Configurational Forces on Elastic Structures

Davide Bigoni [ID]**, Federico Bosi** [ID]**, Francesco Dal Corso** [ID]**, and Diego Misseroni** [ID]

Abstract The discovery of configurational forces acting on elastic structures and its initial applications are reviewed. Configurational forces are related to the possibility that an elastic structure can change its configuration, thus inducing a variation in the potential energy. This concept has already led to several applications (the elastica arm scale, the dripping of an elastic rod, and the torsional actuator), has been shown to strongly affect stability, and to be related to limbless locomotion. It is believed that these results will open a new research territory in mechanics.

Keywords Eshelbian mechanics · Configurational force · Elastica · Structural stability · Locomotion · Deployable structures

1 Introduction

The concept of configurational force was introduced by Eshelby [1–4] to describe the tendency of defects to move inside solids. In particular, it is postulated that the movement occurs in a way that the total potential energy of the mechanical system decreases, until eventually it reaches a minimum, corresponding to a configuration where these forces vanish. The defects can be massless, such as voids, cracks, vacancies, and dislocations, or can possess a mass, such as inclusions. Within this framework, it may be easier to figure out the movement of a massless defect, e.g. a dislocation in a crystal lattice, than a stiff inclusion in a material. However, the latter can be identified as a portion of material belonging to a phase different from that of the surrounding material, so that the boundary of the inclusion will grow or shrink to minimize the energy.

D. Bigoni (✉) · F. Dal Corso · D. Misseroni
DICAM, University of Trento, via Mesiano 77, Trento, Italy
e-mail: davide.bigoni@unitn.it

F. Bosi
Department of Mechanical Engineering, University College London, Torrington Place, London WC1E 7JE, UK

The configurational force, also called 'Eshelbian', or 'material', or 'driving', or 'non-Newtonian', is defined as the negative gradient of the total potential energy V of a body with respect to a parameter κ determining the configuration of the defect, namely, $-\partial V(\kappa)/\partial \kappa$.

Well-known examples of configurational forces are the Peach-Koehler interaction between dislocations, the crack-extension force in fracture mechanics, or the material force developing on a phase boundary in a solid under loading. These forces are the central concept in Eshelbian mechanics, a well-consolidated and famous theory (see, for instance, the monographs by Gurtin [5], Kienzler and Herrmann [6], and Maugin [7, 8], and the journal special issues by Dascalu et al. [9], and Bigoni and Deseri [10]).

Despite the broad diffusion of Eshelbian mechanics, examples of configurational forces developing and acting on elastic structures were unknown before the work by Davide Bigoni, Federico Bosi, Francesco Dal Corso and Diego Misseroni, the authors of this article, who opened this field in [11]. The key concept relies on introducing a particular constraint, which allows a change in configuration of the elastic system. When this change leads to a variation of the total potential energy of the structure, an Eshelby-like force develops. A simple constraint permitting the development of a configurational force is the sliding sleeve, which constrains a portion of an elastic rod inserted into it. This constraint prevents rotation and transverse displacement, but allows for axial frictionless movement. The portion of the rod inside the sliding sleeve can be considered as a 'defect', which might change its position, so inducing a variation in the total potential energy and generating a related configurational force.

The simplest configurational force is developed at the end of a sliding sleeve constraining an initially straight elastic rod, subject to a transverse load P, Fig. 1a.

This structure was analyzed in a fully non-linear regime and the configurational force was demonstrated using two different approaches, namely, a variational technique and an asymptotic method, the latter based on the definition of an imperfection in the constraint. Finally, the structure has been realized and instrumented, so that the configurational force has been validated through experimental measures.

The presence of the configurational force was also theoretically derived and experimentally verified within a dynamic framework. More specifically, Armanini et al. [12] have shown that the configurational force deeply influences the dynamics of an elastic rod with a lumped mass at one end and constrained with a frictionless sliding sleeve at the other, as shown in Fig. 2. Moreover, it has also been proven that configurational forces acting at the ends of an elastic rod can strongly influence band gaps in the wave dispersion diagram and introduce a nonlinear coupling between longitudinal and transverse displacements [13].

In this article, the effects of configurational forces on one-dimensional elastic structures are reviewed, together with applications to soft devices (the elastica arm scale, the dripping of an elastic rod, and the torsional actuator), implications on stability (the penetration on an elastic blade), and connections with limbless locomotion (the snaking rod).

It is suggested that the concept of configurational force may lead to a new and exciting research field, *configurational structural mechanics*, connected to the devel-
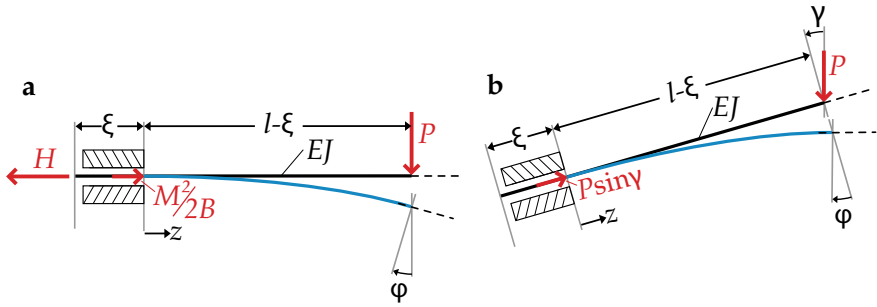
Fig. 1 **a** A simple elastic structure showing the existence of a configurational force. The key to understanding why this force develops is the presence of the sliding sleeve on the left end of an elastic rod of total length l, constraining a length ξ. The elastic rod is subjected to a dead vertical load P on its right end and to an axial dead force H applied at its left end. The presence of the Eshelby-like force $P^2(l - \xi)^2/(2B) = P\phi$ (where $B = EJ$ is the rod's bending stiffness and ϕ is the rotation of the loaded end of the rod) changes the force H at equilibrium, not null whenever the rod is bent. **b** In the absence of the axial force, $H = 0$, the equilibrium of the inclined sliding sleeve (the load has a transverse component equal to $P\cos\gamma$) is obtained only when $\gamma = \phi$. When $\phi < \gamma$ the rod slips inside the sliding sleeve, while, when $\phi > \gamma$, the rod is ejected outside the sliding sleeve
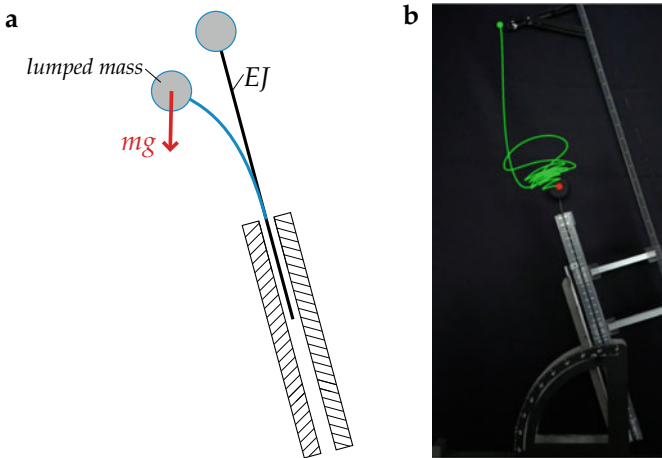

Fig. 2 **a** Schematic of the experiment showing the effects of a configurational force on the dynamics of an elastic rod equipped with a lumped mass. **b** Snapshot of an experiment showing the trajectory of the lumped mass (green line) and evidencing an oscillatory motion during which the elastic rod slips (*into* and *out from* the sliding sleeve) because of the action of a configurational force

opment of devices to be employed in soft robotics, actuation, deployable structures, and deformable sensors.

2 Configurational Force in a Simple Elastic Structure

With reference to Fig. 1a, an elastic rod (straight in its unloaded configuration) of uniform bending stiffness EJ and total length l is considered, constrained by a (frictionless) siding sleeve and subjected to a transverse and an axial force, respectively P and H, applied at the two ends. Under the small rotation hypothesis, the generic configuration of the system is defined by the transverse displacement field $v(z)$, where z is the curvilinear coordinate measured from the sliding sleeve exit. Moreover, the configuration is also described by the configurational parameter ξ, defining the position of the sliding sleeve exit along the rod, and therefore the length $l - \xi$ of the rod undergoing flexural deformations. The total potential energy V can be written as

$$V(v, \xi) = \frac{EJ}{2} \int_0^{l-\xi} \left(v'' \right)^2 dz - P \int_0^{l-\xi} v' \, dz - H\xi, \tag{1}$$

where a dash means derivation with respect to the coordinate z. The transverse displacement field is subjected to the following kinematic conditions at the sliding sleeve exit

$$v(0) = v'(0) = 0. \tag{2}$$

Following a variational approach, the transverse displacement v and the length ξ are subjected to variations $\tilde{v}$ and $\tilde{\xi}$ through a small parameter ϵ as

$$v \longrightarrow v + \epsilon \tilde{v}, \quad \xi \longrightarrow \xi + \epsilon \tilde{\xi}. \tag{3}$$

By considering the kinematic boundary conditions (2), the constraints on the variation $\tilde{v}$ in the transverse displacement field at the sliding sleeve exit are

$$\tilde{v}(0) = \tilde{v}'(0) = 0. \tag{4}$$

Annihilation of the first variation (in ϵ) of the total potential energy V is

$$EJ \int_0^{l-\xi} v'' \tilde{v}'' \, dz - \frac{EJ \left[v''(l - \xi) \right]^2}{2} \tilde{\xi} - P \int_0^{l-\xi} \tilde{v}' \, dz + \left[Pv'(l - \xi) - H \right] \tilde{\xi} = 0. \tag{5}$$

Considering Eq. (4), the first integral in Eq. (5) can be evaluated through integration by parts as

$$\int_0^{l-\xi} v'' \tilde{v}'' \, dz = v''(l - \xi) \, \tilde{v}'(l - \xi) - \int_0^{l-\xi} v''' \tilde{v}' \, dz. \tag{6}$$

and therefore the first variation (5) reduces to

$$-\int_0^{l-\xi} \left(E J \, v''' + P \right) \tilde{v}' \, dz + E J \, v''(l - \xi) \tilde{v}'(l - \xi)$$
$$-\left[H + \frac{E J \left[v''(l - \xi) \right]^2}{2} - P v'(l - \xi) \right] \tilde{\xi} = 0. \tag{7}$$

Taking into account Eq. (4), the integral in Eq. (7) can be further integrated by parts as

$$-\int_0^{l-\xi} \left(E J \, v''' + P \right) \tilde{v}' \, dz = \left[E J \, v'''(l - \xi) + P \right] \tilde{v}(l - \xi) - E J \int_0^{l-\xi} v'''' \, \tilde{v}' \, dz, \tag{8}$$

so that annihilation of the first variation under arbitrary variations $\tilde{v}$ yields to the governing equation of the Euler elastica (in its linearized version)

$$v''''(z) = 0, \qquad z \in [0, l - \xi] \tag{9}$$

complemented, in addition to the kinematic boundary conditions (2), by the static boundary conditions

$$v'''(l - \xi) = -\frac{P}{E J}, \quad \text{and} \quad v''(l - \xi) = 0, \tag{10}$$

representing the shear and moment conditions at the end of the rod, $z = l - \xi$.

The arbitrariness of $\tilde{\xi}$ allows concluding from Eq. (7) that axial equilibrium does not always correspond to $H = 0$ as it would follow by ignoring the configurational force. More specifically, H corresponds to a non-null value when the rod is bent and is expressed by

$$H = P \phi, \tag{11}$$

where ϕ is the rotation at the loaded end, $\phi = v'(l - \xi)$. Through integration of the linearized elastica, Eq. (9), under the boundary conditions (2) and (10), the loaded end rotation ϕ results

$$\phi = \frac{P(l - \xi)^2}{2 E J}, \tag{12}$$

and therefore the equilibrium of the system evidences the presence of a *configurational force*

$$H = \frac{P^2(l - \xi)^2}{2 E J}. \tag{13}$$

If the elastic rod is inclined at an angle γ with respect to the loading P, so that its transverse and axial components are respectively $P \cos \gamma$ and $P \sin \gamma$, the equilibrium equation (11) would change to

$$H + P \sin \gamma = P \phi \cos \gamma, \tag{14}$$

where the rotation ϕ of the loaded end of the rod is now

$$\phi = \frac{P \sin \gamma (l - \xi)^2}{2EJ}. \tag{15}$$

From Eq. (14), the equilibrium in the case of null force $H = 0$ (Fig. 1b) implies the following geometric condition

$$\phi = \tan \gamma, \tag{16}$$

which, assuming the angle γ to be sufficiently small to allow the retainment of the linear term, leads to

$$\phi = \gamma, \tag{17}$$

implying an orthogonality condition between the applied load and the rod's tangent.

The geometric condition (17) to attain equilibrium has been proven to hold also in the case of large rotations and large angles γ and shows that [11]

- when $\phi = \gamma$ the rod is at equilibrium;
- when $\phi < \gamma$ the rod slips inside the sliding sleeve;
- when $\phi > \gamma$ the rod is ejected outside the sliding sleeve.

3   Elastica Arm Scale

With reference to Fig. 3a, the concept underlying the elastica arm scale is the result of nonlinear equilibrium kinematics of a rod constrained centrally by a sliding sleeve, which induces two configurational forces, with outward direction from each sliding sleeve exit and opposite to each other. Therefore, the deflection of the arms becomes necessary for equilibrium, which would be impossible for a rigid system. The rigid arms of ordinary scales are replaced by a flexible elastic lamina, so that the rod can reach a unique equilibrium configuration when two vertical dead loads are applied at its ends. This configuration has been analytically solved by Bosi et al. [14], showing that the knowledge of a given load value P (at the right end) and the measure of the equilibrium length l_{eq} allows for the identification of an unknown load Q (at the left end).

In a sense, the elastica arm scale is realized through the combination of two different mechanical principles, each one underlying the operation of a specific family of classical scales: (i) equilibrium (on which the steelyard scale is based) and (ii)
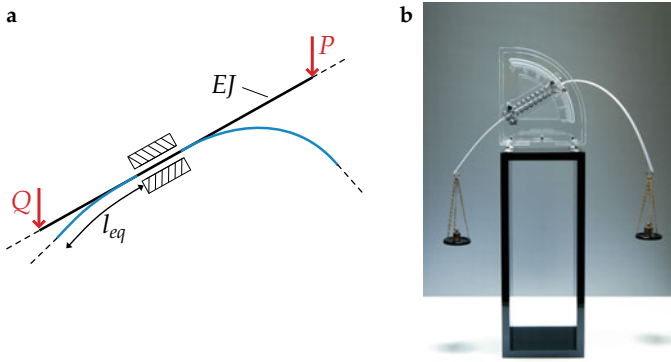
Fig. 3 **a** Schematic of the elastica arm scale based on an inclined sliding sleeve. The equilibrium is attained in the sliding sleeve direction through the contributions of the dead loads P and Q, and the two configurational forces arising from the elastic bending of the rod. **b** A prototype of the elastica arm scale realized by D. Misseroni and donated by the authors to the 'Museo della Bilancia' in Campogalliano (Italy)
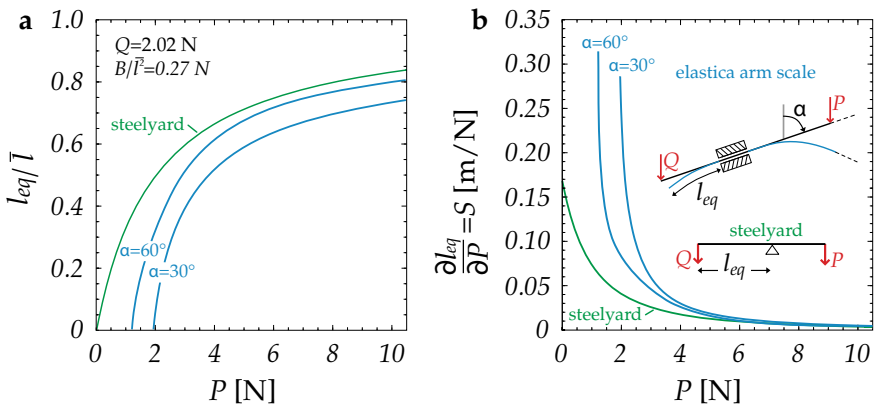

Fig. 4 Comparison between a steelyard scale and the elastica arm scale, the latter inclined at two different angles α. **a** Equilibrium length l_{eq} for different applied loads P. **b** Sensitivity S as a function of the applied load P

deformation (on which the spring scale is based). A comparison between the elastica arm scale and the ordinary steelyard scale is shown in Fig. 4. The equilibrium length l_{eq} and the sensitivity (denoted by S) for different applied loads P are shown in Fig. 4a and b, respectively. It follows that at small given load P, the sensitivity S of the elastica arm scale can become superior to the traditional device.

Prototypes of the elastica arm scale have been designed, realized, and tested at the Instabilities Lab of the University of Trento. The prototype shown in Fig. 3b has been realized by D. Misseroni and donated by the authors to the 'Museo della Bilancia' in Campogalliano (Italy).

The mechanics of the elastica arm scale has been further investigated by O'Reilly [15, 16], who considered the material momentum balance law for rods.

4   Dripping of an Elastic Rod

The application of configurational forces presented in this section addresses the so-called 'self-encapsulation' problem, in which an elastic rod is loaded with a transverse force applied at midspan (between two constraints kept at a fixed distance), with the purpose of reaching a closed deformation loop, thus encapsulating a finite region. Self-encapsulation has connections to micro- or nano-fabrication technologies for deployable structures used in sensors. In this field of application, self-assembly can be achieved through magnetic forces [17], while a self-folding spherical structure has been invented [18] and a dynamic self-encapsulation technique for a thin plate and a rod has already been pointed out [19, 20]. In the former case, only a reduction in the volume of a sphere is achieved, while in the latter self-encapsulation is obtained as a result of both dynamic effects and capillary forces, which are related to the presence of a liquid droplet attached to the rod.

The new idea pursued here is obtained through the use of two sliding sleeves, as illustrated in Fig. 5. These sleeves provide two compressive configurational forces, which tend to 'close' the rod. Since the differential equation of the elastica not only governs the oscillation of a simple pendulum and the deflection of an elastic rod, but also the shape of a droplet, it can be appreciated that the process of encapsulation will be similar to the formation of a droplet, from which the idea of 'dripping of an elastic
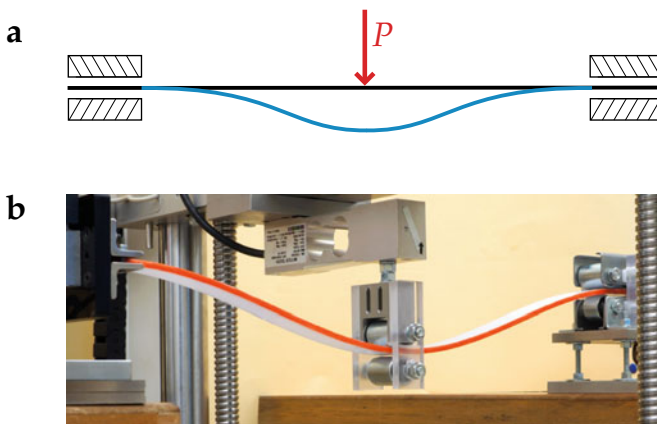

Fig. 5 a The encapsulation of an elastic rod, loaded with a transverse force P applied at midspan, is obtained through the use of two sliding sleeves. The sliding sleeves provide two equal and opposite configurational forces, driving the encapsulation mechanism. **b** Experimental setup designed to perform experiments on self-encapsulation
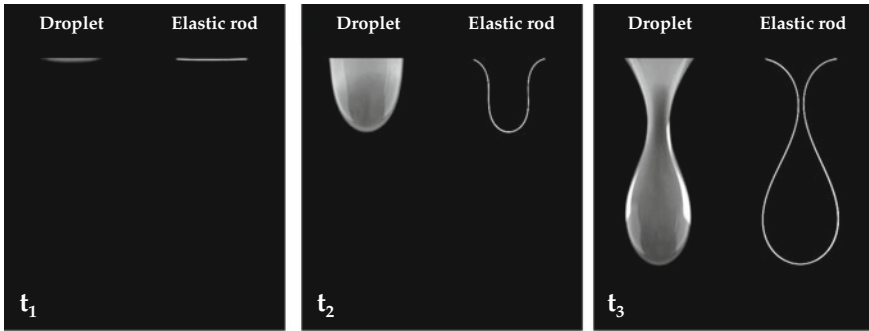
Fig. 6 The dripping of an elastic rod is compared with the progressive formation of an oil drop at three different configurations

rod' arose. The encapsulation driven by Eshelby-like forces was demonstrated to be always possible (for every rod geometrical configuration) by Bosi et al. [21]. The dripping of an elastic rod is shown in Fig. 6, where it is compared with the process of formation of an oil droplet.

5 Penetrating Blades

Configurational forces deeply influence the stability of structures. A paradigmatic example has been presented by Bigoni et al. [22] and is sketched in Fig. 7, where an elastic rod is loaded with a dead vertical force P applied at one end and it can slide inside a sliding sleeve present at the other end, while being axially restrained by a linear spring of stiffness k.

This structure is a generalization of that reported in [23], used to show two counter-intuitive effects: (i) an increase in the stiffness k of the restraining spring *lowers* the buckling load and (ii) the straight configuration of the elastic rod may return to be stable at an axial load higher than that triggering buckling.

During the post-critical deformation of the rod sketched in Fig. 7, a configurational force progressively develops. This force is vertical and directed upwards, so that it lifts the rod and sometimes it can become so large that the rod is expelled from the sliding sleeve. The configurational force also plays an important role in the restabilization problem, when the straight configuration spontaneously returns stable after bifurcation, Bosi et al. [24], or when a rod is injected inside a constraint [25]. Further problems of instability involving configurational forces were analyzed by Liakou et al. [26–28].
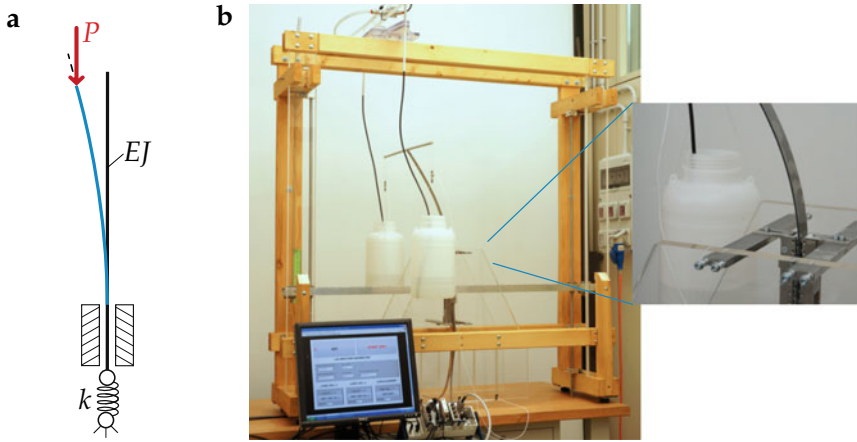
Fig. 7 Instability of a penetrating blade: the elastic rod is loaded axially with a vertical dead force P and it can slide inside a sliding sleeve placed at the other end, while being restrained by a linear spring of stiffness k. Three important features emerge from this problem: (i) an increase in the stiffness of the spring leads to a decrease of the buckling load; (ii) the straight configuration is stable for loads smaller than the buckling load, but can return stable at higher loads; (iii) during the post-critical behaviour a vertical upward configurational force develops, which could even eject the rod from the sleeve. **a** Schematic of the penetrating blade problem. **b** Experimental setup realized for the experimental validation of the features associated with the mechanics of the system

6 Torsional Actuator

The same concept ruling the development of a configurational force in a rod under bending can be applied under a state of torsion. In this case, Bigoni et al. [29] have proven that the configurational force is given by

$$\frac{M^2}{2D},\tag{18}$$

where M is the twisting moment and D is the torsional rigidity of the rod.

This configurational force may be used to create an actuator, turning a twist into a longitudinal motion, as sketched in Fig. 8.

The actuator represents a purely 'elastic machine' or 'soft device', in which the longitudinal trust is obtained without the use of gears or other transmission mechanisms.

Fig. 8 The torsional actuator, based on the development of a configuration force, transforms a twist into a longitudinal trust through a release of the elastic energy, without any use of gears or other transmission mechanisms

7 Snaking of an Elastic Rod

In the problem of 'snaking', an elastic rod, straight when unloaded, is inside a curved channel. The channel is perfectly frictionless, rigid and tight to the rod. In this situation, a propulsion force is developed from a release of elastic flexural energy and the rod's movement is realized. Such a propulsion force is a configurational force, which was theoretically estimated and experimentally demonstrated by Dal Corso et al. [30]. The obtained propulsion force has been suggested to represent the key in the limbless locomotion, typical of serpentine motion [31–34].

8 Conclusions

The presence of configurational forces on structures was discovered for the first time, analyzed and developed by the authors of this article in the last decade. These forces have been theoretically demonstrated, experimentally validated and implemented in the realisation of novel reconfigurable elastic devices. They have been shown to strongly affect the behaviour of elastic structures and led to the discovery of several new effects. These results inspire new interpretations of configurational forces in solids [35] and pave the way to the new territory of configurational structural mechanics, towards innovative applications of reconfigurable mechanisms in several fields ranging from nanomedicine to aerospace.

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