Chapter 12
Numerical Methods Based on Spline Quasi-Interpolating Operators for Hammerstein Integral Equations

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Abstract In this paper, we propose collocation type method, its iterated version and Nyström method based on discrete spline quasi-interpolating operators to solve Hammerstein integral equation. We present an error analysis of the approximate solutions and we show that the iterated solution of collocation type exhibits a superconvergence as in the case of the Galerkin method. Finally, we provide numerical tests, that confirm the theoretical results.

Keywords Quasi-interpolants · Spline functions · Collocation method · Nyström method · Hammerstein integral equation

12.1 Introduction

The issue of integral equations is one of the most useful mathematical tools in pure and applied mathematics. It has huge applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equations (ODEs) and partial differential equations (PDEs) can be transformed into resolution problems of some approximate integral equations (Ref. [16]).

In this paper we are interested in Hammerstein nonlinear integral equation given by

$$u(x) = f(x) + \int_a^b k(x,t)\psi(t,u(t))\,dt, \quad x \in [a,b] \tag{12.1}$$

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D. Barrera et al. (eds.), *Mathematical and Computational Methods for Modelling, Approximation and Simulation*, SEMA SIMAI Springer Series 29,
https://doi.org/10.1007/978-3-030-94339-4_12

where f, k and ψ are continuous functions, with ψ nonlinear with respect to the second variable and u is the function to be determined. This type of equations appears in nonlinear physical phenomena such as electromagnetic fluid dynamics and reformulation of boundary value problems with a nonlinear boundary condition, see [7, 15]. Classical methods of solving (12.1) are projection methods. Within the commonly used projection methods, there are Galerkin and collocation methods based respectively on a series of orthogonal and interpolating projectors in finite dimensional approximation spaces. Both methods have been studied by many authors, see for example [6, 11, 14]. The idea of improving Galerkin and collocation solutions by an iteration technique was introduced by Sloan [23]. Then several authors applied this idea for different types of equations, see for example [8]. In [13], the authors introduced a new collocation-type method for the numerical solution of (12.1) and its superconvergence properties were studied in [12]. In [4], they used superconvergent Nyström and degenerate kernel methods to solve (12.1). More recently, for smooth kernel or less smooth along the diagonal, the authors in [5] introduced superconvergent product integration method to approximate the solution of (12.1). For Hammerstein integral equation with singular kernel, Gelerkin-type/modified Galerkin-type and Kantorovich methods are studied in [3].

Recently, many authors have been interested in using spline quasi-interpolating operators for the approximation of solution of integral equations. In particular, in [21] Fredholm integral equation is solved by using degenerate kernel methods based on quasi-interpolating operators. A new quadrature rule based on integrating spline quasi-interpolant is derived in [20] and used to solve Fredholm integral equation by Nyström method. A modified Kulkarni's method based on spline quasi-interpolating operators is investigated in [1]. The authors in [9], used spline quasi-interpolants that are projectors for the numerical solution of Fredholm integral equation by Galerkin, Kantorovich, Sloan and Kulkarni's schemes. The same operators are used in [10] to solve Urysohn nonlinear integral equations by using collocation and Kulkarni's schemes.

In this paper we investigate collocation and Nyström type methods based on spline quasi-interpolants that are not necessary projectors to solve Hammerstein integral equation. Here is an outline of the paper. In Sects. 12.2 and 12.3 we recall the definitions and main properties of the spline quasi-interpolants and a quadrature formulas associated, and present their convergence properties. In Sect. 12.4 we introduce the collocation and Nyström methods based in spline quasi-interpolants. In Sect. 12.5 we give a general framework for the error analysis of the approximate and the iterated solutions. Finally, in Sect. 12.6 we provide some numerical results, illustrating the approximation properties of the proposed methods.

12.2 A Family of Discrete Spline Quasi-Interpolants

Let $\mathcal{X}_n := \{x_k, 0 \le k \le n\}$ be the uniform partition of the interval $I = [a, b]$ into n equal subintervals, i.e. $x_k := a + kh$, with $h = (b - a)/n$ and $0 \le k \le n$. We consider the space $\mathcal{S}_d(I, \mathcal{X}_n)$ of splines of degree d and class C^{d-1} on this partition.

Its canonical basis is formed by the $n+d$ normalized B-splines $\{B_k, k \in \mathcal{J}\}$, with $\mathcal{J} := \{1, 2, \ldots, n+d\}$. The support of each B_k is the interval $[x_{k-d-1}, x_k]$ if we add multiple knots at the endpoints.

The discrete quasi-interpolants of degree $d > 1$ is a spline operator of the form

$$Q_d f := \sum_{k \in \mathcal{J}} \mu_k(f) B_k, \tag{12.2}$$

where the coefficients $\mu_k(f)$ are linear combinations of values of f on either the set T_n (for d even) or on the set $\mathcal{X}_n$ (for d odd), where

$$T_n := \{t_0 = x_0,\ t_k = \frac{1}{2}\left(x_{k-1} + x_k\right), k = 1, \ldots, n,\ t_{n+1} = b\},$$
$$\mathcal{X}_n := \{x_k,\ k = 0, \ldots, n\}.$$

Therefore, for d even, we set $f\left(T_n\right) = \left\{f_k = f\left(t_k\right), 0 \le k \le n+1\right\}$, and for d odd, we set $f\left(\mathcal{X}_n\right) = \left\{f_k = f\left(x_k\right), 0 \le k \le n\right\}$. The coefficients $\mu_k(f)$ for $d + 1 \le k \le n$, have the following form

$$\mu_k(f) := \begin{cases} \sum_{i=0}^{d} \alpha_{i+1,k} f_{k-d+i+1}, & \text{if } d \text{ is even,} \\ \sum_{i=1}^{d} \alpha_{i,k} f_{k-d+i-1}, & \text{if } d \text{ is odd,} \end{cases}$$

where α_{ik} are calculated such that the quasi-interpolants Q_d reproduce the space $\mathbb{P}_d$ of all polynomials of total degree at most d, i.e.

$$Q_d p = p, \quad \forall p \in \mathbb{P}_d.$$

The extremal coefficients $\mu_k(f)$ for $1 \le k \le d$ and $n+1 \le k \le n+d$ have particular expressions.

The quasi-interpolants Q_d can be written under the following quasi Lagrange form

$$Q_d f = \sum_{j=0}^{\mathtt{n_d}} f_j L j,$$

where $\mathtt{n_d} := n+1$ if d is even, $\mathtt{n_d} := n$ if d is odd and L_j are linear combinations of finite number of B-splines B_j.

Since μ_k are continuous linear functionals, the operator Q_d is uniformly bounded on $C([a, b])$ and classical results in approximation theory provide that for any subinterval $I_k = [x_{k-1}, x_k]$, $1 \le k \le n$ and for any function f, we have

$$\|f - Q_d f\|_{\infty, I_k} \le (1 + \|Q_d\|) dist_{\infty, I_k}(f, \mathbb{P}_d),$$

where

$$dist_{\infty,I_k}(f,\mathbb{P}_d) = \inf_{p\in\mathbb{P}_d} \|f - p\|_{\infty,I_k}.$$

Therefore, if $f \in C^{d+1}([a,b])$, we get

$$\|f - Q_d f\|_\infty \le C_1 h^{d+1} \|f^{(d+1)}\|_\infty, \tag{12.3}$$

for some constant C_1 independent of h. As usual $\|f - p\|_{\infty,I_k} = \max_{x\in I_k} |f(x) - p(x)|$ and $\|f - p\|_\infty = \max_{x\in[a,b]} |f(x) - p(x)|$.

In what follows, we give two examples of spline quasi-interpolants denoted by Q_2 and Q_3.

- Q_2 is the C^1 quadratic spline quasi-interpolant exact on $\mathbb{P}_2$ and defined by (see e.g. [17])

$$Q_2 f := \sum_{k=1}^{n+2} \mu_k(f) B_k, \tag{12.4}$$

where the coefficient functionals $\mu_k(f)$ are given by

$$\begin{aligned} &\mu_1(f) = f_0, \quad \mu_2(f) = -\frac{1}{3} f_0 + \frac{3}{2} f_1 - \frac{1}{3} f_2, \\ &\mu_k(f) = -\frac{1}{8} f_{k-2} + \frac{5}{4} f_{k-1} - \frac{1}{8} f_k, \quad 3 \le k \le n, \\ &\mu_{n+1}(f) = -\frac{1}{3} f_{n-1} + \frac{3}{2} f_n - \frac{1}{3} f_{n+1}, \quad \mu_{n+2}(f) = f_{n+1}. \end{aligned} \tag{12.5}$$

In [17] the author has proved that the quasi-interpolant Q_2 is uniformly bounded and its infinity norm is given by

$$\|Q_2\|_\infty = \frac{305}{207} \approx 1.4734.$$

- Q_3 is the C^2 cubic spline quasi-interpolant exact on $\mathbb{P}_3$ and defined by (see e.g. [18])

$$Q_3 f := \sum_{k=1}^{n+3} \mu_k(f) B_k, \tag{12.6}$$

where the coefficient functionals $\mu_k(f)$ are given by

$$\begin{aligned}
&\mu_1(f) = f_0, \ \mu_2(f) = \frac{1}{18}\big(7f_0 + 18f_1 - 9f_2 + 2f_3\big), \\
&\mu_k(f) = \frac{1}{6}\big(-f_{k-3} + 8f_{k-2} - f_{k-1}\big), \quad 3 \le k \le n+1, \qquad (12.7) \\
&\mu_{n+2}(f) = \frac{1}{18}\big(2f_{n-3} - 9f_{n-2} + 18f_{n-1} + 7f_n\big), \ \mu_{n+3}(f) = f_n.
\end{aligned}$$

The infinity norm of the quasi-interpolant Q_3 is uniformly bounded and it is given by

$$\|Q_3\|_\infty = 1.631.$$

In the case of even degree, the quasi-interpolant Q_d present some interesting properties. The first one concern superconvergence at some evaluation points. These superconvergence points depends upon the degree d. The following theorem provide them explicitly for the case of Q_2.

Theorem 12.1 *If* $f \in C^4([a,b])$*, then*

$$\begin{aligned}
|Q_2 f(x_i) - f(x_i)| &= O(h^4), \quad 0 \le i \le n, \ i \ne 1, n-1. \\
|Q_2 f(t_i) - f(t_i)| &= O(h^4), \quad 0 \le i \le n+1, \ i \ne 1, 2, n-1, n.
\end{aligned} \qquad (12.8)$$

Proof Using the exact values of Bsplines on x_i, t_i and the definition of coefficient functionals, we can show that $e_3(x_i) = e_3(t_i) = 0$ for all points included in (12.8), where e_3 represents the error of Q_2 on the monomial x^3. For the excluded points in (12.8), these values are not zero. Next, following the same logical scheme as in the proof of Lemma 4.1 in [9], we can get (12.8). □

12.3 Quadrature Rules Based on Q_d Defined on a Uniform Partition

Let f be a continuous function on the interval $[a,b]$, we consider the numerical evaluation of the integral

$$\mathcal{I}(f) := \int_a^b f(x)dx$$

by quadrature rules based on Q_d defined on a uniform partition. These rules are defined by

$$\begin{aligned}\mathcal{I}_{Q_d}(f) &:= \int_a^b Q_d f(x)dx \\ &= \begin{cases} h\sum_{j=0}^{n+1}\omega_j f_j & \text{if } d \text{ even} \\ h\sum_{j=0}^{n}\omega_j f_j & \text{if } d \text{ odd}\end{cases}\end{aligned}$$

where $\omega_j = \dfrac{1}{h}\displaystyle\int_a^b L_j(x)dx$ and L_j are quasi-Lagrange functions associated with Q_d.

In particular, for $d = 2$ and $d = 3$ we obtain the following quadrature rules

$$\mathcal{I}_{Q_2}(f) := h\sum_{j=0}^{n+1}\omega_j f_j, \quad \text{et} \quad \mathcal{I}_{Q_3}(f) := h\sum_{j=0}^{n}\omega_j f_j.$$

Where the ω_j weights are given explicitly in the following table.

j	0	1	2	3	4	...	$n-4$	$n-3$	$n-2$	$n-1$	n	$n+1$
ω_j $(d=2)$	$\frac{1}{9}$	$\frac{7}{8}$	$\frac{73}{72}$	1	1	...	1	1	1	$\frac{73}{72}$	$\frac{7}{8}$	$\frac{1}{9}$
ω_j $(d=3)$	$\frac{23}{72}$	$\frac{4}{3}$	$\frac{19}{24}$	$\frac{19}{18}$	1	...	1	$\frac{19}{18}$	$\frac{19}{24}$	$\frac{4}{3}$	$\frac{23}{72}$	–

As Q_d is exact on $\mathbb{P}_d$, we deduce that the associated quadrature formulas $\mathcal{I}_{Q_d}$ are also exact on $\mathbb{P}_d$. Therefore, the error $\mathcal{E}_{Q_d}(f) := \mathcal{I}(f) - \mathcal{I}_{Q_d}(f) = O(h^{d+1})$. However, in the case of an even degree d, this last error is more accurate. Indeed, the following theorem holds true (for the proof see [2]).

Theorem 12.2 *Let d be an even number and let Q_d be the quasi-interpolant defined by (12.2). For any function $f \in C^{d+2}([a,b])$, we have*

$$\int_a^b (f(t) - Q_d f(t))dt = O(h^{d+2}). \tag{12.9}$$

Moreover, for any weight function $g \in \mathcal{W}^{1,1}$ (i.e. $\|g'\|_1$ bounded), we have

$$\int_a^b g(t)(f(t) - Q_d f(t))dt = O(h^{d+2}). \tag{12.10}$$

Particular cases of the rule $\mathcal{I}_{Q_d}$ for $d = 2$ and $d = 4$ are studied in depth in [19] and [20] respectively. The authors in these papers have provided explicit error estimations and they have made comparisons with similar rules of interpolatory type.

12.4 Methods Based on $\mathcal{Q}_d$

Let us consider the Hammerstein integral equation (12.1) given in operator form as

$$u - \mathcal{K}u = f, \tag{12.11}$$

where $\mathcal{K}$ in the Hammerstein integral operator defined on $\mathcal{L}^\infty[a, b]$ by

$$\mathcal{K}u(x) = \int_a^b k(x, t)\psi(t, u(t))dt.$$

Since the kernel k is assumed to be continuous, the operator $\mathcal{K}$ is compact from $\mathcal{L}^\infty[a, b]$ to $C[a, b]$. In what follows, we propose two methods based on the quasi-interpolant operator $\mathcal{Q}_d$ to solve (12.11).

12.4.1 Collocation Type Method and Its Iterated Version

Recall that a spline quasi-interpolant of degree d is an operator defined on $C[a, b]$ by:

$$\begin{aligned} \mathcal{Q}_d : C[a, b] &\longrightarrow \mathcal{S}_d(I, \mathcal{X}_n) \\ f &\longrightarrow \textstyle\sum_{j=1}^{n+d} \mu_j(f)B_j. \end{aligned}$$

We propose to approximate the integral operator $\mathcal{K}$ in (12.11) by $\mathcal{K}_n^c := \mathcal{Q}_d\mathcal{K}$ and the second member f by $\mathcal{Q}_d f$. The approximate equation is then given by

$$u_n^c - \mathcal{Q}_d\mathcal{K}u_n^c = \mathcal{Q}_d f, \tag{12.12}$$

where

$$\mathcal{Q}_d\mathcal{K}u_n^c = \sum_{i=1}^{n+d} \mu_i\left(\mathcal{K}u_n^c\right)B_i.$$

The approximate solution u_n^c is a spline function, then we can write

$$u_n^c = \sum_{i=1}^{n+d} \alpha_i B_i.$$

Replacing u_n^c in Eq. (12.12), we obtain

$$\sum_{i=1}^{n+d}\alpha_i B_i - \sum_{i=1}^{n+d}\mu_i\Big(\mathcal{K}\big(\sum_{j=1}^{n+d}\alpha_j B_j\big)\Big)B_i = \sum_{i=1}^{n+d}\mu_i(f)B_i, \tag{12.13}$$

since the family $\{B_i,\ 1 \le i \le n+d\}$ is a basis of $\mathcal{S}_d(I, \mathcal{X}_n)$, we can identify the coefficients and we obtain the following nonlinear system:

$$\alpha_i - \mu_i\Big(\mathcal{K}\big(\sum_{j=1}^{n+d}\alpha_j B_j\big)\Big) = \mu_i(f), \qquad i = 1, 2, \ldots, n+d. \tag{12.14}$$

Another interesting solution to consider is the following iterated one

$$\hat{u}_n^c := \mathcal{K}\left(u_n^c\right) + f. \tag{12.15}$$

Applying Q_d on both sides, we find

$$Q_d\hat{u}_n^c = Q_d\mathcal{K}\left(u_n^c\right) + Q_d f = u_n^c. \tag{12.16}$$

Replacing in (12.15), we find that $\hat{u}_n^c$ satisfy the following equation

$$\hat{u}_n^c = \mathcal{K}Q_d\hat{u}_n^c + f. \tag{12.17}$$

We show later that the iterated solution $\hat{u}_n^c$ is more accurate than u_n^c.

Remark 12.1 It is important to note the presence of integrals in system (12.14) and also in the expression of iterated solution (12.15). When implementing the method these integrals were calculated numerically using high accuracy quadrature rules, like those defined on [22], to imitate exact integration .

12.4.2 Nyström Method

In the Nyström method the operator $\mathcal{K}$ in (12.11) is approximated by

$$\mathcal{K}_n^N u := \sum_{j=0}^{\mathrm{n_d}}\omega_j k(., \xi_j)\psi(\xi_j, u(\xi_j)),$$

where ξ_j and ω_j are respectively the nodes and the weights of the quadrature rule $\mathcal{I}_{Q_d}$ based on Q_d. More precisely, ξ_j are given by t_i for d even and by x_i for d odd.

Hence, the corresponding approximate equation is given by

$$u_n^N - \sum_{j=0}^{\mathtt{n_d}} \omega_j k(., \xi_j) \psi(\xi_j, u_n^N(\xi_j)) = f. \tag{12.18}$$

Taking this last equation at the points ξ_i, we get the following non linear system

$$u_n^N(\xi_i) - \sum_{j=0}^{\mathtt{n_d}} \omega_j k(\xi_i, \xi_j) \psi(\xi_j, u_n^N(\xi_j)) = f(\xi_i) \ , \quad 0 \le i \le \mathtt{n_d}. \tag{12.19}$$

By solving this system, we obtain the approximate solution u_n^N at points ξ_i. Over all the domain, u_n^N is given by the following interpolation formula

$$u_n^N = \sum_{j=0}^{\mathtt{n_d}} \omega_j k(, \xi_j) \psi(\xi_j, u_n^N(\xi_j)) + f.$$

Remark 12.2 It should be noted that the Nyström method is completely discrete because the system (12.19) does not contain integrals to be evaluated numerically. Which makes this method one of the easiest methods to implement.

12.5 Error Analysis

Let u^* be the unique solution of (12.1), and let $\mathfrak{a}$ and $\mathfrak{b}$ be real numbers such that

$$[\min_{x \in [a,b]} u^*(x), \max_{x \in [a,b]} u^*(x)] \subset [\mathfrak{a}, \mathfrak{b}].$$

Define $\Omega = [a, b] \times [\mathfrak{a}, \mathfrak{b}]$. We assume throughout this paper unless stated otherwise, the following conditions on f, k and ψ:

(C.1) $k \in C\big(\Omega\big)$.

(C.2) $f \in C\big([a, b]\big)$.

(C.3) $\psi\ (t, x)$ is continuous in $t \in [a, b]$ and Lipschitz continuous in $x \in [\mathfrak{a}, \mathfrak{b}]$, i.e., there exists a constant $q_1 > 0$ such that

$$\left|\psi\ (t, x_1) - \psi\ (t, x_2)\right| \le q_1 \left|x_1 - x_2\right|, \quad \text{for all } x_1, x_2 \in [\mathfrak{a}, \mathfrak{b}].$$

(C.4) The partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $q_2 > 0$ for which

$$\left|\psi\ (t, x_1)^{(0,1)} - \psi\ (t, x_2)^{(0,1)}\right| \le q_2 \left|x_1 - x_2\right|, \quad \text{for all } x_1, x_2 \in [\mathfrak{a}, \mathfrak{b}].$$

Condition (C.4) implies that the operator $\mathcal{K}$ is Fréchet-differentiable and that $\mathcal{K}'(u^*)$ is Mq_2-Lipschitz, where

$$\mathcal{K}'(u^*)v(s) = \int_a^b k(s,t)\frac{\partial \psi}{\partial u}\big(t, u^*(t)\big)v(t)dt,$$

and

$$M = \sup_{s\in[a,b]} \int_a^b |k(s,t)|dt.$$

Furthermore, the operator $\mathcal{K}'(u^*)$ is compact.

12.5.1 Collocation and Nyström Solutions

It is easy to see that the operators $\mathcal{K}_n^c$ and $\mathcal{K}_n^N$ are Fréchet differentiables and

$$(\mathcal{K}_n^c)'(u^*)v(s) = \sum_{i=1}^{n+d} \mu_i\Big(\mathcal{K}'(u^*)v\Big)B_i(s),$$

$$(\mathcal{K}_n^N)'(u^*)v(s) = \sum_{j=1}^{n+d} \boldsymbol{\omega}_j k(s,\xi_j)\frac{\partial \psi}{\partial u}(\xi_j, u^*(\xi_j))v(\xi_j).$$

Throughout the rest of this paper, we denote by L the operator $\mathcal{K}'(u^*)$ and by L_n either the operator $(\mathcal{K}_n^c)'(u^*)$ or the operator $(\mathcal{K}_n^N)'(u^*)$.

The following lemmas state some properties needed to prove the existence and the convergence of the approximate solutions. Their proofs are consequences of conditions (C.1)–(C.4), the fact that L_n is linear operator and that Q_d converges to the identity operator pointwise.

Lemma 12.1 *Assume that 1 is not an eigenvalue of* L*. Then for* n *large enough, 1 is not in the spectrum of* L_n *and* $(I - L_n)^{-1}$ *exists as a bounded linear operator, i.e.,*

$$\left\|(I - L_n)^{-1}\right\|_\infty \le C_1,$$

for a suitable constant C_1 *independent of* n*.*

Lemma 12.2 *Assume that 1 is not an eigenvalue of* L*. Then for* n *large enough,* L_n *is Lipschitz continuous on* $B(u^*, \delta)$ *for* $\delta > 0$*.*

Put $Tu = \mathcal{K}u + f$. Equation (12.11) becomes

$$u = Tu, \tag{12.20}$$

Generally, the previous equation is approximated by

$$u_n = T_n u_n, \tag{12.21}$$

where T_n is a sequence of approximating operators. We quote the following theorem from [24] which gives conditions on T_n to ensure the convergence of u_n to the exact solution.

Theorem 12.3 *Suppose that the Eq. (12.20) has a unique solution u^* and the following conditions are satisfied*

(i) T_n is Fréchet-differentiable and $(I - T_n'(u^*))^{-1}$ exists and is uniformly bounded.
(ii) For certain values of $\delta > 0$ and $0 < q < 1$, the inequalities

$$\sup_{\|u-u^*\|\leq\delta} \left\| \left[I - T_n'\left(u^*\right)\right]^{-1} \left(T_n'(u) - T_n'\left(u^*\right)\right) \right\|_\infty \leq q,$$

$$\alpha := \left\| \left[I - T_n'\left(u^*\right)\right]^{-1} \left(T_n u^* - Tu^*\right) \right\|_\infty \leq \delta(1-q),$$

are valid.

Then the approximate equation (12.21) has a unique solution u_n in $B(u^*, \delta)$ such that

$$\frac{\alpha}{1+q} \leq \left\|u_n - u^*\right\|_\infty \leq \frac{\alpha}{1-q}. \tag{12.22}$$

We now give our result about the existence and uniqueness of the approximate solution for collocation and Nyström methods studied in this paper. Recall that for collocation method, $T_n u = \mathcal{K}_n^c u + Q_d f$, and for Nyström method $T_n u = \mathcal{K}_n^N u + f$.

Theorem 12.4 *Let u^* be the unique solution of Eq. (12.20). Under the assumptions (C.1)–(C.4), there exists a real number $\delta_0 > 0$ such that the approximate equation (12.21) has a unique solution u_n in $B(u^*, \delta_0)$ for a sufficiently large n. Moreover, the error estimate (12.22) holds.*

Proof We give the proof in the case of collocation method. The proof is similar for Nyström case. From Theorem 12.3, it suffices to prove that (i) and (ii) are satisfied. Lemma 12.1 ensures that (i) is valid for sufficiently large n say for all $(n > N_1)$. Moreover from Lemma 12.2, for $\left\|u - u^*\right\|_\infty \leq \delta$ and $n > N_1$, we have

$\left\|T_n'(u) - T_n'\left(u^*\right)\right\|_\infty \leq m\delta$, where m is the Lipschitz constant of L_n. Hence

$$\begin{aligned}
&\left\|\left[I - T_n'\left(u^*\right)\right]^{-1}\left(T_n'(u) - T_n'\left(u^*\right)\right)\right\|_\infty \\
&\quad \leq \left\|\left[I - T_n'\left(u^*\right)\right]^{-1}\right\|_\infty \left\|T_n'(u) - T_n'\left(u^*\right)\right\|_\infty \\
&\quad \leq m\delta \left\|\left[I - T_n'\left(u^*\right)\right]^{-1}\right\|_\infty .
\end{aligned}$$

Therefore,

$$\sup_{\|u-u^*\|\leq\delta} \left\|\left[I - T_n'\left(u^*\right)\right]^{-1}\left(T_n'(u) - T_n'\left(u^*\right)\right)\right\|_\infty \leq q,$$

with $q = m\delta \left\|\left(I - T_n'\left(u^*\right)\right)^{-1}\right\|_\infty$. Here we take $\delta = \delta_0$ so small that $0 < q < 1$. Now we have

$$\begin{aligned}
\left\|Tu^* - T_n u^*\right\|_\infty &= \left\|\left(\mathcal{K}u^* - \mathcal{Q}_d\mathcal{K}u^*\right) + \left(f - \mathcal{Q}_d f\right)\right\|_\infty \\
&= \left\|\left(I - \mathcal{Q}_d\right)\mathcal{K}u^* + \left(I - \mathcal{Q}_d\right) f\right\|_\infty \\
&= \left\|\left(I - \mathcal{Q}_d\right)\left(\mathcal{K}u^* + f\right)\right\|_\infty \\
&= \left\|\left(I - \mathcal{Q}_d\right)u^*\right\|_\infty \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

i.e. there exists N_2 such that for $n > N_2$

$$\alpha = \left\|\left[I - T_n'\left(u^*\right)\right]^{-1}\left(Tu^* - T_n u^*\right)\right\| \leq \delta_0(1 - q).$$

Consequently, the condition (ii) is also valid. Hence, for $n > max\{N_1, N_2\}$, using Theorem 12.2, one can conclude that (12.21) has a unique solution in $B(u^*, \delta_0)$ and the inequalities (12.22) hold. □

Using the results obtained in Theorem 12.4 and error estimates (12.3)–(12.10), we give in the following theorem the error explicit estimates of the collocation and Nyström methods based on quasi-interpolants $\mathcal{Q}_d$.

Theorem 12.5 *Let u_n be a unique solution of the approximate equation (12.21) in $B(u^*, \delta_0)$ for a sufficiently large n. Assume that*

(i) $k \in C^{0,d+1}\left([a, b] \times [a, b]\right)$,
(ii) $\psi \in C^{d+1}\left([a, b] \times [a, b]\right)$,

(iii) $f \in C^{d+1}\big([a,b]\big)$.

Then

$$\|u_* - u_n\|_\infty = O(h^{d+1}).$$

Moreover, if d is even and if

(i) $k \in C^{0,d+2}\big([a,b]\times[a,b]\big)$,
(ii) $\psi \in C^{d+2}\big([a,b]\times[a,b]\big)$,
(iii) $f \in C^{d+2}\big([a,b]\big)$,

then, in the case of Nyström method it holds

$$\|u_* - u_n^N\|_\infty = O(h^{d+2}).$$

Proof It is an immediate consequence of preceding result and estimates (12.3)–(12.10). □

12.5.2 Iterated Collocation Solution

Recall that the iterated solution satisfies the following equation:

$$\hat{u}_n^c - \mathcal{K}Q_d\hat{u}_n^c = f. \tag{12.23}$$

Define r_n by

$$r_n = \frac{\left\|\mathcal{K}\left(u^*\right) - \mathcal{K}\left(u_n^c\right) - L\left(u^* - u_n^c\right)\right\|_\infty}{\left\|u^* - u_n^c\right\|_\infty},$$

where $L = \mathcal{K}'\left(u^*\right)$. From Theorem 12.4 and the definition of L, we conclude that

$$r_n \xrightarrow[n\to\infty]{} 0.$$

Moreover, it is possible to see that

$$r_n \le \frac{\xi}{2}\left\|u^* - u_n^c\right\|_\infty,$$

where ξ is a positive constant. We also use the following notation

$$a = \left\|\left(I - L\right)^{-1}\right\|_\infty.$$

The error estimate for the iterated solution is given in the following theorem.

Theorem 12.6 *Let u^* be a unique solution of Eq. (12.20). Assume that the assumptions (C.1)–(C.4) are satisfied. Then for a sufficiently large n, we have*

$$\left\|u^* - \hat{u}_n^c\right\|_\infty \leq \xi \left\|u^* - u_n^c\right\|_\infty^2 + a \left\|L\big(I - Q_d\big)\mathcal{K}u^*\right\|_\infty + a \left\|L\big(I - Q_d\big)L\big(u^* - u_n^c\big)\right\|_\infty$$

Proof We have

$$\begin{aligned}
\big(I - L\big)(u^* - \hat{u}_n^c) &= \mathcal{K}u^* - \mathcal{K}u_n^c - L\left(u^* - u_n^c\right) + L\left(\hat{u}_n^c - u_n^c\right) \\
&= \mathcal{K}u^* - \mathcal{K}u_n^c - L\left(u^* - u_n^c\right) + L\left(I - Q_d\right)\mathcal{K}u_n^c \\
&= \mathcal{K}u^* - \mathcal{K}u_n^c - L\left(u^* - u_n^c\right) \\
&\quad - L\left(I - Q_d\right)\left(\mathcal{K}u^* - \mathcal{K}u_n^c - L\left(u^* - u_n^c\right)\right) \\
&\quad + L\left(I - Q_d\right)\left(\mathcal{K}u^* - L\left(u^* - u_n^c\right)\right) \\
&= \Big(I - L\left(I - Q_d\right)\Big)\left(\mathcal{K}u^* - \mathcal{K}u_n^c - L\left(u^* - u_n^c\right)\right) \\
&\quad + L\left(I - Q_d\right)\mathcal{K}u^* \\
&\quad - L\left(I - Q_d\right)L\left(u^* - u_n^c\right).
\end{aligned}$$

Multiplying by $\big(I - L\big)^{-1}$, we find

$$\begin{aligned}
u^* - \hat{u}_n^c = &\Big(I - \big(I - L\big)^{-1} L Q_d\Big)\left(\mathcal{K}u^* - \mathcal{K}u_n^c - L\left(u^* - u_n^c\right)\right) \\
&+ \big(I - L\big)^{-1} L\left(I - Q_d\right)\mathcal{K}u^* \\
&- \big(I - L\big)^{-1} L\left(I - Q_d\right) L\left(u^* - u_n^c\right).
\end{aligned}$$

Therefore,

$$\left\|u^* - \hat{u}_n^c\right\|_\infty \leq \tilde{\xi} r_n \left\|u^* - u_n^c\right\|_\infty + a \left\|L\big(I - Q_d\big)\mathcal{K}u^*\right\|_\infty + a \left\|L\big(I - Q_d\big)L\big(u^* - u_n^c\big)\right\|_\infty$$

Which completes the proof of the theorem. □

Now, we show a preliminary result before stating the main theorem of this subsection.

Lemma 12.3 *Let Q_d be a quasi-interpolant of even degree d defined on the uniform partition of the interval $[a, b]$ of meshlength h. Assume that $f, g, k \in C[a, b]$ and $\dfrac{\partial\psi}{\partial u} \in C^{d+2}\big([a, b] \times [a, b]\big)$. Then we have*

$$\Big\|L\big(I - Q_d\big)g\Big\|_\infty = O\big(h^{d+2}\big).$$

Proof By definition of L we have:

$$\Big(L\left(I - Q_d\right) g\Big)(s) = \int_a^b k(s,t)\frac{\partial\psi}{\partial u}\big(t, u^*(t)\big)\left(I - Q_d\right) g(t)dt$$
$$= \int_a^b q(s,t)\left(I - Q_d\right) g(t)dt.$$

where $q(s,t) = k(s,t)\dfrac{\partial\psi}{\partial u}\big(t, u^*(t)\big)$.
This error corresponds to the error of the quadrature formula $\mathcal{I}_{Q_d}$ based on a quasi-interpolant Q_d with a certain weight function $q(s,t)$, when from the Sect. 12.3, if d is even, the order of convergence h^{d+2} is obtained. □

Theorem 12.7 *Assume that the assumptions of Theorem 12.6 are satisfied. Then the iterated solution of the collocation type method based on quasi-interpolant Q_d of even degree d satisfies*

$$\|u^* - \hat{u}_n^c\|_\infty = O\big(h^{d+2}\big).$$

Proof From Theorem 12.6, we have

$$\|u^* - \hat{u}_n^c\|_\infty \le \xi\,\|u^* - u_n^c\|_\infty^2 + a\,\Big\|L\big(I - Q_d\big)\mathcal{K}u^*\Big\|_\infty$$
$$+ a\,\Big\|L\big(I - Q_d\big)L\big(u^* - u_n^c\big)\Big\|_\infty.$$

On the one hand, using the error of the approximate solution and the previous lemma, it holds

$$\|u^* - u_n^c\|_\infty^2 = O\big(h^{2d+2}\big), \tag{12.24}$$

$$\Big\|L\big(I - Q_d\big)\mathcal{K}u^*\Big\|_\infty = O\big(h^{d+2}\big). \tag{12.25}$$

On the other hand, we have

$$\Big\|L\big(I - Q_d\big)L\big(u^* - u_n^c\big)\Big\|_\infty \le \xi\,\Big\|\big(I - Q_d\big)L\Big\|_\infty\,\|u^* - u_n^c\|_\infty.$$

Furthermore, it is easy to see that

$$\left\|(I-\mathcal{Q}_d)L\right\|_\infty = \mathcal{O}\big(h^{d+1}\big).$$

Then

$$\left\|L\big(I-\mathcal{Q}_d\big)L\big(u^*-u_n^c\big)\right\|_\infty = \mathcal{O}\big(h^{2d+2}\big). \tag{12.26}$$

Using (12.24), (12.25) and (12.26), we deduce that

$$\left\|u^*-\hat{u}_n^c\right\|_\infty = \mathcal{O}\big(h^{d+2}\big).$$

which completes the proof of theorem. □

We recall that $\mathcal{Q}_2$ is superconvergent on the set of evaluation points $\mathcal{X}_n$ (from Theorem 12.1). Therefore the following corollary holds.

Corollary 12.1 *Let u_n^c be collocation approximate solution obtained by using the spline quasi-interpolant $\mathcal{Q}_2$. Then, we have the following superconvergence properties*

$$\begin{aligned}|u^*(x_i)-u_n^c(x_i)| &= \mathcal{O}(h^4), \quad 0\le i\le n,\ i\ne 1, n-1.\\ |u^*(t_i)-u_n^c(t_i)| &= \mathcal{O}(h^4), \quad 0\le i\le n+1,\ i\ne 1,2,n-1,n.\end{aligned} \tag{12.27}$$

Proof From the Eq. (12.16) and Theorem 12.1 we obtain

$$\begin{aligned}|u^*(\xi_i)-u_n^c(\xi_i)| &= |u^*(\xi_i)-\mathcal{Q}_2\hat{u}_n^c(\xi_i)|\\ &\le |u^*(\xi_i)-\hat{u}_n^c(\xi_i)| + |\hat{u}_n^c(\xi_i)-\mathcal{Q}_2\hat{u}_n^c(\xi_i)|\\ &= \mathcal{O}(h^4),\end{aligned}$$

where ξ_i are either x_i or t_i given in (12.27), hence the result. □

12.6 Numerical Results

In this section, we consider three examples of Hammerstein integral equations to illustrate the theory established in previous sections for collocation-type method, its iterated solution and Nyström method. As quasi- interpolating operators we use those given by (12.4) and (12.5) for the quadratic case, and by (12.6) and (12.7) for the cubic case. Note that the different nonlinear systems were solved using a *Newton-Raphson* algorithm.

For successively doubled values of n, we compute the maximum absolute errors

$$E_\infty^c := \|u^* - u_n^c\|_\infty, \;\; \hat{E}_\infty^c := \|u^* - \hat{u}_n^c\|_\infty, \;\; E_\infty^N := \|u^* - u_n^N\|_\infty,$$

and the maximum absolute error at the superconvergent points given by

$$ES^c = \max_i |u^*(\xi_i) - u_n^c(\xi_i)|.$$

where ξ_i are either x_i or t_i given in (12.27). Moreover, we present the corresponding numerical convergence orders $\mathcal{NCO}$, obtained by the logarithm to base 2 of the ratio between two consecutive errors.

The following table gives the data of the three examples of equations considered. For all theses examples, we note that $[a, b] = [0, 1]$.

	Kernel k	Function ψ	Second member f	Exact solution u^*
Example 1	$\pi x \sin(\pi t)$	$\dfrac{t}{1+(u(t))^2}$	$\sin\left(\dfrac{\pi}{2}x\right) - 2x\ln(24 - 16\sqrt{2})$	$\sin\left(\dfrac{\pi}{2}x\right)$
Example 2	$x + 2\sin\left(\dfrac{\pi t}{4}\right)$	$-\big(u(t)\big)^2$	$\dfrac{4(4-\sqrt{2}) + 3(2+\pi)x}{6\pi} + \cos\left(\dfrac{\pi}{4}x\right)$	$\cos\left(\dfrac{\pi}{4}x\right)$
Example 3	xt	$(u(t))^3$	$\exp(-2x) - \dfrac{1}{36}\left(1 - \dfrac{7}{\exp(6)}\right)$	$\exp(-2x)$

- **Case of quadratic quasi-interpolants $\mathcal{Q}_2$**
  The obtained results are reported in Tables 12.1, 12.2 and 12.3, which confirm the theoretical convergence orders predicted theoretically for each method. Moreover, we notice that the approximate collocation solution is superconvergent at x_i and t_i as stated in Corollary 12.1.
- **Case of cubic quasi-interpolants $\mathcal{Q}_3$**
  The obtained results are reported in Tables 12.4, 12.5 and 12.6. In this case the degree of the quasi-interpolant is odd and the theoretical results obtained previously are well confirmed.

Table 12.1 E_∞^c, ES^c, $\hat{E}_\infty^c$, E_∞^N and corresponding $\mathcal{NCO}$

Example 1								
n	E_∞^c	$\mathcal{NCO}$	ES^c	$\mathcal{NCO}$	$\hat{E}_\infty^c$	$\mathcal{NCO}$	E_∞^N	$\mathcal{NCO}$
8	1.25(−04)	–	2.53(−05)	–	9.09(−06)	–	5.83(−05)	–
16	1.57(−05)	2.99	1.62(−06)	3.96	5.84(−07)	3.96	2.23(−06)	4.71
32	1.97(−06)	3.00	1.02(−07)	3.99	3.61(−08)	4.02	8.82(−08)	4.66
64	2.29(−07)	3.10	6.73(−09)	3.92	1.86(−09)	4.28	3.85(−09)	4.52
Theoretical value	–	03	–	04	–	04	–	04

Table 12.2 E^c_∞, ES^c, $\hat{E}^c_\infty$, E^N_∞ and corresponding NCO

Example 2								
n	E^c_∞	NCO	ES^c	NCO	$\hat{E}^c_\infty$	NCO	E^N_∞	NCO
8	1.08(−05)	–	8.10(−06)	–	6.71(−07)	–	6.17(−06)	–
16	1.37(−06)	2.97	1.06(−06)	2.94	4.36(−08)	3.95	4.08(−07)	3.92
32	1.74(−07)	2.98	1.35(−07)	2.97	2.65(−09)	4.04	2.61(−08)	3.97
64	2.01(−08)	3.10	1.69(−08)	3.00	1.62(−10)	4.03	1.64(−09)	3.98
Theoretical value	–	03	–	04	–	04	–	04

Table 12.3 E^c_∞, ES^c, $\hat{E}^c_\infty$, E^N_∞ and corresponding NCO

Example 3								
n	E^c_∞	NCO	ES^c	NCO	$\hat{E}^c_\infty$	NCO	E^N_∞	NCO
8	2.18(−04)	–	5.63(−05)	–	1.83(−06)	–	5.88(−05)	–
16	2.96(−05)	2.88	4.49(−06)	3.65	2.33(−07)	2.98	5.12(−06)	3.52
32	3.88(−06)	2.93	4.83(−07)	3.22	2.16(−08)	3.43	3.77(−07)	3.76
64	4.63(−07)	3.07	5.68(−08)	3.09	1.15(−09)	4.23	2.56(−08)	3.88
Theoretical value	–	03	–	04	–	04	–	04

Table 12.4 E^c_∞, $\hat{E}^c_\infty$, E^N_∞ and corresponding NCO

Example 1						
n	E^c_∞	NCO	$\hat{E}^c_\infty$	NCO	E^N_∞	NCO
8	7.07(−05)	–	1.15(−05)	–	6.40(−04)	–
16	4.66(−06)	3.92	8.43(−07)	3.77	3.76(−05)	4.09
32	2.93(−07)	3.99	5.49(−08)	3.94	1.48(−06	4.66
64	1.85(−08)	3.98	3.60(−09)	3.93	5.58(−08)	4.73
Theoretical value	–	04	–	04	–	04

Table 12.5 E^c_∞, $\hat{E}^c_\infty$, E^N_∞ and corresponding NCO

Example 2						
n	E^c_∞	NCO	$\hat{E}^c_\infty$	NCO	E^N_∞	NCO
8	4.38(−06)	–	1.18(−06)	–	2.91(−05)	–
16	2.93(−07)	3.90	1.13(−07)	3.38	2.59(−06)	3.49
32	1.91(−08)	3.94	8.29(−09)	3.77	1.80(−07)	3.84
64	1.25(−09)	3.94	5.77(−10)	3.84	1.17(−08)	3.94
Theoretical value	–	04	–	04	–	04

12.7 Conclusions

In this paper we have proposed the Nyström and collocation type methods based on the quasi-interpolants Q_d, and the iterated solution in order to numerically solve the Hammerstein equation, and we also have studied their order of convergence.

Table 12.6 E^c_∞, $\hat{E}^c_\infty$, E^N_∞ and corresponding $\mathcal{NCO}$

Example 3						
n	E^c_∞	$\mathcal{NCO}$	$\hat{E}^c_\infty$	$\mathcal{NCO}$	E^N_∞	$\mathcal{NCO}$
8	1.19(−04)	–	5.98(−06)	–	2.18(−05)	–
16	8.68(−06)	3.76	5.41(−07)	3.46	1.55(−05)	0.49
32	5.81(−07)	3.90	3.77(−08)	3.84	1.85(−06)	3.07
64	3.78(−08)	3.94	2.52(−09)	3.90	1.54(−07)	3.58
Theoretical value	–	04	–	04	–	04

Finally, we have presented some numerical examples, illustrating the approximation properties of the proposed methods.

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