The Conditioning of a Linear Barycentric Rational Interpolant

Jean-Paul Berrut

Abstract We study what a referee has asked us to call the "second Berrut rational interpolant", introduced thirty years ago, which is linear in the interpolated function values and converges exponentially when the nodes are conformal images of Chebyshev points. We show that the Lebesgue constant for this interpolant grows just logarithmically with the number of nodes under reasonable assumptions, extending results of Bos et al. in 2013 for the corresponding first interpolant.

Keywords Linear rational interpolation · Conditioning · Logarithmic growth · Lebesgue constant · Conformally shifted nodes

1 Introduction

Polynomial interpolation has been a standard computational tool in applied mathematics at least since the time of Newton, Waring and Lagrange [12]. In its simplest version, it consists in associating to a vector $\mathbf{x} = [x_0, \ldots, x_n]^T$ of, say, $n + 1$ distinct abscissae, and a vector $\mathbf{f} = [f_0, \ldots, f_n]^T$ of corresponding ordinates, which may be values of a function f or not, the unique polynomial $p_n[\mathbf{f}]$ of degree at most n that takes on the value f_j at x_j, $j = 0, \ldots, n$. The corresponding mapping has several nice properties. It is linear: the polynomial interpolating a linear combination of $\mathbf{f}$-vectors is the linear combination of the corresponding polynomials, i.e., $p_n[\alpha \mathbf{f} + \beta \mathbf{g}] = \alpha p_n[\mathbf{f}] + \beta p_n[\mathbf{g}]$, $\forall \, \alpha, \beta \in \mathbf{R}$; moreover, the uniqueness makes it a projection, a property that permits the study of the conditioning by means of the Lebesgue constant.

But it has its drawbacks, the main one being the fact that it is useful for large n only if the nodes are located in particular ways in the interpolation interval $[a, b] \equiv$

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[min x_j, max x_j] [19, Chap. 5]. Long before the advent of electronic computing, for instance, Runge showed that, for the most important case of equidistant nodes, the interpolant of a meromorphic function converges only if the poles of the latter lie outside a certain region containing $[a, b]$ [12]. And even for functions, such as e^x, for which the interpolant converges for equidistant points, it is extremely ill-conditioned for n large [13, Sect. 5.5].

It is therefore appropriate to look for alternative infinitely smooth interpolants which work well for general sets of nodes. In [3], the author took advantage of the observation by Werner [22] that, by modifying the so-called *weights* λ_j in the barycentric formula

$$p_n[\mathbf{f}](x) = \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j} f_j \Bigg/ \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j}, \qquad \lambda_j := 1 \Bigg/ \prod_{k \neq j} (x_j - x_k), \quad (1)$$

for p_n, one obtains a rational function, say $r_n[f]$, of degree $\leq n$, i.e., with both numerator and denominator degrees at most n; the new suggestion was to choose weights β_j different from the λ_j to obtain a good denominator for the given set of nodes, instead of the unique denominator 1 for all sets of nodes in polynomial interpolation. That way the two properties of linearity and projection are preserved [4].

A good choice of the β_j requires the ordering of the nodes according to

$$a = x_0 < x_1 < \cdots < x_n = b \, ; \tag{2}$$

indeed, the weights should alternate in sign, for otherwise the interpolant will not reproduce all linear functions [14]. In view of this, the simplest possible choice of the weights, as proposed in [3], is

$$\beta_j = (-1)^j, \tag{3}$$

independently of the values of the nodes. We denote the corresponding interpolant by R_0; the referee has suggested that this should be called the "first Berrut rational interpolant" (see [14]). It was shown in [3] that R_0 has no real poles; moreover, its $O(h)$-convergence (where $h := \max_{i=0,\ldots,n-1}(x_{i+1} - x_i)$) and well-conditioning were conjectured. The $O(h)$-convergence was proved in [11] under a local mesh ratio condition.

For an approximation operator which is a projection, the conditioning is best addressed through its Lebesgue constant Λ_n (see Sect. 2). For R_0, the latter has been shown to increase only logarithmically for the most frequently used sets of nodes [9], a behaviour as nice as that of polynomial interpolation with the most favourable nodes (see also [21]).

However, by considering the even rational trigonometric interpolant obtained by transferring the polynomial onto the circle with the transform $x = \cos \phi$ [3], one comes to the conclusion that the first and last terms of the sums in both the numerator

and the denominator of the interpolant should appear only once and the other terms twice. After simplifying by a factor of two, one sees that the weights [3]

$$\beta_j = (-1)^j \delta_j, \qquad \delta_j := \begin{cases} 1/2, & j = 0 \text{ or } j = n, \\ 1, & \text{otherwise}, \end{cases} \tag{4}$$

should be better than (3). (To be precise, this formula for the weights holds in the present case, where $x_0 = a$ and $x_n = b$, i.e., where the interval $[x_0, x_n]$ covers the whole interval of interpolation: when the interpolant is to be used beyond one or both extremities, slightly more complicated formulae involving trigonometric functions should be used, see [3].) *We denote the corresponding interpolant by R_1*, the "second Berrut rational interpolant". When the nodes are the Chebyshev points of the second kind, the weights (4) coincide with the λ_j [13, p. 252], $R_1 \equiv p_n[\mathbf{f}]$, and exponential convergence is achieved if f is analytic in an ellipse containing $[a, b]$ [20, p. 57]. It was shown in [3] that R_1 does not have any poles in $[a, b]$ for any set of nodes, and conjectured in the same work that R_1 is well-conditioned. Additionally, it was proved in [2] that R_1 retains the exponential convergence when the x_j are conformally shifted Chebyshev nodes, and conjectured that the convergence is $O(h^2)$ in general; conditions guaranteeing the latter are given in [5].

In 2007, Floater and Hormann [11] presented a family of interpolants with a higher order of convergence, which they denote by $d + 1$, $d \in \mathbf{N}^0$. For every d and every set of nodes $\mathbf{x}$, these authors give a linear rational interpolant $r_n^{(d)}[\mathbf{f}]$, depending on d, such that $\|r_n^{(d)}[\mathbf{f}] - f\| = O(h^{d+1})$. After constructing the interpolant as a blend of the $n - d$ interpolating polynomials corresponding to the subvectors of $d + 1$ consecutive nodes, they provide the weights of its barycentric representation (which always exists, as shown in [7]). For equidistant nodes, $r_n^{(0)}[\mathbf{f}]$ and $r_n^{(1)}[\mathbf{f}]$ correspond to R_0 and R_1, respectively; however, this is not the case with other sets of nodes.

In the present paper, we study the Lebesgue constant of R_1 for equidistant and well-spaced nodes such as conformally shifted Chebyshev points of the second kind, which are the most important in practice. Our results are extensions of results of Bos et al. [9] for R_0.

2 The Conditioning of the Interpolant R_1

The conditioning of a problem is the sensitivity of its solution to perturbations of its data. (In many contexts, this is called stability, but in numerical analysis the latter term should be reserved for the stability of an algorithm [13, p. 33].) Here the data are the function values $\mathbf{f}$ and the solution is the interpolant

$$R_1(x) = \sum_{j=0}^{n}{}'' \frac{(-1)^j f_j}{x - x_j} \bigg/ \sum_{i=0}^{n}{}'' \frac{(-1)^j}{x - x_j}, \tag{5}$$

where the double prime attached to a sum indicates that its first and last terms are halved.

The fact that R_1 is a linear projection operator into a linear space of functions with a Lagrange basis (i.e., a basis $\{b_j(x)\}$ with the Lagrange property $b_j(x_i) = \delta_{ij}$) leads to the fact that the norm of that operator, called its Lebesgue constant, provides a good characterization of the conditioning. Indeed, if one perturbs each f_j by a number e_j, i.e., perturbing the vector $\mathbf{f}$ with the vector $\mathbf{e} := (e_j)$, then for a linear interpolant $r_n[\mathbf{f}]$

$$r_n[\mathbf{f} + \mathbf{e}](x) = \sum_{j=0}^{n}(f_j + e_j)b_j(x).$$

The corresponding perturbation of the interpolant amounts to

$$r_n[\mathbf{f} + \mathbf{e}](x) - r_n[\mathbf{f}](x) = \sum_{j=0}^{n} e_j b_j(x)$$

so that

$$\left| r_n[\mathbf{f} + \mathbf{e}](x) - r_n[\mathbf{f}](x) \right| \leq \sum_{j=0}^{n} |e_j| \cdot |b_j(x)|$$

$$\leq \|\mathbf{e}\| \sum_{j=0}^{n} |b_j(x)| \qquad (6)$$

where $\| \cdot \|$ denotes the vector infinity norm.

Equation (6) gives a local bound for the perturbation induced by $\mathbf{e}$. The function

$$\lambda_n(x) := \sum_{j=0}^{n} |b_j(x)|$$

is called the *Lebesgue function* of the interpolation operator, its infinity norm

$$\Lambda_n := \|\lambda_n\|$$

the *Lebesgue constant*. (It turns out that Λ_n is the norm of the interpolation operator on the space of continuous functions.) Equation (6) then yields

$$\|r_n[\mathbf{f} + \mathbf{e}] - r_n[\mathbf{f}]\| \leq \Lambda_n \cdot \|\mathbf{e}\|;$$

r_n being linear, this means $\|r_n[\mathbf{e}]\|/\|\mathbf{e}\| \leq \Lambda_n$; the global relative perturbation of the interpolant is bounded by Λ_n.

Definition 1 We say that Λ_n grows at most logarithmically with n, if

$$\Lambda_n \leq a + b \ln n \qquad (7)$$

for some positive constants a and b and every $n > n_0$, $n_0 \in \mathbf{N}$.

The rational interpolant $r_n[\mathbf{f}]$ obtained by replacing the weights λ_j in (1) by some β_j independent of $\mathbf{f}$ can be written as $\sum_{j=0}^n f_j b_j(x)$ with the Lagrange basis functions

$$b_j(x) := \frac{\beta_j}{x - x_j} \Big/ \sum_{i=0}^n \frac{\beta_i}{x - x_i},$$

so that [7]

$$\lambda_n(x) = \sum_{j=0}^n \left| \frac{\beta_j}{x - x_j} \right| \Big/ \left| \sum_{i=0}^n \frac{\beta_i}{x - x_i} \right|. \qquad (8)$$

For R_1 this yields

$$\lambda_n(x) = \sum_{i=0}^n {}'' \frac{1}{|x - x_i|} \Big/ D(x),$$

where

$$D(x) := \left| \sum_{i=0}^n {}'' \frac{(-1)^i}{x - x_i} \right|.$$

If x is one of the x_j's, a limit calculation in (8) shows that $\lambda_n(x) = 1$. Otherwise, let x_k be the node at the immediate left of x, i.e., $x \in (x_k, x_{k+1})$. If k is odd and $\leq n - 5$, then

$$D(x) = \cdots - \frac{\delta_{k-1}}{x - x_{k-1}} + \frac{1}{x - x_k} + \frac{1}{x_{k+1} - x} - \frac{1}{x_{k+2} - x} + \cdots ,$$

where δ_j is defined in (4). Therefore we may write the numerator of $\lambda_n(x)$ as $D(x) + 2N(x)$, where

$$N(x) = \cdots + \frac{\delta_{k-3}}{x - x_{k-3}} + \frac{\delta_{k-1}}{x - x_{k-1}} + \frac{1}{x_{k+2} - x} + \frac{1}{x_{k+4} - x} + \cdots$$

(setting $\delta_j = 0$ for $j < 0$) to obtain

$$\lambda_n(x) = 1 + 2N(x)/D(x).$$

Note that all terms of N are positive. If k is even, one may multiply all terms in D and N by -1 to get the same expression for λ_n.

At first sight, it seems that, in view of the factors $1/2$ arising in the first and last terms, one must study the two extremal intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$ separately from the others. However, a shift by $\frac{x_0+x_n}{2} =: m$ of the interval $[x_0, x_n]$ together with the abscissae x_j, i.e.,

$$\widetilde{x}_j := x_j - m, \qquad \widetilde{x} := x - m$$

(so that m becomes 0) without change of the weights just shifts $\lambda_n(x)$. Moreover, a reflection $\widehat{x}_j := -x_j$ and $\widehat{x} := -x$ reflects $\lambda_n(x)$ as well. Also, $\lambda_n(x)$ is unchanged by a multiplication of all weights by -1. Thus, if there are at least two points on each side of m, we may bound $\lambda_n(x)$ only on $[x_0, m]$: bounds for $x \in (m, x_n]$ are obtained after a reflection with respect to m and a multiplication of the weights by -1, if n is odd. This shows that it suffices to treat just the case $x \in [x_0, x_1]$ separately, and explains why the δ_j's appear to the left only in D and N above.

Lemma 1 *For $x \in (x_k, x_{k+1})$, $1 \leq k \leq \left[\frac{n}{2}\right]$,*

$$D(x) \geq A_1 + A_2, \tag{9}$$

where

$$A_1 = \frac{x_k - x_{k-1}}{(x_{k+1} - x_{k-1})(x_{k+1} - x_k)}, \qquad A_2 = \frac{x_{k+2} - x_{k+1}}{(x_{k+1} - x_k)(x_{k+2} - x_k)},$$

and

$$N(x) \leq B_1 + B_2 \tag{10}$$

where

$$B_1 = \sideset{}{'}\sum_{i \geq 0} \frac{1}{x_k - x_{k-1-2i}}, \qquad B_2 = \sideset{}{'}\sum_{i \geq 0} \frac{1}{x_{k+2+2i} - x_{k+1}}$$

and the prime means that, if present, the terms $1/(x_k - x_0)$, resp. $1/(x_n - x_{k+1})$, are to be multiplied by $1/2$.

Proof For the denominator,

$$D(x) \geq D_1(x) + D_2(x),$$

where

$$D_1(x) := -\frac{\delta_{k-1}}{x - x_{k-1}} + \frac{1}{x - x_k} \geq \frac{x_k - x_{k-1}}{(x - x_{k-1})(x - x_k)},$$

and

$$D_2(x) := \frac{1}{x_{k+1} - x} - \frac{1}{x_{k+2} - x} = \frac{x_{k+2} - x_{k+1}}{(x_{k+1} - x)(x_{k+2} - x)}.$$

(In D, all pairs of terms corresponding to nodes to the left of x_{k-1} are positive, and so is the term corresponding to x_0 if k is even; the same holds accordingly to the

right of x_{k+1}. When $k = 1$, the factor δ_0 in front of $1/(x - x_0)$ makes D even larger.) Since D_1 and D_2 are decreasing, respectively increasing, for $x \in (x_k, x_{k+1})$,

$$D_1(x) \geq D_1(x_{k+1}) = A_1, \qquad D_2(x) \geq D_2(x_k) = A_2.$$

For the numerator,
$$N(x) = N_1(x) + N_2(x),$$

where
$$N_1(x) = \cdots + \frac{\delta_{k-3}}{x - x_{k-3}} + \frac{\delta_{k-1}}{x - x_{k-1}},$$

and
$$N_2(x) = \frac{1}{x_{k+2} - x} + \frac{1}{x_{k+4} - x} + \cdots.$$

Since N_1 and N_2 are decreasing and increasing, respectively, for $x \in (x_k, x_{k+1})$,

$$N_1(x) \leq N_1(x_k) = B_1, \qquad N_2(x) \leq N_2(x_{k+1}) = B_2.$$

∎

One proves in a similar way the following corresponding results for $k = 0$.

Lemma 2 *For $x \in (x_0, x_1)$,*
$$D(x) \geq A_1 + A_2, \tag{11}$$

where
$$A_1 = \frac{1/2}{x_1 - x_0}, \qquad A_2 = \frac{x_2 - x_1}{(x_1 - x_0)(x_2 - x_0)},$$

and
$$N(x) \leq B_2 \tag{12}$$

where
$$B_2 = \sum_{i \geq 0}{}' \frac{1}{x_{2i+2} - x_1}.$$

2.1 Equidistant Nodes

For uniformly spaced points $x_k = a + kh$, $h = (b - a)/n$, the lemmas imply that $D(x) \geq 1/h$ for every $x \in [a, b]$. Moreover,

$$B_1 \leq \frac{1}{h}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{\ell}\right), \qquad \ell = \begin{cases} k - 1, & k \text{ even}, \\ k, & k \text{ odd}, \end{cases}$$

and

$$B_2 \leq \frac{1}{h}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{\ell}\right), \qquad \ell = \begin{cases} n - k - 1, & n - k \text{ even,} \\ n - k - 2, & n - k \text{ odd,} \end{cases}$$

so that $N(x) < \frac{1}{h}L$ with

$$L := \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-k-1}\right). \tag{13}$$

As in similar proofs, we now use the fact that $\ln(2j + 1)$ is an upper bound of the jth sum of the harmonic series, so that

$$L \leq \ln(2k + 1) + \ln(2(n - k - 1) + 1),$$

which is maximized over real k in $[0, n - 1]$ by $k = (n - 1)/2$, implying that

$$L \leq 2\ln(n), \tag{14}$$

and therefore,

$$\lambda_n(x) \leq 1 + 4\ln(n).$$

Theorem 1 *For uniformly spaced nodes, the Lebesgue constant of the interpolant R_1 grows at most logarithmically with n.*

This result is already known as the case $d = 1$ of [8]. However, the above proof of the bound for D does not make use of the original formula for the Floater–Hormann interpolant, merely of the barycentric formula.

We have confirmed this logarithmic growth numerically. The second column of Table 1 displays Λ_n for doubling numbers of interpolation nodes. To check the constant b in the logarithmic growth, we have also computed after every pair $(n, 2n)$ an approximation $\widehat{a}$ and $\widehat{b}$ in the logarithmic growth formula by solving the system of equations

$$\widehat{a} + \widehat{b}\ln(2n + 2) = \Lambda_{2n+2}$$
$$\widehat{a} + \widehat{b}\ln(n + 1) = \Lambda_{n+1}.$$

The corresponding values of $\widehat{b}$ are given in the third column of the table. Compared with the number $2/\pi = 0.6366197723675814$, they are in line with the main term of the asymptotic formula for the Lebesgue constant of R_0 given in [15, 23]. (All the computations described in this article were performed in MATLAB on MacBook Pro computers.)

Table 1 The Lebesgue constant Λ_n for equidistant and shifted equidistant nodes. For each of the two we give the constant itself as well as the empirical value $\widehat{b}$ of the factor of $\ln(n)$

$n+1$	Equidistant		Shifted equidistant	
	Λ_n^{E}	$\widehat{b}^{\mathrm{E}}$	Λ_n^{SE}	$\widehat{b}^{\mathrm{SE}}$
10	2.2220		2.2989	
20	2.6839	0.66644	2.7574	0.66136
40	3.1414	0.65992	3.2185	0.66531
80	3.5904	0.64781	3.6687	0.64944
160	4.0357	0.64244	4.1142	0.64272
320	4.4789	0.63947	4.5575	0.63960
640	4.9212	0.63809	4.9998	0.63808
1280	5.3630	0.63735	5.4416	0.63735
2560	5.8045	0.63698	5.8831	0.63698
5120	6.2459	0.63680	6.3245	0.63680
10240	6.6873	0.63671	6.7658	0.63671

2.2 Well-Spaced Nodes

The article [9] introduces the following property of interpolation points.

Definition 2 For each $n \in \mathbf{N}$, let $X_n := \{x_k\}_{k=0}^n$ be a set of interpolation nodes satisfying property (2) and let

$$h_k := x_{k+1} - x_k, \qquad k = 0, \ldots, n-1.$$

An array $X = (X_n)_{n \in \mathbf{N}}$ of such sets is called *well-spaced* if there exist two constants K and R, $R \geq 1$, both independent of n, such that the three conditions

$$\frac{x_k - x_{k-1}}{x_k - x_j} \leq \frac{K}{k-j}, \qquad k = 1, \ldots, n, \quad j = 0, \ldots, k-1, \tag{15}$$

$$\frac{x_{k+1} - x_k}{x_j - x_k} \leq \frac{K}{j-k}, \qquad k = 0, \ldots, n-1, \quad j = k+1, \ldots, n, \tag{16}$$

$$\frac{h_k}{h_{k-1}} \leq R, \quad \frac{h_{k-1}}{h_k} \leq R, \qquad k = 1, \ldots, n-1, \tag{17}$$

hold for every set of nodes X_n.

We shall indifferently call the array and the nodes well-spaced. Note that the property of well-spacing is invariant with respect to translation or spacing of the nodes.

The authors of [9] established that every such array leads to a logarithmic growth of the Lebesgue constant of R_0:

Theorem 2 *If $X = (X_n)_{n \in \mathbf{N}}$ is an array of well-spaced nodes, then the Lebesgue constant of the interpolant R_0 grows at most logarithmically with n.*

We shall now use the above lemmas to show the same for R_1. Let first $k \geq 1$. Then, according to (17),

$$A_1 = \frac{h_{k-1}}{(x_{k+1} - x_{k-1})h_k} \geq \frac{1}{x_{k+1} - x_{k-1}} \cdot \frac{1}{R}$$

and

$$A_2 = \frac{h_{k+1}}{h_k(x_{k+2} - x_k)} \geq \frac{1}{R} \cdot \frac{1}{x_{k+2} - x_k}$$

so that

$$A_1 + A_2 \geq \frac{1}{R}\left(\frac{1}{x_{k+2} - x_k} + \frac{1}{x_{k+1} - x_{k-1}}\right). \tag{18}$$

Moreover, $x_{k+2} - x_k = h_k + h_{k+1} \leq R(h_{k-1} + h_k) = R(x_{k+1} - x_{k-1})$ and $x_{k+1} - x_{k-1} = h_{k-1} + h_k \leq (R + 1)h_k$, so that

$$A_1 + A_2 \geq \frac{1}{R}\left(\frac{1}{R} + 1\right)\frac{1}{x_{k+1} - x_{k-1}} \geq \frac{R+1}{R^2}\frac{1}{(R+1)h_k} = \frac{1}{R^2 h_k}. \tag{19}$$

If $k = 0$, we have in Lemma 2

$$A_1 + A_2 \geq A_1 = \frac{1}{2h_0}.$$

On the whole, we therefore have for $x \in (x_k, x_{k+1})$

$$D(x) \geq \frac{1}{Sh_k}$$

with

$$S := \begin{cases} 1, & k = 0, \\ R^2, & k > 0. \end{cases}$$

In the upper bound for N, possible factors $1/2$ may be ignored. We compute for $k \neq 0$

$$\begin{aligned}
B_1 &= \frac{1}{x_k - x_{k-1}} + \frac{1}{x_k - x_{k-3}} + \frac{1}{x_k - x_{k-5}} + \frac{1}{x_k - x_{k-7}} + \cdots \\
&= \frac{1}{x_k - x_{k-1}}\left(1 + \frac{x_k - x_{k-1}}{x_k - x_{k-3}} + \frac{x_k - x_{k-1}}{x_k - x_{k-5}} + \frac{x_k - x_{k-1}}{x_k - x_{k-7}} + \cdots\right).
\end{aligned}$$

By (15), this may be bounded as

$$B_1 \leq \frac{1}{h_{k-1}} \left(1 + \frac{K}{3} + \frac{K}{5} + \frac{K}{7} + \cdots \right).$$

Similarly for B_2:

$$
\begin{aligned}
B_2 &= \frac{1}{2} \frac{1}{x_{k+2} - x_{k+1}} + \frac{1}{x_{k+4} - x_{k+1}} + \frac{1}{x_{k+6} - x_{k+1}} + \frac{1}{x_{k+8} - x_{k+1}} + \cdots \\
&= \frac{1}{x_{k+2} - x_{k+1}} \left(\frac{1}{2} + \frac{x_{k+2} - x_{k+1}}{x_{k+4} - x_{k+1}} + \frac{x_{k+2} - x_{k+1}}{x_{k+6} - x_{k+1}} + \frac{x_{k+2} - x_{k+1}}{x_{k+8} - x_{k+1}} + \cdots \right),
\end{aligned}
$$

which by (16) may be majorized as

$$B_2 \leq \frac{1}{h_{k+1}} \left(1 + \frac{K}{3} + \frac{K}{5} + \frac{K}{7} + \cdots \right)$$

with the same number of terms as in Sect. 2.1, so that with (13)

$$N(x) \leq \left(\frac{1}{h_{k-1}} + \frac{1}{h_{k+1}} \right) K L$$

and thus with (19)

$$
\begin{aligned}
\lambda_n(x) &\leq 1 + 2R^2 \left(\frac{h_k}{h_{k-1}} + \frac{h_k}{h_{k+1}} \right) 2K \ln(n) \\
&\leq 1 + 8K R^3 \ln(n).
\end{aligned}
$$

If $k = 0$, $B_1 = 0$ and $B_2 \leq \frac{1}{h_1} \left(1 + \frac{K}{3} + \frac{K}{5} + \cdots \right)$ so that

$$N(x) \leq \frac{2}{h_1} K \ln(n)$$

and

$$\lambda_n(x) \leq 1 + \frac{h_0}{h_1} 4K \ln(n) \leq 1 + 4K R \ln(n).$$

This establishes our first main theorem.

Theorem 3 *If $X = (X_n)_{n \in \mathbf{N}}$ is an array of well-spaced nodes, then the Lebesgue constant of the interpolant R_1 grows at most logarithmically with n.*

2.3   *Conformally Shifted Nodes*

The complexity of the pseudospectral solution of a boundary value problem can often be drastically reduced by a move of the points, for instance to improve the conditioning of the system of equations to be solved, or to have more points close to a steep gradient (front).

In [16], Kosloff and Tal-Ezer suggested applying such point shifts to loosen the step restriction in the solution of the systems of ordinary differential equations arising from the pseudospectral method of lines applied to time evolution problems. The idea is to introduce, besides the interval I on which the function is to be interpolated, another interval J containing the standard points (such as equidistant or Chebyshev). Then, to obtain a particular set of points in I, a (one-to-one) conformal map of a domain containing J, say g, is used, which applies J onto I, thereby determining the desired shifted points. Since g is infinitely differentiable, the composite function $f \circ g$ inherits the differentiability properties of f.

Recall that R_1 coincides with the interpolating polynomial, when the nodes are Chebyshev points of the second kind, so that it converges exponentially to interpolated functions which are analytic. Recall also that R_1 retains the exponential convergence for the same class of functions, when the Chebyshev points of the second kind are shifted conformally [2]. This example of use of R_1 is likely the most important, as it greatly improves the accuracy of pseudospectral solutions of boundary value ordinary and partial differential equations whose solution displays large gradients: see the applications in explosion modeling [18], fluid infiltration [10] and finance [17]. We shall now see how the results of Bos et al. [9] and the above theorems imply the logarithmic growth of the Lebesgue constant for such nodes.

Bos et al. work on the interval [0, 1]. This is indeed general, as one immediately sees that a shift and a contraction/dilation of the interval of interpolation do not modify the Lebesgue function (8) and the Lebesgue constant is thus unaffected by such operations. These authors give the following definition of a (regular) distribution function of nodes.

Definition 3 A function $F \in C[0, 1]$ is a *distribution function*, if it is a strictly increasing bijection of the interval [0, 1]. F is *regular*, if $F \in C^1[0, 1]$ and F' has a finite number of zeros in [0, 1] with finite multiplicities.

Given a distribution function F and some $n \in \mathbf{N}$, the authors define the associated interpolation nodes $X_n = X_n(F)$ as the set of x_k with

$$x_k := F(k/n), \qquad k = 0, 1, \ldots, n. \tag{20}$$

They then prove the following result.

Theorem 4 *If F is a regular distribution function and X_n are the associated interpolation nodes from (20) for every $n \in \mathbf{N}$, then the array of nodes $X = (X_n)_{n \in \mathbf{N}}$ is well-spaced.*

Since a conformal map g cannot have any zero derivative in its domain [1, p. 133], Theorem 2 automatically yields the following results.

Corollary 1 *The Lebesgue constants of R_0 as well as of R_1 with conformally shifted* equidistant *points grow at most logarithmically with n.*

Proof For R_0, the result follows immediately from Theorem 3.5 in [9] and Theorem 2 above, for R_1 from Theorem 3. ∎

Among their examples of regular distributions, Bos and al. give the extrema of the Chebyshev polynomials, which are the Chebyshev points of the second kind.

Lemma 3 *A composition of regular distribution functions is itself a regular distribution function.*

Proof This follows directly from the chain rule. ∎

Corollary 2 *The Lebesgue constant of R_1 with conformally shifted* Chebyshev points *of the second kind grows at most logarithmically with n.*

Proof Let $g(y)$ be the conformal map of the interval containing the Chebyshev points to the interpolation interval. g being conformal, its derivative does not have any zeros, so that g is a regular distribution function on J. Then $g \circ F$ is a composition of regular distribution functions, thus a regular distribution function as well, according to Lemma 3. The corresponding array of nodes is well-spaced according to Theorem 4 and the Lebesgue constant of R_1 with a well-spaced array grows at most logarithmically (Theorem 3). ∎

We have numerically verified this logarithmic behavior of the Lebesgue constant with conformally shifted points. To increase the efficiency of methods for solving differential problems, it is natural to move more points toward the location of a front. A simple map for the case where the front lies in the center of the interval $[-1, 1]$ is that suggested by Kosloff and Tal-Ezer for another purpose and mentioned above,

$$x = g_\alpha(y) = \frac{\arcsin(\alpha y)}{\arcsin(\alpha)}, \qquad 0 < \alpha < 1. \tag{21}$$

See the improvement in accuracy in [6]. In the limit $\alpha \to 0$, the nodes remain the equidistant ones. In the limit $\alpha \to 1$, the singularities at $\pm 1/\alpha$ of g_α tend toward the extremities of the interval.

First we have repeated the experiment documented in columns 2 and 3 of Table 1 for *equidistant points* shifted with the map (21) and $\alpha = 0.9$. The results are in columns 4 and 5: they demonstrate that the constant b in (7) seems quite independent of the shift.

Then we have repeated the experiment of Table 1, but now with Chebyshev points of the second kind and shifted Chebyshev points of the second kind. The results, given in Table 2, seem to show that the main term of the Lebesgue constant does not change much with a conformal shift of Chebyshev nodes, and is very close to the one corresponding to equidistant points.

Table 2 The Lebesgue constant Λ_n for Chebyshev points of the second kind, and the same shifted. The values of $\widehat{b}$ have the same meaning as in Table 1

$n+1$	Chebyshev 2nd kind		Shifted Chebyshev	
	Λ_n^{C}	$\widehat{b}^{\text{C}}$	Λ_n^{SC}	$\widehat{b}^{\text{SC}}$
10	2.3619		2.5474	
20	2.8371	0.68568	3.0219	0.68452
40	3.2948	0.66034	3.4794	0.66008
80	3.7442	0.64829	3.9287	0.64823
160	4.1895	0.64241	4.3740	0.64239
320	4.6328	0.63950	4.8173	0.63950
640	5.0750	0.63806	5.2595	0.63806
1280	5.5168	0.63734	5.7013	0.63735
2560	5.9583	0.63698	6.1428	0.63698
5120	6.3997	0.63680	6.5842	0.63680
10240	6.8410	0.63671	7.0256	0.63671

3   Conclusion

This article has been devoted to the study of the behaviour, as the number of nodes increases, of the second "Berrut" interpolant introduced some decades ago. We have shown that, at least in the instances of practical interest, the interpolant enjoys the same perfect condition of its special case of the usual Chebyshev interpolating polynomial, and likewise of its first-order variant R_0, with a Lebesgue constant growing at most logarithmically. This is true in particular of the important case of the shifted Chebyshev nodes used in practical applications, and justifies their use with arbitrary large numbers of nodes.

References

1. Ahlfors, L.: Complex Analysis, 2nd edn. McGraw-Hill, New York (1966)
2. Baltensperger, R., Berrut, J.-P., Noël, B.: Exponential convergence of a linear rational interpolant between transformed Chebyshev points. Math. Comp. **68**, 1109–1120 (1999)
3. Berrut, J.-P.: Rational functions for guaranteed and experimentally well-conditioned global interpolation. Comput. Math. Appl. **15**, 1–16 (1988)
4. Berrut, J.-P.: Linear rational interpolation of continuous functions over an interval. In: Gautschi, W. (ed.) Mathematics of Computation 1943–1993: A Half–Century of Computational Mathematics. Proceedings of Symposia in Applied Mathematics, pp. 261–264. American Mathematical Society, Providence (1994)

5. Berrut, J.-P.: Properties of a barycentric rational interpolant. Rend. Sem. Mat. Univ. Politec. Torino **76**, 39–45 (2018)
6. Berrut, J.-P., Baltensperger, R.: The linear rational pseudospectral method for boundary value problems. BIT **41**, 868–879 (2001)
7. Berrut, J.-P., Mittelmann, H.D.: Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval. Comput. Math. Appl. **33**, 77–86 (1997)
8. Bos, L., De Marchi, S., Hormann, K., Klein, G.: On the Lebesgue constant of barycentric rational interpolation at equidistant nodes. Numer. Math. **121**, 461–471 (2012)
9. Bos, L., De Marchi, S., Hormann, K., Sidon, J.: Bounding the Lebesgue constant for Berrut's rational interpolant at general nodes. J. Approx. Theory **169**, 7–22 (2013)
10. Cueto-Felgueroso, L., Juanes, R.: Adaptive rational spectral methods for the linear stability analysis of nonlinear fourth-order problems. J. Comput. Phys. **228**, 6536–6552 (2009)
11. Floater, M.S., Hormann, K.: Barycentric rational interpolation with no poles and high rates of approximation. Numer. Math. **107**, 315–331 (2007)
12. Gautschi, W.: Interpolation before and after Lagrange. Rend. Semin. Mat. Univ. Politec. Torino **70**, 347–368 (2012)
13. Henrici, P.: Essentials of Numerical Analysis with Pocket Calculator Demonstrations. Wiley, New York (1982)
14. Hormann, K.: Barycentric interpolation. In: Fasshauer, G.E., Schumaker, L.L. (eds.) Approximation Theory XIV: San Antonio 2013. Springer Proceedings in Mathematics & Statistics, vol. 83, pp. 197–218. Springer, New York (2014)
15. Ibrahimoglu, B.A., Cuyt, A.: Sharp bounds for Lebesgue constants of barycentric rational interpolation at equidistant points. Exp. Math. **25**, 347–354 (2016)
16. Kosloff, D., Tal-Ezer, H.: A modified Chebyshev pseudospectral method with an $\mathcal{O}(N^{-1})$ time step restriction. J. Comput. Phys. **104**, 457–469 (1993)
17. Pindza, E., Patidar, K., Ngounda, E.: Rational spectral collocation method for pricing American vanilla and butterfly spread options. In: Dimov, I., Faragó, I., Vulkov, L. (eds.) Finite Difference Methods, Theory and Applications. Lecture Notes in Computer Science, vol. 9045, pp. 323–331. Springer, Cham (2015)
18. Tee, T.W., Trefethen, L.N.: A rational spectral collocation method with adaptively transformed Chebyshev grid points. SIAM J. Sci. Comput. **28**, 1798–1811 (2006)
19. Trefethen, L.N.: Spectral Methods in MATLAB. SIAM, Philadelphia (2000)
20. Trefethen, L.N.: Approximation Theory and Approximation Practice. SIAM, Philadelphia (2013)
21. Vértesi, P.: On barycentric interpolation. I. (On the T-Lebesgue function and T-Lebesgue constant). Acta Math. Hungar. 147, 396–407 (2015)
22. Werner, W.: Polynomial interpolation: Lagrange versus Newton. Math. Comp. **43**, 205–217 (1984)
23. Zhang, R.-J.: Optimal asymptotic Lebesgue constant of Berrut's rational interpolation operator for equidistant nodes. Appl. Math. Comput. **294**, 139–145 (2017)