A Two-Stage Variational Inequality Formulation for a Game Theory Network Model for Hospitalization in Critic Scenarios

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Abstract In this paper, we introduce the theoretical structure of a stochastic Generalized Nash Equilibrium model describing the competition among hospitals with first aid departments for the hospitalization in a disaster scenario. Each hospital with a first aid department has to solve a two-stage stochastic optimization problem, one before the declaration of the disaster scenario and one after the disaster advent, to determine the equilibrium hospitalization flows to dispatch to the other hospitals with first aid and/or to hospitals without emergency rooms in the network. We define the Generalized Nash Equilibria of the model and, particularly, we consider the Variational Equilibria which is obtained as the solution to a variational inequality problem. Finally, we present a basic numerical example to validate the effectiveness of the model.

Keywords Game theory · Stochastic optimization · Hospitalization dispatching · Variational equilibrium

1 Introduction

Critic scenarios, such as earthquakes, hurricanes, fires, pandemic advents, are situations in which unexpected violent natural events or global events alter normal human activities. In such situations, it is essential that institutions, governments, humanitarian organizations have the possibility to use various tools that can help them in the management of critical situations, to mitigate the consequences.

The unpredictability of such events dictates the need to provide these entities with non-deterministic mathematical models that allow them to estimate, depending on the scenario that may occur, the best strategy to be implemented to assist the multiple phases of the disaster management.

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In the existing literature, several network-based mathematical models, both of a deterministic and non-deterministic nature, have been developed to provide support to the disaster management phases (see, for instance, [2, 4, 5, 9–11]).

More specifically, in [2], authors provide an optimization model consisting of a dynamic supply chain network for personal protective equipment in the COVID-19 pandemic scenario in which they study the associated evolutionary variational inequality (see, for instance, [1]) in the presence of a delay function. In [4], authors propose a stochastic optimization approach for the distribution of medical supplies in emergency situations due to natural disasters, providing a two-stage stochastic programming model for which they derive a two-stage variational inequality formulation. In [5], authors present an evacuation model for which they derive a two-stage stochastic programming model. Finally, in [10], is proposed a two-stage stochastic game theory model describing the behavior of national governments in a healthcare disaster determined by COVID-19 pandemic advent and their competition for essential medical supplies in different phases of disaster preparedness.

The advent of a disaster scenario could cause an uncontrolled increase in requests for hospital care. In the moments following the disaster, the injured or registered victims are cared by the hospitals in the geographical area where the event occurred. The sudden advent of a disaster, the possible huge number of requests for assistance, the lack of medical personnel and the limited capacity of hospitalization cause inevitable overcrowding of the hospital structures and delays in the management of emergencies, as well as a large use of emergency vehicles and related costs increase. On the other hand, not all hospitals located in the geographical area of interest have emergency medical departments and this causes further inconvenience in the management of hospitalization requests. These factors, together with the total unpreparedness of hospitals, could make challenging, expensive and time-consuming the process of responding to the disastrous event.

Disaster management consists of different phases, including the preparedness and the response phases. In particular, when we consider an integrated preparation and response phase, it is of fundamental importance that decision makers make predictions on the uncertain possible disastrous scenarios that may occur and on their associated severity, so as not to be caught unprepared once the event occurs. These reasons led us to propose the two-stage stochastic optimization model described below, in which the decision makers are hospitals with emergency facilities, which seek to minimize both the total time of handling hospitalization requests from different geographical areas and the total costs due to patient transfers to other hospitals.

The paper is organized as follows. In Sect. 2, we develop the disaster stochastic game theory network model for hospitalization. We describe how hospitals with first aid departments compete to minimize their expected disutility, consisting of the total management time of an emergency and the total transfer cost to other hospitals. We describe the minimization problems that hospitals have to solve in the different phases of the management of the disaster scenario and we define the Generalized Nash Equilibrium and Variational Equilibrium of the proposed game theory model. Section 4 is dedicated to conclusion.

2 The Mathematical Model

The two-stage stochastic Supply Chain Network game theory model consists of K geographic areas, with a typical one by k, M hospitals with emergency rooms, with a generic one denoted by i and N hospitals without emergency rooms, with a generic one denoted by j. We denote by $\mathscr{M}_i$ the set of hospitals with first aid departments, except hospital i. In this model, the decision makers are hospitals with emergency rooms. Usually, in non-critical situations, emergency calls from a geographic location that are taken over by a hospital with an emergency room are handled by the receiving hospital, proceeding with the reception of the patient, anamnesis, diagnosis, possible hospitalization and subsequent discharge. In some cases (specialized centers, lack of staff, hospital overcrowding) it is possible that some emergency calls taken in charge by a hospital with first aid departments are subsequently routed to other hospitals with or without first aid departments. If, on the other hand, an emergency situation arises (outbreak of a pandemic, environmental disaster, etc.), it is highly likely that there will be a greater exchange among hospitals (for example, during the Covid-19 pandemic, due to the high number of patients in intensive care in Bolzano, Northen Italy, transfers were made to hospitals in Palermo, South Italy).

In this model, we want to provide a two-stage stochastic model, where in the pre-crisis phase, hospitals with emergency rooms consider several scenarios with different probabilities, so they are not surprised by a subsequent critical phase, trying to minimize the weighted sum between the management of emergency calls and the hospital dispatching times and the transport costs due to the transfers to other hospitals.

The three-tier network that describes the problem is represented in Figs. 1 and 2 which contain, respectively, the network topology for each hospital i and the entire

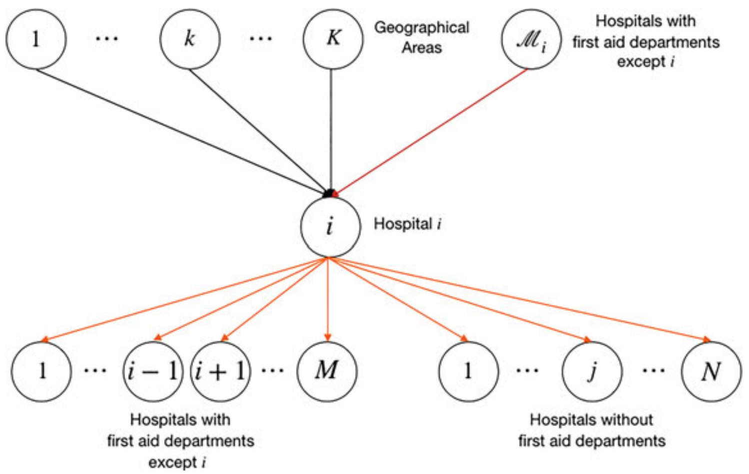

Fig. 1 Network topology for hospital i

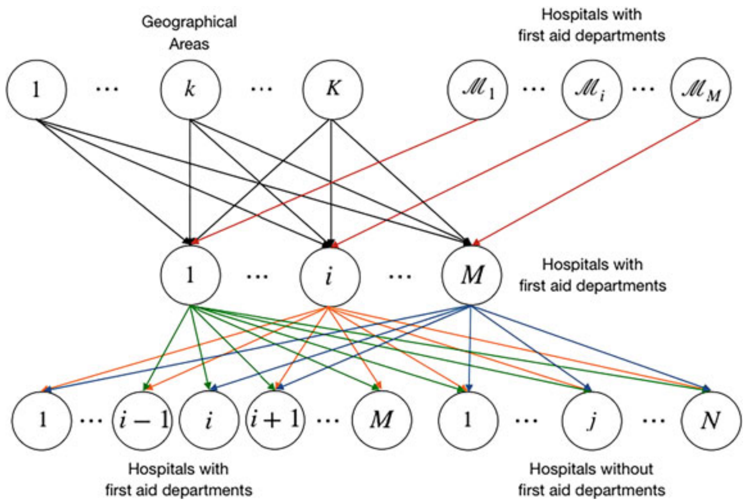

Fig. 2 Network topology

network topology. The variables and the parameters of the model are reported in Tables 1 and 2.

Facing a disaster scenario, the main goal of hospitals with first aid departments is to guarantee the treatment to patients who need medical devices and/or medical care, taking into account that requests for hospital care are all met as closely as possible. However, it is very difficult and time-consuming to ensure a complete management of the huge demand for hospital care, since the disaster scenario we are considering is global in nature.

Each hospital aims at minimizing its expected disutility which consists of the weighted sum between the expected dispatching time caused by the overcrowding of hospitals and the subsequent transportation costs of patients to other hospitals. The actions that the hospital takes in the first stage, before the disaster scenario, and the associated costs, are deterministic. However, the actions that hospitals take in phase 2, as soon as the disaster has occurred, depends on the possible scenario and the realization of probabilistic parameters.

We denote by:

- t^1_{ki} the transport time of an emergency patient from a geographic area k, $k = 1, \dots, K$, to hospital i, $i = 1, \dots, M$, in stage 1 and let us assume that t^1_{ki} is a function of the amount of emergency calls q^1_{ki}, namely:

$$t^1_{ki} := t^1_{ki}(q^1_{ki}), \quad \forall k = 1, \dots, K, \ \forall i = 1, \dots, M; \tag{1}$$

Table 1 Variables for the model

Notation	Variables
q^1_{ki}	The amount of emergency calls from the geographic area k, $k = 1, \dots, K$, handled by hospital i, $i = 1, \dots, M$, in stage 1. We group these quantities for all k into the vector $q^1_i \in \mathbb{R}^K_+$ and, in turn, we group these vectors into the vector $q^1 = (q^1_i)_{i=1,\dots,M} \in \mathbb{R}^{MK}_+$
$\hat{q}^1_i$	The amount of emergencies handled by hospital i, $i = 1, \dots, M$, dispatched by hospital with first aid departments belonging to $\mathscr{M}_i$ set in stage 1. We group these quantities into the vector $\hat{q}^1 \in \mathbb{R}^M_+$
$\tilde{q}^1_{il}$	The amount of emergencies handled by hospital i, $i = 1, \dots, M$, and subsequently dispatched to hospital $l \neq i$ in stage 1. We group these quantities for all $l \neq i$ into the vector $\hat{q}^1_i \in \mathbb{R}^{M-1}_+$ and, in turn, we group these quantities into the vector $\tilde{q}^1 \in \mathbb{R}^{M(M-1)}_+$
$\bar{q}^1_{ij}$	The amount of emergencies handled by hospital i, $i = 1, \dots, M$, dispatched to hospital j, $j = 1, \dots, N$, in stage 1. We group these quantities for all j into the vector $\bar{q}^1_i \in \mathbb{R}^N_+$ and, in turn, we group these quantities for all i into the vector $\bar{q}^1 \in \mathbb{R}^{MN}_+$
$q^{2\omega}_{ki}$	The amount of emergency calls from the geographic area k, $k = 1, \dots, K$, handled by hospital i, $i = 1, \dots, M$, in stage 2 when scenario ω occurs. We group these quantities for all k into the vector $q^{2\omega}_i \in \mathbb{R}^K_+$ and, in turn, we group these vectors into the vector $q^{2\omega} = (q^{2\omega}_i)_{i=1,\dots,M} \in \mathbb{R}^{MK}_+$. Finally, we group these vectors for all scenarios $\omega \in \Omega$ into the vector $q^2 \in \mathbb{R}^{\|\Omega\|MK}_+$
$\hat{q}^{2\omega}_i$	The amount of emergencies handled by hospital i, $i = 1, \dots, M$, arrived by hospital with first aid departments belonging to $\mathscr{M}_i$ set in stage 2 when scenario $\omega \in \Omega$ occurs. We group these quantities for all hospital into the vector in $\hat{q}^{2\omega} \in \mathbb{R}^M_+$ and, inturn, we group these quantities for all scenarios $\omega \in \Omega$ into the vector $\hat{q}^2 \in \mathbb{R}^{M\|\Omega\|}_+$
$\tilde{q}^{2\omega}_{il}$	The amount of emergencies handled by hospital i, $i = 1, \dots, N$, and subsequently dispatched to hospital $l \neq i$ in stage 2 when scenario ω occurs. We group these quantities for all $l \neq i$ into the vector $\tilde{q}^{2\omega}_i \in \mathbb{R}^{M-1}_+$ and, in turn, we group these quantities into the vector $\tilde{q}^{2\omega} \in \mathbb{R}^{M(M-1)}_+$. Finally, we group these vectors for all scenarios $\omega \in \Omega$ into the vector $\tilde{q}^2 \in \mathbb{R}^{\|\Omega\|M(M-1)}_+$
$\bar{q}^{2\omega}_{ij}$	The amount of emergency calls from k handled by hospital i dispatched to hospital j, $j = 1, \dots, N$, in stage 2. We group these quantities for all j into the vector $\bar{q}^{2\omega}_i \in \mathbb{R}^N_+$ and, in turn, we group these quantities for all i into the vector $\bar{q}^{2\omega} \in \mathbb{R}^{MN}_+$. Finally, we group these vectors for all scenarios $\omega \in \Omega$ into the vector $\bar{q}^2 \in \mathbb{R}^{\|\Omega\|MN}_+$
q	The vector $q = (Q^1, Q^2) \in \mathbb{R}^{MK+M+M(M-1)+MN+\|\Omega\|(MK+M+M(M-1)+MN)}_+$, where $Q^1 = (q^1, \hat{q}^1, \tilde{q}^1, \bar{q}^1)$ and $Q^2 = (q^2, \hat{q}^2, \tilde{q}^2, \bar{q}^2)$

- t^1_i the management time of an emergency patient arrived at hospital i, $i = 1, \dots, M$, in stage 1 and let us assume that t^1_i is a function of $\sum_{k=1}^{K} q^1_{ki}$ and $\hat{q}^1_i$,

Table 2 Parameters for the model

Notation	Parameters
$\omega \in \Omega$	The disaster scenario
p_ω	The probability of disaster scenario ω in stage 2, $\omega \in \Omega$
α_i	The weight in [0, 1]
C_i^1	The capacity of hospital i, $i = 1, \dots, M$, in stage 1
$C_i^{2\omega}$	The capacity of hospital i, $i = 1, \dots, M$, in stage 2 under scenario ω, $\forall \omega \in \Omega$
$\tilde{Q}_j^1$	The maximum capacity of hospital j, $j = 1, \dots, N$, in stage 1
$\tilde{Q}_j^{2\omega}$	The maximum capacity of hospital j, $j = 1, \dots, N$, in stage 2 under scenario ω, $\forall \omega \in \Omega$
β_i	The unit penalty encumbed by hospital i, $i = 1, \dots, N$, on the unmet demand
$d_i^{2\omega}$	The total demand for hospital i, $i = 1, \dots, N$, when scenario ω occurs in stage 2, $\forall \omega \in \Omega$

namely:

$$t_i^1 := t_i^1 \left(\sum_{k=1}^{K} q_{ki}^1, \hat{q}_i^1 \right) = t_i^1 \left(q_i^1, \hat{q}_i^1 \right). \tag{2}$$

This assumption suggests that the management time of an emergency call in the hospital i depends on the total flow of requests from each demand market and the total flow of requests transferred from hospitals belonging to $\mathscr{M}_i$, to hospital i;

- $\tilde{t}_{il}^1$, $l \neq i$, the transfer time of an emergency from hospital i, $i = 1, \dots, M$ to hospital $l = 1, \dots, M$, $l \neq i$, in stage 1 and let us assume that $\tilde{t}_{il}^1$ is a function of $\tilde{q}^1$, namely

$$\tilde{t}_{il}^1 := \tilde{t}_{il}^1(\tilde{q}); \tag{3}$$

- $\bar{t}_{ij}^1$ the transfer time of an emergency from hospital i, $i = 1, \dots, M$, to hospital j, $j = 1, \dots, N$, in stage 1 and let us assume that $\bar{t}_{ij}^1$ is a function of $\bar{q}^1$, namely

$$\bar{t}_{ij}^1 := \bar{t}_{ij}^1(\bar{q}^1) \tag{4}$$

- $\tilde{c}_{il}^1$ the transfer cost of an emergency from hospital i, $i = 1, \dots, M$, to hospital $l = 1, \dots, M$, $l \neq i$, in stage 1 and let us assume that $\tilde{c}_{il}^1$ is a function of $\tilde{q}_{il}^1$, namely

$$\tilde{c}^1_{il} := \tilde{c}^1_{il}(\tilde{q}^1_{il}); \tag{5}$$

- $\bar{c}^1_{ij}$ the transfer cost of an emergency from hospital i, $i = 1, \ldots, M$, to hospital j, $j = 1, \ldots, N$, in stage 1 and let us assume that $\bar{c}^1_{ij}$ is a function of $\bar{q}^1_{ij}$, namely

$$\bar{c}^1_{ij} := \bar{c}^1_{ij}(\bar{q}^1_{ij}); \tag{6}$$

Similarly, we define time and cost functions in stage 2, observing that these functions will depend on the variables of the second stage, and, therefore, will be affected by the uncertainty due to the scenario $\omega \in \Omega$.

Each hospital is faced with the following two-stage stochastic optimization model in which, as previously mentioned, it seeks to minimize the weighted sum between total expected dispatching time and the total expected costs (cf. Tables 1 and 2 for a detailed explanation of the role of each variable and parameter):

$$\begin{aligned} \text{Min} \Bigg\{ & \sum_{k=1}^{K} t^1_{ki}(q^1_{ki}) + t^1_i\left(q^1_i, \hat{q}^1_i\right) + \sum_{\substack{l=1,\ldots,M,\\ l\neq i}} \tilde{t}^1_{il}(\tilde{q}^1) + \sum_{j=1}^{N} \bar{t}^1_{ij}(\bar{q}^1) \\ & +\alpha_i \left(\sum_{\substack{l=1,\ldots,M,\\ l\neq i}} \tilde{c}^1_{il}(\tilde{q}^1_{il}) + \sum_{j=1}^{N} \bar{c}^1_{ij}(\bar{q}^1_{ij}) \right) + \mathbb{E}_{\Omega}[T^2_i(Q^2,\omega)] \Bigg\} \end{aligned} \tag{7}$$

subject to:

$$\sum_{\substack{l=1,\ldots,M,\\ l\neq i}} \tilde{q}^1_{il} + \sum_{j=1}^{N} \bar{q}^1_{ij} \leq \sum_{k=1}^{K} q^1_{ki} + \hat{q}^1_i, \tag{8}$$

$$\sum_{k=1}^{K} q^1_{ki} + \hat{q}^1_i \leq C^1_i, \tag{9}$$

$$\sum_{i=1}^{M} \bar{q}^1_{ij} \leq Q^1_j, \quad \forall j = 1, \ldots, N, \tag{10}$$

$$q^1_{ki},\ \hat{q}^1_i,\ \tilde{q}^1_{il},\ \bar{q}^1_{ij} \geq 0, \quad \forall k = 1, \ldots, K;\ \forall l = 1, \ldots, M,\ l \neq i;\ \forall j = 1, \ldots, N. \tag{11}$$

Constraint (8) ensures that the sum of emergencies dispatched by i to all hospitals l plus the sum of emergencies dispatched by i to all hospitals j is not grater than the number of emergencies reaching i from all geographical areas.

Constraint (9) ensures that the sum of emergency calls from all geographical areas plus the transferred emergencies from all others hospitals to hospital i is not greater than the maximum capacity of hospital i.

Constraints (10) are shared constraints and ensure that the sum of transferred emergencies from all hospitals i, $i = 1, \ldots, M$ to a hospital with no first aid department is not greater than the maximum capacity of the latter.

Constraints (11) are non-negative constraints.

The last term of objective function (7) represents the expected value of the loss to hospital i in stage 2. This loss depends also on the unmet demand from all geographical areas, that is

$$d_i^{2\omega} - \sum_{k=1}^{K} q_{ki}^1 - \hat{q}_i^1 - \sum_{k=1}^{K} q_{ki}^{2\omega} - \hat{q}_i^{2\omega}.$$

We have:

$$\mathbb{E}_{\Omega}[T_i^2(Q^2, \omega)] = \sum_{\omega \in \Omega} p_\omega [T_i^2(Q^2, \omega)],$$

where the loss for hospital i in stage 2 is the solution to the following second stage minimization problem:

$$\begin{aligned}
\text{Minimize } T_i^2(Q^2, \omega) = & \sum_{k=1}^{K} t_{ki}^1(q_{ki}^{2\omega}) \\
& + t_i^{2\omega}\left(q_i^{2\omega}, \hat{q}_i^{2\omega}\right) + \sum_{\substack{l=1,\ldots,M, \\ l \neq i}} \tilde{t}_{il}^1(\tilde{q}^{2\omega}) + \sum_{j=1}^{N} \bar{t}_{ij}^{2\omega}(\bar{q}^{2\omega}) \\
& + \alpha_i \left(\sum_{\substack{l=1,\ldots,M, \\ l \neq i}} \tilde{c}_{il}^1(\tilde{q}_{il}^{2\omega}) + \sum_{j=1}^{N} \bar{c}_{ij}^{2\omega}(\bar{q}_{ij}^{2\omega}) \right) \\
& + \beta_i \left[d_i^{2\omega} - \sum_{k=1}^{K} q_{ki}^1 - \hat{q}_i^1 - \sum_{k=1}^{K} q_{ki}^{2\omega} - \hat{q}_i^{2\omega} \right]
\end{aligned} \tag{12}$$

subject to the following constraints:

$$\sum_{\substack{l=1,\dots,M,\\ l\neq i}} \tilde{q}_{il}^{2\omega} + \sum_{j=1}^{N} \bar{q}_{ij}^{2\omega} \leq \sum_{k=1}^{K} q_{ki}^{2\omega} + \hat{q}_i^{2\omega}, \quad \forall \omega \in \Omega \tag{13}$$

$$\sum_{k=1}^{K} q_{ki}^{2\omega} + \hat{q}_i^{2\omega} \leq C_i^{2\omega}, \quad \forall \omega \in \Omega, \tag{14}$$

$$\sum_{i=1}^{M} \bar{q}_{ij}^{2\omega} \leq Q_j^{2\omega}, \quad \forall j = 1, \dots, N, \ \forall \omega \in \Omega, \tag{15}$$

$$q_{ki}^{2\omega},\ \hat{q}_i^{2\omega},\ \tilde{q}_{il}^{2\omega},\ \bar{q}_{ij}^{2\omega} \geq 0, \forall k = 1, \dots, K;\ \forall l = 1, \dots, M,\ l \neq i; \\ \forall j = 1, \dots, N;\ \forall \omega \in \Omega. \tag{16}$$

In stage 2, when the severity of a disaster is declared, each hospital i carries out restorative actions, to complete its first stage. Therefore, each hospital i seeks to minimize the total dispatching time and the total transport costs and the damage due to the unmet demand. The disaster could cause a shortage of staff, a greater demand for medical care and, therefore, a consequent decrease in the number of places available at each hospital, both with and without emergency rooms. Therefore, similarly to stage 1, in the second stage constraints (13)–(16) must be satisfied.

Following [10] and the standard stochastic optimization theory, the two optimization problems of stage 1 and stage 2 can be solved as a unique minimization problem, namely (cf. Tables 1 and 2 for a detailed explanation of the role of each variable and parameter):

$$\begin{aligned}
\text{Min} \Bigg\{ & \sum_{k=1}^{K} t_{ki}^1(q_{ki}^1) + t_i^1\left(q_i^1, \hat{q}_i^1\right) + \sum_{\substack{l=1,\dots,M,\\ l\neq i}} \tilde{t}_{il}^1(\tilde{q}^1) + \sum_{j=1}^{N} \bar{t}_{ij}^1(\bar{q}^1) \\
& + \alpha_i \left(\sum_{\substack{l=1,\dots,M,\\ l\neq i}} \tilde{c}_{il}^1(\tilde{q}_{il}^1) + \sum_{j=1}^{N} \bar{c}_{ij}^1(\bar{q}_{ij}^1) \right) + \sum_{\omega\in\Omega} p_\omega \Bigg[\sum_{k=1}^{K} t_{ki}^1(q_{ki}^{2\omega}) + t_i^{2\omega}\left(q_i^{2\omega}, \hat{q}_i^{2\omega}\right) \\
& + \sum_{\substack{l=1,\dots,M,\\ l\neq i}} \tilde{t}_{il}^1(\tilde{q}^{2\omega}) + \sum_{j=1}^{N} \bar{t}_{ij}^{2\omega}(\bar{q}^{2\omega}) + \alpha_i \left(\sum_{\substack{l=1,\dots,M,\\ l\neq i}} \tilde{c}_{il}^1(\tilde{q}_{il}^{2\omega}) + \sum_{j=1}^{N} \bar{c}_{ij}^{2\omega}(\bar{q}_{ij}^{2\omega}) \right) \\
& + \beta_i \left(d_i^{2\omega} - \sum_{k=1}^{K} q_{ki}^1 - \hat{q}_i^1 - \sum_{k=1}^{K} q_{ki}^{2\omega} - \hat{q}_i^{2\omega} \right) \Bigg] \Bigg\}
\end{aligned} \tag{17}$$

subject to constraints (8)–(11) and (13)–(16).

We define the feasible set of hospital i as follows:

$$\mathscr{K}_i = \Big\{ q \in \mathbb{R}_+^{MK+M+M(M-1)+MN+|\Omega|(MK+M+M(M-1)+MN)} \text{ such that: } (8),\ (9),\ (11),\ (13),\ (14),\ (16) \text{ hold} \Big\}, \tag{18}$$

and $\mathscr{K}^1 = \prod_{i=1}^{M} \mathscr{K}_i$. Moreover, let $\mathscr{S}$ be the set of shared constraints, namely $\mathscr{S} = \{q : (10) \text{ and } (15) \text{ hold}\}$. Finally, we define the feasible set $\mathscr{K}^2 = \mathscr{K}^1 \cap \mathscr{S}$.

The objective function (17) represents the expected disutility of hospital i.

We assume that, for each hospital i, the time and cost functions are convex and continuously differentiable. We have the following definition.

Definition 1 (Generalized Nash Equilibrium) A strategy profile $q^* \in \mathscr{K}^2$ is a Stochastic Generalized Nash Equilibrium if, for each hospital i, $i = 1, \ldots, M$:

$$\begin{aligned} & E(DU_i(q_i^{1*}, \hat{q}_i^{1*}, \tilde{q}_i^{1*}, \bar{q}_i^{1*}, q_i^{2*}, \hat{q}_i^{2*}, \tilde{q}_i^{2*}, \bar{q}_i^{2*}, q_{-i}^{1*}, \hat{q}_{-i}^{1*}, \tilde{q}_{-i}^{1*}, \bar{q}_{-i}^{1*}, q_{-i}^{2*}, \hat{q}_{-i}^{2*}, \tilde{q}_{-i}^{2*}, \bar{q}_{-i}^{2*})) \\ & \leq E(DU_i(q_i^{1}, \hat{q}_i^{1}, \tilde{q}_i^{1}, \bar{q}^{1}, q_i^{2}, \hat{q}_i^{2}, \tilde{q}_i^{2}, \bar{q}_i^{2}, q_{-i}^{1*}, \hat{q}_{-i}^{1*}, \tilde{q}_{-i}^{1*}, \bar{q}_{-i}^{1*}, q_{-i}^{2*}, \hat{q}_{-i}^{2*}, \tilde{q}_{-i}^{2*}, \bar{q}_{-i}^{2*})), \\ & \qquad \forall (q_i^1, \hat{q}_i^1, \tilde{q}_i^1, \bar{q}_i^1, q_i^2, \hat{q}_i^2, \tilde{q}_i^2, \bar{q}_i^2) \in \mathscr{K}_i \cap \mathscr{S}, \end{aligned} \tag{19}$$

where

$$q_{-i}^{1*} = (q_1^{1*}, \ldots, q_{i-1}^{1*}, q_{i+1}^{1*}, \ldots, q_M^{1*}), \quad q_{-i}^{2*} = (q_1^{2*}, \ldots, q_{i-1}^{2*}, q_{i+1}^{2*}, \ldots, q_M^{2*}),$$

$$\hat{q}_{-i}^{1*} = (\hat{q}_1^{1*}, \ldots, \hat{q}_{i-1}^{1*}, \hat{q}_{i+1}^{1*}, \ldots, \hat{q}_M^{1*}), \quad \hat{q}_{-i}^{2*} = (\hat{q}_1^{2*}, \ldots, \hat{q}_{i-1}^{2*}, \hat{q}_{i+1}^{2*}, \ldots, \hat{q}_M^{2*}),$$

$$\tilde{q}_{-i}^{1*} = (\tilde{q}_1^{1*}, \ldots, \tilde{q}_{i-1}^{1*}, \tilde{q}_{i+1}^{1*}, \ldots, \tilde{q}_M^{1*}), \quad \tilde{q}_{-i}^{2*} = (\tilde{q}_1^{2*}, \ldots, \tilde{q}_{i-1}^{2*}, \tilde{q}_{i+1}^{2*}, \ldots, \tilde{q}_M^{2*}),$$

$$\bar{q}_{-i}^{1*} = (\bar{q}_1^{1*}, \ldots, \bar{q}_{i-1}^{1*}, \bar{q}_{i+1}^{1*}, \ldots, \bar{q}_M^{1*}), \quad \bar{q}_{-i}^{2*} = (\bar{q}_1^{2*}, \ldots, \bar{q}_{i-1}^{2*}, \bar{q}_{i+1}^{2*}, \ldots, \bar{q}_M^{2*}).$$

Each hospital seeks to minimize its expected disutility, that depends not only on its own decisions, but also on the strategies of the other players. According to the above definition, hospitals will be in a state of equilibrium if no player can unilaterally change his strategy without obtaining a greater disutility. Moreover, the presence of shared constraints provides an interconnection among feasible sets of players.

This formulation provides a model based on a Generalized Nash Equilibrium (see, for instance, [3]). In general, Generalized Nash Equilibrium problems can be formulated through quasi-variational inequality problems (see [6]). However, a class of Generalized Nash Equilibria, the Variational Equilibria, can be formulated as a variational inequality problem (see, for instance, [8] and [11]). As in [10], we will deal with the variational equilibrium of the model.

Definition 2 (Variational Equilibrium) A strategy vector $q^* \in \mathscr{K}^2$ is a Variational Equilibrium of the above Stochastic Generalized Nash Equilibrium problem if $q^* \in \mathscr{K}^2$ is a solution to the variational inequality:

$$
\begin{aligned}
&\sum_{k=1}^{N}\sum_{i=1}^{M}\left[\frac{\partial t_{ki}^{1}(q_{ki}^{1*})}{\partial q_{ki}^{1}}+\frac{\partial t_{i}^{1}\left(q_{i}^{1*},\hat{q}_{i}^{1*}\right)}{\partial q_{ki}^{1}}-\beta_i\right]\times(q_{ki}^{1}-q_{ki}^{1*})\\
&+\sum_{i=1}^{M}\left[\frac{\partial t_{i}^{1}\left(q_{i}^{1*},\hat{q}_{i}^{1*}\right)}{\partial \hat{q}_{i}^{1}}-\beta_i\right]\times(\hat{q}_{i}^{1}-\hat{q}_{i}^{1*})\\
&+\sum_{i=1}^{M}\sum_{\substack{l=1,\dots,M,\\ l\neq i}}\left[\frac{\partial \tilde{t}_{il}^{1}(\tilde{q}^{1*})}{\partial \tilde{q}_{il}^{1}}+\alpha_i\frac{\partial \tilde{c}_{il}^{1}(\tilde{q}_{il}^{1*})}{\partial \tilde{q}_{il}^{1}}\right]\times(\tilde{q}_{il}^{1}-\tilde{q}_{il}^{1*})\\
&+\sum_{i=1}^{M}\sum_{j=1}^{N}\left[\frac{\partial \bar{t}_{ij}^{1}(\bar{q}^{1*})}{\partial \bar{q}_{ij}^{1}}+\alpha_i\frac{\partial \bar{c}_{ij}^{1}(\bar{q}_{ij}^{1*})}{\partial \bar{q}_{ij}^{1}}\right]\times(\bar{q}_{ij}^{1}-\bar{q}_{ij}^{1*})\\
&+\sum_{\omega\in\Omega}p_\omega\sum_{k=1}^{N}\sum_{i=1}^{M}\left[\frac{\partial t_{ki}^{2\omega}(q_{ki}^{2\omega*})}{\partial q_{ki}^{2\omega}}+\frac{\partial t_{i}^{2\omega}\left(q_{i}^{2\omega*},\hat{q}_{i}^{2\omega*}\right)}{\partial q_{ki}^{2\omega}}-\beta_i\right]\times(q_{ki}^{2\omega}-q_{ki}^{2\omega*})\\
&+\sum_{\omega\in\Omega}p_\omega\sum_{i=1}^{N}\left[\frac{\partial t_{i}^{2\omega}\left(q_{i}^{2\omega*},\hat{q}_{i}^{2\omega*}\right)}{\partial \hat{q}_{i}^{2\omega}}-\beta_i\right]\times(\hat{q}_{i}^{2\omega}-\hat{q}_{i}^{2\omega*})\\
&+\sum_{\omega\in\Omega}p_\omega\sum_{i=1}^{M}\sum_{\substack{l=1,\dots,M,\\ l\neq i}}\left[\frac{\partial \tilde{t}_{il}^{2\omega}(\tilde{q}^{2*})}{\partial \tilde{q}_{il}^{2\omega}}+\alpha_i\frac{\partial \tilde{c}_{il}^{2\omega}(\tilde{q}_{il}^{2\omega*})}{\partial \tilde{q}_{il}^{2\omega}}\right]\times(\tilde{q}_{il}^{2\omega}-\tilde{q}_{il}^{2\omega*})\\
&+\sum_{\omega\in\Omega}p_\omega\sum_{i=1}^{M}\sum_{j=1}^{N}\left[\frac{\partial \bar{t}_{ij}^{2\omega}(\bar{q}^{2*})}{\partial \bar{q}_{ij}^{2\omega}}+\alpha_i\frac{\partial \bar{c}_{ij}^{2\omega}(\bar{q}_{ij}^{2\omega*})}{\partial \bar{q}_{ij}^{2\omega}}\right]\times(\bar{q}_{ij}^{2\omega}-\bar{q}_{ij}^{2\omega*})\geq 0\quad \forall q\in\mathscr{K}^2.
\end{aligned}
\tag{20}
$$

The advantage of detecting a variational equilibrium consists in using the well-known variational inequality theory, for which theorems of existence and uniqueness of the solution are stated (see [7]).

3 An Illustrative Numerical Example

In this section, we solve an illustrative numerical example to validate the effectiveness of the model. We consider $g = 2$ geographical areas, $h = 2$ hospitals with first aid departments, $s = 3$ hospitals without first aid departments and $|\Omega| = 2$ scenarios. In the first scenario $\omega_1 = 1$, we suppose that the consequences of the advent of the disaster scenario are severe while in the second scenario $\omega_2 = 2$ we assume that the consequences are not severe. Consequently, in the first scenario the requests for hospitalization are more than in the second one.

For the computation of the optimal solution we have applied the modified projection method described in [10]. The calculations were performed using the MATLAB program. The algorithm was implemented on a laptop with 1.8 GHz Intel Core i5 dual-core and 8 GB RAM, 1600 MHz DDR3. For the convergence of the method a tolerance of $\epsilon = 10^{-4}$ was fixed. The method has been implemented with a constant step $\alpha = 0.1$.

The numerical data and the size of the problem are constructed for easy interpretation purposes. We have the following data:

$$p_{\omega_1} = 0.8,\; p_{\omega_2} = 1 - p_{\omega_1} = 0.2, \quad \beta_1 = \beta_2 = 20,$$
$$\Gamma_1^1 = 25,\; \Gamma_2^1 = 35, \quad \Gamma_1^{2\omega_1} = 45,\; \Gamma_2^{2\omega_1} = 60, \quad \Gamma_1^{2\omega_2} = 30,\; \Gamma_2^{2\omega_2} = 53,$$
$$\tilde{C}_1^1 = 10,\; \tilde{C}_2^1 = 15,\; \tilde{C}_3^1 = 25, \quad \tilde{C}_1^{2\omega_1} = 18,\; \tilde{C}_2^{2\omega_1} = 20,\; \tilde{C}_3^{2\omega_1} = 30,$$
$$\tilde{C}_1^{2\omega_2} = 15,\; \tilde{C}_2^{2\omega_2} = 18,\; \tilde{C}_3^{2\omega_2} = 25,$$
$$d_1^{2\omega_1} = d_2^{2\omega_1} = 80, \quad d_1^{2\omega_2} = d_2^{2\omega_2} = 50.$$

The equilibrium solution is shown in Table 3.

The computational time needed to calculate the equilibrium solution is 50 seconds. As shown in Table 3, in phase 1, where cost and time functions and demands are deterministic, there is not a huge transfer between hospitals with first aid departments and between hospitals with and without first aid departments.

Table 3 Equilibrium solution

Stage 1		Stage 2: scenario ω_1		Stage 2: scenario ω_2	
Solution	Value	Solution	Value	Solution	Value
q_{11}^{1*}	20.6	$q_{11}^{2\omega_1*}$	46.3	$q_{11}^{2\omega_2*}$	24.3
q_{12}^{1*}	3.8	$q_{12}^{2\omega_1*}$	46.3	$q_{12}^{2\omega_2*}$	21.8
q_{21}^{1*}	9.3	$q_{21}^{2\omega_1*}$	53.6	$q_{21}^{2\omega_2*}$	35.7
q_{22}^{1*}	26.1	$q_{22}^{2\omega_1*}$	53.6	$q_{22}^{2\omega_2*}$	33.2
$\hat{q}_1^{1*}$	0.5	$\hat{q}_1^{2\omega_1*}$	47.6	$\hat{q}_1^{2\omega_2*}$	16.1
$\hat{q}_2^{1*}$	0.5	$\hat{q}_2^{2\omega_1*}$	47.3	$\hat{q}_2^{2\omega_2*}$	15.9
$\tilde{q}_{12}^{1*}$	4.6	$\tilde{q}_{12}^{2\omega_1*}$	18.5	$\tilde{q}_{12}^{2\omega_2*}$	11.9
$\tilde{q}_{21}^{1*}$	7.1	$\tilde{q}_{21}^{2\omega_1*}$	22.7	$\tilde{q}_{21}^{2\omega_2*}$	20.2
$\bar{q}_{11}^{1*}$	3.9	$\bar{q}_{11}^{2\omega_1*}$	7.2	$\bar{q}_{11}^{2\omega_2*}$	5.2
$\bar{q}_{12}^{1*}$	3.4	$\bar{q}_{12}^{2\omega_1*}$	8.2	$\bar{q}_{12}^{2\omega_2*}$	6.2
$\bar{q}_{13}^{1*}$	3.8	$\bar{q}_{13}^{2\omega_1*}$	11.1	$\bar{q}_{13}^{2\omega_2*}$	6.6
$\bar{q}_{21}^{1*}$	3.9	$\bar{q}_{21}^{2\omega_1*}$	10.8	$\bar{q}_{21}^{2\omega_2*}$	9.7
$\bar{q}_{22}^{1*}$	3.5	$\bar{q}_{22}^{2\omega_1*}$	11.8	$\bar{q}_{22}^{2\omega_2*}$	11.8
$\bar{q}_{23}^{1*}$	3.3	$\bar{q}_{23}^{2\omega_1*}$	14.6	$\bar{q}_{23}^{2\omega_2*}$	11.1

When a disaster scenario occurs in stage 2, there is an increase in requests for hospitalization and a consequent increase in transfers between hospital structures. Particularly, under scenario ω_1, the severity of which is higher, hospital 1 fails to satisfy the total demand, having an unmet demand equal to 174.

4 Conclusion

In this paper, we presented a stochastic Generalized Nash Equilibrium model to describe the competition among hospitals with first aid departments for hospitalization in response to the advent of a disaster scenario. We obtained a two-stage stochastic optimization problem and the presence of shared constraints for all hospitals with first aid departments led us to consider a Generalized Nash Equilibrium problem for which we derived the Variational Equilibrium and the associated variational inequality problem. The results in this paper add to the growing literature of game theory and and two-stage stochastic models in disaster management. This theoretical model can be applied to any disastrous event that involves a sudden and nondeterministic increase in hospitalization, such as the recent and still current COVID-19 pandemic.

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