A Short Proof of the Non-biplanarity of K_9

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Abstract. Battle, Harary, and Kodama (1962) and independently Tutte (1963) proved that the complete graph with nine vertices is not biplanar. Aiming towards simplicity and brevity, in this note we provide a short proof of this claim.

Keywords: Biplanar graph · Biplanar drawing · Edge crossing

1 Introduction

An embedding (or drawing) of a graph in the Euclidean plane is a mapping of its vertices to distinct points in the plane and its edges to smooth curves between their corresponding vertices. A planar embedding of a graph is a drawing of the graph such that no two edges cross. A graph that admits such a drawing is called planar. A *biplanar embedding* of a graph $H = (V, E)$ is a decomposition of H into two planar graphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ such that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$, together with planar embeddings of H_1 and H_2. In this case, H is called *biplanar*. In other words, a graph is called biplanar if it is the union of two planar graphs; that is, if its thickness[1] is 1 or 2. The *complete graph* with n vertices, denoted by K_n, is a graph that has an edge between every pair of its vertices. Let G be a subgraph of K_n that has n vertices. The *complement* of G, denoted by $\overline{G}$, is the graph obtained by removing all edges of G from K_n.

As early as 1960 it was known that K_8 is biplanar and K_{11} is not biplanar. There exist several biplanar embeddings of K_8; see e.g. [2] for a self-complementary drawing. The non-biplanarity of K_{11} is easily seen, since it has 55 edges while a planar graph with eleven vertices cannot have more than 27 edges, by Euler's formula. Finding the smallest integer n, for which K_n is non-biplanar, was a challenging question for some time [7]. The following fundamental theorem due to Battle, Harary, and Kodama ([1], 1962) and independently proved by Tutte ([15], 1963) answers this question and implies that K_9 is non-biplanar.

[1] The thickness of a graph G is the minimum number of planar subgraphs whose union equals to G.

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Theorem 1. *Every planar graph with at least nine vertices has a nonplanar complement.*

Both proofs of Theorem 1 involve a thorough case analysis. Battle, Harary, and Kodama gave an outline of a proof through six propositions. Some of these propositions require detailed case analysis, which is not given in the original paper. For example, the authors write: "There are several cases to discuss in order to establish Propositions 4 and 5. In each case, we can prove that $\overline{G}$ contains a subgraph homeomorphic to $K_{3,3}$ or K_5." A detailed proof of these propositions is appeared in the master's thesis of Hearon [9]. Tutte's proof is a 13-page paper, and enumerates all simple triangulations (with no separating triangles) with up to 9 vertices and verifies that the complement of each triangulation is nonplanar (the connection to triangulations will become clear shortly). It seems that Harary was not quite satisfied with any of these proofs as he noted in his Graph Theory book [8] that "this result was proved by exhaustion; no elegant or even reasonable proof is known." We are still unaware of any short proof of this result. (See [10] for a recent attempt towards a new proof.)

The non-biplanarity of K_9 has the same flavor as the well-known theorem of Kuratowski on non-planar graphs (stated in Theorem 3). The *biplanar crossing number* of a graph is the minimum number of crossings over all drawings of the graph in two planes [3]. It is known that K_9 can be drawn in two planes with one crossing (see e.g. [6]). This and Theorem 1 imply that the biplanar crossing number of K_9 is 1. Determining biplanar crossing numbers of K_n for small values of n is important as they lead to better bounds for biplanar crossing numbers of K_n for large values of n; see e.g. [3,4,13], and [6,14] for more recent progress.

2 Our Proof

In this section we present a short proof of Theorem 1. Our proof is complete, self-contained, and only uses Kuratowski's theorem for non-planar graphs. Towards our proof we show (in Theorem 2) that a particularly restricted drawing of K_8 cannot be biplanar (see Fig. 2(a) for an illustration).

Theorem 2. *Let H be an embedded planar graph with eight vertices such that the boundary of its outer face is a 5-cycle and there are no edges between the three vertices that are not on the outer face. Then the complement of H is nonplanar.*

Proof of Theorem 1. Consider a planar graph G with nine vertices. For the sake of contradiction assume that its complement $\overline{G}$ is also planar. Fix a planar embedding of G and a planar embedding of $\overline{G}$. For convenience we use G and $\overline{G}$ for referring to planar graphs and to their planar embeddings. If there are two vertices in G that lie on the same face and are not connected by an edge, then we transfer the corresponding edge from $\overline{G}$ to G and connect the two vertices by a curve in that face. After this operation both G and $\overline{G}$ remain planar. Repeating this process converts G to an edge-maximal planar graph. In particular G becomes a triangulation in which the boundary of every face (including the outer face) is a triangle (i.e. a 3-cycle).

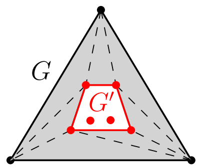
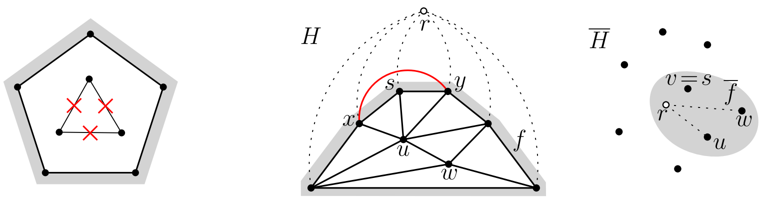


Fig. 1. Seven edges needed to triangulate the shaded polygon.

Claim 1. At least one vertex on the outer face of G has degree larger than four. To prove this claim we use contradiction. Assume that all three vertices on the outer face of G are of degree at most 4. The removal of these three vertices from G results in a 6-vertex graph G'. The region, that is between the boundaries of the outer face of G and the outer face of G' is a polygon with a hole, that is triangulated by at most six edges of G (because every vertex on the outer face of G has at most two edges in the interior of this polygon). The boundary of the outer face of G', i.e. the hole, has three vertices because otherwise (if it has at least four vertices) the polygon would require at least seven edges to be triangulated, as in Fig. 1; this can be verified by a simple counting argument using Euler's formula for planar graphs, see also [12, Proof of Lemma 5.2]. Thus the outer face of G' is a 3-cycle. In this case the other three vertices of G' which are in the interior of this 3-cycle together with the three removed vertices from G form a $K_{3,3}$ in $\overline{G}$, which contradicts its planarity. This proves Claim 1.

In view of Claim 1 we assume that at least one vertex, say r, on the outer face of G has degree $k \geq 5$. Remove r from G and $\overline{G}$ and denote the resulting graphs by H and $\overline{H}$, respectively. Notice that $(H, \overline{H})$ is a biplanar embedding of K_8. Let f and $\overline{f}$ be the faces of H and $\overline{H}$, respectively, that contain the removed vertex r, as in Fig. 2(b). Notice that f is the outer face of H. Since $(G, \overline{G})$ was a biplanar embedding of K_9, in which r was connected to all other 8 vertices, we have the following observation.

Observation 1. Every vertex of the resulting graph K_8 lies on f or on $\overline{f}$.



(a) (b)

Fig. 2. Illustration of (a) the statement of Theorem 2 (b) the proof of Theorem 1.

Since $\overline{G}$ was a simple graph (no multiedges and no loops), the face $\overline{f}$ has at least three vertices; these vertices are not necessarily connected in $\overline{H}$. Since G was a triangulation, the boundary of the outer face f of H is a k-cycle. If $k > 5$ then let s be a vertex of f that also lies on $\overline{f}$; such a vertex exists because $\overline{f}$ has at least three vertices and we have eight vertices in total. Let x and y be the neighbors of s on f. If xy is an edge of H then draw it as a curve in f. If xy is not an edge of H then transfer it from $\overline{H}$ to H and draw it in f, as in Fig. 2(b). Now, the new outer face f of H has $k - 1$ vertices. Repeat the above process until the outer face of H has exactly five vertices.

At this point f has five vertices. Let u, v, w be the vertices of K_8 that are not on f. These three vertices lie on $\overline{f}$, because of Observation 1 and our choices of s (for the case $k > 5$). If any of the edges uv, uw, and vw are not in $\overline{H}$ then transfer them from H to $\overline{H}$ and draw in $\overline{f}$ without crossing other edges. We obtain a planar graph H that satisfies the constraints of Theorem 2 and so that its complement $\overline{H}$ is planar. This contradicts Theorem 2. □

To prove Theorem 2 we use the theorem of Kuratowski [5,11] that "a finite graph is non-planar if and only if it contains a subgraph that is homeomorphic to K_5 or $K_{3,3}$." The following is an alternative statement for Kuratowski's theorem, which is given in [15].

Theorem 3. *A graph G is nonplanar if one of the following conditions hold: (i) G has six disjoint connected subgraphs $A_1, A_2, A_3, B_1, B_2, B_3$ such that for each A_i and B_j there is an edge with one end in A_i and the other in B_j. (ii) G has five disjoint connected subgraphs A_1, A_2, A_3, A_4, A_5 such that for each A_i and A_j, with $i \neq j$, there is an edge with one end in A_i and the other in A_j.*

Proof of Theorem 2. Let the 5-cycle $C = (a_1, a_2, a_3, a_4, a_5)$ be the boundary of the outer face of H, and let u, v, and w be the three vertices that are not on the outer face, i.e., lie on internal faces of H. By the statement of the theorem uv, uw, and vw are edges of the complement graph $\overline{H}$. Except for the three pairs (u, v), (u, w), (v, w), if a pair of vertices lie on the same internal face of H and are not connected by an edge, then we transfer the corresponding edge from $\overline{H}$ to H and connect the two vertices by a curve in the face. After this operation H remains planar. Repeating this process makes H edge-maximal (in the above sense).

Let H' be the embedded planar subgraph of H that is induced by the five vertices of C. The graph H' consists of the cycle C together with zero, one, or two chords as in Fig. 4.

Claim 2. If an internal face f of H' contains u, v, or w then one of them is connected to all boundary vertices of f in H. The shaded region in the figure to the right represents f. To verify the claim, first observe that (by edge-maximality of H) one of the vertices in f, say v, is connected to at least three boundary vertices of f, i.e., v's degree in H is at least three. We argue that v should be connected to all boundary vertices of f. For a contradiction assume that v is not connected to some vertex a_i on f. Let a_j and a_k be the neighbors of v on f that are visited first while walking on boundary of f in clockwise and counterclockwise directions starting from a_i. Since v is not connected to other vertices in the interior of f, we could have moved the edge $a_j a_k$ from $\overline{H}$ to H and draw it in f, as in Fig. 3. This means that H is not edge-maximal, which is a contradiction.

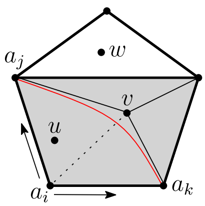


Fig. 3. Moving $a_j a_k$ from $\overline{H}$ to H.

Now we consider three cases depending on the number of chords of H'. In each case we show that $\overline{H}$ is nonplanar.

- H' has no chords. Let v be the vertex of H that (by Claim 2) is connected to each a_i; see Fig. 4(a). By planarity of H, each of u and w can only be adjacent to two consecutive vertices of C. Hence there exists a vertex of C (say a_1) that is adjacent to neither u nor w. In this setting, regardless of the locations of u and w, the five connected subgraphs u, w, a_1, $\{a_2, a_4\}$ and $\{a_3, a_5\}$ from $\overline{H}$ satisfy condition (ii) of Theorem 3. Thus $\overline{H}$ is nonplanar.
- H' has one chord. After a suitable relabeling assume that this chord is (a_2, a_5). Let f denote the face of H' whose boundary is the 4-cycle (a_2, a_3, a_4, a_5); this face is shaded in Fig. 4(b). This face contains some vertices of $\{u, v, w\}$ because otherwise H' should have a chord in f (by maximality of H) which contradicts our assumption that H' has one chord. Let v be the vertex in f that (by Claim 2) is connected to all its boundary vertices. By planarity of H, each of u and w can only be adjacent to two consecutive vertices of f. Therefore, the six connected subgraphs u, w, a_1, v, $\{a_2, a_4\}$, and $\{a_3, a_5\}$ from $\overline{H}$ (partitioned into $\{u, w, a_1\}$ and $\{v, \{a_2, a_4\}, \{a_3, a_5\}\}$) satisfy condition (i) of Theorem 3. Thus $\overline{H}$ is nonplanar.
- H' has two chords. Let a_1 be the vertex that is incident to the two chords as in Fig. 4(c). By planarity of H, each of u, v, and w can only be adjacent to one vertex in $\{a_2, a_4\}$ and to one vertex in $\{a_3, a_5\}$. Thus, the five connected subgraphs u, v, w, $\{a_2, a_4\}$, and $\{a_3, a_5\}$ from $\overline{H}$ satisfy condition (ii) of Theorem 3, and hence $\overline{H}$ is nonplanar. □

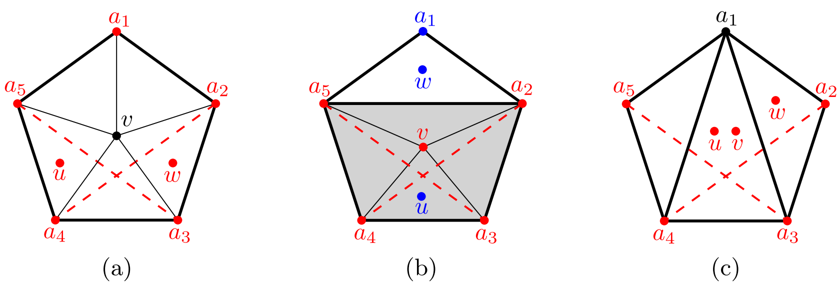


Fig. 4. Solid edges belong to H, bold edges belong to H', dashed edges belong to $\overline{H}$.

3 Conclusions

For any integer $k \geq 1$ let $\nu(k)$ be the smallest integer for which the (edges of the) complete graph with $\nu(k)$ vertices cannot be drawn in k planes without creating a crossing. As the maximum number of (noncrossing) edges that can be drawn in a plane is $3\nu(k) - 6$ and the number of edges of the complete graph is $\binom{\nu(k)}{2}$, a counting argument implies that

$$\nu(k) \leq \left\lfloor \frac{6k + 1 + \sqrt{36k^2 - 36k + 1}}{2} \right\rfloor + 1.$$

This bound implies that $\nu(1) \leq 5$ and $\nu(2) \leq 11$, however for $k \in \{1,2\}$ we already know that $\nu(1) = 5$ and $\nu(2) = 9$. It would be interesting to find exact value of $\nu(k)$ for larger values of k.

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