Model Reduction and Realization Theory of Linear Switched Systems

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Abstract The goal of this chapter is to present an overview of some recent results on model reduction of linear switched systems and their interplay with realization theory of these systems. The emphasis will be on those results on model reduction which are directly related to realization theory, we do not aim at being exhaustive. In particular, we will review some recent results on balanced truncation and moment matching, focusing on the theoretical aspects rather than on the computational ones.

Keywords Hybrid systems · Realization theory · Model reduction · Balanced truncation · Moment matching

1 Introduction

In this chapter we will present an overview of some recent results on model reduction of hybrid systems which rely heavily on realization theory. Both model reduction and realization theory have been central to Prof. Antoulas's work, so we feel that showing the interaction between these two topics for hybrid systems is a fitting tribute to his scientific contribution.

Hybrid systems [13] are non-linear systems which combine continuous and discrete behavior. More precisely, a hybrid system is a finite collection of continuous-state dynamical systems, indexed by a set of so called *discrete modes (or states)*. The state of each dynamical system is governed by a set of differential or difference equations. The discrete mode in any time instant can be chosen arbitrarily or

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it may depend on the value of the continuous state and possibly other constraints, which are referred to as *guards*. The transitions between the discrete states may result in a jump in the state of the underlying continuous dynamical system. This jump is defined by the application of the so called *reset maps*. *Linear switched systems (LSSs)* [19, 37] are the simplest and most widely studied subclass of hybrid systems where the continuous subsystems are linear systems, and the change of the discrete state is externally generated.

While there is a large literature on control of hybrid systems in general, and of LSSs in particular, the computational complexity of the existing algorithms for hybrid systems is high, and hence they cannot be applied to large scale systems. In order to address this problem, model reduction methods were proposed for hybrid systems. Model reduction methods for LSSs can be grouped into the following categories.

LMI-based methods. These methods compute the matrices of the reduced order model by solving a set of linear matrix inequalities (LMIs). The disadvantage is that the proposed conditions are only sufficient, and the trade-off between the dimension of the reduced model and the error bound is not clear. Moreover, the computational complexity of solving those LMIs might be too high. Without claiming completeness, we mention [12, 38–40].

Methods based on local Gramians. These algorithms are based on finding observability/controllability Gramians for each linear subsystem. For these methods often there are no error bounds and the reduced order model need not be well-posed. Examples of such papers include [7–9, 14, 16, 20, 21]. Note that to the best of our knowledge, the only algorithm which always yields a well-posed LSS of the same type as the original one and for which there exists an analytic error bound (which holds only for slow switching) is the one of [14]. For the case of jump-linear systems with a stochastic switching a similar approach was taken in [18] and an error bound was derived.

Methods based on common Gramians. These methods rely on finding the same observability/controllability Gramians for each linear subsystem. These Gramians are derived as solutions of a suitable LMI. Such algorithms were described in [34, 35] and an analytic error bound was derived in [29]. These algorithms apply only to LSSs which have a global quadratic Lyapunov function. Moreover, the computational complexity of solving the corresponding LMIs is high. In order to address this problem [31] proposes to replace LMIs by Lyapunov equations. The downside of the latter approach is that the error bounds of [29] do not always apply. Another approach was proposed in [33], where for a specific subclass of LSSs, the original LSS is replaced by a linear time-invariant (LTI) system and classical balanced truncation is applied.

Moment matching. The idea behind these algorithms is to find a reduced order linear switched system such that certain coefficients of the series expansions of the input-output maps of the original and the reduced order system coincide. The series expansion can be the Taylor series with respect to switching times, in which case a number of the so-called Markov parameters coincide. Alternatively, the series expansion can be a Laurent-series expansion of a multivariate Laplace transform of the input-output map around a certain frequency. The former approach was pursued in [1, 4, 5], the latter in [15]. While those methods do not allow for analytical error bounds, under suitable assumption it can be guaranteed that the reduced model will have the same input-output behavior for certain switching signals [1, 4, 5]. A somewhat different approach is that of [32], which considers LSSs with state-dependent switching and it proposes a model reduction procedure which guarantees that the reduced model has the same steady-state output response to certain inputs as the original model.

In this chapter we discuss some recent results on balanced truncation and moment matching for LSSs. For the sake of simplicity, *we consider LSSs only in continuous time, and we will assume that all the linear subsystems are defined on the same state-space and the reset maps are identity*. Most of the presented results are true for the discrete-time case too, and some can be extended to include LSSs with reset maps which are not identity. We will cite the relevant literature on these extensions.

The model reduction methods to be discussed in this chapter rely heavily on realization theory. The goal of realization theory is to understand the relationship between input-output behaviors and internal (state-space) representations. A fairly complete realization theory was developed for LSSs, see the discussion and references in [22–25, 28]. The results on realization theory of LSSs rely on realization theory of bilinear systems and recognizable formal power series [6, 17, 36].

Realization theory is relevant for model reduction in many ways. First, minimization and realization algorithms can be viewed as simple model reduction algorithms. Moreover, the relationship between span-reachability, observability and minimality is closely related to the existence of Gramians which are used in balanced truncation. Realization theory is even more critical for moment matching, as the latter can be viewed as partial realization algorithm. We will elaborate on the precise relationship later on.

The chapter is structured as follows. In Sect. 2 we present the formal definition of the class of LSSs and the corresponding terminology. In Sect. 3 we present a brief overview of the relevant results on realization theory of switched systems. In Sect. 4 we discuss model reduction: In Sect. 4.1 we present balanced truncation and in Sect. 4.2 we discuss moment matching for LSSs.

2 Linear Switched Systems: Basic Definitions

A *linear switched system* (LSS) is a control system of the form

$$\Sigma \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the continuous-valued state at time t, $\sigma(t) \in Q$ is the discrete mode at time t, $y(t) \in \mathbb{R}^p$ is the output at time t, and $u(t) \in \mathbb{R}^m$ is the continuous-valued input at time t. The set Q is a finite one, and it is referred to as the set of discrete modes or states. Moreover, $A_q \in \mathbb{R}^{n\times n}$, $B_q \in \mathbb{R}^{n\times m}$, $C_q \in \mathbb{R}^{p\times n}$ are the matrices of the linear system in the discrete state $q \in Q$. The number n is called the *dimension (order) of* Σ and will be denoted by $\dim(\Sigma)$. The short-hand notation

$$\Sigma = \{A_q, B_q, C_q\}_{q\in Q}$$

is used for LSSs of the form (1).

Let $\mathbb{R}_+$ be the *real time-axis*, i.e. $\mathbb{R}_+ = [0, +\infty)$. Denote by $\mathrm{AC}(\mathbb{R}_+, \mathbb{R}^k)$ respectively $PC(\mathbb{R}_+, \mathbb{R}^k)$ the set of all absolutely continuous respectively piecewise-continuous functions of the form $h : \mathbb{R}_+ \to \mathbb{R}^k$.[1] Let $\mathcal{X} = AC(\mathbb{R}_+, \mathbb{R}^n)$, $\mathcal{U} = PC(\mathbb{R}_+, \mathbb{R}^m)$, $\mathcal{Y} = PC(\mathbb{R}_+, \mathbb{R}^p)$, and let $\mathcal{Q}$ be the set of all piecewise-constants functions $g : \mathbb{R}_+ \to Q$. A tuple $(x, u, \sigma, y) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Q} \times \mathcal{Y}$ is called a solution, if (x, u, σ, y) satisfy (1). For any switching signal $\sigma \in \mathcal{Q}$, input $u \in \mathcal{U}$ and initial state $x_0 \in \mathbb{R}^n$, there exists a unique solution (x, u, σ, y) of Σ such that $x(0) = x_0$. This prompts us to define the *input-to-state* map $\mathcal{X}_\Sigma : \mathcal{U} \times \mathcal{Q} \to \mathcal{X}$ and the *input-output* map $Y_\Sigma : \mathcal{U} \times \mathcal{Q} \to \mathcal{Y}$ of an LSS Σ as follows: $X_\Sigma(u, \sigma) = x$, and $Y_\Sigma(u, \sigma) = y$ if and only if (x, u, σ, y) is the unique solution of Σ such that $x(0) = 0 \in \mathbb{R}^n$.[2]

Intuitively, an LSS is just a control system which switches among finitely many linear time-invariant systems. The switching signal is part of the input. Whenever a switch occurs, the continuous state remains the same, only the differential equation governing the state and output evolution changes. That is, whenever we switch to a new linear system, we start the new linear system from the state which is the final state of the previous linear system.

We model potential input-output behaviors of LSSs as functions

$$f : \mathcal{U} \times \mathcal{Q} \to \mathcal{Y}, \tag{2}$$

and call them *input-output maps*. They capture the behavior of a black-box, which reacts to piecewise-continuous inputs and switching sequences by generating outputs in $\mathbb{R}^p$. Next, we define what it means that this black-box can be modelled as an LSS, i.e. that an LSS is a realization of f. The LSS Σ is a *realization of an input-output map f of the form* (2) , if $Y_\Sigma = f$, i.e. if the input-output map of Σ coincides with f. If Σ is a realization of f, then Σ is *a minimal realization of* f, if for any LSS realization $\hat{\Sigma}$ of f, $\dim \Sigma \le \dim \hat{\Sigma}$. Two LSSs Σ_1, Σ_2 are said to be *input-output equivalent*, if their input-output maps are equal, i.e. $Y_{\Sigma_1} = Y_{\Sigma_2}$. An LSS Σ is said to be *minimal*, if it is a minimal realization of its own input-output map $f = Y_\Sigma$.

[1] Piecewise-continuous functions have a finite number of discontinuities on each finite interval and at each point of discontinuity, the left- and right-hand side limits exist and are finite.

[2] The definition of the input-to-state and input-output map can be extended to include non-zero initial states [23, 24]. We prefer to stick to zero initial state to avoid excessive notation and terminology.

3 Realization Theory of Linear Switched Systems

Below we present the main results on minimality, existence of a realization, and a Ho-Kalman-like realization algorithm for LSSs. These results are identical for discrete-time LSSs [25], and can be extended to LSSs with linear reset maps [24, 27]. In order to present these results, and later throughout the chapter, we will use the following notation from automata theory [11].

Notation (Q^*, Q^+, ϵ, q^k) Denote by Q^+ the set of all finite sequences of elements of Q, i.e. each element $w \in Q^+$ is of the form $w = a_1 a_2 \cdots a_k$ for some $a_1, a_2, \ldots, a_k \in Q$, $k \in \mathbb{N}, k > 0$. The integer k is called the *length* of w and it is denoted by $|w|$. Let $\epsilon \notin Q^+$ be a symbol, which we will call the *empty sequence or empty word*. By convention, the length of ϵ is defined to be zero. Denote by Q^* the set $Q^+ \cup \{\epsilon\}$. For any two sequences $w, v \in Q^*$, we denote by wv the concatenation of w and v. If $w, v \in Q^+$ are of the form $v = v_1 v_2 \cdots v_k$, $k > 0$ and $w = w_1 w_2 \cdots w_m$, $m > 0$, $v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_m \in Q$, then define $vw = v_1 v_2 \cdots v_k w_1 w_2 \cdots w_m$. If $v = \epsilon$ and $w \in Q^*$, then define $vw = w$, and if $w = \epsilon$ and $v \in Q^*$, then define $vw = v$. For $q \in Q$ and $k \in \mathbb{N}$, $k > 0$, we denote by q^k the sequence $\overbrace{qq \cdots q}^{k-times}$; by convention $q^0 = \epsilon$.

3.1 Minimality of Linear Switched Systems

We start by presenting the main results on minimality of LSSs. To this end, we introduce the notions of observability, span-reachability and isomorphism. Let Σ be an LSS of the form (1). Then Σ is said to be *observable*, if for any two distinct states $x_{1,0} \neq x_{2,0} \in \mathbb{R}^n$, there exists an input u and a switching signal σ, such that if (x_i, u, σ, y_i), $i = 1, 2$ are two solutions of Σ with $x_i(0) = x_{i,0}$, $i = 1, 2$, then $y_1 \neq y_2$, i.e., the outputs induced by the initial states $\{x_{i,0}\}_{i=1,2}$ under the input u and switching signal σ are different. Let $\mathscr{R}_0(\Sigma) \subseteq \mathbb{R}^n$ denote the reachable set of the Σ from the zero initial state, i.e., $\mathscr{R}_0(\Sigma)$ is the set of all vectors x_f such that for some $t \in \mathbb{R}_+, x_f = X_\Sigma(u, \sigma)(t)$ for some $u \in \mathcal{U}, \sigma \in \mathcal{Q}$. We say that Σ is *span-reachable*, if $\mathbb{R}^n = \text{Span}\, \mathscr{R}_0(\Sigma)$, i.e., if $\mathbb{R}^n$ is the smallest vector space containing $\mathscr{R}_0(\Sigma)$. Note that span-reachability and reachability are the same in continuous-time [37].

Two LSSs $\Sigma_1 = \{A_q, B_q, C_q\}_{q \in Q}$, and $\Sigma_2 = \{A_q^a, B_q^a, C_q^a\}_{q \in Q}$ are said to be *isomorphic*, if there exists a non-singular square matrix $\mathcal{S}$ such that $A_q^a = \mathcal{S} A_q \mathcal{S}^{-1}$, $B_q^a = \mathcal{S} B_q$, and $C_q^a = C_q \mathcal{S}^{-1}$ for all $q \in Q$.

Theorem (Minimality, [22, 23]) *An LSS is minimal, if and only if it is span-reachable and observable. If Σ_1 and Σ_2 are two minimal LSSs and they are input-output equivalent, then they are isomorphic.* □

Note that minimality of an LSS does not imply minimality of any of its linear subsystems, see [23] for a counter-example. Hence, realization theory of LSSs cannot be reduced to realization theory of linear subsystems.

We present an algorithm for converting any LSS to a minimal one while preserving its input-output map. To this end, we define the subspaces

$$\mathcal{V}^* = \mathrm{Span}\mathscr{R}_0(\Sigma), \tag{3}$$

$$\mathcal{W}^* = \mathrm{Span}\{x_0 \in \mathbb{R}^n \mid \forall \sigma \in \mathcal{Q} : \exists x \in \mathcal{X} : \tag{4}$$
$$(x, 0, \sigma, 0) \text{ is a solution of } \Sigma \text{ with } x(0) = x_0 \}.$$

It can be shown that Σ is span-reachable if and only if $\mathcal{V}^* = \mathbb{R}^n$, and Σ is observable, if and only if $\mathcal{W}^* = \{0\}$, see [24, 37].

Procedure (*Minimization*) Consider the factor space $\mathcal{V}^*/\mathcal{W}^*$; recall that the elements of this linear space are equivalence classes generated by the following equivalence relation $\approx$ on $\mathcal{V}^*$: $x_1 \approx x_2$ if and only if $x_1 - x_2 \in \mathcal{W}^*$. Let $[x] = \{y \mid \mid x \approx y\}$ be the equivalence class represented by x. The linear vector space $\mathcal{V}^*/\mathcal{W}^*$ is finite dimensional. Define the linear maps $\mathcal{A}_q : \mathcal{V}^*/\mathcal{W}^* \ni [x] \mapsto [A_q x] \in \mathcal{V}^*/\mathcal{W}^*$, $\mathcal{C}_q : \mathcal{V}^*/\mathcal{W}^* \ni [x] \mapsto C_q x \in \mathbb{R}^p$, $\mathcal{B}_q : \mathbb{R}^m \ni u \mapsto [B_q u] \in \mathcal{V}^*/\mathcal{W}^*$, $q \in Q$. It can be shown [22, 23] that $A_q\mathcal{V}^* \subseteq \mathcal{V}^*$, $A_q\mathcal{W}^* \subseteq \mathcal{W}^*$, $\mathrm{Im} B_q \subseteq \mathcal{V}^*$, $\mathcal{W}^* \subseteq \ker C_q$. From this it follows that the linear maps $\mathcal{A}_q, \mathcal{B}_q, \mathcal{C}_q$ are well defined. Choose a finite basis in $\mathcal{V}^*/\mathcal{W}^*$, and let ${}^{\mathrm{m}}A_q, {}^{\mathrm{m}}B_q, {}^{\mathrm{m}}C_q$ be the matrix representations of the linear maps $\mathcal{A}_q, \mathcal{B}_q, \mathcal{C}_q$, in that basis. Then $\Sigma_{\mathrm{m}} = \{{}^{\mathrm{m}}A_q, {}^{\mathrm{m}}B_q, {}^{\mathrm{m}}C_q\}_{q \in Q}$ is a minimal LSS which is input-output equivalent to Σ.

3.2 Existence of a Realization, Ho-Kalman Algorithm, Markov Parameters

We first define the notion of a generalized kernel representation, existence of which is a necessary condition for existence of a realization by an LSS. An input-output map f has a *generalized kernel representation*, if there exists a family of functions $\{G^f_v : \mathbb{R}^{|v|}_+ \to \mathbb{R}^{p \times m}\}_{v \in Q^+}$, such that for all $u \in \mathcal{U}$, $\sigma \in \mathcal{Q}$,

$$f(u, \sigma)(t) = \sum_{i=1}^{k} \int_0^{t_i} G^f_{q_i \cdots q_k}(t_i - s, t_{i+1}, \ldots, t_k) u(s + T_{i-1}) ds$$

where $q_i \in Q$, $0 < t_i \in \mathbb{R}_+$, $i = 1, \ldots, k$ are such that $\sigma(s) = q_i$ for $s \in [T_{i-1}, T_i)$ for some $T_j \in \mathbb{R}_+$, $T_j < T_{j+1}$, $j \in \mathbb{N}$, $T_0 = 0$, and $t \in [T_{k-1}, T_k)$ for some $k > 0$, and $t_i = T_i - T_{i-1}$, for $i = 1, \ldots, k-1$ and $t_k = t - T_{k-1}$. Moreover, $\{G^f_v\}_{v \in Q^+}$ have to satisfy a number of technical conditions [22, 23]. From [22, 23] it follows that Σ of the form (1) is a realization of f, if and only if f has a generalized kernel representation and for all $q_1, \ldots, q_k \in Q$, $t_1, \ldots, t_k \in \mathbb{R}_+$, $k > 0$,

$$G^f_{q_1\cdots q_k}(t_1,\ldots,t_k) = C_{q_k} e^{A_{q_k} t_k} \cdots e^{A_{q_2} t_2} e^{A_{q_1} t_1} B_{q_1}. \tag{5}$$

From the technical conditions in [22, 23] on generalized kernel representations it follows that if f has a generalized kernel representation, then there exist functions $\{S^f_{r,q} : Q^* \to \mathbb{R}^{p\times m}\}_{r,q\in Q}$ such that

$$G^f_{q_1\cdots q_k}(t_1,\ldots,t_k) = \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} S^f_{q_k,q_1}(q_1^{\alpha_1}\cdots q_k^{\alpha_k})\Pi_{i=1}^k \frac{t_i^{\alpha_i}}{\alpha_i!}.$$

That is, the functions $\{S^f_{r,q}\}_{r,q\in Q}$ uniquely determine $\{G^f_v\}_{v\in Q^+}$ and hence f, and conversely, the functions $\{S^f_{r,q}\}_{r,q\in Q}$ can be recovered from f. The values of $\{S^f_{r,q}\}_{r,q\in Q}$ are called the *Markov parameters* of f.

Notation For every $v \in Q^*$ and a collection of $n\times n$ matrices $\{A_q\}_{q\in Q}$ define the matrix A_v as follows: if $v = q_1\cdots q_k$ with $q_1,\ldots,q_k \in Q$, $k > 0$, then $A_v = A_{q_k}A_{q_{k-1}}\cdots A_{q_1}$, and if $v = \epsilon$, then $A_\epsilon = I_n$, where I_n is the $n\times n$ identity matrix.

Lemma ([22, 23]) *An LSS of the form (1) is a realization of f, if and only if f has a generalized kernel representation, and for all $v \in Q^*$, $q, q_0 \in Q$, $S^f_{q,q_0}(v) = C_q A_v B_{q_0}$.*

In order to present a Ho-Kalman-like realization algorithm and a Hankel-rank condition for existence of an LSS realization, *we consider only the SISO case, i.e., $p = m = 1$*, see [4, 26] for the general case, and we adapt the notion of selection from [4, 26]. We call any subset $\alpha \subset Q^*\times Q$ a *selection*. Consider selections α and β, such that α is of finite cardinality n_α and β is of finite cardinality n_β respectively. Fix an enumeration

$$\alpha = \{(u_i, q_i)\}_{i=1}^{n_\alpha},\ \ \beta = \{(v_j,\sigma_j)\}_{j=1}^{n_\beta}. \tag{6}$$

Let us now define the matrix $\mathcal{H}^f_{\alpha,\beta} \in \mathbb{R}^{n_\alpha\times n_\beta}$ as follows:

$$\left[\mathcal{H}^f_{\alpha,\beta}\right]_{i,j} = S^f_{q_i,\sigma_j}(v_j u_i)\ \ i = 1,\ldots,n_\alpha,\ \ j = 1,\ldots,n_\beta. \tag{7}$$

Intuitively, the rows of $\mathcal{H}^f_{\alpha,\beta}$ are indexed by the elements of α, and the columns by the elements of β.

Theorem (Existence [23]) *The input-output map f has a realization by an LSS, if and only if f has a generalized kernel representation, and*

$$\sup_{\alpha,\beta\subseteq Q^*\times Q,\alpha,\beta \textit{ are finite}} \operatorname{rank} \mathcal{H}^f_{\alpha,\beta} = n_{\mathrm{m}} < +\infty. \tag{8}$$

If (8) holds, then n_{m} is the dimension of a minimal LSS realization of f. □

The proof of Theorem 3.2 leads to the following Ho-Kalman-like algorithm. For the selections α, β from (6), define the matrices $\mathcal{H}^f_{q,\alpha,\beta} \in \mathbb{R}^{n_\alpha \times n_\beta}$, $\mathcal{H}^f_{\alpha,q} \in \mathbb{R}^{n_\alpha \times 1}$ and $\mathcal{H}^f_{q,\beta} \in \mathbb{R}^{1 \times n_\beta}$, $q \in Q$ as follows:

$$\left[\mathcal{H}^f_{q,\alpha,\beta}\right]_{i,j} = S^f_{q_i,\sigma_j}(v_j q u_i), \ \left[\mathcal{H}^f_{\alpha,q}\right]_{i,1} = S^f_{q_i,q}(u_i), \ \left[\mathcal{H}^f_{q,\beta}\right]_{1,j} = S^f_{q,\sigma_j}(v_j).$$

for all $i = 1, \ldots, n_\alpha$, $j = 1, \ldots, n_\beta$.

Procedure (*Ho-Kalman algorithm*) Assume that rank $\mathcal{H}^f_{\alpha,\beta} = n_{\mathrm{m}}$. Consider the factorization $H^f_{\alpha,\beta} = O_{n_{\mathrm{m}}} R_{n_{\mathrm{m}}}$ such that $O_{n_{\mathrm{m}}} \in \mathbb{R}^{n_\alpha \times n_{\mathrm{m}}}$, $R_{n_{\mathrm{m}}} \in \mathbb{R}^{n_{\mathrm{m}} \times n_\beta}$, rank $O_{n_{\mathrm{m}}} =$ rank $R_{n_{\mathrm{m}}} = n_{\mathrm{m}}$. Define

$$\hat{A}_q = O^+_{n_{\mathrm{m}}} \mathcal{H}^f_{q,\alpha,\beta} R^+_{n_{\mathrm{m}}}, \ \hat{B}_q = O^+_{n_{\mathrm{m}}} \mathcal{H}^f_{\alpha,q}, \ \hat{C}_q = \mathcal{H}^f_{q,\beta} R^+_{n_{\mathrm{m}}}$$

and $O^+_{n_{\mathrm{m}}}$, $R^+_{n_{\mathrm{m}}}$ is the Moore-Penrose inverse of $O_{n_{\mathrm{m}}}$ and $R_{n_{\mathrm{m}}}$ respectively. Define the LSS $\hat{\Sigma} = \{\hat{A}_q, \hat{B}_q, \hat{C}_q\}_{q \in Q}$.

Lemma ([10, 22, 26]) *If n_{m} is the dimension of a minimal LSS realization of f, then the LSS $\hat{\Sigma}$ defined in Procedure 3.2 is a minimal realization of f.*

From [28] it follows that we can choose $\alpha_N = \beta_N = \{(v, q) \mid v \in Q^*, |v| \leq N, q \in Q\}$, where N is any integer not smaller than the dimension of a minimal LSS realization of f. This choice of the nice selection is not very practical, as the size of the Hankel-matrix $H^f_{\alpha_N,\beta_N}$ grows exponentially with N. Note that the Ho-Kalman algorithm described above can be used to define a smooth (analytic) manifold structure for the space of equivalence classes of minimal LSSs related by isomorphism [26].

4 Model Reduction

In model reduction, we would like to find LSSs of *smaller* dimension, input-output maps of which are *close* (but not necessarily equal) to that of the original LSS. This is in contrast to realization theory, where we were interested in finding a minimal LSS with *exactly* the same input-output map as the original one. The latter is a special case of the former. Model reduction algorithms follow the following general pattern.

Algorithm 1 Model reduction/minimization algorithm
Inputs: $\Sigma = \{A_q, B_q, C_q\}_{q \in Q}$, matrices $V \in \mathbb{R}^{n \times r_1}$, $W \in \mathbb{R}^{r_2 \times n}$.
Output: $\bar{\Sigma} = (\bar{A}_q, \bar{B}_q, \bar{C}_q\}_{q \in Q}$.

1: Let $r =$ rank WV and let $S \in \mathbb{R}^{r \times r_2}$, $T \in \mathbb{R}^{r_1 \times r}$, $SWVT = I_r$.
2: $\bar{A}_q = SWA_qVT$, $\bar{C}_q = C_qVT$, $\bar{B}_q = SWB_q$, $q \in Q$.
3: **return** $\bar{\Sigma} = \{\bar{A}_q, \bar{B}_q, \bar{C}_q\}_{q \in Q}$.

Intuitively, Algorithm 1 restricts the system to the set $\mathrm{Im}\,V$ and then merges those of its states x_1, x_2 for which $x_1 - x_2 \in \ker W$. If $\ker W = \mathcal{W}^*$ and $\mathrm{Im}\,V = \mathcal{V}^*$, with $\mathcal{W}^*$, $\mathcal{V}^*$ from (3), then the LSS $\bar{\Sigma}$ returned by Algorithm 1 is a minimal LSS which is input-output equivalent to Σ, i.e., Algorithm 1 is just an implementation of Procedure 3.1. Algorithms for computing such matrices W, V such that $\ker W = \mathcal{W}^*$ and $\mathrm{Im}\,V = \mathcal{V}^*$ are described in [22, 24].

In case of model reduction, Algorithm 1 can again be used. However, instead of applying it with matrices W and V such that $\ker W = \mathcal{W}^*$ and $\mathrm{Im}\,V = \mathcal{V}^*$, we use matrices W, V such that $\mathcal{W}^* \subseteq \ker W$ and $\mathrm{Im}\,V \subseteq \mathcal{V}^*$, i.e., we restrict the system to a subset of the set of reachable states, or we merge states which do not produce the same input-output behavior. The resulting LSS model will no longer be a realization of f, but its input-output map will approximate f in a suitable sense. Depending on the method we use, we will either be able to provide a global error bound on the difference between the input-output maps of the original model and the reduced one, or state that for certain switching sequences the two input-output maps coincide. We will elaborate on various methods below.

4.1 Model Reduction by Balanced Truncation

Let Σ be an LSS of the form (1), and assume that Σ is *quadratically stable*, i.e., there exists a matrix $P > 0$ such that $\forall q \in Q : A_q^T P + P A_q < 0$. In this case Σ is globally uniformly asymptotically (exponentially) stable [19] with the Lyapunov function $V(x) = x^T P x$. A matrix $\mathcal{Q}$ will be called an *observability Gramian*, if

$$\forall q \in Q : A_q^T \mathcal{Q} + \mathcal{Q} A_q + C_q^T C_q \leq 0, \ \mathcal{Q} > 0. \tag{9}$$

Likewise, a matrix $\mathcal{P}$ will be called a *controllability Gramian*, if

$$\forall q \in Q : A_q \mathcal{P} + \mathcal{P} A_q^T + B_q B_q^T \leq 0, \ \mathcal{P} > 0. \tag{10}$$

Note that in contrast to the linear case, controllability/observability Gramians for LSSs are not unique, since they are solutions of LMIs and not of Lyapunov equations.

The procedure for balanced truncation is as follows. We apply Algorithm 1 with the following choice of W and V. Find U such that $\mathcal{P} = UU^T$ and find an orthogonal L such that $U^T \mathcal{Q} U = L \Lambda^2 L^T$, where $\Lambda = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$. Pick $r \leq n$. Define

$$W = \begin{bmatrix} I_r & 0 \end{bmatrix} \Lambda^{1/2} L^T U^{-1}, \ V = U L \Lambda^{-1/2} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \ .$$

Then $\mathrm{rank}\ W = \mathrm{rank}\ V = r$, $\mathrm{rank}\ WV = r$, $S = T = I_r$.

The intuition behind the procedure above is similar to that of balanced truncation for linear systems: By applying the transformation $\mathcal{S} = \Lambda^{1/2} L^T U^{-1}$ to Σ, we obtain an LSS $\Sigma_{bal} = \{\mathcal{S}A_q\mathcal{S}^{-1}, \mathcal{S}B_q, C_q\mathcal{S}^{-1}\}_{q\in Q}$, such that $\Lambda = \mathcal{S}^{-T}\mathcal{Q}\mathcal{S}^{-1} = \mathcal{S}\mathcal{P}\mathcal{S}^T$ is both an observability and a controllability Gramian. We obtain $\bar{\Sigma}$ from Σ_{bal} by taking the upper-left $r \times r, r \times m, p \times r$, blocks of $A_q, B_q, C_q, q \in Q$ respectively. That is, we discard those states which correspond to small values of the diagonals of Λ. The intuition behind this approach is that the discarded states are either difficult to reach (it requires high energy input to reach them) or difficult to observe (their contribution to the energy of the output is small). More precisely, let us fix an integer $r > 0$ which represents the desired state dimension of the reduced order model. Let $(\tilde{x}, u, \sigma, \tilde{y})$ be a solution of Σ_{bal} such that $\tilde{x}(0) = 0$, assume that for all $t > \tau_0, u(t) = 0$. It then can be shown [29] that

$$\begin{aligned} \sum_{i=1}^{r} \tilde{x}_i^2(\tau_0)\frac{1}{\sigma_i} + \sum_{i=r+1}^{n} \frac{1}{\sigma_i}\tilde{x}_i^2(\tau_0) &\leq \int_0^{\tau_0} \|u(s)\|_2^2 ds, \\ \sum_{i=1}^{r} \tilde{x}_i^2(\tau_0)\sigma_i + \sum_{i=r+1}^{n} \sigma_i\tilde{x}_i^2(\tau_0) &\geq \int_{\tau_0}^{\infty} \|\tilde{y}(s)\|_2^2 ds, \end{aligned} \tag{11}$$

and $\tilde{x}_i(\tau_0)$ denotes the ith component of $\tilde{x}(\tau_0)$. That is, if $\sigma_{r+1}, \ldots \sigma_n$ are small, and the energy of u is small, i.e., $\int_0^{\tau_0} \|u(s)\|_2^2 ds$ is small, then the values $\tilde{x}_{r+1}(\tau_0), \ldots, \tilde{x}_n(\tau_0)$ have to be small due to the first inequality, and they contribute little to the output starting from the time instance τ_0 due to the second inequality.

The numbers $\sigma_1, \ldots, \sigma_n$ are called *singular values* of the pair $(\mathcal{P}, \mathcal{Q})$ and they are the square roots of the eigenvalues of the product $\mathcal{P}\mathcal{Q}$.

Theorem ([29]) *For any $\sigma \in \mathcal{Q}$, $u \in \mathcal{U}$ such that $\int_0^{\infty} \|u(s)\|_2^2 ds < +\infty$,*

$$\int_0^{\infty} \|Y_\Sigma(u, \sigma)(s) - Y_{\bar{\Sigma}}(u, \sigma)(s)\|_2^2 ds \leq (2\sum_{k=r+1}^{n} \sigma_k)^2 \int_0^{\infty} \|u(s)\|_2^2 ds.$$

□

Further extensions The results discussed above also hold for discrete-time LSSs [29]. The assumption that $\mathcal{P}, \mathcal{Q}$ do not depend on $q \in Q$ implies quadratic stability, which is a quite restrictive assumption. In [14] this assumption was replaced by local stability of the linear subsystems. Moreover, an error bound similar to Theorem 4.1 was derived in [14], but it holds only for switching signals with a sufficiently large dwell time. Note that [14] allows for LSSs with non-trivial linear reset maps. In [31] an alternative definition of Gramians was presented which has the advantage of having a tighter relationship with minimality.

Relationship with realization theory First, the existence of positive definite Gramians $\mathcal{P}, \mathcal{Q}$ is a necessary (but not sufficient) condition for minimality of quadratically stable LSSs [29]: the kernels of positive semi-definite observability (resp. controllability) Gramians are contained in the set of unobservable states (resp. states which are not in the span of reachable states). Intuitively, when the Gramians are positive definite, we can bring them to a balanced form and then identify the states corresponding to small singular values with unobservable/unreachable states. Then balanced truncation can be thought of as a numerical implementation of the minimization Procedure 3.1. Theorem 4.1 provides means to evaluate the effect of the discarded "small" singular values on the approximation error.

Realization theory is also necessary to show that balanced truncation is well posed. More precisely, the application of balanced truncation relies on the availability of a quadratically stable LSS and by Theorem 4.1, the quality of the reduced model relies on the singular values of the observability/controllability Gramians. If the original model is not quadratically stable, then one may wonder if there exist input-output equivalent quadratically stable models. Using realization theory it is shown in [29] that in order to decide if balanced truncation can be applied, it is sufficient to transform the original model to a minimal one, and then to check if the minimal model is quadratically stable. Moreover, any minimal model can be used for balanced truncation without introducing more conservativity. Indeed, in [29] it is shown that the singular values of any pair of controllability/observability Gramians for any LSS are not smaller than the singular values of some pair of Gramians of a minimal input-output equivalent LSS. Moreover, due to isomorphism, all controllability/observability Gramians of minimal input-output equivalent LSSs are related by a similarity transform and have the same singular values.

Finally, realization theory and the notion of Hankel-matrix can be used to relate singular values of Gramians to norms of a Hankel-operator [29].

4.2 Moment Matching

Consider an LSS Σ of the form (1), and let us denote its input-output map by f. *For the sake of simplicity, we assume that* $p = m = 1$, *i.e., we deal only with the SISO case*. Recall that since f is realizable by an LSS then f has to have a generalized kernel representation $\{G_v^f\}_{v \in Q^+}$. The idea of moment matching is to find a reduced order LSS $\bar{\Sigma}$ such that for certain sequences $v \in Q^+$, G_v^f is close to $G_v^{Y_{\bar{\Sigma}}}$. Intuitively, this means that the input-output map of $\bar{\Sigma}$ will be close to that f.

In Sect. 4.2.1 we present the approach [1, 4, 5], where we look for a reduced order model $\bar{\Sigma}$ such that certain Taylor-series coefficients of $G_v^{Y_{\bar{\Sigma}}}$ (Markov parameters of $Y_{\bar{\Sigma}}$) and of G_v^f (Markov parameters of f) coincide. In Sect. 4.2.2 we present the approach [15], where we consider multi-variate Laplace transforms H_v^f of G_v^f and we look for reduced order models $\bar{\Sigma}$ such that the Laplace transform $H_v^{Y_{\bar{\Sigma}}}$ of the function $G_v^{Y_{\bar{\Sigma}}}$ coincides with H_v^f for some complex values.

4.2.1 Matching Markov Parameters

Consider an LSS $\bar{\Sigma}$. Let α and β be two selections. Then $\bar{\Sigma}$ is called a (α, β)-*partial realization of* f, if for every $(v, q_0) \in \beta$, $(u, q) \in \alpha$,

$$S^{Y_{\bar{\Sigma}}}_{q,q_0}(vu) = S^{f}_{q,q_0}(vu). \tag{12}$$

That is, $\bar{\Sigma}$ is a (α, β)-partial realization of f, if those Markov parameters of f and of the input-output map $Y_{\bar{\Sigma}}$ of $\bar{\Sigma}$ which are indexed by (α, β) coincide. This means that certain high-order derivatives of f and of $Y_{\bar{\Sigma}}$ are the same. That is, a (α, β)-partial realization of f can be viewed as an LSS, input-output map of which approximates f. If $\alpha = Q^* \times Q$ or $\beta = Q^* \times Q$, then any (α, β)-partial realization of f is a realization of f.

The idea behind moment matching is then to replace an LSS Σ by a reduced order LSS $\bar{\Sigma}$ such that $\bar{\Sigma}$ is a (α, β)-partial realization of the input-output map $f = Y_{\Sigma}$ of Σ. The various algorithms differ in the way the selections α, β are chosen. The moment matching algorithms which produce (α, β)-partial realizations arise from Algorithm 1 by a suitable choice of the matrices W and V. In order to explain these choices in more detail, we introduce the following definitions. Define the subspaces

$$\mathscr{O}_{\alpha}(\Sigma) = \bigcap_{(v,q)\in\alpha} \ker C_q A_v,\ \ \mathscr{R}_{\beta}(\Sigma) = \text{Span}\{A_w B_{q_0} \mid (w, q_0) \in \beta\}.$$

There are 3 choices of matrices W and V.

(A) $\ker W = \mathscr{O}_{\alpha}(\Sigma)$ and $V = I_n$. Then Algorithm 1 returns a $(\alpha, \{\epsilon\} \times Q)$-partial realization of f [4, Theorem 3].
(B) $\text{Im} V = \mathscr{R}_{\beta}(\Sigma)$ and $W = I_n$. Then Algorithm 1 returns a $(\{\epsilon\} \times Q, \beta)$-partial realization of f, [4, Theorem 2].
(C) $\ker W = \mathscr{O}_{\alpha}(\Sigma)$, $\text{Im} V = \mathscr{R}_{\beta}(\Sigma)$, rank W = rank V = rank WV. Then Algorithm 1 returns (α, β)-partial realization of f, [4, Theorem 4].

There are two strategies for choosing α, β.

The first one is to choose α (resp. β) to be of finite cardinality r such that $\dim \mathscr{O}_{\alpha}(\Sigma) = n - r$ (resp. $\dim \mathscr{R}_{\beta}(\Sigma) = r$), and in this case the reduced order model will have dimension r. In this case, $\mathscr{O}_{\alpha}(\Sigma) = \ker O_{\alpha}$ (resp. $\mathscr{R}_{\beta}(\Sigma) = \text{Im} R_{\beta}$), and

$$O_{\alpha} = \left[A^T_{v_1} C^T_{s_1},\ \ldots,\ A^T_{v_r} C^T_{s_r}\right]^T,\ \ R_{\beta} = \left[A_{w_1} B_{q_1},\ \ldots,\ A_{w_r} B_{q_r}\right],$$

where $\alpha = \{(v_i, s_i)\}_{i=1}^r$, and $\beta = \{(w_i, q_i)\}_{i=1}^r$. Using these matrix representations the matrices W and V described above can easily be computed.

The second option for choosing nice selections is to choose (α, β) to be consistent with a certain set of switching signals. In this case, the dimension of the reduced order model cannot be fixed in advance, but it is known that the reduced order model will have the same input-output behavior along those switching sequences which belong to this designated set. More precisely, assume that the switching signal $\sigma \in \mathcal{Q}$ has the property that $\sigma(s) = q_i$, $s \in [T_{i-1}, T_i)$, $T_0 = 0$, $T_i = \sum_{r=1}^{i} t_r$ for some $q_i \in Q$, $0 < t_i \in \mathbb{R}_+$, $0 < i \in \mathbb{N}$. We will say that a selection (α, β) is *consistent with* σ, if for every $i > 0$, for every $\omega_i, \ldots, \omega_k \in \mathbb{N}$,

$$((q_i)^{\omega_i}(q_{i+1})^{\omega_{i+1}} \cdots (q_k)^{\omega_k}, q_i) \in \beta,\ \ ((q_1)^{\omega_1}(q_2)^{\omega_2} \cdots (q_i)^{\omega_i}, q_i) \in \alpha.$$

Theorem ([4]) *Assume that* (α, β) *is consistent with* σ *and* $\bar{\Sigma}$ *is an* (α, β)*-partial realization of* f. *Then* $Y_{\bar{\Sigma}}(u, \sigma) = f(u, \sigma)$, *for all* $u \in \mathcal{U}$. □

Note that for a pair of selections to be consistent with a switching signal (or a set of switching signals), the selections involved have to be infinite sets. If the prefixes of the sequences of discrete modes of the desired switching signals form a regular language, then there exist algorithms to compute matrix representations of $\mathscr{O}_\alpha(\Sigma)$, $\mathscr{R}_\beta(\Sigma)$, see [4, 5].

Further extensions. The model reduction method described above was extended to linear parameter-varying (LPV) models [3] and bilinear systems [30]. In addition, the method above was applied to LSSs arising from asynchronous sampling of linear time-invariant systems [2].

Relationship with realization theory. To begin with, the whole idea of matching Markov parameters relies on the notion of Markov parameters and partial realization, which are integral parts of realization theory. In fact, the result of the Ho-Kalman realization algorithm from Procedure 3.2 is isomorphic to the LSS returned by Algorithm 1 with the choice of the matrices W and V as described in option (C) above. That is, moment matching is just a reformulation of Ho-Kalman algorithm when the latter is applied to finite Hankel-matrices, rank of which is not maximal. The partial realization algorithm of [28] is a particular instance of this model reduction method, if $\alpha = \beta = \{v \in Q^* \mid |v| \leq N\} \times Q$ is chosen. Furthermore, Theorem 4.2.1 and its counterpart for the discrete-time case [5] can be viewed as extensions of realization theory of LSSs with constrained switching [22, 23].

4.2.2 Moment Matching in Frequency Domain

By applying multivariate Laplace transform of the functions $\{G^f_v\}_{v \in Q^+}$ we can define a sequence of functions $\{H^f_v\}_{v \in Q^+}$ of complex variables as follows:

$$H^f_v(s_1, \ldots, s_k) = \int_0^\infty \cdots \int_0^\infty G_v(t_1, \ldots, t_k) e^{s_1 t_1 + \cdots + s_k t_k} dt_1 \cdots dt_k \qquad (13)$$

for all $Re(s_i) > s_0$ for a suitable $s_0 \in \mathbb{R}$, where $k = |v|$. If f has a realization by a LSS Σ of the form (1) then $G^f_{q_1 \cdots q_k}(t_1, \ldots, t_k)$ satisfies (5), and hence

$$H^f_{q_1, q_2, \ldots, q_k}(s_1, s_2, \ldots, s_k) = C_{q_k} \Phi_{q_k}(s_k) \Phi_{q_{k-1}}(s_{k-1}) \cdots \Phi_{q_1}(s_1) B_{q_1}, \qquad (14)$$

where $\Phi_q(s) = (sI_n - A_q)^{-1}, q_j \in Q,\ 1 \leqslant j \leqslant k$. We call the functions $\{H^f_v\}_{v \in Q^+}$ the *generalized transfer functions* of the input-output map f [15].

Let Γ and Θ be finite sets of tuples so that $\Gamma, \Theta \subseteq \{(v, \underline{\mu}) \mid v \in Q^+, \underline{\mu} \in \mathbb{C}^k, k = |v|\}$. We will say that an LSS $\bar{\Sigma}$ is a (Γ, Θ)*-partial realization of* f if for every $(w, \underline{\mu}) \in \Gamma$, $(v, \underline{\lambda}) \in \Theta$, $H^f_{wv}(\underline{\mu}, \underline{\lambda}) = H^{Y_{\bar{\Sigma}}}_{wv}(\underline{\mu}, \underline{\lambda})$.

Our goal is to find an LSS $\bar{\Sigma}$ such that $\bar{\Sigma}$ is a (Γ, Θ)-partial realization of f, and the dimension of $\bar{\Sigma}$ is smaller than that of Σ. To this end, for any $v = q_1 \cdots q_k \in Q^+$, $q_1, \ldots, q_k \in Q$, define

$$\mathbf{r}((v, \underline{\mu})) = \Phi_{q_k}(\mu_k) \cdots \Phi_{q_1}(\mu_1) B_{q_1}, \;\; \mathbf{o}((v, \underline{\mu})) = C_{q_k} \Phi_{q_k}(\mu_k) \cdots \Phi_{q_1}(\mu_1),$$

for any $\underline{\mu} = (\mu_1, \ldots, \mu_k) \in \mathbb{C}^k$. Assume that the cardinality of Γ and Θ are both r and consider an enumeration $\Gamma = \{(w_i, \underline{\mu}_i)\}_{i=1}^r$ $\Theta = \{(v_i, \underline{\lambda}_i)\}_{i=1}^r$ of these sets. Define the matrices

$$R = \left[\mathbf{r}((w_1, \underline{\mu}_1)) \ldots \mathbf{r}((w_r, \underline{\mu}_r))\right], \;\; O = \left[\mathbf{o}((v_1, \underline{\lambda}_1))^T \ldots \mathbf{o}((v_r, \underline{\lambda}_r))^T\right]^T.$$

Assume that rank $OR = r$. We can apply Algorithm 1 with $W = O$, $V = R$ resulting in an LSS $\bar{\Sigma}$ which will have the following property.

Theorem ([15]) *With the notation and assumptions above, the LSS $\bar{\Sigma}$ is a (Γ, Θ)-partial realization of f.* □

This method has an alternative formulation in terms of *generalized Loewner matrices* [15], thus extending the well-known Loewner matrix based model reduction method for linear systems.

Relationship with realization theory The reformulation of this method in terms of generalized Loewner matrices yields a partial realization algorithm, as it depends on data which can directly be obtained from Laplace transforms of the input-output map. In a way, this method is the first step towards a reformulation of realization theory of LSSs in frequency domain.

5 Conclusions

In this chapter we presented a brief overview of some recent results on realization theory and model reduction of linear switched systems. It is well known that for linear systems there is a deep connection between these two disciplines. We hope that this chapter convinces the reader that this remains true for hybrid systems and that it is worthwhile to do further research on this topic. There are many possible directions for future research. A particularly natural one is to extend the results of this chapter to hybrid systems with state-dependent switching, for example to piecewise linear systems. The latter can be viewed as a feedback interconnection of a linear switched system with a discrete event generator, hence we are hopeful that the results of this chapter will be useful for such an extension.

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