The Perturbation Method for Dynamic Analysis of Pole Vaulting

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Abstract. Pole vaulting has progressed slowly in terms of performance since the 1960's. The world record varies by a few centimeters due to the use of flexible poles. In this research, we propose a new stochastic optimization approach to take into account the uncertainties. A simplified mass-spring model has been implemented to model the pole vault. This model is considered as a tool to explore the possibilities of trajectories and performances, in which the parameters associated with this system are uncertain. This approach is based on the association of the perturbation method with the fourth order Runge-Kutta method (RK4). The new stochastic approach is used to optimize the dynamic response of the system and the computation time. The results of the perturbation method (PM) are compared to those of the reference Monte Carlo method (MC).

Keywords: Monte Carlo · Perturbation method · Pole vaulting · Mass-spring · Stochastic parameters

1 Introduction

Pole Vaulting is an Olympic discipline in which the athlete tries to clear the bar as high as possible through a flexible pole transforming kinetic energy into potential energy, therefore, the performance obtained changes considerably with the combination of two determining factors such as the properties of the pole (stiffness, weight,) and the physical ability of the athlete (speed, force…).

The pole vault has been the interest of many scientific works for the last thirty years. Researchers seek to identify performance factors either through experimental studies [1–4] or through numerical simulations based on mechanical or mathematical models [5–12].

Motivated by the mechanics of the pole vault, the aim of this research is to perform a stochastic study on a simplified model of the pole vault consisting of a point mass and a linear spring, then the determination of the nonlinear dynamic response of a vault using the differential equations of motion. In general, assumes that the physical parameters

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of the model are deterministic (material properties, initial and boundary conditions…) but in reality, these parameters are random. To evaluate the variability of the dynamic response with respect to the variability of the uncertain parameters of the model, a Monte Carlo simulation is used [13, 14]. This method is often used as a reference, even if the prohibitive calculation time limits the use of this method. therefore, we use the perturbation method [15–17] associated with the fourth order Runge-Kutta method [18]. This method based on the Taylor series development of the response around the average values of the random variables, and allows to calculate directly the averages and standard deviations of the solutions.

2 Dynamic Modeling

The dynamic modeling of the pole vault is not an easy problem due to the nonlinearity of the system. Therefore, we decided to model the dynamics using a linear spring-mass system which we base on a non-dimensional study thanks to the assumption of the behavior of the vault. The point mass attach to the spring with an initial velocity V_0 and detach when the spring has no energy store and the mass has no horizontal velocity.

2.1 Linear Mass-Spring System

We consider the system composed of a point mass m and a spring of stiffness k and length R which can rotate freely around the x axis and θ denotes its angle with the y axis, measured counterclockwise.

It is assumed that the length of the spring in its initial state is R_0, its initial angle of rotation is θ_0. The mass has an initial velocity V_0 along the negative y axis when it contacts the free end of the spring can detach from the spring at any time.

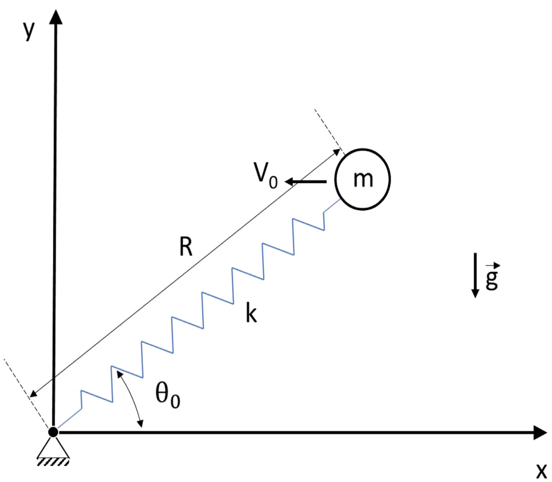

Fig. 1. Point mass system and linear spring in its initial state

2.2 The Equation of Motion

The equation of motion of the mass-spring system in Fig. 1 is determined by the Lagrange method in the polar coordinate base, where R and θ are the generalized coordinates, the equations of motion are as follows:

$$\begin{bmatrix} \ddot{R} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} R\theta^2 + \frac{k}{m}(R_0 - R) - g\sin\theta \\ -\frac{2}{R}\dot{\theta}\dot{R} - \frac{g}{R}\sin\theta \end{bmatrix} \tag{1}$$

And the initial conditions are:

$$\begin{matrix} R(0) = R_0 & \theta(0) = \theta_0 \\ \dot{R}(0) = -V_0\cos\theta & \dot{\theta}(0) = \frac{V_0}{R_0}\sin\theta \end{matrix} \tag{2}$$

We non-dimensionalize Eq. (1) by introducing the following non-dimensional variables of time, position and velocity:

$$\tau = t\sqrt{\frac{k}{m}}, \quad r = \frac{R}{R_0}, \quad v_0 = \frac{V_0}{R_0}\sin\theta \tag{3}$$

The non-dimensional equation is:

$$\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} r\dot{\theta}^2 + (1 - r) - \varepsilon\sin\theta \\ -\frac{2}{r}\dot{r}\dot{\theta} - \frac{\varepsilon}{r}\sin\theta \end{bmatrix} \tag{4}$$

Due to the weight of the mass, the non-dimensional static deflection of the spring is ε, where:

$$\varepsilon = \frac{\delta}{L_0}\delta = \frac{\mathrm{mg}}{\mathrm{k}} \tag{5}$$

The non-dimensional initial condition is:

$$\begin{matrix} r(0) = 1 & \theta(0) = \theta_0 \\ \dot{r}(0) = -\vartheta_0\cos\theta_0, & \dot{\theta}(0) = \vartheta_0\sin\theta_0 \end{matrix} \tag{6}$$

The mass must be released from the spring at a certain time, provided that the kinetic energy of motion of the mass converted into potential energy. Then these two conditions that must be satisfied:

$$\dot{r}(\tau_f) = 1, \qquad \dot{r}(\tau_f)\cos\theta(\tau_f) = r(\tau_f)\dot{\theta}(\tau_f)\sin\theta(\tau_f) \tag{7}$$

Finally, the point mass moves freely under the influence of gravity only, then its dynamics will be given by the relation:

$$\ddot{Z} = -g \to \ddot{z} = -\varepsilon \tag{8}$$

The dynamics of the mass-spring system is described by four state variables $r, \dot{r}, \theta, \dot{\theta}$. Since the final time t_f is also unknown, we have to solve a two-point boundary value problem (TPBVP) where (7) provides two conditions that must be satisfied at the final

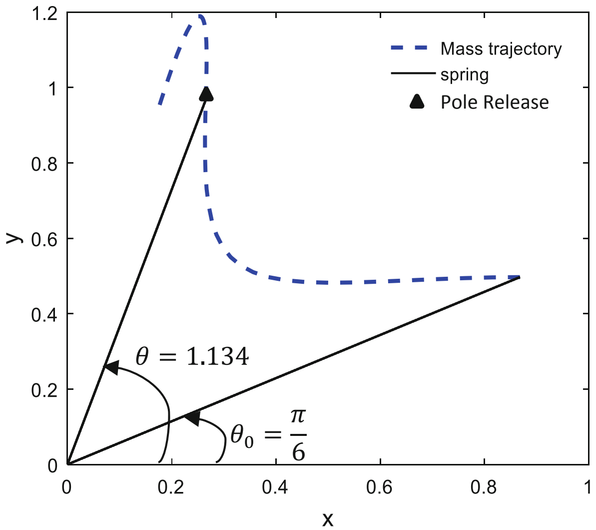

Fig. 2. Trajectory of the mass of Fig. 1 with $\varepsilon = 0.1$ for the initial speed $\vartheta_0 = 0.432$ and the initial angle $\theta_0 = \frac{\pi}{6}$

time. We first solve the TPBVP assuming that ε and θ are constants. A solution was obtained through trial and error for the specific case where $\varepsilon_0 = 0.1$ and $\theta_0 = \frac{\pi}{6}$ rad, Eq. (4) was integrated in time using the initial conditions in (6) for different values of the velocity ϑ_0.

The results are presented in Fig. 2, by transforming the non-dimensional polar coordinate system to the non-dimensional Cartesian coordinate system:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{9}$$

Only $\vartheta_0 = 0.432$ satisfies the five boundary conditions at $t_f = 3.337$ which provides the solution of TPBVP. At this moment the mass detaches from the spring, the maximum height reached by the mass is h = 1.191. The equations of motion of the mass integrated with respect to time using the method of (RK4). The model developed in this section has been implemented in MATLAB 2020.

3 Stochastic Study

The classical method followed when studying mechanical systems is based on the assumption that the model is deterministic i.e., that its parameters are constant. However, if we conduct experiments, we will realize the limits of deterministic modeling. Because there is always a difference between the calculated and measured results, this is due to the uncertainty of (the material properties, and the initial conditions….). These uncertainties have an impact on the dynamic displacement behavior of the mass-spring system. It is therefore necessary to use stochastic methods, in particular the Monte Carlo method and the perturbation method.

3.1 Monte Carlo Method

The estimation of the moment (mean and variance) of the dynamic response function of the mechanical system can be obtained by Monte Carlo simulation. Although its computational cost is high, this method has been widely used by dedicated software (such as MATLAB, ANSYS…) and provides a reference for approximate calculations. The Monte Carlo method has the advantage of taking into account all types of uncertainties on the parameters of a mechanical system. However, one of its main drawbacks is the computational time required due to its iterative nature. The dynamic response function is considered as a random variable image of the base random variable. The simulation involves constructing a sample of random variable $Y_1, Y_2, \ldots, Y_n$ and processing the sample using standard statistical techniques. The n simulations are performed independently according to the distribution law of the basic random variables.

The mean of Y is given by:

$$E[Y] = \frac{1}{n}\sum_{i=1}^{n} Y_i \tag{10}$$

The variance of Y is given by:

$$Var[Y] = \frac{1}{n-1}\sum_{i=1}^{n} [Y_i - E(Y)]^2 \tag{11}$$

3.2 Perturbation Method

The perturbation method is widely used in the field of stochastic finite elements, and is based on the expansion of the Taylor series, which is a function of the basic random physical variables, mechanical properties, geometric characteristics, (the random parameters must clearly appear in the dynamic matrix). The perturbation method allows to calculate the mean and the standard deviation of the displacement of a dynamic system with uncertain parameters. This method has been used in many fields to solve linear and nonlinear problems, for static or dynamic modes.

The perturbation method can be used for mechanical systems with independent random parameters. It is based on the expansion of the first order Taylor series.

In this section we will apply the perturbation method. This method consists of approximating the dynamic function of movements Eq. (4) of random variables by their Taylor expansion around their mean values, depending on the order considered of the Taylor expansion. The method is said to be first order, second order or higher.

For the mechanical system of Eq. (4) with uncertain parameters, we suppose that the static deviation parameters and the initial velocity are functions of the random variables, $\{\beta_p\}_{(p=1,\ldots,P)}$.

The vector of the average parameters is defined by $\{\overline{\beta}\}$, and the quantity:

$$\{d\beta\} = \{\beta\} - \{\overline{\beta}\}$$

The following notation is used to simplify the writing of derivatives:

$$
\begin{aligned}
[A]^0 &= [A](\beta)|_{\beta=\overline{\beta}} \\
[A]^n &= \left.\frac{\partial[A](\beta)}{\partial\beta_n}\right|_{\beta=\overline{\beta}}
\end{aligned}
\tag{12}
$$

$[A]^0$, $[A]^n$ are deterministic corresponding to the derivatives, the repetition of the index " n " twice implies a summation.

The unknown position, velocity and acceleration vectors are also developed by Taylor series as follows:

$$
\begin{aligned}
\{r\} &= \{r\}^0 + \{r\}^n d\alpha_n \\
\{\dot{r}\} &= \{\dot{r}\}^0 + \{\dot{r}\}^n d\alpha_n \\
\{\ddot{r}\} &= \{\ddot{r}\}^0 + \{\ddot{r}\}^n d\alpha_n \\
\{\theta\} &= \{\theta\}^0 + \{\theta\}^n d\alpha_n \\
\{\dot{\theta}\} &= \{\dot{\theta}\}^0 + \{\dot{\theta}\}^n d\alpha_n \\
\{\ddot{\theta}\} &= \{\ddot{\theta}\}^0 + \{\ddot{\theta}\}^n d\alpha_n
\end{aligned}
\tag{13}
$$

Substituting these developments in the equation of motion, writing the terms of the same order, we obtain the following differential systems:

- Equation of order zero:

$$
\begin{aligned}
\{x\}^0 &= \{r\}^0 \cos\{\theta\}^0 \\
\{y\}^0 &= \{r\}^0 \sin\{\theta\}^0
\end{aligned}
\tag{14}
$$

- First order equation:

$$
\{x\}^n = \{r\}^n \cos\{\theta\}^0 + \{r\}^0\{\theta\}^n \sin\{\theta\}^0 \tag{15}
$$

$$
\{y\}^n = \{r\}^n \sin\{\theta\}^0 + \{r\}^0\{\theta\}^n \cos\{\theta\}^0 \tag{16}
$$

The mean is given by:

$$
\begin{aligned}
E[x(t)] &= \{x(t)\}^0 \\
E[y(t)] &= \{y(t)\}^0
\end{aligned}
\tag{17}
$$

The variance is given by:

$$
\begin{aligned}
Var[x(t)] &= \{x(t)\}^{n2} Var(\beta_n) \\
Var[y(t)] &= \{y(t)\}^{n2} Var(\beta_n)
\end{aligned}
\tag{18}
$$

4 Simulation and Results

To evaluate the variability of the nonlinear dynamic motion response with random parameters, we calculated the two moments (mean, standard deviations) of displacement of the point mass.

In this section, numerical results are presented after the formulation derived in Sect. 3, using MATLAB 2020 computer language scripts. The results of the perturbation method are compared with the Monte Carlo results for 3000 simulations. Some parameters are considered as random variables, the first random variable is the static spring deflection ε, and the second random variable is the initial velocity ϑ_0.The stochastic behavior is described using the normal random variables so these parameters are described by the following relations:

$$\varepsilon = \varepsilon_0 + \sigma_\varepsilon \aleph_1, \quad \vartheta = \vartheta_0 + \sigma_\vartheta \aleph_1 \tag{19}$$

ε_0 et ϑ_0 designate the average values, $\aleph_1$,$\aleph_2$ are the normal random variables, σ_ε, σ_ϑ, are the associated standard deviation.

Table 1. Non-dimensional initial parameters and conditions of the system.

ε_0	ϑ_0	θ_0	r_0
0.1	0.432	30°	1

To see the influence of multiple uncertain parameters on the dynamic response of the mass-spring system, it is assumed that the static deflection ε the velocity ϑ_0 are all uncertain parameters and the angle θ_0 and the initial spring length r_0 are constants Table 1.

For $\sigma = \sigma_\varepsilon = \sigma_\vartheta$ the mean values and standard deviations of the dynamic components of the nonlinear displacements along the two directions x and y have been calculated with the method of first order perturbation.

The results obtained are shown in Fig. 3 and Fig. 4 for $\sigma = 5\%$ and in Fig. 5 for $\sigma = 10\%$. These results are compared to those obtained with the Monte Carlo referential technique using 3000 simulations.

The results of the mean value response are very satisfactory, the instantaneous mean values of the displacements and the standard deviations are consistent with the Monte Carlo reference solutions, and the errors are still acceptable.

For the results of the standard deviations of the displacements, we can clearly see that for $\sigma = 5\%$ and $\sigma = 10\%$, the proposed perturbation method provides very similar results with the reference Monte Carlo method as well as the reduced computation time, from other Table 2.

On the other hand, we can see that the error increases when the standard deviation of uncertain parameters increases.

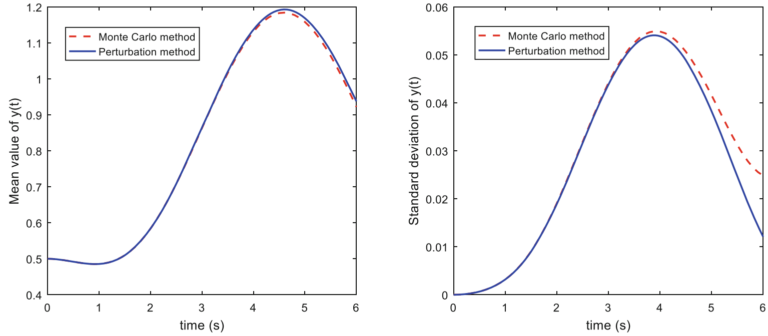

Fig. 3. Instantaneous mean value and standard deviation of x(t) for $\sigma = 5\%$.

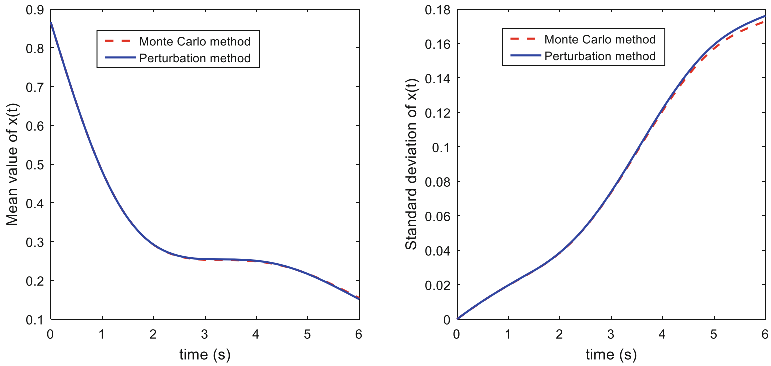

Fig. 4. Instantaneous mean value and standard deviation of y(t) for $\sigma = 5\%$.

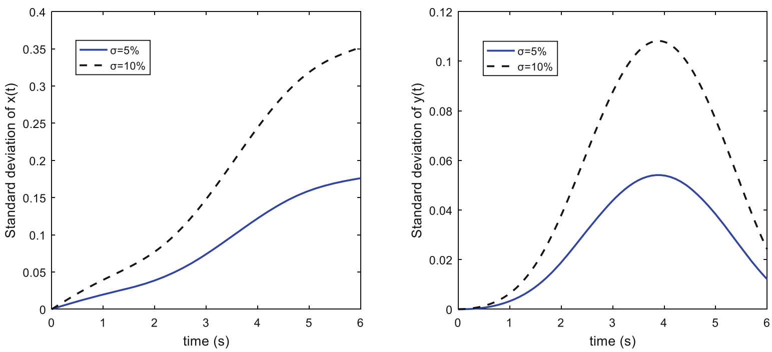

Fig. 5. Standard deviation of x(t) and y(t) for $\sigma = 5\%$ $\sigma = 10\%$

Table 2. Comparison of the CPU time in (s) between the two-simulation methods Monte Carlo method and perturbation method of order 1.

	Monte Carlo Simulation MCS	First order perturbation method
CPU time in (s)	45.45	1.34

5 Conclusion

This paper presents the stochastic method based on the development of the perturbation method by the first order Taylor expansion in combination with the RK4 method.

We have obtained similar results by the perturbation method and the Monte Carlo method. The results showed that these methods are efficient in terms of saving computational time.

This method allowed us to optimize the dynamic response of the mass-spring system and the computation time. We concluded that the two stochastic parameters ε and ϑ_0 significantly affect the performance of the jump.

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