Boole's Symbolized Laws of Thought Facing Empiricism

Marie-José Durand-Richard

Abstract George Boole (1815–1864) is rightly known as a logician, the author of an algebra of logic, even if he did not conceive it quite in the same way as we know it. The calculus on classes and the calculus of propositions, that he set out to be equivalent, were at the core of his algebra of logic. After more than twenty centuries where logic was associated to the analysis of language, it was handled for the first time with mathematical symbols. However, the known Boole's algebra is not the one Boole produced. In this chapter, I would like to move apart from the classical recurring approach to history – often referred to as Whiggist – and to contextualize Boole's work so as to brighten the main trends which guided this renewal of how to deal with logic. Boole was also a mathematician, and since his *Mathematical Analysis of Logic* (1847), his work was very close to the symbolical way of thinking algebra, as it was developed in Cambridge by "The Analytics" in order to found usual algebraic practices. *An Investigation on the Laws of Thought* (1854) enunciated a more systematic view of this project for logic, devised as a whole system, facing what was at stake for the theory of knowledge with the strong rise of experimental sciences. Clearly, Boole wanted to stand out from empiricism, and from metaphysics as well, and to maintain an absolute meaning to formal sciences. And I will scrutinize his philosophical references so as to precise how his enterprise answered these issues, relating to necessary reasoning, and to probable reasoning as well.

Keywords Boole · Symbolical logic · Algebra · Empiricism · Probability

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M.-J. Durand-Richard (✉)
University Paris 8 Vincennes Saint-Denis, SPHERELaboratory, UMR 7219 CNRS-University Paris Cité, Paris, France

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1 Introduction

George Boole (1815–1864) is rightly known as a logician, the author of an algebra of logic introducing two logical calculus, one on classes, one on propositions, that he demonstrated to be equivalent. His work introduced mathematical symbols in a matter devised for more than twenty centuries – since Aristotle in fact – as a framework for a rigorous analysis of reasoning, founded on the examination of language. Even if this bend toward mathematics was a crucial step for structuring logic as we know it nowadays, my interest in this chapter is not to focus on the aspects of Boole's papers and books which looked like its present state, with some recurring approach of history. I am much more concerned with the mathematical and logical trends he met, which led him to build this new system of logic.

Although these references are often passed over,[1] any researcher who knows the history of mathematics of the early nineteenth-century England cannot avoid to be struck by the first words of Boole's *Mathematical Analysis of Logic* (1847). He started by claiming his close proximity with "The Analytics," the students network of Cambridge young algebraists who undertook to found algebra as a science since 1812:

> They who are acquainted with the present state of the theory of Symbolical Algebra, are aware, that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed, is equally admissible, and it is thus that the same process may, under one scheme of interpretation, represent the solution of a question on the properties of numbers, under another, that of a geometrical problem, and under a third, that of a problem of dynamics or optics. This principle is indeed of fundamental importance; and it may with safety be affirmed, that the recent advances of pure analysis have been much assisted by the influence which it has exerted in directing the current ([9], 3).

Indeed, Symbolical Algebra was just this new way of thinking algebra. Here, Boole focused on its main idea issued from this new algebra: the interpretation of symbols did not affect the validity of reasoning, which was only founded on combination laws on arbitrary symbols. In such a frame, calculations on symbols are completely independent from the notions of number, quantity or magnitude, which, historically, were considered as primary notions.

Before proceeding to consider the consequences of this claim on Boole's own view of logic, I would like to recall the elements that testify to his long proximity with "The Analytics." As Boole's biography has already been largely studied [22, 45], I will focus only on these elements, and show how this reforming group was concerned with the renewal of mathematics, both in its teaching and as part of a theory of knowledge. Then, I will analyze at what stakes answered this

[1] Even when Luis Laïta recognized "the influence of [his] research of a universal method in analysis on [his] creation of Logic," his view is still rather anachronistic, as he looked for similitudes by looking backwards from Boole's work on logic, rather than studying first the effective scheme of the Symbolical approach [41].

specific approach of algebra to which Boole's subscribed, separating calculus on meaningless symbols, and interpretation of symbols. Finally, I will show how Boole's approach of logic answered to these stakes, and how he applied it both to necessary reasoning and to probable reasoning.

2 Boole's Relationship with "The Analytics"

The group named "The Analytics" was initiated by Charles Babbage (1791–1871). He brought together a dozen students from Cambridge University, among whom Goerge Peacock (1791–1858) and John F. W. Herschel (1792–1871) are probably the most well known. The initial intention of their Analytical Society was to the differential notation in the infinitesimal calculus – then called *The Calculus* – into the Senate House Examination [54]. With Oxford, these two Anglican institutions stood as "seminairies of sound learning and religious education," and were devoted to constitute and to preserve the stability and permanence of all forms of knowledge. To this end, Oxford curriculum was founded on scholastic logic, and Cambridge one was rooted on Euclidean geometry and Newtonian fluxionary notation of *The Calculus*.

After their first manifest and textbooks entirely written in Leibnizian notation [52], the really political involvement of "The Analytics" in the reforming movement was pursued on two fronts. First, they thought of new foundations for algebra, so that it could replace geometry as a propedeutic science in Cambridge curriculum. *A Treatise on Algebra* (1830), where George Peacock (1791–1858) presented his views on Symbolical Algebra for Cambridge students, was its culmination [48]. It was edited again in two volumes in 1842–45. Followers among his students, particularly Duncan F. Gregory (1813–1844) published numerous papers and books extending his views to a Calculus of Operations [40, 32]. Second, this group of Cambridge algebraists was included in a larger reformist network, "the network of Cambridge," whose main commitment was to create new scientific institutions of national importance, so as to counterbalance those created in the new industrial cities [13, 39]. The whole project was to reconcile the "practical men" of these cities and the "educated men" of the traditional centers of knowledge – primarily Anglican universities and the Royal Society – with science instead of religion as a new conceptual cement. In this perspective, science should remain a general science, but it had also to integrate new experimental data, wherever they came from, as well as what had been developed from them, and experimental sciences as well. The strong and continuous political determination of this network led particularly to the foundation of the *British Association for the Advancement of Science* (1831), and later, more essentially, to the first reform of Cambridge and Oxford statutes since 1570 (1855–1858), a reform which will lead to the complete separation between teaching and religion in 1871 [26].

Boole had very early contacts with "The Analytics." He is often considered as a self-educated man, which had not the same meaning as today, since the universities trained very few students ([22], 13–51). He learned Continental mathematics – which was the main source of "The Analytics" – at the library of the *Mechanics Institute* of Lincoln, whose his father was secretary. But above all, in the early 1830s, in his immediate neighborhood, he met a friend of Babbage, the naturalist and landowner Sir Edward F. Bromhead (1781855), one of the founders of the ephemeral Analytical Society, who introduced him to symbolical methods. They corresponded for a long time. In a letter dated July 24, 1839, Boole wrote to Bromhead:

> I have just obtained by the principle of separation of symbols a series of very important results leading to a direct and uniform method of integration for linear equations both in the Differential Calculus and in the Calculus of finite and mixed differences. The method is so remarkable both as respects the form of the process and its faculty of application that had I not fully tested it both in principle and in practice I should feel disposed to question its truth ([45], p. 48–49).

The method of separation between the symbols of variables and those of operations was initiated by the French analyst Louis F. A. Arbogast (1759–1803) in his *Calcul des Dérivations* (180) and was a major tool for "The Analytics." It was notably relied on by Babbage when he described how data and operations stepped in independently in *The Analytical Engine* ([28], 46), and it founded Gregory's Calculus of Operations.

In 1839, Boole first met Gregory as co-founder with Robert L. Ellis (1817–1859) of the *Cambridge Mathematical Journal,*[2] the first journal specialized in mathematics in Great Britain. It aimed to spread original methods, and to open its pages to young researchers, both as readers and authors, including from abroad. So, its founders perceived mathematics as a science in progress. Numerous contributors will become famous mathematicians and physicists, such as Boole, A. de Morgan (1806–1871), J. J. Sylvester (1814–97), G. G. Stokes (1819–1903), A. Cayley (1821–95), and W. Thomson (1824–1907). Their journal made visible the renewal of mathematics in England and was an important marker in its process of professionalization.

Boole's first two papers, published in the second issue of this journal, dealt with Continental topics: invariants and calculus of variations. They were prepared in 1838 before his contact with Gregory and Cambridge [5, 6]. The second one already separated the symbol of differentiation *d/dx* from the symbol of function *u*, according to Arbogast's method. So, Boole was ready to follow Gregory publishing papers on the resolution of linear differential equations, which extended radically Arbogast's approach. He gave another method than Gregory for the resolution of such equations with constant coefficients [7, 35].

So, during the 1840s, Boole published first as one of "The Analytics," and his work was appreciated enough to receive the Gold Medal of the Royal Society

[2] The journal first appeared in small quarterly fascicules from 1837, and these first fascicules were assembled as a first issue in 1839. After some irregularities in the first years, the publication was quickly stabilized, and took importance, becoming the *Cambridge and Dublin Mathematical Journal* in 1845, and then the *Quarterly Journal of Pure and Applied Mathematics*, from 1855.

in 1844, for an impressive paper, "On a general method of analysis," where he presented a very high level of symbolical abstraction [47]. This general method propounded a generalization of the calculus of operations as exhibited by Gregory, allowing to produce symbolical presentations for the resolution of linear differential equations with variable coefficients, so as for the theory of series, the theory of generating functions, and the theory of equations of finite differences. If Boole introduced this work with historical references to famous Continental analysts – from Euler to Laplace – his work resulted from the maturation of his numerous exchanges, through their papers, with less known Analytics, such as Brice Bronwin and Robert Murphy (1806–43). At this period, he was in a close relationship with numerous algebraists of the second generation of the symbolical network, who will recommend his candidature[3] of the professorship at Queen's College of Cork in 1849.

From this paper, it seems that Boole moved away from mathematics to turn to logic, even if he still published some papers in the *Cambridge Mathematical Journal*. His main books on logic were produced in 1847 – *A Mathematical Analysis of Logic* – and 1854 – *An Investigation on the Laws of Thought*. Their specific contents exemplified how he was looking for a rigorous transfer of his practice of the calculus of operations on logic. Notwithstanding, he never gave up mathematics, as his last publications were two whole treatises dealing with the subjects that guided the research of symbolic algebraists for the first half of the nineteenth century. In *A Treatise on Differential Equations* in 1859 and in *A Treatise on the Calculus of Finite Differences* in 1860, symbolical methods appeared as the achievement tot be reached after presenting the usual methods [11, 12, 34].

In this long intellectual growth of Boole's ideas on the calculus of operations alongside "The Analytics," it would be a serious omission to forget the 22 years of correspondence (1842–1864) between Boole and De Morgan, dealing with all the subjects related to analysis, mainly the calculus of operations, differential equations, probabilities, and logic [53]. This correspondence began a long time before – and made him well aware of – the controversy between De Morgan and the Edinburgh professor of Logic and Metaphysics Sir William Hamilton (1788–1856) on the invention of the quantification of predicate which is abundantly commented in the history of logic[4] ([DEMO], 297–323; [22], 96–105)).

So, the biographical elements just highlighted give evidence of the close relationship, both personal and intellectual, of Boole with the network of "The Analytics." Both of them insisted upon the symbolical way of thinking operations, and upon their scope by working on meaningless symbols. This approach of the role of symbols in thinking and in calculations was particularly discussed in Great

[3] These recommendation letters were from Peacock, Ellis, Thomson, de Morgan, Kelland, and Rev. Charles Graves (1812–99).

[4] This was really a false quarrel, as the quantification of the predicate was first produced by George Bentham (1880–85), which Hamilton did not ignore it, since he wrote a report on the new publications in *The Edniburgh Review* for 1833, which contained this work [4, 37].

Britain at this period [14]. It concerned the whole field of knowledge and was at the core of hard debates. It will be shown how Boole's work wanted to introduce a strong mediation between opposite positions on this issue.

3 The Symbolical Approach of Reasoning and Its Social Stakes

Hard debates on the nature of reasoning took place throughout the whole first half of the nineteenth century. If the Reform Act (1832) alleviated political tensions, it did not fill in the gap widened between the "learned men" and the "practical men" from the Industrial Revolution [46]. The attachment of the Anglican Universities to permanent forms of knowledge made difficult for them to include experimental sciences and prevented them to think the industrial world. These institutions of "sound learning and religious education" were not prepared to understand and regulate the new nascent world, with all its transformations. Many pedagogical attempts were driven, both in Oxford for logic, and in Cambridge for mathematics, sometimes to exclude, sometimes to integrate these new forms of knowledge into the traditional curricula, according to the political inclinations of the protagonists [26, 27, 54]. The most striking example in Oxford was the attempt of Richard Whately (1787–1863), Fellow of Oriel College, and then archbishop of Dublin from 1831, whose *Elements of Logic* (1826) will be a major reference to Boole in 1847. In a long appendix to this treatise, Whately tried to think political economy in the framework of the scholastic logic [17]. For the same purpose of logically structuring the new experimental sciences, other authors rather tried to renovate this logic so as to incorporate new criteria of classification. Hamilton will call them "The New Analytics," referring directly to Aristotle's *Analytics* ([22], 97). In such a way, the quantification of the predicate allowed to privilege the extensional form of the proposition rather than its intentional form, and to write it as an equation. So, on both sides, in logic as in mathematics, the crucial issue was to renew the tools of analysis, and moreover, to ground it on new foundations.

Among all these attempts, the symbolical approach of Cambridge algebraists stood for the most radical search of a fruitful association between theory and practice, with a specific place given to each one, and with an explicit relationship between the two. This research concerned both the existence of adequate institutions and the conceptual structure of mathematics. The first topic has already been analyzed [13, 26], with the numerous commitments of "The Analytics" and the network of Cambridge to establish a scientific community inspired by Bacon's *Novum Organon* ([1], 94). So only the second topic will be plainly examined here, so as to analyze more precisely what seminal structural features Boole inherited from the symbolical approach of mathematics.

3.1 The Main Features of the Symbolical Approach from Peacock to Boole

The introduction of the differential notation in the infinitesimal calculus in Cambridge examinations was not enough for the Dons to accept algebra in place of geometry in the curriculum. While the *Elements* of Euclid structured geometry for centuries, with demonstrations from postulates, axioms and definitions, algebra emerged from successive practices, replacing numbers by letters, while keeping their arithmetic definitions. These practices induced the use of unfounded entities, such as negative or impossible quantities, and the introduction of unsolved paradoxes, for instance when a calculation was implicitly moving through the real field to the complex field, as evidenced by the long controversy that stirred the whole eighteenth century around the logarithms of negative numbers [15].

After a decade of papers by Babbage on notation ([24], 996–176), Peacock's Symbolical Algebra was conceived to resolve these issues. Operations were no more rooted on their technical means to obtain effective results, but on their properties as "laws of combination." Nevertheless, Peacock did not present it in an axiomatic way, but from a constructive epistemology, so as to preserve both practices as the source of any process of abstraction, and the superiority of necessary truths on contingent results as well. So, Symbolical Algebra occurred as the third step of a mental process of abstraction, and the two first ones were as follows:

- Common Arithmetic, "the science of measure and quantity," of which Peacock first wrote an extended history in 1826 for the *Encyclopaedia Metropolitana*, focusing on the underlying presence of mental operations in naming and writing numbers all over the world in the multitude of languages [31, 50],
- Arithmetical Algebra, the symbols of which were "general in their form, but not in their value." It did not correspond to the present state of algebra, the logical difficulties of which stemmed from the usual confusion between the fluctuating meanings of symbols between arithmetic and algebra as it stood. Arithmetical Algebra was rather a logical reconstruction, where operations kept the same limitations of meaning as in arithmetic. For instance, $(a - b)$ and $\sqrt{a-b}$ existed only if $a \geq b$.

But the epistemological status of this Arithmetical Algebra was much more important than the one of a literal arithmetic. As symbols concealed numbers, and therefore, numerical results, this algebra was also a "science of suggestion," as Peacock named it, making explicit – revealing in some way – operational processes by the ways and the rules to combine symbols.

Finally, Symbolical Algebra was the "language of symbolical reasoning," where symbols were "general in their form and in their value," that is perfectly arbitrary. So, arithmetical equality, as only contingent, was replaced by a strictly symbolical equivalence of forms. No interpretation was necessary to get negative and imaginary quantities, which just occurred as symbolical entities from equivalent forms such as:

$$a-(b+c) = a-b-c$$

and $\sqrt{a-(b+c)} = \sqrt{a-b-c}$ when $\boldsymbol{a} = \boldsymbol{b}$. In other words, results in symbolical algebra were perfectly independent of the meaning of symbols. Thereby, the existence of a meaning for symbols was no more even necessary. In any case, it did not affect at all operations and their results: it was logically subordinate and posterior to them.

With such a view, the discovery of these general forms of Symbolical Algebra was no more supported by analogy, by which they were often legitimated during the eighteenth century – implicitly or explicitly according to the periods and the authors – to extend the scope of operations on symbols whose meaning was therefore changing. Rather than prohibiting analogy, Peacock settled it as the perceptive part of a more essential mental law, the "principle of permanence of equivalent forms," by which the results obtained in Arithmetical Algebra – this science of suggestion – could be transferred directly in Symbolical Algebra, as long as their forms were absolutely general:

> (A): Whatever form is algebraically equivalent to another when expressed in general symbols, must continue to be equivalent, whatever those symbols denote.
> (B): Converse Proposition: Whatever equivalent form is discoverable in arithmetical algebra considered as the science of suggestion, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as their form ([49], 198–199).

Although a retrospective view of the history of mathematical thought often induces it, the twofold aspect of this principle precludes to consider that Arithmetical Algebra would appear, in Peacock's view, as a simple application of Symbolical Algebra. Its role as science of suggestion also shows that it could not be so. It seems more relevant to interpret this twofold principle by resorting to philosophy than to mathematics. In fact, the main characteristics of Peacock's construction were strongly supported by the philosophy of Locke in his *Essay Concerning Human Understanding,* which was taught at Cambridge University [25]. It was summoned by algebraists as Robert Woodhouse (1773–1827) and Charles Babbage (1791–1877) since the beginning of the nineteenth century, in order to support a theory of invention founded on the "perception of the operations of our minds" ([44] II.1.4; [2], 59). All these authors abundantly used Locke's vocabulary. They praised attention to perceptions as the main source of our ideas, to the arbitrariness of signs because of the separate existence of ideas and words, and to demonstration defined as a "method of showing the agreement of remote ideas by a train of intermediate ideas" ([55], 107), what algebra was specially suitable to exemplify. It can be said that Symbolical Algebra stood as the logic of arithmetical operations, considered as rooted on Locke's view of the operations of the mind.

Several followers of Peacock quickly noticed that this logic was too much scanty to cover the whole scope of operations, as they were already more varied at this time, for instance through the calculus of functions ([40], Chap. 3). After several papers on symbolical methods for the resolution of differential equations, Gregory carried out the same approach as Peacock to determine what he named the "real nature of symbolical algebra" in 1840, where he identified classes of operations submitted to the same laws of combination [36]. So, he used abundantly analogy between operations inside each class, but he went further than Peacock to overtake it, replacing the principle of permanence of equivalent forms by a very powerful assertion, a better structured principle which I name "the principle of transfer." It precised the conditions which had to be fulfilled to use it. He first presented it in 1839, from three laws – law of indices, commutative law, and distributive law – established for numbers, and transferred to some functions, particularly to the symbol Δ for finite differences, and to the symbol d/dx for differentials:

> For whatever is proved of the latter symbols, from the known laws of their combination, must be equally true of all other symbols which are subject to the same laws of combination.
>
> Now the laws of combination of the symbols *a*, *b*, &c., are:
>
> $a^m.a^n x = a^{m+n}.x$ [index law]
> $a\{b(x)\} = b\{a(x)\}$ [commutative law]
> $a(x) + a(y) = a(x+y)$ [distributive law]
>
> And if f, f_1, &c. be any other general symbols of operation (f and f_1 being of the same kind) subject to the same laws of combination, so that:
>
> $f^m.f^n(x) = f^{m+n}(x)$
> $f\{f_1(x)\} = f_1\{f(x)\}$
> $f(x) + f(y) = f(x+y)$
>
> Then whenever we may have proved of *a, b*, &c. depending on these three laws, must necessarily be true of f, f_1, &c.
>
> Now, we know that the symbol *d* is subject to these laws
>
> $d^m.d^n(x) = d^{m+n}(x)$
> $\frac{d}{dx}\{\frac{d}{dy}(z)\} = \frac{d}{dy}\{\frac{d}{dx}(z)\}$
> $d(x) + d(y) = d(x+y)$
>
> and the same is true for the symbol Δ.
>
> Hence the binomial theorem (to take a particular case) by which have been proved for (*a*) et (*b*), is equally true for $\frac{d}{dx}$ and $\frac{d}{dy}$ ([35] 34).

This long quotation intends to show both the main source of this principle, that is the search of symbolical methods for the resolution of differential equations, and of equations on finite differences by which they were approximated. It was the touchstone of the crucial papers of Gregory on these subjects in 1839, where he systematically relied on the method of separation of symbols of operations from those of quantities. It was fully taken up on the first page of Boole's "General Method of Analysis" in 1844, where he assessed that he considered this principle as granted to present this method:

> There are a number of theorems in ordinary algebra, which, though apparently proved to be true only for symbols representing numbers, admit of a much more extended application. Such theorems depend only on the laws of combination to which the symbols are subject, and are therefore true for all symbols, whatever their nature may be, which are subject to the same laws of combination ([8], 225).

This principle will be continually reiterated to found Boole's whole approach of logic, where algebraical results will be transferred to logical propositions, from his mathematical interpretation of "and" and "or" from natural language.[5] The method of separation of symbols of operations from those of quantities made possible to highlight the logic of operations in mathematics, and so, to investigate the "mathematical principles of natural philosophy" on an algebraical form, as well as to look at logic in a mathematical way.

3.2 Boole's Symbolical Answer to Empiricism

Before introducing his specific symbolical approach of logic,[6] Boole detailed how his own methodology answered the debates related to the nature of logic and beyond, to the nature of knowledge.[7] John Stuart Mill (1806–1874) just published his *System of Logic, Ratiocinative and Inductive*. Following Whately for example ([26], 11–14), he accepted algebra as a simple system of notation, allowing the logicians to use it mechanically, which was a dare for the history of thought. Facing this opposition, Locke's philosophy in his *Essay Concerning Human Understanding* founded knowledge not only on the observation of nature by our perceptions, but also – and essentially I would say – on reflection, which was "the perception of the operations of our own minds" ([44], II.1.4). What I name his moderate empiricism allowed "the Analysts" to maintain a mediating position, and therefore to close the previous debate by rooting algebraical operations on these operations of the mind, grounded on the nature of human being. As soon as 1847, Boole considered algebra was much more than a mechanical means of writing, and refuted its use in logic in the manner supported by the radical empiricism of J. S Mill (1806–1874):

> If the utility of the application of Mathematical forms to the science of Logic were solely a question of Notation, I should be content to rest the defence of this attempt upon a principle which has been stated by an able living writer: "Whenever the nature of the subject permits the reasoning process to be without danger carried on mechanically, the language should be constructed on as mechanical principles as possible ; while in the contrary case it should be so constructed, that there shall be the greatest possible obstacle to a mere mechanical use of it [J.S. Mill, vol ii, p. 292, *System of Logic, Ratiocinative and Inductive*] ([9], 2).

In the very beginning of *An Investigation on the Laws of Thought,* where his symbolical approach of logic was completely developed as a system, Boole

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[5] This principle was also mobilized by Augustus de Morgan (1806–71) in his *Trigonometry and Double Algebra* (1849), in order to generalize what is a symbolical algebra:

> Any system of symbols which obeys these rules and no others, except they be formed by combinations of these rules – and which uses the preceding symbols and no others – except they be new symbols invented in abbreviation of combinations of these symbols – is symbolical algebra ([20] 103-104)

[6] It will be detailed in the next chapter.

[7] He will also conclude on the subject in his whole final Chap. 22 entitled: "On the nature of science and the constitution of the intellect."

announced the large scope of his design, from demonstrative to probable knowledge. Chapters on probabilities covered six chapters, almost one third of the book. More than the sole scholastic doctrine of inferences, logic concerned the whole field of reasoning, in every aspect of life, but essentially, of science, where experimental facts had to be integrated and thought of:

> To enable us to deduce correct inferences from given premises is not the only object of Logic; nor is it the sole claim of the theory of Probabilities that it teaches us how to establish the business of life assurance on a secure basis; and how to condense whatever is valuable in the records of innumerable observations in astronomy, in physics, or in that field of social inquiry which is fast assuming a character of great importance. Both these studies have also an interest of another kind, derived from the light which they shed upon the intellectual powers ([10] I-2).

Bacon and Locke were quoted as the first protests against the traditional logic, and Boole announced clearly that the design of his work was "to investigate the fundamental laws of those operations of the mind by which reasoning is performed," and "to give expression to them in the symbolical language of a Calculus." His inscription in the trend of "The Analysts" cannot be ignored. He asserted that "a science of intellectual faculties is possible," and fixed the requirements for his general method in Logic, where Locke's vocabulary was still sensible, as well as in his distinction between ideas and words as signs of ideas:

> To unfold the secret laws and relations of those high faculties of thought by which all beyond the merely perceptive knowledge of the world and of ourselves is attained or matured,
> [To] exhibit Logic ..... as a system of processes carried on by a system of symbols .... ([10] I-2).

As I announced in § 3.1, Boole's symbolical approach was plainly supported by Gregory's principle of transfer: "if the arithmetical and the logical processes are expressed in the same manner, their symbolical expression will be subject to the same formal law" ([10], 22). He quoted it very frequently in his book, each time he wanted to use – and also to supersede – "the close analogy between the operations of the mind in general and its operations in the particular science of Algebra" ([10], 4). His claim was to constitute logic as a "positive knowledge" ([10], 313), which supposed to separate his system of thought from any metaphysical speculation, and from idealistic and skeptical considerations as well. So, it was supported by an explicit methodology, intended to observe how the operations of the mind were working, apart from any causal assumption. First, in Chapter II, Boole put forward a direct analysis of the language, and of the relations between classes of objects[8] that this language makes possible to isolate. In a second time, in Chapter III, he examined the operation of mental selection which corresponded to the choice of these classes, where Locke's thought can still be identified. In both cases, the results of these analysis were formally equivalent, and Boole considered that this coincidence confirmed the validity of his positive approach. As Boole already wrote in 1847:

[8] Classes of objects corresponded to the general terms in Locke's philosophy.

> That which renders Logic possible, is the existence in our minds of general notions, our ability to conceive of a class, and to designate its individual members by a common name. The theory of Logic is thus intimately connected with that of Language. A successful attempt to express logical propositions by symbols, the laws of whose combinations should be founded upon the laws of the mental processes which they represent, would, so far, be a step towards a philosophical language ([9] 4-5).

But the discovery of the laws of thought by observation would not be the same as the discovery of the laws of nature. Here, Boole referred – without quoting him – to a principle formulated by David Hume (1711–76) in his *Enquiry Concerning Human Understanding*, a principle known as "Hume's fork," where he distinguished between "relations of ideas" and "matters of facts".[9] Boole used this principle so as to assert that only one observation is enough to discover one law of thought. He stated it in this form:

> On the other hand, the knowledge of the laws of the mind does not require as its basis any extensive collection of observations. The general truth is seen in the particular instance, and it is not confirmed by the repetition of instances ...... [that] truth is manifest in all its generality by reflection upon a single instance of its application. ... [The perception of such general truths is not derived from an induction from many instances, but is involved in the clear apprehension of a single instance. In connexion with this truth is seen the not less important one that our knowledge of the laws upon which the science of the intellectual powers rests, whatever may be its extent or its deficiency, is not probable knowledge ([10], I-4).

Even if Hume was never quoted in Boole's *Investigation*, this principle was given under several forms throughout the book. This reference is very important as Hume went on Locke's analysis on the operations of mind, but developed a more skeptical philosophy of understanding, particularly on the relation between cause and effect, which he reduced to the habit to expect phenomena in succession ([38] 30–34). The considerable importance Boole gave to probabilities is equally significant for his special tribute to Hume's probabilistic vision of knowledge, but he used Hume's fork to assert that the laws of thought escaped both to induction and probability: they remain supported by the faculties of the human mind. Once more, it can be seen that Boole was open to the contributions of empiricist philosophy, notably on the operations of the mind, on the arbitrariness of signs, and on the distinction between experiments on nature and discourse on it, based on Locke's distinction between the unknowable substance of things, perception on things, ideas, and finally words produced from these perceptions and reflections on them. Boole's mediating position induced him to support these distinctions in his system of logic. This one, founded on the relations between classes rather than on inferences, testifies perfectly to this approach ([25], 10–17).

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[9] Here is Hume's statement:

> All the objects of human reason or enquiry may naturally be divided into two kinds, to wit, *Relations of Ideas,* and *Matters of Fact.* Of the first kind are the sciences of Geometry, Algebra, and Arithmetic; and in short, every affirmation, which is either intuitively or demonstratively certain..... Propositions of this kind are discoverable by the mere operation of thought, without dependence on what is anywhere existent in the universe ([38] 18).

4 Boole's Symbolical Writing of the Laws of Thought for Demonstrative Knowledge

My specific presentation in this chapter intends to underline how Boole's new view of logic answered the contradictions opposing scholars in the philosophical and mathematical background of his time.

In his first methodological approach, Boole considered language as "an instrument of reasoning," and logic as the grammar of this language – its syntactic structure nowadays. He determined its articulations by considering it as a system combining arbitrary signs, devoid of any interpretation. The characteristics he gave of this symbolical language were founded on a specification of the writing of mathematical operations which he transferred in logic:

> Proposition I:
>
> All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed on the following elements, viz.:
>
> 1st. Literal symbols, as *x*, *y*, &c., representing things as subjects of our conceptions.
>
> 2nd. Signs of operations, as +, –, &c., standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.
>
> 3rd. The sign of identity, =.
>
> And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of algebra [10]., II-4]

He then explored heuristically the structure of this symbolical language by referring the Hume's fork. So, if x represented *white things* and if y represented *sheep*, xy represented *white sheep*. So, the conjunction "and" corresponded to the multiplication in algebra. Besides, if x represented *men* and if y presented *women*, $x + y$ represented both of them. So, the conjunction "or," with its exclusive meaning, corresponded to the addition of algebra. From a few examples, Boole immediately remarked that:

$$xy = yx$$
$$x + y = y + x$$

which he considered as laws expressing general truths. Furthermore, if z represented *Europeans*, il is possible to write:

$$z\,(x + y) = zx + zy$$

Therefore, Boole identified there the commutative law and the distributive laws for literal symbols and algebraical symbols as well. And they could be expressed by these equations in algebra and in logic as well. As soon as he wrote the first of these laws, Boole specified that he was not using any analogy between the two domains, but indeed, the expression of a same formal law:

> In saying this, it is not affirmed that the process of multiplication in Algebra, of which the fundamental law is expressed by the equation
>
> $$xy = yx,$$
>
> possesses in itself any analogy with that process of logical combination which *xy* has been made to represent above; but only that if the arithmetical and the logical process are expressed in the same manner, their symbolical expressions will be subject to the same formal law ([10], II-8).

And he will repeat it on multiple occasions all along the book. This formal identity between laws of algebra and mathematical expression of logical properties allowed him to use abundantly the principle of transfer inherited from Gregory, which he has already so appropriated – at least since his 1844 paper – that he does not even feel the need to name it this time.

> The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established ([10], II-15).

Because of his specific characterization of the conjunction "or" in its exclusive meaning, it appears clearly that what is called Boole's algebra nowadays is not exactly what Boole considered as an Algebra of Logic. Moreover, this Algebra of Logic was a very special Algebra, because it was submitted to a very special law ([10], II-15).). Indeed, when the two symbols x and y had the same meaning, the expression xy became *xx,* that is, x^2, which had the same meaning as *x.* Algebraically speaking,

$$x^2 = x \quad \text{which could be transformed in} \quad x–x^2 = 0$$
$$\text{which gave, by factorizing,} \quad x\,(1–x) = 0$$

where the algebraical solutions of this equation are 0 and 1 ([6], III 11–12). For Boole, this algebraical expression evinced the great superiority of his system of logic on the scholastic logic, as its positive knowledge superseded the "metaphysical principle of contradiction," usually considered as the "fundamental axiom of any philosophy," by this more fundamental law of thought:

> PROPOSITION IV – That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2 = x$ ([10], III-15].

Anyway, his Algebra of Logic was a special one, where the symbols *x*, *y*, *z*, ... admitted only of the values 0 and 1, to which Boole attributed the respective meanings *Nothing* and *Universe*. Both these meanings and the definition of the contrary class could be clearly asserted because of his statement of what was "the universe of discourse,"[10] in which any selection of classes would operate in a given context ([10], III-4). Provided with these algebraical formulations, Boole completed his analysis of terms by specifying the separation of the universe of discourse in its different disjoint constituent parts according to the properties they possess or do not possess. For instance,

> Either x's or y's," be expressed by $x(1-y)+y(1-x)$, when the classes denoted by x and y are exclusive, by $x+y(1-x)$ when they are not exclusive. Similarly let the expression, "Either x's, or y's, or z's," be expressed by $x(1-y)(1-z)+y(1-x)(1-z)+z(1-x)(1-y)$, when the classes denoted by x , y , and z , are designed to be mutually exclusive, by $x+y(1-x)+z(1-x)(1-y)$, when they are not meant to be exclusive, and so on ([10], IV-7).

More generally, what Boole named a "logical function," that he wrote as a mathematical function $f(x)$, would be decomposed in such a partition between such subclasses, from the distinction between each class and its contrary, for each variable designating a possible property of elements.

Afterward, what Boole named "primary propositions," that is "relations between classes," could be expressed by a simple equality in case of universal propositions, or with the adjunction of the symbol v – designating "*some*" – in case of particular propositions:

> 1st. If the proposition is affirmative, form the expression of the subject and that of the predicate. Should either of them be particular, attach to it the indefinite symbol v, and then equate the resulting expressions.
>
> 2ndly. If the proposition is negative, express first its true meaning by attaching the negative particle to the predicate, then proceed as above ([10], IV-14).

This algebraical grounding of logic brought a considerable extension of Aristotle's classification of propositions, of the conversion of propositions and his theory of syllogisms, to which Boole devoted a short Chapter XV. In fact, he intended mainly to show that the issue of the elimination of a mean term between two premises became the issue of the elimination of a term between two equations, or of several terms between several equations. The law of duality $x\ (1-x)=0$, which was shown to be partaken by all logical symbols, made logical elimination more powerful than algebraical elimination. It allowed to eliminate as many terms as necessary to obtain a unique conclusive proposition ([10], VII & VIII).

The whole Boolean algebra of logic was founded on this calculus of classes, on which he built his analysis of propositions, and later, his approach of probabilities. As "primary propositions" expressed relations among things, "secondary propositions" – such as "If the sun shines, the day will be fair," or "If all men are wise, then all men are temperate" – expressed relations among propositions ([10], XI-5). Boole invoked a "close and harmonious analogy" between the theory of primary and secondary propositions ([10], XI-2) which he was going to deal with, by referring to the time during which a proposition is true. For instance, he explained the validity of a statement such as "If the proposition X is true, then, the proposition Y is true," by asserting:

[10] De Morgan previously alluded to the notion of universe in 1846 [19], but Boole's "universe of discourse" was much more precise, anyway precise enough to be generally adopted.

> An undoubted meaning of this proposition is, that the time in which the proposition X is true, is time in which the proposition Y is true. This indeed is only a relation of coexistence, and may or may not exhaust the meaning of the proposition, but it is a relation really involved in the statement of the proposition, and further, it suffices for all the purposes of logical inference ([10], XI-5).

From this reference to time, Boole was able to consider that secondary propositions were submitted to the same logical laws than primary propositions, and so, to establish on them a formal logical calculus equivalent to the first calculus he just elaborated on primary propositions. To find deduction of reasoning on the previous existence of classes of objects, and on a relation of coexistence between propositions – rather than to a possible relation of cause to effect usually associated with logical inference – bears witness to the impact of Hume's skeptical philosophy, which reduced deduction to the habit of an expected succession between phenomena. Here again, Boole was sensitive to Hume's criticism of traditional logic and tried to overcome it by his own approach. Referring to time did not incline him to follow Kant's philosophy. He clearly stood out of it, even if he only alluded to him and to all "those who regard space and time as merely 'forms of the human understanding,' conditions of knowledge imposed by the very constitution of the mind upon all that is submitted to its apprehension," without never naming them. It was all the easier not to follow Kant's philosophy of understanding that contemporaneous analytical procedures of solving geometric problems began to consider the possibility for space to exist with four dimensions or more, "beyond the realm of sensible extension" ([10], XI-16).

5 Boole's Symbolical Writing of the Laws of Thought for Probable Knowledge

Most of the time, Boole's work on the theory of probabilities in his *Investigation on the Laws of Thought* is neglected or forgotten, firstly perhaps because of the shortcut often imposed to the title. This work is however far from being anecdotic or marginal. As soon as 1867, Charles S. Peirce (1839–1914) will see it as the main use of Boolean calculus ([22], 202). Once more, Boole intended to find the superiority of his theory on the laws of operations of the human mind compared to usual approaches of this topic. His approach remained the same. The title of Chapter XVII, "Demonstration of a general method for the resolution of problems in the theory of probabilities," indicated precisely his project and gave the structure of the following chapters. Boole's references testify to a thorough knowledge of his contemporaries' works, essentially those of Condorcet [16], Laplace [42, 43], Poisson [51], Quetelet, and Cournot [18], as well as to the specific interest of the algebraists of the symbolic current for the theory of probabilities,[11] such as De Morgan [21] and W. Donkin [23] (1814–69), then Savilian professor of astronomy at Oxford University, forgotten today. His methodology also remained the same. Boole started from previous results, that is, from acquired practices, which suggested his general method.

First of all, he specified the state of the art in Chapter XVI, taking care to distinguish between the philosophical aspect and the computational aspect of the notion of "probability." He recalled the fundamental definitions given by Poisson – a "distinguished writer" – in his *Recherches sur la probabilité des jugements*:

> - "The probability of an event is the reason we have to believe that it has taken place, or that it will take place".
>
> - "The measure of the probability of an event is the ratio of the number of cases favourable to that event, to the total number of cases favourable or contrary, and all equally possible" (equally likely to happen) . . . . . . ([10], XVI-2).

From these definitions, Boole pursued with statements directly issued form Laplace's views of knowledge, asserting that "Probability is expectation founded upon partial knowledge," and that "A perfect acquaintance with all the circumstances affecting the occurrence of an event would change expectation into certainty, and leave neither room nor demand for a theory of probabilities" ([10], XVI-2). Laplace's deterministic view of knowledge agreed well with the theological background of Locke's philosophy, for whom all knowledge remained knowledge of the work of God, and an approach of its creator ([44], II.7.6). His induced view of probabilities allowed Boole to consider it as a branch of logic, without any incompatibility with the Lockean assertion of the necessarily incomplete character of human knowledge. From this double heritage, however, Boole moved away from the previous practices, which constructed the theory of conditional probabilities from the probability of independent events. He considered this hypothesis of independent events as hazardous, because very often, the truth of such a hypothesis of their independence was not known. Boole preferred to refer to "simple events," which he just combined to define "compound events." He led off an explicit epistemological analysis of "simplicity" attributed to events, which did not concern their nature, but only the state of knowledge:

> I would, in the first place, remark that the distinction between simple and compound events is not one founded in the nature of events themselves, but upon the mode or connexion in which they are presented to the mind. [. . . .] The prescriptive usages of language, which have assigned to particular combinations of events single and definite appellations, and have left unnumbered other combinations to be expressed by corresponding combinations of distinct terms or phrases, is essentially arbitrary ([6], XVII-3).

[11] The quoted works are those of Laplace, *Théorie analytique des probabilities* (1814); Poisson, *Recherches sur la probabilité des jugements en matière criminelle et en matière civile* (1837); Cournot, *Exposition de la Théorie des Chances* (1843); Condorcet, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* (1785); and those of the British De Morgan, Probability, *Encyclopædia Metropolitana*; and Donkin, *Philosophical Magazine* (1851).

According to the Lockean and Symbolical approaches, Boole preferred to rely on operations and their combinations, and therefore, to consider events as combination of events supposedly simple. Thereby, the probability of compound events could be viewed as a combination of probabilities of simple events. But Boole first precised that probabilities did not concern directly the events, but the propositions which related them:

> [Their] form consists in substituting for events the propositions which assert that those events have occurred, or will occur; and viewing the element of numerical probability as having reference to the truth of those propositions, not to the occurrence of the events concerning which they make assertion ([10], XVI-6).

This surreptitious shift from events to probabilities allowed Boole to elude – at least temporarily – the analysis of the adequacy of the second to the first. But chiefly, it allowed him to apply the general method he had developed in previous chapters, and so, to transfer this time to probabilities the results obtained for the calculus of primary propositions, and already transferred to secondary propositions. So, he could recover the results obtained by his predecessors, but in the framework of his own method. For instance ([10], XVII-7):

Events		Probabilities
xy	Concurrence of x and y	pq
$x(1-y)$	Occurrence of x without y	$p(1-q)$
$(1-x)y$	Occurrence of y without x	$(1-p)q$
$(1-x)(1-y)$	Conjoint failure of x and y	$(1-p)(1-q)$

Also according to the symbolical approach, the issue of data was not at all considered as part of the theory of probabilities. Indeed, it supposed to skip from a statistical estimate of observations of any event to the determination of its probability, which supposed to pass to the limit when the number of observations tend toward infinity. So, for Boole as for the symbolical algebraists, the issue of data was part of the numerical field, and it was here the subject of a separate chapter, entitled "Of statistical conditions" ([10] XIX).

The whole Boole's project appeared very clearly in Chapters and XX and XXI: "Problems relating to the connexion of causes to effects" and "Particular application of the previous general method to the question of the probability of judgments." Boole intended to substitute his own general method to the theorem of Bayes on the probabilities of causes [3].[12] Precisely, this theorem involved hypothesis on causes independent of each other, and Boole insisted on the fact that this was a quite fallacious hypothesis, particularly in the example of making a judgment between the members of an assembly or a court of law, by which he concluded his analysis:

[12] Once more, Boole never cited this theorem. Laplace independently retrieved Bayes' theorem in his "Mémoire sur la probabilité des causes par les événements" in 1774. Boole was directly in linée of Laplace's work [42].

> We may collect from the above investigations the following facts and conclusions:
>
> 1st. That from the mere records of agreement and disagreement in the opinions of any body of men, no definite numerical conclusions can be drawn respecting either the probability of correct judgment in an individual member of the body, or the merit of the questions submitted to its consideration.
>
> 2nd. That such conclusions may be drawn upon various distinct hypotheses, as—1st, Upon the usual hypothesis of the absolute independence of individual judgments; 2ndly, upon certain definite modifications of that hypothesis warranted by the actual data; 3rdly, upon a distinct principle of solution suggested by the appearance of a common form in the solutions obtained by the modifications above adverted to.
>
> Lastly. That whatever of doubt may attach to the final results, rests not upon the imperfection of the method, which adapts itself equally to all hypotheses, but upon the uncertainty of the hypotheses themselves.

This conclusive example shows that Boole, as his contemporaneous algebraists reformers, was concerned by the implications of science in society, including here inside his theoretical effort. Nonetheless, he was wary of human decisions, and was looking for guarantees for their accuracy. Therefore, he was keen to propose a purely symbolical method, which depended only on the laws of thought, and which made it possible to establish explicitly when it was essential to introduce additional hypotheses before continuing to solve a problem, so that each could judge, and propose other possible hypotheses if necessary.

6 Conclusion

Even if Gottlob Frege (1848–1925) is often considered as the founder of modern logic – for his formalization of logical inference, and of universal and existential propositions with quantifiers – Boole realized the first essential jump from analysis of language to mathematical formalization of this analysis [29]. His work was supported by the same will as that of the Cambridge algebraists to close the debates between theory and practices in mathematics. Moreover, as evidenced by the whole text, and specially by the last chapter of the *Laws of Thought,* Boole grasped empirical critics on logics, and, as the Cambridge algebraists, he was alarmed at the dangers of a mechanization of thought by algebraic or formal reasoning, and at the frailness and the lack of universality of human practices. He partook the methodology of those who tried to found algebra on symbolical methods, and introduced a median thought so as to close the debates about the competition between logic and mathematics in the system of human knowledge.

The most important point was to focus on formal operating processes, and to show that they did not work only mechanically, by founding them on the faculties of mind. The moderate empiricism of Locke's philosophy of human understanding was a crucial support, with his conceptions of arbitrary signs and his distinctions between things, ideas on things from perceptions, and words expressing them. So, Boole insisted on Locke's idea that axioms, usual principles, or maxims were neither a creation of the human mind, nor the origins of our ideas, but only shortcuts introduced *a posteriori* for pedagogical purposes so as not to recall their elaboration steps ([44], IV-12-2). Boole argued that "there exists in our nature faculties which enable us to ascend from the particular facts of experience to the general propositions...; as well as faculties whose office it to deduce from general propositions asserted as true the particular conclusions which they involve" ([10], XXII-9).

This grounding of the laws of thought on the faculties of mind led Boole, even if he decided to deal only with the efficiency of the operations of mind, always to refer to a principle of order, of harmony, of coherence, which human mind can perceive by analogies revealing it. That is why, in spite of Boole's strong support on Symbolical Algebra, it would be an error to consider his algebra of logic as part of applied mathematics. Boole rather considered that logic and mathematics are both applications of a symbolical calculus, expressing these laws of thought with two different interpretations [33]. Moreover, as mathematics so founded on practices became a profane knowledge, Boole remained eager to show that all this knowledge tends toward the knowledge of God. The sole pretention of human knowledge may be to discover the truth, to comprehend it, but neither for Locke, nor for Peacock or Boole, to create it. It is the reason why they introduced a hierarchy between practices and general method, which disappear today in axiomatic approaches, especially in model theory where analogy is used in reciprocal ways from applications to models [30].

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