Tabular Notations

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Abstract On the basis of the Tractarian distinction between operation and function (and particularly between truth-operation and truth-function), this paper distinguishes "operational" and "tabular" logical notations. An operational logical notation is one that represents some logical operation. A tabular logical notation is one that represents the result of some logical operation without representing the operation. The paper shows that standard notations for sentential calculus and Euler diagrams are operational, while Wittgenstein's truth tables and Venn diagrams are tabular.

Keywords Logical notations · Diagrams · Truth tables · Truth-operation · Truth-function · Wittgenstein

Mathematics Subject Classification: Primary 97E30, 03B05, 03B10

1 Truth-Operations Versus Truth-Functions

In the *Tractatus*, Wittgenstein distinguishes between an operation and a function and particularly between a truth-operation and a truth-function. In the context of the sentential calculus, a proposition is a truth-function of elementary propositions ([1] §5) and is the result of truth-operations on elementary propositions ([1] §5.3). A truth-operation is the way in which a truth-function (a proposition) results from another truth-function; so "~*p*" results from the application of the truth-operation "~" to the elementary proposition "*p*." Truth-operations are iterative; thus, "~~*p*" is the result of two successive applications of the truth-operation "~" to the elementary proposition "*p*." One single truth-operation, the joint denial or Sheffer stroke, is capable of expressing all possible results of truth-operations on elementary

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propositions, i.e., the Sheffer stroke is a sole sufficiently operator for the sentential calculus. A generalization of the Sheffer stroke to n elementary propositions is presented at [1] §5.5 by saying that "every truth-function is a result of the successive application of the operation (– – – – – T) (ξ, ...) to elementary propositions." This generalizes the Sheffer stroke, which is a binary operation, to the simultaneous negation (represented by the left-hand parentheses) of n elementary propositions (represented by the right-hand parentheses).

It is a central claim of the *Tractatus* that an operation does not characterize the *sense* of a proposition ([1] §5.25). The sense of a proposition is its agreement and disagreement with the possibilities of truth and falsity (or truth possibilities) of elementary propositions ([1] §4.2); a proposition is the expression of agreement and disagreement with truth possibilities of elementary propositions ([1] §4.4); the expression of agreement and disagreement with truth possibilities of elementary propositions is the expression of its truth conditions ([1] §4.431); therefore, the sense of a proposition is its truth conditions, and the proposition expresses (i.e., *shows*, [1] §4.022) it and expresses nothing else. An operation does not characterize the sense of a proposition because one and the same sense (agreement and disagreement with truth possibilities of elementary propositions) may be obtained by application of distinct truth-operations on elementary propositions. For example, "~(p & ~ q)" and "$p \supset q$" agree and disagree with the truth possibilities of the elementary propositions "p" and "q" in precisely the same cases, i.e., they have the same truth conditions, and thus the same sense. If "~" really characterized the sense of "~(p & ~ q)," it should equally characterize the sense of "$p \supset q$," because "~(p & ~ q)" and "$p \supset q$" have the same sense. But no operation can be said to characterize the sense of a proposition in which it does not occur. Thus, "~" does not characterize the sense of "$p \supset q$," and therefore, it neither characterizes the sense of "~(p & ~ q)," because the sense of these two propositions is the same ([1] §5.43).

A truth-operation does not contribute to the sense of the proposition in which it occurs. An operation only expresses a difference between logical forms ([1] §5.24) but does not itself express a logical form. An operation asserts nothing; only the result of the operation does, but the result of an operation depends only on the basis of the operation ([1] §5.25), not on the operation itself. One and the same truth-function of elementary propositions may be seen as the result of distinct operations on those propositions. Thus, the result of the truth-operation "~" on "(p & ~ q)," which in turn is the result of the truth-operation "&" on "p" and "~q" (which latter is itself the result of the truth-operation "~" on "q"), is the same as the result of the truth-operation "⊃" on "p" and "q."[1]

[1] On Wittgenstein's distinction between operations and functions, see [2 , pp. 113–121], [3 , pp. 138–152], and [4].

2 Propositional Signs

Since the sense of a proposition is given by its truth conditions and since the expression of the truth conditions of a proposition is the expression of its agreement and disagreement with the truth possibilities of elementary propositions, the sign that expresses agreement and disagreement with the truth possibilities of elementary propositions is a propositional sign, i.e., it expresses the truth conditions of that proposition or its sense. Thus, a truth table like that in Fig. 1, since it expresses, in the right-hand column, the proposition's agreement and disagreement with the truth possibilities of the elementary propositions "*p*" and "*q*" given in the left-hand column, is a propositional sign ([1] §4.442). The truth table in Fig. 1 does not represent the truth-operation on elementary propositions by which the truth-function expressed in the right-hand column has been obtained. It only represents the *result* of some such truth-operation on elementary propositions. Truth-operations on elementary propositions which have one and the same truth-function as a result are not distinguished in a truth table. The reason is that since a truth-operation does not characterize the sense of a proposition but only the result of truth-operations does and this is the sense of the proposition, a propositional sign that should express the truth-operation by which a truth-function is obtained from elementary propositions would express something *foreign to the sense of the proposition*.

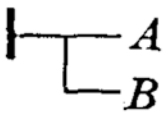
The truth table for a truth-operation is usually considered as the definition of that operation. But the truth table of a proposition is in no obvious sense its definition. Semiotically speaking, truth tables may be taken as *what is signified* by a propositional sign. A sign is an entity generated by the connection between an *expression* (Saussure's *signifiant*) and a *content* (Saussure's *signifié*; cf. [5] §13). In a proposition, an expression, like "$p \supset q$," is connected with a content, which is the particular correlation of truth-values with the truth possibilities of the elementary propositions "*p*" and "*q*" which the truth-operation "$\supset$" is taken to express and

Fig. 1 A truth table

‘p	q	’
T	T	T
F	T	T
T	F	
F	F	T.

which may be represented in a truth table like that in Fig. 1. This is certainly the way Frege understood the matter. Take Frege's explanation of the meaning of the conditional stroke of his *Begriffsschrift*:

> If *A* and *B* stand for contents that can become judgments (§2), there are the following four possibilities:
>
> 1. *A* is affirmed and *B* is affirmed;
> 2. *A* is affirmed and *B* is denied;
> 3. *A* is denied and *B* is affirmed;
> 4. *A* is denied and *B* is denied.
>
> Now



> stands for the judgment that *the third of these possibilities does not take place but one of the three others does* ([6] §5).

Frege clearly considers the indication that of the four truth possibilities of two elementary propositions only the third is negated as the definition of his conditional stroke. This indication is nothing more than the truth table for the conditional stroke. Thus, if we consider a *Begriffsschrift* propositional sign in which the conditional stroke occurs as the expression or *signifiant*, the truth table that expresses its truth conditions is its content or *signifié.* However, Wittgenstein's distinction between truth-operations and truth-functions, and his argument that a truth-operation does not characterize the sense of the proposition in which it occurs, together show that what according to Frege is the content of a propositional sign (or equivalently, the definition of a truth-operator that occurs within a propositional sign) can with right be considered *as the propositional sign itself.* Frege's propositional sign, the conditional stroke, expresses a truth-operation on elementary propositions. But a truth-operation does not characterize the sense of the proposition in which it occurs. A propositional sign is a sign of its truth conditions, not of the truth-operations of which those truth conditions are the result. In other words, the expression of the truth conditions of a proposition *is already a propositional sign*: while Frege considers his conditional stroke to be a propositional sign whose content is a truth table, for Wittgenstein the truth table is itself a propositional sign. In still other words, for Wittgenstein truth tables can constitute the signs of a logical notation, and not simply what is represented by the signs of a logical notation or what defines the truth-operators that occur in those signs.

3 Tabular Versus Operational Logical Notations

The idea that a truth table *is* (rather than, is represented by) a propositional sign and thus that truth tables may constitute the signs of a logical notation is a direct consequence of the Wittgensteinian distinction between an operation and a function. This distinction can constitute the basis of a more general division of logical notations, as I now proceed to show.

Some notations represent operations, and others represent the result of operations, i.e., functions. Consider the difference between Euler and Venn diagrams. In a recent history of logical diagrams, the difference between these two systems is explained as follows: "Euler circles represent the actual relation of the classes. [. . .] instead of representing directly the actual relation of the classes, Venn draws first a primary diagram representing all possible sub-classes that one obtains by intersecting the terms involved in the argument. Then, he adds distinctive marks to represent propositions according to the emptiness or occupation of the sub-classes" ([7], pp. 617–618). In Euler diagrams (Fig. 2), an "actual relation" is represented, the topological disjunction of the circles labeled "A" and "B"; and this topological disjunction directly represents the logical disjunction of the classes for which the circles stand. The actual topological relation of disjunction that holds between the two circles is the propositional sign. By way of contrast, in Venn diagrams (Fig. 3), the propositional sign is obtained in two successive steps. As a first step (Fig. 3a), circles labeled "A" and "B" are drawn in such a way as to divide the space into four regions: one region outside both "A" and "B" (~A~B), one region inside "A" but outside "B" (A~B), one region inside "B" but outside "A" (~AB), and one region inside both "A" and "B" (AB). These regions represent classes: "~A~B" is the class of the objects which are neither "A" nor "B"; "A~B" the class of the objects which are "A" but not "B"; "~AB" the class of the objects which are "B" but not "A"; and "AB" the class of the objects which are both "A" and "B." Given two terms "A" and "B," these four classes exhaust all logical possibilities. As a second step (Fig. 3b), one of the four regions so determined is blackened or shaded in order to represent that the corresponding class is empty. Thus, in Fig. 3b, the region corresponding to "AB" is blackened, meaning that the corresponding class is empty: no object is both "A" and "B."

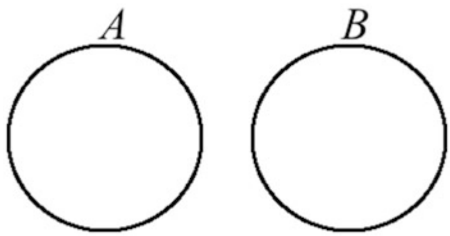


Fig. 2 Euler diagram for *No A is B*

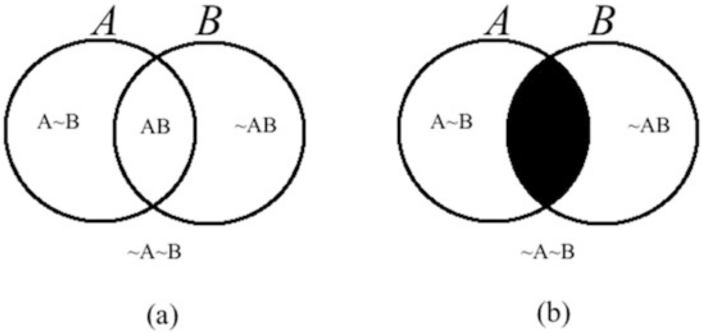


Fig. 3 Venn diagram for *No A is B*

The difference between Euler and Venn diagrams is the difference between notations that represent operations and notations that represent the result of operations. Euler diagrams represent operations. An operation in Euler diagrams is the combination of two or more Euler circles so that one of the five Gergonne relations[2] obtains. I will call such operations Gergonne operations. In Fig. 2, the Gergonne operation of disjunction between classes is actually represented by the disjunction of the circles. By contrast, Venn diagrams represent the result of operations: in Fig. 3b, no operation of disjunction is actually represented but only the result of this operation, namely, that the class of objects belonging to both classes is empty. The four regions determined by the intersecting circles in Fig. 3a function precisely as the left-hand columns in Fig. 1, and the diagram in Fig. 3b functions precisely as the right-hand column in Fig. 1: just like the left-hand columns in Fig. 1 represent, in the context of the sentential calculus, the possibilities of truth and falsity (or truth possibilities) of the elementary propositions "*p*" and "*q*," so the regions in Fig. 3a represent, in the context of the calculus of classes, the possibilities of belonging or non-belonging of an object to the two classes "A" and "B"; and just like the right-hand column in Fig. 1 represents the proposition's agreement and disagreement with the truth possibilities of the elementary propositions "*p*" and "*q*" (i.e., represents the proposition's truth conditions), so the diagram in Fig. 3b represents the agreement and disagreement with the possibilities of belonging or non-belonging of an object to the two classes "A" and "B," i.e., represents which of those possibilities is not realized.

The parallel should be reinforced by the following consideration. We saw above that the occurrence of a truth-operation in a proposition does not characterize the sense of that proposition; only the result of the operation does, and this does not depend on the operation but on the basis of the operation. Consequently, one and

[2] See [8]; the five Gergonne relations that can obtain between two classes "A" and "B" are (1) "A" is strictly included in "B," (2) "A" strictly includes "B," (3) "A" coincides completely with "B," (4) "A" and "B" are completely disjoint, and (5) "A" and "B" partly overlap.

the same truth-function of elementary propositions may be the result of distinct truth-operations on elementary propositions. Thus, for example, the truth-function represented in Fig. 1 may be the result of the truth-operation "~" on "(p & ~ q)" or the result of the truth-operation "⊃" on "p" and "q." Something remarkably similar happens with Euler and Venn diagrams.

Euler diagrams are unable to represent our imperfect knowledge about the actual relations of the classes that they represent. Venn was the first to see this:

> The great bulk of the propositions which we commonly meet with are founded, and rightly founded, on an imperfect knowledge of the actual mutual relations of the implied classes to one another. When I say that all X is Y, I simply do not know, in many cases, whether the class X comprises the whole of Y or only a part of it. And even when I do know how the facts are, I may not intend to be explicit, but may purposely wish to use an expression which leaves this point uncertain. Now one very marked characteristic about these circular diagrams is that they forbid the natural expression of such uncertainty. ([9], p. 2)

In Venn diagrams, by contrast, the sentence "All As are Bs" is perfectly represented by the single diagram in Fig. 5, which only represents that the class of the As which are not Bs is empty, thus leaving undecided whether there are Bs which are not As or not. Unlike Euler diagrams, Venn diagrams are able to represent our imperfect knowledge about the actual relations of the classes that they represent.

This problem, known in literature as the "over-specificity"[3] of certain systems of logical representation, is unsolvable in Euler diagrams without a modification of the basic conventions of the system. In Euler diagrams, the sentence "All As are Bs," whose standard meaning is taken to be "A is included in B" (where "is included in" is a partial order), cannot be represented in one single diagram without representing *either* that there are Bs which are not As *or* that all Bs are As.

This difference in the capability of representing imperfect information is due to the fact that while Euler diagrams represent operations, Venn diagrams represent the result of operations. In Fig. 4a, the Gergonne operation of strict inclusion between the classes "A" and "B" is represented by the strict inclusion of the circles "A" and "B," and in Fig. 4b, the Gergonne operation of complete coincidence between "A" and "B" is represented by the complete overlapping of the circles "A" and "B." By contrast, Venn diagrams represent the result of operations without representing operations: in Fig. 5, no operation of strict inclusion or complete coincidence

[3] [10, 11]. But the phenomenon was already perfectly clear to Johann H. Lambert in the eighteenth century; see [12] §§173–194, [7, 13]. The idea is that in any system of diagrams whatsoever, there exists a set of "operational constraints" which may or may not intervene in the process of encoding and extracting information. Operational constraints will give rise to a "free ride," i.e., a semantically significant fact is obtained by a reasoner in a diagram, while the instructions that the reasoner has followed to construct the diagram do not entail the realization of it. Under some conditions, the operational constraints will produce further information that logically follows from the information contained in the instructions to construct the diagram; under other conditions, the operational constraints produce "overdetermined alternatives," that is, will represent information that do not logically follow from the instructions to construct the diagram. The notion of "free ride" has been further developed and generalized into that of "observational advantage" in [14]; see also [15].

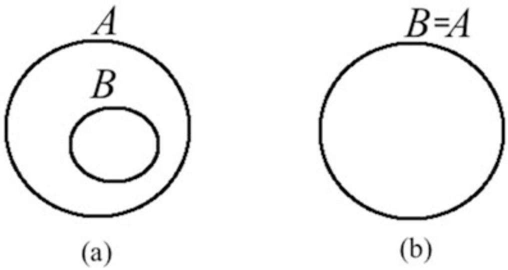


Fig. 4 Euler diagram for *All As are Bs*

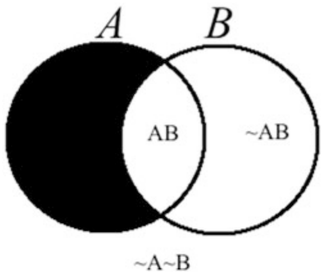


Fig. 5 Venn diagram for *All As are Bs*

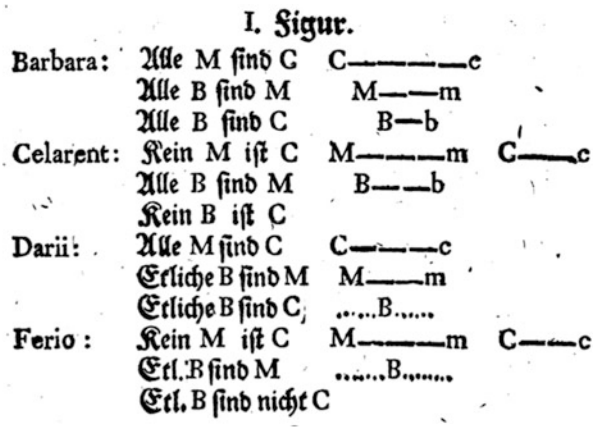
between "A" and "B" is actually represented but only *the common result of either operation*, namely, that the class of objects belonging to "B" but not to "A" is empty. The truth-function represented in Fig. 5 may be the result of the Gergonne operation of strict inclusion between "A" and "B" or the result of the Gergonne operation of complete coincidence between "A" and "B." And since it is both, it is also neither. As the occurrence of a truth-operation in a proposition does not characterize its sense, so the possibility that one and the same truth-function represented in a Venn diagram may be the result of one or more Gergonne operations on classes does not characterize the sense of the proposition represented in that diagram. Thus, another way of saying that Venn diagrams do not suffer from "over-specificity" is that they do not represent Gergonne operations.[4]

The same difference as we have found to hold between, one the one hand, the Russellian language of the sentential calculus and Euler diagrams and, on the other, Wittgenstein's truth-table notation and Venn diagrams can also be found in other systems of logical representation. For example, Leibniz's and Lambert's diagrams for syllogistic are clearly of the first kind, as they represent certain operations on line segments that correspond to operations on the classes that the line segments represent (Figs. 6 and 7; [12, 17]). So are also Frege's *Begriffsschrift* and Peirce's existential graphs. By contrast, James Welton's diagrams and Wittgenstein's *ab*-notation are clearly of the second kind, as they represent the result of truth-operations on elementary propositions (Figs. 8 and 9; [1, 18] §6.1203). So are also Marquand's diagrams (Marquand 1881).

[4] A comparison of Venn diagrams with Wittgensteinian truth tables is made in [16], pp. 107–124.

Fig. 6 Leibniz's diagrams for the four categorical propositions

Fig. 7 Lambert's diagrams for first-figure syllogisms



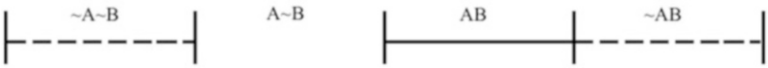
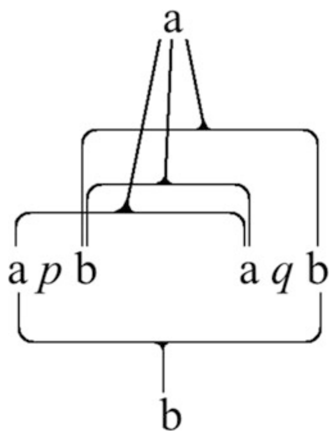


Fig. 8 Welton's diagrams

Fig. 9 Wittgenstein's *ab*-notation



I call "tabular" a logical notation that represents the result of a logical operation (i.e., a logical function or truth-function) without representing the operation; and "operational" or "non-tabular" a logical notation that represents a logical operation. As mentioned above, in their chapter on the history of logical diagrams, Moktefi and Shin express this difference by saying that while Euler diagrams "directly represent classes" and Gergonne relations between classes, Venn diagrams do not "represent the classes at all, but rather compartments, which when marked, tell whether the corresponding class is empty or occupied" ([7], p. 626). In fact, Moktefi and Shin follow Venn, who in attempting to state what the difference between his own system of diagrams and the Eulerian is, wrote that "[t]he most accurate answer is that our diagrammatic subdivisions, or for that matter our symbols generally, stand for compartments and not for classes" ([19], p. 119). This distinction is later in their chapter extended from Euler and Venn diagrams to Euler-like and Venn-like notations: thus, Leibniz's and Lambert's diagrams are Euler-like (they directly represent classes and relations between classes), while Welton's diagrams are Venn-like (they first represent compartments which are then marked to indicate whether the corresponding class is empty or not). My distinction between tabular and operational notations is a *generalization* of Moktefi and Shin's, which in turn was a generalization of Venn's. The generalization is needed in order to account for the fact that not all notations represent classes or compartments corresponding to classes. Wittgenstein's truth tables, for example, do not represent compartments corresponding to classes; they represent, in the left-hand columns, the truth possibilities of elementary propositions and, in the right-hand column, the agreement and disagreement of a given proposition with the truth possibilities of elementary propositions, i.e., the proposition's truth conditions. Conversely, Frege's *Begriffsschrift* and Peirce's existential graphs do not directly represent classes; they represent truth-operations on elementary propositions. Since as I have explained the representation of a Gergonne relation is the representation of an *operation* on classes and since the representation of the emptiness or non-emptiness of compartments corresponding to classes is the representation of the result of an operation on classes (a logical function), Moktefi and Shin's distinction between Euler-like and Venn-like notations can be generalized to that between operational and tabular notations. Unlike Moktefi and Shin's distinction, my distinction is not limited to notations that represent the logic of classes but covers, at least in principle, any notation whatever.

4 Conclusion: A Note on Diagrammatic Typology

Moktefi and Shin (see [7]) implicitly provide a typology of logical diagrams by a sort of dichotomic method. They first present what they call "spatial" diagrams and explain that there are two main varieties of them, i.e., Euler and Venn diagrams. Then they present other "kinds" of logical diagrams, which they call "linear" and "tabular" diagrams, and divide them according as they are of the Euler variety or of the Venn variety. For example, Leibniz's and Lambert's diagrams are linear and Euler-like; Welton's diagrams are linear and Venn-like; and Marquand's diagrams are tabular and Venn-like; Macfarlane used tabular diagrams of both the Euler variety ([20]) and of the Venn variety ([21]). (In fact, Macfarlane's diagrams are deemed by the authors to be of an intermediary kind between spatial and tabular.) Lewis Carroll also used tabular, Venn-like diagrams ([22, 23]). When it comes to

Frege's and Peirce's notations, however, the authors silently abandon the implicit method of classification employed until then in their analysis. Fregean and Peircean diagrams are said to be "another style of diagrams" ([7], p. 649), and it is presumably for their stylistic difference that their method cannot be applied to them.

This method of classification has at least two faults. First, it cannot account for notations, like Frege's *Begriffsschrift* and Peirce's existential graphs, which have a greater expressive power than the logic of classes. Second, it is based on properties of notations which have not been subjected to sufficient notational scrutiny. Let us start with the second fault. Moktefi and Shin concede that tabular diagrams (in their sense of "tabular") "are also spatial" ([7], p. 632). But if this is so, then the spatial/tabular distinction is destined to fail as a criterion of diagrammatic taxonomy. As another example, consider Frege's *Begriffsschrift*. Although this notation is not subjected to the same kind of analysis as are Euler and Venn diagrams, there is no reason to refuse to call such a notation linear (in the authors' sense of "linear"): just like "proper" linear diagrams, such as Leibniz's and Lambert's, *Begriffsschrift* formulas are made out of line segments combined in certain ways. Likewise, Peirce's existential graphs should be considered as a species of the genus of spatial diagrams (in the authors' sense of "spatial"), because the graphs can contain "cuts," and these are closed curves just like Euler circles are.

No one of the three members of Moktefi and Shin's threefold division of notations ("spatial," "linear," and "tabular") seems to me to merit the status of taxonomic category. This division is formed according to the "matter" of which formulas are made. Thus, when the matter is a closed curve, the notation is "spatial"; when it is a line segment, it is "linear"; when it is a quadrangular space, it is "tabular." But what if one comes out with a notation based on quadrangular spaces whose edges are rounded? Should this count as a spatial or as a tabular system of notation? Peirce's existential graphs may contain cuts, and these are closed curves. But nothing will change in the notation if instead of closed curves one were to use quadrangular closed spaces. Would this change existential graphs into a tabular notation? Or, as an even more radical example, consider the color of the ink: ink is in a sense something of which formulas are made; should we then distinguish between "blue" diagrams and "black" diagrams according as they are written in blue or black ink?

As far as the method of diagrammatic taxonomy is concerned, I propose to abandon the admittedly uncertain threefold division of notations into "spatial," "linear," and "tabular" notations and to generalize the distinction between Euler-like and Venn-like notations into that between "tabular" (in my sense) and "operational" notations. This would both get rid of the problems related to the hybrid status of some notations (Are all tabular notations spatial? Is *Begriffsschrift* linear? Could existential graphs become tabular?) and at the same time provide a more general taxonomic framework in which all notations, and not just those that represent the logic of classes, could find a home.

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