0.$$

Here $D_{\Gamma_G}(-) := \mathrm{RHom}_{\Gamma_G}(-,\Gamma_G)$ is the dualizing functor for Γ_G-modules.

The Mellin transformation can be used to translate the question of understanding the cohomology jump loci of a constructible complex to a problem in commutative algebra. Specifically, by the projection formula, cohomology jump loci of $\mathscr{F}^\bullet \in D^b_c(G;\mathbb{C})$ are determined by those of the Mellin transformation $\mathcal{M}_*(\mathscr{F}^\bullet)$ of $\mathscr{F}^\bullet$ as follows (see [74, 77]):

$$\mathcal{V}^i(G,\mathscr{F}^\bullet) = \mathcal{V}^i(\mathcal{M}_*(\mathscr{F}^\bullet)). \tag{10.127}$$

Here, if R is a Noetherian domain and $E^\bullet$ is a bounded complex of R-modules with finitely generated cohomology, the closed points of $\mathcal{V}^i(E^\bullet)$ can be described as

$$\mathcal{V}^i(E^\bullet) := \{\mathfrak{p} \in \operatorname{Spec} R \mid H^i(F^\bullet \otimes_R R/\mathfrak{p}) \neq 0\},$$

with $F^{\cdot}$ a bounded above finitely generated *free* resolution of $E^{\cdot}$. One also has the following:

Proposition 10.6.6 ([9, Lemma 2.8]) *Let R be a regular Noetherian domain and $E^{\cdot}$ a bounded complex of R-modules with finitely generated cohomology. Then $H^i(E^{\cdot}) = 0$ for $i < 0$ if, and only if,* $\operatorname{codim} \mathcal{V}^{-i}(E^{\cdot}) \geq i$ *for* $i \geq 0$.

By using the identification (10.127), together with Proposition 10.6.6 and standard facts from commutative algebra, the following result of [77] is a direct consequence of Theorem 10.6.5:

Theorem 10.6.7 ([77, Theorem 4.7]) *For any $\mathbb{C}$-perverse sheaf $\mathcal{F}^{\bullet}$ on a semi-abelian variety G, the cohomology jump loci of $\mathcal{F}^{\bullet}$ satisfy the following properties:*

(i) Propagation:

$$\mathcal{V}^{-m-g}(G, \mathcal{F}^{\bullet}) \subseteq \cdots \subseteq \mathcal{V}^{0}(G, \mathcal{F}^{\bullet}) \supseteq \mathcal{V}^{1}(G, \mathcal{F}^{\bullet}) \supseteq \cdots \supseteq \mathcal{V}^{g}(G, \mathcal{F}^{\bullet}).$$

Moreover, $\mathcal{V}^{i}(G, \mathcal{F}^{\bullet}) = \emptyset$ *if* $i \notin [-m-g, g]$.

(ii) Codimension lower bound: for all $i \geq 0$,

$$\operatorname{codim} \mathcal{V}^{i}(G, \mathcal{F}^{\bullet}) \geq i \text{ and } \operatorname{codim} \mathcal{V}^{-i}(G, \mathcal{F}^{\bullet}) \geq i.$$

Theorem 10.6.7 is inspired by, and can be viewed as a topological counterpart of, similar properties satisfied by the Green-Lazarsfeld jump loci of topologically trivial line bundles, see [50, 51].

Remark 10.6.8 Let $\mathcal{F}^{\bullet}$ be a $\mathbb{C}$-perverse sheaf so that not all $H^j(G; \mathcal{F}^{\bullet})$ are zero. The propagation property (i) can be restated as saying that the set of integers j for which $H^j(G; \mathcal{F}^{\bullet}) \neq 0$ form an *interval* of consecutive integers. Indeed, let

$$k_+ := \max\{j \mid H^j(G; \mathcal{F}^{\bullet}) \neq 0\} \text{ and } k_- := \min\{j \mid H^j(G; \mathcal{F}^{\bullet}) \neq 0\}.$$

Then it is easy to see that the propagation property (i) is equivalent to: $k_+ \geq 0$, $k_- \leq 0$ and

$$H^j(G; \mathcal{F}^{\bullet}) \neq 0 \iff k_- \leq j \leq k_+.$$

In this form, the result of Theorem 10.6.7(i) provides a generalization of a result of Weissauer [128, Corollary 1] from the abelian context.

A nice consequence of Theorem 10.6.7 is the following generic vanishing result:

Corollary 10.6.9 *For any $\mathbb{C}$-perverse sheaf $\mathcal{F}^{\bullet}$ on a semi-abelian variety G,*

$$H^i(G; \mathcal{F}^{\bullet} \otimes_{\mathbb{C}} L_{\rho}) = 0$$

for any generic rank-one $\mathbb{C}$*-local system* L_ρ *and all* $i \neq 0$*. In particular,*

$$\chi(G, \mathscr{F}^\bullet) \geq 0. \tag{10.128}$$

Moreover, $\chi(G, \mathscr{F}^\bullet) = 0$ *if, and only if,* $\mathcal{V}^0(G, \mathscr{F}^\bullet) \neq \mathrm{Char}(G)$.

The above generic vanishing statement was originally proved by other methods in [67, Theorem 2.1] in the l-adic context and further generalized to arbitrary field coefficients in [75, Theorem 1.1]. For abelian varieties, generic vanishing results were obtained in [68, Theorem 1.1], [112, Corollary 7.5], [127, Vanishing Theorem] or [9, Theorem 1.1]. The signed Euler characteristic property (10.128) is originally due to Franecki and Kapranov [39, Corollary 1.4] (and compare with the Example 10.3.40 for another approach to this signed Euler characteristic property).

Remark 10.6.10 The properties of perverse sheaves from Theorem 10.6.7 and Corollary 10.6.9 are collectively termed the *propagation package* for perverse sheaves on semi-abelian varieties.

10.6.4 Characterization of Perverse Sheaves on Semi-abelian Varieties

In this section, we discuss a complete (global) characterization of perverse sheaves on semi-abelian varieties; see [77] for complete details. Motivation is also provided by the following result:

Theorem 10.6.11 ([107]) *If* A *is an abelian variety and* $\mathscr{F}^\bullet \in D^b_c(A; \mathbb{C})$*, then* $\mathscr{F}^\bullet$ *is perverse if, and only if, for all* $i \in \mathbb{Z}$*,* $\mathrm{codim}\, \mathcal{V}^i(A, \mathscr{F}^\bullet) \geq |2i|$.

Furthermore, as a consequence of Theorem 10.6.4, Proposition 10.6.6 and Artin's vanishing theorem 10.3.59, one gets the following:

Theorem 10.6.12 ([77, Corollary 6.8]) $\mathscr{F}^\bullet \in D^b_c(T; \mathbb{C})$ *is perverse on a complex affine torus* T *if, and only if,*

(i) For all $i > 0$*:* $\mathcal{V}^i(T, \mathscr{F}^\bullet) = \emptyset$*, and*
(ii) For all $i \leq 0$*:* $\mathrm{codim}\, \mathcal{V}^i(T, \mathscr{F}^\bullet) \geq -i$.

In order to unify and generalize the results of Theorems 10.6.11 and 10.6.12 to the semi-abelian context, one can make use of the new notions of (co)dimension for the cohomology jump loci, which were introduced in [77]. First recall the following.

Definition 10.6.13 A closed irreducible subvariety V of $\mathrm{Char}(G)$ is called *linear* if there exists a short exact sequence of semi-abelian varieties

$$1 \to G''(V) \to G \xrightarrow{q} G'(V) \to 1$$

and some $\rho \in \mathrm{Char}(G)$ such that

$$V := \rho \cdot \mathrm{Im}\,(q^{\#}\colon\ \mathrm{Char}(G'(V)) \to \mathrm{Char}(G)).$$

Here $G''(V)$ and $G'(V)$ depend on V, and $q^{\#}$ is induced by $q\colon G \to G'(V)$.

With the above definition, one has the following important structure result:

Theorem 10.6.14 ([17]) *For any $\mathscr{F}^{\bullet} \in D_c^b(G;\mathbb{C})$, each jump locus $\mathcal{V}^i(G, \mathscr{F}^{\bullet})$ is a finite union of linear subvarieties of* $\mathrm{Char}(G)$.

Definition 10.6.15 Let G be a semi-abelian variety and let V be an irreducible linear subvariety of $\mathrm{Char}(G)$. In the notations of Definition 10.6.13, let $T''(V)$ and $A''(V)$ denote the complex affine torus and, resp., the abelian variety part of $G''(V)$. Define:

abelian codimension: $\mathrm{codim}_a\, V := \dim A''(V)$,
semi-abelian codimension: $\mathrm{codim}_{sa}\, V := \dim G''(V)$.

Similar notions can be defined for reducible subvarieties by taking the minimum among all irreducible components.

Remark 10.6.16 Let V be a nonempty linear subvariety of $\mathrm{Spec}\,\Gamma_G$.

1. If $G = T$ is a complex affine torus, then: $\mathrm{codim}_{sa}\, V = \mathrm{codim}\, V$ and $\mathrm{codim}_a\, V = 0$.
2. If $G = A$ is an abelian variety, we have: $\mathrm{codim}_{sa}(V) = \mathrm{codim}_a(V) = \frac{1}{2}\,\mathrm{codim}(V)$.

In the above notations, the following generalization of Schnell's result was obtained in [77]:

Theorem 10.6.17 ([77, Theorem 6.6]) *A constructible complex $\mathscr{F}^{\bullet} \in D_c^b(G;\mathbb{C})$ is perverse on G if, and only if,*

(i) $\mathrm{codim}_a\, \mathcal{V}^i(G, \mathscr{F}^{\bullet}) \geq i$ *for all* $i \geq 0$, *and*
(ii) $\mathrm{codim}_{sa}\, \mathcal{V}^i(G, \mathscr{F}^{\bullet}) \geq -i$ *for all* $i \leq 0$.

Proof The "only if" part is proved by induction on $\dim T$, using Theorem 10.6.11 as the beginning step of the induction process. For the "if" part, one shows that the two codimension lower bounds in the statement of Theorem 10.6.17 are sharp. See [77, Section 6] for complete details. □

10.6.5 Application: Cohomology Jump Loci of Quasi-Projective Manifolds

The results of Theorems 10.6.7, 10.6.17 and Corollary 10.6.9 can be directly applied for the study of cohomology jump loci $\mathcal{V}^i(X) \subseteq \mathrm{Char}(X) = \mathrm{Char}(\mathrm{Alb}(X))$ of a smooth complex quasi-projective variety X. Specifically, one has the following.

Theorem 10.6.18 *Let X be a smooth quasi-projective variety of complex dimension n. Assume that $R\,\mathrm{alb}_*(\mathbb{C}_X[n])$ is a perverse sheaf on $\mathrm{Alb}(X)$ (e.g., alb is proper and semi-small, or alb is quasi-finite). Then the cohomology jump loci $\mathcal{V}^i(X)$ satisfy the following properties:*

(1) Propagation property:

$$\mathcal{V}^n(X) \supseteq \mathcal{V}^{n-1}(X) \supseteq \cdots \supseteq \mathcal{V}^0(X) = \{\mathbb{C}_X\};$$

$$\mathcal{V}^n(X) \supseteq \mathcal{V}^{n+1}(X) \supseteq \cdots \supseteq \mathcal{V}^{2n}(X).$$

(2) Codimension lower bound: for all $i \geq 0$,

$$\mathrm{codim}_{sa}\, \mathcal{V}^{n-i}(X) \geq i \quad \text{and} \quad \mathrm{codim}_a\, \mathcal{V}^{n+i}(X) \geq i.$$

(3) Generic vanishing: $H^i(X, L_\rho) = 0$ for generic $\rho \in \mathrm{Char}(X)$ and all $i \neq n$.
(4) Signed Euler characteristic: $(-1)^n \chi(X) \geq 0$.
(5) Betti property: $b_i(X) > 0$ for all $i \in [0, n]$, and $b_1(X) \geq n$.

Example 10.6.19 Let X be a smooth closed subvariety of a semi-abelian variety G. The closed embedding $i\colon X \hookrightarrow G$ is a proper semi-small map, and hence the Albanese map $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ is also proper and semi-small. Then $R\,\mathrm{alb}_*(\mathbb{C}_X[\dim X])$ is a perverse sheaf on $\mathrm{Alb}(X)$ and the jump loci of X satisfy the properties listed in Theorem 10.6.18.

10.6.6 Application: Topological Characterization of Semi-abelian Varieties

The Structure Theorem 10.6.14 together with the propagation package of Theorem 10.6.7 and Corollary 10.6.9 can be used to give the following topological characterization of semi-abelian varieties (see [77, Proposition 7.7]):

Theorem 10.6.20 *Let X be a smooth quasi-projective variety with proper Albanese map (e.g., X is projective), and assume that X is homotopy equivalent to a real torus. Then X is isomorphic to a semi-abelian variety.*

Proof Assume X has complex dimension n. By the BBD decomposition theorem [6], $R\,\mathrm{alb}_*(\mathbb{C}_X[n])$ is a direct sum of shifted semi-simple perverse sheaves on $\mathrm{Alb}(X)$. Denote by $\mathcal{U}$ the collection of all simple summands $\mathscr{F}^\bullet$ appearing (up to a shift) in $R\,\mathrm{alb}_*(\mathbb{C}_X[n])$. Then, by using Theorem 10.6.7(i) and the identification (10.126), one gets that

$$\bigcup_{i=0}^{2n} \mathcal{V}^i(X) = \bigcup_{\mathcal{U}} \mathcal{V}^0(\mathrm{Alb}(X), \mathscr{F}^\bullet). \tag{10.129}$$

Since X is homotopy equivalent to a real torus, a direct calculation yields that $\bigcup_{i=0}^{2n} \mathcal{V}^i(X)$ is just an isolated point. Hence, by (10.129), for every simple perverse sheaf $\mathscr{F}^\bullet \in \mathcal{U}$, the jump locus $\mathcal{V}^0(\mathrm{Alb}(X), \mathscr{F}^\bullet)$ is exactly this isolated point, so Corollary 10.6.9 gives $\chi(\mathrm{Alb}(X), \mathscr{F}^\bullet) = 0$.

Simple perverse sheaves with zero Euler characteristic on semi-abelian varieties are completely described in [77, Theorem 5.5] by using the structure Theorem 10.6.14 and the propagation package for their cohomology jump loci. In particular, it follows that for all $\mathscr{F}^\bullet \in \mathcal{U}$ one has that $\mathscr{F}^\bullet = \mathbb{C}_{\mathrm{Alb}(X)}[\dim \mathrm{Alb}(X)]$. So $R\,\mathrm{alb}_* \mathbb{C}_X$ is a direct sum of shifted rank-one constant sheaves on $\mathrm{Alb}(X)$. Since X and $\mathrm{Alb}(X)$ are both homotopy equivalent to tori, and since $b_1(X) = b_1(\mathrm{Alb}(X))$, one gets that $b_i(X) = b_i(\mathrm{Alb}(X))$ for all i. Therefore,

$$R\,\mathrm{alb}_* \mathbb{C}_X \simeq \mathbb{C}_{\mathrm{Alb}(X)}.$$

Since alb is proper, it follows that all fibers of alb are zero-dimensional. Then it can be seen easily that alb is in fact an isomorphism. □

Under the assumptions of Theorem 10.6.20, it follows that the Albanese map $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ is an isomorphism. One can similarly prove the following special case of a question of Bobadilla-Kollár [10], see [78] for more general results:

Proposition 10.6.21 *Let X be a projective manifold, and denote by X^{ab} the universal free abelian cover of X, i.e., the covering associated with the homomorphism $\pi_1(X) \to H_1(X;\mathbb{Z})/Torsion$. If X^{ab} is homotopy equivalent to a finite CW complex, then the sheaves $R^i\,\mathrm{alb}_* \mathbb{C}_X$ are local systems on $\mathrm{Alb}(X)$ for all $i \geq 0$.*

Remark 10.6.22 The study of perverse sheaves on semi-abelian varieties has other interesting topological applications, e.g., strong finiteness properties for Alexander-type invariants, generic vanishing of Novikov and L^2-homology, the study of homological duality properties of smooth complex algebraic varieties, etc., see [38, 74, 75, 77] for more details.

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