Chapter 9
Some Properties of Convexity Structure and Applications in b-Menger Spaces

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Abstract We discuss, in Menger spaces, the notion of convexity using the convex structure introduced by Takahashi (Kodai Math Sem Rep 22:142–149, 1970), then we develop some geometric and topological properties. Furthermore, we introduce the notion of strong convex structure and we compare it with the Takahashi convex structure. At the end, we prove the existence and uniqueness of a solution for a Volterra type integral equation.

Keywords Takahashi convex structure · Probabilistic strong convex structure · Volterra type integral equation

9.1 Introduction and Preliminaries

In the paper [1] authors introduced the concept of probabilistic b-metric space (b-Menger space), which the probabilistic b-metric mapping F is not necessarily continuous, and which generalizes the concept of probabilistic metric space (Menger space [8, 9]) and b-metric space. They discussed its topological and geometrical properties and they showed the fixed point and common fixed point property for nonlinear contractions in these spaces [5]. Furthermore, they defined the notion of fully convex structure and established [4] in fully convex b-Menger spaces the existence of common fixed point for nonexpansive mapping by using the normal structure property. Also, they showed a fixed point theorem in b-Menger spaces using B-contraction with cyclical conditions (See [2, 3] and [6]).

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Definition 9.1 ([1]) A b-Menger space is a quadruple (X, F, T, s) where X is a nonempty set, F is a function from $X \times X$ into Δ^+, T is a t-norm, $s \geq 1$ is a real number, and the following conditions are satisfied:

For all $p, r, q \in X$ and $x, y > 0$,

1. $F_{pp} = H$,
2. $F_{pr} = H \Rightarrow p = r$,
3. $F_{pr} = F_{rp}$,
4. $F_{pr}(s(x + y)) \geq T(F_{pq}(x), F_{qr}(y))$.

It should be noted that a Menger space is a b-Menger space with $s = 1$.

Definition 9.2 Let (X, F) be a probabilistic semimetric space (i.e., (1), (2) and (3) of Definition 9.1 are satisfied). For p in X and $t > 0$, the strong t-neighborhood of p is the set

$$N_p(t) = \{q \in X : F_{pq}(t) > 1 - t\}.$$

The strong neighborhood system at p is the collection

$$\wp_p = \{N_p(t) : t > 0\}$$

and the strong neighborhood system for X is the union

$$\wp = \bigcup_{p \in X} \wp_p.$$

In probabilistic semimetric space, the convergence of sequence is defined as follow.

Definition 9.3 Let $\{x_n\}$ be a sequence in a probabilistic semimetric space (X, F). Then

1. The sequence $\{x_n\}$ is said to be convergent to $x \in X$, if for every $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $F_{x_n x}(\epsilon) > 1 - \epsilon$ whenever $n \geq N(\epsilon)$.
2. The sequence $\{x_n\}$ is called a Cauchy sequence, if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $n, m \geq N(\epsilon) \Rightarrow F_{x_n x_m}(\epsilon) > 1 - \epsilon$.
3. (X, F) is said to be complete if every Cauchy sequence has a limit.

Schweizer and Sklar proved that if (X, F, T) is a Menger space with T is continuous, then the family $\Im$ consisting of $\emptyset$ and all unions of elements of this strong neighborhood system for X determines a Hausdorff topology for X (see [11] and [10]).

Lemma 9.1 ([11]) *Let (X, F, T) be a Menger space with T is continuous, X be endowed with the topology $\Im$ and $X \times X$ with the corresponding product topology. Then F is a uniformly continuous mapping from $X \times X$ into Δ^+.*

However, Mbarki et al. [1] showed that in b-Menger space (X, F, T, s), the probabilistic b-metric F is not continuous in general even though T is continuous.

Example 9.1 ([1]) Let $X = \mathbb{N} \cup \{\infty\},\ 0 < a \leq 1$. Define $F^a : X \times X \to \Delta^+$ as follow:

$$F^a_{bc}(t) = \begin{cases} H(t) & \text{if } b = c, \\ H(t-7) & \text{if } b \text{ and } c \text{ are odd and } b \neq c, \\ H(t - |\frac{a}{b} - \frac{a}{c}|) & \text{if } b \text{ and } c \text{ are even or } bc = \infty, \\ H(t-3) & \text{otherwise.} \end{cases}$$

It easy to show that $(X,\ F^a,\ T_M,\ 4)$ is a b-Menger space with T_M is continuous. In the sequel, we take $a = 1$. Consider the sequence $x_n = 2n, n \in \mathbb{N}$. Then $F_{2n\infty}(t) = H(t - \frac{1}{2n})$. Therefore $x_n \to \infty$, but $F_{2n1}(t) = H(t-3) \neq H(t-1) = F_{1\infty}(t)$. Hence F is not continuous at ∞.

In this work we give two notions of structures convex, first one is the probabilistic extension of the Takahashi convex structure defined in ordinary metric spaces, second one is the strong convex structure more general than the probabilistic Takahashi convex structure and we study a properties and relationship of the theory of betweenness in Menger spaces using those convex structures. We finish this work by showing the existence and uniqueness of a solution for a Volterra type integral equation.

9.2 Probabilistic Takahashi Convex Structure

In 1970, Takahashi [12] introduced the concept of convexity in a metric space. We give the probabilistic version of this convex structure.

Definition 9.4 Let $(X,\ F,\ T)$ be a Menger space, and let I be the closed unit interval $[0,\ 1]$. A probabilistic Takahashi convex structure (PTCS) on X is a function $W : X \times X \times I \to X$ which has the property that for every $x,\ y \in X$ and $\lambda \in I$ we have

$$F_{zW(x,\ y,\ \lambda)}(\lambda s + (1-\lambda)t) \geq T(F_{zx}(s),\ F_{zy}(t))$$

for all $z \in X$, and $s,\ t > 0$. If $(X,\ F,\ T)$ is equipped with PTCS, we call X a convex Menger space.

In the sequel of this section we suppose that $ImF \subset \mathcal{D}^+$.

Definition 9.5 Let W be a PTCS on a Menger space $(X,\ F,\ T)$. We say that W is a strict PTCS if it has a property that whenever $w \in X$ and there is $(x,\ y,\ \lambda) \in X \times X \times (0,\ 1)$ for which

$$F_{zw}(2t) \geq T(F_{zx}(\frac{t}{\lambda}),\ F_{zy}(\frac{t}{1-\lambda}))\ \ for\ \ every\ \ z \in X$$

then $w = W(x, y, \lambda)$.

Proposition 9.1 *Let W be a strict PTCS on the Menger space (X, F, T). Then for all $x, y \in X$ and $\lambda \in I$ we have*

$$W(x, y, \lambda) = W(y, x, 1-\lambda).$$

Proof The equality is true for $\lambda = 0$ and $\lambda = 1$. Let $\lambda \in (0, 1)$, we have

$$\begin{aligned} F_{zW(y,\, x,\, 1-\lambda)}(2t) &\geq T(F_{zy}(\frac{t}{1-\lambda}),\ F_{zx}(\frac{t}{\lambda})) \\ &= T(F_{zx}(\frac{t}{\lambda}),\ F_{zy}(\frac{t}{1-\lambda})), \end{aligned}$$

for all $z \in X$.

By strictness we get

$$W(x, y, \lambda) = W(y, x, 1-\lambda).$$

□

Theorem 9.1 *Let W be a strict PTCS on the Menger space (X, F, T_M) under the t-norm T_M. Then for every $x, y \in X$ and $\alpha, \beta \in [0, \frac{1}{2})$, we have*

$$W(W(x, y, \alpha), y, \beta) = W(x, y, 2\alpha\beta).$$

Proof Let $x, y \in X$. The assertion is true for $\alpha = 0$ or $\beta = 0$. Let $\alpha, \beta \in (0, \frac{1}{2})$. For all $z \in X$ we have

$$\begin{aligned} F_{zW(W(x,\, y,\, \alpha),\, y,\, \beta)}(2t) &\geq \min\left(F_{zW(x,\, y,\, \alpha)}\left(\frac{t}{\beta}\right),\ F_{zy}\left(\frac{t}{1-\beta}\right)\right) \\ &\geq \min\left(\min(F_{zx}\left(\frac{t}{2\beta\alpha}\right),\ F_{zy}\left(\frac{t}{2\beta(1-\alpha)}\right),\right. \\ &\quad \left. F_{zy}\left(\frac{t}{1-\beta}\right)\right) \\ &= \min\left(F_{zx}\left(\frac{t}{2\beta\alpha}\right),\ \min\left(F_{zy}\left(\frac{t}{2\beta(1-\alpha)}\right),\right.\right. \\ &\quad \left.\left. F_{zy}\left(\frac{t}{1-\beta}\right)\right)\right) \\ &\geq \min\left(F_{zx}\left(\frac{t}{2\beta\alpha}\right),\ F_{zy}\left(\frac{t}{1-2\beta\alpha}\right)\right). \end{aligned}$$

Because of $\frac{1}{2\beta(1-\alpha)} \geq \frac{1}{1-2\beta\alpha}$ and $\frac{1}{1-\beta} \geq \frac{1}{1-2\beta\alpha}$, it holds

$$W(W(x,\ y,\ \alpha),\ y,\ \beta) = W(x,\ y,\ 2\alpha\beta).$$

□

Definition 9.6 A convex Menger space $(X,\ F,\ T)$ with a probabilistic Takahashi convex structure W will be called strictly convex if, for arbitraries $x,\ y \in X$ and $\lambda \in (0,\ 1)$ the element $W(x,\ y,\ \lambda)$ is the unique element which satisfies

$$F_{xy}(\frac{t}{\lambda}) = F_{W(x,\ y,\ \lambda)y}(t), \quad F_{xy}(\frac{t}{1-\lambda}) = F_{W(x,\ y,\ \lambda)x}(t),$$

for all $t > 0$.

Theorem 9.2 *Let W be a strict PTCS on a strictly convex Menger space $(X,\ F,\ T)$. Then for every $x,\ y \in X$ with $x \neq y$ the mapping $\lambda \mapsto W(x,\ y,\ \lambda)$ is an injective from* $[0,\ \frac{1}{2})$ *into X.*

Proof Let $\alpha,\ \beta \in [0,\ \frac{1}{2})$ such that $\alpha \neq \beta$ and assume, without loss of generality, that $\alpha < \beta$. Let $x,\ y \in X$ such that $x \neq y$. We have

$$\begin{aligned} F_{W(x,\ y,\ \alpha)W(x,\ y,\ \beta)}(t) &= F_{W((W(x,\ y,\ \beta),\ y,\ \frac{\alpha}{2\beta})W(x,\ y,\ \beta)}(t) \\ &= F_{W(x,\ y,\ \beta)y}(\frac{t}{1-\frac{\alpha}{2\beta}}) \\ &= F_{xy}(\frac{t}{\beta(1-\frac{\alpha}{2\beta})}) \end{aligned}$$

for all $t > 0$.

Since $x \neq y$, then $F_{xy} \neq H$, hence there exists $t_0 > 0$ such that $F_{xy}(t_0) < 1$, for $\frac{t}{\beta(1-\frac{\alpha}{2\beta})} < t_0$ we have $F_{W(x,\ y,\ \alpha)W(x,\ y,\ \beta)}(t) < 1$. So $F_{W(x,\ y,\ \alpha)W(x,\ y,\ \beta)} \neq H$ and therefore $W(x,\ y,\ \alpha) \neq W(x,\ y\ \beta)$. □

Theorem 9.3 *Let W be a strict PTCS on a compact Menger space $(X,\ F,\ T)$. Then for each $\lambda \in (0, 1)$, $W_\lambda : (x, y) \mapsto W_\lambda(x, y) = W(x,\ y,\ \lambda)$ is continuous as a mapping from $X \times X$ into X.*

Proof Given $\lambda \in (0,\ 1)$ and let $\{(x_n,\ y_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X$ which converges to $(x,\ y)$ and let w be a cluster point of the sequence $\{W(x_n,\ y_n,\ \lambda)\}_{n=1}^{\infty}$. By using Theorems 2.2-2.5 of [7], select a subsequence $\{W(x_{n_k},\ y_{n_k},\ \lambda)\}_{n=1}^{\infty}$ which converges to w. Then for any $z \in X$, we have

$$F_{zW(x_{n_k},\ y_{n_k},\ \lambda)}(2t) \geq T(F_{zx_{n_k}}(\frac{t}{\lambda}),\ F_{zy_{n_k}}(\frac{t}{1-\lambda}))\ \ for\ \ k = 1,\ 2,\ \ldots.$$

Strictness and using the fact that the set of the points of discontinuity of F is countable now guarantees that

$$w = W(x,\ y,\ \lambda).$$

It follows that $W(x,\ y,\ \lambda)$ is the only cluster point of the sequence

$\{W(x_n,\ y_n,\ \lambda)\}_{n=1}^{\infty}$. Therefore, in view of Theorem 2.4 of [7], $\{W(x_n,\ y_n,\ \lambda)\}_{n=1}^{\infty}$ must converges to $W(x,\ y,\ \lambda)$ which complete the proof. □

9.3 Probabilistic Strong Convex Structure

In this section, we give a relationship between $W(W(x,\ y,\ s_1),\ z,\ s_2)$ and $W(W(x,\ z,\ t_1), y\ ,\ t_2)$ for $s_1,\ s_2,\ t_1,\ t_2 \in [0, 1]$.

Definition 9.7 Let $(X,\ F,\ T)$ be a Menger space, and let $P = \{(\alpha,\ \beta,\ \gamma) \in I \times I \times I : \alpha + \beta + \gamma = 1\}$. A probabilistic strong convex structure (PSCS) on X is a continuous function $K : X \times X \times X \times P \to X$ with the property that for each $(x,\ y,\ z; (\alpha,\ \beta,\ \gamma)) \in X \times X \times X \times P$, $K(x,\ y,\ z,\ (\alpha,\ \beta,\ \gamma))$ is the unique point of X which satisfies

$$F_{wK(x,\ y,\ z,\ (\alpha,\ \beta,\ \gamma))}(\alpha s + \beta t + \gamma r) \geq T(T(F_{wx}(s),\ F_{wy}(t)),\ F_{wz}(r)) \tag{9.1}$$

for every $w \in X$ and for all $s,\ t,\ r > 0$.

Remark 9.1 The uniqueness assumption in last definition guarantees that if p is a permutation of $\{1, 2, 3\}$, then, for $(x_1,\ x_2,\ x_3,\ (\alpha_1,\ \alpha_2,\ \alpha_3)) \in X \times X \times X \times P$, we have

$$K(x_1,\ x_2,\ x_3,\ (\alpha_1,\ \alpha_2,\ \alpha_3)) = K(x_{p(1)},\ x_{p(2)},\ x_{p(3)},\ (\alpha_{p(1)},\ \alpha_{p(2)},\ \alpha_{p(3)})).$$

Proposition 9.2 *Let $(X,\ F,\ T)$ be a strong convex Menger space with $ImF \subset \mathcal{D}^+$ and K its PSCS. Define $W_K : X \times X \times I \to X$ by $W_K(x,\ y,\ \lambda) = K(x,\ y,\ x,\ (\lambda,\ 1-\lambda, 0))$. Then W_K is a PTCS on X.*

Proof Let $s,\ t > 0$, $x,\ y \in X$ and $\lambda \in I$. We have

$$\begin{aligned} F_{wW_K(x,\ y,\ \lambda)}(\lambda s + (1-\lambda)t) &= F_{wK(x,\ y,\ x,\ (\lambda,\ 1-\lambda;0))}(\lambda s + (1-\lambda)t) \\ &\geq T(T(F_{wx}(s),\ F_{wy}(t)),\ F_{wz}(r)), \end{aligned}$$

for all $w \in X$ and $r > 0$.

Setting $r \to \infty$ we get

$$F_{wW_K(x,\ y,\ \lambda)}(\lambda s + (1-\lambda)t) \geq T(F_{wx}(s),\ F_{wy}(t)),$$

for all $w \in X$.

Then W_K is a PTCS on X. □

Theorem 9.4 *For any three points $x,\ y,\ z$ in a strong convex Menger space $(X,\ F,\ T)$, if $\beta \leq \frac{1}{2}$ and $\alpha \in I$. Then,*

$$W(W(x,\ y,\ \alpha),\ z,\ \beta) = K(x,\ y,\ z,\ (\frac{4\beta\alpha}{3},\ \frac{4\beta(1-\alpha)}{3},\ \frac{3-4\beta}{3})).$$

Proof Let $\beta \leq \frac{1}{2}$, $\alpha \in I$ and $x,\ y,\ z \in X$. For all $w \in X$ we have

$$\begin{aligned} F_{wW(W(x,\ y,\ \alpha),\ z,\ \beta)}(3t) &\geq T(F_{wW(x,\ y,\ \alpha)}(\frac{3t}{2\beta}),\ F_{wz}(\frac{3t}{2(1-\beta)})) \\ &\geq T(T(F_{wx}(\frac{3t}{4\beta\alpha}),\ F_{wy}(\frac{3t}{4\beta(1-\alpha)})),\ F_{wz}(\frac{3t}{2(1-\beta)})) \\ &\geq T(T(F_{wx}(\frac{3t}{4\beta\alpha}),\ F_{wy}(\frac{3t}{4\beta(1-\alpha)})),\ F_{wz}(\frac{3t}{3-4\beta})), \end{aligned}$$

because $\frac{3}{2(1-\beta)} \geq \frac{3}{3-4\beta}$.

Since $\frac{4\beta\alpha}{3} + \frac{4\beta(1-\alpha)}{3} + \frac{3-4\beta}{3} = 1$, then by uniqueness

$$W(W(x,\ y,\ \alpha),\ z,\ \beta) = K(x,\ y,\ z,\ (\frac{4\beta\alpha}{3},\ \frac{4\beta(1-\alpha)}{3},\ \frac{3-4\beta}{3})).$$

□

Corollary 9.1 *For any three points $x, y,\ z$ in a strong convex Menger space $(X,\ F,\ T)$, if $\frac{1}{4} \leq \beta \leq \frac{1}{2}$ and $\alpha \leq \frac{4\beta-1}{4\beta}$, then*

$$W(W(x,\ y,\ \alpha),\ z,\ \beta) = W(W(x,\ z,\ \beta\alpha[\frac{3-4\beta(1-\alpha)}{4}]^{-1}), y,\ \frac{3-4\beta(1-\alpha)}{4}).$$

Proof The conditions $\frac{1}{4} \leq \beta \leq \frac{1}{2}$ and $\alpha \leq \frac{4\beta-1}{4\beta}$ imply that $\frac{3-4\beta(1-\alpha)}{4} \leq \frac{1}{2}$ and $\gamma = \beta\alpha[\frac{3-4\beta(1-\alpha)}{4}]^{-1} \in (0,\ 1)$. We apply the Theorem 9.4 we get

$$W(W(x,\ z,\ \gamma),\ y,\ \frac{3-4\beta(1-\alpha)}{4}) = K(x,\ z,\ y,\ (\frac{4\beta\alpha}{3},\ \frac{3-4\beta}{3},\ \frac{4\beta(1-\alpha)}{3})),$$

and by permutation we obtain

$$W(W(x,\ y,\ \alpha),\ z,\ \beta) = W(W(x,\ z,\ \beta\alpha[\frac{3-4\beta(1-\alpha)}{4}]^{-1}),\ y,\ \frac{3-4\beta(1-\alpha)}{4}).$$

□

9.4 Application to An Integral Equation

As an application of the Theorem 4.1 in [1], we will consider the following Volterra type integral equation:

$$x(t) = g(t) + \int_0^t \Omega(t,\ \alpha,\ x(\alpha))d\alpha, \tag{9.2}$$

for all $t \in [0,\ k]$, where $k > 0$.

Theorem 9.5 *Let $\Omega \in C([0,\ k] \times [0,\ k] \times \mathbb{R},\ \mathbb{R})$ be an operator satisfying the following conditions:*

1. $\|\Omega\|_\infty = sup_{t,\ \alpha\in[0,\ k],\ x\in C([0,\ k],\ \mathbb{R})}|\Omega(t,\ \alpha,\ x(\alpha))| < \infty$.
2. *There exists $L > 0$ such that for all $t,\ \alpha \in [0,\ k]$ and $x,\ y \in C([0,\ k], \mathbb{R})$ we obtain*

$$|\Omega(t,\ \alpha,\ fx(\alpha)) - \Omega(t,\ \alpha,\ fy(\alpha))| \leq \frac{L}{\sqrt{2}}|x(\alpha) - y(\alpha)|,$$

where $f : C([0,\ k],\ \mathbb{R}) \to C([0,\ k],\ \mathbb{R})$ is defined by

$$fx(t) = g(t) + \int_0^t \Omega(t, \alpha,\ fx(\alpha))d\alpha, \qquad g \in C([0,\ k],\ \mathbb{R}).$$

Then the Volterra type integral equation (9.2) has a unique solution $x^* \in C([0,\ k],\ \mathbb{R})$.

Proof We define the mapping $F : C([0,\ k],\ \mathbb{R}) \times C([0,\ k],\ \mathbb{R}) \to \mathcal{D}^+$ by

$$F_{xy}(t) = H(t - \max_{t\in[0,\ k]}(|x(t) - y(t)|^2 e^{-2Lt})), \quad t > 0,\ x,\ y \in C([0,\ k],\ \mathbb{R}).$$

From Lemma 3.1 of [1], $(C([0,\ k],\ \mathbb{R}),\ F,\ T_M,\ 2)$ is a complete b-Menger space with coefficient $s = 2$. Therefore, for all $x,\ y \in C([0,\ k],\ \mathbb{R})$, we get

$$\begin{aligned}
&F_{fxfy}(r)\\
&= \mathcal{H}(r - \max_{t\in[0,\ k]}(|fx(t) - fy(t)|^2 e^{-2Lt}))\\
&= \mathcal{H}(r - \max_{t\in[0,\ k]}(|\int_0^t (\Omega(t,\ \alpha,\ fx(\alpha)) - \Omega(t,\ \alpha,\ fx(\alpha)))d\alpha|^2 e^{-2Lt}))\\
&= \mathcal{H}(r - \max_{t\in[0,\ k]}(|\int_0^t (\Omega(t,\ \alpha,\ fx(\alpha)) - \Omega(t,\ \alpha,\ fx(\alpha)))e^{-L\alpha}e^{L(\alpha-t)}d\alpha|^2))\\
&\geq \mathcal{H}(r - \frac{L^2}{2}\max_{t\in[0,\ k]}(|x(t) - y(t)|^2 e^{-2Lt})\max_{t\in[0,\ k]}(\int_0^t e^{L(\alpha-t)}d\alpha)^2)
\end{aligned}$$

$$= \mathcal{H}(r - \frac{1}{2}(1 - e^{-Lk})^2 \max_{t\in[0,\ k]}(|x(t) - y(t)|^2 e^{-2Lt}))$$

$$= \mathcal{H}(r - \frac{c}{2} \max_{t\in[0,k]}(|x(t) - y(t)|^2 e^{-2Lt}))$$

$$= \mathcal{H}(\frac{2r}{c} - \max_{t\in[0,k]}(|x(t) - y(t)|^2 e^{-2Lt}))$$

$$= F_{xy}(\frac{2r}{c}),$$

where $c = (1 - e^{-Lk})^2$.

Therefore, in view of Theorem 4.1 in [1] with $\varphi(r) = cr$, $c \in [0,\ 1]$, we deduce that the operator f has a unique fixed point $x^* \in C([0,\ k],\ \mathbb{R})$, which is the unique solution of the integral equation (9.2). □

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