Chapter 7
The Completely Discretized Problem of the Dual Mixed Formulation for the Heat Diffusion Equation in a Polygonal Domain by the Crank-Nicolson Scheme in Time

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Abstract The purpose of this paper is to prove *a priori* error estimates for the completely discretized problem of the dual mixed formulation for the heat diffusion equation in a polygonal domain. We complete the discretization of the problem (Farhloul et al., Functional Analysis and Evolution Equations. The Günter Lumer Volume, p. 240, Birkhäuser, Basel, 2007) in time by using the Crank-Nicolson scheme and we show the existence, the stability and a priori error estimates for the solution of the completely discretized problem.

Keywords Dual mixed finite element method · Heat diffusion equation · Singularities · Grids refinements · A priori error estimates

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7.1 Introduction

The aim of this paper is to study the completely discretized problem of the dual mixed formulation for the heat evolution equation in a polygonal domain Ω of $\mathbb{R}^2$. Let us note that we have dealt in [2] with the mixed dual semi-discretized formulation in space. In this work, we will complete the discretization of the problem in time by using the Crank-Nicolson scheme. For this completely

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discretized problem, we are going to demonstrate the existence and the uniqueness of the solution. Then, we will show the stability of this scheme. Finally, we will prove, under some conditions of meshing refinement in the reentrant corner of the polygonal domain, a priori error estimates of order 1 in space, and 2 in time for the solution of the completely discretized problem when using the Crank-Nicolson implicit scheme.

7.2 The Model Problem

Let Ω be a bounded polygonal domain of $\mathbb{R}^2$. In particular the boundary of $\Omega : \partial\Omega \equiv \Gamma = \cup_{j=1}^{N} \overline{\Gamma}_j$ for some $N \in \mathbb{N}$, where Γ_j is an open segment of a straight line of the plane $\mathbb{R}^2$, $\forall\ j = 1, 2, \ldots, N$. As it is well known, the geometric singularities of the domain (the angles) induce in general singularities on the solution of the Cauchy problem with Dirichlet boundary condition for the heat diffusion equation (see for example the books of P. Grisvard [3, 4]). As shown in [4] and [3], we may suppose without harming to the generality that Ω has only one nonconvex angle, in other words one reentrant corner, and that its vertex is located at the origin. In the following, we denote by ω ($\omega > \pi$) the measure of that angle. For a fixed $T > 0$, let us set $Q := \Omega\times]0, T[$ and let us denote by $\Sigma := \Gamma\times]0, T[$ the lateral boundary of the cylinder Q. We introduce the following weighted Sobolev space (see [3], definition 8.4.1.1 and lemma 8.4.1.2 p. 388):

$$H^{2,\alpha}(\Omega) = \{v \in H^1(\Omega);\ r^\alpha D^\beta v \in L^2(\Omega),\ \forall \beta \in \mathbb{N}^2 : |\beta| = 2\},$$

which is a Hilbert space for the norm

$$\|v\|_{2,\alpha,\Omega} = (\|v\|_{1,\Omega}^2 + |v|_{2,\alpha,\Omega}^2)^{1/2},$$

where the semi-norm $|.|_{2,\alpha,\Omega}$ is defined by

$$|v|_{2,\alpha,\Omega} = \left(\sum_{|\beta|=2} \left\| r^\alpha D^\beta v \right\|_{0,\Omega}^2 \right)^{1/2},$$

r denotes the distance to the origin of $\mathbb{R}^2$. Let us recall that:

$$\overset{\circ}{H}{}^1(\Omega) = \{v \in H^1(\Omega);\ v_{/\partial\Omega} = 0\}.$$

Let us consider the Cauchy problem for the heat diffusion equation in Ω up to time T: given the right-hand side $f \in L^2\left(0, T; L^2(\Omega)\right)$ and the initial condition $g \in$

$\overset{\circ}{H}{}^1(\Omega)$, find $u \in H^1\left(0, T; L^2(\Omega)\right) \cap L^2(0, T; \overset{\circ}{H}{}^1(\Omega))$ weak solution of:

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) \text{ in } Q, \\ u(x,t) = 0 \text{ on } \Sigma, \\ u(x,0) = g(x)\ , \quad \text{for } x \in \Omega. \end{cases} \tag{7.1}$$

We have seen in Proposition 3 of [2], that Problem (7.1) admits a unique solution:

$$u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^{2,\alpha}(\Omega)) \cap L^2(0, T; \overset{\circ}{H}{}^1(\Omega)),$$

with $\alpha \in \left]1 - \frac{\pi}{\omega}, 1\right[$.

In the following, we will consider the additional unknown heat flux $\mathbf{p} := \nabla u$ (rigorously speaking, the heat flux density vector is the opposite [10]). We have proved in [2, Theorem 4] that the couple $(\mathbf{p}, u) \in L^2(0, T; H(\text{div}, \Omega)) \times H^1(0, T; L^2(\Omega))$ is the unique solution of the dual mixed formulation:

$$\begin{cases} \int_\Omega \mathbf{p}(t).\mathbf{q}\, dx + \int_\Omega u(t)\ \text{div}\, \mathbf{q}\, dx = 0, \quad \forall \mathbf{q} \in H(\text{div}, \Omega), \quad \text{for a.e. } t \in I, \\ \int_\Omega v\ \text{div}\, \mathbf{p}(t) dx = -\int_\Omega (f(t) - u_t(t))\ v\, dx, \quad \forall v \in L^2(\Omega), \quad \text{for a.e. } t \in I, \\ u(0) = g \in \overset{\circ}{H}{}^1(\Omega). \end{cases} \tag{7.2}$$

7.3 The Completely Discretized Problem

Let us consider a family of triangulations $(\mathcal{T}_h)_{h>0}$ on $\overline{\Omega}$. For a triangle K belonging to the triangulation $\mathcal{T}_h$, let us denote by h_K the diameter of K and by ρ_K the interior diameter of K, i.e. the diameter of the largest of disc included in K. As in Theorem 8.4.1.6 p. 392 of [3], we suppose that the family of triangulations $(\mathcal{T}_h)_{h>0}$ has the property that $\max\limits_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K}$ is bounded by a positive constant independent of the parameter h; in that case, one says usually that the family of triangulations is regular (see for example [1] (17.1) p. 131). In accordance with the tradition (see [1] remark 17.1 p. 131) the same letter h may have also another significance: it may denote instead: $h =: \max_{K \in \mathcal{T}_h} h_K$. The true significance of h is always clear from the context.

For the discretization in space, let us recall that we are using the approximant spaces:

$$X_h := \left\{ \mathbf{q}_h \in H(\text{div};\Omega); \forall K \in \mathcal{T}_h \;:\; \mathbf{q}_{h/K} \in RT_0(K) \right\},$$

$$M_h := \left\{ v_h \in L^2(\Omega); \forall K \in \mathcal{T}_h\,,\; v_{h/K} \in P_0\,(K) \right\},$$

where $RT_0(K) := P_0(K)^2 \oplus P_0(K)\binom{x_1}{x_2}$ denotes the real vector space of dimension three of the so called Raviart-Thomas vectorfields of degree 0 on the triangle K ($RT_0(K)$ is denoted $D_1(K)$ in [7, p. 550]), and $P_0\,(K)$ is the real vector space of dimension one of the constant functions on the triangle K. For the complete discretized problem, we use the time subdivision of the interval $[0, T]$ into N subintervals $[t_{n-1}, t_n]$ $(N \geq 2)$, such that: $0 = t_0 \leq \cdots \leq t_n \leq \cdots \leq t_N = T$. $\Delta t = t_n - t_{n-1}$ denotes the fixed time step. Let us denote by u_h^n the approximation of the temperature at time $t_n = n\Delta t$ in M_h. For the approximation of $\frac{\partial u}{\partial t}$ at the time t_n, we use the following formula:

$$\overline{\partial} u_h^n = \frac{(u_h^n - u_h^{n-1})}{\Delta t}.$$

7.4 The Crank-Nicolson Scheme

Before giving the complete discretized mixed formulation for the heat diffusion equation by the Crank-Nicolson scheme, we will first define new variables. We set,

$$t_{n-\frac{1}{2}} = \frac{t_n + t_{n-1}}{2},\;\; \mathbf{p}_h^{n-\frac{1}{2}} = \frac{\mathbf{p}_h^n + \mathbf{p}_h^{n-1}}{2},\; \text{and } u_h^{n-\frac{1}{2}} = \frac{u_h^n + u_h^{n-1}}{2}. \tag{7.3}$$

Let us consider the following problem:

$$\begin{cases} \int_\Omega \mathbf{p}_h^{n-\frac{1}{2}}.\mathbf{q}_h\, dx + \int_\Omega u_h^{n-\frac{1}{2}}\, \text{div}\,\mathbf{q}_h\, dx = 0, \quad \forall \mathbf{q}_h \in X_h,\; \forall n \geq 1, \\ \int_\Omega v_h\, \text{div}\,\mathbf{p}_h^{n-\frac{1}{2}} dx = -\int_\Omega (f(t_{n-\frac{1}{2}}) - \overline{\partial} u_h^n)\, v_h\, dx, \quad \forall v_h \in M_h,\; \forall n \geq 1, \\ u_h^0\,,\ \text{given.} \end{cases} \tag{7.4}$$

Let us note that in (7.4) appear $\mathbf{p}_h^{n-\frac{1}{2}}$, $u_h^{n-\frac{1}{2}}$, u_h^n and u_h^{n-1}. Thus, we can choose as unknowns $\mathbf{p}_h^{n-\frac{1}{2}}$ and u_h^n for $n \geq 1$. Another alternative is to consider that the unknowns are the $\mathbf{p}_h^n$ for $n \geq 0$ and u_h^n for $n \geq 1$. u_h^0 denotes the initial condition that is known and $\mathbf{p}_h^0$ is exceptionally defined by the equation:

$$\int_\Omega \mathbf{p}_h^0 . \mathbf{q}_h \, dx + \int_\Omega u_h^0 \, \operatorname{div} \mathbf{q}_h \, dx = 0, \quad \forall \mathbf{q}_h \in X_h. \tag{7.5}$$

This last choice of unknowns presents the following advantages:

- Traditional unknowns.
- Symmetry of "**p**" and "u" in the problem.

Proposition 7.1 *Problem* (7.4) *admits one and only one solution* $(\mathbf{p}_h^n, u_h^n)_{n \in IN}$.

Proof Let us start by demonstrating the unicity. Therefore, let us show that if $(\mathbf{p}_h^n, u_h^n) \in X_h \times M_h$ verifies :

$$\begin{cases} \int_\Omega \mathbf{p}_h^{n-\frac{1}{2}} . \mathbf{q}_h \, dx + \int_\Omega u_h^{n-\frac{1}{2}} \, \operatorname{div} \mathbf{q}_h \, dx = 0, \quad \forall \mathbf{q}_h \in X_h, \ \forall n \geq 1 \\ \int_\Omega v_h \, \operatorname{div} \mathbf{p}_h^{n-\frac{1}{2}} dx = \int_\Omega \overline{\partial} u_h^n \, v_h \, dx, \quad \forall v_h \in M_h, \ \forall n \geq 1 \text{ and } u_h^0 = 0, \end{cases} \tag{7.6}$$

then $(\mathbf{p}_h^n, u_h^n) = 0$. By taking $\mathbf{q}_h = \mathbf{p}_h^{n-\frac{1}{2}}$ in the first equation of (7.6) and $v_h = u_h^{n-\frac{1}{2}}$ in the second equation of (7.6), we obtain:

$$\int_\Omega \left| \mathbf{p}_h^{n-\frac{1}{2}} \right|^2 \, dx + \int_\Omega \overline{\partial} u_h^n \, u_h^{n-\frac{1}{2}} \, dx = 0. \tag{7.7}$$

On the other hand we have:

$$\overline{\partial} u_h^n \, u_h^{n-\frac{1}{2}} = \frac{1}{2\Delta t} \left(u_h^n - u_h^{n-1} \right) \left(u_h^n + u_h^{n-1} \right) = \frac{1}{2\Delta t} \left(u_h^n \right)^2, \tag{7.8}$$

on the condition of having already demonstrated that $u_h^{n-1} = 0$. So let us take $n = 1$ in (7.8) and since $u_h^0 = 0$ (initial condition), consequently $u_h^1 = 0$, and $\mathbf{p}_h^{\frac{1}{2}} = 0$ by Eq. (7.7) the second term in the left-hand side of Eq. (7.7) being also a priori non-negative due to [6].

According to (7.5), $u_h^0 = 0$ implies that $\mathbf{p}_h^0 = 0$. Knowing already that $\mathbf{p}_h^{\frac{1}{2}} = 0$, it follows that $\mathbf{p}_h^1 = 0$. Thus $\mathbf{p}_h^2 = 0$ and $u_h^2 = 0$ by (7.7) and (7.8) with $n = 2$ and so on for every $n \geq 1$.

For the existence, it is well known by the Riesz Representation Theorem applied to Eq. (7.5) that $\mathbf{p}_h^0$ exists (see Remark 1.5.2 [5, p. 31]). According to the system

(7.4) with $n = 1$, and with the aim of constructing $\mathbf{p}_h^1$ and u_h^1, u_h^0 and $\mathbf{p}_h^0$ being known, we have

$$\begin{cases} \int_\Omega \mathbf{p}_h^1.\mathbf{q}_h\, dx + \int_\Omega u_h^1 \operatorname{div} \mathbf{q}_h\, dx = \int_\Omega \mathbf{p}_h^0.\mathbf{q}_h\, dx + \int_\Omega u_h^0 \operatorname{div} \mathbf{q}_h\, dx, \forall \mathbf{q}_h \in X_h, \\ \int_\Omega v_h \operatorname{div} \mathbf{p}_h^1 dx - \frac{2}{\Delta t}\int_\Omega u_h^1\, v_h\, dx = -\int_\Omega v_h \operatorname{div} \mathbf{p}_h^0 dx - \frac{2}{\Delta t}\int_\Omega u_h^0\, v_h\, dx \\ -2\int_\Omega f\left(t_{1/2}\right)\, v_h\, dx,\ \forall v_h \in M_h. \end{cases} \tag{7.9}$$

Let Φ_h: $X_h \times M_h \longmapsto X'_h \times M'_h$, be the application defined by:

$$\left(\mathbf{p}_h^1, u_h^1\right) \longmapsto \left(\mathbf{q}_h \longmapsto \int_\Omega \mathbf{p}_h^1.\mathbf{q}_h\, dx + \int_\Omega u_h^1\ \operatorname{div} \mathbf{q}_h\, dx,\right.$$
$$\left. v_h \longmapsto \int_\Omega v_h\ \operatorname{div} \mathbf{p}_h^1 dx - \frac{2}{\Delta t}\int_\Omega u_h^1\, v_h\, dx\right).$$

Let us prove that it is an isomorphism. Since Φ_h is linear of $X_h \times M_h$ in its dual, and since the two spaces $X_h \times M_h$ and $X'_h \times M'_h$ have the same dimension, it suffices to show that Φ_h is injective to prouve its bijectivity. Let $\left(\mathbf{p}_h^1, u_h^1\right)$ such that:

$$\int_\Omega \mathbf{p}_h^1.\mathbf{q}_h\, dx + \int_\Omega u_h^1\ \operatorname{div} \mathbf{q}_h\, dx = 0,\ \forall \mathbf{q}_h \in X_h, \tag{7.10}$$

$$\int_\Omega v_h\ \operatorname{div} \mathbf{p}_h^1 dx - \frac{2}{\Delta t}\int_\Omega u_h^1\, v_h\, dx = 0,\ \forall v_h \in M_h. \tag{7.11}$$

In (7.10), let us take $\mathbf{q}_h = \mathbf{p}_h^1$ and $v_h = u_h^1$ in (7.11), it follows that :

$$\int_\Omega \left|\mathbf{p}_h^1\right|^2\ dx + \frac{2}{\Delta t}\int_\Omega \left|u_h^1\right|^2\ dx = 0. \tag{7.12}$$

We obtain $\mathbf{p}_h^1 = 0$ and $u_h^1 = 0$. It follows from that the injectivity and thus the bijectivity of Φ_h. So it suffices to apply Φ_h^{-1} to the couple of linear forms defined by the two members of the right hand side of the system of Eqs. (7.9). In a similar way, we construct $\left(\mathbf{p}_h^2, u_h^2\right)$ by considering the system (7.4), with $n = 2$ and so on. □

7.4.1 Stability

Now we will give the stability result of the Crank-Nicolson scheme.

Theorem 7.1 *Suppose* $\Delta t \leq \frac{1}{2}$. *There exists a constant* $c > 0$ *independent of* h *such that:*

$$\left\|u_h^N\right\|_{0,\Omega}^2 \lesssim \left\|u_h^0\right\|_{0,\Omega}^2 + \Delta t \left\|\mathbf{p}_h^0\right\|_{0,\Omega}^2 + \sum_{n=1}^{N} \Delta t \left\|f(t_{n-\frac{1}{2}})\right\|_{0,\Omega}^2, \tag{7.13}$$

and

$$\left\|\mathbf{p}_h^N\right\|_{0,\Omega} \leq \left\|\mathbf{p}_h^0\right\|_{0,\Omega} + \sqrt{\frac{T}{2}} \max_{n=1,\dots,N} \|f(t_n)\|_{0,\Omega}. \tag{7.14}$$

Proof For the proof of (7.13), we use in particular the discrete inequality of Gronwall [6, p. VI-9], with:

$$\begin{cases} \varphi_n = \|u_h^n\|_{0,\Omega}^2 + \Delta t \, \|\mathbf{p}_h^n\|_{0,\Omega}^2 \\ m_0 = 2 \\ m_1 = \cdots = m_{N-1} = 2\Delta t \\ C = 2\left\|u_h^0\right\|_{0,\Omega}^2 + \Delta t \left\|\mathbf{p}_h^0\right\|_{0,\Omega}^2 + 2\Delta t \sum_{n=1}^{N} \left\|f(t_{n-\frac{1}{2}})\right\|_{0,\Omega}^2 \end{cases}$$

For the complete demonstration of (7.13) see the proof of Theorem 1.5.15 [5, pp. 52–54]. For the proof of (7.14), see the proof of Proposition 1.5.16 [5, p. 55]. □

7.4.2 Error Estimates on the Temperature and on the Heat Flux Density Vector

In order to demonstrate the results related to the error estimate, proceeding similarly as in the semi-discrete case [2], we decompose the error: $u(t_n) - u_h^n$ into the sum of $u(t_n) - \tilde{u}_h(t_n)$ and $\tilde{u}_h(t_n) - u_h^n$, where $(\tilde{\mathbf{p}}_h(t_n), \tilde{u}_h(t_n)) \in X_h \times M_h$ is the solution of the "elliptic projection problem" at time t_n : find $(\tilde{\mathbf{p}}_h(t_n), \tilde{u}_h(t_n)) \in X_h \times M_h$ solution of,

$$\begin{cases} \int_\Omega \tilde{\mathbf{p}}_h(t_n).\mathbf{q}_h \, dx + \int_\Omega \tilde{u}_h(t_n) \ \operatorname{div} \mathbf{q}_h \, dx = 0, \quad \forall \mathbf{q}_h \in X_h, \\ \int_\Omega v_h \ \operatorname{div} \ \tilde{\mathbf{p}}_h(t_n) dx = -\int_\Omega (f(t_n) - u_t(t_n)) \, v_h \, dx, \quad \forall v_h \in M_h. \end{cases} \tag{7.15}$$

The elliptic problem (7.15) being true for n and $n-1$, making the sum, we obtain:

$$\begin{cases} \int_\Omega \frac{\tilde{\mathbf{p}}_h(t_n)+\tilde{\mathbf{p}}_h(t_{n-1})}{2}.\mathbf{q}_h \, dx + \int_\Omega \frac{\tilde{u}_h(t_n)+\tilde{u}_h(t_{n-1})}{2} \operatorname{div} \mathbf{q}_h \, dx = 0, \quad \forall \mathbf{q}_h \in X_h, \\ \int_\Omega v_h \operatorname{div} \frac{\tilde{\mathbf{p}}_h(t_n)+\tilde{\mathbf{p}}_h(t_{n-1})}{2} dx = -\int_\Omega (\frac{f(t_n)+f(t_{n-1})}{2} - \frac{u_t(t_n)+u_t(t_{n-1})}{2}) \, v_h \, dx, \forall v_h \in M_h. \end{cases}$$

We may rewrite the Crank-Nicolson scheme, with u_h^0 , given by:

$$\begin{cases} \int_\Omega \mathbf{p}_h^{n-\frac{1}{2}}.\mathbf{q}_h \, dx + \int_\Omega u_h^{n-\frac{1}{2}} \ \operatorname{div} \mathbf{q}_h \, dx = 0, \quad \forall \mathbf{q}_h \in X_h, \forall n \geq 1 \\ \int_\Omega v_h \ \operatorname{div} \mathbf{p}_h^{n-\frac{1}{2}} dx = - \int_\Omega (f(t_{n-\frac{1}{2}}) - \overline{\partial} u_h^n) \, v_h \, dx, \quad \forall v_h \in M_h, \ \forall n \geq 1 \end{cases}$$

Let us set $\theta_h^n := u_h^n - \tilde{u}_h(t_n)$ and $\boldsymbol{\varepsilon}_h^n := \mathbf{p}_h^n - \tilde{\mathbf{p}}_h(t_n)$. Then, by subtraction we obtain the following system:

$$\begin{cases} \int_\Omega \frac{\boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}}{2}.\mathbf{q}_h \, dx + \int_\Omega \frac{\theta_h^n + \theta_h^{n-1}}{2} \ \operatorname{div} \mathbf{q}_h \, dx = 0, \quad \forall \mathbf{q}_h \in X_h, \\ \int_\Omega v_h \ \operatorname{div} \frac{\boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}}{2} \, dx \\ \quad = - \int_\Omega (f(t_{n-\frac{1}{2}}) - \frac{f(t_n) + f(t_{n-1})}{2} - (\overline{\partial} u_h^n - \frac{u_t(t_n) + u_t(t_{n-1})}{2})) \, v_h dx. \ \forall v_h \in M_h, \end{cases} \tag{7.16}$$

Proposition 7.2 *Assume* $f, \frac{df}{dt} \in H^1(0, T; L^2(\Omega))$ *and* $\Delta g + f(0) \in \overset{\circ}{H}{}^1(\Omega)$ *as well as* $\Delta \left(\Delta g + f(0) \right) + \frac{df}{dt}(0) \in \overset{\circ}{H}{}^1(\Omega)$ *. Then*

$$u_{ttt} \in L^2(0, T; L^2(\Omega)).$$

Proof Let $w \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ be the solution of the heat diffusion equation

$$\begin{cases} \frac{dw}{dt}(t) = \Delta w(t) + \frac{d^2 f}{dt^2}(t), \ \forall' t \in [0, T] \\ w(0) = \Delta \left(\Delta g + f(0) \right) + \frac{df}{dt}(0). \end{cases}$$

Set $v(t) = \int_0^t w(s) \, ds + \Delta g + f(0)$, $\frac{dv}{dt}(t) = w(t)$ and $v(0) = \Delta g + f(0) \in \overset{\circ}{H}{}^1(\Omega)$.

By integrating the equation $\frac{dw}{dt}(s) = \Delta w(s) + \frac{d^2 f}{dt^2}(s)$, $\forall' s \in [0, T]$ from 0 to t, we obtain:

$$w(t) - w(0) = \Delta \left(v(t) - \Delta g - f(0) \right) + \frac{df}{dt}(t) - \frac{df}{dt}(0)$$

$$i.e. \ \frac{dv}{dt}(t) - \Delta \left(\Delta g + f(0) \right) - \frac{df}{dt}(0) = \Delta v(t) - \Delta \left(\Delta g + f(0) \right) + \frac{df}{dt}(t) - \frac{df}{dt}(0).$$

Then, by simplifying the 2 members we obtain that v is the solution of the following Cauchy problem:

$$\begin{cases} \frac{dv}{dt}(t) = \Delta v(t) + \frac{df}{dt}(t) \\ v(0) = \Delta g + f(0) \in \overset{\circ}{H}{}^1(\Omega). \end{cases}$$

According to the proof of Proposition 8 in [2], we have $v = \frac{du}{dt}$. Thus $\frac{d^2u}{dt^2} = \frac{dv}{dt} = w \in H^1(0, T; L^2(\Omega))$. And consequently $u_{ttt} \in L^2(0, T; L^2(\Omega))$. □

Theorem 7.2 *Under the hypotheses of Proposition 7.2, there exists a constant* $c > 0$ *independent of h and of k such that:*

$$\|u_h^n - \tilde{u}_h(t_n)\|_{0,\Omega} \leq ch\left(\|u_0\|_{H^{2,\alpha}(\Omega)} + \int_0^{t_n} \|u_t(s)\|_{H^{2,\alpha}(\Omega)}\, ds\right) + 2\,\Delta t^2 \times \left(\int_0^{t_n} \|u_{ttt}(s)\|_{0,\Omega}\, ds + \int_0^{t_n} \|f_{tt}(s)\|_{0,\Omega}\, ds\right). \tag{7.17}$$

Proof The first step of this demonstration is to bound $\|\theta_h^n\|$ in terms of $\left\|\theta_h^{n-1}\right\|$ and $\|\omega^n\|$. For this, let's take $v_h = \theta_h^n + \theta_h^{n-1}$ in the second equation of the system (7.16) and $\mathbf{q}_h = \boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}$ in the first. We have:

$$\int_\Omega \frac{\left|\boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}\right|^2}{2}\, dx = \int_\Omega \left(f(t_{n-\frac{1}{2}}) - \frac{f(t_n) + f(t_{n-1})}{2} - \left(\overline{\partial} u_h^n - \frac{u_t(t_n) + u_t(t_{n-1})}{2}\right)\right) \times \left(\theta_h^n + \theta_h^{n-1}\right)\, dx. \tag{7.18}$$

And as we have :

$$\begin{aligned} \overline{\partial} u_h^n - \frac{u_t(t_n) + u_t(t_{n-1})}{2} &= \overline{\partial}\theta_h^n + \tilde{u}(t_n) - \tilde{u}(t_{n-1}) - \frac{u_t(t_n) + u_t(t_{n-1})}{2} \\ &= \overline{\partial}\theta_h^n + (R_h - I)\,\overline{\partial} u(t_n) + \overline{\partial} u(t_n) \\ &\quad - \frac{u_t(t_n) + u_t(t_{n-1})}{2}, \end{aligned}$$

where, with analogy to the book of Vidar Thomée [8], "$R_h()$" denotes here the component in M_h of the couple of $X_h \times M_h$ "elliptic projection of" (cf definition 6 of [2]). Thus,

$$\begin{aligned}
&\overline{\partial} u_h^n - \frac{u_t(t_n) + u_t(t_{n-1})}{2} + \frac{f(t_n) + f(t_{n-1})}{2} - f(t_{n-\frac{1}{2}}) \\
&= \overline{\partial}\theta_h^n + (R_h - I)\, \overline{\partial} u(t_n) + \left(\overline{\partial} u(t_n) - u_t(t_{n-\frac{1}{2}})\right) + u_t(t_{n-\frac{1}{2}}) \\
&\quad - \frac{u_t(t_n) + u_t(t_{n-1})}{2} + \frac{f(t_n) + f(t_{n-1})}{2} - f(t_{n-\frac{1}{2}}) \\
&= \overline{\partial}\theta_h^n + (R_h - I)\, \overline{\partial} u(t_n) + \left(\overline{\partial} u(t_n) - u_t(t_{n-\frac{1}{2}})\right) \\
&\quad + \Delta \left[u(t_{n-\frac{1}{2}}) - \frac{1}{2}\left(u(t_n) + u(t_{n-1})\right)\right],
\end{aligned} \tag{7.19}$$

because $\Delta u(t) + f(t) = u_t(t)$, for every $t > 0$.

Let us set $\widetilde{\omega}^n := \widetilde{\omega}_1^n + \widetilde{\omega}_2^n + \widetilde{\omega}_3^n$, with:

$$\begin{aligned}
\widetilde{\omega}_1^n &:= (R_h - I)\, \overline{\partial} u(t_n), \\
\widetilde{\omega}_2^n &:= \left(\overline{\partial} u(t_n) - u_t(t_{n-\frac{1}{2}})\right), \\
\widetilde{\omega}_3^n &:= \Delta \left[u(t_{n-\frac{1}{2}}) - \frac{1}{2}\left(u(t_n) + u(t_{n-1})\right)\right].
\end{aligned}$$

Then, it follows from (7.19) and (7.18), that,

$$\int_\Omega \frac{\left|\boldsymbol{\varepsilon}_h^{\,n} + \boldsymbol{\varepsilon}_h^{\,n-1}\right|^2}{2}\, dx = -\int_\Omega (\overline{\partial}\theta_h^n + \widetilde{\omega}^n)\, \left(\theta_h^n + \theta_h^{n-1}\right)\, dx. \tag{7.20}$$

Thus, we obtain:

$$\int_\Omega \overline{\partial}\theta_h^n \left(\theta_h^n + \theta_h^{n-1}\right)\, dx \le -\frac{1}{2}\left\|\boldsymbol{\varepsilon}_h^{\,n} + \boldsymbol{\varepsilon}_h^{\,n-1}\right\|^2 + \left\|\widetilde{\omega}^n\right\| \left\|\theta_h^n + \theta_h^{n-1}\right\|.$$

As

$$\int_\Omega \overline{\partial}\theta_h^n \left(\theta_h^n + \theta_h^{n-1}\right)\, dx = \frac{\left\|\theta_h^n\right\|^2 - \left\|\theta_h^{n-1}\right\|^2}{\Delta t}, \tag{7.21}$$

we deduce that :

$$\|\theta_h^n\|_{0,\Omega}^2 - \left\|\theta_h^{n-1}\right\|_{0,\Omega}^2 \leq \Delta t \left(-\frac{1}{2}\left\|\boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}\right\|_{0,\Omega}^2 + \|\widetilde{\omega}^n\|_{0,\Omega}\left\|\theta_h^n + \theta_h^{n-1}\right\|_{0,\Omega}\right). \tag{7.22}$$

And so, a fortiori we get

$$\|\theta_h^n\|^2 - \left\|\theta_h^{n-1}\right\|^2 \leq \Delta t \ \|\widetilde{\omega}^n\|_{0,\Omega}\left(\|\theta_h^n\|_{0,\Omega} + \left\|\theta_h^{n-1}\right\|_{0,\Omega}\right). \tag{7.23}$$

Thus,

$$\|\theta_h^n\|_{0,\Omega} \leq \left\|\theta_h^{n-1}\right\|_{0,\Omega} + \Delta t \ \|\widetilde{\omega}^n\|_{0,\Omega}, \tag{7.24}$$

so that it suffices to bound $\widetilde{\omega}^n$. Let us start with $\widetilde{\omega}_2^n$. By definition we have

$$\begin{aligned}
\Delta t \ \|\widetilde{\omega}_2^n\|_{0,\Omega} &= \Delta t \ \left\|\overline{\partial} u(t_n) - u_t(t_{n-\frac{1}{2}})\right\|_{0,\Omega} \\
&= \Delta t \ \left\|\frac{u(t_n) - u(t_{n-1})}{\Delta t} - u_t(t_{n-\frac{1}{2}})\right\|_{0,\Omega} \\
&= \left\|u(t_n) - u(t_{n-1}) - \Delta t \ u_t(t_{n-\frac{1}{2}})\right\|_{0,\Omega}.
\end{aligned}$$

Using Taylor's formula, we get

$$u(t_n) = u(t_{n-\frac{1}{2}}) + \frac{\Delta t}{2} \ u_t(t_{n-\frac{1}{2}}) + \frac{\Delta t^2}{8} \ u_{tt}(t_{n-\frac{1}{2}}) + \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - s)^2 u_{ttt}(s)\, ds.$$

Hence, at the time t_{n-1}, we have

$$u(t_{n-1}) = u(t_{n-\frac{1}{2}}) - \frac{\Delta t}{2} \ u_t(t_{n-\frac{1}{2}}) + \frac{\Delta t^2}{8} \ u_{tt}(t_{n-\frac{1}{2}}) + \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_{n-1}} (t_{n-1} - s)^2 u_{ttt}(s)\, ds. \tag{7.25}$$

Let us consider the difference of these two above equalities, we obtain

$$\begin{aligned}
&u(t_n) - u(t_{n-1}) - \Delta t \ u_t(t_{n-\frac{1}{2}}) \\
&= \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - s)^2 u_{ttt}(s)\, ds - \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_{n-1}} (t_{n-1} - s)^2 u_{ttt}(s)\, ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left\| u(t_n) - u(t_{n-1}) - \Delta t \ u_t(t_{n-\frac{1}{2}}) \right\|_{0,\Omega} \\
&\leq \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - s)^2 \left\| u_{ttt}(s) \right\|_{0,\Omega} ds + \frac{1}{2}\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (t_{n-1} - s)^2 \left\| u_{ttt}(s) \right\|_{0,\Omega} ds \\
&\leq \frac{\Delta t^2}{8}\int_{t_{n-\frac{1}{2}}}^{t_n} \left\| u_{ttt}(s) \right\|_{0,\Omega} ds + \frac{\Delta t^2}{8}\int_{t_n}^{t_{n-1}} \left\| u_{ttt}(s) \right\|_{0,\Omega} ds \\
&= \frac{\Delta t^2}{8}\int_{t_{n-1}}^{t_n} \left\| u_{ttt}(s) \right\|_{0,\Omega} ds.
\end{aligned}$$

Thus we have demonstrated that

$$\Delta t \ \left\| \widetilde{\omega}_2^n \right\|_{0,\Omega} \leq \frac{\Delta t^2}{8}\int_{t_{n-1}}^{t_n} \left\| u_{ttt}(s) \right\|_{0,\Omega} ds. \tag{7.26}$$

Now, let us try to bound $\left\| \widetilde{\omega}_3^n \right\|_{0,\Omega}$. The Taylor formula gives:

$$\begin{aligned}
\frac{1}{2}u(t_n) &= \frac{1}{2}u(t_{n-\frac{1}{2}}) + \frac{\Delta t}{4} \ u_t(t_{n-\frac{1}{2}}) + \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - s)u_{tt}(s)\,ds, \\
\frac{1}{2}u(t_{n-1}) &= \frac{1}{2}u(t_{n-\frac{1}{2}}) - \frac{\Delta t}{4} \ u_t(t_{n-\frac{1}{2}}) + \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_{n-1}} (t_n - s)u_{tt}(s)\,ds.
\end{aligned}$$

By summing these two above equalities we obtain:

$$u(t_{n-\frac{1}{2}}) - \frac{1}{2}\left(u(t_n) + u(t_{n-1})\right) = -\frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - s)u_{tt}(s)\,ds - \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_{n-1}} (t_{n-1} - s)u_{tt}(s)ds.$$

So, by applying the Laplace operator Δ, we obtain

$$\begin{aligned}
&\left\| \Delta \left[u(t_{n-\frac{1}{2}}) - \frac{1}{2}\left(u(t_n) + u(t_{n-1})\right) \right] \right\|_{0,\Omega} \\
&\leq \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - s) \left\| \Delta u_{tt}(s) \right\|_{0,\Omega} ds + \frac{1}{2}\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} \left| t_{n-1} - s \right| \left\| \Delta u_{tt}(s) \right\|_{0,\Omega} ds \\
&\leq \frac{\Delta t}{4}\int_{t_{n-1}}^{t_n} \left\| \Delta u_{tt}(s) \right\|_{0,\Omega} ds.
\end{aligned}$$

Consequently,

$$\Delta t \; \|\widetilde{\omega}_3^n\|_{0,\Omega} \leq \frac{\Delta t^2}{4} \int_{t_{n-1}}^{t_n} \|\Delta u_{tt}(s)\|_{0,\Omega}\, ds. \tag{7.27}$$

We still have to bound $\|\widetilde{\omega}_1^n\|_{0,\Omega}$. Let us recall that

$$\widetilde{\omega}_1^n := (R_h - I)\, \overline{\partial} u(t_n) = (R_h - I)\, \frac{u(t_n) - u(t_{n-1})}{\Delta t}.$$

Thus, by using Proposition 12 in [2, p. 252], there exists a constant $c > 0$ independent of h such that:

$$\begin{aligned}\Delta t \; \|\widetilde{\omega}_1^n\|_{0,\Omega} &\leq ch\, \|u(t_n) - u(t_{n-1})\|_{H^{2,\alpha}(\Omega)} \\ &= ch \left\| \int_{t_n-1}^{t_n} u_t(s)\, ds \right\|_{H^{2,\alpha}(\Omega)}.\end{aligned}$$

Consequently

$$\Delta t \; \|\widetilde{\omega}_1^n\|_{0,\Omega} \leq ch \int_{t_n-1}^{t_n} \|u_t(s)\|_{H^{2,\alpha}(\Omega)}\, ds. \tag{7.28}$$

According to the inequality (7.24), we have that:

$$\begin{aligned}\|\theta_h^n\|_{0,\Omega} &\leq \left\|\theta_h^{n-1}\right\|_{0,\Omega} + \Delta t\; \|\widetilde{\omega}^n\|_{0,\Omega} \\ &\leq \left\|\theta_h^{n-2}\right\|_{0,\Omega} + \Delta t \left(\left\|\widetilde{\omega}^{n-1}\right\|_{0,\Omega} + \|\widetilde{\omega}^n\|_{0,\Omega} \right) \\ &\leq \left\|\theta_h^{n-3}\right\|_{0,\Omega} + \Delta t \left(\left\|\widetilde{\omega}^{n-2}\right\|_{0,\Omega} + \left\|\widetilde{\omega}^{n-1}\right\|_{0,\Omega} + \|\widetilde{\omega}^n\|_{0,\Omega} \right) \\ &\;\;\vdots \\ &\leq \left\|\theta_h^0\right\|_{0,\Omega} + \Delta t \sum_{i=1}^{n} \left\|\widetilde{\omega}^i\right\|_{0,\Omega},\end{aligned}$$

recalling that $\theta_h^0 = u_h^0 - \tilde{u}_h(0) = 0$. By using inequalities (7.26), (7.27) and (7.28), we get

$$\Delta t \; \|\widetilde{\omega}_2^n\|_{0,\Omega} \leq \frac{(\Delta t)^2}{8} \int_{t_{n-1}}^{t_n} \|u_{ttt}(s)\|_{0,\Omega}\, ds, \tag{7.29}$$

$$\Delta t \ \left\|\widetilde{\omega}_3^n\right\|_{0,\Omega} \leq \frac{(\Delta t)^2}{4}\int_{t_{n-1}}^{t_n} \left\|\Delta u_{tt}(s)\right\|_{0,\Omega} ds, \tag{7.30}$$

$$\Delta t \ \left\|\widetilde{\omega}_1^n\right\|_{0,\Omega} \leq ch \int_{t_{n-1}}^{t_n} \left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)} ds. \tag{7.31}$$

Consequently recalling that $\widetilde{\omega}^n = \widetilde{\omega}_1^n + \widetilde{\omega}_2^n + \widetilde{\omega}_3^n$, we obtain

$$\begin{aligned}\left\|\theta_h^n\right\|_{0,\Omega} = \left\|u_h^n - \tilde{u}_h(t_n)\right\|_{0,\Omega} \leq ch \int_{t_0}^{t_n} \left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)} ds + \\ (\Delta t)^2 \left(\int_{t_0}^{t_n} \left\|u_{ttt}(s)\right\|_{0,\Omega} ds + \int_{t_0}^{t_n} \left\|\Delta u_{tt}(s)\right\|_{0,\Omega} ds\right).\end{aligned}$$

And since $\Delta u_{tt}(s) = \frac{d^2}{dt^2}\Delta u(s) = u_{ttt}(s) - f_{tt}(s)$, we can replace $\Delta u_{tt}(s)$ by $u_{ttt}(s) - f_{tt}(s)$ in the above inequality.

Finally, we get the following inequality:

$$\begin{aligned}\left\|u_h^n - \tilde{u}_h(t_n)\right\|_{0,\Omega} \ \leq \ ch \left(\int_0^{t_n} \left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)} ds\right) + \\ 2\,\Delta t^2 \left(\int_0^{t_n} \left\|u_{ttt}(s)\right\|_{0,\Omega} ds + \int_0^{t_n} \left\|f_{tt}(s)\right\|_{0,\Omega} ds\right).\end{aligned}$$

□

Theorem 7.3 *Let $\{\mathcal{T}_h\}$ be a regular family of triangulations on $\overline{\Omega}$, satisfying the properties (i) and (ii) of Proposition 9 of [2, p. 250]. Under the hypotheses of Proposition 7.2, and for $\alpha \in \left]1 - \frac{\pi}{w}, 1\right[$, there exists a constant $c > 0$ independent of h such that for every $n \geq 1$, we have*

$$\begin{aligned}&\left\|u(t_n) - u_h^n\right\|_{0,\Omega} \\ &\leq c\,h \left(|u(t_n)|_{H^1(\Omega)} + |u(t_n)|_{H^{2,\alpha}(\Omega)} + \int_0^{t_n} \left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)} ds\right) \\ &\quad + 2\,(\Delta t)^2 \left(\int_0^{t_n} \left\|u_{ttt}(s)\right\|_{0,\Omega} ds + \int_0^{t_n} \left\|f_{tt}(s)\right\|_{0,\Omega} ds\right).\end{aligned} \tag{7.32}$$

Proof It suffices to apply the triangular inequality:

$$\left\|u(t_n) - u_h^n\right\|_{0,\Omega} \leq \left\|u(t_n) - \tilde{u}_h(t_n)\right\|_{0,\Omega} + \left\|\tilde{u}_h(t_n) - u_h^n\right\|_{0,\Omega}.$$

By the inequality (7.17) and the inequality 5.6 of Proposition 9 in [2], we obtain the result. □

Similarly to the bound obtained in the implicit case, and in order to demonstrate the error estimate on $\mathbf{p}_h^n$, we need an analogous result to Proposition 1-5-10 of [5, p. 42], but for here $\widetilde{\omega}^n := \widetilde{\omega}_1^n + \widetilde{\omega}_2^n + \widetilde{\omega}_3^n$.

Proposition 7.3 *Let us suppose that* $f \in H^1(0,T;L^2(\Omega))$ *and* $\Delta g + f(0) \in \mathring{H}^1(\Omega)$. *Then,*

$$\bar{\partial}\|\boldsymbol{\varepsilon}_h^n\|^2 \leq \|\widetilde{\omega}^n\|^2, \quad \text{with } \boldsymbol{\varepsilon}_h^n := \mathbf{p}_h^n - \tilde{\mathbf{p}}_h(t_n). \tag{7.33}$$

Proof Let us consider the Crank-Nicolson scheme for the mixed method, written in the following form:

$$\begin{cases} \int_\Omega \mathbf{p}_h^n.\mathbf{q}_h\,dx + \int_\Omega u_h^n \,\operatorname{div}\mathbf{q}_h\,dx = 0, \quad \forall \mathbf{q}_h \in X_h, \quad \forall n \geq 1 \\ \int_\Omega v_h \,\operatorname{div}\frac{\mathbf{p}_h^n+\mathbf{p}_h^{n-1}}{2}\,dx = -\int_\Omega (f(t_{n-\frac{1}{2}}) - \overline{\partial} u_h^n)\, v_h\,dx, \;\forall v_h \in M_h, \forall n \geq 1. \end{cases} \tag{7.34}$$

By subtracting member by member from the first equation of (7.34), the first equation defining the elliptic projection:

$$\int_\Omega \tilde{\mathbf{p}}_h(t_n).\mathbf{q}_h\,dx + \int_\Omega \tilde{u}_h(t_n)\,\operatorname{div}\mathbf{q}_h\,dx = 0, \quad \forall \mathbf{q}_h \in X_h,$$

we obtain,

$$\int_\Omega \boldsymbol{\varepsilon}_h^n.\mathbf{q}_h\,dx + \int_\Omega \theta_h^n\,\operatorname{div}\mathbf{q}_h\,dx = 0, \quad \forall \mathbf{q}_h \in X_h, \tag{7.35}$$

where $\boldsymbol{\varepsilon}_h^n := \mathbf{p}_h^n - \tilde{\mathbf{p}}_h(t_n)$ and $\theta_h^n := u_h^n - \tilde{u}_h(t_n)$. Equation (7.35) being true for n and $n-1$, by making the difference member by member and by dividing by the time step, we obtain:

$$\int_\Omega \overline{\partial}\boldsymbol{\varepsilon}_h^n.\mathbf{q}_h\,dx + \int_\Omega \overline{\partial}\theta_h^n\,\operatorname{div}\mathbf{q}_h\,dx = 0, \quad \forall \mathbf{q}_h \in X_h. \tag{7.36}$$

Taking $\mathbf{q}_h = \boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}$ in Eq. (7.36), we obtain:

$$\|\boldsymbol{\varepsilon}_h^n\|_{0,\Omega}^2 - \left\|\boldsymbol{\varepsilon}_h^{n-1}\right\|_{0,\Omega}^2 = -\Delta t \int_\Omega \operatorname{div}\left(\boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}\right)\overline{\partial}\theta_h^n\,dx. \tag{7.37}$$

By the equalities (7.16) and (7.19), it follows that:

$$\int_\Omega v_h \ \operatorname{div} \frac{\boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}}{2}\, dx = \int_\Omega \left(\overline{\partial}\theta_h^n + \widetilde{\omega}^n\right)\, v_h\, dx\,,$$

$\forall v_h \in M_h$. In particular, if we choose $v_h = 1_K$, for any $K \in \mathcal{T}_h$, we obtain

$$\operatorname{div} \frac{\boldsymbol{\varepsilon}_h^n + \boldsymbol{\varepsilon}_h^{n-1}}{2} = P_h^0\left(\overline{\partial}\theta_h^n + \widetilde{\omega}^n\right) = \overline{\partial}\theta_h^n + P_h^0\,\widetilde{\omega}^n\,, \tag{7.38}$$

since $\overline{\partial}\theta_h^n = \frac{\theta_h^n - \theta_h^{n-1}}{\Delta t} \in M_h$. From (7.37) and (7.38), follows that

$$\left\|\boldsymbol{\varepsilon}_h^n\right\|_{0,\Omega}^2 - \left\|\boldsymbol{\varepsilon}_h^{n-1}\right\|_{0,\Omega}^2 = -2\,\Delta t\left\|\overline{\partial}\theta_h^n\right\|^2 - 2\,\Delta t\int_\Omega P_h^0\,\widetilde{\omega}^n\ \overline{\partial}\theta_h^n\, dx \tag{7.39}$$

$$\leq \Delta t\left\|P_h^0\,\widetilde{\omega}^n\right\|_{0,\Omega}^2 + \Delta t\left\|\overline{\partial}\theta_h^n\right\|_{0,\Omega}^2 - 2\,\Delta t\left\|\overline{\partial}\theta_h^n\right\|^2 \tag{7.40}$$

$$\leq \Delta t\left\|\widetilde{\omega}^n\right\|_{0,\Omega}^2. \tag{7.41}$$

By dividing both sides of the above inequality (7.41) by the time step Δt, we obtain: $\bar{\partial}\|\boldsymbol{\varepsilon}_h^n\|^2 \leq \|\widetilde{\omega}^n\|^2$. □

Corollary 7.1 *Under the hypotheses of Proposition 7.2, there exists a constant $c >$ 0 independent of h and Δt such that:*

$$\|\boldsymbol{\varepsilon}_h^n\|^2 \leq ch^2\int_0^{t_n}\left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)}^2 ds + c\,(\Delta t)^4\left(\int_0^{t_n}\left\|u_{ttt}(s)\right\|_{0,\Omega}^2 ds + \int_0^{t_n}\left\|\Delta u_{tt}(s)\right\|_{0,\Omega}^2 ds\right). \tag{7.42}$$

Proof By inequality (7.31), we have:

$$\Delta t\left\|\widetilde{\omega}_1^j\right\|_{0,\Omega} \leq ch\int_{t_j-1}^{t_j}\left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)} ds. \tag{7.43}$$

Thus:

$$\begin{aligned}\Delta t\sum_{j=1}^{j=n}\left\|\widetilde{\omega}_1^j\right\|^2 &\leq c\frac{h^2}{\Delta t}\sum_{j=1}^{j=n}\left(\int_{t_{j-1}}^{t_j}\left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)} ds\right)^2\\ &\leq ch^2\sum_{j=1}^{j=n}\int_{t_{j-1}}^{t_j}\left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)}^2 ds\\ &= h^2\int_{t_0}^{t_n}\left\|u_t(s)\right\|_{H^{2,\alpha}(\Omega)}^2 ds\,.\end{aligned} \tag{7.44}$$

By (7.29), we have $\Delta t \left\| \widetilde{\omega}_2^j \right\|_{0,\Omega} \leq \frac{\Delta t^2}{8} \int_{t_{j-1}}^{t_j} \left\| u_{ttt}(s) \right\|_{0,\Omega} ds.$

It follows from a similar calculus using the inequality of Cauchy-Schwartz:

$$\Delta t \sum_{j=1}^{j=n} \left\| \widetilde{\omega}_2^j \right\|^2 \leq c \Delta t^4 \int_{t_0}^{t_n} \left\| u_{ttt}(s) \right\|_{0,\Omega}^2 ds. \tag{7.45}$$

And also for (7.30), we obtain:

$$\Delta t \sum_{j=1}^{j=n} \left\| \widetilde{\omega}_3^j \right\|^2 \leq c \Delta t^4 \int_{t_0}^{t_n} \left\| \Delta u_{tt}(s) \right\|_{0,\Omega}^2 ds. \tag{7.46}$$

Now, by using Proposition 7.3, we get

$$\begin{cases} \left\| \boldsymbol{\varepsilon}_h^1 \right\|^2 - \left\| \boldsymbol{\varepsilon}_h^0 \right\|^2 \leq \Delta t \left\| \widetilde{\omega}^1 \right\|^2, \\ \left\| \boldsymbol{\varepsilon}_h^2 \right\|^2 - \left\| \boldsymbol{\varepsilon}_h^1 \right\|^2 \leq \Delta t \left\| \widetilde{\omega}^2 \right\|^2, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ \left\| \boldsymbol{\varepsilon}_h^n \right\|^2 - \left\| \boldsymbol{\varepsilon}_h^{n-1} \right\|^2 \leq \Delta t \left\| \widetilde{\omega}^n \right\|^2. \end{cases} \tag{7.47}$$

It follows that by summing up these inequalities, by (7.5) and $u_h^0 = \tilde{u}_h(0)$:

$$\| \boldsymbol{\varepsilon}_h^n \|^2 \leq \Delta t \sum_{j=1}^{j=n} \left\| \widetilde{\omega}^j \right\|^2. \tag{7.48}$$

Since $\widetilde{\omega}^n = \widetilde{\omega}_1^n + \widetilde{\omega}_2^n + \widetilde{\omega}_3^n$, it follows that

$$\| \boldsymbol{\varepsilon}_h^n \|^2 \leq 3\Delta t \sum_{j=1}^{j=n} \left\| \widetilde{\omega}_1^j \right\|^2 + 3\Delta t \sum_{j=1}^{j=n} \left\| \widetilde{\omega}_2^j \right\|^2 + 3\Delta t \sum_{j=1}^{j=n} \left\| \widetilde{\omega}_2^j \right\|^2. \tag{7.49}$$

From inequalities (7.46), (7.45) and (7.44) the assertion follows. □

In conclusion, we get the following result:

Theorem 7.4 *Under the hypotheses of Proposition 7.2, let $\{\mathcal{T}_h\}$ be a regular family of triangulations on $\overline{\Omega}$, satisfying the properties (i) and (ii) of Proposition 9 of [2,* *p. 250]. For* $\alpha \in \left]1 - \frac{\pi}{w}, 1\right[$, *there exists a constant* $c > 0$ *independent of* h *such that for every* $n \geq 1$, *we have*

$$\begin{aligned}
&\left\|\mathbf{p}(t_n) - \mathbf{p}_h^n\right\|_{0,\Omega} \\
&\lesssim h \, \left(|u(t_n)|_{H^{2,\alpha}(\Omega)} + \|u_t\|_{L^2(0,t_n;H^{2,\alpha}(\Omega))}\right) \\
&\quad + 2\,(\Delta t)^2 \left(\sqrt{\int_0^{t_n} \left\|u_{ttt}(s)\right\|_{0,\Omega}^2 ds} + \sqrt{\int_0^{t_n} \left\|\Delta u_{tt}(s)\right\|_{0,\Omega}^2 ds}\right).
\end{aligned}$$

Proof By the triangular inequality:

$$\begin{aligned}
\left\|\mathbf{p}(t_n) - \mathbf{p}_h^n\right\|_{0,\Omega} &\leq \left\|\mathbf{p}(t_n) - \tilde{\mathbf{p}}_h(t_n)\right\|_{0,\Omega} + \left\|\tilde{\mathbf{p}}_h(t_n) - \mathbf{p}_h^n\right\|_{0,\Omega}, \\
&= \left\|\mathbf{p}(t_n) - \tilde{\mathbf{p}}_h(t_n)\right\|_{0,\Omega} + \left\|\boldsymbol{\varepsilon}_h^n\right\|_{0,\Omega}.
\end{aligned}$$

By Farhloul et al. [2, (5.5) p. 250]:

$$\left\|\mathbf{p}(t_n) - \tilde{\mathbf{p}}_h(t_n)\right\|_{0,\Omega} \lesssim h \; |u(t_n)|_{H^{2,\alpha}(\Omega)}.$$

From this estimate and the preceding corollary, the result follows. □

7.5 Conclusion

In this paper we have demonstrated that the completely discretized problem of the mixed formulation for the heat equation using the Crank-Nicolson scheme for time discretization admits one and only one solution. By refining the meshings according to Raugel's rules near the reentrant corners [9], we have established optimal order of convergence for this completely discretized dual mixed method for the heat diffusion equation in a polygonal domain. And this by using for the spatial discretization, Raviart–Thomas vectorfields of degree 0 for the heat flux density vector, locally constant functions for the scalar field of temperatures, and the Crank-Nicolson scheme for the discretization in time.

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