Chapter 2
Steady Systems of PDEs. Two Examples from Applications

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Abstract After some general comments on the tremendous difficulties associated with non-linear systems of PDEs, we will focus on two such illustrative situations where the underlying model depends on the highly nonlinear behavior coming from a system of PDEs. The first one is motivated by inverse problems in conductivity and the process to recover an unknown conductivity coefficient from measurements in the boundary; the second focuses on an optimal control problem for soft robots in which the underlying model comes from hyper-elasticity where the state system models the behavior of non-linear elastic materials capable of undergoing large deformations.

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2.1 Introduction: Systems of PDEs: Why Are They So Difficult?

It is not hard to find hundreds of excellent experts on (single) PDEs. The subject is mature and expanding rapidly in many fundamental and interesting directions. However, it is not so if one is interested in systems of PDEs. As a matter of fact, when one talks about systems of PDEs, some fundamental examples come to mind. Without pretending to be exhaustive, we can mention the following.

- Dynamical:
  - Navier-Stokes, and variants (Stokes, Euler, etc).
  - Systems of conservation laws, first-order hyperbolic systems.
  - Diffusion systems, non-linear waves.
  - Other.

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- Steady:
  - Reaction-diffusion systems.
  - Linear elasticity.
  - Non-linear elasticity, hyper-elasticity.
  - Other situations.

According to its complexity, a possible classification, as in any other area of Differential Equations, could be:

- Linear systems.
- Non-linear systems: non-linearity on lower order terms (most studied).
- Fully non-linear systems: non-linearity on highest order terms.

Though there are some other alternatives based on the implicit function theorem [10, 29], we would like to focus on the steady situation, where the main procedure to show existence of solutions is limited to equations or systems that are variational in nature [5, 10, 24]: there is an underlying energy functional whose Euler-Lagrange equation or system is precisely the one to be solved

$$E(\mathbf{u}) = \int_\Omega F(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))\,d\mathbf{x},$$

$$\mathbf{u} : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N, \quad \mathbf{u}, \text{ subjected to boundary conditions.}$$

The principal strategy for existence of solutions for the underlying Euler-Lagrnage system of PDEs is to try to minimize the functional E among those feasible fields $\mathbf{u}$'s. This method requires a fundamental structural property for the existence of minimizers: convexity of the integrand

$$F(\mathbf{x}, \cdot, \cdot) : \mathbb{R}^N \times \mathbb{R}^{N\times N} \to \mathbb{R}, \tag{2.1}$$

for a.e. $\mathbf{x} \in \Omega$. This convexity, in turn, amounts to ellipticity or monotonicity of the associated optimality system. As a matter of fact, among the systems of PDEs most examined in the literature, we can mention diffusion systems where the principal part of the system is the laplacian operator or some other linear, elliptic variant. This class of problems directly stems from the convexity of the underlying functional. They are non-linear systems where the non-linearity occurs on lower-order terms.

Why is convexity so important? Because existence is based on weak lower semicontinuity

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \Rightarrow E(\mathbf{u}) \le \liminf_{j\to\infty} E(\mathbf{u}_j),$$

and this property rests in a crucial way on the convexity properties of the integrand F in (2.1). In the scalar case, when $\mathbf{u} = u$ is just a scalar function, weak lower semicontinuity is equivalent to usual convexity; but in the real vector case when $\mathbf{u}$ is a true field with several components, convexity is only sufficient, not necessary for weak lower semicontinuity [5, 11, 26]. The simplest example of this surprising and unexpected phenomenon is the determinant function

$$\det : \mathbf{M}^{2\times 2} \to \mathbb{R}.$$

Note that if

$$\mathbf{u}_j(\mathbf{x}) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathbf{u}_j = \mathbf{u}_0 \text{ on } \partial\Omega,$$
$$\mathbf{u}_j \rightharpoonup \mathbf{u} \text{ in } W^{1,p}(\Omega;\mathbb{R}^2) \Rightarrow \mathbf{u} = \mathbf{u}_0 \text{ on } \partial\Omega,$$

then for the area functional

$$\mathrm{A}(\mathbf{u}_0(\Omega)) = \mathrm{A}(\mathbf{u}_j(\Omega)) = \int_\Omega \det \mathbf{u}_j(\mathbf{x})\, d\mathbf{x} = \int_\Omega \det \mathbf{u}(\mathbf{x})\, d\mathbf{x}.$$

This means that the area functional is not only weakly lower semicontinuous but even weak continuous, and yet the function det is NOT convex. For vector problems (systems), there is a more general convexity condition, the “quasiconvexity”, imposed on F to have the equivalence with the weak lower semicontinuity property.

To put this property in perspective, recall that usual convexity can be formulated in the form

$$\phi(\mathbf{F}) : \mathbb{R}^{m\times N} \to \mathbb{R},$$
$$\phi(\mathbf{F}) \le \int_\Omega \phi(\mathbf{F} + \mathbf{U}(\mathbf{x}))\, d\mathbf{x}, \quad \mathbf{U}(\mathbf{x}) : \Omega \subset \mathbb{R}^N \to \mathbb{R}^{m\times N}, \int_\Omega \mathbf{U}(\mathbf{x})\, d\mathbf{x} = \mathbf{0}.$$

From this viewpoint, quasiconvexity is then determined through

$$\phi(\mathbf{F}) : \mathbb{R}^{m\times N} \to \mathbb{R},$$
$$\phi(\mathbf{F}) \le \int_\Omega \phi(\mathbf{F} + \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \quad \mathbf{u}(\mathbf{x}) : \Omega \subset \mathbb{R}^N \to \mathbb{R}^m, \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega,$$
$$\mathbf{U}(\mathbf{x}) = \nabla\mathbf{u}(\mathbf{x}) : \Omega \to \mathbb{R}^{m\times N}.$$

It can be understood by saying that quasiconvexity is convexity with respect to gradient fields. The main source of difficulties for systems of PDEs is that existence theorems require the quasiconvexity property, and this condition is difficult to manipulate. Indeed, it is still quite a mysterious concept.

One could think that if, after all, convexity is sufficient for weak lower semicontinuity one could be dispensed with dealing with quasiconvexity. The point is that usual convexity cannot be taken for granted in many of the situations of interest. A paradigmatic situation happens in hyper-elasticity, for a non-quadratic F, where convexity is incompatible with frame indifference [5, 10]. A typical situation is then to face a system of PDEs of interest that corresponds to the Euler-Lagrange system of a certain functional, but it is NOT convex. Two scenarios may demand attention:

1. either there are minimizers, in spite of this lack of convexity (though uniqueness might be compromised); or
2. there are no such minimizers, and then we do not have tools to decide whether the underlying system admits solutions.

We plan to examine two distinct situations in which one is forced to face a certain non-linear system of PDEs of interest. The first one deals with an alternative method to explore inverse problems in conductivity. The second one focuses on a control problem governed by a non-linear system of PDEs in the context of hyper-elasticity.

In the analysis of these two problems, several collaborators have contributed to its success: F. Maestre from U. de Sevilla (Spain), and J. Martínez-Frutos, R. Ortigosa and F. Periago from U. Politécnica de Cartagena (Spain).

Though we specifically focus on the two already-mentioned examples, there are a number of other fundamental problems where either the underlying topology of spaces changes, or the lack of convexity, though not fully and systematically treated yet, could be of interest. In particular, because it has been a dominant field of research in the last 30 years, we would like to bring reader's attention towards optimal transport problems [13, 27, 30]), among other possibilities.

2.2 Inverse Problems in Conductivity in the Plane

Suppose we are facing the problem of determining the conductivity coefficient $\gamma(\mathbf{x})$ of a media occupying a region $\Omega \subset \mathbb{R}^2$, by testing and measuring how it reacts to some electric stimulus. More precisely, we provide boundary datum $u_0(\mathbf{x})$ around $\partial\Omega$, and through the unknown solution $u(\mathbf{x})$ of the Dirichlet problem

$$\operatorname{div}[\gamma(\mathbf{x})\nabla u(\mathbf{x})] = 0 \text{ in } \Omega, \quad u(\mathbf{x}) = u_0(\mathbf{x}) \text{ on } \partial\Omega,$$

one can measure the response of the media by measuring the Neumann datum

$$v_0(\mathbf{x}) \equiv \gamma(\mathbf{x})\nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \text{ on } \partial\Omega.$$

The problem consists in recovering the coefficient $\gamma(\mathbf{x})$ from the pair of data (u_0, v_0). Typically, one has a broader set of measurement-pairs (u_i, v_i), $i = 1, 2, \ldots, m$. This is by now a classic and very well-studied problem, for which fundamental contributions have already been made. Starting from [8], from the mathematical point of view this kind of problems have led to deep and fundamental results (see [1, 4, 7, 9, 12, 16, 28], and references therein). The book [15] is a basic reference in this field. The main issues concerning inverse problems are *uniqueness* in various scenarios [4, 23], *stability* [2, 6] and *reconstruction* [3, 22].

From the conductivity equation

$$\operatorname{div}[\gamma(\mathbf{x})\nabla u(\mathbf{x})] = 0 \text{ in } \Omega \subset \mathbb{R}^2,$$

one can derive the algebraic identity

$$\gamma\nabla u + \mathbf{R}\nabla v = \mathbf{0}, \quad \mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.2}$$

for a certain function v which is unique up to additive constants. The interplay between u and v is classically understood through the Beltrami equation, though we will stick to (2.2). The algebraic equation (2.2) furnishes three important pieces of information:

1. a conductivity equation for v:

$$\operatorname{div}\left[\frac{1}{\gamma(\mathbf{x})}\nabla v(\mathbf{x})\right] = 0 \text{ in } \Omega;$$

2. a direct expression for the conductivity coefficient

$$\gamma(\mathbf{x}) = \frac{|\nabla v(\mathbf{x})|}{|\nabla u(\mathbf{x})|};$$

3. Dirichlet boundary datum for v around $\partial\Omega$: if we multiply (2.2) by the outer normal vector $\mathbf{n}(\mathbf{x})$ to $\partial\Omega$, we see that

$$\gamma(\mathbf{x})\nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \nabla v(\mathbf{x}) \cdot \mathbf{t}(\mathbf{x}) \text{ on } \partial\Omega, \tag{2.3}$$

   where $\mathbf{t}(\mathbf{x})$ is the counter-clockwise, unit tangent vector to $\partial\Omega$. Note that (2.3) provides Dirichlet boundary condition v_0 for v upon integration around $\partial\Omega$ of the Neumann condition for u, and so it is a datum of the problem once a measurement pair (u_0, v_0)

Hence, if $\gamma(\mathbf{x})$ is unknown, we can envision to recover it through the pair (u, v) solution of the system of PDEs

$$\operatorname{div}\left(\frac{|\nabla v|}{|\nabla u|}\nabla u\right) = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega, \tag{2.4}$$

$$\operatorname{div}\left(\frac{|\nabla u|}{|\nabla v|}\nabla v\right) = 0 \text{ in } \Omega, \quad v = v_0 \text{ on } \partial\Omega. \tag{2.5}$$

In fact, we are using v_0 for two different things: for the second component of a measurement, and for the Dirichlet boundary condition of the second component of the non-linear systems of PDEs. One is the tangential derivative of the other around

$\partial\Omega$. We hope this will not create confusion, as it is clear that once a measurement pair (u_0, v_0) is known, the corresponding Dirichlet boundary condition for the second component v of our system of PDEs is easily calculated by integration around $\partial\Omega$ of v_0. We are also using the same notation v_0 for this Dirichlet datum.

We therefore focus on how to find and compute a solution pair (u, v) for our above system of PDEs. It is easily checked that, at least formally, it is the Euler-Lagrange system corresponding to the functional

$$I(\mathbf{u}) = \int_\Omega |\nabla u_1(\mathbf{x})|\,|\nabla u_2(\mathbf{x})|\,d\mathbf{x}, \quad \mathbf{u} = (u_1, u_2), u = u_1, v = u_2. \tag{2.6}$$

But the integrand for I

$$\phi(\mathbf{F}) : \mathbf{M}^{2\times 2} \to \mathbb{R}, \quad \phi(\mathbf{F}) = |\mathbf{F}_1|\,|\mathbf{F}_2|, \quad \mathbf{F} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix},$$

is not convex, not even quasiconvex. This means that a typical existence theorem for minimizers for I cannot be used to show existence of minimizers for I in appropriate function spaces, and so our unique possibility to prove existence of solutions of our system of PDEs, and to recover the conductivity coefficient γ, is absolutely blocked from the outset. Moreover, ϕ is not coercive either. We are left with no general alternative to show existence of solutions for our system of PDEs. As a matter of fact, existence of solutions depend on properties of boundary data (u_0, v_0).

2.2.1 A Situation in Which Existence Can be Achieved

We describe how to build pairs of data (u_0, v_0), closely connected to the underlying inverse problem, for which existence of solutions for our system of PDEs can be achieved. Generate pairs of Dirichlet data in the following way:

1. $\gamma(\mathbf{x}) \geq \gamma_0 > 0$ in Ω, and $u_0 \in H^{1/2}(\partial\Omega)$ freely.
2. Solve the problem

$$\operatorname{div}(\gamma\nabla u) = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega, \tag{2.7}$$

   and compute the tangential derivative $w_0 = \nabla u \cdot \mathbf{t}$ around $\partial\Omega$.
3. Find a solution of the Neumann problem

$$\operatorname{div}\left(\frac{1}{\gamma}\nabla v\right) = 0 \text{ in } \Omega, \quad \frac{1}{\gamma}\nabla v \cdot \mathbf{n} = w_0 \text{ on } \partial\Omega. \tag{2.8}$$

4. Take $v_0 = v|_{\partial\Omega}$.

This mechanism can be used to generate synthetic pairs of compatible boundary measurements to be used in numerical experiments.

Proposition 2.1 *If data pairs* (u_0, v_0) *are built in the above way, the system*

$$\operatorname{div}\left(\frac{|\nabla v|}{|\nabla u|}\nabla u\right) = 0, \quad \operatorname{div}\left(\frac{|\nabla u|}{|\nabla v|}\nabla v\right) = 0 \quad \textit{in } \Omega,$$

under $u = u_0$, $v = v_0$ *on* $\partial\Omega$, *admits, at least, one solution.*

Proof The proof is elementary. Equation (2.7) implies the existence of a function $w \in H^1(\Omega)$, unique up to an additive constant, such that

$$\gamma \nabla u + \mathbf{R}\nabla w = \mathbf{0}.$$

This vector equation can be rewritten in the form

$$\mathbf{R}\nabla u = \frac{1}{\gamma}\nabla w,$$

and hence we can conclude that

$$\operatorname{div}\left(\frac{1}{\gamma}\nabla w\right) = 0 \text{ in } \Omega, \quad \frac{1}{\gamma}\nabla w \cdot \mathbf{n} = \nabla u \cdot \mathbf{t} = w_0 \text{ on } \partial\Omega.$$

If we compare this information with (2.8), we can conclude that the auxiliary function w is, except for an additive constant, the function v. As a consequence, we see that the pair (u, v) is a solution of our non-linear system of PDEs. □

Much more information about this problem can be found in [18].

2.2.2 *The Multi-Measurement Case*

In practice, the determination of an unknown conductivity coefficient $\gamma(\mathbf{x})$ can be facilitated in one is allowed to make several measurements corresponding to pairs

$$(u_{1,0}^{(j)}, u_{2,0}^{(j)}), \quad j = 1, 2, \ldots, N.$$

In this case the system for an unknown field

$$\mathbf{u}(\mathbf{x}) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^{2N}$$

becomes

$$\operatorname{div}\left(\frac{|\nabla \mathbf{u}_2|}{|\nabla \mathbf{u}_1|}\nabla u_1^{(j)}\right) = 0 \text{ in } \Omega, \quad u_1^{(j)} = u_{1,0}^{(j)} \text{ on } \partial\Omega, \tag{2.9}$$

$$\operatorname{div}\left(\frac{|\nabla \mathbf{u}_1|}{|\nabla \mathbf{u}_2|}\nabla u_2^{(j)}\right) = 0 \text{ in } \Omega, \quad u_2^{(j)} = u_{2,0}^{(j)} \text{ on } \partial\Omega, \tag{2.10}$$

for $j = 1, 2, \dots, N$, where

$$\mathbf{u} = (\mathbf{u}^{(j)})_{j=1,2,\dots,N} = (u_1^{(j)}, u_2^{(j)})_{j=1,2,\dots,N},$$

$$\mathbf{u}^{(j)} = (u_1^{(j)}, u_2^{(j)}), \quad \mathbf{u}_i = (u_i^{(j)})_{j=1,2,\dots,N}, \quad i = 1, 2.$$

We also have an underlying functional whose Euler-Lagrange system is precisely (2.9)–(2.10), namely

$$I_N(\mathbf{u}) = \int_\Omega |\nabla \mathbf{u}_1|\, |\nabla \mathbf{u}_2|\, d\mathbf{x}. \tag{2.11}$$

This functional, as in the one-measurement case, is neither convex nor quasiconvex, and once again one runs into the same difficulties concerning existence of solutions. In those cases in which there is a solution $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ of our system, the recovered $\gamma(\mathbf{x})$ would be

$$\gamma(\mathbf{x}) = \frac{|\nabla \mathbf{u}_2(\mathbf{x})|}{|\nabla \mathbf{u}_1(\mathbf{x})|} \quad \text{a.e. } x \in \Omega,$$

and thus, we see that the information coming from every measurement for $j = 1, 2, \dots, N$ is taken into account.

2.2.3 Some Numerical Tests

For those situations in which synthetic data sets have been determined through the technique of Sect. 2.2.1, one can proceed to approximate the solution of the corresponding non-linear system of PDEs, and consequently to find an approximation of the conductivity coefficient. We have explored three possible approximation procedures:

1. a typical Newton-Raphson method applied directly to the non-linear system either for the one-measurement or for the multi-measurement cases;
2. a descent algorithm based on minimizing the corresponding functional either (2.6) or (2.11);

3. a fix-point strategy consisting in iterating the action of the operation

$$(u, v) \mapsto (U, V) = \mathbf{T}(u, v)$$

where

$$\operatorname{div}\left(\frac{|\nabla v|}{|\nabla u|}\nabla U\right) = 0 \text{ in } \Omega, \quad U = u_0 \text{ on } \partial\Omega,$$

$$\operatorname{div}\left(\frac{|\nabla u|}{|\nabla v|}\nabla V\right) = 0 \text{ in } \Omega, \quad V = v_0 \text{ on } \partial\Omega,$$

for the one-measurement case, and a similar one for the multi-measurement situation.

All three methods worked fine for our experiments, though the descent method is the slowest in converging. See in Figs. 2.1, 2.2, 2.3, and 2.4.

It is worth-while to point out that we have a fine certificate of convergence for our numerical simulations. Indeed, the modified corresponding functional, adding

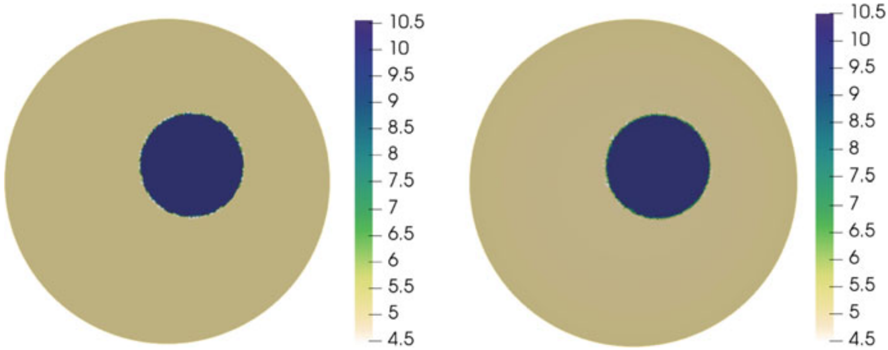

Fig. 2.1 Left. The target function γ with one inclusion. Right. The computed one

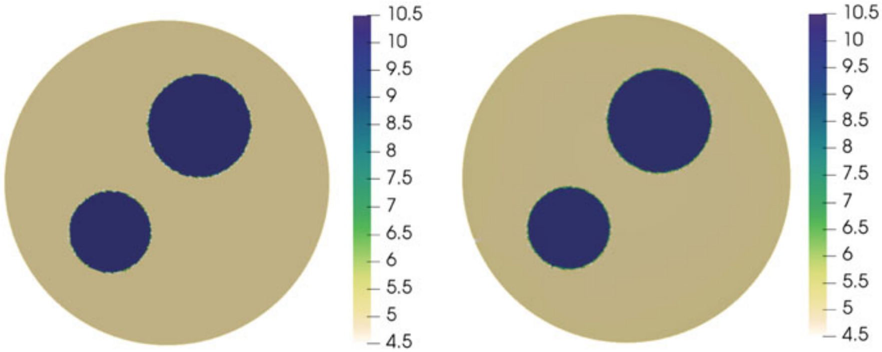

Fig. 2.2 Left. The target function γ with two disjoint inclusions. Right. The approximation

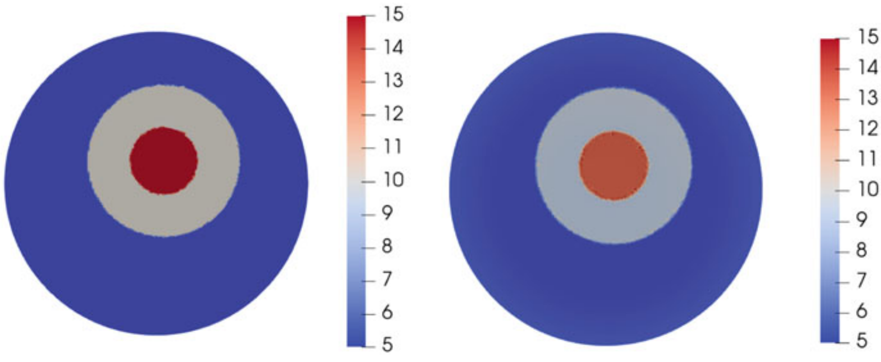

Fig. 2.3 Left. The target function γ with one inclusion inside another. Right. The approximated one

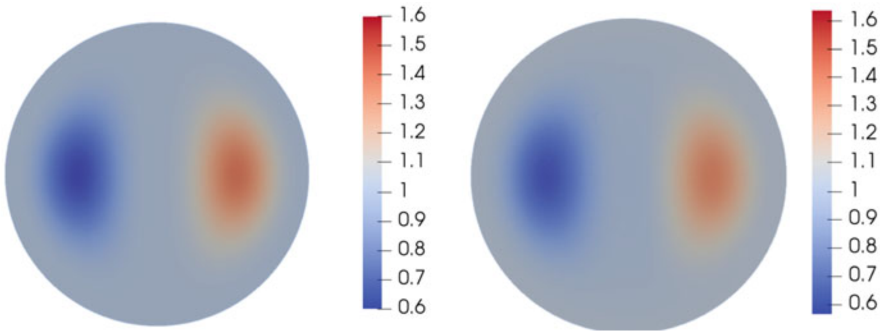

Fig. 2.4 Left. A continuous target γ. Right. The computed one

an additional term in (2.6) and in (2.11), become

$$I_1^*(\mathbf{u}) = \int_\Omega [|\nabla u_1(\mathbf{x})|\,|\nabla u_2(\mathbf{x})| - \det \nabla \mathbf{u}]\, d\mathbf{x}, \quad \mathbf{u} = (u_1, u_2),$$

$$I_N^*(\mathbf{u}) = \int_\Omega \left[|\nabla \mathbf{u}_1|\,|\nabla \mathbf{u}_2| - \sum_j \det \nabla \mathbf{u}^{(j)} \right] d\mathbf{x},$$

$$\mathbf{u} = (\mathbf{u}^{(j)})_{j=1,\ldots,N}, \mathbf{u}_i = (u_i^{(j)})_{j=1,\ldots,N}.$$

Because we are adding a null-lagrangian in both cases, the underlying Euler-Lagrange systems of PDEs are exactly the same that the ones for the old functionals, namely (2.4)–(2.5) and (2.9)–(2.10), respectively. The advantage of these two new functionals over the old ones is that the infimum-minimum value for I_1^* and I_N^* should vanish for the solutions we are seeking. Hence, we can be sure that approximations $\mathbf{u}_k$ are fine provided

$$I_1^*(\mathbf{u}_k) \searrow 0, \quad I_N^*(\mathbf{u}_k) \searrow 0.$$

This fact was true in all of our above simulations.

More information is available in [18, 25].

2.3 Some Optimal Control Problems for Soft Robots

There is recently an ever-increasing interest in simulating and controlling the behavior of robots that may, eventually, undergo large deformations as opposed to the more traditional setting in which they are considered essentially as rigid structures, or allowed small deformations. For instance, one might be interested in simulating the heliotropic effect in plants where the stem under the flower aligns with the direction of the sun through an osmotic potential (inner pressure) inside the cells [14]. To pursue this objective, one should move from the linear-elasticity setting to the more complex non-linear or hyper-elastic framework. Another reference worth mentioning is [17] where existence of optimal boundary forces are shown to exist for a control problem arising in facial surgery models. A different viewpoint that is almost mandatory to examine at some point, but that we overlook here, is that of allowing uncertainties in the measurement of material parameters. Apparently, rather large variabilities have been reported in bulk and shear moduli measurements of hyper-elastic materials that are commonly used to simulate rubber-like or biological tissues. This possibility demands to address stochastic versions of corresponding optimal control problems (see [19]).

The following features are important for such models of material behavior:

- material properties: finite strain-elasticity with internal energy density of the form

$$W(\mathbf{F}) = a\|\mathbf{F}\|^2 + b\|\operatorname{cof}\mathbf{F}\|^2 + c(\det\mathbf{F})^2 - d\log(\det\mathbf{F}) + e \tag{2.12}$$

  where a, b, c, d, and e are material constants;
- reference configuration $\Omega \subset \mathbb{R}^3$;
- boundary conditions: displacement condition in a part of the boundary $\mathbf{v} = \mathbf{0}$ on $\Gamma_D \subset \partial\Omega$, traction-free on the complement $\Gamma_N = \partial\Omega \setminus \Gamma_D$;
- control: inner pressure $t(\mathbf{x})$ acting on every point in Ω;
- objective: given two fields $\mathbf{s}(\mathbf{x})$ and $\mathbf{s}_{des}(\mathbf{x})$ (identifying the direction of the stem and the direction indicating the sun, respectively), choose the inner pressure $t(\mathbf{x})$ so as to align, as closely as possible, $\mathbf{s}(\mathbf{x})$ with $\mathbf{s}_{des}(\mathbf{x})$.

Under such conditions, equilibrium equations for feasible solution $\mathbf{u}$ for the direct problem become:

$$\int_\Omega W_{\mathbf{F}}(\nabla\mathbf{u}) : \nabla\mathbf{v}\, d\mathbf{x} = 0$$

for every test $\mathbf{v}$ with $\mathbf{v} = \mathbf{0}$ on Γ_D. This is indeed the weak form of

$$\text{div}[W_{\mathbf{F}}(\nabla\mathbf{u})] = \mathbf{0} \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0} \text{ on } \Gamma_D,\ W_{\mathbf{F}}(\nabla\mathbf{u})\mathbf{n} = \mathbf{0} \text{ on } \Gamma_N.$$

But such a energy density W is not quadratic nor convex. The corresponding system of PDEs is non-linear, and non-elliptic in the classical sense. Existence of solutions is achieved through polyconvexity [5], and the remarkable properties of minors. Uniqueness does not hold in general. More specifically, one can show the existence of minimizers for the direct variational problem

$$\text{Minimize in } \mathbf{u} \in \mathcal{A}: \quad E(\mathbf{u}) = \int_\Omega W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x},$$
$$\mathcal{A} = \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ in } \Gamma_D\}.$$

Such minimizer will be a solution of the above system of PDEs.

2.3.1 The Control Problem

The control problem we would like to explore can be formulated in the following terms

$$\text{Minimize in } t(\mathbf{x}): \quad \frac{1}{2}\int_\Omega \|(\mathbf{id} + \nabla\mathbf{u})\mathbf{s} - \mathbf{s}_{des}\|^2\, d\mathbf{x} + \frac{M}{2}\int_\Omega t^2\, d\mathbf{x}$$
$$+ \frac{\epsilon}{2}\int_\Omega |\nabla t|^2\, d\mathbf{x},$$

subject to

$$\int_\Omega \nabla_{\mathbf{F}} W(\mathbf{id}+\nabla\mathbf{u}(\mathbf{x})) : \nabla\mathbf{v}(\mathbf{x})\, d\mathbf{x} = \int_\Omega t(\mathbf{x})\, \text{cof}(\mathbf{id}+\nabla\mathbf{u}(\mathbf{x})) : \nabla\mathbf{v}(\mathbf{x})\, d\mathbf{x} \tag{2.13}$$

for all $\mathbf{v}$ in

$$\mathbb{V}_D := \left\{\mathbf{v} \in H^1\left(\Omega; \mathbb{R}^N\right) : \mathbf{v} = 0 \text{ on } \Gamma_D\right\}.$$

The internal energy density W is assumed of the form (2.12). The state system is the non-linear system of PDEs

$$\text{div}[W_{\mathbf{F}}(\mathbf{id} + \nabla\mathbf{u}) - t\, \text{cof}(\mathbf{id} + \nabla\mathbf{u})] = \mathbf{0},$$

that corresponds to a minimizer of the associated integral functional with density

$$W(\mathbf{F}) - t \det \mathbf{F}, \quad W(\mathbf{F}) = a\|\mathbf{F}\|^2 + b\|\operatorname{cof} \mathbf{F}\|^2 + c(\det \mathbf{F})^2 - d \log(\det \mathbf{F}) + e,$$

whose weak formulation is precisely (2.13).

Theorem 2.1 *Under the conditions explained above, there is an optimal pressure for our control system.*

To understand the main difficulty in the proof of such a result, consider a minimizing sequence $\{t_j\}$ with corresponding states $\{\mathbf{u}_j\}$, i.e.

$$\int_\Omega \nabla_\mathbf{F} W(\mathbf{id} + \nabla \mathbf{u}_j(\mathbf{x})) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x} = \int_\Omega t_j(\mathbf{x}) \operatorname{cof}(\mathbf{id} + \nabla \mathbf{u}_j(\mathbf{x})) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x} \tag{2.14}$$

for $\mathbf{v} \in \mathbb{V}_D$. Under these circumstances, it is easy to find that $t_j \to t$, $\mathbf{u}_j \to \mathbf{u}$ (weak convergences), but the whole point is to check whether limits t and $\mathbf{u}$ are such that

$$\int_\Omega \nabla_\mathbf{F} W(\mathbf{id} + \nabla \mathbf{u}(\mathbf{x})) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x} = \int_\Omega t(\mathbf{x}) \operatorname{cof}(\mathbf{id} + \nabla \mathbf{u}(\mathbf{x})) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x}$$

for every $\mathbf{v} \in \mathbb{V}_D$. That is to say, whether t and $\mathbf{u}$ are related through the state system.

If we take limits in j in (2.14), we realize that we need to have

$$\nabla_\mathbf{F} W(\mathbf{id} + \nabla \mathbf{u}_j) \to \nabla_\mathbf{F} W(\mathbf{id} + \nabla \mathbf{u}), \quad \operatorname{cof}(\mathbf{id} + \nabla \mathbf{u}_j) \to \operatorname{cof}(\mathbf{id} + \nabla \mathbf{u}),$$
$$t_j \operatorname{cof}(\mathbf{id} + \nabla \mathbf{u}_j) \to t \operatorname{cof}(\mathbf{id} + \nabla \mathbf{u}).$$

Surprisingly, these three convergences are correct if $t_j \to t$, (strong) and $\mathbf{u}_j \rightharpoonup \mathbf{u}$ (weak), due essentially to the remarkable convergence properties of the subdeterminants

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \Rightarrow M(\nabla \mathbf{u}_j) \rightharpoonup M(\nabla \mathbf{u})$$

for every subdeterminant M, and the presence of the quadratic term in the derivatives of $t(\mathbf{x})$ in the cost functional. The proof of the existence result is finished by standard weak lower semicontinuity through convexity.

2.3.2 The Fiber Tension Case

This is another fundamental control problem in the field of soft robots, in which given a field

$$\mathbf{a} = \mathbf{a}(\mathbf{x}) : \Omega \to \mathbb{R}^N,$$

a control

$$m = m(\mathbf{x}) : \Omega \to \mathbb{R}$$

is sought to minimize the cost functional

$$\frac{1}{2}\int_\Omega \|\mathbf{u}(\mathbf{x}) - \mathbf{u}_{\text{des}}(\mathbf{x})\|^2 \, d\mathbf{x} + \frac{M}{2}\int_\Omega m(\mathbf{x})^2 \, d\mathbf{x} + \frac{\epsilon}{2}\int_\Omega |\nabla m(\mathbf{x})|^2 \, d\mathbf{x}$$

where

$$\int_\Omega \nabla_{\mathbf{F}} W\left(\mathbf{id} + \nabla\mathbf{u}\right) : \nabla\mathbf{v} \, d\mathbf{x} = \int_\Omega m\mathbf{a}^T \left(\mathbf{id} + \nabla\mathbf{u}\right)^T \nabla\mathbf{v}\,\mathbf{a} \, d\mathbf{x}$$

for all $\mathbf{v}$ in

$$\mathbb{V}_D := \left\{\mathbf{v} \in H^1\left(\Omega; \mathbb{R}^N\right) : \mathbf{v} = 0 \text{ on } \Gamma_D\right\}.$$

The underlying state system is the weak formulation of the non-linear system of PDEs which is the Euler-Lagrange system corresponding to the functional

$$\int_\Omega \left[W\left(\mathbf{id} + \nabla\mathbf{u}\right) - \frac{1}{2}m|\left(\mathbf{id} + \nabla\mathbf{u}\right)\mathbf{a}|^2\right] d\mathbf{x},$$

for an inner density $W(\mathbf{F})$ as before. A similar existence result can be shown in this context as well.

2.3.3 Some Simulations

We present some simulations on both control problems described in the previous section. More information can be found in [20] and [21].

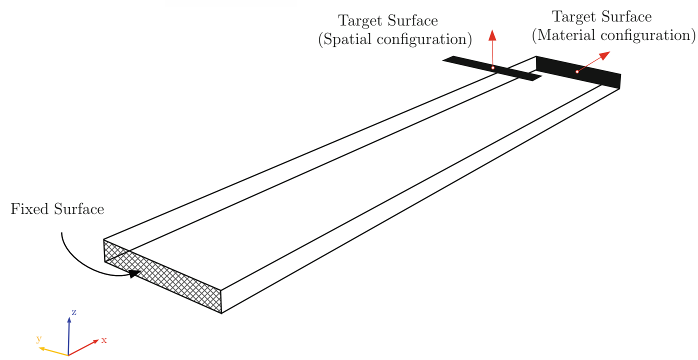

Fig. 2.5 Objective and boundary conditions for the control problem

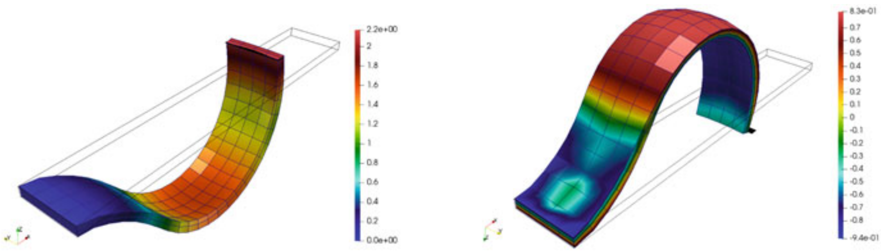

Fig. 2.6 Actuated control and pressure distribution

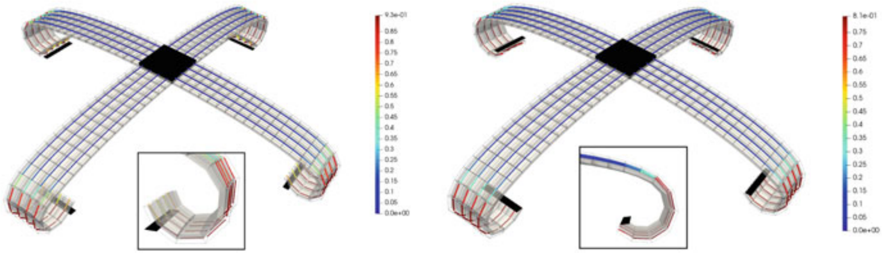

Fig. 2.7 A control situation for the fiber case. Left: desired effect. Right: mechanism activated

For the first one, we consider the situation depicted in Fig. 2.5. It is a clamped elastic plate whose free end surface is desired to be aligned vertically through an optimal use of an inner pressure field. Figure 2.6 shows the optimal result, and the optimal distribution of pressure.

Concerning the fiber problem, one desires to activate optimally a griping mechanism as in Fig. 2.7.

2.4 Conclusions

When facing a steady, non-linear systems of PDEs, one needs to care about the structure of the problem. If it is not variational, there is no much one can do.

Assuming it is variational, i.e. it corresponds to the Euler-Lagrange system for a certain integral functional with an integrand $W(\mathbf{F})$, the success of the analysis depends on the properties of $W : \mathbb{R}^{N\times n} \to \mathbb{R}$.

- If W is convex in the usual sense, under additional technical hypotheses like coercivity, there should be no problem in proving existence of solutions.
- If convexity is not correct, or it cannot be easily shown, two situations may happen:
  - Quasiconvexity holds: existence without convexity (optimal control for soft robots).
  - Quasiconvexity does not hold: results heavily depend on ingredients of problem (inverse problems in conductivity).

The major trouble is that we do not fully understand this quasiconvexity condition as it may be very difficult to decide if a given W enjoys such property, unless we have designed it so that it does. Much remains to be understood about non-linear systems of PDEs.

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