Strain-Difference Based Nonlocal Elasticity Theories: Formulations and Obtained Results

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Abstract The contributions to nonlocal elasticity given by the authors in the last two decades are reported in this article. To better illustrate the above contributions and their pertinence to the nowadays research framework, we start with a scrutiny of the inconsistencies encountered within the Eringen's purely nonlocal model and the remedies required to overcome shortcomings and paradoxical situations known from the literature. It is shown that the so-called strain-difference based nonlocal theories encompassing the mentioned contributions provide effective methods to address boundary-value problems. Applications to plates by nonlocal finite elements and size effects analysis of beams in bending have been reported as illustrative examples of previously obtained results.

Keywords Elasticity theory · Nonlocal elasticity · Eringen's integral nonlocal theory · Eringen's differential nonlocal theory

1 Introduction

The roots of nonlocal elasticity can be traced back to the continuum theories by [1–3], as well as to the multipolar elasticity theory by [4]. Eringen and Edelen [5–9] formulated nonlocal elasticity theories featured by the presence of quantities called residuals with which the nonlocal nature of fields as body forces, mass, stress, entropy, etc., is assessed which makes them rather cumbersome for application purposes.

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Eringen and co-workers (see e.g. [10–13]) simplified the above elasticity theories in such a way that the nonlocal theory and the classic one differ from each other only in the stress-strain constitutive equation, but leaving unaltered the equilibrium equations and the kinematic relationships. Notably, with the nonlocal constitutive equation the stress at a point is expressed as a weighted mean value over a neighbor region of that point. For this purpose, a convolution operation upon the strain field is used, which is featured by a nonlocal kernel being a function of the Euclidean distance between the stress and strain points. Rogula, [14], provided a mathematical definition of the nonlocality concept based upon the existence of an internal length scale as a material parameter. Variational principles within this nonlocal elasticity theory were developed by Polizzotto [15].

The above simplified Eringen's nonlocal model was widely used for both theoretical and engineering applications not only within elasticity, but also in many other fields as plasticity, crack and damage mechanics, dislocation theory, etc. There exists a huge literature on these matters, for which reference is made to more specific works as: [16–18] for plasticity, crack mechanics and damage mechanics; [19] for nonlocal theories of elasticity, thermo-elasticity and electro-magneto-elasticity. The above nonlocal elasticity theory developed by Eringen and co-workers will be referred to as the Eringen's (integral) nonlocal theory, or Eringen's (integral) nonlocal model, in the following.

An important aspect of a nonlocal continuum theory with respect to the classical local one is that, on addressing a boundary-value problem, the former leads to governing integro-differential equations carrying in more computational difficulties than the governing differential equations to which one arrives with the latter. Eringen, [20], provided a method to address an integral nonlocal boundary-value problem by means of a differential equation of which the Green function coincides with the kernel of the integro-differential equation. After this step by Eringen, a new branch of nonlocal elasticity grew up with the name of *differential nonlocal elasticity*. This was widely applied to beam, plate and shell models simulating sensor and actuator devices within micro- and nano-technologies in the purpose to solve various engineering problems as buckling, vibrations and wave propagation problems, along with size effect analysis, for which reference is made to the review papers [21–23].

1.1 Inconsistencies of the Eringen's Nonlocal Model. Remedies

Notwithstanding the notable success of the Eringen's nonlocal theory, some inconsistencies were soon discovered and discussed [24], as described hereafter.

One such inconsistency originates from the Eringen's integral nonlocal model leading to ill-posed boundary-value problems. In fact, the inherent integro-differential governing equation, viewed as integral equation, falls into the category of Fredholm integral equations of the first kind which —as known from integral equation theory

[25]—admits multiple solutions, or no solutions at all. After the contributions by [26, 27], it is known nowadays that a necessary and sufficient condition in order that an integral nonlocal model admits a (unique) solution is that some special boundary conditions (called also "constitutive" BCs, or even "nonlocality" BCs) must be not in contrast with the imposed traction boundary conditions. A remedy to the above drawback was proposed by Eringen, [5, 28], and implemented by many others [15, 29–37]. With this formulation the fully nonlocal model is replaced with a two-phase local/nonlocal mixture model, which leads to Fredholm integral equations of the second kind, hence to well-posed boundary-value problems. Another remedy to the Eringen's integral nonlocal model was proposed by [38] with a new formulation in which the basic concepts of the Eringen's integral nonlocal model are saved, but the strain and stress play therein interchanged roles.

A second inconsistency of the Eringen's integral nonlocal model is more directly related to the differential form of it. Eringen [20] likely proposed this model to address problems with infinite domain (like crack tip singularities, wave propagation, and the like). Peddieson and co-workers [39] first used the differential nonlocal model for size effects analysis of micro- and nano-beams. It was found that the model generally predicts softening effects on the stiffness with increasing the length scale parameter of the beam, but may also predict hardening (as for a cantilever beam under uniform load), or even no size effects at all (as in the so-called "paradox" case, namely, a cantilever beam under point load at the free end), all apparently without a precise rule. The right motivation for which the above anomalous behavior gets out is likely due to the conjugate governing differential equation having a degree equal to that of the integro-differential equation. Therefore the differential-based solution *cannot in general coincide* with the integral-based solution (if it exists), due to the impossibility to implement the nonlocality BCs as side conditions.

A third inconsistency of the Eringen's nonlocal model has been identified with its property of not saving uniform local fields. This implies that, in the case of homogeneous nonlocal elastic material, the stress corresponding to a uniform strain is not uniform in general, except in an infinite domain. In other words, the Eringen nonlocal model does not comply with the *locality recovery condition* [40], that is, the condition for which the material behaves as a local material under a uniform strain field. A milder form of this condition is the *local stress recovery condition* [41], in which the stress is uniform under a uniform strain, but the material still saves some nonlocality features. A remedy to this kind of inconsistency, proposed by Polizzotto and co-workers [41], is in the form of a two-phase local/nonlocal model in which the nonlocal phase is driven by the strain difference measured at the generic point with respect to the reference point where the stress is evaluated. This model called *strain-difference based nonlocal model*, automatically satisfies the local stress recovery condition; it also leads to Fredholm integral equations of the second kind and thus to well-posed boundary-value problems. Another form of strain-difference based nonlocal model was also proposed in [40], which complies with the more stringent locality recovery condition, that is, under uniform strain, not only the stress is uniform, but the inherent Helmholtz energy potential looses its dependence on the length scale parameter.

1.2 Objectives and Outline

The purpose of the present paper is to give an insight over the family of strain-difference based nonlocal models developed by the authors in the last two decades within elasticity. To be concise, the mentioned family is reduced to two basic models, of which one (called "first type") complies with the local stress recovery condition, the other (called "second type") complies with the more stringent locality recovery condition and both models lead to well-posed boundary value problems. The paper is organized as follows. After the introductory arguments in Sect. 1, some preliminary considerations are reported in Sect. 2. Sections 3 and 4 constitute the central part of the paper, in which the two strain-difference based models are presented. Applications of these models to engineering problems are reported in Sect. 5. Section 6 concludes the paper.

2 Preliminaries to Eringen's Nonlocal Elasticity

In this section, the Eringen's nonlocal constitutive model of continuum elasticity is recalled in its fully nonlocal integral form. For this purpose, a 3D (finite) solid body of domain V is considered within a $\mathbb{R}^3$ space, which before deformation is referred to a Cartesian orthogonal co-ordinate system, say $\mathbf{x} = (x_1, x_2, x_3)$. The body is constrained at a portion, say S_c, of its boundary surface $S = \partial V$, in such a way as to impede any rigid motion. The body is also subjected to external actions, which are assumed in the form of body forces, say $\mathbf{b}(\mathbf{x})$ within V (N/m^3) and surface forces, or tractions, say $\mathbf{p}(\mathbf{x})$, applied on the free portion $S_f = S \backslash S_c$ (N/m^2). All these forces vary in time in a quasi-static manner.

Eringen and co-workers, [10–13, 19], proposed a nonlocal constitutive model for elastic materials, which in the common case of homogeneous material, is expressed as

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C} : \int_V g(\mathbf{x}, \mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V' \quad \forall \mathbf{x} \in V \tag{1}$$

where the colon product denotes double index contraction operations (e.g. $(\mathbf{C} : \boldsymbol{\varepsilon})_{ij} = C_{ijkl}\varepsilon_{kl}$) and $\mathrm{d}V' = \mathrm{d}V(\mathbf{x}')$. The stress $\boldsymbol{\sigma}(\mathbf{x})$ of Eq. (1) is the "nonlocal" stress arising at the field, or reference, point $\mathbf{x} \in V$, $\boldsymbol{\varepsilon}(\mathbf{x}')$ is the "local" strain acting at the generic source point $\mathbf{x}' \in V$. Also, the two-point symmetric function $g(\mathbf{x}, \mathbf{x}')$ is the *kernel* function (called also *attenuation* or *influence* function after Eringen), which is positive definite and generally is assumed to depend on the field and source points $\mathbf{x}, \mathbf{x}'$ through the Euclidean distance $r = |\mathbf{x} - \mathbf{x}'|$. For mathematical convenience, often in the literature the kernel g is taken in the form of an exponential as

$$g(\mathbf{x}, \mathbf{x}') = k_0 \exp\Big(-\frac{r}{c}\Big), \quad r = |\mathbf{x} - \mathbf{x}'| \tag{2}$$

where c denotes the *length scale parameter*. The coefficient k_0 is determined through the *normalization condition*

$$\int_{V_\infty} g(\mathbf{x}, \mathbf{x}') \, \mathrm{d}V' = 1 \tag{3}$$

where V_∞ is the infinite domain ($\mathbb{R}^3$) in which V is embedded, [16, 19]. In the Introduction, it was pointed out that the Eringen's nonlocal constitutive stress-strain relation (1) leads to ill-posed boundary-value problems whereby there may be either a multiple solution, or more likely no solution at all, due to the impossibility to conciliate the boundary traction conditions with the nonlocality BCs, [26, 27].

In our treatment of nonlocal elasticity, we make reference to a two-phase local/nonlocal model cast in the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C} : \left[\xi \boldsymbol{\varepsilon}(\mathbf{x}) + (1 - \xi) \underbrace{\int_V g(\mathbf{x}, \mathbf{x}') \boldsymbol{\varepsilon}(\mathbf{x}') \, \mathrm{d}V'}_{\mathbf{E}(\mathbf{x})} \right] \tag{4}$$

This relation strongly appeals to a two-phase local/nonlocal mixture model with the coefficient ξ playing the role of local phase parameter, but also that of an (essentially positive) material constant, not necessarily less than 1. The two-phase model (4) leads to well-posed boundary-value problems. The stress $\boldsymbol{\sigma}(\mathbf{x})$ is there expressed linearly in terms of the local strain $\boldsymbol{\varepsilon}(\boldsymbol{x})$ along with the *strain integral* $\mathbf{E}(\mathbf{x})$ given by the formula

$$\mathbf{E}(\mathbf{x}) = \mathscr{R}(\boldsymbol{\varepsilon})(\mathbf{x}) := \int_V g(\mathbf{x}, \mathbf{x}') \boldsymbol{\varepsilon}(\mathbf{x}') \, \mathrm{d}V' \tag{5}$$

where $\mathscr{R}(\cdot)$ denotes the nonlocal operator acting on $(\cdot)$.

For a correct thermodynamic treatment of the considered two-phase model let us assume, in agreement with the stress-strain relation (4), the existence of an internal energy potential, say $u = u(\boldsymbol{\varepsilon}, \mathbf{E}, \eta)$, where η is the entropy density, [15, 19, 42]. Assuming isothermal conditions for simplicity, the energy balance principle (or first thermodynamics principle) can be cast in a point-wise form as

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + P \qquad \text{in } V \tag{6}$$

where P is the (nolocality) *energy residual*, that is, the energy density transmitted to the generic particle from all other particles within the body as a consequence of the nonlocal nature of the material, [9, 15, 19, 42]. The following *insulation condition* has to be satisfied, [9], that is

$$\int_V P(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) \, \mathrm{d}V = 0 \tag{7}$$

holding for any deformation mechanism and signifying that the particle system is constitutively insulated within V, that is, no long distance energy is transmitted to the body from the exterior environment due to the nonlocal behaviour of the material.

Let the Helmholtz free energy $\psi = \psi(\boldsymbol{\varepsilon}, \mathbf{E})$ be introduced using the Legendre transform $\psi = u - T\eta$, with $T > 0$ being the (constant) absolute temperature. The energy balance (6) may then be rewritten as

$$T\dot{\eta} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} + P \geq 0 \qquad \text{in } V \tag{8}$$

Here, the non-negativity sign on the r.h.s. is introduced to enforce the second thermodynamics principle. In this way, inequality (8) identifies itself with the Clausius-Duhem inequality, [43, 44], which differs from its classical counterpart only for the presence of the energy residual P. Was P identically vanishing for any deformation mechanism, then the material would be a simple material.

Let inequality (8) be integrated over V to obtain, recalling (7) and using the appropriate Green identity, the following inequality

$$\int_V \left[\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \mathscr{R}\left(\frac{\partial \psi}{\partial \mathbf{E}} \right) \right] : \dot{\boldsymbol{\varepsilon}} \, \mathrm{d}V \geq 0 \tag{9}$$

As this has to hold for arbitrary deformation mechanism $\dot{\boldsymbol{\varepsilon}}$, a necessary and sufficient condition of (9) is the state equation

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} + \mathscr{R}\left(\frac{\partial \psi}{\partial \mathbf{E}} \right) \tag{10}$$

where $\boldsymbol{\sigma}$ represents the Cauchy stress work-conjugate to $\dot{\boldsymbol{\varepsilon}}$. Equation (10) implies that (9) is satisfied as an equality and therefore

$$P = \dot{\psi} - \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = \frac{\partial \psi}{\partial \mathbf{E}} : \mathscr{R}(\dot{\varepsilon}) - \mathscr{R}\left(\frac{\partial \psi}{\partial \mathbf{E}} \right) : \dot{\varepsilon} \qquad \text{in } V \tag{11}$$

which is the constitutive equation for P.

3 Strain-Difference Based Nonlocal Models of First Type

In this section, we start to provide strain-difference based nonlocal models. Two types of such models will be presented, that is, first type models in the present section, second type models in the next section. In this section the material is characterized by a Helmholtz free energy function of the strain $\boldsymbol{\varepsilon}$ and strain integral $\mathbf{E}$ specified by (5), cast as a quadratic form as

$$\psi = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbf{C}_\infty : \boldsymbol{\varepsilon} + \frac{1}{2} \boldsymbol{\varepsilon} : \mathbf{C}_1 : \mathbf{E} + \frac{1}{2} \mathbf{E} : \mathbf{C}_2 : \mathbf{E} \tag{12}$$

which constitutes an extension of the treatment reported in [41]. Since $\mathbf{E} \to \mathbf{0}$ as the length scale parameter $c \to \infty$, the moduli tensor $\mathbf{C}_\infty$ characterizes the asymptotic behavior of the material; the tensors $\mathbf{C}_1$, $\mathbf{C}_2$ are defined in terms of the moduli tensor of classic anisotropic elasticity, $\mathbf{C}$, that is, $\mathbf{C}_1 = \alpha_1\mathbf{C}$, $\mathbf{C}_2 = \alpha_2\mathbf{C}$, with α_1, α_2 being suitable scalar coefficients. Both $\mathbf{C}_\infty$ and $\mathbf{C}$ are considered nonhomogeneous within V.

Let the body be subjected to external quasi-static actions as specified at the beginning of Sect. 2. We go to determine the governing equations of the inherent boundary-value problem within the framework of small displacements and linearized elasticity using the principle of the minimum total potential energy of nonlocal elasticity, [15], through the functional $\Sigma[\mathbf{u}]$ here cast in the form

$$\Sigma[\mathbf{u}] := \int_V \psi(\boldsymbol{\varepsilon}, \mathbf{E})\,\mathrm{d}V - \int_V \mathbf{b}\cdot\mathbf{u}\,\mathrm{d}V - \int_{S_f} \mathbf{p}\cdot\mathbf{u}\,\mathrm{d}S \tag{13}$$

Here, $\psi(\boldsymbol{\varepsilon}, \mathbf{E})$ is the functional (12), whereas $\mathbf{u}$ denotes the inherent displacements. The functional (13) has to be minimized under the kinematic restrictions relating $\boldsymbol{\varepsilon}$ to $\mathbf{u}$, as well as the imposed displacements $\mathbf{u} = \mathbf{u}_c$ on S_c. Following a straightforward procedure, [15], one easily arrives at the *total Cauchy stress* $\boldsymbol{\sigma}(\mathbf{x})$ cast in the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}_\infty(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) + \int_V \mathbf{k}(\mathbf{x}, \mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V' \tag{14}$$

The (symmetric) kernel $\mathbf{k}(\mathbf{x}, \mathbf{x}')$ here above is the *nonlocal anisotropic moduli tensor*, that is,

$$\mathbf{k}(\mathbf{x}, \mathbf{x}') = \alpha_1\mathbf{k}_1(\mathbf{x}, \mathbf{x}') + \alpha_2\mathbf{k}_2(\mathbf{x}, \mathbf{x}') \tag{15}$$

where

$$\left.\begin{aligned} \mathbf{k}_1(\mathbf{x}, \mathbf{x}') &:= \frac{1}{2}\Big[\mathbf{C}(\mathbf{x}) + \mathbf{C}(\mathbf{x}')\Big]g(\mathbf{x}, \mathbf{x}') \\ \mathbf{k}_2(\mathbf{x}, \mathbf{x}') &:= \int_V \mathbf{C}(\mathbf{z})g(\mathbf{x}, \mathbf{z})g(\mathbf{x}', \mathbf{z})\mathrm{d}V(\mathbf{z}) \end{aligned}\right\} \tag{16}$$

The Cauchy stress $\boldsymbol{\sigma}(\mathbf{x})$ of (14) collects a local contribution at $\mathbf{x} \in V$, along with two types of long distance contributions, one of which is the result of the interaction of every two particles $(\mathbf{x}', \mathbf{x})$ $\forall\mathbf{x}' \in V$, the other of every three particles $(\mathbf{x}', \mathbf{z}, \mathbf{x})$ $\forall(\mathbf{x}', \mathbf{z}) \in V$.

Since the $\boldsymbol{\sigma}(\mathbf{x})$ of (14) does not comply with the locality recovery condition, Eq. (14) is replaced with

$$\boldsymbol{\sigma}(\mathbf{x}) = \alpha_0\mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) + \int_V \mathbf{k}(\mathbf{x}, \mathbf{x}') : [\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x})]\,\mathrm{d}V' \tag{17}$$

Equation (17) shows that $\boldsymbol{\sigma}(\mathbf{x}) = \alpha_0\mathbf{C}(\mathbf{x}) : \bar{\boldsymbol{\varepsilon}}$ for $\boldsymbol{\varepsilon}(\mathbf{x}) = \bar{\boldsymbol{\varepsilon}}$ in V. Indeed, the *local stress recovery condition* is satisfied, but not the *locality recovery condition*, since in fact for $\boldsymbol{\varepsilon}(\mathbf{x}) = \bar{\boldsymbol{\varepsilon}}$, ψ can be shown to still have a *nonlocal* form as

$$\psi = \frac{1}{2}\,\bar{\boldsymbol{\varepsilon}} : \mathbf{C}_\infty(\mathbf{x}) : \bar{\boldsymbol{\varepsilon}} + \Big[\alpha_1\gamma(\mathbf{x}) + \alpha_2\gamma^2(\mathbf{x})\Big]\frac{1}{2}\,\bar{\boldsymbol{\varepsilon}} : \mathbf{C}(\mathbf{x}) : \bar{\boldsymbol{\varepsilon}} \tag{18}$$

which contains c through the function

$$\gamma(\mathbf{x}) = \int_V g(\mathbf{x}, \mathbf{x}')\,\mathrm{d}V' \tag{19}$$

The variational principle invoked previously led us to two forms of the stress-strain relation, namely (14) pertaining to the original *strain-integral model* characterized by an asymptotic moduli tensor $\mathbf{C}_\infty(\mathbf{x})$, and (17), pertaining to the *strain-difference based model* characterized by an asymptotic moduli tensor $\alpha_0\mathbf{C}$, $\alpha_0 > 0$. The variational principle also leads us to the equilibrium equations with which the stress $\boldsymbol{\sigma}$ is required to comply, that is,

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{b} = 0 \quad \text{in} \quad V, \qquad \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{p} \quad \text{on} \quad S_f \tag{20}$$

where $\mathbf{n}$ denotes the unit external normal to $S = \partial V$, along with the kinematic equations, that is,

$$\boldsymbol{\varepsilon} = \frac{1}{2}\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\Big) \quad \text{in} \quad V, \qquad \mathbf{u} = \mathbf{u}_c \quad \text{on} \quad S_c \tag{21}$$

Equation (20) and (21), typical of linearized elasticity, associated with (17) form a consistent set of equations governing a well-posed boundary-value problem. It in fact leads to an integro-differential equation of the second differential order in the displacement $\mathbf{u}$ constituting a Fredholm integral equation of the second kind. This admits a unique solution free from any paradoxical condition and additionally satisfies the local stress recovery condition.

4 Strain-Difference Based Nonlocal Models of Second Type

In this section, a strain-difference based nonlocal model complying with the locality recovery condition is reported. Here, the material is characterized by a Helmholtz free energy $\psi(\boldsymbol{\varepsilon}, \mathbf{E}_d)$ defined as

$$\psi = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\,\mathbf{E}_d : (\alpha\mathbf{C}) : \mathbf{E}_d \tag{22}$$

where $\mathbf{E}_d = \mathscr{R}(\mathscr{D}\boldsymbol{\varepsilon})$ = *strain difference integral* in which

$$\mathscr{D}\boldsymbol{\varepsilon}(\mathbf{x}, \mathbf{x}') = \boldsymbol{\varepsilon}(\mathbf{x})' - \boldsymbol{\varepsilon}(\mathbf{x}) \quad \forall(\mathbf{x}, \mathbf{x}') \quad \in V \tag{23}$$

and then

$$\mathbf{E}_d = \mathscr{R}(\mathscr{D}\boldsymbol{\varepsilon})(\mathbf{x}) = \int_V g(\mathbf{x}, \mathbf{x}')\big[\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x})\big]\,\mathrm{d}V' \tag{24}$$

The strain-difference based nonlocal model here considered is a phenomenological model accounting for inhomogeneity of the moduli tensor $\mathbf{C}(\mathbf{x})$ and of the length scale parameter $c = c(x)$ along with the additional attenuation effects produced by the latter inhomogeneities. As reported in [40], where the mentioned model was developed, the kernel function $g(\mathbf{x}, \mathbf{x}')$ is taken in the form

$$g(\mathbf{x}, \mathbf{x}') = k_0\ \exp\big(-\frac{r_{\mathrm{eq}}}{c_0}\big) \tag{25}$$

where c_0 is the largest value of $c(\mathbf{x})$ in V, whereas r_{eq} denotes the *equivalent distance* defined as

$$r_{\mathrm{eq}} = r + r^* \tag{26}$$

Here, r is the so called *geodetical distance*, that is, the length of the shortest path between $\mathbf{x}$ and $\mathbf{x}'$ without intersecting the boundary surface. For a non-convex domain it is $r \geq |\mathbf{x} - \mathbf{x}'|$, but $r = |\mathbf{x} - \mathbf{x}'|$ for a convex one (no holes, nor rientrant angles, nor cracks). The scalar r^* is a fictitious (non-negative) distance accounting for the additional attenuation effects produced by inhomogeneities of both $\mathbf{C}(\mathbf{x})$ and $c(\mathbf{x})$. However, in the present short review, convex domains are considered, therefore $r = |\mathbf{x} - \mathbf{x}'|$ and $c(\mathbf{x}) = c =$ constant.

The constitutive stress-strain relation can be obtained as in Sect. 3, but using the Helmholtz potential (22). So we obtain the total Cauchy stress as

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} + \alpha\,\mathscr{R}\big(\mathbf{C} : \mathscr{R}(\mathscr{D}\boldsymbol{\varepsilon})\big) \tag{27}$$

After some mathematical operations (not reported here for brevity), one obtains two possible forms for the stress-strain relation, that is, either

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) - \alpha \int_V \mathbf{J}(\mathbf{x}, \mathbf{x}')[\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x})]\,\mathrm{d}V' \tag{28}$$

or, equivalently,

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V \mathbf{S}(\mathbf{x}, \mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V' \tag{29}$$

Here above, $\mathbf{J}$ and $\mathbf{S}$ denote the *nonlocal stiffness tensors* expressed as

$$\mathbf{J}(\mathbf{x}, \mathbf{x}') = \big[\gamma(\mathbf{x})\mathbf{C}(\mathbf{x}) + \gamma(\mathbf{x}')\mathbf{C}(\mathbf{x}')\big]g(\mathbf{x}, \mathbf{x}') - \mathbf{k}_2(\mathbf{x}, \mathbf{x}') \tag{30}$$

$$\mathbf{S}(\mathbf{x}, \mathbf{x}') = \frac{1}{2}\big[\gamma^2(\mathbf{x})\mathbf{C}(\mathbf{x}) + \gamma^2(\mathbf{x}')\mathbf{C}(\mathbf{x}')\big]\delta(\mathbf{x} - \mathbf{x}') - \mathbf{J}(\mathbf{x}, \mathbf{x}') \tag{31}$$

where $\gamma(\mathbf{x})$ is given by (19) $(0 < \gamma(\mathbf{x}) \leq 1)$, whereas the tensor $\mathbf{k}_2$ is given by $(16)_2$. Also, $\mathbf{J}$ and $\mathbf{S}$ satisfy the equalities

$$\int_V \mathbf{J}(\mathbf{x}, \mathbf{x}')\, \mathrm{d}V' = \gamma^2(\mathbf{x})\mathbf{C}(\mathbf{x}), \qquad \int_V \mathbf{S}(\mathbf{x}, \mathbf{x}')\, \mathrm{d}V' = \mathbf{0} \qquad \forall\, \mathbf{x} \in \mathbf{V} \tag{32}$$

It is readily seen that for a uniform strain field, say $\boldsymbol{\varepsilon}(\mathbf{x}) = \bar{\boldsymbol{\varepsilon}} = \text{const.}$, Eq. (28) gives $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \bar{\boldsymbol{\varepsilon}}$, that is, the local stress condition is recovered correspondingly. Additionally, since $\mathbf{E}_d\mathscr{R}(\mathscr{D}\bar{\boldsymbol{\varepsilon}}) = 0$ identically, the Helmholtz free energy loses its dependence on the length scale parameter c, which means that the locality recovery condition is satisfied with the strain-difference based nonlocal model of second type. Combining (28), or (29), with the equilibrium equation (20) and the kinematic equations (21) leads, again as in Sect. 3, to an integro-differential equation of the second differential order in the displacement $\mathbf{u}$. This constitutes a Fredholm integral equation of second kind, which admits a unique solution free from paradoxical condition and in addition complying with the locality recovery condition.

5 Numerical Results

The strain-difference based nonlocal model of second type has been implemented into a nonlocal version of the finite element method (NL-FEM) in [45]. Such formulation is based on a nonlocal total potential energy principle given in [40], where the strain-difference based NL-FEM was proposed starting from a variational treatment of a BVP which is a straightforward extension to the strain-difference-based nonlocal model of the general variational principles conceived in [15]. Some numerical findings, related to a nonhomogeneous plate under tension, are given hereafter as an effective application of the NL-FEM. The above quoted papers are referred for theoretical and computational details. The discussed model has been also applied in [46] to the analysis of small-scale Euler-Bernoulli beams in bending. The relevant beam problem is reduced to a set of three mutually independent Fredholm integral equations of the second kind (each independent of the beam's ordinary boundary conditions, only one depends on the given load), which can be routinely solved numerically. Some results concerning a benchmark beam case are given next as a second numerical example, while referring to the above quoted paper for details.

5.1 A Nonhomogeneous Square Plate Under Tension

The square plate under tension depicted in Fig. 1 is analyzed via the NL-FEM. Geometry, material data, boundary and loading conditions are specified in Fig. 1 whose sketch shows that the plate is fixed along the edge at $x = 0$ with assigned displacements $\bar{u}_x = \bar{u}_y = 0$, and suffers uniform given displacements $\bar{u}_x = 0.001\text{cm}$

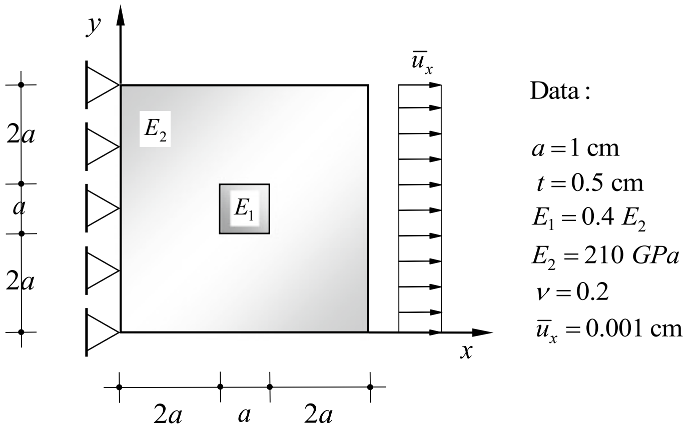

Fig. 1 Nonhomogeneous square plate under tension, after [45], with piecewise constant Young modulus. Geometry (t= thickness), boundary and loading conditions, material data

along the free edge at $x = 5a$. The attenuation function is the bi-exponential of Eq.(2) with $k_0 = 1/2\pi c^2 t$ with an *influence distance* (i.e. the distance beyond which $g(\mathbf{x}, \mathbf{x}') \approx 0$) equal to $11c$. C^0 quadratic, 8−nodes isoparametric Serendipity nonlocal finite elements (NL-FEs) have been used for the numerical analyses. The peculiarity of such NL-FEs is that each element, say the $n-$th one, aside from the standard (local) element stiffness matrix, say $\mathbf{k}_n^{\text{loc}}$, is equipped with element matrices of *nonlocal nature*, say $\mathbf{k}_n^{\text{nonloc}}$ and $\mathbf{k}_{nm}^{\text{nonloc}}$. The first one accounts for the influence exerted on the $n-$th element by the nonlocal diffusive processes over the whole domain, it contains the operator $\gamma(\mathbf{x})$ given by (19). The second one is a set of element matrices, all pertaining to the $n-$th element, precisely: a self-stiffness matrix, obtained for $m = n$, plus all the cross-stiffness matrices with $m \neq n$ being m the generic element *neighbor* of element n. Each $\mathbf{k}_{nm}^{\text{nonloc}}$ (with $n \neq m$) contains the matrices of the Cartesian derivatives of the shape functions of elements m and n so accounts for the influence on the element n of the neigbor elements. Further details are given in [45] and are here omitted for brevity.

In Fig. 2 the plate size a has been proportionally varied, assuming $a = 0.5, 1, 2$ cm, so defining three proportional sized plates with the boundary conditions of Fig. 1 and suffering the displacements $\bar{u}_x$ which have been accordingly proportionally varied with the plate dimension such that $\bar{u}_x = a/1000$. The three solutions obtained with local elasticity coincide, each FE model is a scaled version of the other two. In contrast, the three nonlocal elastic solutions given by the NL-FEM show a decreasing – a flattening – for decreasing specimen dimensions confirming the capacity of the NL-FEM to capture size effects.

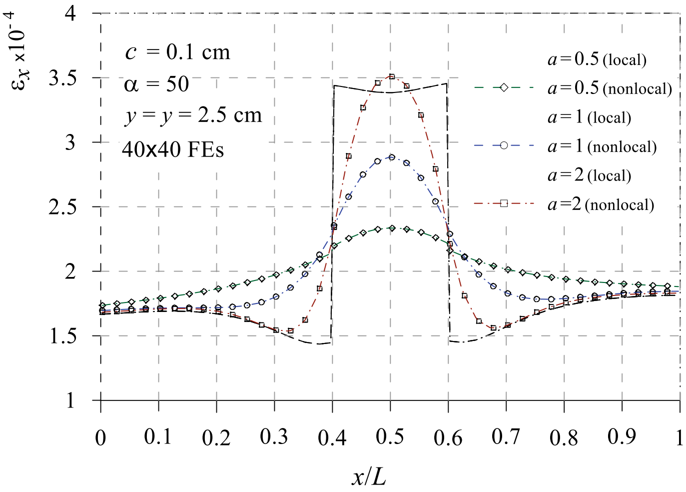

Fig. 2 Nonhomogeneous plate under tension, after [45]. Strain profiles ε_x versus x/L at $y = 2.5$ cm, $c = 0.1$ cm, $\alpha = 50$, mesh of 40×40 FEs and for three different proportional sized plates given by $a = 0.5$, 1 and 2 cm. Local (lines without markers) and nonlocal (lines with markers) solutions

5.2 *Small-Scale Euler-Bernoulli Beams in Bending*

A simple benchmark Euler-Bernoulli beam in bending has been analyzed, precisely a cantilever beam under point load P at the free end. The beam has been assumed homogenous, of length L and referred to orthogonal co-ordinates (x, y, z). The x axis coincides with the beam axis, z is oriented along the beam height, y is in the width direction. The bending plane coincides with the plane (x, z), the (y, z) axes coincide with principal inertia axes of the cross section. The only meaningful strain component is ε_{xx} and the transverse attenuation effects are assumed to be negligible such that the attenuation function g can be considered to be a function of the x co-ordinate only, i.e. $g = g(x, x')$ herein assumed in the bi-exponential form of Eq. (2). The fundamental bending moment/curvature relation featuring the strain-difference based nonlocal model for beams proves to be:

$$M(x) = EI\chi(x) - \alpha \int_0^L J(x, x')\big[\chi(x') - \chi(x)\big]\,\mathrm{d}x' \tag{33}$$

where $M(x)$ and $\chi(x)$ are the bending moment and the curvature, respectively, E is the Young modulus and I the second area moment of the cross-section. Equation (33), counterpart of (28) for the EB beam model, leads to a solving equation for the beam problem which is a Fredholm integral equation of the second kind. The latter, as shown in [46] (see also [47, 48]), can be solved by a splitting strategy that provides a set of three mutually independent Fredholm integral equations of the second kind, all of which holding no matter how the beam is constrained. The solution for the

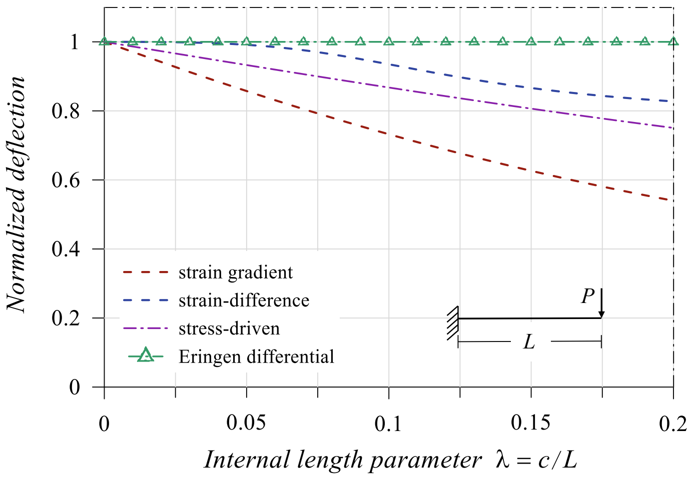

Fig. 3 Cantilever beam subjected to a point load at the free end, after [46]. Normalized deflection at the free end cross section *versus* internal length parameter λ for strain gradient (Polizzotto 2014, [49], dashed line), strain-difference integral (present model, solid line), stress-driven (Barretta et al. 2018, [50], dash dot line) and Eringen differential (Peddieson et al. 2003, [39], solid line with triangles) constitutive behavior

beam problem is indeed achieved to within four constants to be determined by the ordinary boundary conditions characterizing the analyzed specific beam case.

In Fig. 3 the normalized deflection versus the normalized internal length parameter is reported. The obtained result is plotted against the ones given by the strain gradient model [49], the stress-driven model [50] and the Eringen differential [39]. The obtained results show that, with the exception of the Eringen's nonlocal model which is affected by the discussed inconsistencies, the predicted size effects are of stiffening type, a circumstance which seems to confirm the well-known *smaller-is-stiffer* phenomenon. It is worth noting that for "small" values of the internal length c the three methods are in substantial agreement with one another, while for $c \to \infty$, at difference with the other models, the asymptotic behavior predicted by the strain-difference (here not shown for brevity) is of local type, a result in agreement with the expected (local) behavior of a size dependent nonlocal beam model, which for $c \to \infty$ behaves like an atomic lattice model, [20, 51].

6 Conclusion

An overview of a family of strain-difference based nonlocal elasticity theories has been presented. These essentially include two model types, of which one complies with the "local stress recovery condition", the other with the more stringent "locality recovery condition", but both models lead to well-posed boundary-value-problems without computational drawbacks or other paradoxical conditions.

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