Thoughts on Using the History of Mathematics to Teach the Foundations of Mathematical Analysis

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Abstract This paper discusses ideas for a different approach to teaching the foundations of mathematical analysis. The main idea is to avoid the use of what Keith Devlin in 2005 called "formal definitions," which are definitions that nobody can understand without working with them. For students without mathematical maturity, these definitions can be difficult to understand and use. This paper discusses an approach that uses the history of mathematics to first develop fundamental concepts and only introduces formal definitions after the concepts are understood. The audience for this approach is third-year undergraduate students. Several examples are provided in the paper.

1 The Problem

Our long experience of teaching mathematicians and computer scientists indicates to us that many students of moderate ability have trouble understanding mathematics. A significant part of this problem appears to be that many students think that mathematics is a game of manipulating symbols without worrying about their meaning. They then try to learn mathematics by memorizing the manipulations. Students who try to learn this way will be lost if they forget one detail. This problem seems to start with beginning algebra and becomes more serious after the transition from computational courses like calculus and elementary linear algebra to more advanced courses, which are more theoretical.

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At a meeting of the Canadian Mathematical Society in 2005, Keith Devlin gave the plenary address (Devlin 2005) in which he discussed what he called *formal definitions*, which are definitions that nobody can completely understand without first working with them. This contrasts with dictionary definitions of words which are understood easily by anyone who looks up the definition. These formal definitions are common in advanced mathematics courses, including analysis, but for students without mathematical maturity, they can be difficult to learn and understand.

This is the motivation for the authors of this paper writing a book, tentatively entitled *A Non-formal Introduction to Mathematical Analysis*, which is intended for a third-year undergraduate course to introduce students to analysis while avoiding the use of formal definitions as much as possible. Some reactions to a preliminary draft of this book suggest that it might be useful to precede this book with content that provides a bridge between computational courses and the more theoretical courses which follow in the normal mathematical curriculum.

Many textbooks for this type of bridge course begin with formal logic, apparently assuming that this should indicate to students what proofs are and why they are needed. Starting this way appears to place too much of a burden on the students. It would be better for the formal logic to occur later in the book when examples from earlier parts of the course can be used to illustrate how formal logic can serve to analyze them. We believe that this is the best approach for students to recognize the *meaning* of the logical symbols.

For example, analysis textbooks often start with the axioms for a complete ordered field, give the $\varepsilon - \delta$ definition of the limit of a function and the $\varepsilon - N$ definition of the limit of a sequence, and then start deriving theorems. This seems to be the wrong order for many students, since it does not help the students learn why these axioms and theorems are important and what analysis is really about. For both the bridge and analysis courses, it seems that approaching the material in a historical order rather than a logical order may help student learning. Even if we do not have an account of the history that is beyond controversy, we think taking a historical approach to the material will make sense to students and will enable students to visualize what they will be studying.[1]

This paper contains some examples of this approach to learning analysis via a historical ordering of the concepts.

[1] For the importance of visualization in the learning of mathematics, see Nardi (2014) as well as the publications of eric.ed.gov on the Role of Visualization in the Teaching and Learning of Mathematics and especially Mathematical Analysis as for example in the article of Miguel de Guzman in https://files.eric.ed.gov/fulltext/ED472047.pdf.

2 Mathematics as *Reasoning*

Mathematics beyond elementary arithmetic is about *reasoning*. This can be demonstrated by showing that simple algebraic equations can be solved in words without using modern algebraic notation. For example, suppose that the problem is to find a number such that five more than three times the number is twenty. We can solve this as follows:

Suppose: Five more than three times the number is twenty.

Then: Three times the number is fifteen.

Hence: The number is five.

This shows that modern algebraic notation is not necessary to do algebra. The key point is that whereas problems in arithmetic tell the student what operations to perform on what numbers, problems in algebra require that reasoning is used to determine those operations and numbers. Furthermore, before the European Renaissance, this was the only way to do algebra, since the modern notation used today began as a kind of shorthand during the Renaissance.

This solution is actually a short proof of the proposition: *If five more than three times the number is twenty, then the number is five.* When the solution is checked, the converse: *if the number is five, then five more than three times the number is twenty* is proved.

To pass from this solution to one using the modern notation involves two steps: first, use the standard notations for arithmetic operations in the sentences in the solution:

$$\text{Suppose}:\ 5 + 3 \times (\text{the number}) = 20.$$

$$\text{Then}:\quad 3 \times (\text{the number}) = 15.$$

$$\text{Hence}:\quad \text{The number} = 5.$$

For the final step, introduce a letter to represent the number: let n be the number. Then

$$\text{Suppose}:\ 5 + 3n = 20.$$

$$\text{Then}:\quad 3n = 15.$$

$$\text{Hence}:\quad n = 5.$$

Note that the role of the notation is to help students notice the patterns needed to complete the reasoning successfully.

It is customary to leave out the logical words out of these solutions, so that the solution might be written as:

$$5 + 3n = 20.$$

$$3n = 15.$$

$$n = 5.$$

This may be part of the reason that there are students who do not realize that there is reasoning involved here.

Leaving out the logical words may also lead students to confusion if the equation has no solution.

Consider, for example, the equation $2\sqrt{x+1} = \sqrt{4x+5}$. If the steps of the solution are written without the logical words, the computation appears as:

$$2\sqrt{x+1} = \sqrt{4x+5}$$

$$4\,(x+1) = 4x+5.$$

$$4x+4 = 4x+5.$$

$$4 = 5.$$

Students who do not understand that this is supposed to be reasoning may feel completely lost at this point. But now inserting the logical words gives:

$$\text{Suppose}:\ 2\sqrt{x+1} = \sqrt{4x+5}$$

$$\text{Then}:\quad 4\,(x+1) = 4x+5.$$

$$\text{Thus}:\quad 4x+4 = 4x+5.$$

$$\text{Hence}:\quad 4 = 5.$$

This may help students realize that what has been shown is *if* $2\sqrt{x+1} = \sqrt{4x+5}$ *then* $4 = 5$. Since $4 \neq 5$ this shows that $2\sqrt{x+1} \neq \sqrt{4x+5}$ for every x.[2]

The conclusion of this example is the result of a *proof by contradiction*: when a contradiction is deduced from an assumption, the conclusion is that the assumption

[2] This example comes from the Preface of Middlemiss (1952).

is false. The idea is closely related to the debating tactic of *reductio ad absurdum*, or *reduction to an absurdity*.

3 Why Is There a Problem About Calculus?

Consider the function $y = f(x) = x^2$ and the problem of finding its derivative $f'(2)$ at $x = 2$. The process for finding this derivative is to consider first the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 2^2}{x - 2}$$

This quotient is evaluated as follows:

$$\frac{x^2 - 2^2}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$$

where the last step is valid only if $x \neq 2$, but when "the limit is taken" at $x = 2$ by substituting 2 for x yields 4. Since the conclusion that the difference quotient is equal to x + 2 is based on the assumption that $x = 2$, it appears that an illegal step was taken.

Calculus textbooks justify this last step by saying that the difference quotient is not being *evaluated* at x = 2, but instead, the computation is a limit as $x \to 2$, thus

$$\frac{dy}{dx} = \lim_{x \to 2} \frac{\Delta y}{\Delta x}$$

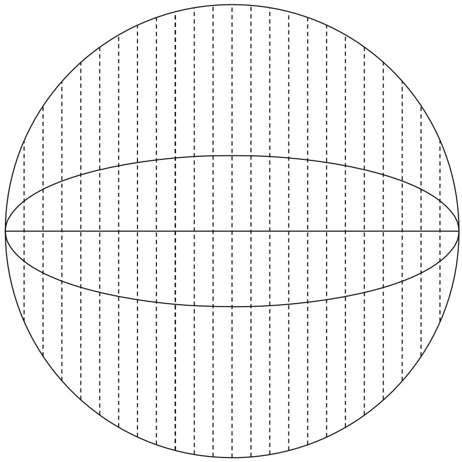
and that all this really means is that if x is *near* 2, then the difference quotient is *near* 4, and further, that is possible to get the difference quotient as close to 4 as desired, simply by choosing x sufficiently close to 2. These standard textbooks typically give a number of rules for evaluating limits with very little justification for these rules.

In the seventeenth century, although the Greek's geometric methods were admired and applied to calculate quadratures, fear of the infinite was abandoned. For example, Johannes Kepler used infinitesimals to determine the area of an ellipse and viewed the circumference of a circle as an infinite sided regular polygon.

Kepler noted that for a circle of radius a and an ellipse of radiuses a and b: (see Fig. 1)

- The ratio of each vertical line within the circle to the vertical line within the ellipse is b/a.
- The area of each of the circle/ellipse is the infinite sum of vertical lines contained in the circle/ellipse.

Fig. 1 Circle and Ellipse



Hence, the ratio of the area of the circle to that of the ellipse is also b/a. This means that

$$\text{the area of the ellipse} = \left(\pi a^2\right) \times (b/a) = \pi ab.$$

Further uses of the infinitesimal as in Kepler's method were developed to calculate more advanced tangents and quadratures and to relate tangents with quadratures. However, there was still a lack of a formal development of the concepts and rules in question. Part of this was filled independently by the work of Newton and Leibniz on the calculus. Infinite series/sequences were crucial for their calculus methods, and they in this way legitimized the use of infinite processes. Although Newton's and Leibniz's work had introduced differentiation and its reverse (integration), they were still short of formulating the definition of limit. Although both Newton and Leibniz invented the calculus independently in the late seventeenth century, they did not have a good explanation of what they were doing. Newton, for example, described what we call

$$\lim_{x \to 2} \frac{\Delta y}{\Delta x}$$

as its "ultimate value," or its value at "the instant of its disappearance." This explanation does not satisfy the normal requirements of a mathematical definition.

In fact, in the 1730s, the inadequacy of this definition was brought home by George Berkeley in Berkeley (1951). Berkeley was an Anglican bishop, and he had become disturbed for the soul of Edmond Halley (the astronomer for whom Halley's Comet is named) who was proclaiming himself an atheist on the basis of Newton's physics. Berkeley happened to be an excellent satirist, and he jumped on Newton's explanation of the limit of the difference quotient as its value at "the instant of its disappearance" by asking why it should not be called "the ghost of a departed quantity." His idea was to suggest that anybody who was prepared to accept Newton's calculus should also have no trouble accepting theology. Despite all this criticism, infinite processes and calculus as developed by Newton and Leibniz continued to be very much in use.

New developments by Euler on the generalization of function were followed by attempts at explaining the notion of limit. This was followed in the nineteenth century by Cauchy's work that successfully combined the new ideas of function and limit in order to give a rigorous formulation of the calculus explaining convergence, divergence, and continuous functions. More rigor was put into explaining the calculus, its notion of limit and continuous function, and even the core on which it is based (the real numbers). This theoretical work is called *analysis*, or the *arithmetization of analysis*. In fact, all analysis can be derived from a set of axioms about the real numbers.

The arithmetization of analysis is one of the greatest intellectual achievements of human history. However, it does not seem possible to appreciate its greatness just by looking at the latest version of the theory itself. It is necessary to consider the entire process by which this theory developed. This process involved major changes in the way mathematicians looked at their subject, the sorts of things they studied, and even at how mathematics could be justified. It really began over 2000 years ago in ancient Greece. In order to fully understand and appreciate analysis, it is important to begin with the ideas of the ancient Greeks.

4 Proofs as Sequences of Statements

In all the solutions of the equations of Sect. 2 of this paper, the arguments consist of sequences of statements. However, proofs were not always sequences of statements. In ancient Greek mathematics, there was a time when a proof consisted of a diagram; understanding the proof meant seeing the diagram the right way.

For example, knowing that the area of a rectangle is the product of its 2 adjacent sides, consider the following proof that the area of a parallelogram is the base times the altitude:

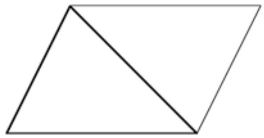
Fig. 2 Area of the parallelogram

This diagram (Fig. 2) shows that the area of the parallelogram is the same as the area of the rectangle with the same base and altitude.

Next, consider the following proof of the formula for the area of a triangle:

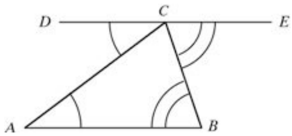
This diagram (Fig. 3) shows that the area of the triangle is half of the area of the parallelogram.

Fig. 3 Area of a triangle



Consider also the following proof (Fig. 4) that the three angles of a triangle (in a Euclidean plane) add up to two right angles:

Fig. 4 Angles of a triangle



Similarly, the following diagram (Fig. 5) shows the Pythagorean Theorem:

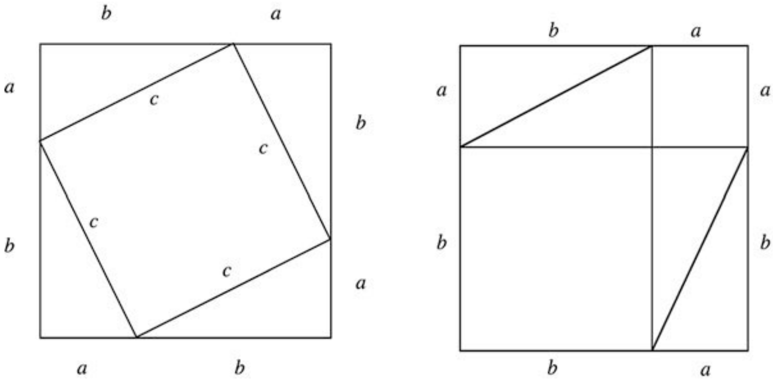


Fig. 5 The Pythagorean Theorem

For this last one, if A is the area of the right triangle with legs a and b and hypotenuse c, then the diagrams show that

$$4A + c^2 = 4A + a^2 + b^2,$$

from which $c^2 = a^2 + b^2$ follows easily.

Proofs by diagrams were prevalent in early mathematics. So how and when did proofs become sequences of statements? One possible way this might have happened is given by Wilbur Knorr (1975, pp. 179–180). It is based on Knorr's reconstruction of how the ancient Greeks might have first proved that the side and diagonal of a square are incommensurable; i.e., there is no length that evenly divides both. This is equivalent to the irrationality of $\sqrt{2}$. It is known that the Ancient

Greeks did arithmetic using diagrams of pebbles arranged in certain ways, so these proofs of arithmetic properties can be illustrated by *pebble diagrams*.

Figure 6 is a pebble diagram illustrating the result that *every even square number is a multiple of 4*:

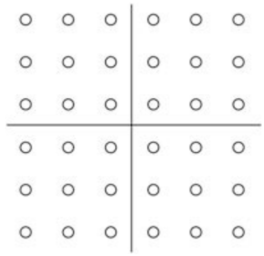


Fig. 6 Pebble diagram for even squares

This only shows one case ($6^2 = 4 \times 3^2$), but it is clear that, at least in principle, a diagram like this one can be created for every even square number. In modern algebraic symbols, this can be shown by $(2n)^2 = 4n^2$.

Figure 7 shows in a similar way that *every odd square number is one more than a multiple of eight*:

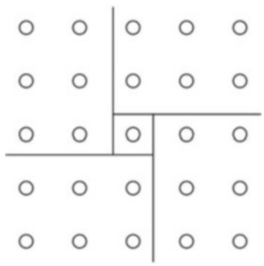


Fig. 7 Pebble diagram for odd squares

That every odd square is one more than a multiple of four is immediate from this diagram; that it is one more than a multiple of 8 follows because each of the four rectangular areas is one number times the next one, and of those two numbers one is always even. In algebraic symbols, we have

$$(2n+1)^2 = 4n^2 + 4n + 1 = 4\left(n^2 + n\right) + 1 = 4n\,(n+1) + 1.$$

Now, by the last of these two numerical results, it follows that the sum of two odd squares is never a perfect square, since it must have the form $(8n+1) + (8m+1) = 8(m+n) + 2$, which cannot be a perfect square since it is divisible by 2 but not divisible by 4.

Now, consider an isosceles right triangle (i.e., a right triangle whose two legs are equal). Such a triangle can be seen as part of a square with a diagonal. We know by the Pythagorean Theorem that if the legs are both l and the hypotenuse is h, then

$$h = \sqrt{2}l.$$

If h and l are commensurable, then there is a unit for which both h and l are positive integers. Assume that h and l are positive integers. Incommensurability can be proven by showing that this assumption leads to a contradiction. So the assumption implies that h is either even or odd.

- If h is even, then so is l, and this means that the unit can be doubled and still have a length which evenly divides h and l. This is equivalent to cutting each of h and l in half. Obviously, this cannot be continued indefinitely, so eventually there must be a length which divides h an odd number of times.
- But if h is odd, then one leg must be even and the other must be odd, so the triangle is not isosceles.

This is a contradiction, since an even number cannot equal an odd number. So the hypothesis that there is a length that exactly divides both h and l an integral number of times must be false.

Note that there is no way to illustrate this result with one or more diagrams. The argument is a sequence of statements. This is Knorr's explanation of how proofs became sequences of statements. Further details are given in Seldin (1990).

5 Bridge Course: Mathematical Theories

Many students are confused by abstract mathematical theories when they are introduced to them for the first time. They may be able to follow some of the proofs, but they seem not to understand what is really going on and why mathematicians find proofs interesting.

For this reason, we believe that a bridge course should begin with an example. One which students at this stage in their mathematical development are supposed to know but often do not: the mathematics of fractions as formal quotients of positive integers. Students are supposed to know that cross-multiplication is the method to determine whether or not two fractions are equal:

$$\frac{m}{n} = \frac{p}{q} \text{ if and only if } mq = np.$$

The idea is to take the relation determined by cross-multiplication not as equality, which means identity of fractions, but as a relation of "having the same value." This gives

$$\frac{m}{n} = \frac{p}{q} \text{ if and only if } m = p \text{ and } n = q.$$

Now define the relation $\sim$ between fractions by

$$\frac{m}{n} \sim \frac{p}{q} \text{ if and only if } mq = np.$$

It is easy to show that ~ is an equivalence relation. A (positive) rational number as an equivalence class can be defined from the equivalence relation ~, and it is easy to prove that the usual properties follow from this definition, including the fact that unlike in the positive integers, each (positive) rational number has an inverse with respect to multiplication.

Double-entry bookkeeping can be used in the form of accounts, each of which has a credit and a debit to look at formal differences of positive integers. Thus, an account with credit c and debit d will represent the formal difference $c\ominus d$. Two accounts $c\ominus d$ and $e\ominus f$ will be equal if and only if they are identical:

$$c \ominus d = e \ominus f \text{ if and only if } c = e \text{ and } d = f,$$

while the relation $\cong$ for "having the same value" will be defined by

$$c \ominus d \cong e \ominus f \text{ if and only if } c + f = d + e.$$

This theory with respect to addition is very similar to fractions (formal quotients) with respect to multiplication. The main difference between this theory and the theory of fractions discussed earlier is that the positive integers have an identity element with respect to multiplication but not with respect to addition. This means that in this case, an identity and an inverse are being added with this theory.

While these theories are not isomorphic, they are extremely similar. It is instructive for the students to compare them. The desire is for the students to see that most of the words in the results (in fractions with respect to multiplication and in accounts with respect to addition) are identical.

From here it is possible to define a commutative cancellation semigroup to consist of a set S together with a binary operation which satisfies closure, commutativity, associativity, and the cancellation law. We then define a binary operation on ordered pairs of elements of S by

$$(x, y) \approx (z, u) \text{ if and only if } x \circ u = y \circ z.$$

This definition corresponds to the definition of "having the same value" for fractions with multiplication and accounts with addition, and the resulting theory is so close to both of those theories that it can be "instantiated" to either of them. This shows that this formal theory allows us to add an identity (if there is not one already present in S) and an inverse for every element.

The hope is that this collection of examples will help students understand why mathematicians study formal theories and find them useful.

6 Bridge Course: Set Theory and Logic

For the rest of the bridge course, we plan to begin by using equivalence relations and equivalence classes to explain cardinal numbers of sets,[3] including infinite sets, and derive some of the elementary properties of transfinite cardinal numbers. We would, of course, take the equivalence relation on sets to be the relation of the existence of a bijection between the sets. The properties of transfinite cardinal numbers include sets of real numbers which are countable and some which are uncountable. The specific countable sets that should be included are the integers, rational numbers, and algebraic numbers. The important uncountable sets to be studied include the real numbers between 0 and 1 and the larger set of single-valued real-valued functions of real numbers.

From here it is possible to use the language of set theory to define functions. We believe that putting the results on transfinite cardinal numbers before these definitions is, in fact, the historical order in which these ideas were introduced and these definitions entered mathematics.

The next step is to use formal logic to evaluate proofs and informal arguments that have occurred previously in the bridge course.

7 Analysis: Limits

Analysis is based on limits, so any course on analysis must begin by considering them.

First, students should be reminded of some of what they (are supposed to) know about limits of functions and sequences. It would be possible to create the first theories of limits of functions and sequences by taking some of the standard limit theorems as axioms and from those deriving the rest. However, doing this formally may be confusing to some students.

Another approach is to begin with looking at two examples of proofs of limit results from ancient Greek geometry. The results are about circles. The theorems of interest are as follows:

1. A theorem of Euclid which says that the areas of circles are to each other as the squares of their radii
2. Archimedes' theorem giving the area of a circle

According to Knorr (1975, pp. 311–312), both theorems were proved by Eudoxus, using a method that came to be known as the *method of exhaustion*.

[3] Given the lack of a proper universal set in axiomatic set theory, we cannot formally define cardinal numbers as equivalence classes, but we can indicate how the idea behind this is approximately that of such equivalent classes to be covered.

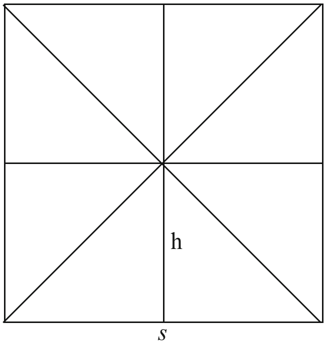


Fig. 8 Square for the proof of the area

For both theorems, a general formula for the area of a regular polygon is needed. To see how this formula is developed, let us look at a square in a new way (Fig. 8). Instead of finding the area simply by taking s^2, start with the bottom of the four triangles obtained by taking the diagonals. The altitude of each triangle is $h = \frac{1}{2}s$, so the area of the triangle is given by $\frac{1}{2}hs$. If we add the areas of all four triangles, we get

$$A = \frac{1}{2}h(4s) = \frac{1}{2}hp$$

where $p = 4s$ is the perimeter. Substituting the values of h and p in terms of s into this formula yields s^2, which is what was expected.

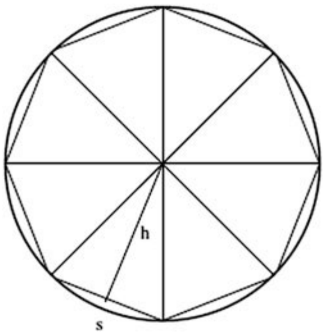


Fig. 9 Octagon for the proof of the area of a circle

Now consider a regular octagon (Fig. 9). If the octagon is divided into eight triangles, each of whose area is $\frac{1}{2}hs$. If all eight triangles are combined noting that $p = 8s$, the result is

$$A = \frac{1}{2}h(8s) = \frac{1}{2}hp.$$

This should be enough to establish the result that the area of any regular polygon is one-half the altitude to a side times the perimeter, or $\frac{1}{2}hp$.

Now what about a circle? If the number of sides of a regular polygon is repeatedly increased, the perimeter will approach the circumference of a circle, and the altitude will approach the radius of the circle. This suggests that the formula for the area of a circle should be

$$A = \frac{1}{2}rC$$

And since π is defined to be the ratio of the circumference of a circle to its diameter, or what amounts to the same thing, the ratio of the circumference to twice its radius, we have

$$\pi = \frac{C}{2r}$$

from which our familiar formula $C = 2\pi r$ follows. If this is substituted into the formula above for the area of a circle, the formula becomes the familiar

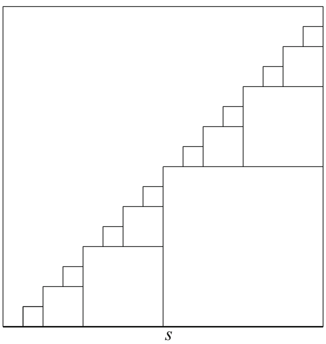
$$A = \frac{1}{2}r\,(2\pi r) = \pi r^2.$$

This must have seemed obvious to the ancient Greeks from an early period in the history of their geometry. But how could they prove it?

At one time, some of them argued that a circle is a regular polygon with infinitely many sides, but they eventually decided that this kind of reasoning is inadequate for mathematical proofs. For just because regular polygons with an increasing number of sides seem to be approaching a circle, deducing the formula for the area is not automatically justified. Evidence like this can be misleading.

Consider the following example: The length of the stepped line (Fig. 10) is clearly $2s$ no matter how many steps there are. But as the number of steps increases, the stepped line seems to approach the diagonal, and the length of the diagonal is $\sqrt{2}s \neq 2s$.

Fig. 10 The stepped line



Thus, although it must have been obvious to the ancient Greeks that the area of a circle is, in our terms, given by the formula

$$A = \frac{1}{2} r C$$

where r is the radius and C is the circumference, it was a long time before a proof was given of this fact. And before that proof was given, the following result, attributed to Eudoxus by Knorr (1975), appeared in Euclid's *Elements* (Heath 1926) as Proposition 2 of Book XII:

Proposition *(The areas of) circles are to one another as the squares on their diameters.*

The original proof of this result, along with a translation into a modern proof, is given in Seldin (1991).

Getting back to the area of a circle, the result on its area was finally published by Archimedes in a book called (in English) "Measurement of a Circle," which can be found in Heath (1912). The statement of this result, which in this book is Proposition 1, is as follows:

Proposition *The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius and the other to the circumference of the circle.*

This original proof, along with its transformation into a modern proof, is given in Seldin (1991).

At the meeting of the Canadian Society for History and Philosophy of Mathematics (CSHPM) in 1990, Judith Grabiner presented as an invited address (Grabiner 1997) on the work of the Scottish mathematician Colin Maclaurin in which she showed a page from a treatise on analysis in which he proved a limit result using exactly this approach.

He began by saying that if the limit were not the value given in his theorem, it must either be greater or less, and then deriving a contradiction from both of the latter two assumptions.

As Seldin showed (Seldin 1990), if these proofs are rewritten in algebraic notation and the absolute value function is used, the proofs are transformed into $\varepsilon - N$ proofs about the limits of sequences.

Once the $\varepsilon - N$ definition is available for the limit of a sequence, it is fairly easy to justify the $\varepsilon - \delta$ definition of the limit of a function. For saying that $|x - c| < \delta$ for some, δ is roughly like saying that $x > N$ for some N; i.e., that x is close to infinity.

Additional work with proofs using the $\varepsilon - \delta$ and $\varepsilon - N$ definitions for limits provides a foundation for considering the completeness axiom for the real numbers.

8 The Real Numbers

When Newton and Leibniz first published on calculus, they talked about "quantities," which were some-thing of an amalgam of the numbers and magnitudes of the ancient Greeks. These quantities included the positive integers and fractions. To get to the modern approach, the rational numbers need to be extended to the real numbers. This requires adding the Axiom of Completeness to the rational numbers.

To explain the Axiom of Completeness, it is useful to use the $\varepsilon - N$ definition of the limit of a sequence and to consider a strictly monotonically increasing infinite sequence with an upper bound. Intuitively, such a sequence must have a limit.

Now if a candidate for the limit is not an upper bound of all the terms of this sequence, it is not the limit of the sequence; for let ε be the difference between this candidate and a term of the sequence that is greater, and then all terms after this latter term will be further from the number than ε.

On the other hand, if the candidate is an upper bound and there is a lower upper bound, then let ε be the difference between these two upper bounds, and then all the terms of the sequence are further than ε away from the given number.

It follows that the limit of this sequence *must* be the least upper bound of the terms of the sequence.

But then, if we are limited to the rational numbers, there will be sequences like this that do not have limits. Dedikind (1965, p. 13) gives a strictly monotonically increasing sequence of rational numbers converging to $\sqrt{2}$. In the rational numbers, this sequence has no limit.

This makes it clear that to guarantee that every sequence of this kind has a limit, it is sufficient to have the Axiom of Completeness, which says that *every nonempty set of numbers that has an upper bound has a least upper bound*.

From this point, students can study some material on uniform convergence and then consider the Riemann integral (which involves a different kind of limiting process from that for functions and sequences).

9 Conclusion

The goal of our research project is to create materials (a book or pair of books) that exposes students to analysis while avoiding introducing definitions before they are used. This is part of our effort to avoid what Devlin called *formal definitions*.

To make the book as useful as possible, we have considered adding material on the topology of the real line. We also plan to add appendices on two methods of constructing the real numbers from the rational numbers: the first method using Dedekind cuts and the second method using Cauchy sequences. These examples are intended to motivate students to study more complex topics in a real analysis course.

We believe that this way of approaching the introductory analysis topics material has a good chance of helping some students understand this material who would not previously have been able to master it.

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