Comments on the Solutions Set of Equilibrium Problems Governed by Topological Pseudomonotone Bifunctions

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Abstract. A recent assertion given by Sadeqi, I., Salehi Paydar, M., in: A comparative study of Ky Fan hemicontinuity and Brezis pseudomonotonicity of mappings and existence results, J. Optim. Theory Appl. (2015), that the set of solutions of the variational inequality problem governed by a pseudomonotone operator is closed, is obtained as a particular case of a result from Bogdan, M., Kolumbán, J.: Some regularities for parametric equilibrium problems, J. Global Optim. (2009). An example of a set in a Hilbert space where $\nabla\|\cdot\|$ is not Fan-hemicontinuous is given. A different approach to show that $\nabla\|\cdot\|$ is indeed topologically pseudomonotone, is expressed. From the corresponding definitions, Fan-hemicontinuity implies topological pseudomonotonicity, but the reverse implication does not hold in general. This strict relationship is strengthened by $\nabla\|\cdot\|$. Moreover, another counterexample is given in a Lebesgue space, instead of the Sobolev space $W_0^{1,3}$ used in Steck, D.: Brezis pseudomonotonicity is strictly weaker than Ky Fan hemicontinuity. J. Optim. Theory Appl. (2019). Stability of pseudomonotonicity with respect to the composition with a linear operator is formulated as open question.

Keywords: Fan-hemicontinuity · Topological pseudomonotonicity · Solutions set · Equilibrium problems · Positively oriented set · Simple pendulum

1 Introduction

Equilibrium problems unify many mathematical models among them variational inequalities and optimization problems (see [1]). These class of problems have been found useful in the study of many problems from the fields of economics, mechanics, and engineering sciences [2–4]. The mathematical tools involved are mainly algebraic ones such as convexity and topological ones such as continuity ([5]), in particular topological pseudomonotonicity for a special class [6,8]. The notion of pseudomonotonicity is related to the class of semi-linear operators (see

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[3,4,7,8]). Existence result for the case of topological pseudomonotone bifunction is given in [9] (see also [10], Theorem 2.3).

It was noted in [11] that the notion of pseudomonotonicity is not equivalent to Fan-hemicontinuity, as it was affirmed in [12]. It is worth to be noted that in [12], sufficient conditions were imposed on a function to obtain pseudomonotonicity for its gradient, and the weak closedness of the solution set of a variational inequality governed by pseudomonotone operators was proved.

In this paper are added some comments and remarks on the two papers mentioned above. Some restrictions on the domain are expressed in such a way the gradient of the norm defined on a Hilbert space to be Fan-hemicontinuous.

The setting is the usual one, $(X, \|\cdot\|)$ a real Banach space with dual X^*, the duality pairing between X^* and X being denoted by $\langle\cdot,\cdot\rangle$; the strong and the weak convergence is indicated by $\longrightarrow$ and $\rightharpoonup$, respectively. Let $K \subseteq X$ be nonempty and convex. The following two notions are discussed.

Definition 1. *$F : K \to X^*$ is said to be* Fan-hemicontinuous *if for every $y \in X$, $x \longmapsto \langle F(x), x-y\rangle$ is weakly sequentially lower semicontinuous on K, that is $\forall (x_k) \subset K$ with $x_k \rightharpoonup x \in K$, one has*

$$\liminf_k \langle F(x_k), x_k - y\rangle \geq \langle F(x), x-y\rangle.$$

This notion was used in the existence results for problems in equilibrium theory ([1]). Since the norm of a Hilbert space is weakly sequentially lower semicontinuous, it is straightforward that the identity operator on a Hilbert space is Fan-hemicontinuous.

Definition 2. *$F : K \to X^*$ is said to be* pseudomonotone *if whenever*

$$(x_k) \subset K \text{ with } x_k \rightharpoonup x \in K, \text{ and } \limsup_k \langle F(x_k), x_k - x\rangle \leq 0, \tag{1}$$

then

$$\liminf_k \langle F(x_k), x_k - y\rangle \geq \langle F(x), x-y\rangle, \text{ for all } y \in K. \tag{2}$$

Recently, an example of nonlinear operator, defined on a Sobolev space which is pseudomonotone but not Fan-hemicontinuous was given in [11]. A result from [12] on the solution set of a variational inequality governed by topological pseudomonotone operators is remarked.

This present content is organized as follows. Subsection 1.1 contains an example of a closed, convex but not positively oriented set (see Definition 3) where $\nabla\|\cdot\|$ is not Fan-hemicontinuous. Also, an alternative proof that $\nabla\|\cdot\|$ is Fan-hemicontinuous on a restrictive set is given. Subsection 1.2 contains a remark about the closedness of the solution set of a variational inequality problem governed by a pseudomonotone operator, stated in [12]. It is a particular case of a result given in [6]. Section 2 is concerned with another example of a pseudomonotone operator that it is not Fan-hemicontinuous. The content ends with an open question related to the composition of a pseudomonotone operator with a linear one.

1.1 Fan-hemicontinuity of the Gradient of the Norm

In this section, some properties of Fan-hemicontinuity and pseudomonotonicity, respectively are emphasized for $\nabla\|\cdot\|$ (in particular, if $\nabla\|\cdot\|$ is Fan-hemicontinuous, then it is pseudomonotone). These led to the property for its domain to ensure that it is Fan-hemicontinuous.

In [12] it was claimed that the gradient of a convex Gateaux differentiable is Fan-hemicontinuous. It is proved that this does not hold in general.

Definition 3. ([13]) *The set $K \subset X$ is said to be positively oriented if $\forall x, y \in K, \langle x, y\rangle \geq 0$.*

Imposing this condition on the set K, one obtains the Fan-hemicontinuity of $\nabla\|\cdot\|$.

Proposition 1. ([13]) *Let X be a Hilbert space, $K \subset X$ closed, convex, with $0_X \notin K$. If K is positively oriented, then $\nabla\|\cdot\|$ is Fan-hemicontinuous on K.*

The positive orientation property of the set K is sufficient in order to obtain the Fan-hemicontinuity of $\nabla\|\cdot\|$. However, there are sets that are not positively oriented.

Example 1. The set

$$K = \{x = (x^j) \in \ell^2 \,\big|\, x^1 \in \mathrm{IR}, x^2 = 1/2, (-1)^j \cdot x^j \geq 0, \text{for } j \geq 3\}$$

is closed, convex but not positively oriented. Moreover, $\nabla\|\cdot\|$ is not Fan-hemicontinuous.

Consider $\bar{x} = (-1, 1/2, 0, 0, ..., 0, ...)$ and $\bar{y} = (1, 1/2, 0, 0, ..., 0, ...)$ that belong to K but $\langle \bar{x}, \bar{y}\rangle = -3/4 < 0$.

Now, it is shown that $\nabla\|\cdot\|$ is not Fan-hemicontinuous on K. For this, it is needed $(x_n) \subset K$, $x_n \rightharpoonup x \in K$, and $y \in K$ such that

$$\liminf_n \left[\|x_n\| - \frac{1}{\|x_n\|} \cdot \langle x_n, y\rangle\right] < \|x\| - \frac{1}{\|x\|}\langle x, y\rangle. \tag{3}$$

Let be defined $x = (-1, 1/2, 0, 0, ..., 0, ...) \in K$, $(x_n) \subset \ell^2$, for $n \in \mathrm{IN}$,

$$x_n^j := \begin{cases} -1, & j = 1 \\ \frac{1}{2}, & j = 2 \\ (-1)^j \frac{1}{2}, & j = n+2 \\ 0, & \text{otherwise}, \end{cases}$$

and $y = (\frac{13}{4}, \frac{1}{2}, 0, 0, ..., 0, ...) = (y^j) \in K$. Observe that $\|x\| = \frac{\sqrt{5}}{2} =: b$, $\|x_n\| = \frac{\sqrt{6}}{2} = a > b$, and $\langle x_n, y\rangle = \langle x, y\rangle = -3$. One has $(x_n) \subset K$, $x_n \rightharpoonup x \in K$. For the values indicated, (3) is checked, that becomes

$$a - b < \left(\frac{1}{a} - \frac{1}{b}\right) \cdot (-3),$$

or equivalently $\frac{\sqrt{30}}{4} = ab < 3$, that is true.

It is true that for f convex and Gateaux differentiable, ∇f is pseudomonotone ([12]). Just for considering one different approach, it is remarked below that $\nabla\|\cdot\|$ is indeed pseudomonotone, in particular, if X is a Hilbert space, $K \subset X, 0_X \notin K, F : K \to X, F(x) = \nabla\|x\|$.

Remark 1. $\nabla\|\cdot\|$ is pseudomonotone if whenever

$$(x_k) \subset K \text{ with } x_k \rightharpoonup x, \text{ and } \limsup_k \frac{1}{\|x_k\|} \cdot \langle x_k, x_k - x\rangle \leq 0, \tag{4}$$

then (2) holds.

It is sufficient to show that

$$\liminf_k \|x_k\| + \liminf_k \left(- \frac{1}{\|x_k\|} \cdot \langle x_k, y\rangle \right) \geq \|x\| - \frac{1}{\|x\|}\langle x, y\rangle. \tag{5}$$

Let be denoted $a_k = \langle x_k, x_k - x\rangle$ and $b_k = \|x_k\|$. The sequence (x_k) is bounded since it is weakly convergent. Let $b > 0$ be such that $0 < b_k \leq b$ for all $k \in \mathrm{IN}$. For an $\varepsilon > 0$ arbitrary, there exists $k_\varepsilon \in \mathrm{IN}$ such that

$$\frac{a_k}{b_k} < \varepsilon \Longleftrightarrow a_k < \varepsilon b_k \leq \varepsilon b,$$

for all $k \geq k_\varepsilon$. It results $\limsup_k a_k \leq \varepsilon b$. Since $\varepsilon > 0$ was arbitrary this implies $\limsup_k a_k \leq 0$. By the inequality (4) one has

$$0 \geq \limsup_k \frac{1}{\|x_k\|} \cdot \langle x_k, x_k - x\rangle = \limsup_k \frac{\|x_k\|^2 - \langle x_k, x\rangle}{\|x_k\|},$$

thus

$$\begin{aligned}\limsup_k \|x_k\|^2 - \|x\|^2 &= \limsup_k \|x_k\|^2 - \lim_k \langle x_k, x\rangle \\ &= \limsup_k (\|x_k\|^2 - \langle x_k, x\rangle) = \limsup_k a_k \leq 0.\end{aligned}$$

Hence $\limsup_k \|x_k\| \leq \|x\|$, therefore $\lim_k \|x_k\| = \|x\|$, that together with $x_k \rightharpoonup x$ give $x_k \longrightarrow x$, since the Kadec-Klee property stands in the Hilbert space setting. Remark that, if (x_n) strongly converges to $x \neq 0$, then obviously $\frac{1}{\|x_n\|} \cdot \langle x_n, y\rangle \to \frac{1}{\|x\|}\langle x, y\rangle$, and consequently (5) is verified. Consequently (2) holds, meaning that $\nabla\|\cdot\|$ is pseudomonotone.

1.2 The Solution Set for the Variational Inequality

Let $K \subseteq X$ be a nonempty convex set and $F : K \to X^*$ be a given mapping. The variational inequality problem is the following:

$$VI(F, K) \text{ find } x \in K \text{ such that } \langle F(x), y - x\rangle \geq 0, \forall y \in K.$$

Denote by $S_{VI} = S_{VI(F,K)}$ the set of its solutions.

The result in [12], Proposition 4.1, is the following: "Let K be a closed and convex subset of X and let $F : K \to X^*$ be pseudomonotone such that S_{VI} is nonempty. Then, S_{VI} is weakly closed." The mentioned assertion is a particular case of Lemma 1 from [6].

Remark 2. Theorem 1 from [6] states the closedness of the solution map for some equilibrium problems, in particular for $n \in \mathrm{IN} \longmapsto S_{VI(F_n,K_n)}$, on a general setting of a Hausdorff topological space X, endowed with two comparable topologies.

Let $f_n, f : X \times X \to \mathrm{IR}$ be the bifunctions given by $f_n(x,y) = \langle F_n(x), x-y\rangle$ and $f(x,y) = \langle F(x), x-y\rangle$, respectively. The hypotheses of the theorem mentioned above include the convergence for (K_n) to K in some sense, a condition that relates F_n and F, and the pseudodomotonicity of F. If these hold, then S_{VI} is closed at ∞, i.e. for each sequence (x_n) of solutions to $VI(F_n, K_n), (x_n)$ convergent with respect to the less strong topology to x in X, imply $x \in S(\infty) = S_{VI(F,K)}$. If $(F(x_n))$ is bounded, it can be applied for the variational form of the bifunction, a constant sequence of operators, $F_n = F$, and for the constant domains $K_n = K$. If one takes the less strong topology be induced by $\rightharpoonup$ on X, then obtains the following result.

Corollary 1. ([6], Lemma 1) *If K is weakly closed and F is pseudomonotone, then $S_{VI(F,K)}$ is weakly closed.*

2 Another Example of a Pseudomonotone Operator That It Is Not Fan-hemicontinuous

The subject of [11] was to prove that there exists a pseudomonotone operator that is not Fan-hemicontinuous. Recall that the error pointed out in the nonequivalence between these two notions was in the proof of Proposition 3.5 from [12], where the implication $(x_k) \subset X, x_k - x \rightharpoonup 0$, in $X, F(x_k) \rightharpoonup f \in X^*$, then $\langle F(x_k), x_k - x\rangle \to 0$ is not correct.

In this section, another example of a pseudomonotone operator that it is not Fan-hemicontinuous is given. Let $X = L^3(0,1)$ be the Lebesgue space and let $F : X \to X^*$ be defined by

$$\langle F(u), v\rangle = \int_0^1 |u(t)| \cdot u(t) \cdot v(t)\, dt,\ \ u, v \in X. \tag{6}$$

By a direct consequence of Proposition 2.3 from [3] (p. 41), F is pseudomonotone since it is monotone and continuous. Indeed, let $S : \mathrm{IR} \to \mathrm{IR}, S(\xi) = |\xi| \cdot \xi$. Since $S'(\xi) = 2|\xi| \geq 0, S$ is increasing. Actually since $[S(\xi) - S(\eta)] \cdot (\xi - \eta) \geq \frac{1}{2}(|\xi| + |\eta|) \cdot (\xi - \eta)^2, \forall \xi, \eta \in \mathrm{IR}$, it follows,

$$\begin{aligned}\langle F(u) - F(v), u - v\rangle &= \int_0^1 \big[S(u(t)) - S(v(t))\big] \cdot [u(t) - v(t)]\, dt \\ &\geq \int_0^1 \frac{1}{2}(|u(t)| + |v(t)|) \cdot [u(t) - v(t)]^2\, dt.\end{aligned}$$

The continuity holds by the following inequalities. One has $|S(\xi) - S(\eta)| \leq 3(|\xi| + |\eta|) \cdot |\xi - \eta|$, $\forall \xi, \eta \in \mathrm{IR}$, so that, by Hölder's inequality

$$\begin{aligned} \big|\langle F(u) - F(u_k), v\rangle\big| &\leq \int_0^1 \big|S(u(t)) - S(u_k(t))\big| \cdot |v(t)|\, dt \\ &\leq 3 \int_0^1 \big[|u(t)| + |u_k(t)|\big] \cdot |u(t) - u_k(t)| \cdot |v(t)|\, dt \\ &\leq 3\big[\|u\|_{L^3} + \|u_k\|_{L^3}\big] \cdot \|u - u_k\|_{L^3} \cdot \|v\|_{L^3}. \end{aligned}$$

Hence $F(u_k) \to F(u)$ as $u_k \to u$, in $L^3(0,1)$.

To prove that F is not Fan-hemicontinuous, one needs to show that there exist $v \in X$ and $(u_k) \subset X$ with $u_k \rightharpoonup u$ in X, such that

$$\liminf_k \langle F(u_k), u_k - v\rangle < \langle F(u), u - v\rangle.$$

To achieve this, it is defined

$$u_k(t) = \begin{cases} l \cdot a, \text{ if } t \in \big(\frac{i}{k}, \frac{i+\theta}{k}\big), i = 0, ..., k-1 \\ -a, \quad \text{if } t \in \big(\frac{i+\theta-1}{k}, \frac{i}{k}\big), i = 1, ..., k, \end{cases}$$

where $a > 0, l > 1$, and $\theta \in (0,1)$ are such that $m := \theta \cdot la + (1-\theta) \cdot (-a) = 0$, hence $\theta = \frac{1}{l+1}$. For this,

$$\begin{aligned} \int_0^1 u_k(t)\, dt &= \sum_{i=0}^{k-1} \int_{i/k}^{\frac{i+\theta}{k}} u_k(t)\, dt + \sum_{i=1}^{k} \int_{\frac{i+\theta-1}{k}}^{i/k} u_k(t)\, dt \\ &= la \cdot \frac{\theta}{k} \cdot k + (-a) \cdot \frac{1-\theta}{k} \cdot k = 0. \end{aligned}$$

One has $u_k \rightharpoonup u := 0 = m$, in $L^3(0,1)$, since there exists $M > 0$ with $\|u_k\|_{L^3} \leq M$ and $\lim_{k\to\infty} \int_D u_k(t)\, dt = 0, \forall D = [c,d] \subset (0,1)$ (see [14], Lemma 1.4). Indeed,

$$\begin{aligned} \|u_k\|_{L^3}^3 &= \sum_{i=0}^{k-1} \int_{i/k}^{\frac{i+\theta}{k}} |u_k(t)|^3\, dt + \sum_{i=1}^{k} \int_{\frac{i+\theta-1}{k}}^{i/k} |u_k(t)|^3\, dt \\ &= \ell^3 a^3 \cdot \frac{\theta}{k} \cdot k + a^3 \cdot \frac{1-\theta}{k} \cdot k = a^3 \cdot (\ell^3 \cdot \theta + 1 - \theta) \\ &= a^3 \cdot l \cdot (\ell^2 + 1)/(l+1) =: A, \end{aligned}$$

thus $M = \sqrt[3]{A}$, can be taken. For the second condition, the computations are the following. If $c < d$, there exists $k, i = [c \cdot k] + 1 \in \mathrm{IN}$, such that $c < \frac{i}{k} < \frac{i+1}{k} < d$. Denote $i_d = \max\{p \in \mathrm{IN} \,|\, \frac{i+p}{k} < d\}$. On the interval $\big[\frac{i}{k}, \frac{i+1}{k}\big]$, one has $\int_{i/k}^{\frac{i+1}{k}} u_k(t)\, dt = \int_{i/k}^{\frac{i+\theta}{k}} u_k(t)\, dt + \int_{\frac{i+\theta}{k}}^{\frac{i+1}{k}} u_k(t)\, dt = \frac{m}{k} = 0$. Similarly on $\big[\frac{i+j}{k}, \frac{i+j+1}{k}\big], 1 \leq j \leq p-1$. On the interval $\big[c,d\big] \setminus \big[\frac{i}{k}, \frac{i_d+1}{k}\big]$, one has $\big|\int_c^{i/k} u_k(t)\, dt\big| \leq la \cdot (i/k) \to 0$, as $k \to \infty$; similarly on $\big[\frac{i_d+1}{k}, d\big]$.

For $v(t) = \alpha > 0, t \in (0,1)$, so $v \in X$, one has $\langle F(u), u - v\rangle = 0$. Compute

$$\begin{aligned}\langle F(u_k), u_k - v\rangle &= \int_0^1 |u_k(t)| \cdot u_k(t) \cdot [u_k(t) - v(t)]\, dt \\ &= \int_0^1 |u_k(t)| \cdot u_k^2(t)\, dt - \int_0^1 |u_k(t)| \cdot u_k(t) \cdot v(t)\, dt \\ &= \|u_k\|_{L^3}^3 - \alpha \cdot \int_0^1 |u_k(t)| \cdot u_k(t)\, dt \\ &= A - \alpha \cdot B, \end{aligned} \tag{7}$$

where the value B from (7) is given by

$$\begin{aligned} B = \int_0^1 |u_k(t)| \cdot u_k(t)\, dt &= \sum_{i=0}^{k-1} \int_{i/k}^{\frac{i+\theta}{k}} \ell^2 a^2\, dt + \sum_{i=1}^{k} \int_{\frac{i+\theta-1}{k}}^{i/k} a \cdot (-a)\, dt \\ &= \ell^2 a^2 \cdot \theta + (-a^2) \cdot (1-\theta) = a^2 \cdot l \cdot (l-1)/(l+1). \end{aligned}$$

Now, chose α such that $A - \alpha \cdot B < 0$, in (7), thus $\liminf_k \langle F(u_k), u_k - v\rangle < 0$. This means that F is not Fan-hemicontinuous.

3 Open Question

In this section, stability of pseudomonotonicity with respect to the composition to a linear operator is questioned. Let us consider two examples of pseudomonotone operator that are not Fan-hemicontinuous.

Let $F : X \to X^*$ be given in (6). Let $\tilde{F} : Y \to Y^*$ be defined on the Sobolev space $Y = W_0^{1,3}(0,1)$, given by

$$\langle \tilde{F}(u), v\rangle = \int_0^1 |\nabla u(t)| \cdot \nabla u(t) \cdot \nabla v(t)\, dt, \ u, v \in Y,$$

that it is known as being pseudomonotone [11]. Let $S : \mathrm{IR} \to \mathrm{IR}, S(\xi) = |\xi| \cdot \xi$. Since

$$\langle F(\nabla u), \nabla v\rangle = \langle -div\, S(\nabla u), v\rangle = \langle \tilde{F}(u), v\rangle, \ \ v \in Y,$$

one can ask about the pseudomonotonicity of $F \circ \nabla$. In this matter, for an arbitrary F, it is formulated the following question.

Q: What (sufficient) conditions should be imposed on $L : Y \to X$ such that for $F : X \to X^*$ pseudomonotone, then $F \circ L$ to be pseudomonotone as well?

References

1. Kassay, G., Rădulescu, V.D.: Equilibrium Problems and Applications, Mathematics in Science and Engineering. Academic Press, Cambridge (2019)
2. Panagiotopoulos, P.D., Fundo, M., Rădulescu, V.: Mathematical Theory of Hemivariational Inequalities, Dekker, New-York (1999)

3. Showalter, R.E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, vol. 49. American Math. Society, USA (1997)
4. Zeidler, E.: Nonlinear Functional Analysis and its Applications, vol. III. Springer Verlag, Berlin, Germany (1990)
5. Bigi, G., Castellani, M., Pappalardo, M., Passacantando, M.: Existence and solution methods for equilibria. Eur. J. Oper. Res. **227**, 1–22 (2013)
6. Bogdan, M., Kolumbán, J.: Some regularities for parametric equilibrium problems. J. Glob. Optim. **44**, 481–492 (2009). https://doi.org/10.1007/s10898-008-9345-3
7. Hartman, P., Stampacchia, G.: On some non linear elliptic differential functional equations. Acta Math. **115**, 271–310 (1966). https://doi.org/10.1007/BF02392210
8. Leray, J., Lions, J.L.: Quelques résultats de Visik sur les problémes elliptiques non linéaires par les méthodes de Minty-Browder. Bull. Soc. Math. Fr. **93**, 97–107 (1965)
9. Gwinner, J.: On fixed points and variational inequalities-a circular tour. Nonlinear Anal. **5**, 565–583 (1997)
10. Gwinner, J., Ovcharova, N.: From solvability and approximation of variational inequalities to solution of nondifferentiable optimization problems in contact mechanics. Optimization **64**, 1–20 (2015)
11. Steck, D.: Brezis pseudomonotonicity is strictly weaker than Ky Fan hemicontinuity. J. Optim. Theory Appl. **181**(1), 318–323 (2019). https://doi.org/10.1007/s10957-018-1435-x
12. Sadeqi, I., Salehi Paydar, M.: A comparative study of Ky Fan hemicontinuity and Brezis pseudomonotonicity of mappings and existence results. J. Optim. Theory Appl. **165**(2), 344–358 (2015). https://doi.org/10.1007/s10957-014-0618-3
13. Bogdan: Fan-hemicontinuity for the gradient of the norm in Hilbert space (under review)
14. Dacorogna, B.: Direct Methods in the Calculus of Variations, Applied Mathematical Sciences 78, vol. III. Springer Verlag, Berlin, Germany (1989)