Observers with Unknown Inputs of Linear Systems

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Abstract This chapter introduces the concepts of observability and observer for linear systems, as well as the concept of sliding mode. It is demonstrated how to build an observer for a linear dynamical system with unknown inputs. The determination of the observer's gain in order to ensure convergence leads to the solution of an LMI issue (bilinear matrix inequalities). The technique based on a change of variables is used to resolve these LMI restrictions, allowing the matrices characterizing the observer to be determined.

1 Introduction

The complete or partial knowledge of the state of the considered system is an important requirement in the fields of control, diagnosis and monitoring of systems. This requirement is difficult to satisfy in most cases. This is due, on the one hand, to the fact that the state variables do not always have a physical meaning and their direct measurement is impossible to achieve. On the other hand, the sensors needed to measure the state variables are unavailable or of insufficient accuracy. Moreover, from an economic point of view, it is desirable to install a minimum of sensors in order to reduce the costs of instrumentation and maintenance.

The measurements made at the output of the system do not give complete information on the internal states of this system. It is therefore essential to reconstruct the unmeasured state variables. The idea used for several years, is the replacement of

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hardware sensors by software sensors or by state observers, which allow to reconstruct the internal information (states, unknown inputs) of the system from the system model, the known inputs and the measured outputs.

A physical system is often subject to disturbances, such as measurement noise, measurement uncertainties, system faults and external disturbances. These noises have an adverse effect on the normal behavior of the process, and their estimation can be used to design a control system capable of minimizing these effects. These disturbances are called unknown inputs when they appear as additional inputs to the process, and their presence can make the estimation of system states difficult.

Several works have been devoted to the synthesis of observers for linear systems with unknown inputs [1–7] The first results on linear state estimation date back to the 1970's. They can be grouped into two categories. The first category assumes a priori knowledge of information about these non-measurable inputs; in particular, Johnson [1] proposes a polynomial approach and Meditch [8] suggests to approximate the unknown inputs by the response of a known dynamical system. The second category proceeds either by estimating the unknown input [7] or by its complete elimination from the system equations [9].

Reduced order observers have been considered by several authors during the last years [10, 11]. However, Yang and Wilde [10] demonstrated that the full order unknown input observer can have a faster convergence speed than the reduced order observer.

The use of observers with unknown inputs for fault diagnosis and process monitoring systems has also attracted much attention [9, 11, 12]. In this chapter, we present some basic notions of observability and observers as well as some methods for reconstructing the states and unknown input of linear systems in the presence of unknown input.

2 Observability

In the literature, it is shown that an observer exists if and only if the state realization of the system in question is observable. Indeed, the observability of a system expresses the possibility of reconstructing the state from the sole knowledge of the input and output signals.

3 Observability of Linear Systems

The observability criteria of a linear system are described in many references [13, 14]. Let us consider the continuous linear time-invariant dynamical system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where t $\geq$ 0; x(t) $\in \mathbb{R}^n$; u(t) $\in \mathbb{R}^m$; y(t) $\in \mathbb{R}^p$, are the state vector, the input and the output of the system (1), respectively. A, B, C are the matrices of the system of appropriate dimensions, and the initial conditions are defined by$x(t_0) = x_0$: Let us recall some definitions and results on observability.

Definition 1 (*Borne* [14]) The system (1) is said to be observable if there exists a time $t_f \geq t_0$ such that the knowledge of the input $u(t)$ and the output $y(t)$ on the interval t $\in \left[t_0 t_f\right]$ is sufficient to determine the initial condition x_0 in a unique way.

For linear systems, the information produced at the output is the superposition of that generated by the input and that generated by the initial condition. If we assume the free regime (u = 0) then we can adopt the following definition.

Definition 2 (*Borne* [14]) The system (1) is observable if and only if, in the free regime ($u(t) = 0$; ∀t$\geq t_0$), the observation of a uniformly zero output $y(t)\ t \in \left[t_0\ t_f\right]$ is possible only for an initial state $x(t_0)$ zero.

Remarks 1 When all state variables are observable, then the system is said to be completely observable, otherwise it is said to be partially observable.

The observability condition is a necessary and sufficient condition to be able to estimate the state of the system from the information collected on the inputs and outputs. Note that the knowledge of x_0 and the state model of the system is sufficient to reconstruct the state x(t) at any time t $\geq t_0$. The observability property of a linear time invariant system is a structural property and depends only on the matrices A and C of the model. The most used criterion to check this property is the Kalman rank criterion formulated by the observability matrix below.

The system described by (1) is completely observable if and only if rank (O) = n such that (O) is the observability matrix defined by:

$$O = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} \tag{2}$$

4 Synthesis of Observers for Linear Systems Without Unknown Inputs

A solution to the problem of state estimation of linear systems has been proposed by Luenberger [15] in the deterministic framework and by Kalman [16] in the stochastic framework. Sliding mode observers are also used for linear systems even if they are themselves of nonlinear structure.

4.1 Luenberger Observer

The theory of observation is essentially based on pole placement techniques. Let $\widehat{x}(t)$ be the estimate of x(t); and $\widehat{y}(t)$ the estimate of y(t).

The observer proposed by Luenberger for the system (1) is described by the following equations:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - \hat{y}(t)); \hat{x}(t_0) = \hat{x}_0 \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{3}$$

where $K \in \mathbb{R}^{n*p}$ is the gain of the observer (3). The block diagram of the observer is illustrated by Fig. 1. The estimation error is given by

$$e(t) = x(t) - \hat{x}(t)$$

The dynamics of this error is governed by the following equation:

$$\dot{e}(t) = (A - KC)e(t); e(t_0) = e_0 = x_0 - \hat{x}_0$$

If the gain is chosen such that the matrix (A−KC) is Hurwitz, i.e., has strictly negative eigenvalues, then the estimation error converges asymptotically to zero. As the observer replaces the sensor, we must therefore ensure a convergence of the estimation error to zero very fast, at least ten times faster than the dynamics of the system. If the couple (A, C) is observable, then it is possible to determine the gain K to have a convergence dynamics chosen beforehand. The problem of constructing the observer is therefore equivalent to solving a pole placement problem. We choose a desired dynamics (choice of the desired eigenvalues of (A−KC), then using the pole placement principle, we determine the gain K.

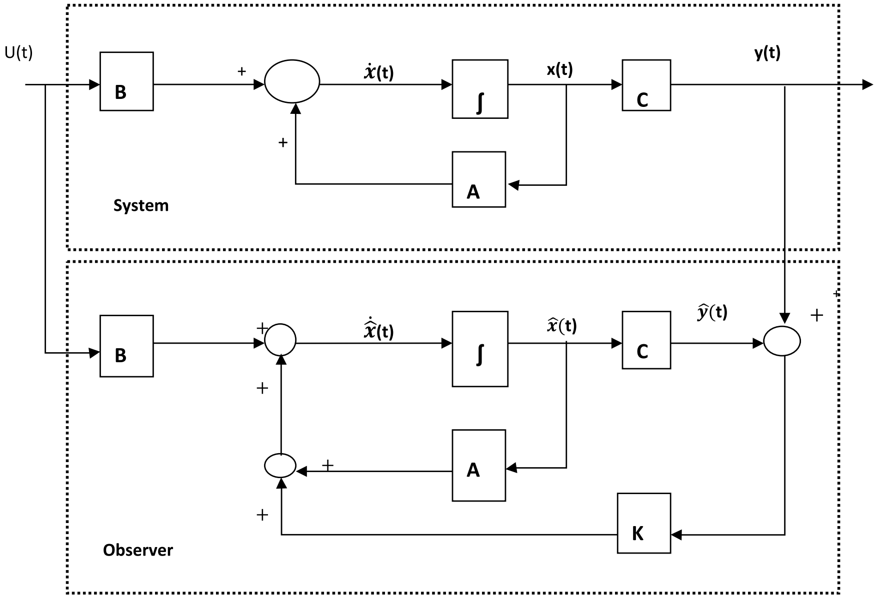

Fig. 1 Block diagram of the Luenberger observer

4.2 Sliding Mode Observer

Techniques based on the theory of variable structures, ensuring the robustness of the controller or the observer, are currently the subject of several research works. One of the best known classes of robust observers is that of sliding mode observers [17–20].

In [18], the principle of sliding mode observers consists in constraining, by means of discontinuous functions, the dynamics of a system of order n to converge to a sliding surface S of dimension (n−p) (p being the dimension of the measurement vector y). The attractiveness of this surface is ensured by conditions called sliding conditions. If these conditions are satisfied, the system converges towards the sliding surface and evolves there according to a dynamics of order (n-p). In the case of sliding mode observers, the dynamics concerned are those of the observation errors $e(t) = x(t) - \hat{x}(t)$. From their initial values e_0, these errors converge to the equilibrium values in two steps:

In the first stage, the trajectory of the observation errors evolves towards the sliding surface on which the errors between the observer output and the real system output (the measurements) $e_y = y - \hat{y}$ are zero. This stage is called the attainment mode.

In the second phase, the trajectory of the observation errors slides on the sliding surface with imposed dynamics so as to cancel all observation errors. This last mode is called sliding mode.

Consider a nonlinear state system of order n:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = Cx(t) \end{cases} \tag{4}$$

where: x(t) $\in \mathbb{R}^n$; u(t) $\in \mathbb{R}^m$; y(t) $\in \mathbb{R}^p$, f represent the state vector, the input or control vector, the output vector, the sufficiently differentiable vector field, respectively.

The different steps for the synthesis of the sliding mode observer are identified in [17]. The first order sliding mode observer allowing to reconstruct the estimated state vector $\hat{x}(t)$ is defined by the structure (5)

$$\begin{cases} \dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + \lambda sign(y - \hat{y}) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{5}$$

where the input u is locally bounded and measurable.

Sign: Represents the usual sign function.

λ: is the observation gain matrix of dimension (n–p). The correction term used is proportional to the discontinuous function sign applied to the output error.

For the estimated state to converge to the true state, the sliding mode observer must satisfy two conditions:

The first condition concerns the mode of reaching and guarantees the attractiveness of the sliding surface S = 0 of dimension p.

The sliding surface is attractive if the Lyapunov function $V(t) = S^T S$ satisfies the condition: $\dot{V}(t) \prec 0$.

The second one concerns the sliding mode, during this step, the corrective gain matrix acts so as to satisfy the following invariance condition:

$$\begin{cases} S = 0 \\ \dot{S} = 0 \end{cases}$$

During this mode, the dynamics of the system are reduced and the system of order n becomes an equivalent system of order (n–p). These criteria allow the synthesis of the sliding mode observer and determine its operation.

Phenomenon of reluctance

In practice, the discontinuous term on the right-hand side of the equation can excite unmodelled high-frequency dynamics that lead to the appearance of what is known as "reticence" or "chattering", which is characterized by strong oscillations around the surface.

5 Synthesis of Observers for Linear Systems with Unknown Inputs

5.1 Utkin Sliding Mode Observer with Unknown Input

Let us consider the continuous linear system time invariant with delay on the measurement

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Rw(t) \\ y(t) = Cx(t) \end{cases} \tag{6}$$

x(t) $\in \mathbb{R}^n$; u(t) $\in \mathbb{R}^m$; y(t) $\in \mathbb{R}^p$, $w(t) \in \mathbb{R}^q$ Are the state vectors, the vector of known inputs, the vector of measurable outputs, the vector of unknown inputs of the system (6), respectively. $A\in \mathbb{R}^{n*n}$; B $\in \mathbb{R}^{n*m}$; C $\in \mathbb{R}^{p*n}$, $R \in \mathbb{R}^{n*q}$ are the state matrix, the matrix of known inputs, the influence matrix of unknown inputs and the output matrix of the system (6), respectively. It is assumed that R is of full column rank and the pair (A; C) is observable. The reconstruction of the state variables is based on the measured outputs; a coordinate change can be performed to obtain the regular form [21].

By respecting these conditions a non-singular transformation matrix allows to rewrite respectively the output, state and control matrices in the new coordinates.

$$\tilde{A} = T_1 A T_1^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tilde{B} = T_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tilde{C} = C T_1^{-1}, \tilde{R} = T_1 R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

*o*ù

$$T_1 = \begin{bmatrix} Q \\ C \end{bmatrix}$$

$$\begin{bmatrix} \tilde{x}_1(t) \\ y(t) \end{bmatrix} = T_1 x(t), \tilde{x}_1(t) \in \mathbb{R}^{(n-p)}, R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

where I_p is the identity matrix of dimension p.

$Q = \begin{bmatrix} 0 & I_{n-p} \end{bmatrix}$ the system (6) in the new coordinates is written as follows

$$\begin{cases} \dot{\tilde{x}}_1(t) = A_{11}\tilde{x}_1(t) + A_{12}y(t) + B_1 u(t) + R_1 w(t) \\ \dot{y}(t) = A_{21}\tilde{x}_1(t) + A_{22}y(t) + B_2 u(t) + R_2 w(t) \end{cases} \tag{7}$$

We note that $CR = R_2$: so $CR \neq 0$ and there exists the pseudo-inverse matrix R_2^+ of the matrix R_2 such that $R_2 R_2^+ = I_{m_1}$, $m_1 = rang(CR) = rang(R)$.

The following transformation is applied to the model given in (7)

$$\begin{bmatrix} \overline{x}_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} I_{n-p} & -R_1 R_2^+ \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ y(t) \end{bmatrix}$$

with

$$T_2 = \begin{bmatrix} I_{n-p} & -R_1 R_2^+ \\ 0 & I_p \end{bmatrix}, \overline{A} = T_2 \tilde{A} T_2^{-1} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}, \overline{B} = T_2 \tilde{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}$$

$$\overline{C} = \tilde{C} T_2^{-2}, \overline{R} = T_2 \tilde{R} = \begin{bmatrix} \overline{R}_1 \\ \overline{R}_2 \end{bmatrix}$$

Where

$$\overline{R}_1 = 0$$

$-R_1 R_2^+$ is the pseudo-inverse of $-R_1 R_2$ and $x_1(1) \in R^{n-p}$.
The system (7) in the new coordinates is given by (8)

$$\begin{cases} \dot{\overline{x}}_1(t) = \overline{A}_{11}\overline{x}_1(t) + \overline{A}_{12} y(t) + \overline{B}_1 u(t) \\ \dot{y}(t) = \overline{A}_{21}\overline{x}_1(t) + \overline{A}_{22} y(t) + \overline{B}_2 u(t) + \overline{R}_2 w(t) \end{cases} \tag{8}$$

The pair ($\overline{A}_{11}$;$\overline{A}_{21}$) is observable because the pair (A; C) is observable [22], the gain L is chosen such that the eigenvalues of the matrix $\overline{A}_{11} - L\overline{A}_{21}$ are in the left half plane of the complex plane.

5.2 Structure of the Utkin Sliding Mode Observer with Unknown Input

The sliding mode observer structure considered for this system is:

$$\begin{cases} \dot{\hat{x}}_1(t) = \overline{A}_{11}\hat{x}_1(t) + \overline{A}_{12}\hat{y}(t) + \overline{B}_1 u(t) + \overline{L} v(t) \\ \dot{\hat{y}}(t) = \overline{A}_{21}\hat{x}_1(t) + \overline{A}_{22}\hat{y}(t) + \overline{B}_2 u(t) + I v(t) \end{cases} \tag{9}$$

where $\hat{y}(t)$ and $\hat{x}_1(t)$ are the estimates of $y(t)$ and $x_1(t)$ respectively, L is the observer gain and $v(t)$ is the discontinuous function given by:

$$v(t) = M sign(\hat{y}(t) - y(t))$$

With M > 0. The state and output estimation errors

$$\begin{cases} e_1(t) = \hat{x}_1(t) - x_1(t) \\ e_y(t) = \hat{y}(t) - y(t) \end{cases} \tag{10}$$

Subtracting (8) from (9), the dynamics of the estimation errors are written as follows:

$$\begin{cases} \dot{e}_1(t) = \overline{A}_{11}e_1(t) + \overline{A}_{12}e_y(t) + \overline{L}v(t) \\ \dot{e}_y(t) = \overline{A}_{21}e_1(t) + \overline{A}_{22}e_y(t) - v(t) + R_2w(t) \end{cases} \tag{11}$$

We perform the following change of variable

$$\begin{bmatrix} \tilde{e}_1(t) \\ e_y \end{bmatrix} = \begin{bmatrix} I_{n-p} & -\overline{L} \\ 0 & I_p \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_y \end{bmatrix}$$

The dynamics of the estimation errors will be written as follows:

$$\begin{cases} \dot{\tilde{e}}_1(t) = \tilde{A}_{11}\tilde{e}_1(t) + \tilde{A}_{12}e_y(t) + \overline{LR}_2w(t) \\ \dot{e}_y(t) = \tilde{A}_{21}\tilde{e}_1(t) + \tilde{A}_{22}e_y(t) - v(t) + \overline{R}_2w(t) \end{cases} \tag{12}$$

$\tilde{e}_1(t) = \tilde{e}_1(t) + Le_y(t)$ and $\tilde{A}_{11} = \overline{A}_{11} - L\overline{A}_{21}$.
$\tilde{A}_{12} = \overline{A}_{12} - L\overline{A}_{22} + \tilde{A}_{11}\overline{L}$ and $\tilde{A}_{22} = \overline{A}_{22} - \overline{A}_{21}\overline{L}$.

Utkin [21] has shown using the theory of singular perturbations, for a large enough gain M the sliding regime can be established on the error (12). So after a finite time the error e(t) and its derivative will be zero and we have from Eq. (12).

$$\dot{\tilde{e}}_1(t) = \tilde{A}_{11}e_1(t)$$

The gain M is chosen such that $\tilde{A}_{11}$ is stable and thus the system of Eqs. (12) converges asymptotically to zero, and $\tilde{e}_1(t) \to 0$ when $t \to \infty$.

The equivalent control method is used to obtain the estimated unknown input. It is assumed that the error of the system (12) is in the slip along $e_y = 0$: thus $\dot{\tilde{e}}_1 = 0$ and $\dot{\tilde{e}}_y = 0$. The solution of the system of Eq. (12) for $w(t)$ gives us the following estimate of $w(t)$:

$$\hat{w} \approx \left((I + \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{L})\tilde{R}_2\right)^{+} v_{eq} \tag{13}$$

where v_{eq} is the equivalent command.

6 Luenberger Observer with Unknown Input

In this section, we present the synthesis of large gain type observers for the class of uniformly observable nonlinear systems to which we have added unknown inputs. The proposed observers jointly estimate the entire state of the system as well as all unknown inputs under sufficient conditions that will be given. Their synthesis neither assumes nor adopts any mathematical model for the unknown inputs. We simply assume that the first derivative with respect to time of each of the unknown inputs is bounded.

Before presenting the class of nonlinear systems that will be the object of our study, we propose to recall the necessary and sufficient conditions for the synthesis of an observer with unknown inputs for linear systems. This will allow us to better understand the sufficient conditions that we will adopt for the synthesis of the proposed observers.

Note the necessary and sufficient conditions that will be recalled for linear systems concern the synthesis of an observer allowing the estimation of states (via a full or reduced order observer) without any knowledge about the unknown inputs. These conditions can be relaxed if certain assumptions about these inputs, such as the boundedness of their derivatives with respect to time, are adopted. We will come back to this point later in this part.

- **Reminders on observers with unknown input synthesis for linear systems**:

We consider the following linear time invariant system:

$$\sum \begin{cases} \dot{x} = Ax + Bu + Gv \\ y = Cx \end{cases} \tag{14}$$

where state x(t) $\in \mathbb{R}^n$, known input u(t) $\in \mathbb{R}^\mu$, unknown input v(t) $\in \mathbb{R}^m$, output y $\in \mathbb{R}^p$, A, B, G, and C are known constant matrices of appropriate dimensions, and matrix G is assumed to be full rank in columns, i.e.

$$Rang(G) = m \tag{15}$$

Without detracting from generality, we assume that the matrix C has the following structure:

$$C = \left[\, I_p\ 0\ ...\ 0 \,\right] \tag{16}$$

In the same way, we will pose

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \tag{17}$$

where $G_1 \in \mathbb{R}^{p*m}$ and $G_2 \in \mathbb{R}^{(n-p)\times m}$. Note that with this notation, we have

$$G_1 = CG \tag{18}$$

An observer with unknown inputs exists for this system if and only if the following two rank conditions are satisfied: [3–5].

$$Rang(CG) = m \tag{19}$$

$$Rang\begin{pmatrix} sI_n - A & G \\ C & 0 \end{pmatrix} = n + m, , \forall s \in C, \Re e(s) \geq 0 \tag{20}$$

We propose in the following to give some developments to show how these conditions are obtained. These developments will be mainly used to bring some complements on the synthesis of the observer when the number of unknown inputs is equal to the number of outputs.

The results we will present are described in [4] [3]. We will repeat them with more details here in the case where the matrix C has the particular structure (but not restrictive) (16).

The objective is to synthesize an observer that is written in the following form:

$$\sum \begin{cases} \dot{z} = Nz + Ly + Du \\ \hat{x} = z - Ey \end{cases} \tag{21}$$

where the observer state $z \in R^n$, $\widehat{x} \in \mathbb{R}^n$ is the estimated state of the system x, $N \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{n\times \mu}$, and $E \in \mathbb{R}^{n\times p}$ are matrices that must be chosen so that the observation error $e = \widehat{x} - x$ converges asymptotically to 0.

To do this, let's pose

$$P = I_n + EC \tag{22}$$

The observation error is then written

$$\begin{aligned} e &= \hat{x} - x \\ &= z - Ey - x \\ &= z - (I_n + EC)x \\ &= z - Px \end{aligned}$$

It follows that

$$\begin{aligned}\dot{e} &= \dot{z} - P\dot{x} \\ &= Nz + Ly + Du - PAx - PBu - PGv \\ &= N(e + Px) + LCx - PAx - (PB - D)u - PGv \\ &= Ne + (NP - PA + LC)x - (PB - D)u - PGv\end{aligned} \tag{23}$$

If the matrices N, L, D and E are chosen so that the following conditions are satisfied

$$\text{N is Hurwitz matrix} \tag{24}$$

$$PG = (I_n + EC)G = 0 \tag{25}$$

$$D = PB = (I_n + EC)B \tag{26}$$

$$LC - PA = -NP \tag{27}$$

Then Eq. (23) becomes

$$\dot{e} = Ne \tag{28}$$

And the observation error converges asymptotically to 0.

It is now necessary to study under which conditions the choice of matrices N, L and E verifying (24), (25) and (27) is possible. Note that the matrix D is determined from E by the relation (26).

Equation (27) can be rewritten as follows

$$\begin{aligned}0 &= NP + LC - PA \\ &= N(I_n + EC) + LC - PA \Rightarrow \\ N &= PA - LC - NEC \\ &= PA - (L + NE)C \\ &= PA - KC\end{aligned} \tag{29}$$

$$K = L + NE \tag{30}$$

With

If we replace N by its expression (29) in Eq. (30), we obtain

$$K = L + (PA - KC)E$$

Or in an equivalent way

$$\begin{aligned} L &= K - (PA - KC)E \\ &= K(I_p + CE) - PAE \end{aligned} \tag{31}$$

The dynamics of the observer (21) then becomes:

$$\dot{z} = (PA - KC)z + Ly + Du \tag{32}$$

where the matrices P (or equivalently the matrix E), K, L and D are given by Eqs. (25), (30), (31) and (26). The problem of synthesizing the observer consists in finding a matrix E satisfying (25) and a vector K so that the matrix PA–KC is a Hurwitz matrix. This is a similar problem to that of the synthesis of classical observers. The eigenvalues of the matrix PA–KC can be chosen arbitrarily if and only if the pair (PA, C) is observable. Otherwise, a vector K such that the observation error (28) converges asymptotically to 0, exists if and only if the pair (PA, C) is detectable.

We will now discuss the conditions under which the matrix E (or equivalently the matrix P) exists.

Taking into account the particular structures considered for the C and G matrices (Eqs. (16) and (17)), Eq. (25) becomes

$$EG_1 = -G \tag{33}$$

The solution of Eq. (33) depends on the rank of the matrix G1 = CG. Note that since C is of rank plain with Rank C = p and Rank G = m, we have Rank G1 = min(p, m). There are two cases to consider:

1. $Rank(G_1) = p \prec m$
2. $Rank(G_1) = m \leq p$

Case 1: $Rank(G_1) = p \prec m$.

In this case, there is no solution for the matrix E. Indeed, the equality (33) cannot take place since on the one hand we have:

$$Rank(EG_1) \leq Rank(G_1) \prec m$$

And on the other hand, we have

$$Rank(-G) = Rank(G) = m$$

Since two equal matrices have trivially the same rank, the equation in E (33) does not admit any solution.

Case 2: $Rank(G_1) = m \leq p$.

The general solution of (33) is

$$E = -GG_1^+ + Y(I_p - G_1G_1^+) \tag{34}$$

where G_1^+ is the left inverse of G_1 and $Y \in \mathbb{R}^{n\times p}$ is an arbitrary matrix. The matrix P can then be expressed as follows:

$$\begin{aligned} P &= I_n + EC \\ &= I_n - GG_1^+C + Y(I_P - GG_1^+)C \\ &= I_n + YC - GG_1^+C - YCGG_1^+C \\ &= I_n + YC - (I_n + YC)GG_1^+C \\ &= (I_n + YC)(I_n - GG_1^+C) \end{aligned} \tag{35}$$

Note that the maximum rank of the matrix,$n - m$, is obtained when the matrix $(I_n + YC)$ is non-singular [3].

We can now summarize the results obtained by the following theorem [3]:

Theorem 1 *An observer of type* (32) *exists for the system* (14) *if and only if*:

(1) $Rank(CG) = Rank(G_1) = m$

(2) $Rank\begin{bmatrix} sI_n - PA \\ C \end{bmatrix} = n, \forall s \in \Re e(s) \geq 0$

We now give a second theorem which shows that the results of Theorem (14) *correspond to the conditions generally adopted for the synthesis of the observer* (32) [3]:

Theorem 2 *It is assumed that* $Rank(CG) = Rank(G_1) = m$ *and that* $Rank(P) = n - m$. *Then the following four conditions are equivalent*:

(1) The pair (PA,C) is detectable (observable);

(2) $Rank\begin{bmatrix} sP - PA \\ C \end{bmatrix} = n, \forall s \in \mathbb{C}; \Re e(s) \geq 0, (\forall s \in \mathbb{C})$;

(3) $Rank\begin{bmatrix} sI_n - PA \\ C \end{bmatrix} = n, \forall s \in \mathbb{C}; \Re e(s) \geq 0$;

4) $Rank\begin{bmatrix} sI_n - A & G \\ C & 0 \end{bmatrix} = n + m, \forall s \in \mathbb{C}, \Re e(s) \geq 0, (\forall s \in \mathbb{C})$;

We will now take a closer look at the case m = p, i.e. when the number of outputs is equal to the number of unknown inputs. This case has been discussed in [4] *and addressed in* [3]. *In this case we will give very simple conditions for the synthesis of the observer and we will give more details on the choice of the poles of the observer, when it exists.*

Special case: Rank(G1) = m and p = m.

We will look directly for the matrices N, L and E satisfying the conditions (24), (25) and (27).

Note that by multiplying each of the members of equality (27) on the right by G, and taking into account equality (25), we obtain

$$LCG = PAG$$

or equivalently, taking into account the structures of C and G,

$$LG_1 = PAG \tag{36}$$

Which then becomes the new equation fixing the choice of L.

As $m = p$ and the matrix G_1 is square and is invertible. From (33) and (36), we obtain:

$$E = -GG_1^{-1}$$
$$L = PAGG_1^{-1}$$

It remains now to study the choice of the matrix N. By noticing that:

$$\begin{aligned} LC - PA &= PAGG_1^{-1}C - PA \\ &= PA(GG_1^{-1}C - I_n) \\ &= -PA(EC + I_n) \end{aligned} \tag{37}$$

$$= -PAP \tag{38}$$

Equation (27) becomes

$$PAP = NP \tag{39}$$

Given the particular structures of C and G, the matrix

$$P = I_n - GG_1^{-1}C$$

Has the following topology

$$P = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} - \begin{bmatrix} I_m & 0 \\ G_2G_1^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ -G_2G_1^{-1} & I_{n-m} \end{bmatrix}$$

Now considering the following partitions of A and N:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

where $A_{11}, N_{11} \in \mathbb{R}^{m\times m}$, $A_{12}, N_{12} \in \mathbb{R}^{m\times(n-m)}$, $A_{21}, N_{21} \in \mathbb{R}^{(n-m)\times m}$ and $A_{22}, N_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$, we obtain

$$PAP = \begin{bmatrix} 0 & 0 \\ -(A_{22} - G_2 G_1^{-1} A_{12}) G_2 G_1^{-1} & A_{22} - G_2 G_1^{-1} A_{12} \end{bmatrix}$$
$$NP = \begin{bmatrix} -N_{12} G_2 G_1^{-1} & N_{12} \\ -N_{22} G_2 G_1^{-1} & N_{22} \end{bmatrix}$$

Equality (39) thus becomes imposes the following relations:

$$\begin{aligned} N_{12} &= 0 \\ N_{22} &= A_{22} - G_2 G_1^{-1} A_{12} \end{aligned} \tag{40}$$

We note that the matrix N_{12} and N_{22} are imposed by the relations (40). It follows that a necessary condition for the matrix N to be Hurwitz is that the matrix $A_{22} - G_2 G_1^{-1} A_{12}$ is also Hurwitz. Thus, observer synthesis is only possible if this matrix is Hurwitz. In this case, only m poles of the observer can be chosen arbitrarily through the choice of the matrix N_{11} (we can take $N_{21} = 0$). The other poles of the observer are equal to the eigenvalues of the matrix $A_{22} - G_2 G_1^{-1} A_{12}$.

In what follows we will consider a class of nonlinear systems and we will propose some sufficient conditions that allow either to simply estimate all the states of the system without any knowledge about the unknown inputs, or to jointly estimate all the states of the system and all the unknown inputs under the additional assumption that the first derivative with respect to time of each unknown input is bounded.

- **Class of non-linear systems considered**:

Let be the following class of multi-input/multi-output nonlinear systems:

$$\begin{cases} \dot{x} = f(u, x) + G(u, s)v \\ y = Cx = x^1 \end{cases} \tag{41}$$

where the state of the system$x \in \mathbb{R}^n$, $x^1 \in \mathbb{R}^p$ is the output of the system, $X \in \mathbb{R}^{n-p}$ is the part of x containing all unmeasured states; the known input $u(t) \in$ U the set of absolutely continuous functions ‘with bounded derivatives from $\mathbb{R}^+$ into U a compact of $\mathbb{R}^v$;v $\in \mathbb{R}^m$ is the unknown input with $m \leq p$; $f(u, x) = \begin{pmatrix} f^1(u, x) \\ f_X(u, x) \end{pmatrix} \in \mathbb{R}^n$, $f^1(u, x) \in \mathbb{R}^p$, $f_X(u, x) \in \mathbb{R}^{n-p} and G(u, s) = \begin{pmatrix} G^1(u, s) \\ G_X(u, s) \end{pmatrix}$ is a matrix of dimension $n \times m$ where $G^1(u, s) and G_X(u, s)$ are respectively of dimension $p \times m and (n - p) \times m$ matrices; $s(t)$ is a known bounded signal whose first derivative with respect to time is also bounded; finally$\overline{C} = \begin{bmatrix} I_p & 0_{p \times (n-p)} \end{bmatrix}$.

- **Observer synthesis procedure**:

For observer synthesis, we adopt the following assumptions:

(H1) The matrix $G^1(u, s(t))$ is full rank in columns for all $u \in$ U and for all $t \geq 0$.

(H2) The derivative with respect to time of the unknown input $v(t)$ is a completely unknown function,$\varepsilon(t)$, which is uniformly bounded, i.e., $\sup_{t \geq 0} \|\varepsilon(t)\| \leq \beta_\varepsilon$ where $\beta_\varepsilon > 0$ is a strictly positive unknown real.

7 Conclusion

In this chapter, the notion of observability and observer for linear systems is presented, also the notion of sliding mode is introduced. how to design an observer for a linear dynamical system under the influence of unknown inputs is shown. The determination of the gain of the observer to guarantee its convergence leads to the resolution of a problem of the LMI type (bilinear matrix inequalities). The resolution of these LMI constraints is performed by the method based on a change of variables and which allows to determine the matrices describing the observer.

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