# 4

# On the Essential Unity of Mathematics, Science, and Art: The Justice Evaluation Function and the Golden Number

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*It is a glorious feeling to perceive the unity of a complex of phenomena which appear as completely separate entities to direct sensory observation.*

Albert Einstein (Quotation from a letter to Marcel Grossmann, written in Milan on 14 April 1901 [1].)

**Summary**

This paper describes how some basic scientific ideas—fairness/comparison/reference-dependence, deficiency and excess, loss and gain—coalesce, unexpectedly revealing a link to the Golden Number, which itself links mathematics and art. Fairness, comparison, and reference-dependence are mathematically equivalent. So are deficiency aversion—deficiency relative to the just amount is felt more keenly than comparable excess—and loss aversion—loss relative to the reference amount looms larger than comparable gain. Representing the outcomes (the justice evaluation $J$ and the value $V$) by zero (for zero deficiency/excess or loss/gain), negative numbers (for deficiency or loss), and positive numbers (for excess or gain) leads naturally to a contrast between the negative and positive outcomes (deficiency and excess in one case, loss and gain in the other). This contrast can be expressed as a difference or a ratio. The ratio of the absolute value of the negative outcome to the positive outcome thus expresses the deficiency aversion coefficient or loss aversion coefficient. This ratio increases as the deficiency/excess or loss/gain $k$ increases, crossing 2—deficiency is felt twice as keenly as excess and loss looms twice as large as gain—when $k$ equals $\left(\sqrt{5}-1\right)/2$ (≈0.618) of the just or reference amount.

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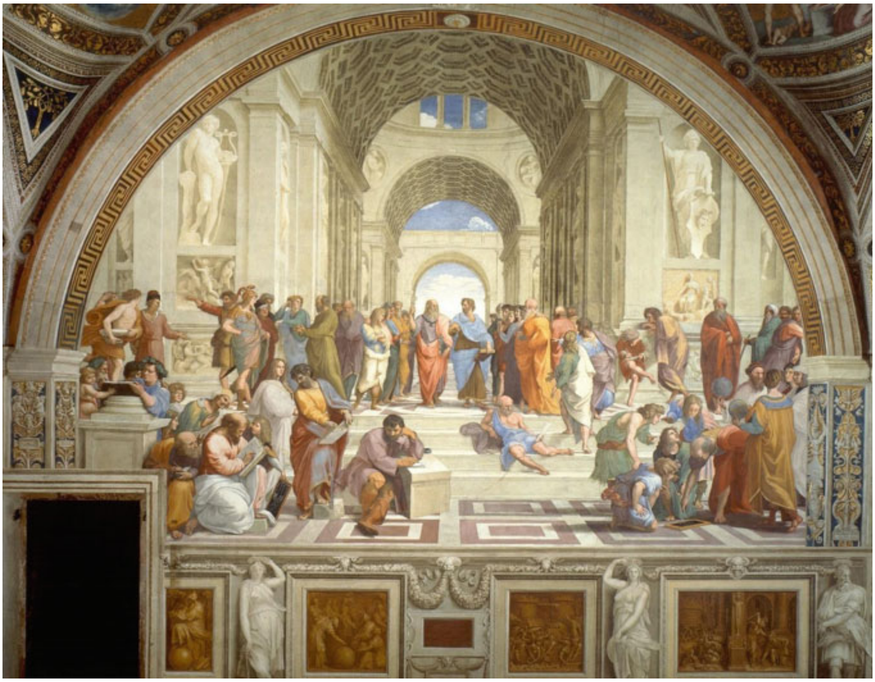

School of Athens painted by Raphael in 1509–1511 in the Apostolic Palace in the Vatican. (Adapted with permission from https://upload.wikimedia.org/wikipedia/commons/4/49/%22The_School_of_Athens%22_by_Raffaello_Sanzio_da_Urbino.jpg).

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## 1 Introduction

In the history of mathematics, science, and art, three words stand out: beauty, unity, surprise. The principles, discoveries, products, and other achievements radiate beauty. As understood at least since Pythagoras (c. 570–c. 495 BCE) and Plato (428/427 or 424/423–348/347 BCE), part of the beauty is the revealed unity. Moreover, these achievements come as a surprise, as do their beauty and unity. They may occur within or across domains, they may be large or small, monumental or modest, yet all delight by their beauty, unity, and surprise [2].[1]

[1] It is illuminating to remember that the subtitle of Thompson's [3] great mathematical classic *Calculus Made Easy* is *"Being a very-simplest introduction to those beautiful methods of reckoning which are generally called by the terrifying names of the differential calculus and the*

This chapter describes some small steps that end in a link across social science and mathematics. In the core story of this chapter, a simple question that arises both in the study of fairness and in the study of decisionmaking—When is deficiency relative to the just reward (or loss relative to the reference amount) felt twice as keenly as excess relative to the just reward (or gain relative to the reference amount)?—is unexpectedly answered by the Golden Number of mathematics. Specifically, deficiency (or loss) is felt twice as strongly as excess (or gain) when the deficiency (or loss) and the excess (or gain) are approximately 0.618 of the just reward (or reference amount).

This surprise signals a deep unifying element between the human sense of justice and mathematics. It signals also a deep bond between justice and beauty [13–18], a bond especially tight in languages where a single word signifies both justice and beauty—*fair*.

The chapter is organized as follows: Sect. 2 describes the justice evaluation and the justice evaluation function, summarizing justice theory and lingering on properties of the justice evaluation function, which will play a key part in the link to the Golden Number. Section 3 briefly reviews the history of the Golden Number and its universal esthetic appeal. Section 4 presents the justice approach to deficiency aversion and loss aversion, drawing out links between the justice evaluation function and the value function, mathematizing deficiency aversion and loss aversion, and representing the contrast between deficiency and excess and between loss and gain via both a difference and a ratio. Section 5 then analyzes the case in which the ratio of the absolute value of the negative outcome to the positive outcome equals 2—the case in which deficiency is felt twice as keenly as excess and loss is felt twice as strongly as gain—finding that this occurs when the deficiency/excess or loss/gain $k$ equals $\left(\sqrt{5}-1\right)/2$ ($\approx 0.618$) of the just or reference amount. Section 6 discusses additional recent social science research which builds on the Golden Number. A short note concludes, followed by a short list of core messages.

## 2 The Justice Evaluation and the Justice Evaluation Function

The justice evaluation and the justice evaluation function are part of justice theory, a social science effort to understand the operation of the human sense of justice. Accordingly, we begin with a brief overview of justice theory.

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*integral calculus."* For brief social science overviews of beauty, unity, and surprise, see [4–10]. For examples from philosophy and literature, see [11, 12].

## 2.1 The Human Sense of Justice: Justice Theory

Around the clock and around the world, the sense of justice is at work. People form ideas about what is just, and they assess the justice or injustice of what they see around them. Both the ideas of justice and the assessments of injustice trigger a variety of individual and social processes, reaching virtually every area of the human experience. Justice concerns are pervasive, from the fierce children's cry of "It's not fair" to the melancholy reflection of heads of state that "Life is unfair" [19, 20]. It is therefore not surprising that understanding the sense of justice is a basic goal in social science, as pointed out in [21, p. 43–44], [22, p. xi–xii], and [23, 24].[2]

The framework for justice theory begins with four central questions, proceeds to identify three basic actors and four basic quantities, and finally embeds them in four basic processes represented by four basic functions. The four functions are deployed both theoretically and empirically. Theoretically, they play parts in both deductive and nondeductive theories, leading to testable predictions, including novel predictions, and testable propositions [28, 29]. Empirically, besides playing their obvious part in estimation and testing of the predictions and propositions, the four functions provide a foundation for measurement, for new data collection designs, for new data analysis protocols, and for interpretation of results.

### 2.1.1 Four Central Questions

There are four central questions in the study of justice:

i. What do individuals and societies think is just, and why?
ii. How do ideas of justice shape determination of actual situations?
iii. What is the magnitude of the injustice associated with given departures from perfect justice?
iv. What are the behavioral and social consequences of injustice?

Each question covers a family of questions, and each can be addressed both theoretically and empirically. The set of four questions, compiled by Jasso and Wegener [30, p. 398], integrates two earlier rival lists of three questions each [31, p. 1400] and [32, p. 155, 174].

### 2.1.2 Three Basic Actors

Three actors play fundamental parts in the sense of justice:

- Observer: The observer forms ideas of justice and judges the justice or injustice of specific actual situations;

[2] However, the sense of justice is not the only driver of behavior; as noted by Homans [23], status and power also play foundational parts. Thus, the sense of justice may not be universal—some people may be justice-oblivious, or may experience, or express, justice concerns only in some domains and not in others [25, 26]. Using Rayo and Becker's [27] evocative words, we may say that justice, status, and power are "carriers of happiness" and that the extent to which each occupies the mind and heart varies across people.

- Rewardee: The rewardee receives an amount of a good or bad being distributed or has a rank in the distribution of the good or bad; and
- Allocator: The allocator makes the distribution.

Some situations have only an observer and a rewardee. Sometimes one person plays all three parts or two of them. For example, when schoolchildren judge the fairness of the grades they receive in school, they are simultaneously observer and rewardee.

### 2.1.3 Four Basic Quantities

As is already evident from the four central questions, justice theory highlights four basic quantities:

i. Actual reward: The actual reward, denoted $A$, is the amount or level of the reward received by the rewardee;
ii. Just reward: The just reward, denoted $C$, is the amount or level of the reward the observer thinks just for the rewardee;
iii. Justice evaluation: The justice evaluation, denoted $J$, is the observer's assessment that the rewardee is justly or unjustly rewarded, and, if unjustly rewarded, whether underrewarded or overrewarded and to what degree; and
iv. Justice consequences: The observer's justice evaluation triggers many consequences, at both individual and social levels and touching vast areas of human experience.

Rewards may be cardinal, like earnings and wealth, or ordinal, like beauty and skills. Rewards of which more is preferred to less are called *goods*; rewards of which less is preferred to more are called *bads*. For example, for most observers, beauty and wealth are goods, and taxes and time in prison are bads. Cardinal goods are also called positive resources. Bads include both burdens (like chores) and punishments (the stuff of retributive justice). When the reward is cardinal, it is represented in the reward's own units (say, money or land or head of cattle); when the reward is ordinal, it is represented by relative ranks within a group or collectivity. People who value cardinal things are called *materialistic*; people who value ordinal things are called *nonmaterialistic*. Societies, too, are called materialistic and nonmaterialistic.[3]

The justice evaluation is represented by the full real-number line, with zero representing the point of perfect justice, negative numbers representing degrees of underreward, and positive numbers representing degrees of overreward.

[3] Elaborating footnote 2, if, as currently understood [25], status notices only the ordinal dimension of rewards, while justice and power notice both cardinal and ordinal dimensions, then there are five types of societies—justice-materialistic, justice-nonmaterialistic, status, power-materialistic, and power-nonmaterialistic—echoing Plato's insight. Thus, justice is active in two of the five types of societies.

### 2.1.4 Four Basic Functions

The actors and quantities are embedded in functions that address the basic questions:

i. Actual reward function: The allocator, guided by allocation rules, uses rewardee characteristics and other inputs to generate the actual reward for the rewardee;
ii. Just reward function: The observer, guided by justice principles, uses rewardee characteristics and other inputs to generate the just reward for the rewardee;
iii. Justice evaluation function: The observer compares the actual reward to the just reward, generating the justice evaluation; and
iv. Justice consequences function: The justice evaluation triggers a long train of justice consequences, possibly incorporating non-justice factors—stretching out to all domains of human behavior and the social life and giving distributive justice the character of a basic sociobehavioral force.

### 2.1.5 Remarks About Justice Theory

Before ending this brief overview of justice theory, it is useful to highlight several features.

First, note the parallel structure between the actual reward function and the just reward function. In the actual reward function, the allocator is guided by allocation rules to generate the actual reward, while in the just reward function, the observer is guided by principles of justice to generate the just reward;

Second, following Brickman et al. [33], the principles of justice include both principles of *microjustice*—pertaining to who should get what and why—and principles of *macrojustice*—pertaining to what the overall reward distribution should look like;

Third, the just reward and the justice evaluation are always observer-specific and rewardee-specific;

Fourth, observers demonstrate independence of mind, as enshrined in the fundamental principle owed independently to Hatfield [24, p. 152] and Friedman [34]: Justice is in the eye of the beholder;

Fifth, the actual reward, just reward, and justice evaluation lead not only to special functions (as above) but also to special distributions, such as the just reward distribution and the justice evaluation distribution, both of which can be succinctly summarized via observer-by-rewardee matrices [10, 35];

Sixth, though in this paper the rewardee is discussed for the case of an individual person, the justice apparatus scales up to the case in which the rewardee is a collectivity and the reward any of its characteristics, such as its resource endowment or its inequality. That is, the rewardee can be either a natural person or a corporate person;

Seventh, the justice evaluation function is tightly linked to two inequality measures:

- Atkinson's inequality (one minus the ratio of the geometric mean to the arithmetic mean); and
- Theil's mean logarithmic deviation (the log of the ratio of the arithmetic mean to the geometric mean);

Eighth, the justice evaluation function is used in the proof for a theorem stating that "*inequality in the distribution of a good is a bad, and inequality in the distribution of a bad is a good*";

Ninth, algebraic manipulation of the arithmetic mean of the justice evaluation distribution shows that overall injustice can be decomposed into injustice due to poverty and injustice due to inequality [35];

Tenth, justice theory has been generalized to the broader set of comparison processes, which includes not only relative deprivation, which may be thought of as fairness by another name, but also self-esteem, which pertains to the special case in which the rewardee is the same as the observer [36]; and

Lastly, justice/comparison theory yields a large number of testable predictions, including novel predictions, for example:

1. *Parents of two or more nontwin children will spend more of their toy budget at an annual giftgiving occasion than at the children's birthdays;*
2. *A thief's gain from theft is greater when stealing from a fellow group member than from an outsider, and this premium is greater in poor groups than in rich groups;*
3. *Veterans of wars fought away from home are more vulnerable to posttraumatic stress than veterans of wars fought on home soil;*
4. *Blind persons are less at risk of eating disorders than are sighted persons;*
5. *Inheritance tempers grief;*
6. *In groups where husbands earn more than their wives, marital cohesiveness increases with husbands' earnings inequality and wives' mean earnings and decreases with wives' earnings inequality and husbands' mean earnings; and*
7. *In materialistic societies with two warring subgroups, conflict severity increases as inequality increases.*

In this set of predictions, all were novel predictions at the time they were obtained except for Prediction 5, which exactly mirrors Cervantes' observation at the end of *Don Quixote*. As for empirical test, Prediction 1 is consistent with toy sales figures in the United States [37, p. 263], Prediction 3 with journalistic observations that Vietnamese veterans of the Vietnam War appear to be better adjusted than American veterans of the Vietnam War [38], and Prediction 6 with Bellou's [39]) finding that as male wage inequality increases, the divorce rate decreases. These and other predictions are amenable to a wide variety of tests, e.g., across cultures and historical periods. Thus, much is yet to be learned about the operation of the human sense of justice and of comparison processes more generally.

Figure 1 provides visualization of the world of distributive justice.

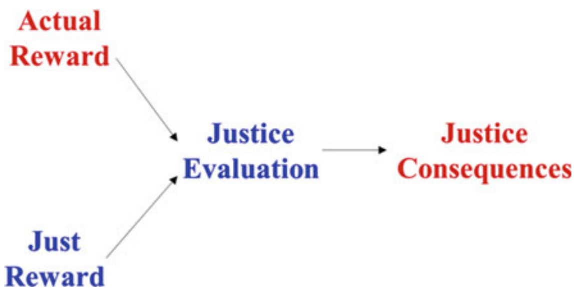


**Fig. 1** The world of distributive justice. (Reproduced from Jasso et al. [29])

## 2.2 A Closer Look at the Justice Evaluation Function

The justice evaluation function (JEF) specifies the justice evaluation $J$ as the logarithm of the ratio of the actual reward $A$ to the just reward $C$ (where "just" always means "just in the eyes of the observer"):

$$J = \theta \ln \left( \frac{A}{C} \right) \tag{1}$$

When the actual reward equals the just reward, the justice evaluation equals zero, the point of perfect justice. The parameter θ is called the *signature constant*. Its sign, called the *framing coefficient*, is positive for goods and negative for bads. Its absolute value, called the *expressiveness coefficient*, represents the observer's expressiveness. Things are framed as goods or bads by the observer. Formally, in the observer's eyes, a thing is a good if more is preferred to less, and a thing is a bad if less is preferred to more.

Thus, in the case of a good, when the actual reward is greater than the just reward, $J$ is positive, indicating overreward, and when the actual reward is smaller than the just reward, $J$ is negative, indicating underreward. Values of $J$ close to zero indicate low degrees of injustice, and values far from zero indicate high degrees of injustice.

The hallmark of the justice evaluation function, embedded in Eq. (1), is that the justice evaluation depends on two variables, the actual reward and the just reward, and that the two have an opposite operation, such that, ignoring theta (or equivalently, for the case of a good), the first and second partial derivatives of $J$ with respect to $A$ are positive and negative, respectively, and the first and second partial derivatives of $J$ with respect to $C$ are negative and positive, respectively. Thus, in this case of a good, as $A$ increases, holding $C$ constant, the justice evaluation function is increasing and concave down, while as $C$ increases, holding $A$ constant, it is decreasing and concave up.[4]

The justice evaluation function has useful properties:

1. *exact mapping from combinations of A and C to J;*
2. *the outcome it yields is in justice units (not reward units);*
3. *integration of rival conceptions of J as a ratio and as a difference;*
4. *deficiency aversion,* viz., *deficiency is felt more keenly than comparable excess (and loss aversion,* viz., *losses are felt more keenly than gains);*
5. *scale invariance;*
6. *additivity, such that the effect of A on J is independent of the level of C, and conversely;*
7. *symmetry, such that interchanging A and C changes only the sign of J;*
8. *the log-ratio form of the justice evaluation function is the limiting form of the difference between two power functions, which both strengthens integration of the ratio and difference views and also integrates power-function and logarithmic approaches; and*
9. *a link between the justice evaluation function and the Golden Number, approximately 0.618, such that the loss aversion ratio equals 2 when the actual reward equals the just reward plus or minus the product of the just reward and the Golden Number.*

The first four properties were described in the original research report in 1978 [31]. Properties 5 and 6 were established in 1990 [36], properties 7 and 8 in 1996 [43]. Property 9 was first mentioned in 2006 [44] and more fully discussed in 2015 [45], but not comprehensively analyzed until this chapter.

The framing coefficient is always critically important, but the expressiveness coefficient, which is critically important in empirical analysis, can safely be ignored in purely theoretical analysis. Much of the work in this chapter is purely theoretical, and for simplicity, focuses on goods. Accordingly, the signature constant can be safely set to one. Only much later in the chapter will it be necessary to examine bads and as well to invoke the distinction between the *experienced* and the *expressed* justice evaluation [28, 30].

In the generalization to comparison processes, the actual reward is sometimes called the actual holding, the just reward the comparison holding (or more commonly the reference point), and the justice evaluation the comparison outcome and denoted $Z$. The comparison holding can arise from myriad quantities, such as self or

[4] The log-ratio specification of the justice evaluation function has rich intellectual roots, going back to Bernoulli's [40] utility function and Fechner's [41, 42] sensation function. However, the justice evaluation function differs in at least two important ways from its predecessors. Recall that in Bernoulli's function, utility is a logarithmic function of wealth, and in Fechner's function, subjective magnitude is a logarithmic function of the ratio of a physical stimulus to the (constant) lower absolute threshold for that stimulus. Thus, both Bernoulli's and Fechner's functions are functions of one variable, and they are concave functions. In contrast, the justice evaluation function is a function of two variables, and, as can be shown by inspection of the principal minors of its Hessian second derivative matrix, it is neither concave nor convex.

other's actual reward or function thereof, something in the past or envisioned or desired, a parameter of the actual reward distribution, and so on [46]. For example, in the foundational accounts owed to Marx and James, respectively, satisfaction arises from comparing the hut to the palace [47, pp. 84–85], and self-esteem arises from comparing success to pretensions [48, p. 200].

For comprehensiveness and clarity, the justice evaluation function and the comparison function are sometimes displayed in a fourfold classification, with both the general expression and the specific log-ratio specification for both the global version covering both goods and bads in a single expression and the conditional version with two branches, for goods and bads, respectively [35, p. 139]. Thus, for example, one of the four cells (Cell B.2) contains Eq. (1), the log-ratio form for both goods and bads. When only goods or only bads are of interest, the usual expressions are either adapted from the global expressions or use only one branch from the two-branch expression, as in Falk and Knell's [49, p. 418] and Jasso's [36, p. 380] general form for utility and comparison, respectively, in the case of a good, which also forms part of Cell A.1 in [35, p. 139].

Figure 2 displays graphs of the justice evaluation function, separately for goods and bads, and showing for each the graphs of $J$ on the $\ln(A)$ and $\ln(C)$ components. As shown, the graphs of the two components are reflections of each other about the $x$-axis. Moreover, the graphs of the $\ln(A)$ components for a good and a bad are reflections of each other about the $x$-axis, as are the graphs of the $\ln(C)$ components for a good and a bad.

Note that when the just reward $C$ is held constant, the justice evaluation function is fully depicted by the graph of $\ln(A)$, and this graph crosses the $x$-axis at the magnitude of $A$ that equals $C$.[5]

The individual's time series of justice evaluations, called the justice profile, provides a picture of the interior justice life, with variation over a unit of time in the number and duration of distinct justice evaluations, their range, the jumps and dips between them, the gaps when the sense of justice is asleep, the means of the underrewarded and overrewarded truncates, etc. New questions arise concerning periodicity and links to specific rewards, to age and experience, and to the larger happiness profile insightfully discussed by Layard [25, p. 417, 50, pp. 367–370, 51].

## 3 The Golden Number

One idea permeates the Golden Number: beauty. There is beauty in proportions, there is beauty in numbers, and there is unrivaled beauty in the special number that represents a certain special proportion. To see this special proportion, consider a

[5] As shown, in the case of a good, the justice evaluation increases at a decreasing rate with the actual reward. Elaborating further on footnotes 2 and 3, the distinguishing feature of the three foundational engines of behavior is their rate of change, such that, continuing with the case of a good, as the actual reward increases, status increases at an increasing rate and power at a constant rate [25].

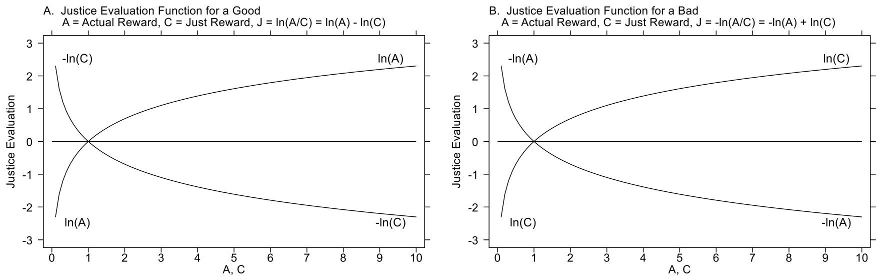


**Fig. 2** Justice evaluation function for goods and bads

line AB. Suppose it is divided at an interior point P such that AP/PB equals PB/AB—that is, the ratio of the smaller to the larger line segment equals the ratio of the larger line segment to the whole line.

Ancient philosophers speculated that this relation among a quantity and two components of the quantity represented ideal proportions. As Plato [52, p. 448] writes in the *Timaeus*: "... the fairest bond is that which makes the most complete fusion of itself and the things which it combines; and proportion is best adapted to effect such a union. For whenever in any three numbers,... there is a mean, which is to the last term what the first term is to it; and again, when the mean is to the first term as the last term is to the mean—then the mean becoming first and last and the first and last both becoming means, they will all of them of necessity come to be the same, and having become the same with one another will be all one".

And mathematicians since the ancients used their tools to approximate the requisite point P for achieving that ideal set of proportions [13, 53, 54]. The first formal definition appears in Euclid's [55, p. 99] *Elements*, Book VI: "A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment, so is the greater to the less."

The solution for P turns out to be the positive solution of a breathtakingly simple and beautiful quadratic equation, viz., $x^2 + x = 1$ [15–17, 56, p. 104]:

$$\frac{\sqrt{5}-1}{2} \approx .618 \tag{2}$$

termed the Golden Number. Thus, if AB has a length of 1, the line is cut at 0.382, and the ratio of the smaller to the larger segment (0.382/0.618) equals 0.618, as does the ratio of the larger segment to the whole (0.618/1).

The history of the Golden Number has rich highlights. For example, it has acquired a portfolio of names, including Golden Ratio, Golden Section, Golden Mean (not to be confused with Aristotle's idea of the golden mean, in which moral virtue is a position between two extremes), Division in Extreme and Mean Ratio (DEMR), and Divine Proportion. Similarly, both the Golden Number and its reciprocal (approximately equal to 1.618) are widely used. At least since Plato, the relation is seen to be like a palindrome—it works the same whether we go from left to right or from right to left. It attracted the great astronomer Kepler, and its first decimal approximation seems to have been made by Michael Maestlin, Kepler's teacher. Reading its history is like reading the history of mathematics, the amazing growth of knowledge laid out before our eyes, a parade of geometric figures, the new Hindu-Arab numerals, algebra, sequences, and so on. Importantly, the Golden Number touches virtually every area of the human and physical worlds, where it is both found in nature and used purposefully in art, science, and engineering, among other endeavors [57, 58].

The Golden Number enthralls by its beauty and, echoing Plato, unifies everything it touches.

## 4 The Justice Approach to Deficiency Aversion and Loss Aversion

This section uses the justice approach to study two distinct situations: (i) the situation involving deficiency or excess relative to the just reward; and (ii) the situation involving loss or gain in the actual reward from Time 1 to Time 2.

### 4.1 Deficiency and Excess in Actual Reward, Relative to Just Reward

As before, let $A$ denote the actual reward and $C$ the just reward. Now let the actual reward equal the just reward plus or minus a constant $k$,

$$A = C_0 \pm k \tag{3}$$

where $k$ is positive and less than $C_0$ ($0 < k < C_0$). The constant $k$ is called the *deficiency* or *excess*.

It has been known since the first report of the justice evaluation function that it possesses the property that deficiency is felt more keenly than comparable excess [20], a property we may call *deficiency aversion*. Indeed, this is the property most closely associated with the JEF and the one that first attracted notice. The idea that underreward is felt more strongly than overreward goes back at least to Adams [59, p. 426], 15 years before the JEF was introduced. As Wagner and Berger [10, p. 719] observe, this was "a phenomenon all justice theorists assumed occurred, but one that had not been incorporated in their theories".

Formally, the deficiency aversion property is stated:

$$\left|J^{\text{Deficiency}}\right| > J^{\text{Excess}} \tag{4}$$

Its mathematical statement is written:

$$\left|\ln\left(\frac{C_0 - k}{C_0}\right)\right| > \ln\left(\frac{C_0 + k}{C_0}\right) \tag{5}$$

The discrepancy in $J$ between deficiency and excess can be expressed in two ways, as a difference and as a ratio.

Define the difference $D^J$ as the absolute value of the justice evaluation $J$ in the deficiency case minus $J$ in the excess case:

$$D^J = \left|\ln\left(\frac{C_0 - k}{C_0}\right)\right| - \ln\left(\frac{C_0 + k}{C_0}\right) \tag{6}$$

Upon algebraic manipulation, the difference $D^J$ becomes:

$$D^J = \ln\left(C_0^2\right) - \ln\left(C_0^2 - k^2\right) \tag{7}$$

The difference $D^J$ is always a positive quantity (as visible in the inequality in Eq. (4)), and it ranges to infinity.

Similarly, define the ratio $R^J$, which can also be called the deficiency aversion coefficient, as the ratio of abs ($J$) in the deficiency case to $J$ in the excess case:

$$R^J = \frac{\left|\ln\left(\frac{C_0 - k}{C_0}\right)\right|}{\ln\left(\frac{C_0 + k}{C_0}\right)} \tag{8}$$

By algebraic manipulation, the ratio $R^J$ becomes:

$$R^J = \frac{\ln\left(\frac{C_0}{C_0 - k}\right)}{\ln\left(\frac{C_0 + k}{C_0}\right)} \tag{9}$$

The ratio ranges from one to infinity.

Many special cases of both the difference $D^J$ and the ratio $R^J$ can be fruitfully examined. Below we focus on one particular question about the ratio $R^J$.

## 4.2 Loss and Gain in Actual Reward, Relative to Time 1 Actual Reward

A second situation arises when the actual reward $A$ changes from Time 1 to Time 2, producing a change in the justice evaluation $J$ from Time 1 to Time 2, denoted $CJ$. Denote the actual reward at Time 1 by $A_0$, let the just reward $C$ remain constant from Time 1 to Time 2, and let the actual reward at Time 2 equal the Time 1 actual reward plus or minus a constant $k$:

$$A = A_0 \pm k \tag{10}$$

where $k$ is positive and less than $A_0$ $(0 < k < A_0)$. The constant $k$ is called the *loss* or *gain*.

In this situation, we obtain the result, parallel to Eq. (4), that loss is felt more keenly than gain. This property is called *loss aversion,* and it was introduced by Kahneman and Tversky [60, 61], who observed that "losses loom larger than gains" [60, p. 279].

Formally, the loss aversion property is stated:

$$\left|CJ^{Loss}\right| > CJ^{Gain} \tag{11}$$

In the justice approach, its mathematical statement is written:

$$\left|\ln\left(\frac{A_0 - k}{A_0}\right)\right| > \ln\left(\frac{A_0 + k}{A_0}\right) \tag{12}$$

As with the discrepancy in $J$ between deficiency and excess, the discrepancy in $J$ between loss and gain can be expressed in two ways, as a difference and as a ratio.

Define the difference $D^{CJ}$:

$$D^{CJ} = \left|\ln\left(\frac{A_0 - k}{A_0}\right)\right| - \ln\left(\frac{A_0 + k}{A_0}\right) \tag{13}$$

Upon algebraic manipulation, the difference $D^{CJ}$ becomes:

$$D^{CJ} = \ln\left(A_0^2\right) - \ln\left(A_0^2 - k^2\right) \tag{14}$$

The difference $D^{CJ}$ is always a positive quantity (as visible in the inequality in (5.11), and it ranges to infinity.

Similarly, define the ratio $R^{CJ}$, which can also be called the loss aversion coefficient, as the ratio of abs ($J$) in the loss case to $J$ in the gain case:

$$R^{CJ} = \frac{\left|\ln\left(\frac{A_0 - k}{A_0}\right)\right|}{\ln\left(\frac{A_0 + k}{A_0}\right)} \tag{15}$$

By algebraic manipulation, the ratio $R^{CJ}$ becomes:

$$R^{CJ} = \frac{\ln\left(\frac{A_0}{A_0 - k}\right)}{\ln\left(\frac{A_0 + k}{A_0}\right)} \tag{16}$$

The ratio $R^{CJ}$ ranges from one to infinity.

### 4.3 Summary of Justice Approach to Deficiency Aversion and Loss Aversion

The justice approach yields predictions for two distinct situations: (i) the situation involving deficiency or excess relative to the just reward; and (ii) the situation involving loss or gain in the actual reward from Time 1 to Time 2. The mathematical outcomes are identical. Justice theory predicts both that (i) deficiency is felt more keenly than comparable excess and (ii) loss is felt more keenly than comparable gain.

Moreover, the justice approach predicts the exact magnitudes by which deficiency (or loss) is felt more keenly than excess (or gain), doing so for both a difference representation of the discrepancy and a ratio representation of the discrepancy.

Of course, the interpretation is context-specific. The deficiency/excess results pertain to assessments of the actual reward relative to the just reward at a point in time. The loss/gain results pertain to assessments of the actual reward as it changes between two points in time.

For ease in contrasting results, using them, and building on them, Table 1 collects the main terms in the justice approach to deficiency aversion and loss aversion.

The sections below examine the case where the magnitudes of the ratios $R^J$ and $R^{CJ}$ equal 2. As will be seen, that is when the Golden Number appears.

## 5 Deficiency Aversion, Loss Aversion, and the Golden Number

### 5.1 Deficiency Aversion and the Golden Number

We turn now to examine the case in which deficiency is felt twice as keenly as comparable excess. This case is important because of empirical evidence that loss is felt twice as keenly as comparable gain [62, p. 1288, 63, pp. 1053–1054].

To analyze this case, we set the ratio $R^J$ equal to 2 and solve for $k$. At the first step, we write:

$$\frac{\ln\left(\frac{C_0}{C_0-k}\right)}{\ln\left(\frac{C_0+k}{C_0}\right)} = 2 \tag{17}$$

Re-arranging terms we obtain:

$$\ln\left(\frac{C_0}{C_0-k}\right) = 2\,\ln\left(\frac{C_0+k}{C_0}\right). \tag{18}$$

**Table 1** Main terms for studying deficiency aversion and loss aversion via the justice evaluation function

| | Difference-based | Ratio-based |
|---|---|---|
| A. Deficiency Aversion | | |
| General statement | $\left\|J^{\text{Deficiency}}\right\| > J^{\text{Excess}}$ | |
| Mathematical statement | $\left\|\ln\left(\frac{C_0-k}{C_0}\right)\right\| > \ln\left(\frac{C_0+k}{C_0}\right)$ | |
| Label | $D^J$ | $R^J$ |
| | $\left\|\ln\left(\frac{C_0-k}{C_0}\right)\right\| - \ln\left(\frac{C_0+k}{C_0}\right)$ | $\frac{\left\|\ln\left(\frac{C_0-k}{C_0}\right)\right\|}{\ln\left(\frac{C_0+k}{C_0}\right)}$ |
| | $\ln\left(C_0^2\right) - \ln\left(C_0^2 - k^2\right)$ | $\frac{\ln\left(\frac{C_0}{C_0-k}\right)}{\ln\left(\frac{C_0+k}{C_0}\right)}$ |
| Range | $0 < D^J < \infty$ | $1 < R^J < \infty$ |
| B. Loss Aversion | | |
| General statement | $\left\|C\,J^{\text{Loss}}\right\| > C\,J^{\text{Gain}}$ | |
| Mathematical statement | $\left\|\ln\left(\frac{A_0-k}{A_0}\right)\right\| > \ln\left(\frac{A_0+k}{A_0}\right)$ | |
| Label | $D^{CJ}$ | $R^{CJ}$ |
| | $\left\|\ln\left(\frac{A_0-k}{A_0}\right)\right\| - \ln\left(\frac{A_0+k}{A_0}\right)$ | $\frac{\left\|\ln\left(\frac{A_0-k}{A_0}\right)\right\|}{\ln\left(\frac{A_0+k}{A_0}\right)}$ |
| | $\ln\left(A_0^2\right) - \ln\left(A_0^2 - k^2\right)$ | $\frac{\ln\left(\frac{A_0}{A_0-k}\right)}{\ln\left(\frac{A_0+k}{A_0}\right)}$ |
| Range | $0 < D^{CJ} < \infty$ | $1 < R^{CJ} < \infty$ |

*Notes* Deficiency and excess are defined relative to the just reward $C$. Loss and gain are defined relative to the Time 1 actual reward $A$.

which leads to the following quadratic equation in $k$:

$$k^2 + C_0 k - C_0^2 = 0 \tag{19}$$

Solving Eq. (19), we find two real roots, one of them positive at:

$$k = C_0\left(\frac{\sqrt{5}-1}{2}\right) \tag{20}$$

This root is quickly seen to include the Golden Number, approximately equal to 0.618. Thus, deficiency is felt twice as keenly as comparable excess when the deficiency (or excess) equals approximately 61.8% of the just reward.

It is extraordinarily pleasing to see the Golden Number appear in justice research, signaling again the deep involvement of nature in the sense of justice. The Golden Number joins logarithms and the beautiful number $e$ in playing a part in the scientific description of the sense of justice, together with smaller but no less beautiful numbers and results. For example, the limit of the geometric mean in the distribution of relative ranks, which embeds two stalwarts—roots and factorials—approaches $1/e$:

$$\lim_{N \to \infty} \frac{\sqrt[N]{N!}}{N+1} = \frac{1}{e} \approx 0.368 \tag{21}$$

Thus, in a nonmaterialistic society that values one ordinal good and views justice as equality, as the population size increases, the average of the justice evaluation distribution moves leftward, attaining progressively larger absolute values of negative magnitudes and approaching its limit of −0.307 [64, p. 13][6]:

$$\lim_{N \to \infty} \ln\left(\frac{2\sqrt[N]{N!}}{N+1}\right) = \ln\left(\frac{2}{e}\right) = \ln(2) - 1 \approx -0.307 \tag{22}$$

Finally, the emergence of the Golden Number in the study of justice reinforces and illuminates the bond between justice and beauty [13–18]—a bond especially tight in languages like English, where the word "fair" signifies both "just" and "beautiful".[7]

Of course, the analysis shows that deficiency is felt twice as keenly as excess only for given special magnitudes of deficiency and excess (namely 0.618 of the just reward). Other magnitudes of deficiency and excess will yield other magnitudes of the ratio $R^J$. In general, the magnitude of $R^J$ depends jointly on the magnitudes of the just reward $C$ and the deficiency or excess $k$. Taking first partial derivatives of $R^J$ with respect to $C$ and $k$ yields negative and positive quantities, respectively. Thus, the greater the just reward, the smaller the deficiency aversion coefficient, and the greater the amount of the deficiency or excess, the greater the deficiency aversion coefficient.

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[6] This society belongs to a family of justice societies that satisfy two conditions, sometimes called the "Primitive alternatives" [52, 64 p. 9–10]: first, following Socrates in Plato's *Gorgias*, the members view justice as equality; and second, following Augustine's definition in *City of God*, they are " a people... bound together by a common agreement as to the objects of their love" and thus value the same goods or bads.

[7] Not all languages distinguish between "justice" and "fairness"—considered by Rawls [65] an important distinction. Thus, his phrase "justice as fairness" is rendered in French as "la justice comme équité" and in Spanish as "la justicia como imparcialidad." Both raise the new challenge of establishing the relation between the French "justice" and "équité" and between the Spanish "justicia" and "imparcialidad" and their relation to the Rawlsian English-language concepts of "justice" and "fairness" as well as "equity" and "impartiality".

## 5.2 Loss Aversion and the Golden Number

Paralleling the work above on deficiency aversion, we turn to the special case of loss aversion in which the ratio $R^{CJ}$ equals 2. As discussed above, this case is important because of empirical evidence that loss is felt twice as keenly as comparable gain [62 p. 1288, 63 p. 1053–1054].

To analyze this case, we set the ratio $R^{CJ}$ equal to 2 and solve for $k$. Paralleling exactly the work above on deficiency and excess, at the first step, we write:

$$\frac{\ln\left(\frac{A_0}{A_0 - k}\right)}{\ln\left(\frac{A_0 + k}{A_0}\right)} = 2 \tag{23}$$

Re-arranging terms we obtain:

$$\ln\left(\frac{A_0}{A_0 - k}\right) = 2\ln\left(\frac{A_0 + k}{A_0}\right) \tag{24}$$

which leads to the following quadratic equation in $k$:

$$k^2 + A_0 k - A_0^2 = 0 \tag{25}$$

Solving Eq. (25), we find two real roots, one of them positive at:

$$k = A_0\left(\frac{\sqrt{5} - 1}{2}\right) \tag{26}$$

As before, this root is quickly seen to include the Golden Number, approximately equal to 0.618. Thus, loss is felt twice as keenly as comparable gain when the loss (or gain) equals approximately 61.8% of the actual reward at Time 1.

Also as before, the analysis shows that loss is felt twice as keenly as gain only for given special magnitudes of loss and gain (namely 0.618 of the actual reward). Other magnitudes of loss and gain will yield other magnitudes of the ratio $R^{CJ}$. In general, the magnitude of $R^{CJ}$ depends jointly on the magnitudes of the actual reward $A$ and the loss or gain $k$. Taking first partial derivatives of $R^{CJ}$ with respect to $A$ and $k$ yields negative and positive quantities, respectively. Thus, the greater the Time 1 actual reward, the smaller the loss aversion coefficient, and the greater the amount of the loss or gain, the greater the loss aversion coefficient.

Table 2 reports a simple numerical example that illustrates these results. Panel A displays the justice evaluations and the loss aversion coefficient for three magnitudes of the Time 1 actual reward (80, 100, and 120), shown as three sets of columns, and four magnitudes of the loss or gain $k$ (0, 25, 50, and 75), each displayed in a row. Thus, for all three levels of the Time 1 actual reward, when

$k$ equals zero, there is neither loss nor gain, and the change in the justice evaluation $CJ$ equals zero.

Each of the three sets of columns representing a magnitude of the Time 1 actual reward has five columns, two for the loss case, two for the gain case, and the loss aversion coefficient. For example, in the row for $k = 25$, looking at the set for the Time 1 actual reward equal to 80, the two columns for the loss case report the numeric representation of the log-ratio and the numerical approximation to $CJ$, and similarly for the gain case. Accordingly, the numeric log-ratio has a numerator of (80 − 25 =) 55 in the loss case and (80 + 25 =) 105 in the gain case, and a denominator of 80 in both cases. The numeric approximations to $CJ$ are thus −0.375 in the loss case and 0.272 in the gain case. The loss aversion coefficient is then the absolute value of -0.375 divided by 0.272, or approximately 1.38.

As expected from the first partial derivative, the loss aversion coefficient declines as the Time 1 actual reward increases, diminishing in each row as the Time 1 actual reward increases from 80 to 100 to 120—for example, when $k = 25$, diminishing from 1.38 to 1.29 to 1.23. Similarly, the loss aversion coefficient increases as $k$ increases, increasing in each column as $k$ increases from 25 to 50 to 75—for example, when the Time 1 actual reward is 80, increasing from 1.38 to 2.02 to 4.19.

Table 2 also reports, in Panel B, the value of $k$ when the loss aversion coefficient equals 2. As shown in Eq. (26), this value equals the product of the Golden Number and the Time 1 actual reward. Thus, the requisite values are approximately 49.4, 61.8, and 74.2 for the Time 1 actual rewards of 80, 100, and 120, respectively.

## 5.3 Remarks on Loss Aversion

As discussed above, the term "loss aversion" was introduced by Kahneman and Tversky [60, 61], who observed that "losses loom larger than gains" [60, p. 279]. Moreover, they found that losses are felt twice as strongly as gains [63, pp. 1053–1054]. Meanwhile, justice theory also predicts that losses are felt more strongly than gains, but the associated loss aversion coefficient is not constant and indeed takes on the value 2 only in the very special case involving the Golden Number.

In the spirit of this chapter, it is exciting to contrast the two approaches, which could be special cases of a larger framework. To that end, this section takes some steps to lay out explicitly the correspondence between them. First, both approaches are embedded in theory, specifically in prospect theory and justice theory. Second, the main driver in both approaches is a function, specifically the value function in prospect theory and the justice evaluation function in justice theory. Third, both the value function and the justice evaluation function emerged from empirical work. Fourth, both the value function and the justice evaluation function are reference-dependent. Fifth, in the value function gains or losses are assessed relative to a reference point, while in the justice evaluation function (i) deficiency and excess are assessed relative to the just reward and (ii) gain and loss are assessed relative to the actual reward at Time 1. Sixth, the outcomes (the value $V$ and the justice evaluation $J$) range from negative numbers for the deficiency/loss condition,

**Table 2** Loss aversion and loss aversion coefficient (LAC), by time 1 amount $A_0$ and loss/gain $k$

| | $A_0 = 80$ | | | | | $A_0 = 100$ | | | | | $A_0 = 120$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A. Change in the justice evaluation, by time 1 amount and loss or gain $k$, and loss aversion coefficient (LAC) | | | | | | | | | | | | | | | |
| $k$ | $\ln\left(\frac{A_0-k}{A_0}\right)$ | ≈ | $\ln\left(\frac{A_0+k}{A_0}\right)$ | ≈ | LAC | $\ln\left(\frac{A_0-k}{A_0}\right)$ | ≈ | $\ln\left(\frac{A_0+k}{A_0}\right)$ | | LAC | $\ln\left(\frac{A_0-k}{A_0}\right)$ | ≈ | $\ln\left(\frac{A_0+k}{A_0}\right)$ | ≈ | LAC |
| 0 | $\ln\left(\frac{80}{80}\right)$ | 0 | $\ln\left(\frac{80}{80}\right)$ | 0 | – | $\ln\left(\frac{100}{100}\right)$ | 0 | $\ln\left(\frac{100}{100}\right)$ | 0 | – | $\ln\left(\frac{120}{120}\right)$ | 0 | $\ln\left(\frac{120}{120}\right)$ | 0 | – |
| 25 | $\ln\left(\frac{55}{80}\right)$ | -0.375 | $\ln\left(\frac{105}{80}\right)$ | 0.272 | 1.38 | $\ln\left(\frac{75}{100}\right)$ | -0.288 | $\ln\left(\frac{125}{100}\right)$ | 0.223 | 1.29 | $\ln\left(\frac{95}{120}\right)$ | -0.234 | $\ln\left(\frac{145}{120}\right)$ | 0.189 | 1.23 |
| 50 | $\ln\left(\frac{30}{80}\right)$ | -0.981 | $\ln\left(\frac{130}{80}\right)$ | 0.486 | 2.02 | $\ln\left(\frac{50}{100}\right)$ | -0.693 | $\ln\left(\frac{150}{100}\right)$ | 0.405 | 1.71 | $\ln\left(\frac{70}{120}\right)$ | -0.539 | $\ln\left(\frac{170}{120}\right)$ | 0.348 | 1.55 |
| 75 | $\ln\left(\frac{5}{80}\right)$ | -2.77 | $\ln\left(\frac{155}{80}\right)$ | 0.661 | 4.19 | $\ln\left(\frac{25}{100}\right)$ | -1.39 | $\ln\left(\frac{175}{100}\right)$ | 0.560 | 2.48 | $\ln\left(\frac{45}{120}\right)$ | -0.981 | $\ln\left(\frac{195}{120}\right)$ | 0.486 | 2.02 |
| B. Magnitude of $k$ when loss aversion coefficient equals 2: $A_0\left(\frac{\sqrt{5}-1}{2}\right) \approx .618A_0$ | | | | | | | | | | | | | | | |
| | ≈ 49.4 | | | | | ≈ 61.8 | | | | | ≈ 74.2 | | | | |

*Note* The loss aversion coefficient is defined as the ratio of the absolute value of the justice evaluation for a loss divided by the justice evaluation for a comparable gain

through zero for a neutral point in prospect theory and the point of perfect justice in justice theory, to positive numbers for the excess/gain condition.

To this point, there is a perfect correspondence between the two approaches. However, at first blush, the perfect correspondence seems to end, for the Kahneman and Tversky [61, p. S258-S259] losses and gains are represented by negative and positive numbers, respectively, while the actual rewards embedding both deficiency and excess are represented by positive numbers. Thus, the graph of the value function occupies Quadrants I and III in the Cartesian plane, while the graph of the justice evaluation function occupies Quadrants I and IV. Moreover, a hallmark of the value function is that it is S-shaped, with the response to losses convex and the response to gains concave, while the justice evaluation function is concave throughout.

However, it may be possible for the JEF to approximate the value function by a simple procedure: treat losses as bads and gains as goods, and represent bads by negative numbers.[8] The resulting graph is an S-shaped curve. But the approximation is not complete, for the S-shaped graph of the value function is asymmetric, with greater steepness for losses than for gains, while the S-shaped graph of the transformed justice evaluation function is symmetric. Put differently, the asymmetry of the value function ensures that losses loom larger than gains; but the symmetric transformed JEF loses loss aversion.

Can loss aversion be restored to the transformed JEF? Recall from Sect. 2 that justice theory distinguishes between the experienced justice evaluation and the expressed justice evaluation, via the expressiveness coefficient, the absolute value of the Signature Constant θ in Eq. (1). If losses elicit greater expressiveness than gains (more shouting, say, or less whispering), then the transformed justice evaluation function becomes asymmetric and can approximate the value function. This avenue has a further advantage, namely, incorporating the expressiveness

[8] Framing is powerful, and it is possible that a loss is framed as receiving a bad rather than a smaller amount of a good.

coefficient renders the loss aversion coefficient constant. It can be constant at two simply by setting the expressiveness coefficient to two. However, the asymmetric transformed JEF loses the connection to the Golden Number.

To see this way of attempting to bring the JEF into alignment with the value function, Table 3 provides a numerical example and Fig. 3 provides visualization. The basic numerical example is taken from the middle set of justice evaluations in Table 2, namely, the set in which the Time 1 amount is 100. This set of five columns becomes the leftmost set in Table 3. The second and third sets of five columns each present the symmetric and asymmetric transformed justice evaluations, respectively. In both sets, losses are treated as bads and gains as goods, and in both sets, losses are represented by negative numbers. The two sets differ, however, in the expressiveness coefficient for losses, set at one in the second set and two in the third (it could, of course, be any other positive number, but two has the advantage that it corresponds to the Tversky and Kahneman empirical finding).

As shown in Table 3, the original justice evaluations in the leftmost set display the loss aversion inherent in the justice evaluation function. However, there is no loss aversion in the middle set of figures corresponding to the symmetric transformed JEF. Moreover, in the rightmost set of figures corresponding to the asymmetric transformed JEF, loss aversion is introduced via the expressiveness coefficient and is thus a constant, losing the connection to the Golden Number.

Figure 3 provides a visualization of the original and transformed justice evaluations. As shown, Panels A and B depict the figures in the first set of columns in Table 3. They differ, however, in that in Panel A the JEF is shown in the original coordinates, while in Panel B it has been translated leftward so that losses are represented by negative numbers and the loss or gain is no longer relative to the Time 1 amount of 100 but rather to the zero point between losses and gains.

Panels C and D correspond to the second and third sets of figures in Table 3. In both, losses are treated as bads and goods as gains. There is no loss aversion in Panel C, which depicts the symmetric transformed JEF. Meanwhile, the loss aversion in Panel D, which depicts the asymmetric transformed JEF, is constant, with the loss aversion coefficient fixed at two, and importantly, the link to the Golden Number is lost.

**Table 3** Loss aversion and loss aversion coefficient (LAC), by time 1 amount $A_0$ and loss or gain $k$: Unifying the prospect theory and justice theory approaches

| | Losses and gains treated as goods<br>$k > 0$ for both losses and gains<br>$A_0 = 100$ | | | | | Losses treated as bads, gains as goods<br>$\theta = -1$ for bads, $+1$ for goods<br>$k < 0$ for losses, $k > 0$ for gains<br>$A_0 = 100$ | | | | | Losses treated as bads, gains as goods<br>$\theta = -2$ for bads, $+1$ for goods<br>$k < 0$ for losses, $k > 0$ for gains<br>$A_0 = 100$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\ln\left(\frac{A_0-k}{A_0}\right)$ | $\approx$ | $\ln\left(\frac{A_0+k}{A_0}\right)$ | $\approx$ | LAC | $-\ln\left(\frac{A_0-k}{A_0}\right)$ | $\approx$ | $\ln\left(\frac{A_0+k}{A_0}\right)$ | | LAC | $-2\ln\left(\frac{A_0-k}{A_0}\right)$ | $\approx$ | $\ln\left(\frac{A_0+k}{A_0}\right)$ | $\approx$ | LAC |
| 0 | $\ln\left(\frac{100}{100}\right)$ | 0 | $\ln\left(\frac{100}{100}\right)$ | 0 | – | $-\ln\left(\frac{100}{100}\right)$ | 0 | $\ln\left(\frac{100}{100}\right)$ | 0 | – | $-2\ln\left(\frac{100}{100}\right)$ | 0 | $\ln\left(\frac{100}{100}\right)$ | 0 | – |
| 25 | $\ln\left(\frac{75}{100}\right)$ | −0.288 | $\ln\left(\frac{125}{100}\right)$ | 0.223 | 1.29 | $-\ln\left(\frac{125}{100}\right)$ | −0.223 | $\ln\left(\frac{125}{100}\right)$ | 0.223 | 1 | $-2\ln\left(\frac{125}{100}\right)$ | −0.446 | $\ln\left(\frac{125}{100}\right)$ | 0.223 | 2 |
| 50 | $\ln\left(\frac{50}{100}\right)$ | −0.693 | $\ln\left(\frac{150}{100}\right)$ | 0.405 | 1.71 | $-\ln\left(\frac{150}{100}\right)$ | −0.405 | $\ln\left(\frac{150}{100}\right)$ | 0.405 | 1 | $-2\ln\left(\frac{150}{100}\right)$ | −0.810 | $\ln\left(\frac{150}{100}\right)$ | 0.405 | 2 |
| 75 | $\ln\left(\frac{25}{100}\right)$ | −1.39 | $\ln\left(\frac{175}{100}\right)$ | 0.560 | 2.48 | $-\ln\left(\frac{175}{100}\right)$ | −0.560 | $\ln\left(\frac{175}{100}\right)$ | 0.560 | 1 | $-2\ln\left(\frac{175}{100}\right)$ | −1.12 | $\ln\left(\frac{175}{100}\right)$ | 0.560 | 2 |

*Note* The loss aversion coefficient is defined as the ratio of the absolute value of the justice evaluation for a loss divided by the justice evaluation for a comparable gain

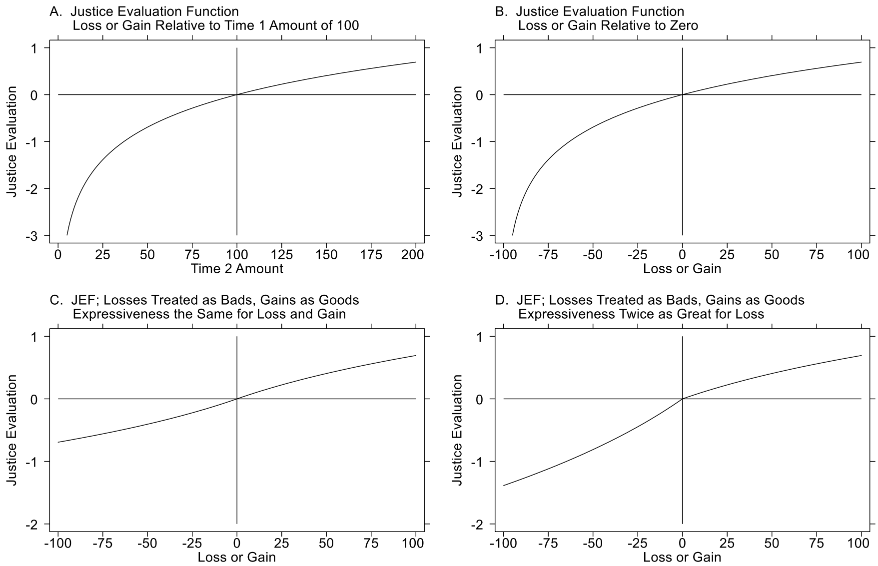


**Fig. 3** Justice evaluation function approximating the value function

As can be readily appreciated, the stage is set for further research in several directions, inclusive of understanding more deeply the underlying differences between the loss aversion process in the value function and in the justice evaluation function as well as in the two (symmetric and asymmetric) transformed justice evaluation functions. As well, further research can assess whether and how researchers and respondents may be unconsciously replicating the "divine proportions" of the Golden Number.

## 6 A Few Words on Money, Games, and Couples

Recently, several scholars have suggested that the special beauty of the Golden Number proportions may extend to matters of allocation and distribution, including salary schedules and classic games like the Ultimatum Game and the Common Pool Resource Game [15–17, 66–70]. Langen [67, 68] observes that salary schedules in hierarchical organizations might replicate Golden Number proportions in the progression from step to step. Thus, salaries in civil service, military service, private firms, public corporations, athletic teams, and dramatic and musical performing ensembles, may usefully be examined for evidence or traces of the Golden Number. Similarly, the many historical accounts of distribution—such as the national systems for allocating prize money from the capture of enemy ships in naval warfare—provide rich territory for assessing actual and preferred proportions.

In the realm of dyads, Langen [66–68] suggested that the Golden Number might be a solution to the Ultimatum Game, and Schuster [15] and Suleiman [16], working independently and using different approaches, both obtained the Golden Number as a new solution for the Ultimatum Game, with a division of 0.618 for the proposer and 0.382 for the responder. Schuster [15] used ideas of optimality and infinite continued fractions to derive the solution at the Golden Number. Suleiman [16] proposed a model of "economic harmony" in which utility is defined as a function of the ratio between actual and aspired payoffs and then derived the solution at the Golden Number. Suleiman [17] subsequently extended the argument to bargaining games with alternating offers, again finding the Golden Number as the solution.

Meanwhile, Vermunt [70] suggested that a fair allocation to self and other may lie between equal division and Golden Number division, with 0.618 for self and 0.382 for other. Of course, for a votary of both Francis of Assisi and the Golden Number, the division might be 0.382 for self and 0.618 for other. This possibility that fair allocation may lie between equal division and Golden Number division may be interpreted—or occur—in several ways. First, the observer's idea of the just reward may be *a specific number* between 0.382 and 0.5 or between 0.5 and 0.618, depending on whether the observer's selection for the rewardee (self or other) is the smaller or the larger. Second, the observer's idea of the just reward may be *any number* between 0.382 and 0.5 or between 0.5 and 0.618, generating a justice zone. More generally, a third-party observer's ideas of the just reward for the two prospective recipients could be either (1) a specific set of complementary proportions between 0.382–0.618 and 0.618–0.382 (e.g., 0.4 & 0.6, 0.5 & 0.5, 0.6 & 0.4, etc.), or (2) any set of complementary proportions between 0.382–0.618 and 0.618–0.382.

Now consider couples of a special kind, romantic couples and married couples. How might they react to various allocations, including both allocations made inside and outside the dyad? Imagine a couple where the partners are close to identical in age, ability, education, aspirations, and so on. The two partners think they merit equal salaries so that each partner's justice evaluation equals the log of the ratio of that partner's earnings to the average of both partners' earnings. However, for unexplained reasons, the bride earns more than the groom. Thus, the bride has a positive justice evaluation and the groom a negative one.

Developing the link between individuals' justice evaluations and cohesiveness (in both dyads and larger groups), justice theory embeds the idea, based on Aristotle's notion that love is possible only to equals, that cohesiveness requires at the very least what may be called an equality fantasy. Accordingly, the partners' cohesiveness declines with disparity in their justice evaluations. Formally, the couple's cohesiveness reduces to the logarithm of the ratio of the smaller to the larger earnings. Thus, cohesiveness is lowest when the ratio of the smaller to the larger earnings is very small and reaches its maximum when the two earnings amounts are equal.

Justice theory also derives the cohesiveness for dyads in which neither partner has earnings, or only one partner has earnings, based on the idea that in such case, the valued good used to drive their self-worth is ordinal. The ordinal-case cohesiveness is the logarithm of one-half.

A number of testable predictions follow immediately, concerning, for example, whether a marriage is strengthened, weakened, or left unaltered if one partner takes up employment or becomes unemployed or retires, etc. The general prediction is that if the spouses' earnings ratio exceeds one-half, marital cohesiveness is greater when both partners are employed, but if their earnings ratio is less than one-half, marital cohesiveness is lower when both partners are employed.

Returning to the special case in which the two partners' salaries are unequal, the couple's cohesiveness equals the logarithm of the ratio of the smaller to the larger earnings. If the earnings ratio assumes the Golden Number proportions, the ratio of the smaller to the larger earnings is approximately 0.618. Thus, it lies between one-half (the ratio when only one partner is employed) and one (when the two salaries are equal). Accordingly, the Golden Number salary ratio is within a "stable" zone, as it does not lead the couple to alter their employment situations.

## 7 Conclusion

- The core story of this chapter asked a simple question—When is deficiency (loss) felt twice as keenly as comparable excess (gain)? The answer—When the deficiency (loss) or excess (gain) are approximately 0.618 of the just amount (Time 1 amount)—was a great surprise. For 0.618 is the Golden Number, the "divine proportion" in mathematics. Thus, the core story exemplifies the beauty, unity, and surprise of mathematics, science, and art.
- The chapter went on to consider whether and how the key driver in this appearance of the Golden Number, namely, the justice evaluation function of justice theory, can approximate the value function of prospect theory.
- The chapter also considered additional exciting recent developments about the Golden Number in social science, specifically in games and dyads.
- Embedded in the work reported in this chapter are several directions for future research. For example, the fact that empirical results indicate that loss is felt twice as keenly as gain suggests that researchers and/or respondents may be (unconsciously) choosing magnitudes of loss and gain that hover about the Golden Number, that is, themselves in thrall to this magic quantity.
- This work raises the question, what is the connection between loss and gain, on the one hand, and bads and goods, on the other?
- A further new question is, How is this connection itself linked to the choice of numbers that both ordinary people and researchers use "in their minds" to represent loss and gain, bads and goods?

- To this mix, we add one final set of questions: What does it mean when the Golden Number suddenly appears in scientific work? Is it a signal from nature? To be sure, it brings beauty, and it brings surprise, and it certainly serves to link mathematics, science, and art. But is there something else?

**Core Messages**

- Mathematics, science, and art share three hallmarks: beauty, unity, surprise.
- The Golden Number of mathematics exemplifies the three hallmarks, both internally and in its relations with science and art.
- The justice evaluation function of social science exemplifies the three hallmarks, both internally and in its relations with mathematics.
- The Golden Number answers the answer to a simple question arising from the justice evaluation function.
- Is the Golden Number a signal from nature?

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