# A Loewner Matrix Approach to the Identification of Linear Time-Varying Systems

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**Abstract** We show that if enough "sufficiently informative" input-output trajectories generated by a linear, time-varying, finite-dimensional system and its dual are given, then *state* trajectories corresponding to them can be computed by factorizing a time-varying matrix directly constructed from the data. From such input-state-output trajectories an "unfalsified" linear time-varying model can be obtained solving a system of functional equations. Our approach is particularly relevant when the data-producing system is self-dual (e.g. if it is conservative).

**Keywords** Interpolation · Loewner matrices · Duality · Time-varying linear systems

## 1 Introduction

It is a pleasure for me to contribute to Prof. Antoulas' 70th birthday *Festschrift*. Thanos was a member of the Reading Committee of my Ph.D. dissertation, and I vividly recall meeting him for the first time on the occasion of my Ph.D. defence. His personal warmth and infectious enthusiasm for research have been a pleasant ingredient of all our interactions, and the elegance and powerful simplicity of his work have strongly influenced me in all stages of my career.

The present contribution is an *hommage* to some of Thanos's own pioneering ideas. In his work on rational interpolation and data modeling (see [1–5]) and part of his *opus* about model-order reduction (see e.g. [6]) he introduced the concept of mirroring of vector-exponential trajectories, and the all-important Loewner matrix. Many years later, Thanos and I showed in [7–9] that such concepts can also be expressed in the language of bilinear- and quadratic differential forms, and pointed out the role played by the concept of *duality* of trajectories and of systems in the Loewner approach. Such intuition makes explicit the connection between the approach to data modeling

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and model order reduction that I have independently pursued with various co-authors (see [10–15]) and Thanos's own.

With Thanos I applied Loewner and duality concepts to the modelling of two-dimensional vector-exponential trajectories and of parametric systems (see [16–18]). In this paper I present a further application of ideas inspired by his Loewner approach. Building on previous work by myself and collaborators (see [19, 20]), I consider the modelling of input-output data produced by linear, time-varying systems. In doing this I will emphasise the essential coherence of my recent and ongoing work with Thanos's original idea, thus paying a modest, but hopefully fitting tribute to his far-sighted intellect and to the strength of his intuition.

## 2 Problem Statement

We are given $N$ (a fixed, "large" number) of input-output trajectories

$$\begin{bmatrix} u_k(\cdot) \\ y_k(\cdot) \end{bmatrix} : \mathbb{R} \to \mathbb{R}^{m+p} \ , \ k = 1, \ldots, N \ , \tag{1}$$

generated by an unknown linear, time-varying state-space system

$$\begin{aligned} \frac{d}{dt} x(\cdot) &= A(\cdot)x(\cdot) + B(\cdot)u(\cdot) \\ y(\cdot) &= C(\cdot)x(\cdot) + D(\cdot)u(\cdot) \ . \end{aligned} \tag{2}$$

In (2) the matrix functions $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are respectively $n \times n$, $n \times m$, $p \times n$ and $p \times m$, with analytic entries. In the following we denote the space of $j \times k$ matrices with real analytic entries by $\mathcal{A}^{j \times k}$. In the rest of this paper we assume that the representation (2) is *controllable* and *observable*, see [25].

We are also given $N$ input-output trajectories

$$\begin{bmatrix} u'_k(\cdot) \\ y'_k(\cdot) \end{bmatrix} : \mathbb{R} \to \mathbb{R}^{p+m} \ , \ k = 1, \ldots, N \ , \tag{3}$$

of the *dual* system of (2), a state-space system with state variable $x'$ associated with analytic matrix functions $(A'(\cdot), B'(\cdot), C'(\cdot), D'(\cdot))$ defining similar equations as (1), such that the following property holds.

**Definition 1** (*Duality*) Two linear, time-varying state-space systems are *dual* if there exists a $n \times n$ matrix-function $Q(\cdot)$ with det $Q(t) \neq 0$ for all $t \in \mathbb{R}$, such that for every pair of trajectories $(x, u, y)$ satisfying (2) and $(x', u', y')$ satisfying the equations defined by $(A'(\cdot), B'(\cdot), C'(\cdot), D'(\cdot))$ it holds that

$$u^\top y' + y^\top u' = \frac{d}{dt}\left(x^\top Q x'\right) \ . \tag{4}$$

In the following we call (4) the *duality* relation.

**Remark 1** Definition 1 is analogous to the characteristic property of *adjoint* nonlinear system stated in Lemma 2.1 of [21]; see also [22] for the definition of this concept in the linear, time-invariant case, and Definition 1 and Proposition 1 p. 605 of [23] for the linear, time-varying case. It is a *power-conservation relation* such as that occurring in lossless systems. Consider for example the relation between the voltage and current pairs at the ports of LC electrical circuits (where the bilinear product on the left-hand side of (4) is electrical power), and the capacitor voltages and inductor currents (where the bilinear product on the right-hand side of (4) is the stored energy). Power-conservation relations are also at the core of the notion of port-Hamiltonian system (see [24]) and the approach presented here can also be applied to linear, time-varying, *dissipative* systems, provided that the bilinear form associated with the dissipation rate is known (see [20]). In Sect. 6 of this paper we illustrate several current and future research directions related to this aspect of our work.

We want to identify from the data (1) and (3) an *unfalsified* i-s-o model for the primal system, i.e. matrices $\widehat{A}(\cdot)$, $\widehat{B}(\cdot)$, $\widehat{C}(\cdot)$, $\widehat{D}(\cdot)$ such that input-state-output Eq. (2) hold for *some* (to be computed) state trajectories $x_k(\cdot)$, $k = 1, \ldots, N$ and the given $(u_k(\cdot), y_k(\cdot))$.

In this problem formulation it is assumed that data from both the primal system (2) *and* from its dual are known. While we recognize that it is unrealistic that measurements from the dual system are available, in order to emphasize the connections with the Loewner framework we shall find it convenient to work on the problem as stated. It is worthwhile to remark that for many classes of systems, *self-duality* is either implied by energy conservation or by other physical properties, for example those implied by port-Hamiltonicity.

## 3 A Loewner Matrix for Time-Varying Systems

We begin with the following fundamental result.

**Proposition 1** *Let $t \in \mathbb{R}$, and let $(x, u, y)$ and $(x', u', y')$ be trajectories of the primal, respectively of the dual system. Assume that $x(-\infty) := \lim_{t\to-\infty} x(t) = 0$ and $x'(-\infty) := \lim_{t\to-\infty} x'(t) = 0$; then*

$$\int_{-\infty}^{t} u(\tau)^\top y'(\tau) + y(\tau)^\top u'(\tau)\, d\tau = x(t)^\top Q(t)x'(t) = x(t)'^\top Q(t)x(t)\ . \quad (5)$$

***Proof*** Integrating both sides of (4), obtain

$$\int_{-\infty}^{t} u(\tau)^\top y'(\tau) + y(\tau)^\top u'(\tau)\, d\tau = x(t)^\top Q(t)x'(t) - x(-\infty)^\top Q(0)x'(-\infty)\ .$$

Now apply the assumption $x(-\infty) = 0 = x'(-\infty)$. □

**Definition 2** Assume that for every $t \in \mathbb{R}$ all trajectories $(x_k, u_k, y_k)$ and $(x_i', u_i', y_i')$ of the data (1), (3) satisfy $x_k(-\infty) = 0 = x_i'(-\infty)$. The *time-varying Loewner matrix* associated with the data (1), (3) is the $N \times N$ symmetric matrix function $E(\cdot) : (-\infty, t] \to \mathbb{R}^{N\times N}$ defined by

$$[E(t)]_{i,k=1,\dots N} := \int_{-\infty}^{t} u_k(\tau)^\top y_i'(\tau) + y_k(\tau)^\top u_i'(\tau)\, d\tau \ . \tag{6}$$

## 3.1 Relation with the Classical Loewner Matrix

In order to justify the terminology introduced in Definition 2, let us consider the case in which the data-generating system (and its dual) are *time-invariant*, and the primal and dual data consists of *vector-exponential* trajectories:

$$\begin{bmatrix} u_i'(t) \\ y_i'(t) \end{bmatrix} =: \begin{bmatrix} \overline{u}_i' \\ \overline{y}_i' \end{bmatrix} e^{\mu_i t} \text{ and } \begin{bmatrix} u_k(\cdot) \\ y_k(\cdot) \end{bmatrix} =: \begin{bmatrix} \overline{u}_k \\ \overline{y}_k \end{bmatrix} e^{\lambda_k t} \ , \tag{7}$$

with $\overline{u}_i', \overline{y}_k$ are $p$-dimensional constant real vectors, $\overline{y}_i', \overline{u}_k$ are $m$-dimensional constant real vectors, and $\mu_i, \lambda_k$ are positive real numbers, $i, k = 1, \dots, N$. The case of complex vectors and exponentials is easily dealt with, but the notation is slightly more involved and we do not consider it here. Note that the input-output trajectories (7) satisfy the condition $x_k(-\infty) = 0 = x_i'(-\infty)$, $i, k = 1, \dots, N$. In the rest of this chapter we assume that

$$\lambda_k + \mu_i \neq 0 \ , \ i, k = 1, \dots, N \ .$$

**Proposition 2** *Assume that the data-generating system and its dual are time-invariant, and that the data is of the form (7), with $\overline{u}_i', \overline{y}_k \in \mathbb{R}^p$, $\overline{y}_i', \overline{u}_k \in \mathbb{R}^m$, and $\mu_i, \lambda_k \in \mathbb{R}_+$, $i, k = 1, \dots, N$. Then*

$$[E(t)]_{i,k} = e^{(\lambda_k+\mu_i)t} \frac{\overline{u}_k^\top \overline{y}_i' + \overline{y}_k^\top \overline{u}_i'}{\lambda_k + \mu_i} \tag{8}$$

***Proof*** From the definition of the data, for every $k, i = 1, \dots, N$ it holds that

$$\begin{aligned} E_{k,i}(t) &= \int_{-\infty}^{t} e^{\lambda_k \tau} \overline{u}_k^\top \overline{y}_i' e^{\mu_i \tau} + e^{\lambda_k \tau} \overline{y}_k^\top \overline{u}_i' e^{\mu_i \tau} d\tau = \int_{-\infty}^{t} e^{(\lambda_k+\mu_i)\tau} \left(\overline{u}_k^\top \overline{y}_i' + \overline{y}_k^\top \overline{u}_i'\right) d\tau \\ &= \frac{e^{(\lambda_k+\mu_i)t}}{\lambda_k + \mu_i} \mid_{-\infty}^{t} \left(\overline{u}_k^\top \overline{y}_i' + \overline{y}_k^\top \overline{u}_i'\right) = e^{(\lambda_k+\mu_i)t} \frac{\overline{u}_k^\top \overline{y}_i' + \overline{y}_k^\top \overline{u}_i'}{\lambda_k + \mu_i} \ . \end{aligned}$$

□

Denote the transfer function of the primal system by $Y(s)$, and let $Y(s) = N(s)D(s)^{-1} = P(s)^{-1}Q(s)$ be respectively a right-coprime and a left-coprime factorization of $Y(s)$. The following result holds.

**Proposition 3** *The transfer function of the dual is*

$$Y'(s) := -Y(-s)^\top = -D(-s)^{-\top}N(-s)^\top = -Q(-s)^\top P(-s)^{-\top} .$$

***Proof*** The proof follows from the material in Sect. 10 of [26], in particular from Proposition 10.1 p. 1730. □

Observe that for $t = 0$ the result of Proposition 2 implies that

$$[E(0)]_{i,k} = \frac{\overline{u}_k^\top \overline{y}'_i + \overline{y}_k^\top \overline{u}'_i}{\lambda_k + \mu_i} \ , i, k = 1, \ldots, N \ . \tag{9}$$

Assume now that $m = p = 1$, i.e. the single-input, single-output case (the general case is considered in Sect. 2.3 of [7]); then one can assume without loss of generality that $\overline{u}_k = 1 = \overline{u}'_i$, $i, k = 1, \ldots, N$. It follows that the matrix $E(0)$ in (9) is the Loewner matrix $\mathbb{L}$ associated with the interpolation data

$$Z = \{-\mu_i\}_{i=1,\ldots,N} \cup \{\lambda_k\}_{k=1,\ldots,N} \text{ and } Y = \{y(-\mu_i)\}_{i=1,\ldots,N} \cup \{y(\lambda_k\}_{k=1,\ldots,N} \ ,$$

see p. 639 of [4].

Moreover, note that in the time-invariant case, since for each pair $(i, k)$, $i, k = 1, \ldots, N$ the associated frequencies $-\mu_i$ and $\lambda_k$ are known, the Loewner matrix (9) and its time-varying version (8) embody the same information. Observe also that from the result of Proposition 3 it follows that the trajectories of the dual system associated with a frequency $\mu_i$ can be computed by *mirroring*: namely, if

$$\begin{bmatrix} u_i \\ y_i \end{bmatrix} \in \mathbb{C}^{m+p}$$

arises from the primal transfer function evaluated at $-\mu_i$, then

$$\begin{bmatrix} u'_i \\ y'_i \end{bmatrix} := \begin{bmatrix} y_i \\ u_i \end{bmatrix} \in \mathbb{C}^{p+m}$$

arises from the dual transfer function evaluated at $\mu_i$. It follows that in the linear, time-invariant case there is no need to assume that the dual trajectories are available from measurements of the dual system, since they can be generated by mirroring.

## 4 Properties of the Time-Varying Loewner Matrix

In this section we follow the structure of Sect. 3: we first establish some properties of the time-varying Loewner matrix (6), and subsequently we show that they are generalisations of well-known properties of the time-invariant case established by Thanos and his collaborators.

The first result is a straightforward consequence of the definition.

**Proposition 4** *Let* $(x_k, u_k, y_k)$ *and* $(x'_i, u'_i, y'_i)$, $i, k = 1, \ldots, N$ *be i-s-o trajectories of the primal, respectively dual system. Assume that all trajectories* $(x_k, u_k, y_k)$ *and* $(x'_i, u'_i, y'_i)$ *satisfy the condition* $x_k(-\infty) = 0 = x'_i(-\infty)$, $i, k = 1, \ldots, N$. *Then for all* $t \in \mathbb{R}$ *it holds that*

$$E(t) = \underbrace{\begin{bmatrix} x'_1(t)^\top \\ \vdots \\ x'_N(t)^\top \end{bmatrix}}_{=:X'(t)^\top} Q(t) \underbrace{\begin{bmatrix} x_1(t) & \ldots & x_N(t) \end{bmatrix}}_{=:X(t)} . \tag{10}$$

***Proof*** The result follows in a straightforward way from Definition 2 and from Proposition 1. □

An important consequence of Proposition 4 is that for "almost all choices" of the input-output trajectories $(u_k, y_k)$ and $(u'_i, y'_i)$, $i, k = 1, \ldots, N$, and "almost all" $t \in \mathbb{R}$ the time-varying Loewner matrix has rank $n$, the dimension of the state space of the primal and dual system. Before proving this result, we need to formalize the concept of "almost all choices"; in order to do this, we use the algebraic notion of *algebraic genericity*.

Let $\mathcal{L}$ be some linear space of finite-dimension $d$; then given a basis $\{\ell_i\}_{i=1,\ldots,d}$ for $\mathcal{L}$, every $\ell \in \mathcal{L}$ can be written as $\ell = \sum_{i=1}^d x_i \ell_i$ for some coefficients $x_i$ in the field on which $\mathcal{L}$ is defined. A map $p : \mathcal{L} \to \mathbb{R}$ is a *polynomial* if $p(\ell)$ is a polynomial in the variables $x_i$, $i = 1, \ldots, d$. An *algebraic variety* is a subset $\mathcal{V}$ of $\mathcal{L}$ consisting of all zeroes of some polynomial $p$. A subset $\mathcal{S} \subset \mathcal{L}$ is called *generic* if there is a proper algebraic variety $\mathcal{V} \subsetneq \mathcal{L}$ such that $\mathcal{S} \supset (\mathcal{L} \setminus \mathcal{V})$.

With these definitions, we can now state the following important result.

**Theorem 1** *Assume that* $N > n$; *then for all* $t \in \mathbb{R}$ *it holds generically that rank* $E(t) =$ *rank* $E_{11}(t) = n$.

***Proof*** We show that rank $E(t) = n$; the equality rank $E(t) =$ rank $E_{11}(t)$ is a straightforward consequence of the fact that det $E_{11}(t)$ is one of the $n \times n$ minors of the $N \times N$ rank $n$ matrix $E(t)$. This minor is generically nonzero if rank$E(t) = n$.

Conclude from Eq. (10) and Definition 1 that

$$\text{rank } E(t) \leq \min \left\{ \text{rank } \begin{bmatrix} x_1(t) & \ldots & x_N(t) \end{bmatrix}, \text{rank } \begin{bmatrix} x'_1(t) & \ldots & x'_N(t) \end{bmatrix} \right\} .$$

We now show that generically rank $\begin{bmatrix} x_1(t) \ldots x_N(t) \end{bmatrix} = n$; an analogous argument yields that generically also rank $\begin{bmatrix} x_1'(t) \ldots x_N'(t) \end{bmatrix} = n$. Using the genericity assumption we will then prove the claim.

In order to show that generically rank $\begin{bmatrix} x_1(t) \ldots x_N(t) \end{bmatrix} = n$, we prove the following, stronger result, which will be useful later. □

**Lemma 1** *Let* $\begin{bmatrix} x_i(\cdot) \\ u_i(\cdot) \end{bmatrix}$, $i = 1, \ldots, N > n + m$ *be trajectories satisfying the state equation in (2). Then generically for all* $t \in \mathbb{R}$ *it holds that*

$$rank \begin{bmatrix} x_1(t) \ldots x_N(t) \\ u_1(t) \ldots u_N(t) \end{bmatrix} = n + m$$

***Proof*** To simplify the argument, we use a nonsingular transformation of the state-space basis as in [25] to transform the Eq. (2) to a more manageable version. Denote by $X(\cdot)$ a fundamental matrix solution of the free response, i.e. a matrix having full rank almost everywhere, such that solves the matrix differential equation $\frac{d}{dt}X(t) = A(t)X(t)$ for every $t \in \mathbb{R}$. Now let $\tau$ be a fixed, but otherwise arbitrary real number, and denote by $\phi(t, \tau)$ the transition matrix of (2) in the interval $[\tau, t]$, i.e. $\phi(t, \tau) = X(t)X(\tau)^{-1}$. Apply the transformation

$$x(\cdot) \to \phi(\cdot, \tau)^{-1}x(\cdot) = \phi(\tau, \cdot)x(\cdot) =: z(\cdot) ,$$

to the state variable $x$, and verify with straightforward manipulations that in this new basis of the state space, the state equation in (2) becomes

$$\frac{d}{dt}z(\cdot) = \phi(\tau, \cdot)B(\cdot)u(\cdot) . \tag{11}$$

We now prove that generically

$$\text{rank} \begin{bmatrix} z_1(t) \ldots z_N(t) \\ u_1(t) \ldots u_N(t) \end{bmatrix} = \begin{bmatrix} \phi(\tau, t) & 0_{n\times m} \\ 0_{m\times n} & I_m \end{bmatrix} \begin{bmatrix} x_1(t) \ldots x_N(t) \\ u_1(t) \ldots u_N(t) \end{bmatrix} = n + m ; \tag{12}$$

note that such equality implies the claim of the Lemma, since det $\begin{bmatrix} \phi(\tau, t) & 0_{n\times m} \\ 0_{m\times n} & I_m \end{bmatrix} \neq 0$.

To prove (12), we use the concept of *controllability* matrix of the matrix pair $(A(\cdot), B(\cdot))$ (see p. 66 of [25]). Define the sequence of $n \times m$ matrices

$$P_0(\cdot) := B(\cdot)$$
$$P_{k+1}(\cdot) := -A(\cdot)P_k(\cdot) + \frac{d}{dt}\left(P_k(\cdot)\right) ;$$

and from such sequence define the $n \times n \cdot m$ *controllability* matrix of the matrix pair $(A(\cdot), B(\cdot))$ by

$$C\left(A(\cdot),B(\cdot)\right) := \left[P_0(\cdot)\ P_1(\cdot)\ \dots\ P_{n-1}(\cdot)\right] \ . \tag{13}$$

It is straightforward to verify that from (11) it follows that

$$\frac{d^k z}{dt^k}(\cdot) = \phi(\tau,\cdot)\left[P_0(\cdot)\ \dots\ P_{k-1}(\cdot)\right]\begin{bmatrix}\frac{d^{k-1}u}{dt^{k-1}}(\cdot)\\ \vdots \\ u(\cdot)\end{bmatrix} ,$$

$k = 1, \dots$. The Wronskian matrix of $\mathrm{col}(z(\cdot), u(\cdot))$ (see p. 67 of [25]) is defined by

$$\mathcal{W}(z(\cdot),u(\cdot)) := \begin{bmatrix} z(\cdot) & \frac{dz}{dt}(\cdot) & \dots & \frac{d^{n+m-1}z}{dt^{n+m-1}}(\cdot) \\ u(\cdot) & \frac{d}{dt}u(\cdot) & \dots & \frac{d^{n+m-1}u}{dt^{n+m-1}}(\cdot)\end{bmatrix} .$$

Note that

$$\begin{aligned}&\mathcal{W}(z(\cdot),u(\cdot)) \\ &= \begin{bmatrix} z(\cdot) & \phi(\tau,\cdot)P_0(\cdot) & \phi(\tau,\cdot)P_1(\cdot) & \dots & \phi(\tau,\cdot)P_{n+m-1}(\cdot) \\ 0 & I_n & 0 & \dots & 0\end{bmatrix}\begin{bmatrix} I_n & 0 & 0 & \dots & 0 \\ 0 & u(\cdot) & \frac{du}{dt}(\cdot) & \dots & \frac{d^{n+m-1}u}{dt^{n+m-1}} \\ 0 & 0 & u(\cdot) & \dots & \frac{d^{n+m-2}u}{dt^{n+m-2}} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & u\end{bmatrix} .\end{aligned} \tag{14}$$

Since $u(\cdot)$ is arbitrarily chosen in the space of $m$-dimensional time functions, generically the second matrix on the right-hand side of (14) has full column rank at $t$. The first $n$ rows of the first matrix on the right-hand side of (14) contain as submatrix a nonsingular transformation (via $\phi(\tau,\cdot)$) of the controllability matrix (13), which has full row rank $n$ on a set of points everywhere dense on any finite interval of $\mathbb{R}$ (see Theorem 4 p. 69 of [25]). Consequently, it has generically full row rank.

From this argument it follows that for every $i = 1, \dots, n$, at every $t \in \mathbb{R}$ generically the Wronskian $\mathcal{W}(\mathrm{col}(z_i(t), u_i(t)))$ has rank $n+m$. It follows that generically the Wronskian of $\begin{bmatrix} z_1(t) & \dots & z_N(t) \\ u_1(t) & \dots & u_N(t)\end{bmatrix}$ also has rank $n+m$. □

We resume the proof of Theorem 1. Lemma 1 implies that the trajectories $z_i(\cdot)$ are linearly independent (see Lemma 3 p. 68 of [25]). Since they are obtained from a nonsingular transformation of the state trajectories $x_i(\cdot)$ of the primal system, it follows that generically

$$\mathrm{rank}\ \left[x_1(t)\ \dots\ x_N(t)\right] = n \ .$$

A result analogous to that of Lemma 1 can be established for the matrix constructed from the state and input trajectories of the dual system, namely that generically

$$\text{rank} \begin{bmatrix} x_1'(t) & \dots & x_N'(t) \\ u_1'(t) & \dots & u_N'(t) \end{bmatrix} = n + m \ ,$$

from which it follows that generically

$$\text{rank} \begin{bmatrix} x_1'(t) & \dots & x_N'(t) \end{bmatrix} = n \ .$$

Now observe that rank $E(t) < n$ if and only if

$$\text{im} \left( \begin{bmatrix} x_1'(t) & \dots & x_N'(t) \end{bmatrix} \right)^{\perp} \cap \ \text{im} \left( \begin{bmatrix} x_1(t) & \dots & x_N(t) \end{bmatrix} \right) \neq \{0\} \ ,$$

which is generically satisfied. Consequently, generically also rank $E(t) = n$, as was to be proved. □

The following result is a straightforward consequence of Propositions 4 and 1.

**Corollary 1** *Define*

$$\begin{aligned} U(t) &:= \begin{bmatrix} u_1(t) & \dots & u_N(t) \end{bmatrix} \ , \ Y(t) := \begin{bmatrix} y_1(t) & \dots & y_N(t) \end{bmatrix} \\ U'(t) &:= \begin{bmatrix} u_1'(t) & \dots & u_N'(t) \end{bmatrix} \ , \ Y'(t) := \begin{bmatrix} y_1'(t) & \dots & y_N'(t) \end{bmatrix} \ . \end{aligned}$$

*Then $E(\cdot)$ satisfies the matrix differential equation*

$$\frac{d}{dt} E(\cdot) = \begin{bmatrix} U'(\cdot)^{\top} & Y'(\cdot)^{\top} \end{bmatrix} Q(\cdot) \begin{bmatrix} U(\cdot)^{\top} \\ Y(\cdot)^{\top} \end{bmatrix} \tag{15}$$

***Proof*** The claim is a straightforward consequence of Propositions 4 and 1. □

The result of Proposition 4 and Theorem 1 can be used to compute an *unfalsified* first-order linear, time-varying model for the primal and dual data.

**Proposition 5** *The trajectories* $\begin{bmatrix} x_k \\ u_k \\ y_k \end{bmatrix}$, $k = 1, \dots, N$ *satisfy the first-order equations*

$$K(t) \frac{d}{dt} X(t) + F(t) X(t) + G(t) \begin{bmatrix} Y(t) \\ U(t) \end{bmatrix} = 0 \ , \tag{16}$$

*where*

$$\begin{aligned} K(\cdot) &:= X'(\cdot)^{\top} Q(\cdot) \\ F(\cdot) &:= \left( \frac{d}{dt} X'(\cdot) \right)^{\top} Q(\cdot) + X'(\cdot)^{\top} \left( \frac{d}{dt} Q(\cdot) \right) \\ G(\cdot) &:= \begin{bmatrix} -U'(t)^{\top} & -Y'(t)^{\top} \end{bmatrix} \ . \end{aligned}$$

*Moreover, generically for all $t \in \mathbb{R}$ the matrix $K(t) = X'(t)^\top Q(t)$ has a left inverse $K(t)^\dagger$, and the trajectories $\begin{bmatrix} x_k \\ u_k \\ y_k \end{bmatrix}$, $k = 1, \ldots, N$ also satisfy the equations*

$$\frac{d}{dt} X(t) = \left[-K(t)^\dagger F(t)\right] X(t) - K(t)^\dagger G(t) \begin{bmatrix} Y(t) \\ U(t) \end{bmatrix} .$$

***Proof*** Equation (16) follows by differentiating the left-hand side of the Eq. (15), using the equality (10).

The claim on the generic existence of a left inverse for $K(t)$ follows from the nonsingularity of $Q(t)$, and the fact that $X'(t)$ has generically full column rank because of the dual version of Lemma 1. The claim follows pre-multiplying both sides of (16) with such left-inverse matrix. □

## 4.1 Relation with the Time-Invariant Case

We consider the linear, time-invariant case, with the same notation and under the same assumptions as in Sect. 3.1, and we review the results illustrated in the linear, time-varying case.

The result of Proposition 4 follows in a straightforward way from an argument similar to that used in Proposition 10.1 p. 1730; note that the assumption therein is that the bases of the state-space of the primal and of the dual system are *matched*, i.e. that $Q(t) = I_n$. In the time-invariant case for every $k, i = 1, \ldots, N$ there exist constant vectors $\overline{x}_k, \overline{x}'_i$ such that

$$x_k(t) = \overline{x}_k e^{\lambda_k t} \text{and} \quad x'_i(t) = \overline{x}'_i e^{\mu_i t} \ ,$$

and consequently (10) can be written as

$$E(t) = \underbrace{\operatorname{diag}\left(e^{\mu_i t}\right)_{i=1,\ldots,N} \begin{bmatrix} \overline{x}'^\top_1 \\ \vdots \\ \overline{x}'^\top_N \end{bmatrix}}_{=:X'(t)^\top} \underbrace{Q\left[\overline{x}_1 \ \ldots \ \overline{x}_N\right] \operatorname{diag}\left(e^{\lambda_k t}\right)_{k=1,\ldots,N}}_{=:X(t)} . \qquad (17)$$

The result of Theorem 1 implies, for $t = 0$, that generically $E(0) = \mathbb{L}$ has rank $n$; in the classical Loewner framework this conclusion was proved in Lemma 2.1 p. 639 of [4].

The result of Proposition 5 is equivalent to that of Proposition 3.1 p. 641 of [4]. Indeed, for the time-invariant case it holds that

$$\frac{d}{dt}\left(X'(t)^\top\right) = \frac{d}{dt}\left(\text{diag}\left(e^{\mu_i t}\right)_{i=1,\ldots,N}\begin{bmatrix}\overline{x}'^\top_1\\ \vdots \\ \overline{x}'^\top_N\end{bmatrix}\right) = \underbrace{\text{diag}\left(\mu_i\right)_{i=1,\ldots,N}}_{=:M} X'(t)^\top$$

$$\frac{d}{dt}\left(X(t)\right) = \frac{d}{dt}\left(\begin{bmatrix}\overline{x}_1 & \ldots & \overline{x}_N\end{bmatrix}\text{diag}\left(e^{\lambda_k t}\right)_{k=1,\ldots,N}\right) = X(t)\underbrace{\text{diag}\left(\lambda_k\right)_{k=1,\ldots,N}}_{=:\Lambda},$$

and the "shifted Loewner matrix" equals $\frac{d}{dt}E(t)\mid_{t=0}$.

## 5 From Loewner Matrix to State Equations

In the classical, time-invariant approach, the Loewner matrix is used to compute a transfer-function model for the data (i.e. a rational interpolant). If $\mathbb{L}$ is of full rank this can be performed directly (see e.g. Theorem 4.1 p. 642 of [4]); in the general case, the factorization of a matrix computed from the Loewner one is necessary (see assumption (20) p. 645 and Theorem 5.1 p. 646 of [4]). We illustrated analogous procedures in [13–16, 18]. We now show that also in the time-varying case similar results hold.

First, some terminology. Given $E(\cdot):\mathbb{R}\to\mathbb{R}^{N\times N}$, we call $E(\cdot) = Z'(\cdot)^\top R(\cdot)Z(\cdot)$ a *rank-revealing* factorisation if $Z':\mathbb{R}\to\mathbb{R}^{r\times N}$, $Z:\mathbb{R}\to\mathbb{R}^{r\times N}$, $R:\mathbb{R}\to\mathbb{R}^{r\times r}$, with $r :=$ rank $E(t) =$ rank $Z'(t) =$ rank $Z(t) =$ rank $R(t)$ on a set of points everywhere dense on any finite interval of $\mathbb{R}$.

We now show that minimal state trajectories corresponding to given input-output data can be computed from a rank-revealing factorisation of the Loewner matrix.

**Theorem 2** *Let* (2) *be a minimal representation of a LTV system and let* $(u_k, y_k, x_k)$ *satisfy* (2)*,* $k = 1,\ldots,N$*. Assume that all trajectories* $(x_k, u_k, y_k)$ *and* $(x'_i, u'_i, y'_i)$ *are such that* $x_k(-\infty) = 0 = x'_i(-\infty)$*, and define* $E(\cdot)$ *by* (6)*. Assume that rank* $E(t) = n$ *for all* $t\in\mathbb{R}$*. Define*

$$\mathcal{B} := \left\{\begin{bmatrix}u\\y\end{bmatrix} \mid \exists\, x \text{ such that (2) holds}\right\}.$$

*Let* $E(\cdot) = Z'(\cdot)^\top R(\cdot)Z(\cdot)$ *be a rank-revealing factorisation of* $E(\cdot)$*. Denote the kth column of Z by* $z_k$*; then there exists an i-s-o representation with state variable z such that* $(u_k, y_k, z_k)$ *satisfies the equations,* $k = 1,\ldots,N$*.*

***Proof*** The claim is evidently correct for the factorisation (10) of $E(t)$ in Proposition 4. Use a standard linear-algebraic argument to conclude that for every $t\in\mathbb{R}$, any rank-revealing factorisation $E(t) = Z'(t)^\top R(t)Z(t)$ is related to (10) by

$$Z(t) = T(t)X(t) \text{and } Z'(t) = T'(t)Z(t),$$

where $T, T'(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n\times n}$ are nonsingular almost everywhere, and $T'(t)^\top R(t)T(t) = R(t)$. This proves that the $k$th column $z_k$ of $Z(\cdot)$ is a state trajectory corresponding to $(u_k, y_k)$, $k = 1, \ldots, N$. It can be verified that the i-s-o representation (2) corresponding to such state variable is induced by the matrices

$$\left(T(\cdot)(A(\cdot)T(\cdot)^{-1} - \frac{d}{dt}\left(T(\cdot)^{-1}\right), T(\cdot)B(\cdot), C(\cdot)T(\cdot)^{-1}, D(\cdot)\right) .$$

□

In the time-invariant case, matrix factorization can be performed with standard numerical algorithms (e.g. the singular value decomposition); however, in the time-varying case, the factorization of a *matrix function* of dimension $N$ is necessary. This is a non-trivial problem with considerable implications of a theoretical and computational nature. We now show that in the generic case, the result of Theorem 1 can be used to achieve a factorization of $E(\cdot)$ by inverting its $(1, 1)$ $n \times n$ submatrix. As in Theorem 1, we partition the Loewner matrix as

$$E(t) = \begin{bmatrix} E_{11}(t) & E_{12}(t) \\ E_{21}(t) & E_{22}(t) \end{bmatrix} , \tag{18}$$

where $E_{11}(t)$ is $n \times n$, with $n = \text{rank}(E(t))$, $E_{12}(t)$ is $n \times (N - n)$, $E_{21}(t)$ is $(N - n) \times n$, and $E_{22}(t)$ is $(N - n) \times (N - n)$. The following result holds.

**Theorem 3** *Let $E(\cdot)$ be partitioned as in (18); generically at $t$ the following equation holds:*

$$E(t) = \begin{bmatrix} I_n \\ E_{21}(t)E_{11}(t)^{-1} \end{bmatrix} E_{11}(t) \begin{bmatrix} I_n & E_{11}(t)^{-1}E_{12}(t) \end{bmatrix} . \tag{19}$$

***Proof*** The fact that $E_{11}(t)$ is generically nonsingular is stated in Theorem 1; consequently, the claim follows if we prove that $E_{22}(t) = E_{21}(t)E_{11}(t)^{-1}E_{12}(t)$. Consider the equality

$$\begin{bmatrix} I_n & 0_{n\times N} \\ -E_{21}(t)^{-1}E_{11}(t) & I_{N-n} \end{bmatrix} \begin{bmatrix} E_{11}(t) & E_{12}(t) \\ E_{21}(t) & E_{22}(t) \end{bmatrix} \begin{bmatrix} I_n & -E_{11}(t)^{-1}E_{12}(t) \\ 0_{(N-n)} \times n & I_{N-n} \end{bmatrix}$$
$$= \begin{bmatrix} E_{11}(t) & 0_{n\times n} \\ 0_{(N-n)\times n} & E_{22}(t) - E_{21}(t)E_{11}(t)^{-1}E_{12}(t) \end{bmatrix} .$$

Now apply Theorem 1 to conclude that since the matrix on the right-hand side of the equation has rank $n = \text{rank}(E_{11}(t))$, the $(2, 2)$-block must be zero. □

From Theorems 2 and 3 it follows that the $k$th column of any of the two right-factors

$$\begin{bmatrix} E_{11}(\cdot) & E_{12}(\cdot) \end{bmatrix} \text{ or } \begin{bmatrix} I_n & E_{11}(\cdot)^{-1}E_{12}(\cdot) \end{bmatrix}$$

of $E(\cdot)$ can be used to associate a state trajectory to the input-output data $\begin{bmatrix} u_k(\cdot) \\ y_k(\cdot) \end{bmatrix}$; without loss of generality in the following we discuss on the basis of the first choice. The derivatives of these state trajectories are the columns of the matrix

$$\left[\tfrac{d}{dt}E_{11}(\cdot)\ \tfrac{d}{dt}E_{12}(\cdot)\right]\ ;$$

define also the matrices

$$\begin{aligned} U(\cdot) &:= \left[u_1(\cdot)\ \dots\ u_N(\cdot)\right] =: \left[U_1(\cdot)\ U_2(\cdot)\right] \\ Y(\cdot) &:= \left[y_1(\cdot)\ \dots\ y_N(\cdot)\right] =: \left[Y_1(\cdot)\ Y_2(\cdot)\right]\ , \end{aligned}$$

with $U_1$ a $m \times n$ matrix function, $U_2$ $m \times (N-n)$ , $Y_1$ $p \times n$, and $Y_2$ $p \times (N-n)$. Theorem 2 implies that matrix functions $\widehat{A}(\cdot)$, $\widehat{B}(\cdot)$, $\widehat{C}(\cdot)$, $\widehat{D}(\cdot)$ exist such that the following equation holds true:

$$\begin{bmatrix} \frac{d}{dt}E_{11}(\cdot) & \frac{d}{dt}E_{12}(\cdot) \\ Y_1(\cdot) & Y_2(\cdot) \end{bmatrix} = \begin{bmatrix} \widehat{A}(\cdot) & \widehat{B}(\cdot) \\ \widehat{C}(\cdot) & \widehat{D}(\cdot) \end{bmatrix} \begin{bmatrix} E_{11}(\cdot) & E_{12}(\cdot) \\ U_1(\cdot) & U_2(\cdot) \end{bmatrix}\ . \tag{20}$$

The following result states how such an unfalsified model can be computed.

**Theorem 4** *Generically for every $t \in \mathbb{R}$ there exists a right-inverse of*

$$\begin{bmatrix} E_{11}(t) & E_{12}(t) \\ U_1(t) & U_2(t) \end{bmatrix}\ , \tag{21}$$

*denoted in the following by*

$$\begin{bmatrix} E_{11}(t) & E_{12}(t) \\ U_1(t) & U_2(t) \end{bmatrix}^{\dagger}\ .$$

*Consequently, for every $t \in \mathbb{R}$ the values $\widehat{A}(t)$, $\widehat{B}(t)$, $\widehat{C}(t)$, $\widehat{D}(t)$ of an unfalsified state-space model (20) for the data (1) can be computed via the equation*

$$\begin{bmatrix} \widehat{A}(t) & \widehat{B}(t) \\ \widehat{C}(t) & \widehat{D}(t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}E_{11}(t) & \frac{d}{dt}E_{12}(t) \\ Y_1(t) & Y_2(t) \end{bmatrix} \begin{bmatrix} E_{11}(t) & E_{12}(t) \\ U_1(t) & U_2(t) \end{bmatrix}^{\dagger}\ .$$

***Proof*** The first part of the claim follows from Theorem 2 and Lemma 1. The second part follows multiplying both sides of Eq. (20) on the right by the right-inverse of (21). □

## 6 Conclusions and Ongoing Research

We have generalised the rational interpolation approach based on the classical (constant) Loewner matrix, originally developed by Thanos and his collaborators, to the case of data generated by a time-varying system, by introducing a time-varying Loewner matrix. Time and again (see Sects. 3.1 and 4.1) we have shown that the results we obtain in our framework are *mutatis mutandis* generalisations of those already known in the classical one. The festive occasion dictated that we concentrate on illustrating such common features, but a couple of remarks are in order, if only to emphasise that Thanos's intuition is more far-reaching than perhaps even he realises.

Firstly, we note that our identification approach is conceptually analogous to that of *subspace identification* (see [27, 28]); from data we first compute state trajectories, and on the basis of those and the measurements we compute state equations. In the rational interpolation setting the intermediate step is neither conceptually nor computationally necessary, since the problem formulation is different. However, an explicit computation of state trajectories through the factorisation of Loewner matrices opens up several interesting opportunities. Since different factorizations generate different state trajectories and consequently different bases of the state space, "special" factorizations can be performed to achieve special (e.g. balanced) state representations. Moreover, besides an "exact" factorization, also an approximate one (obtained for example selecting only a lower rank approximation of a matrix, as happens in the case of singular value decompositions) can be performed. This opens up the possibility of performing model-order reduction of time-varying systems from data (see [13] for the linear, time-invariant case).

Secondly, we emphasize that *duality* plays an essential role in our approach, whether explicitly (as in the present contribution and in [7, 16, 19]) or implicitly, where we have investigated the identification of conservative (see [13–15]) or port-Hamiltonian systems (see [20]). In the latter cases duality arises as a consequence of either *energy conservation* or *power* relations and their effect on the system dynamics. Such properties arise in a variety of systems, for example nonlinear ones, on which our current research is focused. Many of the current techniques to compute nonlinear models from data seem to fall in the class of *parametric*, rather than *system* identification methods: the structure of the model is known, and system identification is reduced to determining numerical values for the undetermined parameters, so that the resulting model explains the given measurements. The potential of a duality (equivalently: energy-, or power relation-based) point of view lies in providing procedures that can identify the *structure* of a system.

Finally, we point out the interesting problems arising from the application of the abstract mathematical approach outlined here to the real world. Computing continuous-time models from sampled data while maintaining fundamental system properties and structures is an important current area of research in system identification (see [29]). In the framework presented here, analogous issues arise in the use of suitable numerical procedures to compute integrals, and in the factorization of matrices of functions, that are guaranteed to produce e.g. self-dual systems.

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