# Structure-Preserving Interpolatory Model Reduction for Port-Hamiltonian Differential-Algebraic Systems


**Christopher Beattie, Serkan Gugercin, and Volker Mehrmann**

*Dedicated to Thanos Antoulas on the occasion of his 70th birthday*

**Abstract** We examine interpolatory model reduction methods that are particularly well-suited for treating large-scale port-Hamiltonian differential-algebraic systems. We are able to take advantage of underlying structural features of the system in a way that preserves them in the reduced model, using approaches that incorporate regularization and a prudent selection of interpolation data. We focus on linear time-invariant systems and present a systematic treatment of a variety of model classes that include combinations of index-1 and index-2 systems, describing in particular how constraints may be represented in the transfer function so that the polynomial part can be preserved with interpolatory methods. We propose an algorithm to generate effective interpolatory models and illustrate its effectiveness on a numerical example.

**Keywords** Port-Hamiltonian descriptor system · Model reduction · Tangential interpolation · Regularization of descriptor system · Staircase form

C. Beattie · S. Gugercin
Department of Mathematics, Virginia Tech, Blacksburg 24061-0123, USA
e-mail: beattie@vt.edu

S. Gugercin
e-mail: gugercin@vt.edu

V. Mehrmann (✉)
Institut für Mathematik, MA 4-5, Technische Universität Berlin, 10623 Berlin, Germany
e-mail: mehrmann@math.tu-berlin.de



C. Beattie et al. (eds.), *Realization and Model Reduction of Dynamical Systems*,
https://doi.org/10.1007/978-3-030-95157-3_13

## 1 Introduction

Port-Hamiltonian (pH) systems are network-based models that arise within a modeling framework in which a physical system model is decomposed into hierarchies of submodels interconnected principally through the exchange of energy. The submodels typically reflect one of a variety of core modeling paradigms that describe phenomenological aspects of the dynamics having different physical character, such as e.g. electrical, thermodynamic, or mechanical. The pH framework is able to knit together submodels featuring dramatically different physics through a disciplined focus on energy flux as a principal mode of system interconnection. pH structure is inherited via power conserving interconnection and a variety of physical properties and functional constraints (e.g., passivity and energy and momentum conservation) are encoded directly into the structure of the model equations [5, 30]. Interconnection of submodels may create further constraints on system behavior and evolution, originating as conservation laws (e.g., Kirchoff's laws or mass balance), or as position and velocity limitations in mechanical systems. As a result, system models are often naturally posed as combinations of dynamical system equations and algebraic constraint equations, i.e., as *port-Hamiltonian descriptor systems* or *port-Hamiltonian differential-algebraic equations* (pHDAE).

When a pHDAE system is linearized around a stationary solution, one obtains a linear time-invariant pHDAE with specially structured coefficient matrices, see [5]. Although the approach we develop here can be extended easily to more general settings, we narrow our focus to a particular formulation offered as one of the simpler among a variety of formulations presented in [5, 22].

**Definition 1** A linear time-invariant DAE system of the form

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\,\mathbf{x} + (\mathbf{B}-\mathbf{P})\,\mathbf{u}, \\ \mathbf{y} &= (\mathbf{B}+\mathbf{P})^T\,\mathbf{x} + (\mathbf{S}+\mathbf{N})\,\mathbf{u}, \end{aligned} \qquad \mathbf{x}(t_0) = 0, \tag{1}$$

with $\mathbf{E}$, $\mathbf{J}$, $\mathbf{R} \in \mathbb{R}^{n\times n}$, $\mathbf{B}, \mathbf{P} \in \mathbb{R}^{n\times m}$, $\mathbf{S} = \mathbf{S}^T$, $\mathbf{N} = -\mathbf{N}^T \in \mathbb{R}^{m\times m}$, on a compact interval $\mathbb{I} \subset \mathbb{R}$, is a pHDAE system if the following properties are satisfied:

1. The differential-algebraic operator

$$\mathbf{E}\frac{d}{dt} - \mathbf{J} : C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n)$$

   is *skew-adjoint*, i.e., $\mathbf{J}^T = -\mathbf{J}$ and $\mathbf{E} = \mathbf{E}^T$,
2. $\mathbf{E}$ is positive semidefinite, i.e., $\mathbf{E} \geq 0$, and
3. the *passivity* matrix

$$\mathbf{W} = \begin{bmatrix} \mathbf{R} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{S} \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$

   is symmetric positive semi-definite, i.e., $\mathbf{W} = \mathbf{W}^T \geq 0$.

The *Hamiltonian function* $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ of the system is $\mathcal{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{E}\mathbf{x}$. $\mathcal{H}$ is a quadratic function of state describing energy storage within the system. The matrix $\mathbf{E}$, which is the Hessian matrix of $\mathcal{H}(\mathbf{x})$, is the *energy matrix*; $\mathbf{R}$ is the *dissipation matrix*; $\mathbf{J}$ is the *structure matrix* describing the energy flux among internal energy storage elements; and $\mathbf{B} \pm \mathbf{P}$ are *port matrices* describing how energy enters and leaves the system. $\mathbf{S}$ and $\mathbf{N}$ are matrices associated with a *direct feed-through* from input $\mathbf{u}$ to output $\mathbf{y}$. If the system has a solution $\mathbf{x} \in C^1(\mathbb{I}, \mathbb{R}^n)$ in $\mathbb{I}$ for a given input function $\mathbf{u}$, then the dissipation inequality

$$\frac{d}{dt}\mathcal{H}(\mathbf{x}) = \mathbf{u}^T\mathbf{y} - \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix}^T \mathbf{W} \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} \leq \mathbf{u}^T\mathbf{y}$$

must hold. The system is both *passive* and *Lyapunov stable*, as $\mathcal{H}$ defines a *Lyapunov function* when $\mathbf{u} = 0$, see [5].

In practice, network-based automated modeling tools such as MODELICA, MATLAB/SIMULINK, or 20-sim[1] may produce system models that are overdetermined as a consequence of redundant modeling. Thus, automated generation of a system model usually must be followed by reformulation and regularization. Evidently, these subsequent steps should preserve the pH structure and retain compatibility with standard simulation, control, and optimization tools.

pHDAE models may be very large and complex, e.g., models that arise from semi-discretized (in space) continuum models in hydrodynamics [11, 19, 29], or mechanics [25]. In such cases, model reduction techniques are necessary to apply control and optimization methods. Preservation of the pH structure, the constraint structure, and the interconnection structure is necessary to maintain model integrity. For linear time-invariant pH systems with positive-definite $\mathbf{E}$, such methods are well-developed, including tangential interpolation approaches [15, 16] and moment matching [26, 28, 31], as well as effort and flow constraint reduction methods [27]. Interpolatory approaches have been extended to nonlinear pH systems as well, see [3] and [9].

In the case of singular $\mathbf{E}$, only recently in [11, 18] have model reduction techniques been introduced that are able to preserve pH structure. Note that for unstructured DAEs, constraint-preserving reduction methods were introduced in [17, 19, 25, 29].

In this work, we discuss particular structure-preserving model reduction methods that incorporate both regularization and interpolation for linear pHDAE systems having the form (1). For different model classes, we describe how constraints are represented in the transfer function and how the polynomial part can be preserved with interpolatory model reduction methods. The key step identifies, as in [5, 24], all redundancies as well as both explicit and implicit system constraints, partitioning the system equations into redundant, algebraic, and dynamic parts. Then, only the dynamic part need be reduced in a way that preserves the structure.

In Sect. 3, we consider pHDAEs of the form (1) and note important simplifications of the general regularization procedure. Structure preserving model reduction of

[1] https://www.modelica.org/, http://www.mathworks.com, https://www.20sim.com.

pHDAEs using interpolatory methods is discussed in Sect. 4 for several specific model structures. We propose an algorithm to generate effective interpolation data in Sect. 5 followed by a numerical example that demonstrates its effectiveness.

## 2 General Differential-Algebraic Systems

In this section, we review some basic properties of linear constant coefficient DAE systems having the general form,

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \end{aligned} \qquad \mathbf{x}(t_0) = 0, \tag{2}$$

where $\mathbf{E}$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$; for details, see, e.g., [7].

The matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is said to be *regular*, if $\det(\lambda\mathbf{E} - \mathbf{A}) \neq 0$ for some $\lambda \in \mathbb{C}$. For regular pencils, the *finite eigenvalues* are the values $\lambda \in \mathbb{C}$ for which $\det(\lambda\mathbf{E} - \mathbf{A}) = 0$. If the *reversed pencil* $\lambda\mathbf{A} - \mathbf{E}$ has the eigenvalue 0, then this is the *infinite eigenvalue* of $\lambda\mathbf{E} - \mathbf{A}$.

With zero initial conditions, $\mathbf{x}(t_0) = 0$ as in (2), and a regular pencil, $\lambda\mathbf{E} - \mathbf{A}$, we obtain in the frequency domain, the *transfer function*, $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$. $\mathbf{H}(s)$ is a rational function mapping the Laplace transform of the input function $\mathbf{u}$ to the Laplace transform of the output function $\mathbf{y}$. If the transfer function is written as

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \mathbf{G}(s) + \mathbf{P}(s),$$

where $\mathbf{G}(s)$ is a proper rational matrix function and $\mathbf{P}(s)$ is a polynomial matrix function, then the finite eigenvalues are the poles of the proper rational part, $\mathbf{G}(s)$, while the infinite eigenvalues are associated with the polynomial part, $\mathbf{P}(s)$.

Regular pencils can be analyzed via the Weierstraß Canonical Form; see, e.g., [13]. The *index*, $\nu$, of the pencil $\lambda\mathbf{E} - \mathbf{A}$ is the size of the largest block associated with the eigenvalue $\infty$ in the Weierstraß canonical form; if $\mathbf{E}$ is nonsingular, then $\nu = 0$.

To analyze general DAEs and to understand the properties of the transfer function, conditions on controllability and observability are needed, see, e.g., [7, 10]. Let $S_\infty(\mathbf{E})$ and $T_\infty(\mathbf{E})$ be two matrices with orthonormal columns spanning, respectively, the right and left nullspace of $\mathbf{E}$. A system is called *strongly controllable* if it satisfies

C1: $\operatorname{rank}[\lambda\mathbf{E} - \mathbf{A},\ \mathbf{B}] = n$ for all $\lambda \in \mathbb{C}$ and C2: $\operatorname{rank}[\mathbf{E},\ \mathbf{A}S_\infty(\mathbf{M}),\ \mathbf{B}] = n$.

Analogously, a system is called *strongly observable* if it satisfies

O1: $\operatorname{rank}\begin{bmatrix} \lambda\mathbf{E}^T - \mathbf{A}^T & \mathbf{C}^T \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$ and O2: $\operatorname{rank}\begin{bmatrix} \mathbf{E}^T & \mathbf{A}^T T_\infty(\mathbf{E}) & \mathbf{C}^T \end{bmatrix} = n$.

If a system is strongly controllable and strongly observable, then it is called *minimal*.

Conditions (C1) and (C2) are preserved under non-singular equivalence transformations as well as under state and output feedback. Note, however, that regularity (or non-regularity) of the pencil, of the index, and of the polynomial part of the transfer function are generally not preserved under state or output feedback. For systems that satisfy (C2), there exists a suitable linear state feedback matrix, $\mathbf{F}_1$, such that $\lambda\mathbf{E} - (\mathbf{A} + \mathbf{B}\mathbf{F}_1)$ is regular and of index at most one. Also if conditions (C2) and (O2) hold, then there exists a linear output feedback matrix, $\mathbf{F}_2$, so that the pencil $\lambda\mathbf{E} - (\mathbf{A} + \mathbf{B}\mathbf{F}_2\mathbf{C})$ has this property, see [7].

## 3 Regularization of PHDAE Systems

In general, it cannot be guaranteed that a system, generated either from a realization procedure or an automated modeling procedure, has a regular pencil $\lambda\mathbf{E} - \mathbf{A}$. Therefore, a regularization procedure typically must be applied, see [6, 8, 20]. For pHDAEs, pH structure helps simplify these regularization procedures significantly, since the pencil $\lambda\mathbf{E} - (\mathbf{J} - \mathbf{R})$ is associated with a free *dissipative Hamiltonian* DAE system (i.e., $\mathbf{u} = 0$). Such systems have many nice properties [22]: The index $\nu$ is at most two; all eigenvalues are in the closed left half plane; and all eigenvalues on the imaginary axis are semi-simple (except for possibly the eigenvalue 0, which then may have Jordan blocks of size at most two). A singular pencil can occur only when $\mathbf{E}$, $\mathbf{J}$, and $\mathbf{R}$ have a common nullspace [23], so if one is able to compute efficiently this common nullspace, it is possible to remove explicitly the singular part.

**Lemma 1** *For the pHDAE in (1), there exists an orthogonal basis transformation matrix* $\mathbf{V} \in \mathbb{R}^{n\times n}$ *such that in the new variable* $\widetilde{\mathbf{x}} = \begin{bmatrix}\widetilde{\mathbf{x}}_1^T & \widetilde{\mathbf{x}}_2^T & \widetilde{\mathbf{x}}_3^T\end{bmatrix}^T = \mathbf{V}^T\mathbf{x}$*, the system has the form*

$$\begin{aligned}\begin{bmatrix}\mathbf{E}_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\widetilde{\mathbf{x}}}_1\\ \dot{\widetilde{\mathbf{x}}}_2\\ \dot{\widetilde{\mathbf{x}}}_3\end{bmatrix} &= \begin{bmatrix}\mathbf{J}_1 - \mathbf{R}_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{x}}_1\\ \widetilde{\mathbf{x}}_2\\ \widetilde{\mathbf{x}}_3\end{bmatrix} + \begin{bmatrix}\mathbf{B}_1 - \mathbf{P}_1\\ \mathbf{B}_2\\ 0\end{bmatrix}\mathbf{u}, \\ \mathbf{y} &= \begin{bmatrix}(\mathbf{B}_1 + \mathbf{P}_1)^T & \mathbf{B}_2^T & 0\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{x}}_1\\ \widetilde{\mathbf{x}}_2\\ \widetilde{\mathbf{x}}_3\end{bmatrix} + (\mathbf{S} + \mathbf{N})\mathbf{u},\end{aligned} \tag{3}$$

*where* $\lambda\mathbf{E}_1 - (\mathbf{J}_1 - \mathbf{R}_1)$ *is regular and* $\mathbf{B}_2$ *has full row rank. Also, the subsystem*

$$\begin{aligned}\begin{bmatrix}\mathbf{E}_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\widetilde{\mathbf{x}}}_1\\ \dot{\widetilde{\mathbf{x}}}_2\end{bmatrix} &= \begin{bmatrix}\mathbf{J}_1 - \mathbf{R}_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{x}}_1\\ \widetilde{\mathbf{x}}_2\end{bmatrix} + \begin{bmatrix}\mathbf{B}_1 - \mathbf{P}_1\\ \mathbf{B}_2\end{bmatrix}\mathbf{u}, \\ \mathbf{y} &= \begin{bmatrix}(\mathbf{B}_1 + \mathbf{P}_1)^T & \mathbf{B}_2^T\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{x}}_1\\ \widetilde{\mathbf{x}}_2\end{bmatrix} + (\mathbf{S} + \mathbf{N})\mathbf{u},\end{aligned} \tag{4}$$

*obtained by removing the third equation and the variable* $\mathbf{x}_3$*, is still a pHDAE.*

***Proof*** First determine an orthogonal matrix $\mathbf{V}_1$ such that

$$\mathbf{V}_1^T(\lambda\mathbf{E}-(\mathbf{J}-\mathbf{R}))\mathbf{V}_1=\lambda\begin{bmatrix}\mathbf{E}_1 & 0\\ 0 & 0\end{bmatrix}-\begin{bmatrix}\mathbf{J}_1-\mathbf{R}_1 & 0\\ 0 & 0\end{bmatrix},\ \mathbf{V}_1^T(\mathbf{B}-\mathbf{P})=\begin{bmatrix}\mathbf{B}_1-\mathbf{P}_1\\ \tilde{\mathbf{B}}_2-\tilde{\mathbf{P}}_2\end{bmatrix}.$$

Such $\mathbf{V}_1$ exists since $\mathbf{E}$, $\mathbf{J}$, $\mathbf{R}$ have a common nullspace when the pencil $\lambda\mathbf{E}-(\mathbf{J}-\mathbf{R})$ is singular [23]. Then a row compression of $\tilde{\mathbf{B}}_2-\tilde{\mathbf{P}}_2$ via an orthogonal matrix $\tilde{\mathbf{V}}_2$ and a congruence transformation with $\mathbf{V}_2=\mathsf{diag}(\mathbf{I},\tilde{\mathbf{V}}_2)$ is performed, so that with $\mathbf{V}=\mathsf{diag}(\mathbf{V}_1,\mathbf{V}_2)$, we obtain the zero pattern in (3). Updating the output equation accordingly and using the fact that the transformed passivity matrix

$$\tilde{\mathbf{W}}=\begin{bmatrix}\mathbf{V}^T\mathbf{R}\mathbf{V} & \mathbf{V}^T\mathbf{P}\\ \mathbf{P}^T\mathbf{V} & \mathbf{S}\end{bmatrix}\in\mathbb{R}^{(n+m)\times(n+m)}$$

is still semidefinite; it follows that $\mathbf{P}_2=0$ and $\mathbf{P}_3=0$, giving the desired form. □

The next result presents a condensed form, showing that conditions (C2) and (O2) are equivalent and hold for the system in (4).

**Lemma 2** *For the* pHDAE *in (4) there exists an orthogonal basis transformation* $\widehat{\mathbf{V}}$ *in the state space and* $\mathbf{U}$ *in the control space such that in the new variables* $\widehat{\mathbf{x}}=\begin{bmatrix}\widehat{\mathbf{x}}_1^T & \widehat{\mathbf{x}}_2^T & \widehat{\mathbf{x}}_3^T\widehat{\mathbf{x}}_4^T & \widehat{\mathbf{x}}_5^T & \widehat{\mathbf{x}}_6^T\end{bmatrix}^T=\widehat{\mathbf{V}}^T\begin{bmatrix}\widetilde{\mathbf{x}}_1^T & \widetilde{\mathbf{x}}_2^T\end{bmatrix}^T$, *and* $\begin{bmatrix}\mathbf{u}_1^T & \mathbf{u}_2^T & \mathbf{u}_3^T\end{bmatrix}^T=\mathbf{U}^T\mathbf{u}$ *the system has the form*

$$\begin{bmatrix}\mathbf{E}_{11} & \mathbf{E}_{12} & 0 & 0 & 0 & 0\\ \mathbf{E}_{21} & \mathbf{E}_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\widehat{\mathbf{x}}}_1\\ \dot{\widehat{\mathbf{x}}}_2\\ \dot{\widehat{\mathbf{x}}}_3\\ \dot{\widehat{\mathbf{x}}}_4\\ \dot{\widehat{\mathbf{x}}}_5\end{bmatrix}=\begin{bmatrix}\mathbf{J}_{11}-\mathbf{R}_{11} & \mathbf{J}_{12}-\mathbf{R}_{12} & \mathbf{J}_{13}-\mathbf{R}_{13} & \mathbf{J}_{14} & \mathbf{J}_{15} & 0\\ \mathbf{J}_{21}-\mathbf{R}_{21} & \mathbf{J}_{22}-\mathbf{R}_{22} & \mathbf{J}_{23}-\mathbf{R}_{23} & \mathbf{J}_{24} & 0 & 0\\ \mathbf{J}_{31}-\mathbf{R}_{21} & \mathbf{J}_{32}-\mathbf{R}_{32} & \mathbf{J}_{33}-\mathbf{R}_{33} & 0 & 0 & 0\\ \mathbf{J}_{41} & \mathbf{J}_{42} & 0 & 0 & 0 & 0\\ \mathbf{J}_{51} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{x}}_1\\ \widehat{\mathbf{x}}_2\\ \widehat{\mathbf{x}}_3\\ \widehat{\mathbf{x}}_4\\ \widehat{\mathbf{x}}_5\\ \widehat{\mathbf{x}}_6\end{bmatrix}$$
$$+\begin{bmatrix}\mathbf{B}_{11}-\mathbf{P}_{11} & \mathbf{B}_{12}-\mathbf{P}_{12} & \mathbf{B}_{13}-\mathbf{P}_{13}\\ \mathbf{B}_{21}-\mathbf{P}_{21} & \mathbf{B}_{22}-\mathbf{P}_{22} & \mathbf{B}_{23}-\mathbf{P}_{23}\\ \mathbf{B}_{31}-\mathbf{P}_{31} & \mathbf{B}_{32}-\mathbf{P}_{32} & \mathbf{B}_{33}-\mathbf{P}_{33}\\ 0 & \mathbf{B}_{42} & \mathbf{B}_{43}\\ 0 & 0 & \mathbf{B}_{53}\\ 0 & 0 & \mathbf{B}_{63}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \mathbf{u}_3\end{bmatrix}, \tag{5}$$

$$\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \mathbf{y}_3\end{bmatrix}=\begin{bmatrix}(\mathbf{B}_{11}+\mathbf{P}_{11})^T & (\mathbf{B}_{21}+\mathbf{P}_{21})^T & \mathbf{B}_{31}+\mathbf{P}_{31})^T & 0 & 0 & 0\\ (\mathbf{B}_{12}+\mathbf{P}_{12})^T & (\mathbf{B}_{22}+\mathbf{P}_{22})^T & \mathbf{B}_{32}+\mathbf{P}_{32})^T & \mathbf{B}_{42}^T & 0 & 0\\ (\mathbf{B}_{13}+\mathbf{P}_{13})^T & (\mathbf{B}_{23}+\mathbf{P}_{23})^T & \mathbf{B}_{33}+\mathbf{P}_{33})^T & \mathbf{B}_{43}^T & \mathbf{B}_{53}^T & \mathbf{B}_{63}^T\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{x}}_1\\ \widehat{\mathbf{x}}_2\\ \widehat{\mathbf{x}}_3\\ \widehat{\mathbf{x}}_4\\ \widehat{\mathbf{x}}_5\\ \widehat{\mathbf{x}}_6\end{bmatrix}$$
$$+\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13}\\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23}\\ \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \mathbf{u}_3\end{bmatrix},$$

*where* $\mathbf{E}_{22}, \mathbf{J}_{33} - \mathbf{R}_{33}, \mathbf{J}_{15}$, *and* $\mathbf{B}_{42}$ *and* $\mathbf{B}_{63}$ *are invertible. Furthermore, the subsystem obtained by deleting the first, fifth, and sixth block row and column satisfies* (C2) *and equivalently* (O2).

***Proof*** The proof follows again by a sequence of orthogonal transformations. Starting from (4) in the first step one determines an orthogonal matrix $\mathbf{V}_1$ (via a spectral decomposition of $\mathbf{E}_1$) such $\widetilde{\mathbf{V}}_1^T \mathbf{E}_1 \widetilde{\mathbf{V}}_1 = \begin{bmatrix} \mathbf{E}_{11} & 0 \\ 0 & 0 \end{bmatrix}$ with $\mathbf{E}_{11} > 0$, and then forms a congruence transformation with $\widehat{\mathbf{V}}_1 = \mathsf{diag}(\widetilde{\mathbf{V}}_1, \mathbf{I})$ yielding

$$\widehat{\mathbf{V}}_1^T (\mathbf{J} - \mathbf{R}) \widehat{\mathbf{V}}_1 = \begin{bmatrix} \tilde{\mathbf{J}}_{11} - \tilde{\mathbf{R}}_{11} & \tilde{\mathbf{J}}_{12} - \tilde{\mathbf{R}}_{12} \\ \tilde{\mathbf{J}}_{21} - \tilde{\mathbf{R}}_{21} & \tilde{\mathbf{J}}_{22} - \tilde{\mathbf{R}}_{22} \end{bmatrix}, \ \widehat{\mathbf{V}}_1^T (\mathbf{B} - \mathbf{P}) = \begin{bmatrix} \tilde{\mathbf{B}}_1 - \tilde{\mathbf{P}}_1 \\ \tilde{\mathbf{B}}_2 - \tilde{\mathbf{P}}_2 \end{bmatrix}.$$

Next compute a full rank decomposition $\widetilde{\mathbf{V}}_2^T (\widetilde{\mathbf{J}}_{22} - \widetilde{\mathbf{R}}_{22}) \widetilde{\mathbf{V}}_2 = \begin{bmatrix} \widehat{\mathbf{J}}_{22} - \widehat{\mathbf{R}}_{22} & 0 \\ 0 & 0 \end{bmatrix}$, where $\widehat{\mathbf{J}}_{22} - \widehat{\mathbf{R}}_{22}$ is invertible and $\widehat{\mathbf{R}}_{22} \geq 0$. This exists, since $\widetilde{\mathbf{J}}_{22} - \widetilde{\mathbf{R}}_{22}$ has a negative semidefinite symmetric part. Then an appropriate congruence transformation with $\widehat{\mathbf{V}}_2 = \mathsf{diag}(\mathbf{I}, \widetilde{\mathbf{V}}_2, \mathbf{I})$ yields

$$\widehat{\mathbf{V}}_2^T \widehat{\mathbf{V}}_1^T \mathbf{E} \widehat{\mathbf{V}}_1 \widehat{\mathbf{V}}_2 = \begin{bmatrix} \mathbf{E}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\mathbf{V}}_2^T \widehat{\mathbf{V}}_1^T (\mathbf{J} - \mathbf{R}) \widehat{\mathbf{V}}_1 \widehat{\mathbf{V}}_2 = \begin{bmatrix} \widehat{\mathbf{J}}_{11} - \widehat{\mathbf{R}}_{11} & \widehat{\mathbf{J}}_{12} - \widehat{\mathbf{R}}_{12} & \widehat{\mathbf{J}}_{13} & 0 \\ \widehat{\mathbf{J}}_{21} - \widehat{\mathbf{R}}_{21} & \widehat{\mathbf{J}}_{22} - \widehat{\mathbf{R}}_{22} & 0 & 0 \\ \widehat{\mathbf{J}}_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \widehat{\mathbf{V}}_2^T \widehat{\mathbf{V}}_1^T (\mathbf{B} - \mathbf{P}) = \begin{bmatrix} \widehat{\mathbf{B}}_1 - \widehat{\mathbf{P}}_1 \\ \widehat{\mathbf{B}}_2 - \widehat{\mathbf{P}}_2 \\ \widehat{\mathbf{B}}_3 \\ \widehat{\mathbf{B}}_4 \end{bmatrix},$$

where $\widehat{\mathbf{J}}_{22} - \widehat{\mathbf{R}}_{22}$ is invertible and $\widehat{\mathbf{B}}_4$ has full row rank.

Then one performs an orthogonal decomposition

$$\widetilde{\mathbf{V}}_3^T \begin{bmatrix} \widehat{\mathbf{B}}_3 \\ \widehat{\mathbf{B}}_4 \end{bmatrix} \mathbf{U} = \begin{bmatrix} 0 & \mathbf{B}_{42} & \mathbf{B}_{43} \\ 0 & 0 & \mathbf{B}_{53} \\ 0 & 0 & \mathbf{B}_{63} \end{bmatrix}$$

with $\mathbf{B}_{42}$ and $\mathbf{B}_{63}$ square nonsingular, where the number of rows in $\mathbf{B}_{63}$ is that of $\widehat{\mathbf{B}}_4$ and applies an appropriate congruence transformation with $\widehat{\mathbf{V}}_3 = \mathsf{diag}(\mathbf{I}, \mathbf{I}, \widehat{\mathbf{V}}_3, \mathbf{I})$ so that one obtains block matrices

$$\begin{bmatrix} \mathbf{E}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \bar{\mathbf{J}}_{11} - \bar{\mathbf{R}}_{11} & \bar{\mathbf{J}}_{12} - \bar{\mathbf{R}}_{12} & \bar{\mathbf{J}}_{13} - \bar{\mathbf{R}}_{13} & \bar{\mathbf{J}}_{14} & 0 \\ \bar{\mathbf{J}}_{21} - \bar{\mathbf{R}}_{21} & \bar{\mathbf{J}}_{22} - \bar{\mathbf{R}}_{22} & 0 & 0 & 0 \\ \bar{\mathbf{J}}_{31} & 0 & 0 & 0 & 0 \\ \bar{\mathbf{J}}_{41} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

As a final step one computes a column compression of the full row rank matrix $\bar{\mathbf{J}}_{41}$ and applies an appropriate congruence transformation. This yields the desired form.

The fact that the several $\mathbf{P}$ blocks do not occur follows again from the semidefiniteness of the transformed passivity matrix. Since $\mathbf{J}_{51}$, $\mathbf{B}_{42}$ and $\mathbf{B}_{63}$ are invertible, it follows immediately that $\mathbf{u}_3 = 0$ and $\hat{\mathbf{x}}_1 = 0$ and that $\hat{\mathbf{x}}_5$ is uniquely determined by all the other variables. Considering the subsystem obtained by removing the first, fifth, and sixth block row and column, it follows from the symmetry structure that the condition (C2) holds if and only (O2) holds.

The procedure to compute the condensed form (3) immediately separates the dynamical part (given by the first block row), the algebraic index-1 conditions (with and without dissipation, given by the second and third block rows), and the index-2 conditions (given by the fourth block row). The last row is the singular part of the free system. However, since the conditions (C2) and (O2) hold, the system can be made regular by output feedback. Note that this is already displayed in the subsystem (4), which, for ease of notation, we denote by

$$\begin{aligned} \mathbf{E}_r \dot{\mathbf{x}}_r &= (\mathbf{J}_r - \mathbf{R}_r)\,\mathbf{x}_r + (\mathbf{B}_r - \mathbf{P}_r)\,\mathbf{u}, \\ \mathbf{y}_r &= (\mathbf{B}_r + \mathbf{P}_r)^T\,\mathbf{x}_r + (\mathbf{S}_r + \mathbf{N}_r)\,\mathbf{u}, \end{aligned} \qquad \mathbf{x}_r(t_0) = 0.$$

If we apply an output feedback $\mathbf{u} = -\mathbf{K}_r\mathbf{y}_r$ with $\mathbf{K}_r = \mathbf{K}_r^T > 0$ that makes the (2,2) block in the closed loop system invertible, then this corresponds to a feedback for the state-to-output map in (4) of the form $\mathbf{y}_r = (\mathbf{I} + (\mathbf{S}_r + \mathbf{N}_r)\mathbf{K}_r)^{-1}(\mathbf{B}_r + \mathbf{P}_r)^T\mathbf{x}_r$, which we can insert into the first equation to obtain

$$\begin{aligned} \mathbf{E}_r\dot{\mathbf{x}}_r &= (\mathbf{J}_r - \mathbf{R}_r)\mathbf{x}_r + (\mathbf{B}_r - \mathbf{P}_r)\mathbf{K}_r\mathbf{y}_r \\ &= \left((\mathbf{J}_r - \mathbf{R}_r) - (\mathbf{B}_r - \mathbf{P}_r)(\mathbf{K}_r^{-1} + \mathbf{S}_r + \mathbf{N}_r)^{-1}(\mathbf{B}_r + \mathbf{P}_r)^T\right)\mathbf{x}_r. \end{aligned}$$

The system matrix is the Schur complement of

$$\begin{bmatrix} \mathbf{J}_r & \mathbf{B}_r \\ -\mathbf{B}_r^T & -\mathbf{N}_r \end{bmatrix} - \begin{bmatrix} \mathbf{R}_r & \mathbf{P}_r \\ \mathbf{P}_r^T & \mathbf{K}_r^{-1} + \mathbf{S}_r \end{bmatrix},$$

which has a symmetric part given by $-\widetilde{\mathbf{W}}_r$, with $\widetilde{\mathbf{W}}_r = \begin{bmatrix} \mathbf{R}_r & \mathbf{P}_r \\ \mathbf{P}_r^T & \mathbf{K}_r^{-1} + \mathbf{S}_r \end{bmatrix} \geq 0$. Hence, the closed loop system is regular and the closed loop system is still pH.

The procedures described above are computationally demanding, since they typically require large-scale singular value decompositions or spectral decompositions. Fortunately, in many practical cases the condensed form is already available directly from the modeling procedure, so that the transfer function can be formed and the model reduction method can be directly applied.

## 4 Interpolatory Model Reduction of pHDAEs

Given an order-$n$ pHDAE as in (1), we want to construct an order-$r$ reduced pHDAE, with $r \ll n$, having the same structured form

$$\begin{aligned} \widehat{\mathbf{E}}\dot{\mathbf{x}}_r &= (\widehat{\mathbf{J}} - \widehat{\mathbf{R}})\,\mathbf{x}_r + (\widehat{\mathbf{B}} - \widehat{\mathbf{P}})\,\mathbf{u}, \quad \mathbf{x}_r(t_0) = 0, \\ \mathbf{y}_r &= (\widehat{\mathbf{B}} + \widehat{\mathbf{P}})^T\,\mathbf{x}_r + (\widehat{\mathbf{S}} + \widehat{\mathbf{N}})\,\mathbf{u}, \end{aligned} \tag{6}$$

such that $\widehat{\mathbf{E}}, \widehat{\mathbf{J}}, \widehat{\mathbf{R}} \in \mathbb{R}^{r\times r}$, $\widehat{\mathbf{B}}, \widehat{\mathbf{P}} \in \mathbb{R}^{r\times m}$, $\widehat{\mathbf{S}} = \widehat{\mathbf{S}}^T$, $\widehat{\mathbf{N}} = -\widehat{\mathbf{N}}^T \in \mathbb{R}^{m\times m}$ satisfy the same requirements as in Definition 1 and that the output $\mathbf{y}_r(t)$ of (6) is an accurate approximation to the original output $\mathbf{y}(t)$ over a wide range of admissible inputs $\mathbf{u}(t)$. We will enforce accuracy by constructing the reduced model (6) via interpolation.

Let $\mathbf{H}(s) = \mathcal{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathcal{B} + \mathbf{D}$ and $\widehat{\mathbf{H}}(s) = \widehat{\mathcal{C}}(s\widehat{\mathbf{E}} - \widehat{\mathbf{A}})^{-1}\widehat{\mathcal{B}} + \widehat{\mathbf{D}}$ denote the transfer functions of (1) and (6), where $\mathcal{C} = (\mathbf{B} + \mathbf{P})^T$, $\mathbf{A} = \mathbf{J} - \mathbf{R}$, $\mathcal{B} = \mathbf{B} - \mathbf{P}$, $\mathbf{D} = \mathbf{S} + \mathbf{N}$, and similarly for the reduced-order ("hat") quantities. Given right-interpolation points $\{\sigma_1, \sigma_2, \ldots, \sigma_r\} \in \mathbb{C}$ together with the corresponding right-tangent directions $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r\} \in \mathbb{C}^m$ and left-interpolation points $\{\mu_1, \mu_2, \ldots, \mu_r\} \in \mathbb{C}$ together with the corresponding left-tangent directions $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \ldots, \boldsymbol{\ell}_r\} \in \mathbb{C}^m$, we wish to construct a reduced model, $\widehat{\mathbf{H}}(s)$, that tangentially interpolates $\mathbf{H}(s)$, i.e.,

$$\mathbf{H}(\sigma_i)\boldsymbol{b}_i = \widehat{\mathbf{H}}(\sigma_i)\boldsymbol{b}_i \quad \text{and} \quad \boldsymbol{\ell}_i^T\mathbf{H}(\mu_i) = \boldsymbol{\ell}_i^T\widehat{\mathbf{H}}(\mu_i), \quad \text{for } i = 1, 2, \ldots, r.$$

These tangential interpolation conditions can be enforced easily via a Petrov-Galerkin projection [1, 4, 12]. In particular, construct $\mathbf{V} \in \mathbb{C}^{n\times r}$ and $\mathbf{Z} \in \mathbb{C}^{n\times r}$ using

$$\begin{aligned} \mathbf{V} &= \left[(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathcal{B}\boldsymbol{b}_1, \;\; (\sigma_2\mathbf{E} - \mathbf{A})^{-1}\mathcal{B}\boldsymbol{b}_2, \;\; \cdots \; (\sigma_r\mathbf{E} - \mathbf{A})^{-1}\mathcal{B}\boldsymbol{b}_r\right] \text{ and} \\ \mathbf{Z} &= \left[(\mu_1\mathbf{E} - \mathbf{A})^{-T}\mathcal{C}^T\boldsymbol{\ell}_1, \;\; (\mu_2\mathbf{E} - \mathbf{A})^{-T}\mathcal{C}^T\boldsymbol{\ell}_2, \;\; \cdots \; (\mu_r\mathbf{E} - \mathbf{A})^{-1}\mathcal{C}^T\boldsymbol{\ell}_r\right]. \end{aligned}$$

Then an interpolatory reduced model can be defined via the projection:

$$\widehat{\mathbf{E}} = \mathbf{Z}^T\mathbf{E}\mathbf{V}, \;\; \widehat{\mathbf{A}} = \mathbf{Z}^T\mathbf{A}\mathbf{V}, \;\; \widehat{\mathcal{B}} = \mathbf{Z}^T\mathcal{B}, \;\; \widehat{\mathcal{C}} = \mathcal{C}\mathbf{V}, \;\; \text{and} \;\; \widehat{\mathbf{D}} = \mathbf{D}. \tag{7}$$

In the setting of pHDAEs, two fundamental issues arise: First, the reduced quantities in (7) are no longer guaranteed to have pH structure. This is most easily seen by examining the reduced quantity, $\widehat{\mathcal{A}} = \mathbf{Z}^T\mathcal{A}\mathbf{V} = \mathbf{Z}^T\mathbf{J}\mathbf{V} - \mathbf{Z}^T\mathbf{R}\mathbf{V}$. If $\widehat{\mathcal{A}}$ is decomposed into its symmetric and skew-symmetric parts, it can no longer be guaranteed that the symmetric part is positive semi-definite. This could be resolved by using a Ritz-Galerkin projection, taking $\mathbf{Z} = \mathbf{V}$, but then only interpolation conditions associated with right interpolation data are satisfied. Even then, there will still be a second issue that persists. In the generic case when $r < \operatorname{rank}(\mathbf{E})$, the reduced quantity $\widehat{\mathbf{E}}$ will generally be nonsingular; thus the reduced system will be an ODE and

the polynomial parts of $\mathbf{H}(s)$ and $\widehat{\mathbf{H}}(s)$ will not match, leading then to unbounded errors.

Structure-preserving interpolatory reduction of pH systems in the most general setting of tangential interpolation has been studied in [15, 16]. However, this work focused on the ODE case. On the other hand, [17] developed the tangential interpolation framework for reducing *unstructured* DAEs with guaranteed polynomial matching. Only recently in [11, 18], the combined problem has been investigated. We now develop a treatment of structure-preserving interpolatory model reduction problem for index-1 and index-2 pHDAEs in the general setting of tangential interpolation.

### 4.1 Semi-explicit Index-1 pHDAE Systems

The simplest class of pHDAEs are *semi-explicit index*-1 pHDAEs of the form

$$\begin{aligned}\begin{bmatrix}\mathbf{E}_{11} & 0\\ 0 & 0\end{bmatrix}\dot{\mathbf{x}}(t) &= \begin{bmatrix}\mathbf{J}_{11}-\mathbf{R}_{11} & \mathbf{J}_{12}-\mathbf{R}_{12}\\ -\mathbf{J}_{12}^T-\mathbf{R}_{12}^T & \mathbf{J}_{22}-\mathbf{R}_{22}\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}\mathbf{B}_1-\mathbf{P}_1\\ \mathbf{B}_2-\mathbf{P}_2\end{bmatrix}u(t),\\ \mathbf{y}(t) &= \begin{bmatrix}\mathbf{B}_1^T+\mathbf{P}_1^T & \mathbf{B}_2^T+\mathbf{P}_2^T\end{bmatrix}\mathbf{x}(t)+(\mathbf{S}+\mathbf{N})\mathbf{u}(t),\end{aligned} \tag{8}$$

where $\mathbf{E}_{11}$ and $\mathbf{J}_{22}-\mathbf{R}_{22}$ are nonsingular. We have the following interpolation result.

**Theorem 1** *Consider the pHDAE system in (8). Let the interpolation points* $\{\sigma_1, \sigma_2, \ldots, \sigma_r\} \in \mathbb{C}$ *and the corresponding tangent directions* $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r\} \in \mathbb{C}^m$ *be given. Construct the interpolatory model reduction basis* $\mathbf{V}$ *as*

$$\mathbf{V} = \begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} = \left[(\sigma_1\mathbf{E}-\mathbf{A})^{-1}\mathcal{B}\boldsymbol{b}_1, \quad \cdots, \quad (\sigma_r\mathbf{E}-\mathbf{A})^{-1}\mathcal{B}\boldsymbol{b}_r\right] \in \mathbb{C}^{n\times r}, \tag{9}$$

*where* $\mathbf{V}$ *is partitioned conformably with the system, and define the matrices*

$$\mathbb{B} = \left[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots\ \boldsymbol{b}_r\right] \in \mathbb{C}^{m\times r} \text{ and } \mathcal{D} = \mathbf{D} - (\mathbf{B}_2^T+\mathbf{P}_2^T)(\mathbf{J}_{22}-\mathbf{R}_{22})^{-1}(\mathbf{B}_2-\mathbf{P}_2) \in \mathbb{C}^{m\times m}.$$

*Let* $\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix} = \mathbf{J}-\mathbf{R}$, *partitioned accordingly to (8). Then, the transfer function* $\widehat{\mathbf{H}}(s)$ *of the reduced model*

$$\widehat{\mathbf{E}}\dot{\mathbf{x}}_r(t) = (\widehat{\mathbf{J}}-\widehat{\mathbf{R}})\mathbf{x}_r(t)+\widehat{\mathcal{B}}\mathbf{u}(t), \qquad \mathbf{y}_r(t) = \widehat{\mathcal{C}}\mathbf{x}_r(t)+\widehat{\mathbf{D}}\mathbf{u}(t), \tag{10}$$

*with*

$$\widehat{\mathbf{E}} = \mathbf{V}_1^T \mathbf{E}_{11} \mathbf{V}_1, \qquad \widehat{\mathcal{C}} = \mathcal{C}\mathbf{V} + (\mathbf{B}_2^T + \mathbf{P}_2^T)\mathbf{A}_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2)\mathbb{B},$$
$$\widehat{\mathbf{A}} = \mathbf{V}^T \mathcal{A}\mathbf{V} + \mathbb{B}^T(\mathcal{D} - \mathbf{D})\mathbb{B}, \qquad \widehat{\mathbf{D}} = \mathcal{D} = \mathbf{D} - (\mathbf{B}_2^T + \mathbf{P}_2^T)\mathbf{A}_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2),$$
$$\widehat{\mathcal{B}} = \mathbf{V}^T \mathcal{B} + \mathbb{B}^T(\mathbf{B}_2^T + \mathbf{P}_2^T)\mathbf{A}_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2),$$

*matches the polynomial part of* $\mathbf{H}(s)$ *and tangentially interpolates it, i.e.,*

$$\mathbf{H}(\sigma_i)\boldsymbol{b}_i = \widehat{\mathbf{H}}(\sigma_i)\boldsymbol{b}_i, \quad \textit{for } i = 1, 2, \ldots, r.$$

*Define* $\widehat{\mathbf{P}} = \frac{1}{2}(-\widehat{\mathcal{G}} + \widehat{\mathcal{C}})$*, and decompose* $\widehat{\mathbf{D}} = \widehat{\mathbf{S}} + \widehat{\mathbf{N}}$ *and* $\widehat{\mathbf{A}} = \widehat{\mathbf{J}} - \widehat{\mathbf{R}}$ *into their symmetric and skew-symmetric parts. Then, the reduced model* (10) *is a* pHDAE *system if the reduced passivity matrix* $\widehat{\mathbf{W}} = \begin{bmatrix} \widehat{\mathbf{R}} & \widehat{\mathbf{P}} \\ \widehat{\mathbf{P}}^T & \widehat{\mathbf{S}} \end{bmatrix}$ *is positive seimdefinite.*

***Proof*** We employ a Galerkin projection using the interpolatory model reduction basis $\mathbf{V}$ to obtain an *intermediate* reduced model

$$\widetilde{\mathbf{E}} = \mathbf{V}_1^T \mathbf{E}_{11} \mathbf{V}_1, \quad \widetilde{\mathbf{A}} = \widetilde{\mathbf{J}} - \widetilde{\mathbf{R}} = \mathbf{V}^T \mathbf{J}\mathbf{V} - \mathbf{V}^T \mathbf{R}\mathbf{V}, \quad \widetilde{\mathcal{B}} = \mathbf{V}^T \mathcal{B}, \quad \widetilde{\mathcal{C}} = \mathcal{C}\mathbf{V}, \quad \text{and} \quad \widetilde{\mathbf{D}} = \mathbf{D}.$$

This reduced model is a pHDAE system due to the use of a one-sided Galerkin projection, and likewise it also satisfies the desired tangential interpolation conditions. However, it will *not* generally match the transfer function, $\mathbf{H}(s)$, at $s = \infty$. Indeed, $\mathbf{H}(s)$ has a polynomial part is given by

$$\lim_{s\to\infty} \mathbf{H}(s) = \mathcal{D} = \mathbf{D} - (\mathbf{B}_2^T + \mathbf{P}_2^T)\mathbf{A}_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2) \neq \widetilde{\mathbf{D}} = \lim_{s\to\infty} \widehat{\mathbf{H}}(s)$$

A remedy to this problem, proposed in [2, 21] and employed in the general DAE setting in [17], is to modify the $\mathbf{D}$-term in the reduced model to match the polynomial part while at the same time *shifting* other reduced quantities appropriately so as to retain tangential interpolation. Using this approach, we obtain a modified reduced model

$$\begin{aligned}
\widehat{\mathbf{E}} &= \widetilde{\mathbf{E}} = \mathbf{V}_1^T \mathbf{E}_{11} \mathbf{V}_1, \\
\widehat{\mathbf{A}} &= \widetilde{\mathbf{A}} + \mathbb{B}^T(\mathcal{D} - \mathbf{D})\mathbb{B} = \mathbf{V}^T \mathbf{J}\mathbf{V} - \mathbf{V}^T \mathbf{R}\mathbf{V} - \mathbb{B}^T(\mathbf{B}_2^T + \mathbf{P}_2^T)A_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2)\mathbb{B}, \\
\widehat{\mathcal{B}} &= \widetilde{\mathcal{B}} + \mathbb{B}^T(\mathbf{B}_2^T + \mathbf{P}_2^T)A_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2) \\
&= \mathbf{V}_1^T(\mathbf{B}_1 - \mathbf{P}_1) + \mathbf{V}_2^T(\mathbf{B}_2 - \mathbf{P}_2) + \mathbb{B}^T(\mathbf{B}_2^T + \mathbf{P}_2^T)A_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2), \\
\widehat{\mathcal{C}} &= \widetilde{\mathcal{C}} + (\mathbf{B}_2^T + \mathbf{P}_2^T)A_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2)\mathbb{B} \\
&= (\mathbf{B}_1^T + \mathbf{P}_1^T)\mathbf{V}_1^T + (\mathbf{B}_2^T + \mathbf{P}_2^T)\mathbf{V}_2^T + (\mathbf{B}_2^T + \mathbf{P}_2^T)A_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2)\mathbb{B}, \quad \text{and} \\
\widehat{\mathbf{D}} &= \mathcal{D} = \mathbf{D} - (\mathbf{B}_2^T + \mathbf{P}_2^T)A_{22}^{-1}(\mathbf{B}_2 - \mathbf{P}_2),
\end{aligned}$$

which satisfies the original tangential interpolation conditions while also matching the polynomial part with a modified $\widehat{\mathbf{D}}$-term. For this system to be pH, we need to check that the associated passivity matrix is still positive semidefinite. After rewriting

the input and output matrix in the usual way, this is exactly the condition on $\widehat{\mathbf{W}}$ in the assertion. We then have that the reduced model not only satisfies the interpolation conditions and matches the polynomial part at $s = \infty$, but also is pH. □

**Remark 1** Note that if the input does not influence the algebraic equations, i.e., if $\mathbf{B}_2 = \mathbf{P}_2 = 0$, then the shift of the constant term is not necessary and the formulas simplify significantly, i.e., $\widehat{\mathbf{A}} = \mathbf{V}^T \mathcal{A} \mathbf{V}$, $\widehat{\mathcal{B}} = \widetilde{\mathcal{B}} = \mathbf{V}^T \mathcal{B}$, $\widehat{\mathcal{C}} = \mathcal{C}\mathbf{V}$, and $\widehat{\mathbf{D}} = \mathcal{D} = \mathbf{D}$.

Another solution to preserving pH structure via interpolation can be obtained through the following theorem.

**Theorem 2** *Consider a full-order pHDAE system of the form* (8)*. Let interpolation points* $\{\sigma_1, \sigma_2, \ldots, \sigma_r\} \in \mathbb{C}$ *and the corresponding tangent directions* $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r\} \in \mathbb{C}^m$ *be given. Construct the interpolatory model reduction basis* $\mathbf{V}$ *as in* (9)*. Then the reduced model*

$$\begin{bmatrix} \mathbf{V}_1^T \mathbf{E}_{11} \mathbf{V}_1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{V}_1^T (\mathbf{J}_{11} - \mathbf{R}_{11}) \mathbf{V}_1 & \mathbf{V}_1^T (\mathbf{J}_{12} - \mathbf{R}_{12}) \\ (-\mathbf{J}_{12}^T - \mathbf{R}_{12}^T) \mathbf{V}_1 & \mathbf{J}_{22} - \mathbf{R}_{22} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{V}_1^T (\mathbf{B}_1 - \mathbf{P}_1) \\ \mathbf{B}_2 - \mathbf{P}_2 \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}_r(t) = \left[ (\mathbf{B}_1^T + \mathbf{P}_1)^T \mathbf{V}_1 \ \mathbf{B}_2 + \mathbf{P}_2^T \right] \mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \tag{11}$$

*retains the pH structure, tangentially interpolates the original model, and matches the polynomial part.*

***Proof*** We first note that the subspace spanned by the columns of $\mathbf{V} = \left[\mathbf{V}_1^T \ \mathbf{V}_2^T\right]^T$ is contained in the subspace spanned by the columns of $\widehat{\mathbf{V}} := \mathsf{diag}(\mathbf{V}_1, \mathbf{I})$. Then, the system in (11) results from reducing the original system in (8) via $\widehat{\mathbf{V}}$. Since $\mathsf{span}(\mathbf{V}) \subseteq \mathsf{span}(\widehat{\mathbf{V}})$, this reduced DAE automatically satisfies the interpolation conditions and since $\widehat{\mathbf{V}}$ does not alter the matrix $\mathbf{J}_{22} - \mathbf{R}_{22}$ and the matrices $\mathbf{B}_2, \mathbf{P}_2$, the polynomial part of the transfer function is $\mathbf{D} - (\mathbf{B}_2^T + \mathbf{P}_2^T)(\mathbf{J}_{22} - \mathbf{R}_{22})^{-1}(\mathbf{B}_2 - \mathbf{P}_2)$, matching that of the original model. The reduced system in (11) is pH as well since the one-sided projection that is used retains the original pH structure. □

**Remark 2** Theorem 2 appears to present an easier alternative to the more complicated Theorem 1 for structure-preserving interpolatory model reduction of index-1 pHDAEs. However, the construction of Theorem 2 may not achieve the maximal reduction possible because redundant algebraic conditions cannot be removed; see [25]. Note also that further orthogonalization of $\mathbf{V}_1$ may be necessary, see [11].

## 4.2 Semi-explicit pHDAE Systems with Index-2 Constraints

The next class of interest will be semi-explicit index-2 systems. We first consider the case that the input does not affect the algebraic equations.

**Theorem 3** *Consider an index-2 pHDAE system of the form*

$$\begin{bmatrix} \mathbf{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{11} - \mathbf{R}_{11} & \mathbf{J}_{12} \\ -\mathbf{J}_{12}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 - \mathbf{P}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t),$$
$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{B}_1^T + \mathbf{P}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \mathbf{D}\mathbf{u}(t), \tag{12}$$

*with* $\mathbf{E}_{11} > 0$ *and set* $\mathbf{A}_{11} = \mathbf{J}_{11} - \mathbf{R}_{11}$. *Given interpolation points* $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ *and associated tangent directions* $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r\}$, *let the vectors* $\mathbf{v}_i$, *for* $i = 1, 2, \ldots, r$, *be the first block of the solution of*

$$\begin{bmatrix} \mathbf{A}_{11} - \sigma_i \mathbf{E}_{11} & \mathbf{J}_{12} \\ -\mathbf{J}_{12}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} (\mathbf{B}_1 - \mathbf{P}_1)\boldsymbol{b}_i \\ \mathbf{0} \end{bmatrix}. \tag{13}$$

*Define* $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r]$. *Then, the reduced model*

$$\widehat{\mathbf{E}}\dot{\mathbf{x}}_r = (\widehat{\mathbf{J}} - \widehat{\mathbf{R}})\mathbf{x}_r + \widehat{\mathcal{B}}\mathbf{u}(t), \quad \mathbf{y}_r = \widehat{\mathcal{C}}\mathbf{x}_r + \widehat{\mathbf{D}}, \tag{14}$$

*with*

$$\widehat{\mathbf{E}} = \mathbf{V}^T\mathbf{E}_{11}\mathbf{V}, \quad \widehat{\mathbf{J}} = \mathbf{V}^T\mathbf{J}_{11}\mathbf{V}, \quad \widehat{\mathbf{R}} = \mathbf{V}^T\mathbf{R}_{11}\mathbf{V},$$
$$\widehat{\mathcal{B}} = \mathbf{V}^T\mathbf{B}_1 - \mathbf{V}^T\mathbf{P}_1, \quad \widehat{\mathcal{C}} = \mathbf{B}_1^T\mathbf{V}^T + \mathbf{P}^T\mathbf{V}_1^T, \quad \textit{and} \quad \widehat{\mathbf{D}} = \mathbf{D}, \tag{15}$$

*is still pH, matches the polynomial part of the original transfer function, and satisfies the tangential interpolation conditions,*

$$\mathbf{H}(\sigma_i)\boldsymbol{b}_i = \widehat{\mathbf{H}}(\sigma_i)\boldsymbol{b}_i, \quad \textit{for} \quad i = 1, 2, \ldots, r.$$

***Proof*** Note first that the regularity of $\lambda\mathbf{E} - (\mathbf{J} - \mathbf{R})$ and the index-2 condition imply that $-\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{J}_{12}$ is invertible, see [5, 20]. Following [17], we write (12) as

$$\begin{aligned} \mathbf{\Pi}\mathbf{E}_{11}\mathbf{\Pi}^T\dot{\mathbf{x}}_1(t) &= \mathbf{\Pi}\mathbf{A}_{11}\mathbf{\Pi}^T\mathbf{x}_1(t) + \mathbf{\Pi}\mathbf{B}_1\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}_1\mathbf{\Pi}^T\mathbf{x}_1(t) + \mathbf{D}\mathbf{u}(t), \end{aligned} \tag{16}$$

and conjoin this with the algebraic equation,

$$\mathbf{x}_2(t) = -(\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{J}_{12})^{-1}\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{A}_{11}\mathbf{x}_1(t) - (\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{J}_{12})^{-1}\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{B}_1\mathbf{u}(t).$$

The skew projector, $\mathbf{\Pi}$, in (16) is defined as $\mathbf{\Pi} = \mathbf{I} - \mathbf{E}_{11}^{-1}\mathbf{J}_{12}(\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{J}_{12})^{-1}\mathbf{J}_{12}^T$, and (16) is now an implicit ODE pH system that can be reduced with standard model reduction techniques. As it stands, this would appear to require computing the projector $\mathbf{\Pi}$ explicitly, see [25]. For general index-2 DAE systems one can avoid this computational step through interpolatory model reduction, see [17].

To adapt this idea to pHDAE systems, we construct $\mathbf{V}$ using (13) and then compute the reduced-order quantities via one-sided projection as in (15). This construction of $\mathbf{V}$, as in [17], guarantees that the reduced model in (14) tangentially interpolates the original pHDAE system in (16) and the polynomial part of the transfer function in (12) is given by $\mathbf{D} = \mathbf{S} + \mathbf{N}$, separated with respect to its symmetric and skew-symmetric parts. Since (14) is an implicit ODE pH system with an exact $\mathbf{D}$-term, it matches the polynomial part of the original transfer function $\mathbf{H}(s)$.

It remains to show that (14) is pH. By construction in (15), $\widehat{\mathbf{J}}$ is skew-symmetric, $\widehat{\mathbf{R}}$ is symmetric positive semidefinite, and $\mathbf{E}_{11}$ is symmetric positive definite. Moreover,

$$\begin{bmatrix} \mathbf{V}^T\mathbf{R}_{11}\mathbf{V} & \mathbf{V}^T\mathbf{P}_1 \\ \mathbf{P}_1^T\mathbf{V} & \mathbf{S} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^T & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{11} & \mathbf{P}_1 \\ \mathbf{P}_1^T & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{V} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \geq 0,$$

since the original model is pH. Therefore, the pH-structure is retained. □

The situation becomes more complicated when the second block in $\mathcal{B}$ is nonzero, that is, when the system has the form

$$\begin{aligned} \begin{bmatrix} \mathbf{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{J}_{11} - \mathbf{R}_{11} & \mathbf{J}_{12} \\ -\mathbf{J}_{12}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 - \mathbf{P}_1 \\ \mathbf{B}_2 - \mathbf{P}_2 \end{bmatrix} \mathbf{u}(t), \\ \mathbf{y}(t) &= \begin{bmatrix} \mathbf{B}_1^T + \mathbf{P}_1^T & \mathbf{B}_2 + \mathbf{P}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \mathbf{D}\mathbf{u}(t). \end{aligned} \tag{17}$$

**Theorem 4** *Consider a pHDAE system of the form* (17) *and define the matrices*

$$\begin{aligned} \mathcal{Z} &= -(\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{A}_{12})^{-1} = \left(\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{J}_{12}\right)^{-1}, \\ \mathcal{B} &= (\mathbf{B}_1 - \mathbf{P}_1) + (\mathbf{J}_{11} - \mathbf{R}_{11})\mathbf{E}_{11}^{-1}\mathbf{J}_{12}\mathbf{Z}(\mathbf{B}_2 - \mathbf{P}_2), \\ \mathcal{C} &= (\mathbf{B}_1^T - \mathbf{P}_1^T) - (\mathbf{B}_2^T - \mathbf{P}_2^T)\,\mathcal{Z}\,\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}(\mathbf{J}_{11} - \mathbf{R}_{11}), \\ \mathcal{D}_0 &= \mathbf{D} - (\mathbf{B}_2^T + \mathbf{P}_2^T)\,\mathcal{Z}\,\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}(\mathbf{B}_1 - \mathbf{P}_1), \quad \textit{and } \mathcal{D}_1 = -(\mathbf{B}_2^T + \mathbf{P}_2^T)\,\mathcal{Z}\,(\mathbf{B}_2 - \mathbf{P}_2). \end{aligned}$$

*Given interpolation points* $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ *and associated tangent directions* $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r\}$, *let the vectors* $\mathbf{v}_i$ *be the first blocks of the solutions of*

$$\begin{bmatrix} \mathbf{J}_{11} - \mathbf{R}_{11} - \sigma_i\mathbf{E}_{11} & \mathbf{J}_{12} \\ -\mathbf{J}_{12}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{z} \end{bmatrix} = \mathbf{B}\boldsymbol{b}_i, \quad \textit{for } i = 1, 2, \ldots, r, \tag{18}$$

*and set* $\mathbf{V} := [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r]$, $\mathbf{u}_1 := \mathbf{u}$, $\mathbf{u}_2 := \dot{\mathbf{u}}$, *and* $\widehat{\mathbf{D}} := \begin{bmatrix} \mathcal{D}_0 & \mathcal{D}_1 \end{bmatrix} = \widehat{\mathbf{S}} + \widehat{\mathbf{N}}$. *Then, the reduced model*

$$\widehat{\mathbf{E}}\dot{\mathbf{x}}_r = (\widehat{\mathbf{J}} - \widehat{\mathbf{R}})\mathbf{x}_r + \begin{bmatrix} \widehat{\mathbf{B}} - \widehat{\mathbf{P}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}, \quad \mathbf{y}_r = (\widehat{\mathbf{B}} + \widehat{\mathbf{P}})^T\mathbf{x}_r + \widehat{\mathbf{D}} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}, \tag{19}$$

$$\textit{with} \qquad \begin{aligned} &\widehat{\mathbf{E}} = \mathbf{V}^T\mathbf{E}_{11}\mathbf{V}, \;\; \widehat{\mathbf{J}} = \mathbf{V}^T\mathbf{J}_{11}\mathbf{V}, \;\; \widehat{\mathbf{R}} = \mathbf{V}^T\mathbf{R}_{11}\mathbf{V}, \\ &\widehat{\mathbf{B}} = \frac{1}{2}\left(\mathbf{V}^T\mathcal{B} + \mathbf{V}^T\mathcal{C}^T\right), \;\; \textit{and} \;\; \widehat{\mathbf{P}} = \frac{1}{2}\left(\mathbf{V}^T\mathcal{C}^T - \mathbf{V}^T\mathcal{B}\right), \end{aligned} \tag{20}$$

*satisfies the interpolation conditions, matches the polynomial part of the transfer function, and preserves the* pH *structure, provided that the reduced passivity matrix* $\widehat{\mathbf{W}} = \begin{bmatrix} \widehat{\mathbf{R}} & \widehat{\mathbf{P}} \\ \widehat{\mathbf{P}}^T & \widehat{\mathbf{S}} \end{bmatrix}$ *is positive semidefinite.*

***Proof*** The proof is similarly to that of Theorem 3. Following [17], the state $\mathbf{x}_1$ can be decomposed as $\mathbf{x}_1 = \mathbf{x}_c + \mathbf{x}_g$, where $\mathbf{x}_g = \mathbf{E}_{11}^{-1}\mathbf{A}_{12}(\mathbf{J}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{J}_{12})^{-1}(\mathbf{B}_2 - \mathbf{P}_2)\mathbf{u}(t)$ and $\mathbf{x}_c(t)$ satisfies $\mathbf{J}_{12}^T\mathbf{x}_c = 0$. Then, one can rewrite (17) as

$$\begin{aligned} \mathbf{\Pi}\mathbf{E}_{11}\mathbf{\Pi}^T\dot{\mathbf{x}}_c(t) &= \mathbf{\Pi}\mathbf{A}_{11}\mathbf{\Pi}^T\mathbf{x}_c(t) + \mathbf{\Pi}\mathbf{B}_1\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{\Pi}^T\mathbf{x}_c(t) + \mathcal{D}_0\mathbf{u}(t) + \mathcal{D}_1\dot{\mathbf{u}}(t). \end{aligned} \tag{21}$$

As before, the ODE part can be reduced with usual model reduction techniques. Following [17], however, we achieve this without computing the projector $\mathbf{\Pi}$ explicitly, instead by constructing $\mathbf{V}$ using (18) and then applying one-sided model reduction with $\mathbf{V}$ to obtain the reduced model

$$\widehat{\mathbf{E}}\dot{\mathbf{x}}_r = (\widehat{\mathbf{J}} - \widehat{\mathbf{R}})\mathbf{x}_r + \widetilde{\mathbf{B}}\mathbf{u}(t), \;\; \mathbf{y}_r = \widetilde{\mathbf{C}}^T\mathbf{x}_r + \mathcal{D}_0\mathbf{u}(t) + \mathcal{D}_1\dot{\mathbf{u}}(t), \tag{22}$$

where $\widehat{\mathbf{E}} = \mathbf{V}^T\mathbf{E}_{11}\mathbf{V}$, $\widehat{\mathbf{J}} = \mathbf{V}^T\mathbf{J}_{11}\mathbf{V}$, $\widehat{\mathbf{R}} = \mathbf{V}^T\mathbf{R}_{11}\mathbf{V}$, $\widetilde{\mathbf{B}} = \mathbf{V}^T\mathcal{B}$, and $\widetilde{\mathbf{C}} = \mathcal{C}\mathbf{V}$. This reduced model, by construction, satisfies the tangential interpolation conditions. Note that the reduced model in (22) has exactly the same realization as the reduced model in (19) except for the reduced $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ terms. The reduced terms $\widehat{\mathbf{E}}$, $\widehat{\mathbf{J}}$, and $\widehat{\mathbf{R}}$ in (22) already have the pH structure. To recover the port symmetry, we determine matrices $\widehat{\mathbf{B}}$ and $\widehat{\mathbf{P}}$ such that $\widetilde{\mathbf{B}} = \mathbf{V}^T\mathcal{B} = \widehat{\mathbf{B}} - \widehat{\mathbf{P}}$ and $\widetilde{\mathbf{C}} = \mathcal{C}\mathbf{V} = (\widehat{\mathbf{B}} + \widehat{\mathbf{P}})^T$ via

$$\widehat{\mathbf{B}} = \frac{1}{2}\left(\widetilde{\mathbf{B}} + \widetilde{\mathbf{C}}^T\right) = \frac{1}{2}\mathbf{V}^T\left(\mathcal{B} + \mathcal{C}^T\right) \quad \text{and} \quad \widehat{\mathbf{P}} = \frac{1}{2}\left(\widetilde{\mathbf{C}}^T - \widetilde{\mathbf{B}}\right) = \frac{1}{2}\mathbf{V}^T\left(\mathcal{C}^T - \mathcal{B}\right),$$

recovering (20). The final requirement to retain the pH structure is, then, again that $\begin{bmatrix} \widehat{\mathbf{R}} & \widehat{\mathbf{P}} \\ \widehat{\mathbf{P}}^T & \frac{1}{2}\left(\mathbf{D} + \mathbf{D}^T\right) \end{bmatrix} \geq 0$, which is the final condition in the statement of the theorem. □

This approach has the disadvantage that $\mathbf{u}_2 = \dot{\mathbf{u}}$ is introduced as an extra input and this, in turn, may lead to difficulties when applying standard control and optimization methods. For this reason it is usually preferable to first perform index reduction via an appropriate output feedback (see Sect. 3) and then apply the results from Sect. 4.1. Note that this will change the polynomial part of the transfer function.

### 4.3 *Semi-explicit* pHDAE *Systems with Index-*1 *and Index-*2 *Constraints*

Finally, we consider semi-explicit index-2 systems that also have an index-1 part, see [11, 18, 25]. We will only consider the special case,

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{22} & 0 \\ \mathbf{E}_{21} & \mathbf{E}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{11} - \mathbf{R}_{11} & \mathbf{J}_{12} - \mathbf{R}_{12} & \mathbf{J}_{13} \\ \mathbf{J}_{21} - \mathbf{R}_{21} & \mathbf{J}_{22} - \mathbf{R}_{22} & 0 \\ \mathbf{J}_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 - \mathbf{P}_1 \\ \mathbf{B}_2 - \mathbf{P}_2 \\ 0 \end{bmatrix} \mathbf{u}, \tag{23}$$

$$\mathbf{y} = \begin{bmatrix} (\mathbf{B}_1 + \mathbf{P}_1)^T & (\mathbf{B}_2 + \mathbf{P}_2)^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} + (\mathbf{S} + \mathbf{N})\mathbf{u}, \text{ where } \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{22} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} > 0,$$

and both $\mathbf{J}_{22} - \mathbf{R}_{22}$ and $\mathbf{J}_{31}$ are nonsingular.

**Theorem 5** *Consider an index-2* pHDAE *system of the form (23) and construct an interpolatory model reduction basis given by*

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1^T & \mathbf{V}_2^T & \mathbf{V}_3^T \end{bmatrix}^T = \begin{bmatrix} (\sigma_1 \mathbf{E} - \mathbf{A})^{-1} \mathcal{B} \boldsymbol{b}_1, & \cdots & (\sigma_r \mathbf{E} - \mathbf{A})^{-1} \mathcal{B} \boldsymbol{b}_r \end{bmatrix} \in \mathbb{C}^{n \times r},$$

*partitioned to conform with the system. Define* $\widehat{\mathbf{V}} := \mathit{diag}(\mathbf{I}, \mathbf{V}_2, \mathbf{I})$.
*Then, the reduced system*

$$\widehat{\mathbf{E}} \dot{\mathbf{x}}_r = (\widehat{\mathbf{J}} - \widehat{\mathbf{R}}) \mathbf{x}_r + \widehat{\mathcal{B}} \mathbf{u}(t), \quad \mathbf{y}_r = \widehat{\mathcal{C}} \mathbf{x}_r + \widehat{\mathbf{D}}, \tag{24}$$

*with*

$$\widehat{\mathbf{E}} = \widehat{\mathbf{V}}^T \mathbf{E} \widehat{\mathbf{V}}, \quad \widehat{\mathbf{J}} = \widehat{\mathbf{V}}^T \mathbf{J} \widehat{\mathbf{V}}, \quad \widehat{\mathbf{R}} = \widehat{\mathbf{V}}^T \mathbf{R} \widehat{\mathbf{V}},$$
$$\widehat{\mathcal{B}} = \widehat{\mathbf{V}}^T \mathcal{B} = \widehat{\mathbf{V}}^T \mathbf{B} - \widehat{\mathbf{V}}^T \mathbf{P}, \quad \widehat{\mathcal{C}} = \mathbf{C} \widehat{\mathbf{V}} = \mathbf{B}^T \widehat{\mathbf{V}}^T + \mathbf{P}^T \widehat{\mathbf{V}}^T, \quad \textit{and} \quad \widehat{\mathbf{D}} = \mathbf{D}, \tag{25}$$

*is still* pH*, matches the polynomial part of the transfer function, and satisfies the tangential interpolation conditions, i.e.,* $\mathbf{H}(\sigma_i)\boldsymbol{b}_i = \widehat{\mathbf{H}}(\sigma_i)\boldsymbol{b}_i$*, for* $i = 1, 2, \ldots, r$.

***Proof*** It follows from the definitions of $\mathbf{V}$ and $\widehat{\mathbf{V}}$ that $\mathsf{span}(\mathbf{V}) \subseteq \mathsf{span}(\widehat{\mathbf{V}})$. Therefore, the resulting reduced system automatically satisfies the interpolation conditions. Since $\widehat{\mathbf{V}}$ does not alter the algebraic constraints, the polynomial part of its transfer function is still $\mathbf{D}$, matching that of the original model. The reduced system in (24)-(25) is a pHDAE as this one-sided projection retains the original pH structure. □

## 5 Algorithmic Considerations

The preceding analysis presumed that interpolation points and tangent directions were specified beforehand. We consider now how one might make choices that generally produce effective approximations with respect to the $\mathcal{H}_2$ system measure.

### 5.1 $\mathcal{H}_2$-inspired Structure-Preserving Interpolation

The $\mathcal{H}_2$ *distance* between the full model $\mathbf{H}(s)$ and the reduced model $\mathbf{H}_r(s)$ is

$$\left\|\mathbf{H}-\mathbf{H}_r\right\|_{\mathcal{H}_2}=\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\|\mathbf{H}(\imath\omega)-\mathbf{H}_r(\imath\omega)\right\|_F^2 d\omega\right)^{1/2},$$

where $\imath^2=-1$ and $\|\mathbf{M}\|_F$ denote the Frobenius norm of a matrix $\mathbf{M}$. To have a finite $\mathcal{H}_2$ error norm the polynomial parts of $\mathbf{H}_r(s)$ and $\mathbf{H}(s)$ need to match. To make this precise, write $\mathbf{H}(s)=\mathbf{G}(s)+\mathbf{P}(s)$ and $\mathbf{H}_r(s)=\mathbf{G}_r(s)+\mathbf{P}_r(s)$ where $\mathbf{G}(s)$ and $\mathbf{G}_r(s)$ are strictly proper transfer functions, and $\mathbf{P}(s)$ and $\mathbf{P}_r(s)$ are the polynomial parts. Therefore, if $\mathbf{H}_r(s)$ is the $\mathcal{H}_2$-optimal approximation to $\mathbf{H}(s)$, then $\mathbf{P}_r(s)=\mathbf{P}(s)$ and $\mathbf{G}_r(s)$ is the $\mathcal{H}_2$-optimal approximation to $\mathbf{G}(s)$. This suggests that one decompose $\mathbf{H}(s)$ into its rational and polynomial parts $\mathbf{H}(s)=\mathbf{G}(s)+\mathbf{P}(s)$ and then apply $\mathcal{H}_2$ optimal reduction to $\mathbf{G}(s)$. However, this requires the explicit construction of $\mathbf{G}(s)$. This problem was resolved in [17] for *unstructured* index-1 and index-2 DAEs. On the other hand, for the ODE case, [16] proposed a pH structure preserving algorithm for minimizing the $\mathcal{H}_2$ norm. In this section, we aim to unify these two approaches.

First, we briefly revisit the interpolatory $\mathcal{H}_2$ optimality conditions; for details we refer the reader to [1, 4, 14] and the references therein. Since $\mathcal{H}_2$ optimality for the DAE case boils down to optimality for the ODE part, we focus on the latter. Let $\mathbf{G}_r(s)=\sum_{i=1}^{r}\frac{\mathbf{c}_i\mathbf{b}_i^T}{s-\lambda_i}$ be the pole-residue decomposition of $\mathbf{G}_r(s)$. For simplicity we assume simple poles. If $\mathbf{G}_r(s)$ is an $\mathcal{H}_2$ optimal approximation to $\mathbf{G}(s)$, then $\mathbf{G}(-\lambda_i)\mathbf{b}_i=\mathbf{G}_r(-\lambda_i)\mathbf{b}_i$, $\mathbf{c}_i^T\mathbf{G}(-\lambda_i)=\mathbf{c}_i^T\mathbf{G}_r(-\lambda_i)$, and $\mathbf{c}_i^T\mathbf{G}'(-\lambda_i)\mathbf{b}_i=\mathbf{c}_i^T\mathbf{G}_r'(-\lambda_i)\mathbf{b}_i$, for $i=1,2,\ldots,r$ where $'$ denotes the derivative with respect to $s$. Therefore an $\mathcal{H}_2$ optimal reduced model is a bitangential Hermite interpolant where the interpolation points are the mirror images of its poles and the tangent directions are the residue directions. Since the optimality conditions depend on the reduced model to be computed, this requires an iterative algorithm, such as the Iterative Rational Krylov Algorithm [14]. For other approaches to $\mathcal{H}_2$ optimal approximation, see, e.g., [1, 4].

Following [16], to preserve the structure, we will satisfy only a subset of these conditions. We will enforce interpolation points to be the mirror images of the reduced order poles and enforce either left or right-tangential interpolation conditions ($\mathbf{Z}=\mathbf{V}$) without the derivative conditions. However, intuitively, one might expect that in the pH setting this may not cause too much deviation from true optimality, since the input-to-state matrix $\mathbf{B}$ and the state-to-output matrix $\mathbf{C}$ are related.

For simplicity, consider the semi-explicit index-2 case. The other scenarios follow similarly. Starting with the interpolation points $\{\sigma_1,\sigma_2,\ldots,\sigma_r\}$ and the tangent directions $\{\boldsymbol{b}_1,\boldsymbol{b}_2,\ldots,\boldsymbol{b}_r\}$, construct the interpolatory reduced pHDAE $\widehat{\mathbf{H}}(s)$ as in Theorem 3. Let $\widehat{\mathbf{H}}(s)=\sum_{i=1}^{r}\frac{\widehat{\boldsymbol{\ell}}_i\widehat{\boldsymbol{b}}_i^T}{s-\lambda_i}+\mathbf{D}$ be the pole-residue decomposition. Since initially the optimality condition $\sigma_i=-\lambda_i(\widehat{\mathbf{J}}-\widehat{\mathbf{R}},\widehat{\mathbf{E}})$ is not (generally) satisfied, choose

$-\lambda_i(\widehat{\mathbf{J}} - \widehat{\mathbf{R}}, \widehat{\mathbf{E}})$ as the next set of interpolation points and $\widehat{\boldsymbol{b}}_i$ as the next set of tangent directions. This is repeated until convergence upon which the reduced model $\mathbf{H}_r(s)$ is not only a structure-preserving pHDAE, but also satisfies $\sigma_i = -\lambda_i(\widehat{\mathbf{J}} - \widehat{\mathbf{R}}, \widehat{\mathbf{E}})$.

## 5.2 Numerical Example

We illustrate the discussed procedure with the incompressible fluid flow model of the Oseen equations, from [18, §4.1]:

$$\begin{aligned} \partial_t \mathsf{v} &= -(\mathbf{a}\cdot\nabla)\mathsf{v} + \mu\Delta\mathsf{v} - \nabla p + f \quad \text{in } \Omega\times(0,T], \qquad \mathsf{v} = 0, \quad \text{on } \partial\Omega\times(0,T], \\ 0 &= -\operatorname{div}\mathsf{v}, \qquad\qquad\qquad\qquad \text{in } \Omega\times(0,T], \qquad \mathsf{v} = \mathsf{v}^0, \qquad\quad \text{in } \Omega\times 0, \end{aligned}$$

where $\mathsf{v}$ and $p$ are the velocity and pressure variables, $\mu = 1$ is the dynamic viscosity, $\mathbf{a} = [\,1\ 1\,]^T$ is the convective velocity, and $\Omega = (0,1)^2$ with boundary $\partial\Omega$. $f$ is an externally imposed body force that, for simplicity, is assumed to be separable: $f(x,t) = b(x)u(t)$. A finite-difference discretization on a staggered rectangular grid leads to a single-input/single-output index-2 pHDAE of the form (12), see [18]. In our model, we used a uniform grid with 50 grid points yielding a descriptor system pHDAE of order $n = 7399$, of which $n_1 = 4900$ degrees of freedom are for velocity and $n_2 = 2499$ for pressure. We initialize our $\mathcal{H}_2$-based approach with logarithmically spaced interpolation points in the interval $[10^{-2}, 10^4]$ and construct a reduced model of the form (14) with orders $r = 1, 2, \ldots, 10$. For every $r$ value, we compute $\|\mathbf{H} - \widehat{\mathbf{H}}\|_\infty / \|\mathbf{H}\|_\infty$ where $\|\mathbf{H}\|_\infty = \sup_{\omega\in\mathbb{R}} \mid \mathbf{H}(\imath\omega) \mid$. Figure 1 shows that the reduced transfer function accurately approximates the original one; for $r = 10$, the relative error is around $10^{-5}$.

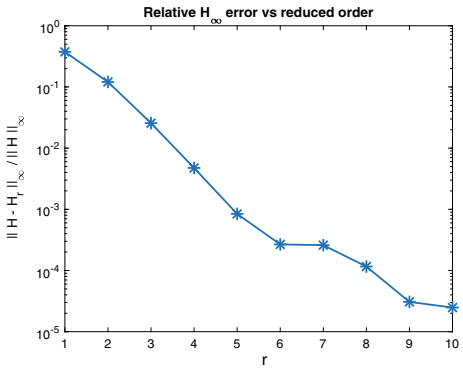


**Fig. 1** Evolution of the model reduction error for Oseen example as $r$ varies

## 6 Conclusions

We have presented interpolatory model reduction methods for several classes of large scale linear time-invariant port-Hamiltonian differential-algebraic systems. We have shown how constraints can be represented in the transfer function in such a way that the polynomial part can be preserved with interpolatory methods while still retaining important system structure. The results were illustrated with a numerical example from flow control.

**Acknowledgements** The works of Beattie and Gugercin were supported in parts by NSF through Grant DMS-1819110. The work of Mehrmann was supported by Deutsche Forschungsgemeinschaft through CRC 910 within project A02. Parts of this work were completed while Beattie and Gugercin visited TU Berlin, for which Gugercin acknowledges support through CRC 910 and Beattie acknowledges support through CRC TRR 154 as well as DFG Research Training Group DAEDALUS.

## References

1. Antoulas, A.C., Beattie, C.A., Gugercin, S.: Interpolatory methods for model reduction. Comput. Sci. Eng. 21. SIAM, Philadelphia (2020)
2. Beattie, C.A., Gugercin, S.: Interpolatory projection methods for structure-preserving model reduction. Syst. Control Lett. **58**(3), 225–232 (2009)
3. Beattie, C.A., Gugercin, S.: Structure-preserving model reduction for nonlinear port-Hamiltonian systems. In: 50th IEEE Conference on Decision and Control and European Control Conference, pp. 6564–6569 (2011)
4. Beattie, C.A., Gugercin, S.: Model reduction by rational interpolation. In: Benner, P., Cohen, A., Ohlberger, M., Willcox, K. (eds.) Model Reduction and Approximation: Theory and Algorithms, CSE 15, Chap. 7, pp. 297–334. SIAM, Philadelphia (2017)
5. Beattie, C.A., Mehrmann, V., Xu, H., Zwart, H.: Linear port-Hamiltonian descriptor systems. Math. Control, Signals, Syst. **30**(4), 1–27 (2018)
6. Binder, A., Mehrmann, V., Miedlar, A., Schulze, P.: A `Matlab` toolbox for the regularization of descriptor systems arising from generalized realization procedures. Preprint 24–2015, Institut für Mathematik, TU Berlin (2015)
7. Bunse-Gerstner, A., Byers, R., Mehrmann, V., Nichols, N.K.: Feedback design for regularizing descriptor systems. Linear Algebra Appl. **299**(1–3), 119–151 (1999)
8. Campbell, S.L., Kunkel, P., Mehrmann, V.: Regularization of linear and nonlinear descriptor systems. In: Biegler, L.T., Campbell, S.L., Mehrmann, V. (eds.) Control and Optimization with Differential-Algebraic Constraints, Advances in Design and Control, vol. 23, pp. 17–36. SIAM (2012)
9. Chaturantabut, S., Beattie, C., Gugercin, S.: Structure-preserving model reduction for nonlinear port-Hamiltonian systems. SIAM J. Sci. Comput. **38**(5), B837–B865 (2016)
10. Dai, L.: Singular Control Systems. Lecture Notes in Control and Information Sciences, vol. 118. Springer, Berlin (1989)
11. Egger, H., Kugler, T., Liljegren-Sailer, B., Marheineke, N., Mehrmann, V.: On structure-preserving model reduction for damped wave propagation in transport networks. SIAM J. Sci. Comput. **40**(1), A331–A365 (2018)
12. Gallivan, K., Vandendorpe, A., Van Dooren, P.: Model reduction of MIMO systems via tangential interpolation. SIAM J. Matrix Anal. Appl. **26**(2), 328–349 (2005)

13. Gantmacher, F.R.: The Theory of Matrices, vol. II. Chelsea Publishing Company, New York, N.Y. (1959)
14. Gugercin, S., Antoulas, A.C., Beattie, C.A.: $\mathcal{H}_2$ Model Reduction for Large-Scale Linear Dynamical Systems. SIAM J. Matrix Anal. Appl. **30**(2), 609–638 (2008)
15. Gugercin, S., Polyuga, R.V., Beattie, C., van der Schaft, A.: Interpolation-based H2 model reduction for port-Hamiltonian systems. In: Proceedings 48th IEEE Conference on Decision and Control, pp. 5362–5369. IEEE (2009)
16. Gugercin, S., Polyuga, R.V., Beattie, C.A., van der Schaft, A.J.: Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems. Automatica **48**, 1963–1974 (2012)
17. Gugercin, S., Stykel, T., Wyatt, S.: Model reduction of descriptor systems by interpolatory projection methods. SIAM J. Sci. Comput. **35**(5), B1010–B1033 (2013)
18. Hauschild, S.-A., Marheineke, N., Mehrmann, V.: Model reduction techniques for linear constant coefficient port-Hamiltonian differential-algebraic systems. Control and Cybernetics, To appear (2020). arXiv:1901.10242
19. Heinkenschloss, M., Sorensen, D.C., Sun, K.: Balanced truncation model reduction for a class of descriptor systems with application to the Oseen equations. SIAM J. Sci. Comput. **30**(2), 1038–1063 (2008)
20. Kunkel, P., Mehrmann, V.: Differential-Algebraic Equations. European Mathematical Society (EMS), Zürich (2006)
21. Mayo, A.J., Antoulas, A.C.: A framework for the solution of the generalized realization problem. Linear Algebra Appl. **425**(2–3), 634–662 (2007)
22. Mehl, C., Mehrmann, V., Wojtylak, M.: Linear algebra properties of dissipative port-Hamiltonian descriptor systems. SIAM J. Matrix Anal. Appl. **39**(3), 1489–1519 (2018)
23. Mehl, C., Mehrmann, V., Wojtylak, M.: Distance problems for dissipative Hamiltonian systems and related matrix polynomials. Technical Report 01-2020, Institut für Mathematik, TU Berlin (2020)
24. Mehrmann, V., Morandin, R.: Structure-preserving discretization for port-hamiltonian descriptor systems. In: 58th IEEE Conference on Decision and Control (CDC), Nice, pp. 6863–6868 (2019)
25. Mehrmann, V., Stykel, T.: Balanced truncation model reduction for large-scale systems in descriptor form. In: Benner, P., Mehrmann, V., Sorensen, D.C. (eds.) Dimension Reduction of Large-Scale Systems. Lecture Notes in Computational Science and Engineering, vol. 45, pp. 83–115. Springer, Berlin/Heidelberg, Germany (2005)
26. Polyuga, R.V., van der Schaft, A.: Structure-preserving moment matching for port-Hamiltonian systems: Arnoldi and Lanczos. IEEE Trans. Autom. Control **56**(6), 1458–1462 (2011)
27. Polyuga, R.V., van der Schaft, A.: Effort- and flow-constraint reduction methods for structure-preserving model reduction of port-Hamiltonian systems. Sys. Control Lett. **61**(3), 412–421 (2012)
28. Polyuga, R.V., van der Schaft, A.J.: Structure preserving model reduction of port-Hamiltonian systems by moment matching at infinity. Automatica **46**, 665–672 (2010)
29. Stykel, T.: Balanced truncation model reduction for semidiscretized Stokes equation. Linear Algebra Appl. **415**(2–3), 262–289 (2006)
30. van der Schaft, A.: Port-Hamiltonian differential-algebraic systems. In: Surveys in Differential-Algebraic Equations. I, pp. 173–226. Springer, Heidelberg (2013)
31. Wolf, T., Lohmann, B., Eid, R., Kotyczka, P.: Passivity and structure preserving order reduction of linear port-Hamiltonian systems using Krylov subspaces. Eur. J. Control **16**(4), 401–406 (2010)