Michele Emmer
Marco Abate *Editors*

Imagine Math 8

Dreaming Venice

Springer

Imagine Math 8

This volume is dedicated to
the Liberty of Ukrainian people.
—Rome, 27 February 2022

Massimiliano Botti, *PI MACHINE Architects*, Paris (February 2022)

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Imagine Math 8

Dreaming Venice

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To
Luca Boschi

To
Olga Filipovna Tcherny
Kostantin Kostantinovic Tcherny
Ekaterina Kostantinova Tcherny
Otto Michalovic Grauding
Tatiana Grauding Emmer

Preface

Imagine all the people
Sharing all the world . . .
John Lennon

Imagine building mathematical models that make it possible to manage our world better, imagine solving great problems, imagine new problems never before thought of, imagine combining music, art, poetry, literature, architecture, theatre, and cinema with mathematics. Imagine the unpredictable and sometimes counterintuitive applications of mathematics in all areas of human endeavour.

Imagination and mathematics, imagination and culture, culture and mathematics. For some years now, the world of mathematics has penetrated deeply into human culture, perhaps more deeply than ever before, even more than in the Renaissance. In theatre, stories of mathematicians are staged; in cinema, Oscars are won by films about mathematicians; all over the world museums and science centres dedicated to mathematics are multiplying. Journals have been founded to explore the relationships between mathematics and contemporary art and architecture. Exhibitions are mounted to present mathematics, art and mathematics, and images related to the world of mathematics.

The volumes in the series *Imagine Math* are intended to help readers grasp how much that is interesting and new is happening in the relationships between mathematics, imagination, and culture.

This eighth volume of *Imagine Math* is different from all the previous ones, including those of the *Mathematics and Culture* series. The reason is very clear: in the last two years the world changed, and we still do not know what the world of tomorrow will look like. Difficult to make predictions. It is difficult to say if and when we will begin to meet and talk to each other again, exchanging ideas and opinions in person. This volume is different because it is not the Proceedings of a conference in Venice since the editions of 2020 and 2021 did not take place. It probably would not be held anymore for many reasons. Years go by not just for people.

This volume has a subtitle *Dreaming Venice*. Venice, the dream city of dreams, that miraculous image of a city on water that resisted for hundreds of years, has become in the last two years a truly an unreachable dream as Shakespeare said (*The Tempest*):

> We are such stuff as dreams are made on,
> and our little life is rounded with a sleep.

We dreamed of Venice, and not being able to meet in person, we created a new volume in which Venice is present but as a desired dream. Without Venice, this series of meetings that began in 1987 would have made no sense.

Many things tie this book to the previous ones. Once again, this volume too starts like *Imagine Math 7,* with a homage to the Italian artist Mimmo Paladino who created exclusively for the *Venice Conference* and the *Imagine Math 8* volume a new series of ten original and unique works of art dedicated to Piero della Francesca.

A memory of Napoleon could not be missing not only for the anniversary of his death on May 5, 1821, but above all because it is Napoleon, passionate of mathematics, who actually gave life to the *Istituto Veneto di Scienze, Lettere ed Arti* (location of the Venice Conference) as the Emeritus Chancellor Fraschini explains.

An intervention on the protection system of Venice, a very fragile city, could not be missing either, a system that seems to be working well.

Many artists, art historians, designers, and musicians are involved in the new book, among others Linda D. Henderson and Marco Pierini, Claudio Ambrosini, and Davide Amodio. Space also for comics and mathematics in a Disney key. Many applications, from Origami to mathematical models for world hunger. Particular attention to classical and modern architecture, with Tullia Iori.

As usual the topics are treated in a way that is rigorous but captivating, detailed, and full of evocations. This is an all-embracing look at the world of mathematics and culture.

Rome, Italy
June 16, 2021

Michele Emmer

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Editors and Contributors

About the Editors

Michele Emmer is professor of mathematics at the University of Rome "Sapienza" until 2015 and member of IVSLA, Istituto Veneto Scienze, Lettere ed Arti, Venice. His areas of interest include PDE and minimal surfaces, relationships between mathematics and arts, architecture, cinema, and culture. Since 1992, he is member of the board of the journal *Leonardo: Art, Science and Technology*, MIT Press. He is a filmmaker and author of the film series "Art and Math" distributed in many countries. He organizes since 1997 the annual conference on "Mathematics and Culture" in Venice. He is editor of the series *Mathematics and Culture* and *Imagine Math*, Springer, as well as the series *The Visual Mind*, MIT press. His books include *Bolle di sapone tra arte e matematica*, 2009, Viareggio Award Best Italian essay 2010*; Numeri immaginari: cinema e matematica*, Bollati Boringhieri, 2011; *Imagine Math 6*, Springer 2018; *Imagine Math 7*, Springer, 2020; *Racconto Matematico*, Bollati Boringhieri, 2019; M. Emmer, M. Pierini, ed. "Soap Bubbles from Vanitas to Architecture", Exhibition, Catalogue, 16 March–9 June 2019, Galleria Nazionale Umbria, Perugia; M. Emmer, ed., "Mimmo Paladino Mathematica", catalogue exhibition, Venice March 2019; *Persone Racconto Russo*, to appear 2021.

Marco Abate is a Full Professor of Geometry at the University of Pisa. He has written more than one-hundred scientific papers and textbooks, as well as several papers on the popularization of mathematics. His interests include holomorphic dynamics, geometric function theory, differential geometry, writing (comic books and more), photography, origami, and travelling (having already visited Antarctica his next destination is the Moon).

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Part I
Homage to Mimmo Paladino

8 Works by Mimmo Paladino

Michele Emmer

In the frescoes of the Basilica of San Francesco in Arezzo there are many beautiful conceptions and attitudes worthy to be extolled; above all other consideration about brilliance and art . . . This is done with very great thought, for Piero gives us to know in this darkness how important it is to copy things as they are and to ever take them from the true model; which he did so well that he enabled the moderns to attain, by following him, to that supreme perfection wherein art is seen in our own time.
Giorgio Vasari, Le vite [1].

Sulla Mathematica, an exhibition with Italian artist Mimmo Paladino' works opened in March 2019 in Palazzo Loredan in Venice, location of the *Istituto Veneto di Scienze, Lettere ed Arti IVSLA*, on the occasion of the International Conference on *Mathematics and Culture* "Imagine Math 7" [2, 3].

Three months later, on June 15th 2019, a large exhibition of the works of the artist Mimmo Paladino was organized in Arezzo by the title *Paladino. La regola di Piero (Piero's Rule)*". The Curator Luigi Maria Di Corato wrote in the catalogue: [4].

"The symbolic work of the exhibition is *Suonno (d'après Piero della Francesca)* (1983). The scene (Fig. 1) attracted the attention of Paladino who decided to transcribe it in his pictorial language, also stated by the artist's choice of a Neapolitan title (*suonno* means dream). In the painting Paladino does not insist on the announcing angel, but on the figure of the valet sitting on the emperor's bed. He surrounds the head of the sleeping man, who has lost the features of Costantine to acquire those of the typical archetypal characters of the artist. As in Piero's works the story is structured in two moments, divided by a vertical element: next to the main scene, a second moment takes shape, as in a following frame of the same

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Fig. 1 Piero della Francesca, *Il sogno di Costantino*, *Storie della vera Croce*, Affresco Basilica San Francesco, Arezzo (1458–1466) Su concessione del Ministero dei beni e attività Culturali, Direzione Regionale musei della Toscana

movie. Here the servant seems to have gotten up to start an incomprehensible gesture dialogue with the emperor." (Fig. 2).

In 1948 Luciano Emmer decided to make a documentary series on art and one was dedicated to Piero's frescoes in Arezzo, by the title *L'invenzione della croce* (The invention of the cross) (Fig. 3) [5]. I remember well the scene of Constantine's dream in the film, filmed with the technique he had invented to never show the location of the figure but always remain inside the work to tell the story from within the universe invented by the artist, telling the story with images commented almost only with music.

That scene came back to me when I wrote an article on Piero and mathematics in the *Notices of the AMS* [6] on the book by J. V. Field *Piero della Francesca. A Mathematician's Art* [7].

"Piero della Francesca (c. 1429–1492) was the painter who 'set forth the mathematical principles of perspective in fairly complete form . . . Piero was the painter mathematician and the scientific artist par excellence, and his contemporaries so regarded him. He was the best geometer of his time." So wrote Morris Kline in his monumental work *Mathematical Thought from Ancient to Modern Times* [8].

Fig. 2 M. Paladino, *Suonno (d'après Piero della Francesca)*, oil on canvas, dim 220 × 205 (1983)

Fig. 3 Frame from the movie *Invenzione della Croce*, (1948) by L. Emmer, © Cineteca Nazionale, Bologna & M. Emmer

Field concurs with this opinion, observing that [9] "the assertion that Piero's pictures are mathematical is usually so vague that it is understandable that some art historians have preferred largely to ignore it."

In the catalogue of the Arezzo exhibition the curator added [10]: "Piero sees mathematics as a sort of religion and by following this he exceeds the limit of two dimensional space creating both a conceptual representation of the world, parallel to reality, with practically infinite potentials, and coding his experiments so that they can be shared and become common art language.

Paladino finds new sources of inspiration in numbers and in the continuous search for space. Two elements that give shape to a language made of iconic signs, which often become part of the work or turn into work themselves as, for example, in the Great Geometries."

As Paladino recently claims: "Piero della Francesca for me is un unlimited source of discoveries. His ability to create shapes from light, space from mathematics, color from gray, his almost heraldic iconicity, are a constant point of reference, almost a rule, that's why I decided to narrate it in Arezzo."

Piero and Paladino's works have several contact points. Both seek an art that is strongly anti-expressionistic, so that the emotional sphere of both artists and the character represented is deliberately tamed thanks to a rigorous process that tends to mediate to *slow down* feelings, in search of that harmony that develops from silence, between correspondences and counterpoints. "Hence, it is no coincidence that Piero's Rule takes shape in Arezzo: an exhibition in six locations, a tribute to confirm how much the painter and the mathematician of Sansepolcro was decisive for Paladino, not only on an aesthetic level, but also on a methodological and theoretical level.

Paladino decided to made a series of paintings for the volume *Imagine Math 8*, a sort of addition to the art works shown in Arezzo, sending 8 works by the general title *Il Principio della prospettiva* for the proceedings of the i*maginary* Venice conference. My best thanks to him for his kindness.

Mimmo Paladino

Il Principio Della Prospettiva

M. Paladino, *Senza titolo*, codice MP P20039 (2020) tecnica mista su tela, 120.00 × 100 cm. Incorniciata misura 104 × 124 cm

M. Paladino, *Senza titolo*, codice MP P20040 (2020) tecnica mista su tela, 120.00 × 100 cm. Incorniciata misura 104 × 124 cm

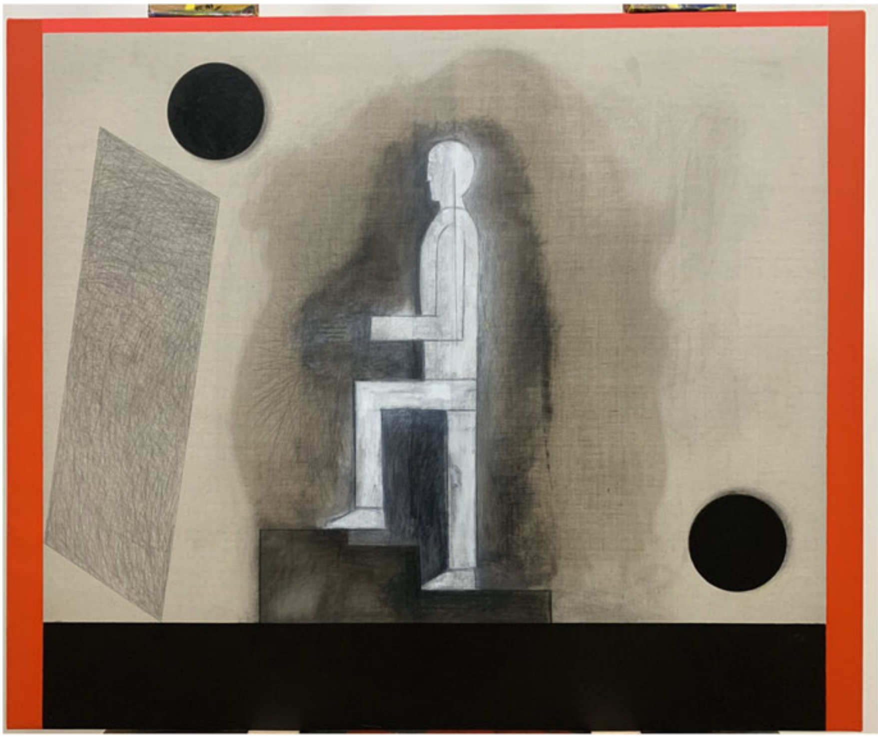
M. Paladino, *Senza titolo*, codice MP P20041 (2020) tecnica mista su tela, 120.00 × 100 cm. Incorniciata misura 104 × 124 cm

M. Paladino, *Senza titolo*, codice MP P21003 (2021) acrilico e matita su tela, 120.00 × 100.00 cm

M. Paladino, *Senza titolo*, codice MP P21007 (2021) acrilico e matita su tela, 120.00 × 100.00 cm

M. Paladino, *Senza titolo*, codice MP P21005 (2021) acrilico e matita su tela, 120.00 × 100.00 cm

M. Paladino, *Senza titolo*, codice MP P21006 (2021) acrilico e matita su tela, 120.00 × 100 cm



M. Paladino, *Senza titolo*, codice MP P21007 (2021) acrilico e matita su tela, 120.00 × 100.00 cm

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9. in [7], pp. 3–4
10. in [4], pp. 17–18

Part II
Dreaming in Venice

Dreaming Venice

Michele Emmer

> *These two little squares linked*
> *by this hidden lane called the "Narrow Passage of Nostalgia" were the fabled center at which two secret worlds melted into one another: one came from the disciplines of the Talmud, the other from Judeo-Greek-Oriental esoteric philosophical disciplines.*
> *This maze of staircases, lanes, yards, and little square was known as the "Serraglio of Beautiful Ideas".*
>
> Hugo Pratt, *Favola di Venezia,* 1977

The *Mathematics and Culture* conference was conceived in Venice, not by chance. It was 1997 and it was a dramatic time for my family. We lived in Turin and our dream, both Valeria's and mine, was to return to Venice, to our small house. We knew that we had little hope but it was the dream we had, as anybody would be entitled to having, just through a little imagination. We decided in 1987 that we wanted to live in Venice. I accepted the position they offered me at the University of Ca′ Foscari. We thought that in a few years' time all the family would be there. Things went differently and after seven years I went back to teach at Sapienza in Rome, at the Faculty of Architecture. In 1996 came the disease, with the consequent transfer to

A different version was published in the book E. Loew ed., *Math at the Time of Corona* by the title *Soap Bubbles Vanitas Venice*, Springer, 2021 [1]

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Turin. But the dream remained and so we invented a conference that would be held always and only in Venice, at the Santa Margherita auditorium of the University of Ca′ Foscari.

It is worth noting that words like dream, creativity, imagination often appear when mathematicians think about their activity.

> We are such stuff as dreams are made on,
> and our little life is rounded with a sleep.
>
> William Shakespeare, *The Tempest,* 1610

Mathematicians' relationship with dreams has an interesting story. Many papers have been published on mathematics and dreams. In november 2019 a conference on *Auto/biographie: prémonitions, rêves, cauchemars* took place at the Cyprus University in Nicosia organized by the *Département d'Etudes Françaises et Européennes* and by the journal *Mediapolis* published by the Presses Universitaires de Louvain. The session *En deça et au-delà des humanités* was dedicated to mathematicians and dreams with talks by Alessio Porretta *What do mathematicians say (or don't say) about dream*s? and M. E. *Rêve, Images Bulles* [2].

In addition to our personal dreams and desires, there were other reasons, which also were very personal. Venice has always been a reference point for us, we went there often, every year, and our great friends live in Venice, Lili Sene and Silvano Gosparini at the *Centro Internazionale della Grafica*. We met them through a book whose review we had written in the Italian newspaper *L'Unità*. It was about the *Teriaca*, a sort of universal medicine produced during the golden years of the Serenissima Venetian Republic, in the city on the lagoon, and only in very few places authorized by the Venetian authorities.

Lili and Silvano's atelier, *Venezia Viva,* was where an important part of the Mathematics and Culture books, catalogues and brochures were to be developed, mostly by hand, and published, only for conference participants. My transfer to the University of Ca 'Foscari in 1991 allowed me to realize my first dream: a book entirely dedicated to the city of Venice, a city that is not only romantic, aquatic, unique, unimaginable but also geometric, magically structured, designed as it was built over the centuries.

My book's title was *La Venezia perfetta* (The Perfect Venice) [3]. A revised and reduced version of the book was published in English with the title *Venetian Geometry, or the Perfect Venice* in one of the three volumes dedicated to the city by Alain Vircondelet *Venice Art and Architecture* published in 2006 by Flammarion [4]. A second edition of my book with a new cover and introduction was published in 2019 [5].

Venice survived precisely because of its fragility, a bet against nature and the history of humanity. Its architectural inlays similar to glass and lace seem to have to disappear with every gust of wind or high water but instead have r esisted through the centuries. It was the right place, dreamed, hoped for to start dreaming in a different way, outside the rigid academic structures (which also rage in Venice).

There was another precise reason to have chosen Venice. Being a city without cars, you know exactly how long it takes to get to from one place to another. Being a small city, it can be covered from one end to the other in an hour. You can take, if you wish, the gondola ferry to cross the Grand Canal or the vaporetto through the Giudecca Canal, or to Lido, to reach the open sea. A mysterious city, sometimes shrouded in the splendid light that could only be painted by Canaletto. A mysterious city, sometimes immersed in fog, or *caligo* in venetian dialect, with the sound of rising and falling water.

Venice's mysterious character is accentuated by the fact that, if you are unfamiliar with the town, with its *sestieri* (the six districts) and *calli* (narrow streets), it is practically impossible to find a given location—even if you know the address. Each *sestiere* has its numbering system and even among Venetians, few know the names of the *calli* and the *campi* (public squares), except for the most famous ones or those near their own homes.

Some years earalier, in the eighties, I had started making the film series *Art and Mathematics* on the links between mathematics and art, with many of the films shot in Venice. I asked myself about the existence of objects, places or artwork of mathematical, geometrical and architectonic interest in Venice. The answer was resounding yes, on two different levels: as the city-theatre *par excellence*, it suffices to walk about Venice for architectural structures with geometric and mathematical forms of certain consequence to meet the eye. Obviously, though, this is true of practically every other city in Italy. However, there are elements specific to Venice that are of particular interest for mathematics as well as for architecture. Consequent with the fact that some of these architectonic-geometric works were erected by the major artists of various periods, it is clearly far from misguided to choose Venice as a privileged place to understand the relationship between art, architecture, mathematics and, more generally, culture. Thus, one acquires a point of view somewhere between the mathematical and the architectural, searching out hidden places and seeing the most famous places of the city in a new light.

This peculiar perspective has many surprises in store. Anyone who frequents Venice very soon realizes that the shortest distance between two points is not always, almost never, the straight line joining them. The shortest route through Venice is always tortuous, labyrinthine. "There exists no natural labyrinthine structure in which the work of man has been progressively overlaid in such a determined manner as to morph into a kind of initiatory reading" wrote the composer Giuseppe Sinopoli. For Sinopoli "the manifestation of the sacred in Venice is wrapped in a paramount symbol that represents it, that determines it, fixes it and holds it fast: the natural double spiral of the Grand Canal around the city coils itself." [6].

I was professor at the *Ca 'Foscari University* in Venice for 7 years. I had my studio in the *Ca′ Dolfin* palace, famous for the 10 canvases by Gianbattista Tiepolo which are currently at the *Metropolitan Museum* in New York, the *Hermitage* in St. Petersburg and the *Kunsthistorisches Museum* in Vienna. The mathematics department, which no longer exists, was located on the so-called Canal Piccolo which joins the Grand Canal right where the two spirals of the Grand Canal invert their direction (Fig. 1).

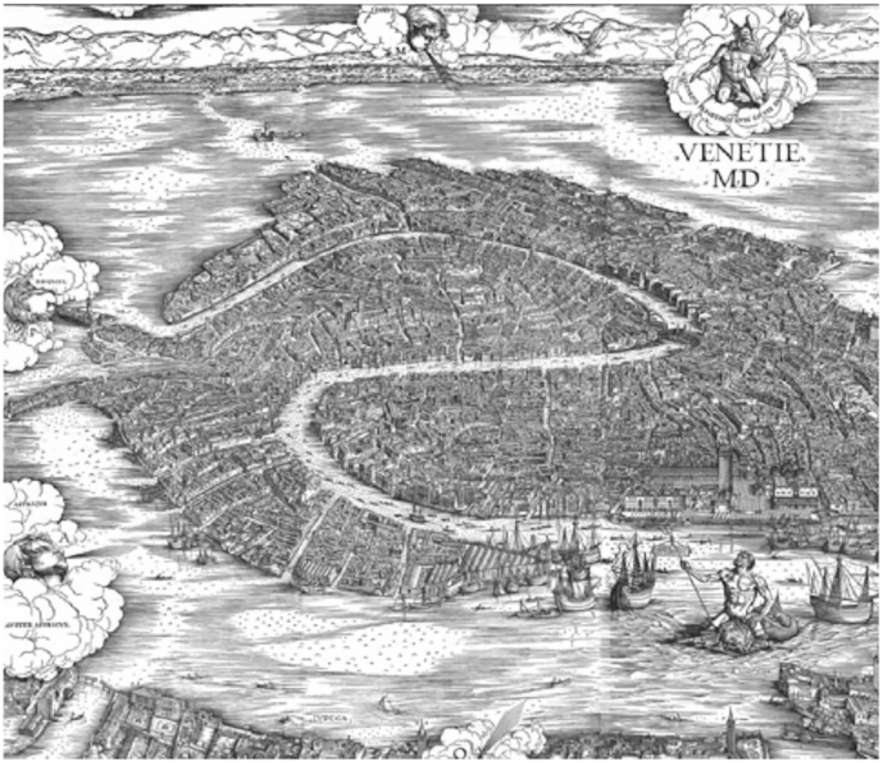

Fig. 1 Jacopo de' Barbari, *Map of Venice*, a. 1500

In my book on Venice I talked about the labyrinthine structure of the city, its topology (years later at the 2008 *Venice Biennale of Art* there was the exhibition *Topological Gardens* [7] by the US artist Bruce Naumann, who had studied mathematics for three years), of the forms found in churches, palaces, squares: helices, symmetries, geometric decorations, crossing of spirals as in glass works called *reticello*. The famous starshaped solid, officially invented by Kepler in 1619 in the volume *Harmonices Mundi*, an important shape in mathematics, had been made in mosaic many years earlier on the floor of the Basilica of San Marco based on drawings by the artist Paolo Uccello (Fig. 2).

Since 2013 I have been an elected a member of the *Instituto Veneto di Scienze, Lettere e Arti* (IVSLA) founded in Venice by Napoleon. Since then I organized my yearly conferences on *Mathematics and Culture* at the Institute which is housed in Palazzo Franchetti on the Canal Grande, one of the most famous Venetian palaces, unique for its history and its architectural features (Fig. 3).

In March 2019 I organized the last *Imagine Math, Mathematics and Culture* conference (the first was in 1997) at IVSLA and at the same time in the other building of the Institute, Palazzo Loredan in Campo Santo Stefano, two hundred

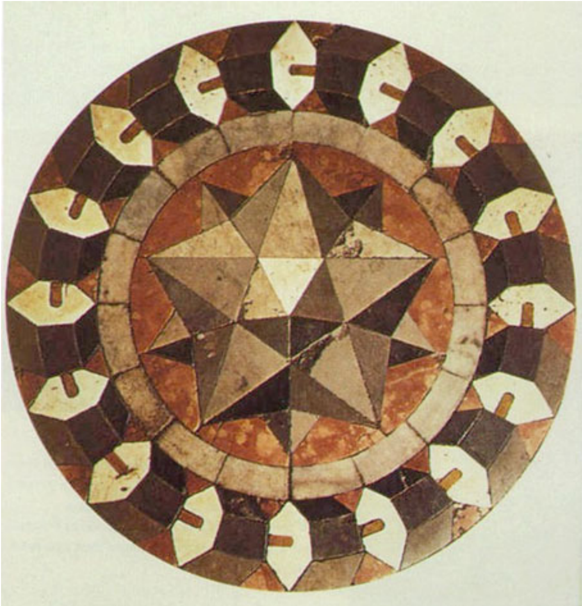

Fig. 2 P. Uccello, drawing, Starshaped Dodecahedron, mosaic, San Marco

Fig. 3 Palazzo Franchetti, IVSLA, Venice, April 2020, photo © C. Morucchio

Fig. 4 M. Paladino, *Without title*, sketch for the conference *Mathematics and culture*, 2018, work on paper, mixed media, mm. 1000 × 800. Private collection by permission

meters from Palazzo Franchetti, I organized a exhibition of the famous Italian artist Mimmo Paladino, with the ten posters he created just for the conference [8]. One became the cover of *Imagine Math 7* published in October 2020 [9] (Fig. 4).

On March 16, 2019, a large exhibition on soap bubbles in art and science opened in *Palazzo dei Priori,* home of the *National Gallery of Umbria* in Perugia, Italy. Marco Pierini, the director of the *Galleria Nazionale dell'Umbria*, curator together with myself of the exhibition, wrote in the *Introduction* to the catalogue [10]:

The exhibition *Soap Bubbles. Forms of Utopia Between Vanitas, Art and Science* was initially intended to be a kind of *mise en scène* of the volume *Bolle di sapone. Tra arte e matematica* [11], physically bringing together the images accompanying the text. However, as organization and research efforts progressed, variations, additions, and deviations to the framework were implemented, modifying the guidelines and rendering the initial catalogue, which was already abundantly vast and well structured, even richer. The plunge into the iconographic universe of soap bubbles was, in fact, full of surprises and continuous discoveries, to the extent that the team involved in the project was genuinely surprised that a thematic exhibition on the subject had not yet been created. While the presence of soap bubbles seems like it would be a unique and sporadic occurrence in the panorama of art history, they actually appear with an unexpected frequency and an uninterrupted continuity. The fragile sphere of soap film was introduced into art at the end of the sixteenth century by Hendrick Goltzius). It has arrived to the present without ever losing popularity, having been adapted to the changing times and iconographic needs.

"Pierini ends his introduction reaching the twenty-first century:

As part of an unremitting tradition, the intriguing and silent bubble has managed to arrive at the other end of the twentieth century through the works of the young Max Beckmann and Cagnaccio di San Pietro. It was at the centre of Man Ray's and Naum Gabo's experimental research, and was appeared at the end of the avant-garde period in the works of Joseph Cornell and Enrico Prampolini. A fascination with bubbles has pervaded even the second half of the twentieth century and the beginning of the twenty-first (Giulio Paolini, Mariko Mori, Jiri Georg Dokoupil). They are now not only depicted but even physically part of the work, like in Nothing (1999), Pipilotti Rist's soap bubble-making device.

A significant part of the exhibition was devoted to links with science, in particular physics, chemistry, biology and of course mathematics. Many references to contemporary architecture and even cinema were present, since the beginning of the history of cinema, with the screening at the exhibition of some silent films of the early twentieth century. The exhibition touched on all these aspects, without exhausting the topic, as it would have been absolutely impossible, due to the extraordinary history of soap bubbles between art and science" [12] (Fig. 5).

The subject of soap bubbles has interested me since I graduated in mathematics in 1970 at the University of Rome presenting a thesis on the works of Renato Caccioppoli. Some of his ideas were later used by the famous Italian mathematician Ennio De Giorgi to introduce the *Theory of Perimeters*, one of the ways to solve problems of calculus of variations related to minimal surfaces. I started my activity at the University of Ferrara as an assistant to Mario Miranda, one of De Giorgi's main collaborators.

Since soap films and soap bubbles are not only models for 3D minimal surfaces and surfaces with assigned mean curvature but are also very beautiful and colorful objects, I immediately became interested in images of soap bubbles in art, starting in the meantime my personal small collection of paintings and objects. Very important was the influence of my father who since 1938 started and invented a new way of making art documentaries on art, among the most famous ones the one on *Giotto* [13], *Leonardo da Vinci* [14], which won a Golden Lion at the *Venice International Film Festival* in 1952, *Goya*, best art film festival in Berlin, 1951 [15] and *Picasso* [16], a film made with the Catalan artist in 1953.

I had a large collection of art books at home and I was lucky enough to meet many artists personally. At the beginning of the eighties I started the series of my films on *Mathematics and art*, they would be 20 at the end, and one of the themes was precisely soap bubbles. The film was made in part at Princeton University's Math Department with Fred Almgren and Jean Taylor [17]. Jean had just proved the correctness of Plateau's observations on the singularities of soap films [18]. Jean and Fred had published an article about their research in the *Scientific American* in 1976, [19] which included beautiful color images of soapy structures.

Fig. 5 A. Romako, (copy from) *Zwei mit Seifenblasen spielende Kinder*, oil on canvas, end XIX Century, private collection by permission

In Art history, the theme of soap bubbles becomes a sub-genre of the more general theme of the Vanitas, *Vanitas Vanitatum et Omnia Vanitas* as written in *Ecclesiastes* 1: 2. In Goltzius' most famous and widespread work of 1594 *Quis evadet?* (who escapes) appear as symbols of Vanitas bubbles, smoke, dried flowers, a skull. In preparing the exhibition a work by Agostino Carracci was identified. With the same title and practically identical to that of Goltzius, the exact date is not known (Fig. 6).

At the exhibition in Perugia I did not want the Vanitas theme to be the prevalent one, and therefore there was only one painting explicitly dedicated to Vanitas, a still life with bubbles and skull. I wanted the playful, fun aspect to be prevalent, present since the beginning of the spread of soap bubbles in art. And the other scientific and artistic aspects. The exhibition was titled *Soap Bubbles. Forms of utopia between Vanitas, Art and Science.* The word Vanitas could not be missing in the title of the exhibition to give a correct location in the history of art.

Almost two years have passed since then. I worked on the publication of the volumes *Imagine Math 7* and *8*. This last volume is not connected to any conference in Venice, since nothing could be organized due to the Covid-19 virus. And

Fig. 6 A. Carracci, *Quis evadet*, bulino, around 1590, Reggio Emilia, Italy, Biblioteca Panizzi, Gabinetto delle stampe “A. Davoli”. By permission

streaming was unthinkable. The venue, Venice, is an essential part of the conference. Streaming meant destroying the idea.

In Italy we were in lockdown for three months. It was not possible to move, to see anyone, not even children and grandchildren who live far away. Writing, even more than using the computer as a video phone, was an essential way to survive and to try to maintain a delicate mental balance. And writing, loneliness led to reflect. To reflect on life, on work, on loved ones, on those who died in this period.

So the Vanitas I wanted to remove from the soap bubbles exhibition resumed its role. Soap bubbles, the fragility of life, and therefore of art, of everyone’s role, everyone’s work and research. Is it all Vanitas, do we just try to forget, trying to build our own space like a soap bubble destined to burst? Research on minimal surfaces, on the problem of Plateau, on those fragile forms which become the symbol of the uselessness of life? And does research in this field, and in that of the history of art, have a meaning? Are not our efforts useless, irrelevant when only health research, biology, vaccinations seem to have a real importance and all the rest are

Fig. 7 Canal Grande near Rialto, Venice. April 2020. Photo C. Morucchio © by permission

vain chimeras as Mathilde Wesendonck wrote for the lieder of Richard Wagner? [20] (Fig. 7)

Sag, welch wunderbare Träume
Halten meinen Sinn umfangen,
Daß sie nicht wie leere Schäume
Sind in ödes Nichts vergangen?
(What wondrous dreams are these/Holding my mind in thrall,/That they, like insubstantial foam,/ Don't barren emptiness recall?)

Telles, les demeures disposées des deux côtés du chenal faisaient penser à des sites de la nature, mais d'une nature qui aurait créé ses œuvres avec une imagination humaine.
(The mansions arranged along either bank of the canal made one think of objects of nature, but of a nature which seemed to have created its works with a human imagination)

Marcel Proust, *La recherche, Albertine disparue, 1925*

And the town, that town that was for hundreds of years the ideal scenario for many events of human civilization, that Venice celebrated, told, saw, revised, reworked, decadent, (leaving apart *the Death in Venice* by Thomas Mann), geometric, fragile, she too was a looming symbol of Vanitas. Decadent (leaving aside *the Death in Venice* by Thomas Mann), geometric, fragile, she too was a looming symbol of Vanitas (Fig. 8). As demonstrated once again by the tragic high water that flooded the city on November 12, 2019, lower only than the devastating one of 1966 (Fig. 9).

Fig. 8 Piazza San Marco, Venice, April 2020. Photo C: Morucchio © by permission

Today, october 3rd, 2020, when I am writing these words, the system MOSE (Modulo sperimentale elettromeccanico, a system of mobile) of protection of the town from high tides started working for the first time. And the water did not invaded the town! (Fig. 10).

In Venice where you could not go due to the virus, Venice where the quarantine was invented in response to the plague. The term originates from the isolation for an indefinite number of days which was imposed on the crews of ships as a preventive measure against the diseases that raged in the fourteenth century, including the plague. A document of 1377 states that before entering Ragusa, today's Dubrovnik in Croatia, it was necessary to spend 30 days (about thirty) in an isolated place, usually the nearby islands off the coast, waiting for any symptoms of plague to appear. In 1448 the Venetian Senate extended the period of isolation up to 40 days giving rise to the term quarantine (originally, Venetian word for forty). Venice was the first to issue provisions to stem the spread of the plague, appointing three guardians of public health in the early years of the Black Plague (1347). The first hospital was founded by Venice in 1403, on a small island adjacent to the town. [based on Wikipedia].

Those scenographic geometries have become empty, fascinating and tragic. The empty, deserted city. And that exhibition of bubbles, in which the reference to Vanitas was just an expedient to stay in the wake of the great exhibitions of the year, seems to have taken place years ago, far in time, in another era.

Years ago Peter Greenaway came to one of the Mathematics and Culture conferences. He had to arrive in Venice on March 24, 1999 from Amsterdam where

Fig. 9 *Acqua alta (High tide) in Venice*, November 2019. Photo C. Morucchio © by permission

Fig. 10 MOSE Project, Venezia, © by permission. October 3, 2020. At 9:52 all MOSE 78 gates isolated the lagoon from the sea. In the town, the tide level settled at around 70 centimeters, while the gates blocked the sea at 125 centimeters. To the left the open sea, to the right the lagoon

he lived. That day the air space of northern Italy was closed because of the bombing of the former Yugoslavia. I don't know how Greenaway found the phone number of the Santa Margherita auditorium where the conference was being held. He told me he would try everything to get to Venice but he didn't know when the plane would leave. He was supposed to come and talk on his film *Drowing by numbers*, [22] which was to be shown with his presence. Hours passed by and the situation remained unchanged. The meeting in a strange atmosphere continued (the bombing planes departed from the Aviano air base, not far from Venice).

Finally in the evening the airspace was reopened, Greenaway who was supposed to return to Amsterdam the next morning said he wanted to come: "I do not want the war to stop me". But he asked to be sure that when he arrive he would had found someone in the hall to listen to him. I asked the audience if they wanted to stay until his arrival which was scheduled for 11 pm and the transfer to the auditorium around midnight. Everyone said they would stay. Including the technical staff. Greenaway spoke at length of his numerical link with cinema. He wrote a long article on the *Proceedings* of the conference with the title *Come costruire un film*, [23] article republished in English [24] with the title *Some organizing Principles.*

A few years later, in 2006, an exhibition of Greenaway's watercolours was organized at the *Venezia Viva gallery*. It was curated by Luca Massimo Barbero with the title *92 Drawings of Water*, a title drawn by hand by the film maker with a catalog created by Lili and Silvano [25] (Fig. 11). It was organized with the help of Domenico De Gaetano who had founded the Volumina publishing house with Greenaway to create large-format art books with famous directors and musicians including Greenaway himself. One of the volumes was *Tulse Luper in Venice* published in 2004 [26]. Years earlier Greenaway had organized a major exhibition on the theme of water, a journey through water entitled *Watching Water* at Palazzo Fortuny in Venice with Luca Massimo Barbero curator. From 2020 Domenico De Gaetano became director of the *Cinema Museum* in Turin, the city to which Greenaway dedicated the other art book created with Volumina in 2002, *Tulse Luper in Turin* [27]. Greenaway invented the character of Tulse Luper, a sort of alter ego, and made him the protagonist of the film *The Tulse Luper Suitcases* in three parts [28] during the years 2003/04.

Did it make sense to keep trying to write, to bring to terms books that will only be virtual, as virtual as the town of Venice observed only by drones? I don't have an answer to these questions, everyone will have their own. But I continued to write and will continue to write, as I am doing even now. Because writing testifies, tells, remains, the memory, the memory. And the very title of the Springer series that I invented is obviously taken from John Lennon's *Imagine,* which imagines a better future.

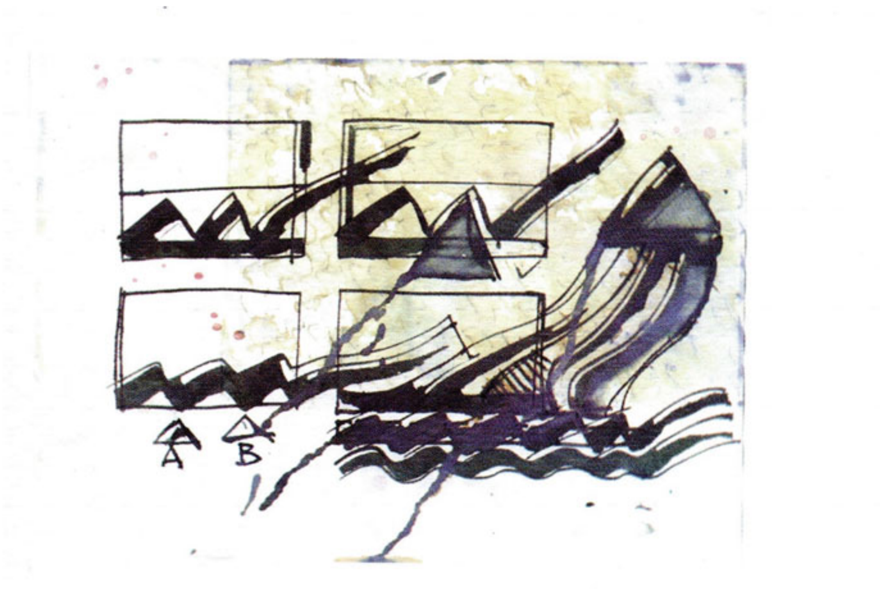

Fig. 11 P. Greenaway, 92 *Drawings of Water*, catalogue, Centro internazionale della Grafica Venezia, drawing n. 10 [21]. Private collection by permission

True, everything is Vanitas, but fortunately we never think about it for too long.

Imagine all the people
Living life in peace
You, you may say I'm a dreamer
But I'm not the only one
I hope someday you will join us
And the world will be as one.
…..
John Lennon, *Imagine,* 1964.

PS: To have a look at the exhibition in Perugia: https://youtu.be/fFHh9hi5fwM.

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The Napoleonic Fresco in Palazzo Loredan, Thinking of the Bicentennial

Sandro G. Franchini

Sometimes we invoke *fate* when trying to identify some *logic* in facts that might seem related to each other despite the obvious randomness of their occurrence. And what happened in Palazzo Loredan, the headquarter of the *Istituto Veneto di Scienze, Lettere ed Arti a Venezia* (the Venetian Institute of Sciences, Letters and Arts) in the spring of 2009 could indeed seem a trick of *fate* (Fig.1).

We had almost reached the end of several major restoration and adaptation works, lasting about 3 years. The last finishing touches remained, and a scrupulous painter had diligently cleaned and repainted the rooms on the mezzanine, when, by scratching and scraping a wall of the corridor leading to the chancellor's office, traces of paintings emerged, from under who knows how many layers of old paint, which at first sight were indecipherable: a line, a pale green leaf, a star, but then also a hand, the fold of a dress. Specialists were immediately called in and, proceeding with great caution in testing the surface here and there for samples, they brought out faces, hands, and figures that were at first barely discernible in the scattered tesserae that gradually reappeared. Immediately, memories emerged of old stories of frescoes in honour of Napoleon (in a corridor?) that Austria had destroyed as soon as it regained possession of the city: stories which we had read, but which had seemed distant, with no relationship to the present (Fig. 2).

A large allegorical scene full of characters unfolds on the wall: Napoleon returns victorious from Austerlitz and lays down his sword, greeted by the dignitaries

These notes refer in particular to the studies by Giuseppe Gullino on the Loredan family and on the history of the Institute; by Gian Domenico Romanelli on Palazzo Loredan; by Giuseppe Pavanello on the work of Giovanni Carlo Bevilacqua; and by Alberto Craievich on the decorations of Palazzo Loredan.

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Fig. 1 Palazzo Loredan, Campo santo Stefano, Venice. The manneristic-style facade, given to Giovanni Grapiglia (1572–1621) photo Beppe Raso, with kind permission

Fig. 2 Giovanni Carlo Bevilacqua, *Allegoria Napoleonica*, affresco (1808/09), photo Beppe Raso, with kind permission of the Istituto Veneto di Scienze, Lettere ed Arti, Venezia

of his court and by cheering crowds, crowned by *Fame*, with *Glory* and *Peace* at his side, in the act of receiving from France and Italy the crowns that he left behind before leaving for war. The fact that Palazzo Loredan once had frescoes dedicated to Napoleon's exploits was known to scholars thanks to the testimony of their author, Giovanni Carlo Bevilacqua (1775–1849). In his autobiography, he recalled the commission he had received from the military governor of Venice to decorate his residence in Palazzo Loredan with scenes and allegories appropriate to the position of the master of the house. However, it was also widely believed that they had been destroyed. Instead, it was with great excitement that we watched the images being slowly recovered; these testified to events and people that those rooms had once witnessed, starting with general Louis Baraguay d'Hilliers (1764–1813), recognisable by the prominence given to him in the composition and by his thick brown hair, portrayed in the act of holding the reins of the Emperor's horse.

With the fall of the *Serenissima*, the already precarious economic conditions of the San Vidal branch of the Loredan family led to the sale of the palace to a building contractor, Giacomo Berti, who in turn sold it to the Austrian government around 1805, so that the building became State property.

With the victory in Austerlitz on 2 December 1805 and the subsequent peace of Pressburg, Venice and Veneto were added to the Kingdom of Italy, and in 1808 general Baraguay d'Hilliers was appointed military governor of the city. D'Hilliers actually knew Venice well, as he was in command of the troops that, sailing from Marghera and landing at the Zattere, had occupied the city in the days immediately following that fateful 12 May 1797. During the brief period in which he remained in Venice, before leaving to join Bonaparte, who was engaged in the final stages of the Italian Campaign, d'Hilliers lodged with the Pisani family in their palace in Santo Stefano; he must have liked the area if, just over 10 years later, he decided to return there and settle in Palazzo Loredan.

Once he had taken possession of the new residence, the general called in one of the greatest artists in Venice at the time to give it an air of military grandeur, decorating what were then the private apartments and the reception hall with two large celebratory scenes and allegorical friezes, among which the five-pointed star with the imperial "N" stamped on it stood out. Bevilacqua's testimony is fundamental and also very clear, as he refers to the two scenes (one is lost) that he painted on that occasion:

> I was called to paint in the Loredan Palace in S. Stefano, which was to be the lodging of general Barague-d'Illiers, the first French Governor in Venice, a very courteous, humane, and gentle person. In the reception chamber, I painted two side paintings in fresco; in the first one, Napoleon, in the act of leaving for a military expedition, delivers to France and Italy the crowns of the Empire and the Kingdom for safekeeping. Mars, who stands at his side, urges him on, and points at the awaiting army. In the other, he returns victorious, accompanied by *Victory*, and *Fame*, flying through the air, announces his triumph. In the middle of the ceiling, the god Mars. General Barague-d'Illiers was always behind me, and with the kindest of manners he pointed out to me the features of Napoleon's face, whom I had not yet seen, his decorations, and the clothes he was wearing. This work, which cost me time, study, and effort, was destroyed with a hammer by order of the Germans (!) when they regained possession of Venice in 1814.

Scholars of Bevilacqua's work were familiar with this writing, just as they were familiar with the two sketches, preserved in the Museo Correr, which the artist had used, with some modifications, to gain the preliminary approval of the client and for the subsequent execution of the paintings. But it was precisely some of Bevilacqua's own statements that put us off track when the question arose as to where they might have been painted. First, the first and most important error: they had not been destroyed by hammering, or at least not all of them. But it was also the mention of a *reception chamber* that confused our understanding, suggesting a room on the *piano nobile* rather than a room on the mezzanine floor. It was only after this fortuitous discovery that it became clear that the distribution of the rooms in that part of the palace had undergone various changes and that what is now a wide corridor was, instead, part of a large hall at the time.

Bevilacqua's account is a vivid and lively testimony of the progress of the work and of the assiduous presence of d'Hilliers, who also succeeded in having the initial project modified by having himself portrayed on the right side, full-length and recognisable from his facial features, while Napoleon's features remained less precise. The fresco should, therefore, be dated after 1808 and not later than 1809, when Baraguay d'Hilliers was forced to leave Venice again, this time forever: first to deal with some anti-French uprisings by the population of Veneto, and then to join the troops commanded by Viceroy Eugene of Beauharnais who rushed to Friuli to oppose the Austrian attempt to regain its lost territories: finally, he honourably commanded a division in the victory of Raab, in Hungary. In 1811, he was appointed governor of Catalonia, having played an important role in the conquest of Spain. He then participated in the disastrous Russian Campaign, where he fell prisoner and drew the anger of Napoleon. He died in Berlin in January 1813, aged forty-eight, by which time even the star of his Emperor had begun to wane.

With the return of the Austrians to Venice in 1813, Palazzo Loredan was used as the seat of the city's military command, as is clearly indicated by the still legible inscription on the lintel of the entrance portal, *K.K. STADT UND FESTUNGS COMMANDO*, which reminded some of the ancient town of Kakania described by Musil. Thus, Palazzo Loredan, in addition to the Venetian Pantheon in the entrance hall, can be considered the place where the symbols of almost the entire history of Venice are represented: the aristocratic Republic, with the Loredan heraldic roses; the Napoleonic period with the frescoes by Bevilacqua; the Austrian domination with the fearsome inscription; up to our century, well represented by the Istituto Veneto.

But what is most interesting to note here is the coincidental presence of the only surviving evidence of the Napoleonic domination in a Venetian building right where the Istituto Veneto, which owes its origin to Napoleon, is today, and which in 1893 moved to its new residence in Campo Santo Stefano. As has recently been recalled, Napoleon established the *Istituto di Scienze, Lettere e Arti*, from which our own Istituto Veneto derives, with a decree issued from the Tuileries on 25 December 1810. It was a strange date to found a scientific institution, but in reality, Napoleon was not celebrating Christmas, but rather his election as a member of the *Institut National de France*, which took place on 5th Nivôse of year VI, which actually corresponded to the 25th of December. In using that date, Napoleon wished to signify that it was not so much the Emperor of France who gave life to the Italian National Institute, but the member of the *Institut,* who had been elected for his studies in the fields of physics, mathematics, and ballistics.

Perhaps, it is worth recalling here the letter written by Bonaparte to President Camus the day after his election as a member of the *Institut*:

> Citizen President,
>
> I am honoured by the vote of the distinguished members of the Institute. I feel that well before I become their equal, I will be for a long time their pupil.
>
> If there was a more expressive manner to let them know the esteem in which I hold them, I would use it.
>
> True victories, the only ones that give one no regrets, are those made over ignorance. The most honourable occupation, the most useful for all peoples, is to contribute to the aggrandisement of human ideas. The true power of the French Republic must from now on consist in not allowing there to be a single new idea which does not belong to it.
>
> Bonaparte

The Istituto Veneto can thus commemorate not only the bicentenary of the death of a dethroned Emperor (Napoleon died on 5 May 1821) but also the bicentenary of the death of its own founder and of an associate, whose legacy of ideas, as far as we are concerned, is still alive, always capable of renewal, animated by the same values that originally inspired it.

The discovery of the Napoleonic fresco? Well, maybe, after all, it was just fate![1]

[1] The original article was published in Italian, S. G. Franchini *L'affresco napoleonico a palazzo Loredan con un pensiero al bicentenario*, La Polifora, n. 11, 25 May 2021, IVSLA, Venice.

MOSE: The Defence System to Safeguard Venice and Its Lagoon

Giovanni Zarotti

1 Introduction

All environments are a constant repetition of processes where nothing is static, nothing is definitive, where there is no long-lasting equilibrium.

The Venice lagoon is the living proof of this. In it, numerous factors interact, each linked to the other by intimate relationships of cause and effect—highly particular chemical, biological, physical and morphological processes; natural phenomena typical of alluvial areas (such as subsidence) and general dynamics even on a planetary scale (eustatism, for example), further complicated by the massive presence of man with his economic and social activities (see Fig. 1).

Today the lagoon is thus the result of the combined action of all these elements and dynamics and its conservation has always required constant action to regulate and safeguard this huge area. But a system modified by man also demands constant and patient work to maintain the result achieved or cope with new emergencies.

Safeguarding and management of the ecosystem are essential to maintaining and defending a precious and unrepeatable asset, admits unexpected difficulties, expected or unhoped for successes and inevitable errors.

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Fig. 1 Landscape of the Venice lagoon. In the foreground, a salt marsh; Venice and the mainland on the background (ph. G. Marcoaldi © Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

2 The Venice Lagoon

Covering 550 km^2, the Venice lagoon is one of the Mediterranean's largest and most important wetlands. It is separated from the Adriatic Sea by a narrow strip of barrier islands, the littoral, that runs for about 60 km, interrupted by the lagoon inlets of Lido (800 m wide), Malamocco (400 m wide) and Chioggia (380 m wide). The tide flows in and out of the inlets twice a day, reaching two high points and two low points (see Fig. 2).

Inside the lagoon basin are Venice, Chioggia and more than 50 islands, among which are Murano, Burano and Torcello; comprises also a series of morphological structures typical of the lagoon environment (salt marshes, mud flats, channels, tidal creeks, shallows, etc.), which perform fundamental hydrodynamic and ecological functions and guarantee the biodiversity of the ecosystem.

The lagoon area also includes important manufacturing sites, economic activities and infrastructure such as the industrial zone of Porto Marghera, the airport and commercial and tourist ports (see Fig. 3).

Over the last few centuries, a series of natural phenomena and factors due to man's interventions have profoundly altered the Venice lagoon environment. Over the course of time, eustasy and subsidence have drastically modified the relationship between land and water with a loss of land level of about 25 cm just in the last 100

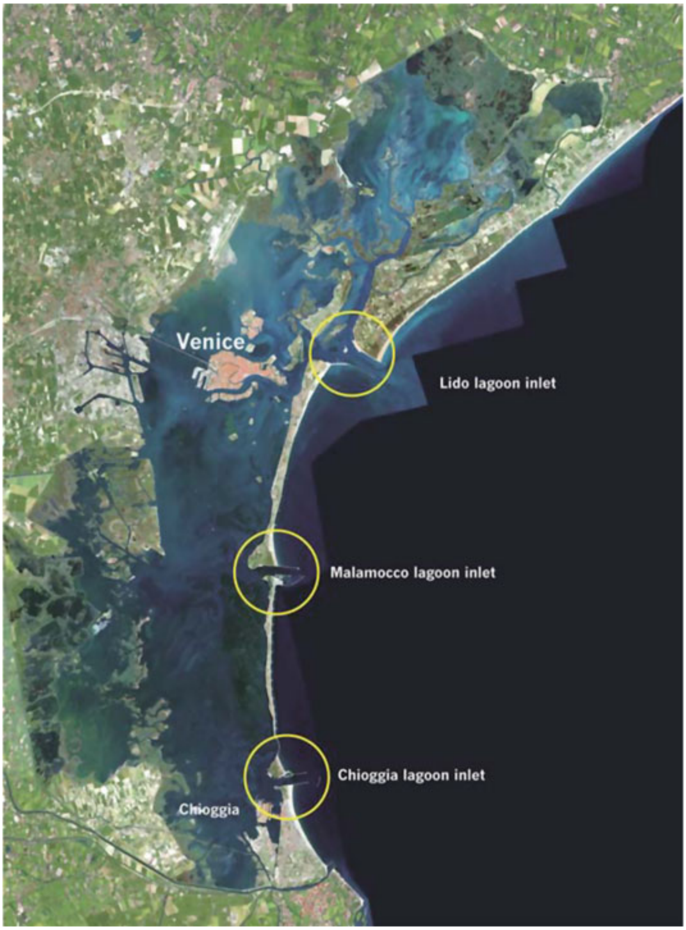

Fig. 2 The Venice lagoon. Location of the inlets (© Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

years. Although just a few centimetres, this is actually quite a lot for a city that rests on the surface of the water.

The lagoon is an unstable place, incessantly contended between the land and the sea. Its survival is thus under perennial threat. With immense work, Venetians have delivered the lagoon from the dominion of the rivers and defended it from the sea. For centuries, they have sought and found, lost then re-conquered an "impossible" equilibrium. With unflagging and complex tenacity, they have "artificially" maintained a natural place.

The interventions for the deviation of the rivers from the lagoon (from the fourteenth to the nineteenth centuries), carried out in order to confront the problem

Fig. 3 The industrial zone of Porto Marghera (© Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

of sedimentation, almost completely eliminated the re-nourishment of sand and sediments from the hinterland. The construction of the outer breakwaters at the inlets, which occurred between 1800 and 1900 with the aim of ensuring the passage of modern ships, also reduced the quantity of sediments brought in by the sea. Over the course of the twentieth century, the creation of the petrochemical centre of Porto Marghera and the excavation of deep navigation canals provoked the emission of a sizeable quantity of pollutants and profound modifications to the lagoon hydrodynamic.

At the end of the twentieth century, the lagoon system therefore had to face a multiplicity of problems with ancient origins or recent causes: the increase in high tides level, causing Venice, Chioggia and other towns and villages in the lagoon to be completely flooded ever more frequently; erosion of the littorals, with the gradual disappearance of the beaches, essential to protect build-up areas on the coast from sea storm; environmental deterioration due to a worsening of water and sediment quality and loss of the typical habitats of the ecosystem, such as salt marches and shallows.

As time goes by, the city is flooded ever more frequently and with ever greater intensity. The relative level of the land dropped by 23 cm with respect to the sea (see Fig. 4).

Fig. 4 Venice. High water: Piazza San Marco flooded (© Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

3 A Wounded Lagoon

On 4 November 1966, Venice was struck by the most violent and dramatic flood ever seen. The water reached the level of 194 cm and Venice was invaded by more than a metre of water. It seemed that the city and other historic towns and villages in the lagoon and along the coast were about to be swept away.

After that dramatic event, the Italian State undertook to save Venice and its lagoon. Through the Special Law for Venice, it defined the areas and methods of action, together with the implementing bodies: regarding the activities carried out by the Italian State, given the complexity of the problem, planning and coordination were essential to ensure coherent and systemic action.

This was entrusted to the *Consorzio Venezia Nuova*, a consortium of Italian construction and engineering companies acting on behalf of the State (Ministero delle Infrastrutture e della Mobilità Sostenibili and its decentralized technical body, the Provveditorato Interregionale per le Opere Pubbliche per il Veneto, Trentino Alto Adige, Friuli Venezia Giulia). *Consorzio Venezia Nuova* has implemented an integrated plan of interventions combining defence from floods and environmental measures (see Fig. 5).

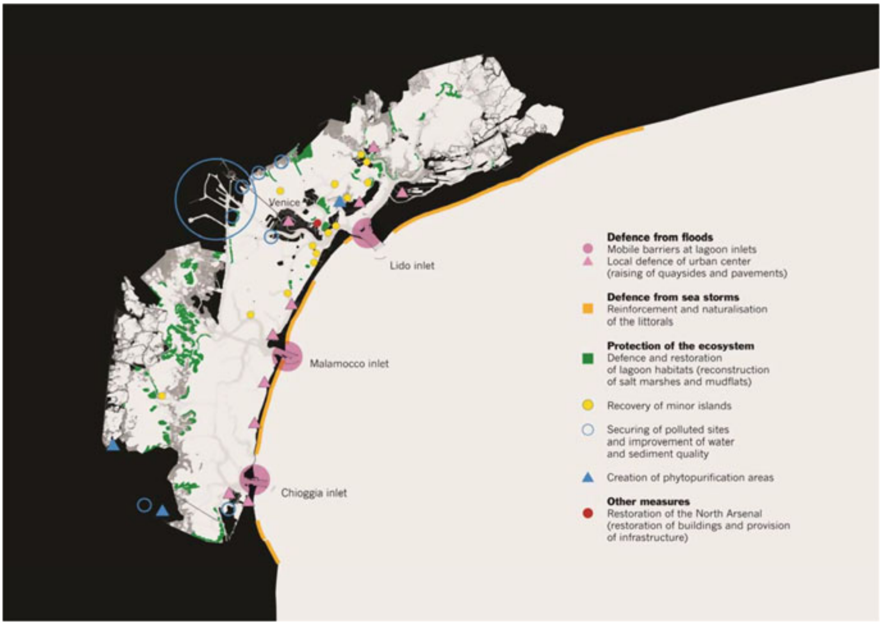

Fig. 5 The General Plan of interventions carried out by the Italian State through Consorzio Venezia Nuova (© Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

4 A Long Story

MOSE took shape after being loomed over by a number of consultations and controversies. The proposal to provide a safe measure from flooding dates back to the 1970s. In 1973, a Special Law was enacted, under which six project proposals were accepted after invitations from *Consiglio Nazionale delle Ricerche* (CNR) and later taken up by the Ministry of Public Works in 1980.

The feasibility study for the proposals was completed in 1981 under a project named *Progettone*, which proposed the setting up of fixed barriers at the inlets, including mobile defence structures.

The second Special Law of Venice to provide criteria and strategies took shape under a committee known as the *Comitatone*, which enabled the Ministry of Public Works to grant a single concession for the companies agreed upon by private negotiation.

In 1982, *Consorzio Venezia Nuova* was entrusted by the Water Authority to design and implement the measures to safeguard the city, which was presented in 1989 under a project named Riequilibrio E Ambiente (REA), which translates as Rebalancing and the Environment. It provided an abstract design of the mobile barriers at the lagoon inlets and was finally approved in 1994 by the Higher Council of Public Works.

The first environmental impact study was accepted in 1998 and was improved in 2002. Construction work of MOSE began simultaneously in 2003. The project is expected to be fully completed by the end of 2021.

5 Defence from Floods: Mobile Barriers at the Lagoon Inlets

The programme of work has no equal anywhere in the world for the size of the area concerned, nature of the problems to be tackled and extent and characteristics of the measures implemented: from defence from sea storms and flooding to environmental protection of the ecosystem. The decision to construct the mobile flood barriers was made after collaboration between all levels of government and consideration of various other coastal defence measures.[1]

MOSE is an integrated plan of interventions implemented across the entire lagoon area to provide a response to the crisis elements, in respect of the identity and capacity for adaptation of the environment (see Fig. 6).

The MOSE barriers are the central element in this great defence and environmental rebalancing system. The solution adopted is based on design constraints associated with precise legislative and governmental strategies.

[1] https://www.mosevenezia.eu/lagoon/?lang=en

Fig. 6 Valle Zappa fish farm (ph. G. Marcoaldi © Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

The flood defence system could not significantly modify flushing between the sea and lagoon, create any visual impact or interfere with the landscape and local economic activities.

The response led to identification of a solution allowing Venice and other urban areas in the lagoon to be protected from all floods, including devastating events, in respect of the hydrological and morphological balance of the ecosystem.

To define the solution, the various alternatives examined concerned both the concept of defence and analysis of systems to regulate tides at the inlets.

The project is also the outcome of an exchange of views with other organizations, authorities and institutions which has become an integral part of the solution adopted, making it a marine and environmental engineering project of absolute excellence.

MOSE is an acronym and stands for "Modulo Sperimentale Elettromeccanico" (Experimental Electromechanical Module). The name aptly alludes to the story of MOSEs parting the Red Sea. The project is an integrated system consisting of four mobile barriers closing off three inlets in the Venice lagoon: Lido, Malamocco and Chioggia. The barriers themselves are made up of 78 flap gates that are installed at the bottom of the inlets to separate the lagoon from the sea when raised. The barrier at the Malamocco inlet even has a lock system installed to allow merchant and industrial ships to cross while the MOSE system is in operation to

Fig. 7 Chioggia inlet. The mobile gates in operation (ph. DronExplore © Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

reduce interference on port activities. Together with other measures, such as coastal reinforcement, the raising of quaysides and the paving and improvement of the lagoon, once raised, the barriers are able to withstand three metres of high tide (see Fig. 7).

The role of the MOSE is twofold: to prevent a flood incident that will disturb everyday life and put people to risk and to protect the city's infrastructure (including its iconic buildings) which gradually erodes during consecutive flood incidents.

The MOSE system in its entirety is an innovation response to the threat of coastal flooding and erosion, both from a construction and coordination standpoint. The hydrological and geophysical profile of the Venice lagoon needed to be fully considered when designing the barriers and their final locations.

MOSE project also employs other smaller scale measures to optimize the overall goal of flood risk reduction in the lagoon. These local defences consist of raising quaysides, roads, walkways and installing smaller gates in the urban canals in the lagoon settlements, for example "Baby MOSE" gates in the small city of Chioggia. This holistic and comprehensive approach to encouraging protection for the entire lagoon, aside from that which is provided by MOSE, is also innovative.

6 How Do the Barriers Work

At the heart of MOSE are rows of gates. When inactive, the gates are full of water and lie completely invisible in special foundation structures in the bed of the inlet channels. When a high tide which could potentially cause flooding is forecast, the barriers enter into operation to temporarily prevent the tide from flowing into the lagoon. They remain in operation only for the duration of the high tide event. The floodgates at each inlet will function independently depending on the force of the tide expected (see Fig. 8).

When the tide drops and the sea and lagoon return to the same level, the gates are filled with water again and return to the housing structures.

Each inlet is equipped with a system to allow vessels to pass through even when the barriers are raised.

Both the coastal defences and mobile barriers are dimensioned to protect the area even if floods intensify as a result of the predicted rise in sea level. This makes the Venice lagoon one of the first areas in the world equipping itself to face the possible effects of climate change.

Thanks to the system's flexibility, the barriers can be used in different ways depending on the characteristics of the tidal event.

The gates are watertight box-shaped structures installed in special foundation structures which form the base in the seabed. The gates are made from sheet steel from 8 mm to 13 mm thick. They are internally reinforced by longitudinal and transverse steel structural elements.

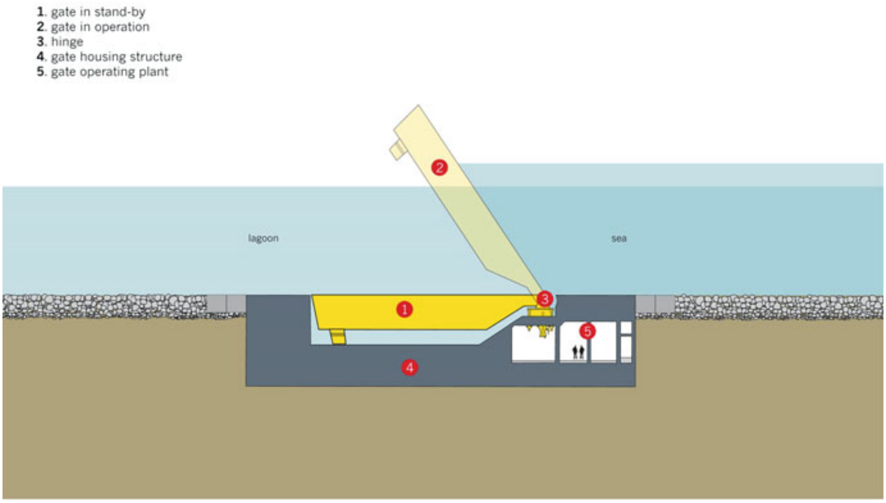

Fig. 8 The MOSE system. Cross-section of a gate and its constituent elements (© Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

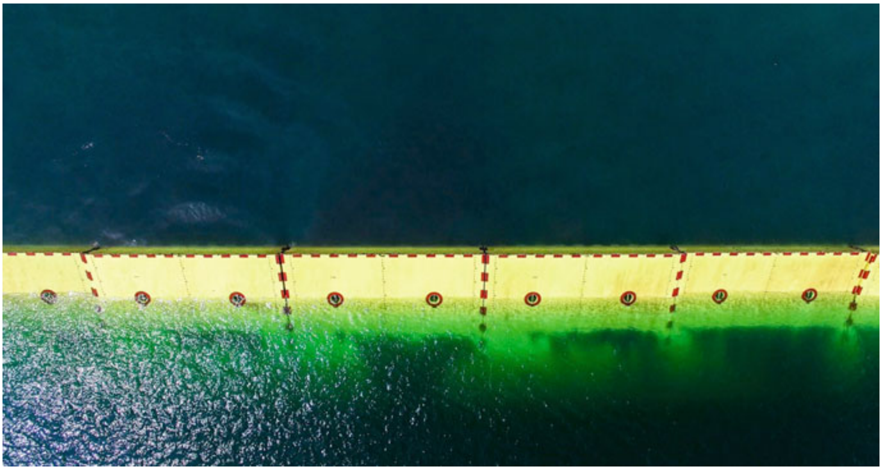

Fig. 9 The North Lido barrier in operation (ph. DronExplore © Ministero delle Infrastrutture e della Mobilità Sostenibili—Provveditorato Interregionale per le Opere Pubbliche per il Veneto—Trentino Alto Adige—Friuli Venezia Giulia, già Magistrato alle Acque di Venezia—*concessionario* Consorzio Venezia Nuova)

Each gate is equipped with pipes for the introduction and expulsion of the compressed air, instruments to detect the angle of inclination, anti-corrosion anodes and other elements necessary for correct functioning. The gates are connected to the foundation structures by means of hinges which also allow them to move. The inner and outer surfaces of the gates are treated with special biocide-free anti-corrosion and anti-fouling paints (see Fig. 9).

7 Recent Times

The city of Venice frequently experiences floods which have recently been exacerbated due to climate change. In particular, on 12 November 2019, Venice experienced its highest floods in 53 years: the high tide that hit the city late at night reached 1.87 m in height, just shy of the record 1.94 m measured in 1966. The water level caused severe damage to infrastructure. Saint Mark's Basilica was severely damaged and many buildings were affected. The city was flooded several more times over the following days.

The Italian Government came under pressure to put MOSE into operation for the safeguarding of Venice, although the project is not expected to be fully functional until the end of 2023.

On the occasion of the recent high waters, the gates were put into operation. On 3 October 2020, MOSE was activated for the first time in the occurrence of a high tide event, preventing some of the low-lying parts of the city (in particular Piazza San Marco) from being flooded. The 78 mobile barriers of the MOSE project were activated after forecasts that the tide would reach up to 135 cm. Without the barrier, a tide at that level would have flooded half of the city. According to officials, if the system had not been activated, Venice would have been flooded for several hours. Nevertheless, the barriers operated properly and the city remained dry. They have been shown to be effective in preventing high tide. MOSE has already saved twenty times Venice from flooding.

8 Since 2006 a Comparison with Other Countries Involved in Coastal Defence

With the MOSE system, Italy has been listed among the member countries of the international network of operators of mobile barriers: I-Storm "International Network for Storm Surge Barrier Management". The I-Storm is a network involving England, Italy, Netherlands, Russia, Germany and the USA (Louisiana—New Orleans) and whose fundamental objective is to share information, experiences and good practices (both in the exercise of construction) between the managers of the barriers associated.[2]

Every year it organized a general meeting during which the representatives of the barriers meet to discuss issues of mutual interest, in which they are purchased and exchanged data and information that become common heritage and shared. So this is an example of dialogue between different countries united by the problem of the defence of the territory from the water, as well as an important tool to increase knowledge and skills.

On many occasions the Provveditorato Interregionale per le OO.PP. and the *Consorzio Venezia Nuova* have been invited to discuss the possibility of adopting the protective measures implemented in Venice. World shows a great interest not only for the mobile barriers, but also for the multidisciplinary model and the procedures implemented by the *Consorzio Venezia Nuova*, on behalf of the Italian State, through the extensive plan of interventions that affected the entire lagoon of Venice.

MOSE allows living standards to be improved in general and the areas of the city most threatened by the water to be re-valued. Management of the MOSE barrier is flexible, thanks to their modularity, the efficiency of the decision-making system and the speed at which the gates can be moved. The system will also be effective in the event of a significant rise in sea level over the next decades.

It is not easy to construct such large-scale defences in fragile environments, but sometimes this approach is necessary to provide significant long-term protection.

[2] https://www.i-storm.org/

Venice represents an especially vulnerable coastal city with globally significant heritage sites and a very active tourism industry. The implementation of local defences diversifies the resilience of the settlements in the lagoon and increases the rate of success for the MOSE project.[3]

[3] https://coastal-management.eu/

Part III
Art and Mathematics

The Rise of Abstractionism: Art and Mathematics

Marco Andreatta

1 "Breaking the Ties with the Concrete and Tangible"

In the second half of the nineteenth century, many spectres were haunting Europe. Besides the spectre of communism, celebrated by the German philosophers K. Marx and F. Engels, there was the one of non-Euclidean geometry. It was first glimpsed by the Russian mathematician N.I. Lobačevskij and the Hungarian J. Bolyai who are considered the pioneers of the field, under the influential supervision of the *Princeps mathematicorum* F. Gauss. These two anarchists of the philosophy of geometry were actually preceded by earlier visionaries, the most significant was probably the Italian Jesuit priest Girolamo Saccheri, who wrote the book "Euclides ab omni naevo vindicatus" (Euclid Freed of Every Flaw (1733)) which languished in obscurity until it was rediscovered by Eugenio Beltrami.

The Manifesto of a new science of the space, which includes non-Euclidean geometry as well, is the celebrated lecture of Bernhard Riemann (1826–1866) in 1856, published posthumously in 1867, titled *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the Hypotheses which lie at the Bases of Geometry). Together with the *Manifesto of the Communist Party* (1848, Marx and Engel), *the Origin of Species* (1859, C. Darwin) and *the Interpretation of Dreams* (1899, S. Freud), it is one of the most influential papers in all fields of human cultural activities.

Riemann concludes the lecture as follows: *The question of the validity of the hypotheses of geometry is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in*

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a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek **the ground of its metric relations outside it, in binding forces which act upon it.** ... *This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go today.*

The lecture can be considered as a starting point for a more abstract study of geometry; it is full of philosophical, in a broader sense, implications. First of all, the observation, originally attributed to Leibniz, that geometry is a **science of the space**, rather than a science of objects contained in it. The objects are described by their reciprocal distance, their measures and shapes, using the *metric relations* defined on the space. Space can be of higher dimension and not necessarily Euclidian but possibly curved by the choice of a non-flat Riemann metric.

Alexander Grothendieck in *Recoltes et semailles* remarks the following: *He* [Riemann] *observed that the ultimate structure of the space is probably discrete; the continuous representation that we do could be an ultra simplification of a more complex reality. That is, for human mind continuous is easier to understand rather than discrete; as a consequence the first is used as an approximation to understand the second. This was an incredibly sharp observation for a mathematician, in a period in which the Euclidian model of the space was not questioned.*

The sentences that conclude the lecture are really astonishing, and one could see an anticipation of Einstein's idea of General Relativity, a geometric theory of the universe which aims to determine the metric relations from the law of gravitation, i.e. the physical mass of the bodies.

The impact that these ideas had on many cultural activities in subsequent years up to nowadays is evident, in particular on visual art. It is more subtle to sort out the full reciprocal interplay between art and mathematics, i.e. to see the influence of art in the developing of mathematical ideas. Which is however extremely useful, as David Mumford suggested [6]: *The saga of mathematics is unknown outside a narrow coterie; the high points of art are basic ingredients of a liberal education. Can we use our knowledge of the latter to open up the former?* (Notice that the title of this section has been taken from Mumford's presentation.)

As an example I like to consider the English painter William Turner (1775–1851). Generally regarded as a precursor of abstract painting, he was a master of the use of light as the ground for the representation of space. Under the influence of the theory of colors of I. Newton and W. Goethe, in his painting, the role of light in determining the colors and the structure of the space is, in my opinion, the analogous of the role of the binding forces in Riemann. Who, by the way, gave the first example of higher dimensional manifold using the space of colors.

The two paintings in Fig. 1 give an idea of what I mean.

The concept of the space as a discrete structure was developed later by visual arts, mainly by the artistic movements Impressionism and Pointillism. The two paintings in Fig. 2 are examples of these techniques.

Fig. 1 William Turner, Venice, la foce del Canal Grande, 1840; Yale Center for British Art. Light and colour, Goethe's Theory, 1843; Tate Gallery

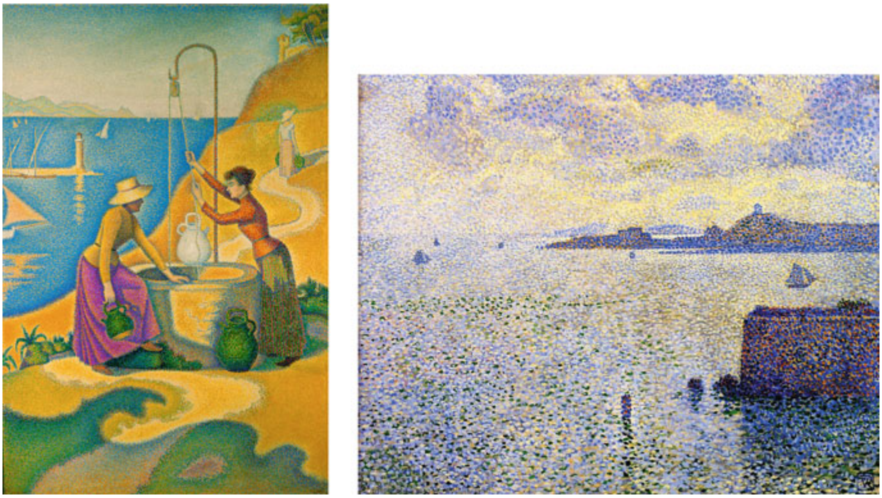

Fig. 2 Paul Signac, Femmes au Puits, 1892; Musée d'Orsay. Theo van Rysselberrghe, Sailboats and Estuary, 1892; Musée d'Orsay

2 Italian Mathematicians Contribute to Construct a European Culture

The work of Riemann gave a dramatic impulse to science, and very soon many mathematicians began to study and develop his ideas. Among them, for the purpose of this short essay, I like to point out two Italians, namely Felice Casorati (1835–1890) and Eugenio Beltrami (1835–1900). They were born in 1835, the first in Pavia and the second in Cremona, subjects of the Lombardo-Veneto Kingdom, a crown land of the Austrian Empire. In 1853, they were students at the University of Pavia, where their long-lasting friendship very likely started.

Casorati, from a middle-class family, graduated in 1856 in engineering and very soon, under the guidance of F. Brioschi, became professor of Algebra and Analytic Geometry in Pavia. Beltrami's life was much more adventurous and troubled, at least in the first part. His father, as well as his grandfather, was a painter; his participation to the Risorgimento uprisings against Austrian Army forced him to leave Italy. The young Eugenio followed the patriotic ideas of his father and as a student of Collegio Ghisleri in Pavia promoted some protests against the Rector, a pro Empire priest. He was removed from the College and, because of lack of money, he could not complete the studies and never got the "Laurea". After working for some years at the Lombardo-Veneto railway system, when Lombardia becomes a land of the kingdom of Italy he was nominated professor of Algebra and Analytic Geometry in Bologna and subsequently of Geodesic in Pisa by the vice minister of education, the mathematician F. Brioschi. In the end of his life, he was politically influent: he was Presidente of the Accademia dei Lincei and Senatore of the Italian Kingdom.

2.1 Felice Casorati

Casorati met Riemann in 1858 in Gottingen, during a famous scientific trip with Betti and Brioschi, and he was so struck by his ideas that he dedicated most of his mathematical activities to disclose them to the mathematical community. In particular, in a famous book in 1868, [4], and later in a paper of 1887, he carefully described the concept of a **Riemann surface** (Casorati called it "Riemanniana"). This is a very abstract concept which relates a geometric object as a surface with a function of a complex variable. It is a perfect interplay between geometry and analysis, between space and metaphysics. In short, to construct a Riemann surface, you take some planes, cut them transversally along prescribed segments (Riemann called these cuts *Querschnitte*), put them one over the other and glue together edges of the cuts in different planes, following a precise mathematical procedure determined by the chosen function, in order to obtain a connected surface without border.

The instructions given by Riemann and later by Casorati represent first sketches of abstract art. They aim to translate mind games into geometric objects; a main difficulty is that some of these Riemann surfaces cannot be constructed in the normal three-dimensional space, unless one introduces branch or singular points.

The instructions and a first drawing of a Riemann surface can be seen in Fig. 3; they are taken from the books by Casorati and a similar one by Carl Neumann. In Fig. 4, one can find a Riemann surface printed in a Fab Lab with a 3D printer.

There is (at least) another famous Felice Casorati (1883–1963), a painter who was the nephew of the mathematician. Referring to his uncle, he said: "My ancestors could explain the scientific order of my paintings, the rationality which pushes me towards an extreme precision, as it is for philosophers, mathematicians and some musicians". His painting *Gli scolari* (1927–1928) is represented in Fig. 5. I saw it in an exhibition at MART in Rovereto titled "Realismo Magico" (Magic Realism),

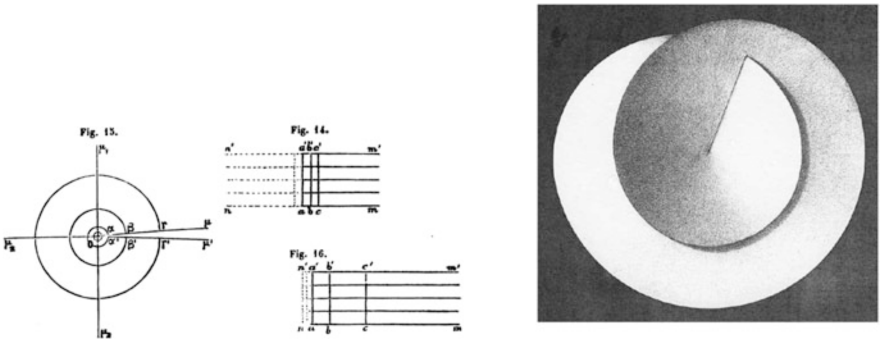

Fig. 3 Felice Casorati, Teorica delle Funzioni di Variabili Complesse, Pavia: Fusi 1868 and Carl Neumann, Vorlesungen über Riemann's Theorie der Abelschen Integralen. Leipzig: Teubner 1865

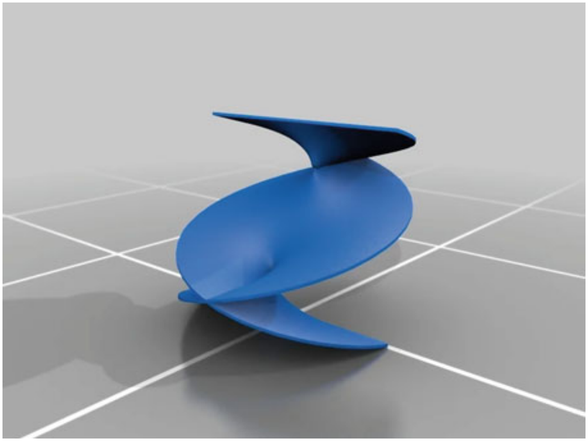

Fig. 4 3D printing of a Riemann surface

the name of a style used by some painters at the beginning of the last century to perform a realistic view of the modern world while also adding magical elements. I was impressed by the description given by one of the curators, Valerio Terraroli, who pointed out "the peculiar geometry of the floor, given through an unusual perspective which suggests a sort of unfolding". To me, this was a clear indication that both Casorati, the mathematician and the painter, were fascinated by the magic realism of the unfolding planes of Riemann surfaces.

Lucio Fontana (1899–1968) was a painter and sculptor who performed masterfully the Riemann technique of Querschnitte, although he probably never knew about Riemann. He was a founder of the artistic movement *Spatialism* and produced a series of monocrome paintings with transversal cuts denominated *Spatial Concepts or Attentes*. He described them as the "Art for the Space Age", where "the figures seem to leave the plane and enter into the space"; this seems to be a good

Fig. 5 Felice Casorati, Gli scolari, 1927; courtesy Museo di Arte Moderna di Palermo "Empedocle Restivo"

interpretation of Riemann's and Casorati's instructions. Figure 6 represents two of Fontana's spatial concepts.

Lucio Fontana realized also many paintings with a finite number of holes, possibly of different size but with a precise spatial order (for instance, displaced along a line) (see Fig. 7). This is an alternative way of describing a Riemann surface used by mathematicians, i.e. assigning a finite number of points on a plane, each with a fixed multiplicity, displayed along a curve.

I like to point out a more recent drawing by the New York visual artist Lun-Yi London Tsai. He is a mathematically trained artist able to talk with mathematicians and to visualize their scientific achievements. Figure 8 is a drawing he created in a dialogue with the mathematician Sandor Kovacs, and it represents a bunch of Riemann surfaces parametrized by another Riemann surface. It can be used to introduce a conjecture in Algebraic Geometry stated by the Russian mathematician Igor Shafarevich (1923–2017), solved and generalized in higher dimension by several mathematicians, including S. Kovacs.

2.2 Eugenio Beltrami

Eugenio Beltrami met Riemann in Pisa, a university frequently visited by the German mathematician in the last years of his life. Besides direct conversations,

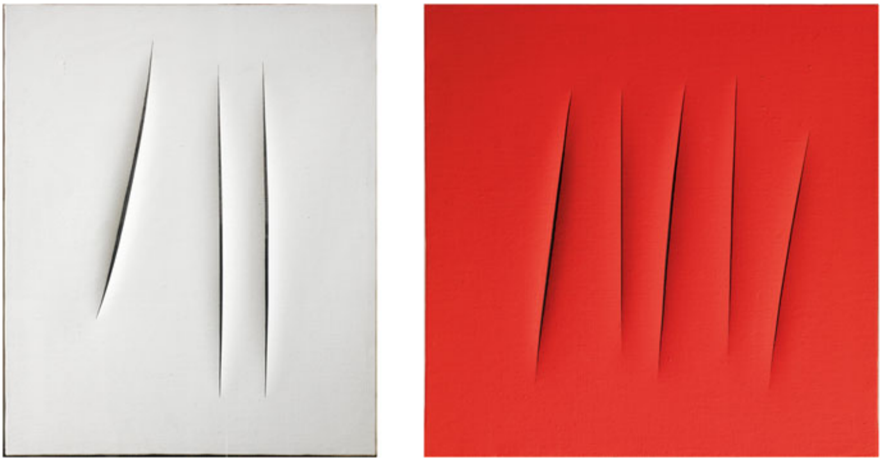

Fig. 6 Lucio Fontana, Concetto Spaziale-Attesa, 1966, courtesy Tornabuoni Art

Beltrami was a careful reader of Riemann's papers, in particular, of the fundamental lecture on geometry; together with Casorati, he ran several seminars on these subjects. In his two most famous papers, *Saggio di Interpretazione della Geometria non-Euclidea* e *Spazi di Curvatura Costanti*, both published in 1868, [2], he provided the first explicit model of non-Euclidian geometry. This is a surface of constant negative curvature, which he called *the pseudosphere*, on which all postulates of Euclidean geometry hold except the fifth: given a line (geodesic) on the surface and a point not on the line, there are infinitely many lines passing through that point and parallel to the first line. This example was a sort of holy grail which philosophers and mathematicians had been searching for hundred years; its discovery changed the way of looking at space. Beltrami gave much credit to Riemann, saying that he hoped *le mie ricerche possano aiutare l'intelligenza di alcune parti di questo profondo lavoro* (my researches can improve the understanding of some part of (Riemann) work). For those who can read Italian, Beltrami's papers are of great interest, a popularization of them can be read in my book [1].

Between 1869 and 1872, Beltrami, whose father was a miniaturist under the guidance of Francesco Hayez, constructed some paper models of his Pseudosphere, the ones in Fig. 9 are displayed at the Department of Mathematics in Pavia. The first is made of 124 pieces and was used by Casorati during the opening Lecture of the Academy year 1873–74 in Pavia.

The paper models represent a piece of the Pseudosphere, since no surface of constant negative curvature can be "embedded" in the Euclidian space, as stated by a famous theorem of Hilbert some years later. Beltrami constructed three different mathematical models of *hyperbolic geometry*; they are complete models (not a piece), and he moreover showed how to transform one into another. Two of them were later reintroduced by F. Klein and H. Poincaré, without mentioning the birthright of Beltrami.

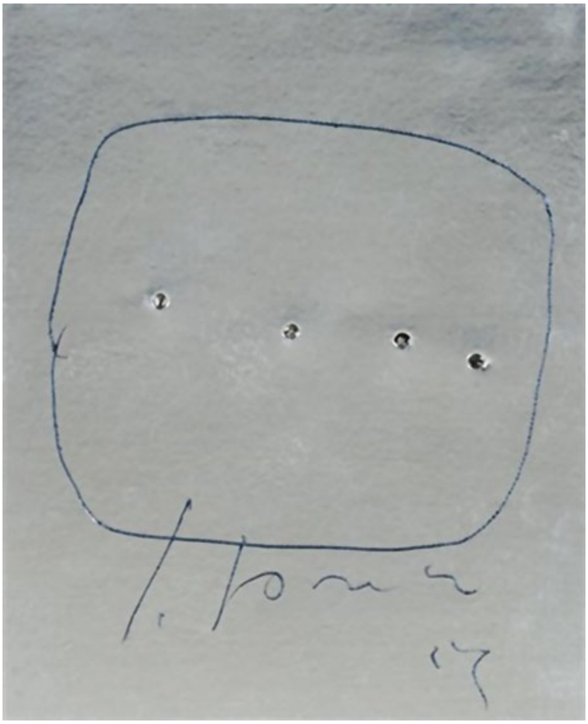

Fig. 7 Lucio Fontana, Concetto Spaziale-Buchi, 1968

Klein, who called the models *hyperbolic geometry*, was very fond of concrete models. Probably motivated by the activities of humanist colleagues who were collecting plaster copies of exemplary statues, as a professor at the Munich Institute of Technology between 1875 and 1880, he collaborated with his colleague Alexander Brill to create plaster models of mathematical surfaces; they were sold by Brill's brother Ludwig. Klein popularized a veritable zoo of models when he toured the United States in 1893, and after his visit, many American universities bought Brill's product. In 1922, Klein proudly claimed "Today, no German university any longer lacks such a collection". The qualities of seriality and eeriness made the plaster casts attractive to artists like Marcel Duchamp, Picasso, De Chirico, Carlo Carrà and Man Ray. He used the German adjective *Anschaulich*, intuitive, to describe his approach to mathematics and supported the spatial Anschauung for mathematical pedagogy. Figure 10 represents two of these surfaces reproduced by the Italian mathematician Luigi Campedelli for the Museo Nazionale di Scienza e Tecnologia, Leonardo da Vinci, Milano.

Nowadays, with a 3D printer, it is very easy to reproduce these surfaces; one can also buy them directly on the web, see, for instance, https://oliverlabs.net/math-objects/ or http://www.3dprintmath.com/, where one can buy the model of the Pseudosphere in Fig. 11.

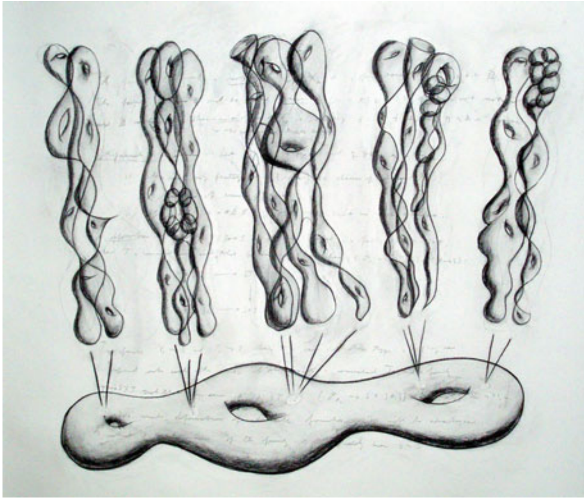

Fig. 8 Lun-Yi London Tsai, Shafarevich's Conjecture, 2007, https://www.londontsai.com/drawing

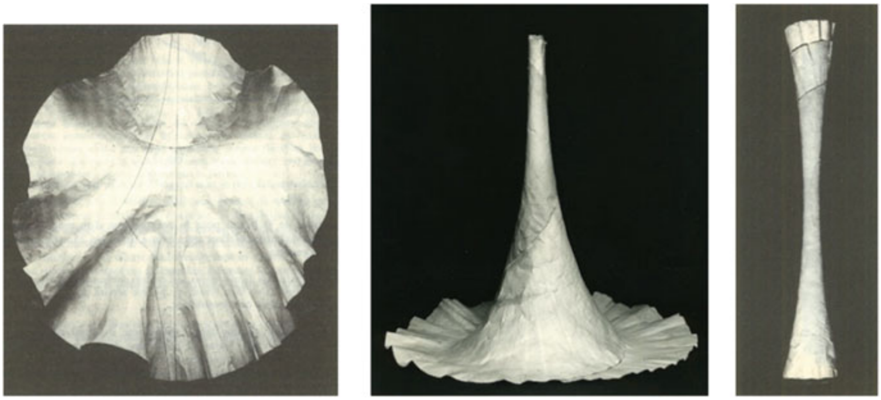

Fig. 9 Eugenio Beltrami, Pseudosphere models, 1869–1872; Department of Mathematics-Pavia

A nice hyperbolic model can be admired in the Seville city park denominated Metropol Parasole (Fig. 12, left); it resembles the shape of a mushroom or of a coral (Fig. 12, right).

One of Beltrami's models consists of a disc in the plane, and the lines are semicircles perpendicular to the border; this is the same model developed later by Klein. The Dutch graphic artist M. Cornelius Escher was fascinated by it. He was first attracted by the tessellation of the hyperbolic disk made by the mathematician H. Coxeter (Fig. 13). Out of Coxeter's drawn, Escher produced four famous engravings, titled "Circle Limits" and printed out of wooden blocks.

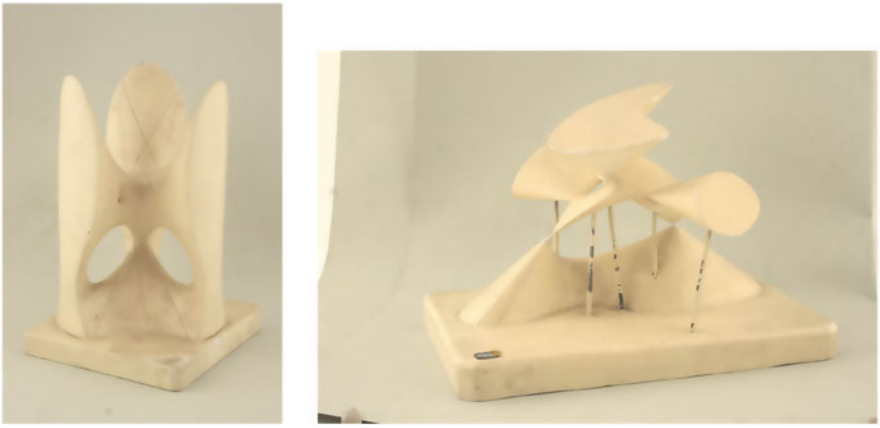

Fig. 10 Clebsch surface and Kummer surface by Campedelli, 1951; Museo Nazionale Scienza e Tecnologia, Leonardo da Vinci, Milano

Fig. 11 Henry Segerman, Tilings and curvature

The tiling proposed by Coxeter consists of hexagonal tiles, each subdivided in twelve black and white triangles; such tiling is possible also in the Euclidean space. In the hyperbolic disc, thanks to the negative curvature of the space, more tilings are allowed; in particular, one can think at a heptagonal one. Physicists Alicia J. Kollár et al. in [5], starting from a heptagonal tiling of the hyperbolic disc, constructed a finite section of it in the Euclidian plane consisting of one central heptagon and two shells of neighbouring tiles. They fabricated it in a 200 nm niobium film using photolithography, see Fig. 14. The length of the sides of the heptagon is kept equal, thanks to their curved shape, a very clever solution.

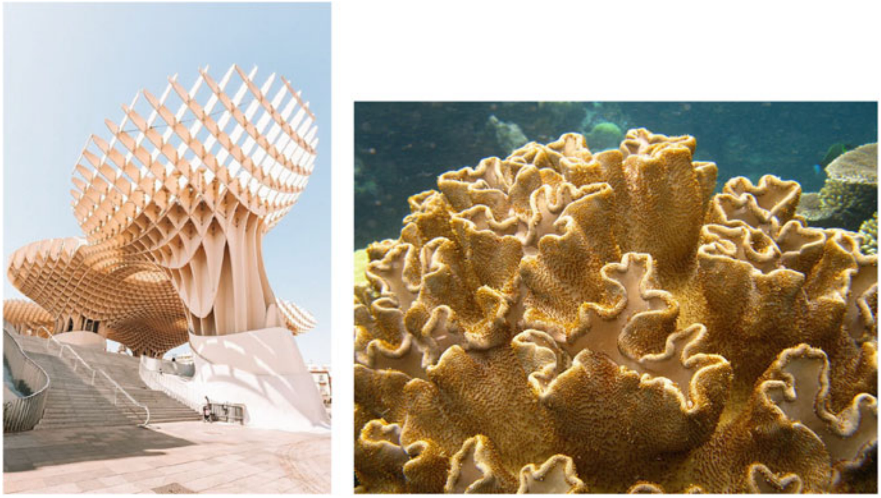

Fig. 12 Seville city park denominated Metropol Parasole, 2009. A coral

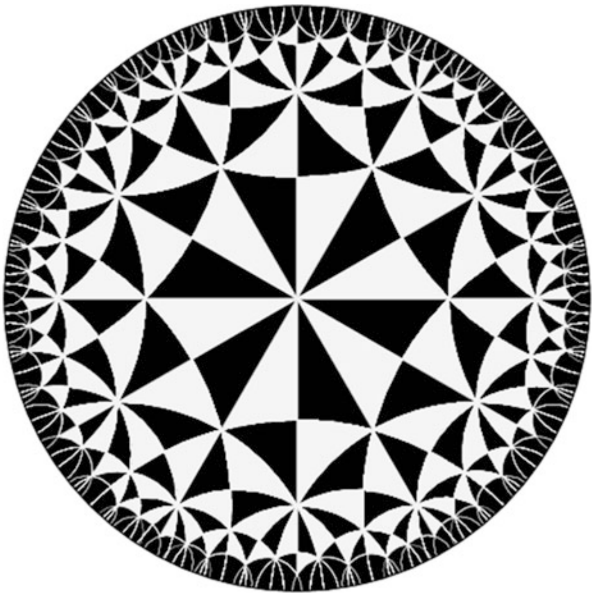

Fig. 13 H. Coxeter's tessellation of the hyperbolic disk

The funny thing is that this is not simply art; it has been proposed as a part of a future superconducting circuit, to perform quantum computation and quantum simulation.

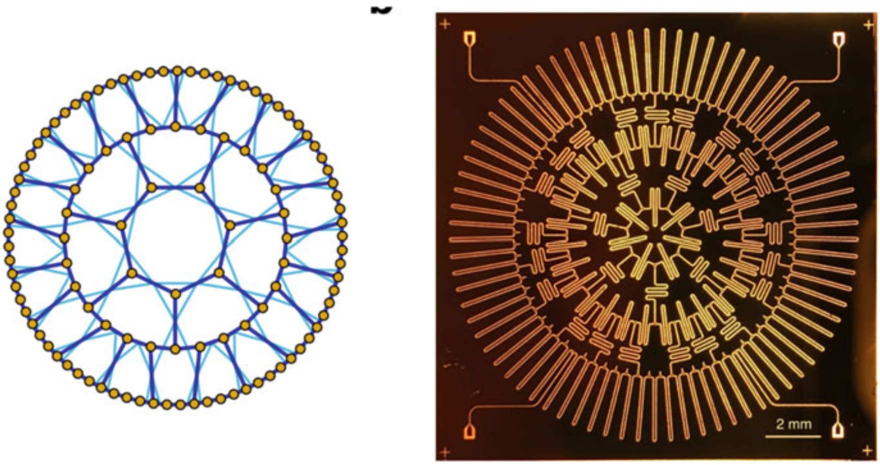

Fig. 14 Alicia J. Kollár et oth., The heptagon-kagome device, 2019; taken from [5]

Engineers and physicists, with the theoretical assistance of mathematicians, are trying to construct graphene's tissues shaped as a Pseudosphere. Graphene is an allotrope of carbon; it is one atom thick, and hence it is the closest in nature to a two-dimensional object. This is very much in the spirit of a later paper by Beltrami, [3], in which he proves that the Theory of Elasticity can be better performed on his Pseudosphere, a first taste that curved spaces are more suitable for science and art.

3 From Edgard Degas to Shigefumi Mori

The French writer and philosopher Paul Valéry (1871–1945) was first interested in Edgard Degas (1834–1917) to complete his "collection of brains", because he admired his mathematical intelligence. During their long-lasting attendance (almost 20 years), Valéry wrote the book "Degas Danse Dessin" (1938). Describing Degas's work, he wrote: *There is a huge difference between seeing something without the pencil in your hand and seeing it while drawing it. Or rather you are seeing two quite different things. Even the most familiar object becomes something else entirely, when you apply yourself to drawing it: you become aware that you did not know it-that you had never truly seen it... It dawns on me that I did not know what I knew: my best friend nose.*

I find it very pertinent to the work of mathematicians: although we are not artists, when we draw a diagram, even digitally, very often we see a different thing, maybe something we had never truly seen before.

In November 2019, the Fields medalist Shigefumi Mori gave a "Lezione Leonardesca" in Milano. During the lecture, he made a parallelism between the study of geometry and the art of painting. He started showing a drawing, Fig. 15, of

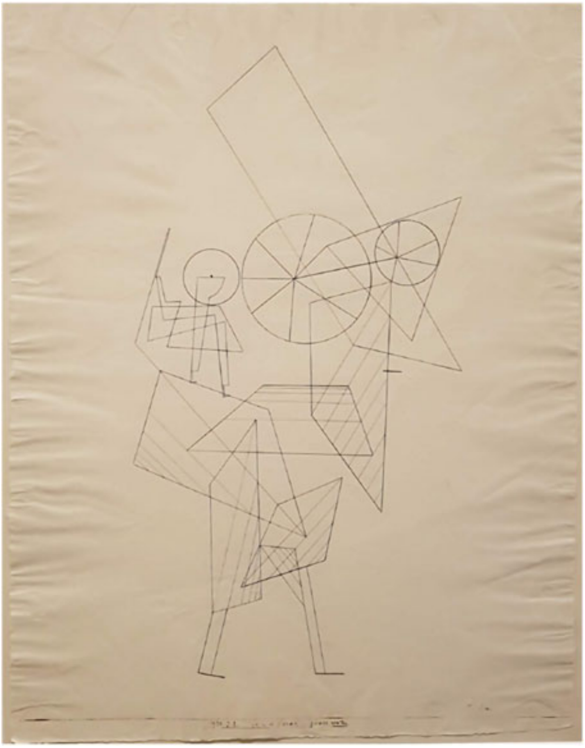

Fig. 15 Paul Klee, Steerable Grandfather (1930), Zentrum Paul Klee

Paul Klee (1879–1940), in the meantime quoting the artist: "Art does not reproduce what one can see but makes seeable what one cannot see".

S. Mori is a leading figure of the geometry movement whose Manifesto is called "Minimal Model Program". This is a scientific programme which aims to classify algebraic varieties, living in a projective space, of any dimension. To reach the classification, one needs to define a number of models and a precise procedure to connect any variety to one of these models. Working in projective spaces and in higher dimension is a very abstract matter which deals with making seeable what cannot be seen. This has many roots in the work of the Italian painters of the Early Renaissance which invented projective geometry in order to make seeable in the plane models living in the space.

In the lecture, he explained that *Geometers study figures via **invariants**, which correspond to the object used in Cubism painting. The use of invariants is similar to abstract paintings. An invariant is defined for each figure; unlike an artistic objects an invariant needs objectivity and reproductivity.* He himself created and contributed to describe many invariants associated with projective varieties. Among others, the *Cone of Curves*, made by all the curves which lie on a variety, was a key tool in his theory, as he explained in the lecture, see Fig. 16.

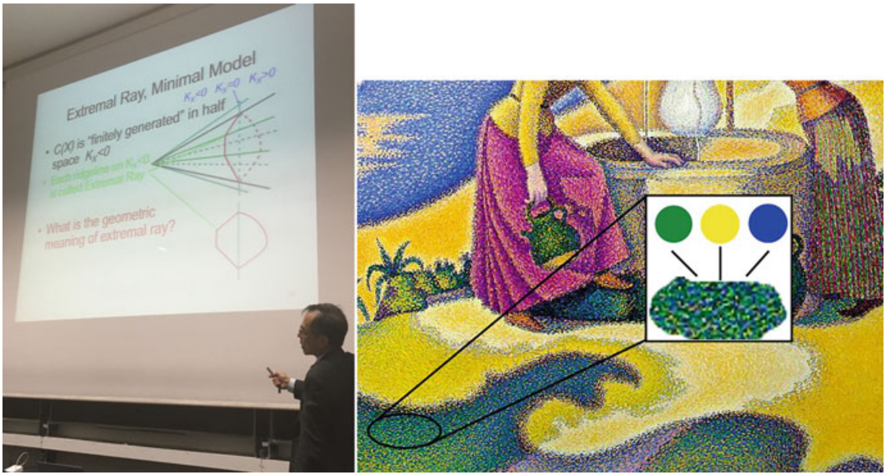

Fig. 16 Shigefumi Mori, Lezione Leonardesca, 2019; Milano. From Paul Signac1892, Wikimedia Commons

The second picture in Fig. 16 is an attempt to describe a "cone of points" associated with the Pointillism painting of Paul Signac, an artistic analogue of the geometric "cone of curves" studied by Mori and his school.

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Aestheticizing an Einsteinian World: The Idea of Space-Time in Russian Literary Theory and in Art Criticism

Clemena Antonova

When, in 1920, Russia's most celebrated living poet, Vladimir Mayakovsky, expressed his intention of sending Einstein "a salutary radio: to the science of the future from the art of the future," [1] he played on ideas that had fascinated his entire generation, both in Russia and in Western Europe. Indeed, in Russia, as elsewhere, Einstein's relativity theory was actively debated among the scientific community. Scientific texts were being translated almost immediately after coming out in their original language, as, for instance, in the respected publications *Novye idei v matematike* (New Ideas in Mathematics) [2] and *Novye idei v fizike* (New Ideas in Physics) [3]. The intense interest in relativity theory spilled over into fields outside the realm of the hard sciences and it carried along artists, poets, and philosophers [4].

The period in the immediate aftermath of the October Revolution (1917) accelerated, in many ways, these developments. The cult of science was, of course, very consistent with a Marxist ideology. The victorious Marxist Revolution fed into the belief of the actual realization, largely through science, of utopian projects, many of which predated the Revolution itself. Thus, what had drawn Mayakovsky's attention to relativity theory was the relevance of "reverse time" to the avant-garde dream of human immortality, itself the heir of the occultist-mystical and uniquely Russian tradition of Cosmism. Another Russian Futurist poet, Velimir Khlebnikov, who had training in mathematics, used the most recent scientific ideas, especially connected to modern notions of time, in the most unexpected ways, in his own work.

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For Khlebnikov, Einstein was an "artist who works with ideas" [5] and these ideas were naturally suitable to avant-garde artistic experiments.

In this chapter, I will draw attention to what I will call an "aesthetization of an Einsteinian world" in the work of two Russian thinkers, Mikhail Bakhtin (1895–1975) and Pavel Florensky (1882–1937). Under "Einsteinian world," an expression I borrow from Bakhtin, I will understand the uniquely modern intuition that all objects of experience exist in a unity of space-time characteristics. The "aesthetization" of this intuition, typical of modern man and postulated by relativity theory, refers to its application to the field of the arts, i.e., the aesthetical worlds, created by the arts. In the two cases here, it was the notion of the unity of space-time that lay in the background of Bakhtin's concept of the *chronotope* (literally, space-time) in literature and Florensky's "reverse perspective" and "reverse time" in the art of the icon. The chronotope has become a hugely popular term in literary theory. Florensky's terms are much less known. In fact, the Russian writer's specific use of "reverse time" has hardly received any attention, while "reverse perspective" has been much misunderstood. Here, I will discuss Florensky's concepts by drawing on some of my own research over a number of years.

Ultimately, what interests me in Bakhtin's chronotope and Florensky's "reverse perspective" and "reverse time" is the idea that we have conceptual tools applied to the analysis of literature and visual art, which do not just borrow modern scientific terminology in a loose metaphorical fashion. In fact, it is the theory of relativity, which *made possible* these notions. The two Russian thinkers' ideas on the relationship between space and time in literature and in the art of the icon respectively could have been advanced only after Einstein's ideas became widely known. In other words, there is a much closer link between the scientific idea and its "aesthetization" than in the common instances of borrowing terminology from one field of knowledge and applying it to another without any intrinsic connection between the two.

1 The Chronotope in Literary Theory and Beyond: Bakhtin

One of the most interesting and influential examples of the application of Einstein's scientific ideas to an aesthetic context is surely Mikhail Bakhtin's literary theory. The expression "Einsteinian world" appears several times in Bakhtin's writings, as, for instance, when he describes how in Dostoevsky's novels "a world of multiple systems is revealed with not one but several reference points (as in an Einsteinian world)" (Bakhtin in a notebook, cited in [6]). In this chapter, I will be concerned with Bakhtin's notion of the *chronotope,* which he introduced first in 1937–1938 in his essay "Forms of Time and of the Chronotope in the Novel" [7]. This is how the Russian author defines the "chronotope": "We will give the name *chronotope* (literally 'time space') to the intrinsic connectedness of temporal and spatial relationships that are artistically expressed in literature" [7, p. 84]. He acknowledged that the term derives from Einstein, but he says that "the special meaning it has in relativity theory is not important for our purposes," as it is borrowed by literary criticism "almost as a metaphor (*almost but not entirely*)" (my italics). Bakhtin emphasized that the main importance of Einstein and his use of space-time terminology is, for aesthetics, the notion of "the inseparability of space and time (time as the fourth dimension of space)" [7, p. 84].

The term "chronotope" is, in fact, the Greek rendition of the expression "space-time coordinates" (*RaumZeit-Koordinaten* in German), which Einstein himself had used in 1912 in a manuscript for a paper on the special theory of relativity. Einstein's term was popularized in Russia through the translation of Ernst Cassirer's book on relativity [8]. Bakhtin very likely heard the term "chronotope" in a lecture by the Russian physiologist A.A. Ukhtomsky, who used it in a behavioral sense [see 6, p. 411ff]. Bakhtin borrowed it from there to use as a tool of literary criticism. The chronotope is meant to describe the mutual interdependence of *artistic* time and space in the work of art. As Tzvetan Todorov mentions, Bakhtin's works are about the relation between text and world, where the text puts forward a model of a world, the constitutive elements of which are time and space [9]. Thus, each literary genre that Bakhtin analyzes, from the classical Greek novel to Rabelais and Dostoevsky, represents a specific spatio-temporal configuration, i.e., a specific chronotope. In this sense, a genre becomes a synonym of a chronotope.

Let us consider for a moment Bakhtin's obviously vague "almost as a metaphor (almost but not entirely)," as it seems important. It was common, especially among poets, artists, and philosophers in Russia, at the beginning of the twentieth century to use scientific terminology in a non-scientific, metaphorical sense. Elsewhere, I drew attention to the metaphor of non-Euclidean geometry in Russian critiques of the pictorial space of the icon [10]. The situation here seems rather different. In fact, it would not be an exaggeration to say that Bakhtin's concept was made possible by Einstein's relativity theory and it could not have been proposed without it. In this sense, this is not a case of just borrowing terminology, which happens to illustrate well a certain idea. With Bakhtin, the very idea depends on the thoroughly modern scientific notion of the inseparability of time and space. In other words, the chronotope is the artistic, literary equivalent of an Einsteinian world, which is fundamentally different from the older Newtonian universe. This is why, I believe, Bakhtin's "almost but not entirely" is significant, as it implies a much closer connection, indeed dependence, of the literary term "chronotope" and the scientific theory of relativity.

Anyone familiar with Bakhtin's work would notice that the Russian author applies the chronotope (time-space) to his analyses of literary works, predating Einstein's relativity theory and the modern idea of the unity of time-space. I still remember my excitement when I discovered, in my last year of secondary school, Bakhtin's literary criticism on Dostoevsky. His work also considers Rabelais, the novel as a literary genre, the epic, ancient forms of the novelistic traditions, etc. In other words, he uses the chronotope, a concept predicated upon twentieth-century science, to describe pre-twentieth-century literary genres. This approach can be problematic. At the same time, what it exemplifies is an analysis of the literature of the past, done from an explicitly modern position. The question is not just what

Dostoevsky's novels meant to his contemporaries, but what they mean to modern man, who lives in an Einsteinian world.

Now, while Bakhtin employs the chronotope exclusively as a category of literary criticism and states that he will not be applying it to other spheres of culture, his descriptions of it often carry more general overtones as in: "In the literary artistic chronotope, spatial and temporal indications are fused into one carefully thought-out, concrete whole. Time, as it were, thickens, takes on flesh, becomes artistically *visible*; likewise, space becomes charged and responsive to movements of time, plot and history" [7, p. 84; my italics].

Put in those terms, the category of the chronotope could be stretched out to cover a larger field and it could be applicable to other arts. Somewhat surprisingly, there have been few attempts to use the term as a tool of analysis in the spheres of painting and sculpture. Deborah Haynes's *Bakhtin and the Visual Arts* stands largely on its own as a systematic attempt to assess the relevance of Bakhtin's ideas in the field of painting and sculpture. Haynes, however, concentrates mainly on Bakhtin's theory of creativity and related issues and mentions only in passing the concept of the chronotope [11]. In an article, Jay Ladin has raised the possibility of the usefulness of the chronotope in the analyses of other arts outside literature. The author believes that "the formal language of the chronotope could be significantly enriched by surveying the existing criticism of other media for insights regarding the construction of space and time" [12, p. 228]. Ladin pays particular attention to the relevance of the chronotope as a critical tool in the sphere of film, while Laurin Porter applies it to theater studies [13]. Ladin's thesis is that as the relations between chronotopes are graphically demonstrable they would yield much easier to identification in non-verbal media. Thus, the application of the chronotope to the visual arts could be rewarding at least on two grounds. Firstly, the chronotope as "a powerful but underdeveloped critical tool" [12, p. 230] might be further elaborated by such an application, addressing its natural propensity for visualization. Secondly, the chronotope might help to illustrate a fundamental aspect of the visual arts.

The possibility of understanding the chronotope as a more general aesthetic category comes across in Bakhtin's analyses of literary works. In any literary structure, the workings of time determine the functional type of space. For example, in *The Work of Francois Rabelais and Folk Culture in the Middle Ages and the Renaissance* (1965), the Russian critic discusses the chronotope particularly in relation to images in Rabelais. Ideas of time and space are closely fused in the interpretation of the grotesque human body. The body's spatial positionings build an image of the defeat of time and death [14]. Thus, when Joseph Frank, for instance, suggested that "spatial form" was a central category of literary modernism [15], he referred, however implicitly, to a major theme in Bakhtin's writings. It seems to me that the field was left open to see the visual arts in terms of the chronotope and so explore their usually neglected temporal aspect in its interconnectedness with the more obvious spatial dimension.

2 The Icon as a Chronotope: Florensky

Visual art is usually taken to be a matter of the manipulation of material in space. It seems evident, on an intuitive level, that it has "no natural temporal dimension" [16] and is "a medium, which, by definition, lacks the dimension of time" [17]. Already, in the eighteenth century, Lessing in his *Laocoön* popularized the traditional division of the arts into arts of time and those of space. In the 1920s, Paul Klee, in his *Creative Credo*, had the following to say: "In Lessing's *Laokoon*, on which we squandered study time when we were young, much fuss is made about the difference between temporal and spatial art. Yet, looking into the matter more closely, we find that this is but a scholastic delusion. For space, too, is a temporal concept" [18]. So, what had happened between Lessing's treatise, which defended a widely accepted, even self-evident, position, and Klee's? Even without understanding thoroughly Einstein's relativity theory, by the 1920s, the implications of that theory had become widely known. Indeed, no idea was better known than the notion that any object of experience presents a unity of space and time and, therefore, space, including pictorial space, is a "temporal concept."

It would, therefore, be quite legitimate to bring together the two notions of "reverse perspective" and "reverse time" by the Russian polymath Pavel Florensky and suggest that, even though they were developed at a slightly different time in different texts, they were, in fact, conceived as part of an organic unity. Florensky, who was initially trained in mathematics and physics before becoming a priest and a theologian, is mostly known for his essay "Reverse Perspective" (written in 1919). At the same time, his idea of "reverse time" at the beginning of his *Iconostasis* (1922) has attracted much less attention. Neither the author himself nor Florensky scholars have drawn an explicit connection between these two notions. What I will suggest here is that "reverse time" and "reverse perspective" should be viewed together as the artistic space-time coordinates of the art of the icon. In other words, the icon, defined by "reverse time" (its temporal dimension) and "reverse perspective" (its spatial dimension), represents a specific chronotope.

It may appear fanciful to apply Bakhtin's term, first advanced in the 1930s, to ideas that Florensky had addressed a decade earlier. However, while there is certainly no question of an influence by Bakhtin on Florensky, an influence the other way is perfectly conceivable. Indeed, the contemporary Russian scholar V.F. Egorov has claimed that Bakhtin's literary theory and especially his concepts of dialogism, the chronotope, etc. owe a debt to Florensky [19]. I will suggest that this influence—direct or indirect—was along the lines of an aesthetization of the Einsteinian world in the sense suggested here; i.e., the modern scientific notion of the unity of space and time underlining the literary chronotope, as Bakhtin explicitly acknowledged, is implied in Florensky's earlier interpretation of the spatial and temporal dimensions of the icon. The connection is most obviously suggested by the terminology and the idea of reversal. Here, I will summarize, firstly, Florensky's definition—rather, definitions—of "reverse perspective" and, then, of "reverse time" before giving one possible interpretation of the temporal dimension of "reverse perspective."

Let us consider, firstly, Florensky's understanding of "reverse perspective," i.e., the principle of the construction of pictorial space in the icon. I have written on this, in greater detail, elsewhere [20], so I will only briefly summarize here. In his essay "Reverse Perspective," the Russian thinker switched among several definitions of the term. In fact, I have identified six different, some of them contradictory, views.

Firstly, Florensky borrows Oskar Wulff's understanding, proposed in an article, in German, of 1907 [21]. According to Wulff, space in the icon is constructed from an inner point of view, i.e., that of the central figure of representation. From this inner perspective, space functions according to the laws of natural vision. As a result, objects and the parts of the objects, which are further away in the distance, are smaller in size, when seen from an inner view, but larger in size in relation to someone with a viewpoint, which is external to the painting. Thus, for instance, in Fig. 1, the side of the coffin of the saint, which is further away, is longer than the side, nearer to the beholder.

At times, however, Florensky follows another very different definition, very likely also via Wulff—space in icons is constructed in such a way as to adjust, usually by a process of elongation, the proportions of figures, depicted above eye level and make them look "right" to the viewer. A third definition comes up with the notion of hierarchical size, i.e., the size of the depicted figures depends on their hierarchical importance. Significantly, this idea was first advanced by Karl Doehlemann in opposition to Wulff's theory [22]. In Florensky's essay, we also come across the argument that space in the icon is more faithful to human vision than linear space—clearly, a notion, which is far away from the hierarchical size thesis. To complicate matters further, Florensky also suggested that the pictorial space of the icon is curved in a manner similar to that of non-Euclidean geometry. In simple words, imagine objects and lines as if drawn on a concave surface, as the building on your left in Fig. 2 (compare to the reconstruction of the same building as it would appear under normal circumstances, Fig. 3) (see [10]). Finally, Florensky tells us that space in the icon is constructed according to the principle of supplementary planes, i.e., icons frequently show aspects of an object, which cannot be seen simultaneously from a fixed position. For instance, the depiction of the coffin of St. Nicholas in Fig. 1 can be only explained as the result of synthesizing several points of view into the image.

In short, Florensky has left a significant amount of confusion about what "reverse perspective" means exactly. Scholars, working on the topic, very rarely notice the fundamental contradictions in Florensky's theory. Further, "reverse perspective" (any of the six definitions above) has never been considered in its relationship with "reverse time," a notion that Florensky proposed a few years later in another text,

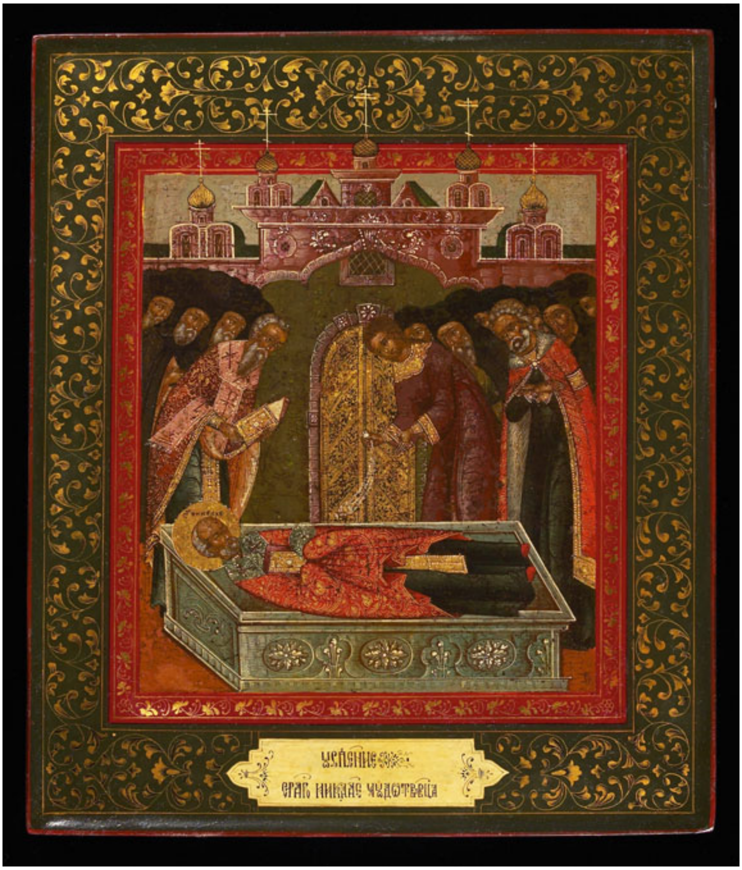

Fig. 1 *The Death of St. Nicholas*, Russian, Mstera School, late nineteenth c., 31.5 × 27.0 cm, Temple Gallery, London

Iconostasis (1922). There are two sources of Florensky's notion of "reverse time"—one is Einstein's and Minkowski's scientific theories and the other is psychology (see [23]).

Studies in psychology, going back to the nineteenth century, describe time in dreams as "reversing" both the direction and the speed of time of woken consciousness. In other words, dream time unfolds from the future to the past, while the events in a dream, lasting a few seconds or minutes, would take a much longer period, sometimes years, in everyday life. For example, imagine that you dream of a succession of events that lead to you taking part in the Battle of Stalingrad, during which you get shot at. The cause of the dream could be the ringing of your alarm clock (the shot in your dream), what psychologists call an external stimulus. The dream, recounting events over a long period of time, has occurred during a split second. Once awake, your consciousness rearranges the events of the dream in the

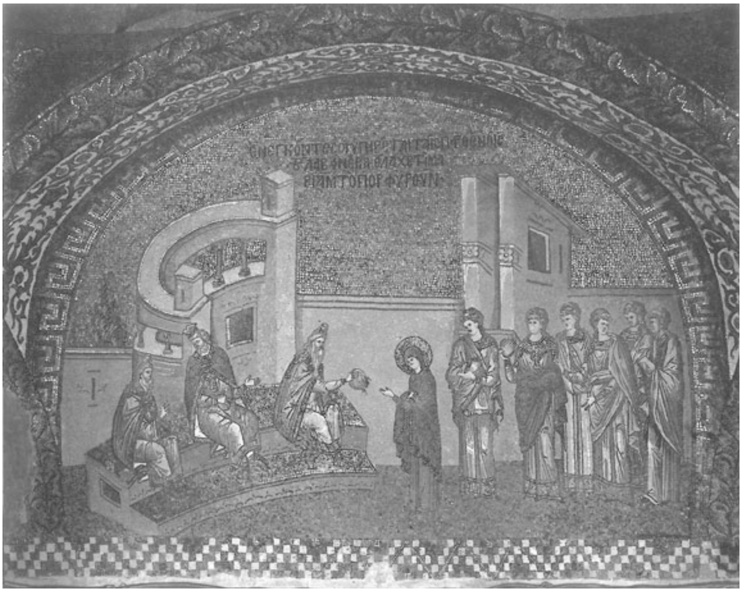

Fig. 2 *Mary Receiving the Purple,* Kariye Camii, Constantinople, c.1304, inner narthex, bay 3, wall lunette; permission by Dumbarton Oaks Library

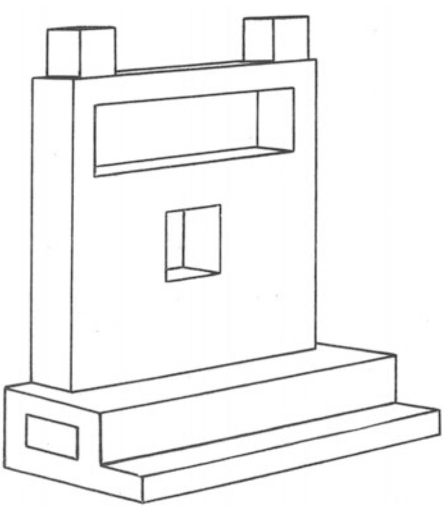

Fig. 3 Reconstruction of the building to the viewer's left in Fig. 2; my drawing

order we are used to, i.e., from the past to the future. So, the shot/the ringing of the alarm clock, rather than being a cause and a beginning, is a denouement. Florensky borrowed this notion of dream time and claimed that it also described pictorial time in the icon. What was important to him was to distinguish between two systems, adhering to different laws, i.e., our world with its linear time and the reverse time of the icon.

I will finish by giving an example, from my own work, of a possible interpretation of the temporal dimension of the construction of pictorial space in the icon. In *Space, Time, and Presence in the Icon: Seeing the World with the Eyes of God* (2010) I offered the following hypothesis: If we understand "reverse perspective" along the lines of "supplementary planes" (Florensky's sixth definition, see above), this principle of the construction of space can be interpreted as the visual analogue of the Christian dogma of divine timeless eternity. To a being that exists outside time and space and is, therefore, viewpointless, objects in our world would appear from all points of view simultaneously. So "God's eye" would "see" things very much as in Fig. 1, where side views and frontal views have been added on the same picture plane (admittedly, not *all* sides of the object have been depicted, but several, which cannot be seen at the same time from a fixed position, have). In other words, we have an example of a Bakhtinian chronotope, which has been visualized and depends on the modern intuition that any configuration in space implies a temporal dimension.

3 Conclusion

In this chapter, I looked, firstly, at Mikhail Bakhtin's "chronotope," a concept of literary theory, with attention to its potential application to the study of the visual arts. Secondly, I considered Pavel Florensky's "reverse perspective" and "reverse time," two notions, meant to describe the art of the icon. My suggestion was that, even though they were advanced in different texts with no explicit connection, these two concepts should be brought together to constitute a Bakhtinian chronotope. Both Bakhtin's and Florensky's writings on this topic are a reaction to Einstein's relativity theory and the idea of space-time unity that it implies. They demonstrate the profound impact of a scientific worldview in the field of literary theory and art criticism.

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Cagli, Olson, Coxeter

Michele Emmer

In 1923 the mathematician Harold Scott MacDonald (Donald) Coxeter started his work on more than three-dimensional geometry. Coxeter writes in the preface to the book *Regular Polytopes*, published in 1948, [1] "the book grew out of an essay on Dimensional Analogy begun in February 1923. It is thus the fulfillment of 24 years' work", with an explicit reference to Edwin Abbott Abbott's book *Flatland: A Romance of Many Dimensions* [2] published without the author's name in 1884.

In chapter VII of his book *Ordinary Polytopes in Higher Space* Coxeter wrote: [3] "Polytope is the general terms of the sequence *point, segment, polygon, polyhedron*... A Polytope is a geometrical figure bounded by portions of lines, planes or hyperplanes: e. g. in two dimensions it is a polygon, in three a polyhedron. The word polytope seems to have been coined by Hoppe in 1882, and introduced into English by Mrs. Stott about twenty years later. Many simple properties of polytopes may be inferred by pure analogy: e. g. 2 points bound a segment, 4 segments bound a square, 6 squares a cube, 8 cubes a hyper-cube and so on." He added: "Only one or two people have ever attained the ability to visualize hyper-solids as simply and naturally as we ordinary mortals visualize solids; but a certain facility in that direction may be acquired by contemplating the analogy between one and two dimensions, then two and three, and so (by a kind of extrapolation) three and four." Coxeter recalls that when we try to understand the idea of a four-dimensional Euclidean space we are helped by imagining the efforts that a hypothetical two-dimensional being would make to visualize the three-dimensional world, exactly what happens in Flatland.

And pointed out that: "Little, if anything, is gained by representing the fourth Euclidean dimension as time. In fact, this idea, so attractively developed by H. G.

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Wells in *The Time Machine*, [4] has led such authors as J. W. Dunne (*An Experiment with Time* [5]) into a serious misconception of the theory of Relativity. Minkowski's geometry of space-time is *not Euclidean*, and consequently has no connection with the present investigation."

An even more effective way, again based on analogy, was suggested by Poincaré in 1891 [6]:

"In the same way that we draw the perspective of a three-dimensional figure on a plane, so we can draw that of a four-dimensional figure on a canvas of three (or two) dimensions. To a geometer this is but child's play. We can even draw several perspectives of the same figure from several different points of view. We can easily represent to ourselves these perspectives, since they are of only three dimensions. Imagine that the different perspectives of one and the same object occur in succession . . . There is nothing, then, to prevent us from imagining that these operations are combined according to any law we choose—for instance, by forming a group with the same structure as that of the movements of an invariable four-dimensional solid. In this there is nothing that we cannot represent to ourselves, and, moreover, these sensations are those which a being would experience who has a retina of two dimensions, and who may be displaced in space of four dimensions. In this sense we may say that we can represent to ourselves the fourth dimension."

The one who first studied and determined the six regular solids of four-dimensional space was Ludwig Schläfli. Using the Coxeter nomenclature they are regular simplex {3, 3, 3}, hypercube {4, 3, 3}, 16-cell {3, 3, 4}, 24-cell {3, 4, 3}, 120- cell {5, 3, 3}, 600-cell {3, 3, 5}. Schläfli's work was not at all appreciated and almost all of his works were not accepted for publication. Only six years after his death, in 1901, the *Theorie der vielfachen Kontinuität* was published, [7] in which Schläfli dealt with n-dimensional geometry and in particular with four-dimensional solids (which he called Polyschem). Some excerpts from this work was published in English and French in 1855 and 1858 but went completely unnoticed, probably due to the fact that, as Coxeter observed, "their dry-sounding titles tended to hide the geometrical treasures that they contain, like the art of van Gogh" [8].

Coxeter wrote: [9] "The discoverers and earlier rediscoverers of the regular polytopes (Schläfli, Stringham, Forchhammer, Rudel and Hoppe) all observed that the total number of even-dimensional elements and the total number of odd-dimensional elements are either equal (as in the case of polygon) or differ by 2 (as in the case of convex polyhedron)".

Many looked for a general formula valid in every dimension. It was Poincaré in 1893 who wrote on the subject a short note which he expanded six years later. In 1893 Poincarè published the first work dedicated to Topology (or analysis situs) and Coxeter recalls that we are exactly in the field of topology with these types of results [9]:

"It must be emphasized that the theorem 9-11is a theorem of topology, which is more general than the ordinary geometry in that it is not concerned with measurement, nor even straightness." Theorem 9–11 is the proof of Euler's formula for Polytopes in all dimensions. (9 is the chapter of Coxeter's book, 1 indicates the paragraph). The lack of attention to Schläfli's works was the reason why

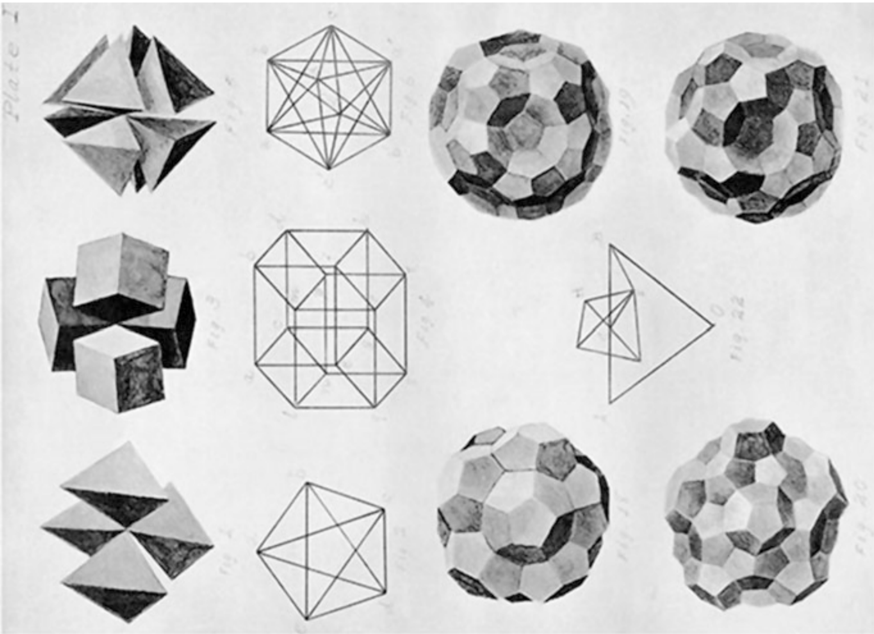

Fig. 1 W. I. Stringham, Regular Figures in n-Dimensional Space, 1880 [10]

many believe that Washington Irving Stringham was the first to determine the regular figures of four-dimensional space in his article *Regular Figures in n-Dimensional Space*, [10] published nearly thirty years after Schläfli's work. The figure of the hypersolids in four dimensions created by Stringham identifies the three-dimensional elements that make up the hypersolids and approaches them in a more or less random way. It was difficult to have a precise idea of their structure, however Stringham's work achieved indisputable success (Fig. 1).

It is no coincidence that the art historian Linda D. Henderson, in her extensive essay dedicated to the influences on art of ideas inspired by the fourth dimension and non-Euclidean geometry, despite having also thoroughly studied the mathematical aspects of the issues dealt with, as shown by extensive specialized bibliography, do not cite Schläfli's work. Speaking of Stringham, Henderson notes that the impact of the article was remarkable, so much so that there are numerous references to it in the writings of mathematicians and non-mathematicians of the early twentieth century [11]. Between 1900 and 1910 the different notions on the fourth dimension, developed in the previous century, spread more and more, even outside the scholars' circle. This phenomenon became more widespread in the United States, where a large number of popular magazines provided ample space to discuss the novelty, and in Russia. Interest peaked in 1909, when *Scientific American* sponsored "the best explanation of four-dimensional geometry", receiving 245 contributions from around the world. As Henderson pointed out, the fourth dimension was interpreted by all participants as a purely spatial phenomenon; time was never mentioned as a fourth dimension.

Linda Henderson makes it clear that, in the literature on the fourth dimension in the late nineteenth and early twentieth centuries, between the two possible interpretations of the fourth dimension, time was always the least important. In a more philosophical and mystical view of the fourth dimension, the role of time was to visualize a higher dimensional space, but time itself was not interpreted as a fourth dimension. Rather, it was the geometry of higher-dimensional spaces, along with non-Euclidean geometries, that fascinated the public in the early twentieth century.

An important role in the popularization of the fourth dimension was played by Abbott's volume, which was immediately a great success; a second edition was published in 1884 and it had nine reprints up to 1915. Mathematicians and writers alike cited *Flatland* on several occasions. Among others, Charles Howard Hinton, a great enthusiast of the philosophy of the fourth dimension, who published several books dedicated to it between 1880 and 1904. In 1907, he published *An Episode of Flatland*, [12] a sort of reworking of Abbott's novella: this was not the only attempt at such rewriting, although no imitator has achieved Abbott's inventiveness or humour.

There was already some confusion between four-dimensional Euclidean space and space-time with time as fourth dimension in Abbott's book, even if the theory of relativity obviously did not exist in 1884. Surely, the encounter between the Sphere and the Square in *Flatland* will contribute a lot to the misunderstanding: the encounter is described by Arthur Eddington in the book *Space, Time and Gravitation* (1920) [13], a classic in the popularization of the theory of relativity, as the best popular exposition of the fourth dimension. Eddington was thinking about the four-dimensional space of space-time, and wondered to what extent the world imagined by Abbott agreed with the space-time of relativity. There are three points in the narrative that Eddington was highlighting. First of all, the fact that when a four-dimensional body moves, its three-dimensional section can vary; in this way it is possible for a rigid body to alter its shape and dimensions. Moreover, a four-dimensional body can enter a completely enclosed three-dimensional room, just as a three-dimensional being can place a pencil anywhere within a two-dimensional square without intersecting its sides. This is how the Sphere behaves when it visits Flatland; the Square naturally fails to see the visitor. Finally, it becomes possible to see the inside of a solid in three dimensions just as a three-dimensional being can see the inside of a square by looking at it from a point outside the plane on which it lies.

The mathematician Jouffret wrote the two volumes *Traité élémentaire de géométrie à quatre dimensions* in 1903 [14] and *Mélanges de géométrie à quatre dimensions* in 1906 [15]. The method used by Jouffret to visualize objects in four dimensions was a kind of descriptive geometry in which these objects were projected onto the two-dimensional plane, i.e. the paper on which they were drawn. In many cases, the objects were rotated in order to obtain additional images that gave more information about their dimensions.

Of course, it should be immediately stated that in no way is a direct cause-and-effect relationship suggested between *n-dimensional* geometry and the development of the art of Picasso and Braque. The main sources of Cubism are to be found in art itself, first of all, in African art and in the paintings by Cézanne.

There is another mathematician who played a role in the development of some Cubists, and particularly in that of Cubism theorists Gleizes and Metzinger. It is Metzinger himself who explains the role of Maurice Princet, who worked in an insurance company. He conceived mathematics as an artist might, and he evoked *n-dimensional* space as a scholar of aesthetics would. He wanted to push painters towards the new ideas about space opened up by Victor Schlegel. Schlegel was one of the mathematicians who had contributed most to the emergence of *n-dimensional* geometry at the end of the nineteenth century, and who had produced three-dimensional models of hypersolids in four dimensions. Coxeter mentions him in his book: [1] "The theory of regular honeycombs in hyperbolic space but I have resisted the temptation to add a fifteenth chapter on that subject."

In the final version of *Les peintres cubistes* Apollinaire writes [16]:

"Les nouveaux peintres ne se sont proposé d'être des géomètres. Mais on peut dire que la géométrie est aux arts plastiques ce que la grammaire est à l'art de l'écrivain. Or, aujourd'hui, les savants ne s'en tiennent plus aux trois dimensions de la géométrie euclidienne. Les peintres ont été amenés tout naturellement et, pour ainsi dire, par intuition, à se préoccuper de nouvelles mesures possibles de l'étendue que dans le langage des ateliers modernes on désignait toutes ensemble et brièvement par le terme de *quatrième dimension*... Elle est l'espace même, la dimension de l'infini ; c'est elle qui doue de plasticité les objets." ("The new painters do not claim to be scholars of geometry. But it is safe to say that geometry is to visual arts as grammar is to the art of writing. Nowadays, scholars are no longer limited to Euclid's three dimensions. Painters have very naturally, one might say instinctively, explored the new possibilities of space which, in the language of modern art, are referred to as the fourth dimension. The fourth dimension is space itself, the dimension of infinity; the fourth dimension gives objects plasticity.")"

Henderson remarks that, apart from specific applications, the fourth dimension played an important role in the development of an idealism suited to Cubist philosophy.

Umberto Boccioni in 1913 discussed in detail the role of the fourth dimension in Futurist art. In 1914 he collected his observations in *Pittura scultura futuriste (Dinamismo Plastico)* (Futurist painting and sculpture: plastic dynamism): [17] "Dynamism is a lyrical conception of shapes interpreted as part of the infinite manifestation of the relationship between their absolute and relative motion, between environment and object, until they form the appearance of a whole: *environment +object*... Between rotation and revolution, in short, is life itself, captured in the shape that life creates in its *infinite succession*... We come to this succession... through an intuitive search for a unique shape that gives continuity in space... to dynamic continuity as a unique shape. And it is not by chance that I say shape and not line, because dynamic shape is a kind of fourth dimension in painting and sculpture, which cannot live perfectly without the complete affirmation of the three dimensions that determine volume.”

Boccioni recalls that the Cubists claimed to fully understand the idea of the fourth dimension:

“I remember having read that Cubism with his breaking up of the object and unfolding of the parts of the object on the flat surface of the picture approached the fourth dimension . . . If with artistic intuition it is ever possible to approach the concept of the fourth dimension, it is we Futurists who are getting there first. In fact, with the unique form that gives continuity in space we create a form that is the sum of the potential unfolding of the three kwon dimensions. Therefore, we cannot make a measured and finite fourth dimension, but rather a continuous projection of forces and forms intuited in their infinite unfolding.”

Boccioni was interested in the passage of a higher-dimensional form through our three-dimensional space, in obtaining a continuous shape via this passage, as in his famous 1913 sculpture *Unique Forms in the Continuity of Space* (Fig. 2).

In 1917, in *La peinture d’avant-garde* [18] Gino Severini clarified how the links between Futurist art and geometry were to be understood, and looked at Poincaré: [19] “*L’espace ordinaire se base en général sur la convention inamovible des 3 dimensions; les peintres, dont les aspirations sont illimitées, ont toujours trouvé trop étroite cette convention. C’est-à-dire qu’aux 3 dimensions ordinaires, ils tâchent d’ajouter une 4e dimension qui les résume et qui est différemment exprimée, mais que constitue le but de l’art des toutes les époques. Boccioni, en définissant ce qu’il appelle le “dynamisme”, fait allusion à une sorte de 4e dimension, qui serait “la forme unique donnant continuité dans l’espace”, . . . Il s’agit de trouver une définition le plus possible simple et vraie, au point de vue artistique. C’est pourquoi, j’ai cherché dans la géométrie qualitative (*Analysis Situs de Poincaré*) la démonstration plus évident de cette 4e dimension, en sachant d’avance, que la science géométrique ne pourrait que soutenir des convention déjà établies par l’intuition artistique de nous tous. Si j’aime chercher souvent un appui sur les vérités de la science, c’est que je vois là un excellent moyen de contrôle et d’ailleurs aucun de nous saurait négliger les notions que la science met à notre portée pour intensifier notre sens du réel.*”

Given the difficulty of drawing three- and two-dimensional projections, not all hypersolids have been equally successful in literature and art. The most successful is definitely the hypercube, also called *tesseract*.

Among the images of the hypercube, the *Divine Cube of the Fourth Dimensions* of Flatland’s Square, Henry Parker Manning’s 1914 images (Fig. 3) became well-known even outside the circle of mathematicians. They represent two of the possible projections of the hypercube in three-dimensional space.

Manning’s images are what Theo Van Doesburg uses in his four-dimensional architectural projects. The magazine *De Stijl*, founded by Theo Van Doesburg and Piet Mondrian in 1917, reprinted in 1923 an article by mathematician Henri Poncaré entitled *Pourquoi l’espace a trois dimensions?* [20] with the sentence *De Beteekenis der 4e Dimensie voor de Nieuwe Beelding* (“The significance of the 4th dimension for the New Design-Plasticism”, the latter being the artistic movement to which

Fig. 2 U. Boccioni, *Forme uniche della continuità nello spazio*, 1913. Museo del Novecento, Milano © Comune di Milano—tutti i diritti di legge riservati

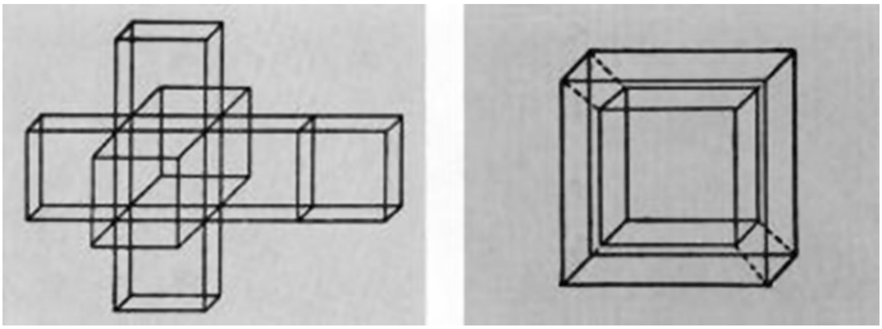

Fig. 3 H. P. Manning, *Hypercube*, 1914

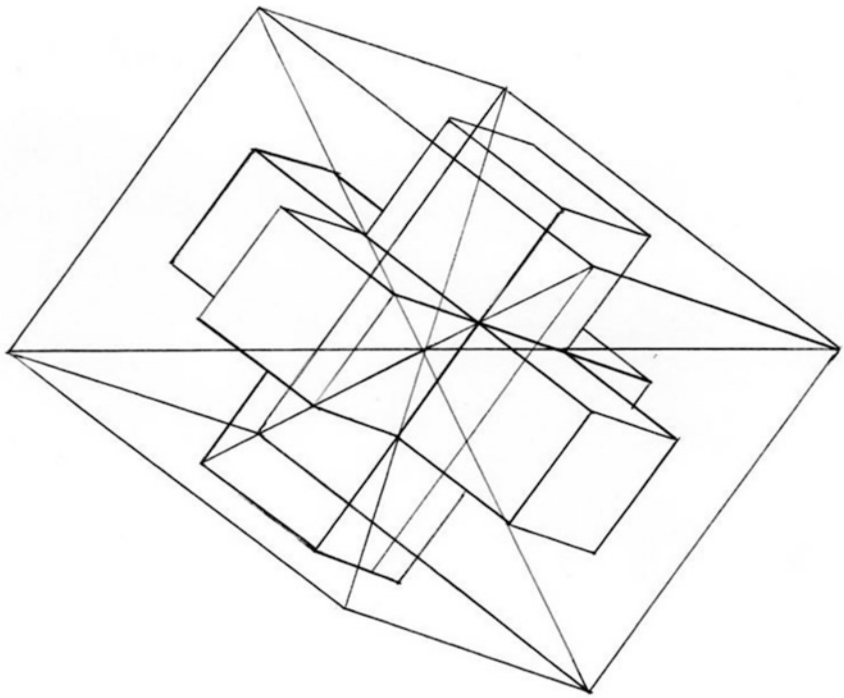

Fig. 4 T. M. Villarreal, drawing, based on T. van Doesburg, *Une nouvelle dimension pènètre notre conscience scientifique et plastique,* 1927 [21]

Mondrian and Van Doesburg had given rise) inserted as a preface. In 1927 Van Doesburg published in *De Stijl* a drawing of the four dimensional cube with the phrase *Une nouvelle dimension pènètre notre conscience scientifique et plastique* ("A new dimension penetrates our scientific and plastic understanding") (Fig. 4) [21].

Between the 1930s and the 1960s, with a few exceptions, interest in the geometry of the fourth dimension declined, both in the mathematical and artistic fields. One of those exceptions is Salvador Dalí, whose painting *Crucifixion (Corpus Hypercubus)* is from 1954.

A few years earlier, both Poincaré's research on topology and Riemann's research on non-Euclidean geometry, and the publication in 1947 of Coxeter's book on polytopes had aroused the interest of an Italian painter living in the USA and of a North American poet. They will create a kind of synthesis between Euclidean and non-Euclidean geometry, multi-dimensional space and topology.

The Moebius Strip

Upon a Moebius strip
materials and weights of pain
their harmony

A man within himself upon an empty ground.
His head lay heavy on a huge right hand
itself a leopard on
his left and angled shoulder.
His back a stave, his side a hole into the bosom of a sphere.

His head passed down the sky (as suns the circle of a year).
His other shoulder, open side and thigh maintained,
by law of conservation of
the graveness of his center,
their clockwise fall.

Then he knew, so came to apogee
and earned and wore himself as amulet.

I saw another man lift up a woman in his arms
he helmeted, she naked too, protected as Lucrece by her alarms.
Her weight tore down his right and muscled thigh

but they in turn returned upon the left
to carry violence outcome in her eye.
It was his shoulder that sustained, the right,
bunched as by buttocks or by brests,
and gave them back the leisure of their rape.

And three or four who danced,
so joined as triple-thighed and bowed and arrowed folk
who spilled their pleasure once as yoke
on stone-henge plain.
Their bare and lovely bodies sweep, in round
of viscera, of legs
of turned-out hip and glance, bound
each to other, nested eggs
of elements in trance.

This poem was written by US poet Charles Olson in November 1946 [22]. Corrado Cagli was an Italian artist who was interested in the topological surface of Moebius's strip, so much so that in 1946 he produced a drawing dedicated to the surface (Fig. 5) and a painting entitled A Moebius (Fig. 6). He was also a friend of Charles Olson's. Cagli's interest in the Moebius strip, in non-Euclidean geometry, and the fourth dimension of space began in the pre-war years, as early as 1939, when he arrived in the USA fleeing the Italian racial laws. He had always been in

Fig. 5 C. Cagli, *Anello di Moebius,* 1946, 51 × 33 cm, India ink on paper, private collection—courtesy Archivio Corrado Cagli, Rome

contact with other Italian and French artists and, when he held his first exhibition in the USA, he became part of the American cultural environment.

He enlists as a volunteer in the war, participates in the Normandy landings, and follows the front through Europe until he assists at the liberation of the Büchenwald camp on 16 April 1945. He will bring back a series of drawings based on what he saw in the concentration camp. Works of great realism and impact (Fig. 7).

The war had interrupted his relationship with Olson. On his return from the war, contact between the artist and the poet resumed. The publication of Coxeter's book comes at the right time. Although non-Euclidean geometries, the fourth dimension, topology and the Moebius strip are not entirely related topics, they are certainly linked to a concept that interests any artist: the idea of space. Cagli always cultivated a parallel interest in abstract and geometric forms in addition to his interest in figurative art. This led him to learn about and read books on mathematics, becoming interested in Riemann's geometry and in topology. In 1947, with Coxeter's book,

Fig. 6 C. Cagli, *A Moebius*, 1947, 50 × 80 cm, oil on canvas, private collection—courtesy Archivio Corrado Cagli, Roma

he discovered four-dimensional shapes. He was fascinated by them as he was fascinated by the Moebius strip, invented (or discovered) by German astronomer August Ferdinand Möbius in 1848.

In Cagli's mind, topology, the Moebius strip, the four dimensions, and the new non-Euclidean geometries became the elements of a renewed interest in geometries and mathematical surfaces. In particular, Cagli suggests that Olson read Coxeter's book. They discuss the Moebius strip and Olson asks Cagli to contribute with his drawings to a small book in which he will include some of his poems. These include the final version of *The Moebius Strip*. This short book is published in 1948 under the title *Y & X*. It includes five poems by Olson and five drawings by Cagli (Fig. 8) [23]. In 1948 Cagli returned to Italy.

Carlotta Castellani wrote in *Corrado Cagli e Charles Olson: la ricerca di nuovi linguaggi tra esoterismo e geometria non euclidea* ("Corrado Cagli and Charles Olson: the search for new languages between esotericism and non-Euclidean geometry"): [24].

"In order to express a multidimensional reality of complex simultaneity through his compositions, Olson undertakes a radical revolution in his writing, disrupting the composition of verse and the organization of language."

The first attempt to make his poetry into a spatial field is found in *The Moebius Strip* written in November 1946 but published as an introduction to Cagli's catalogue on the occasion of his exhibition at the Knoedler Gallery in New York in March 1947, and inspired by Cagli's drawing of the same title. Olson translates the distortions in place when projecting language onto a hypothetical Moebius strip:

Fig. 7 C. Cagli, *Bambino nel campo di concentramento*, drawing, Büchenvald, April 1945, courtesy Archivio Corrado Cagli

'The distortions + movements are intended to force language to do a like job in its dimensions as a painter would operating on a strip'".

Frank Moore recalls Cagli's constant interest in this geometric form: 'Cagli was obsessed with the enigmatic shape which had only one side and one edge and yet occupied physical space. He wanted to make painting on it… Cagli made drawings for Olson's words, Olson wrote words for Cagli's drawings. The poem about Moebius strip was one.'

The equivalence between Olson's poem and Cagli's drawing is acknowledged by the painter himself, who states, referring to Olson's poem: [22] "It seems to me that there is something very mysterious going on if, in the field of dimensions, drawings turn out to be poems and poems blow back sudden changes to the source of the drawings."

Starting from the assumption that it is almost impossible for anyone—scientist or artist—to visualize the fourth dimension, Olson perfectly described Cagli's way of proceeding by taking up Poincaré and Coxeter's analogy and reminding us that once

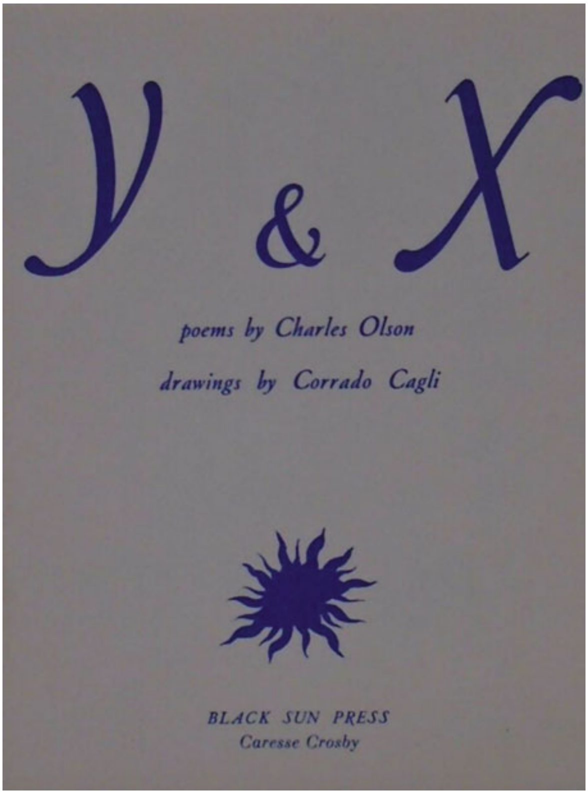

Fig. 8 Cover of the volume *Y&X drawings by Corrado Cagli/poems by Charles Olson*—courtesy Archivio Corrado Cagli, Roma

we understand the analogy between one-dimensionality and two-dimensionality, between two-dimensionality and three-dimensionality, it is possible to contemplate the analogy between three-dimensionality and four-dimensionality. The resulting image seems to describe a field of opposing forces in continuous tension, something similar to what Charles Olson was trying to convey through words. That these reflections were at the heart of the Italian painter's activity is also indicated by the works presented the following year in his personal exhibition in Rome, at the Galleria del Secolo (Fig. 9).

As Cagli recounted in his introduction, May 1949: "By drawings of the fourth dimension I mean those, including mine, which obey the spirit and optical taste of the projective that Donchian employed to represent fourth dimensional solids", [25] for the creation of these works the artist had been inspired by the four-dimensional solids of the self-taught mathematician Paul Samuel Donchian, which he had been able to see in Hartford, Connecticut."

The exhibition will move to the USA in December 1949 at the Watkins Gallery of the American University in Washington under the title of *Drawings in the 4th Dimension*, with a lecture by Charles Olson and a note by Cagli [26] "When I speak of drawings in the 4th dimensions I am referring to those of my own which obey

Fig. 9 C. Cagli, *Catalogue of the exhibition,* Galleria del Secolo, Roma, (1949), courtesy Archivio Corrado Cagli, Roma

the optic spirit and taste, which the mathematician Donchian has expressed in his projections of solids in the 4th dimension. To be elementary that which appears as a cube in three-dimensional space, will in the space of four dimensions, take the form of a hypercube. Since the antithetic space significance of these two solids is understood it becomes possible to see them as measures of two different pictorial system-the cube as the rule and the measure of all paintings in three dimensions, the hypercube as the rule and the measure of paintings in the 4th dimension . . .

On a page of two dimensions, a drawing in the 4th only takes an allusive, not representational force, and I strongly suspect that we will not be able to adventure into the slightly explored field of the n-dimensions until we are prepared to give up both the frame and canvas . . .

I have, within the limits of my research, made some experiments with the Moebus (!) strip and it offers a pure shape and a continuous surface no less suggestive that the circle, no less impressive that the sphere."

The exhibition featured *Eleven Hypercube Drawings*, among others. There should have been some of Donchian's models in the exhibition, and on this subject Davide Colombo writes in his extensive essay *Non-Euclidean Geometry and the Fourth Dimension in the Intellectual Exchange between Charles Olson and Corrado Cagli* [27]. "In a letter dated December 8 1949, Cagli complains about the decision to exhibit Donchian's models of four-dimensional solids because they are too theoretical and programmatic, citing some provocation and risk of confusion... The doubt arises that Cagli's reluctance may be due to the risk that his drawings will be judged as a direct transposition of Donchian's models, thus denying their autonomy and artistic value. In the end, as Olson later recalled, the Donchian solids were not exhibited because they were not in good condition." The exhibition was then taken to *Black Mountain College*, where Olson had been invited by Josef Albers. Olson would be head of the famous College Olson from 1951 to 1956.

Colombo adds: [28] "Neither Olson nor Cagli had specific mathematical and geometrical training: they rely on intuition for their perception and understanding of mathematical concepts, although this does not negate the careful study of mathematical principles. In Cagli's case, these are also influenced by some explanations on non-Euclidean geometry given by his brothers-in-law Oscar Zarinski (husband of his sister Iole) professor of mathematics at Harvard, and Abraham Seidenberg (husband of his sister Ebe), professor of mathematics at Berkeley University and then at Harvard, and by other mathematicians he met, as he remembers in a letter of May 12, 1946. But it was the encounter with Donchian's models that provided Cagli with the inspiration to radically evolve his research."

Donchian is mentioned by Coxeter in *Regular Polytopes* [29]. He also writes a brief biography of him and recalls that section 13.2 *Orthogonal projection onto a hyperplane* [30] of his book is directly inspired by Donchian's models, two of which were photographed and included in Coxeter's book. Tables IV on page 160 and Table VIII on page 273 show the {3, 3, 5} and the {5, 3, 3} polytopes, respectively. (Fig. 10).

"Paul S. Donchian was an American of Armenian descent. His great-grandfather was a jeweler at the court of the Sultan of Turkey, and many of his ancestors were oriental jewelers and handcraftsmen. He was born in Hartford, Connecticut, in 1895. His mathematical training ended with high school geometry and algebra, but he was also interested in scientific subjects . . . He made a thorough analysis of the geometry of hyper-space. His aim was to reduce the subject to its simplest terms, so that anyone like himself with only elementary mathematical training could follow every step . . . Their constructions required all the patience and delicate craftmanship that could be provided by his oriental background . . . To quote Donchian's own words: 'The models are fortunately fool-proof, because if a mistake is made it is immediately apparent and further work is impossible.' In 1934 the models were exhibited at the *Century of Progress Exposition* in Chicago and at the *Annual Exhibit of the American Association for the Advancement of Science* in Pittsburgh. He died in 1967."

Coxeter met Donchian in Chicago at the show and photographed the models he would publish in his book. Together they will write a July 1935 article *An n-Dimensional Extension of Pythagoras' Theorem* [31].

Returning to Olson and Cagli, "Just as the scientist had restored through an open form of sculpture the idea of the fourth dimension, in the same way the artist sought to achieve this representation by analogy in the two-dimensional space of the canvas." [32].

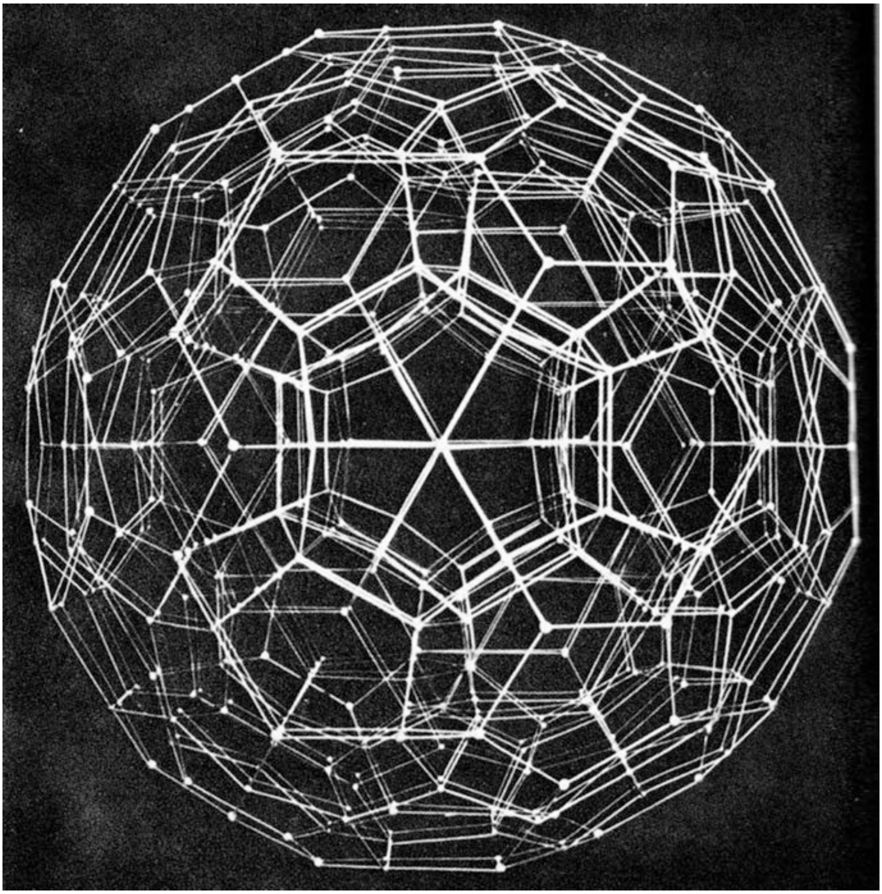

Fig. 10 {5, 3, 3} Model of P. S. Donchian, photo by H.S.M. Coxeter, [1], p. 272

In 1946, the American poet formulated his first ideas about the possible application of the new mathematical theories to poetry. The geometric model offered the possibility of inserting an object belonging to a Euclidean space into a non-Euclidean space. Castellani adds that "It is difficult to understand exactly what the application of these geometric rules means in terms of language." [32].

Olson recognized in his friend Cagli's art the most advanced experimentation with the fourth dimension. In the letter written by Cagli to Olson on December 9, 1946, there are some important considerations about the poem on Moebius and about their common purpose: [33].

"Upon a Moebius strip. I think you are going strong. The all business is wonderful. The poem is up to your best poems, isn't it? And brings me a new wind of inspiration, it seems to me that there is something very mysterious going on there if, in the field of dimensions, drawings turn out to be poems and poems blow back sudden changes to the source of the drawings. (...)any time you show me another poem another door is opened and it looks the way it should look: as the initial point, the beginning, the primordial way of thinking and feeling. (that would be in terms of the tarots: the *Bagatto*) [24]. And I feel very strong about us now, as we have found together a mine of gold."

Colombo notes: [34] "The great novelty that projective geometry and, in general, modern physics and science can offer to Cagli and to art is the possibility of a type of abstract thinking that sees the world as a whole and, at the same time, makes sense of the structure and function of the world as a whole; projective geometry gives the possibility of imagining a completely new space and world, that is, of finding new (and better) solutions at the level of art and thought and at the level of morality."

In his 1949 lecture at the *Watkins Gallery*, Olson traces an intense and fruitful intellectual and human history. The idea is to explain the new concept of space, on which Cagli and he are working, which leads to a new art and a redefinition of man, to realize a morally new humanity [35]. "What I want to do tonight, to justify my appearing before you, is to illuminate for you in what a new conception of space (which is, I think, what Cagli & I keep working towards) leads toward a new art & thus toward a redefinition of man, accomplish, in the moral sense of a new humanitas."

Mark Byers also talks about this *redefinition of Man* in *Charles Olson and American Modernism* [36]. "While Coxeter noted the *psychological value* of the models (since they offered at least a metaphorical visualization of the fourth dimension) Olson thought much more of them. 'What is involved here, he suggests, is something which both science and art have long been capable of, the act of taking a point of vantage from which reality can be freshly seen."

Colombo adds at the end of his article: [37] "Cagli's own research method responds well to a trend that gradually emerged during the 1950s and which saw science not only as a kaleidoscope of new images and aesthetic suggestions that risk becoming mere decorative motifs if not supported by reasoning and solid foundations, but as the bearer of an experimental and operational methodology and approach, of a more open vision."

Corrado Cagli died in Rome on March 28, 1976. In 1978 I visited an exhibition of his work at the *Ca′ d'Oro* gallery in Rome and was struck by an untitled painting that was accompanied in the catalogue by Charles Olson's poem on the Moebius strip. (Fig. 11) I wrote about the Moebius strip and about some of the artists who had been interested in that form, first of all Max Bill and Maurits Cornelis Escher; the article was published in 1981 [38] in the magazine *Leonardo*, then printed by Elsevier. It was edited by Frank Malina, a North American kinetic artist who had moved to Paris from the USA after leaving his job as a rocket engineer. I mentioned it again in 1983 at a conference at the *School of Epistemics* of the University of Edinburgh in November 1981, where Frank Malina was one of the speakers; however, he died the day before the conference opened [39].

At that time I started to work on my film on the Moebius surface, [40] in which Max Bill and the works of Escher were involved, so I phoned Cagli's atelier and made an appointment to go and film the painting I had seen. The atelier was in

Fig. 11 C. Cagli, *Without title*, oil on canvas, (1947) Collection Ebe Cagli Seidenberg. Frame from the movie *Moebius Strip* [40]

Rome, in Via *della Fonte di Fauno 12* (near the Circus Maximus) and is now the headquarters of the *Cagli Foundation*. I remember the day very well, it was March 16, 1978. When I entered the atelier, I was informed that Aldo Moro, president of the *Democrazia Cristiana* (Christian Democrats), several times minister and prime minister of the Italian government, had been kidnapped by the Red Brigades. He was killed on May 9, 1978. One of the great tragedies of the Italian Republic.

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A Fault in the Order: Thoughts on Frayed Strings

Emanuela Fiorelli

Has your heart ever "skipped a beat" over something you've seen, read, or even just imagined? I said "something" not "someone" ... I'm not talking about sentimental issues, but something that excites you at the cerebral level. Sometimes, when I read, see, or imagine, I suddenly perceive that my horizon, just at that moment, is widening; I can almost focus on and give a name to the beach where my waves have always crashed. This may happen while I'm reading a scientific text or while I'm watching a documentary on nature or architecture or a film; it may happen while I'm weaving my threads into plexiglass boxes, or while I'm cooking ... this "skipping a beat" is unpredictable, but one thing I know for sure: I experienced it while reading the poems of Claudio Zanini and listening to the philosophical reflections of Lucio Saviani.

When you meet exceptional people you remember exactly the situation that brought you together. I never met Claudio Zanini in person because of the distancing requirements caused by Covid-19, but I had read his poems "Anxious geometries" in the booklet "Opera prima" published by a well-known literary magazine to which I have subscribed for years. I was immediately struck by the parallels between my artwork and his poems. The booklet sat on my bedside table for years until I decided to contact its author. Immediately we clicked, and the video "I'm Staying Home" was born from our cooperation during the lockdown, in which I read the poems that Claudio wrote while thinking about my works of art resulted in a lovely interweaving of words and images. Recently, as we read each other's bios, we discovered that there was a time when we were close by without knowing it. Michele Emmer had invited us to the same "Mathematics and Culture" conference back in 2013. So, who was it that spun this web?

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M. Emmer, M. Abate (eds.), *Imagine Math 8*,
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I met Lucio Saviani in 2006 while he was presenting the catalogue of the "Bodily silence" exhibition, featuring the works of Paolo Radi. I wanted to meet him because it was the first time I could follow a philosophical conversation without noticing what the ceiling looked like. Philosopher Lucio Saviani is one of the greatest experts of hermeneutics in Italy. It is a pleasure to listen to him during his lectures, not only for his fascinating voice, but especially for his ability to unravel what are otherwise very complicated concepts. And here is the affinity with my works, in which the intricate labyrinth is both the space in which to get lost and the way out. It was therefore natural that I asked him to write some philosophical thoughts related to my works and, in this period, to the lockdown.

The result is the combination of words and images created for this book and I thank Michele Emmer for making it possible. When, 15 years ago (September 1, 2006), I decided to write to him at his university mailbox, I did not expect—although I hoped very much—that he would answer me, nor that, together with his beautiful wife, Marcela, he would soon come to see me in my atelier. I had just started my journey as an artist but he wasn't interested in my resume ... he wanted to see my works in real life! He had sensed that they displayed a strong correspondence with the language of mathematics, and perhaps he saw potential developments that even I had not yet imagined. Had his heart skipped a beat too? Maybe it had, because in 2007, on the occasion of my solo exhibition at Galleria Marchetti, he honored me with a piece in the "Emanuela Fiorelli/Caosmo" catalogue (Fig. 1). A year later, he invited me to the "Mathematics and Culture 2008" conference and later to the "Mathematics and Culture 2013" conference. Recently we met on the occasion of the presentation of the performance "From 1848 to infinity" at the Museum of San Salvatore in Lauro in Rome, in which dancer Katia di Rienzo moved inside my sculpture, made of aluminium and elastic cords, displacing and dilating both real space and the virtual space created and projected by videomaker Massimo Cappellani. This was a lovely weave too!

Using strings, I connect different people, things, surfaces, and spaces. They start with a knot and end with a knot, and between one and the other there is polarized, intimate, poetic, vibrant space ... our life? But this is threatened by something that undermines their continuity, that takes them to different, unexpected planes and confounds their direction ... maybe Covid-19? I thought to publish here both the real, existing, tangible works, and their computer retouched, virtual images, threatened by the things we see only in their consequences. Lucio Saviani and Claudio Zanini were inspired by them for their poetic and philosophical compositions and I thank them sincerely for having joined this project. Again a huge thank you to Michele Emmer who made the publication possible.

Fig. 1 Emanuela Fiorelli, self portrait

1 Poems for Collapse from COVID

Works by Emanuela Fiorelli (Figs. 2–13)
Poems by Claudio Zanini

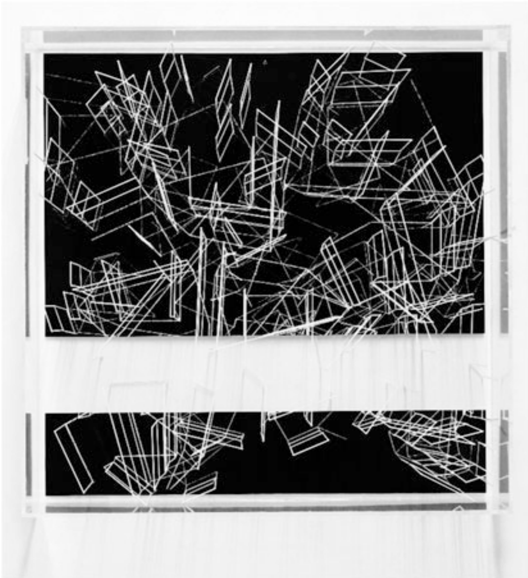


Fig. 2 Emanuela Fiorelli, 25 m—50 × 50 × 14 cm plexiglass, elastic thread, vinyl, 2015

Un rovinar di frammenti

Un rovinar di frammenti
come se dita staccate
in rallentata presa
giù dal baratro incavato
si schiantassero sul fondo.

Sebbene resistano
ancora in sospensione
i fianchi forti nel volo
e impavidi oppongano
inconsapevole diniego,

è il centro che crolla,
franando rapido collassa
lasciando ammutolite ceneri
nel totale e repentino
annientamento d'ogni senso.

A Ruin of Fragments

A ruin of fragments
as though detached fingers
slowly grasping
down from the sunken abyss
smashed on the bottom.

Although the strong flanks
still suspended
hold up in flight
and fearlessly oppose
unwitting denial,

it is the centre that crumbles,
quickly sliding, it collapses
leaving silenced ashes
in the total and sudden
annihilation of all senses.

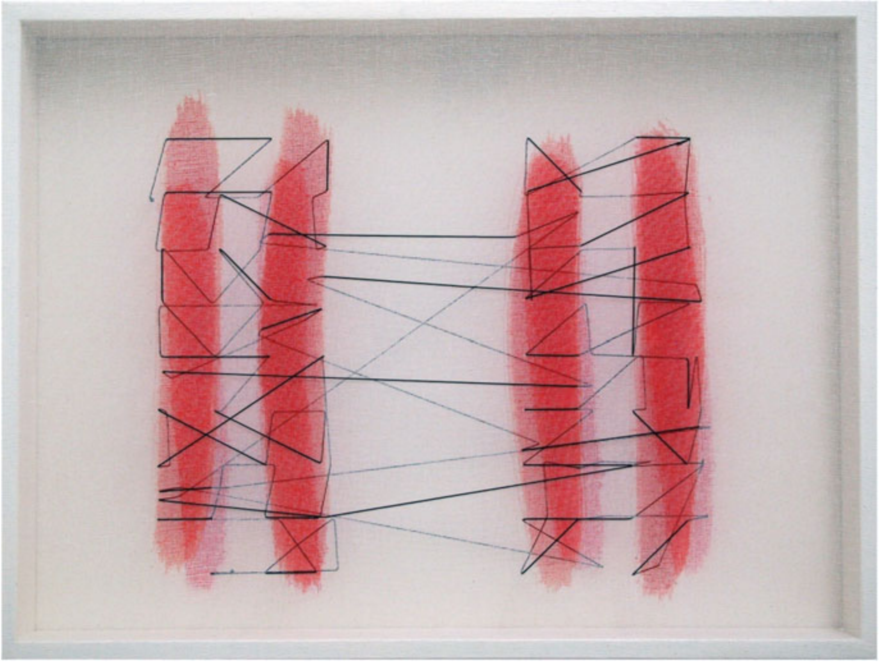


Fig. 3 Emanuela Fiorelli, Legami di sangue, 100 × 70 × 11 cotton thread, tarlatan, and fabric dyeing, 2008

Separazione

Divora, il vuoto della faglia
in profonda frattura verticale
l'arcana macchina del volo
sospesa nell'incurvata onda,
ne soffoca il respiro,
ne succhia avida lo spazio
ne viola l'ordine connesso.

Franti i legami d'annodati fili
scissa l'ordita trama
s'incunea prepotente il vuoto
allontanando mutilati i nodi
l'uno dall'altro respinti
nell'inarrestabile deriva.
Tuttavia, ancora palpitava
l'arcana macchina sospesa
d'un tremore commovente
e breve, prima di morire.

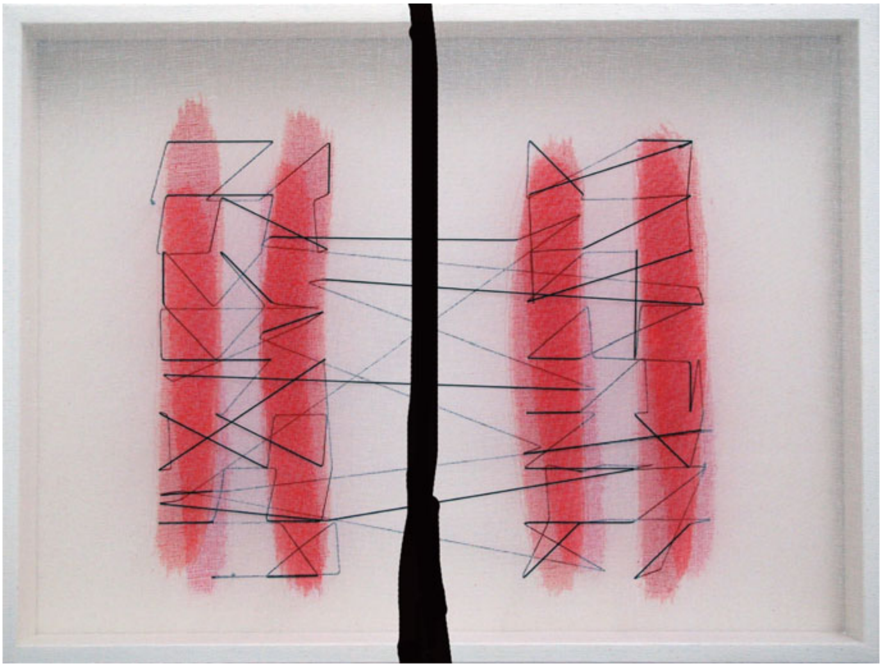


Fig. 4 Emanuela Fiorelli, Legami di sangue, Covid effect

Separation

The emptiness of the fault-line
devours in profound vertical fracture
the arcane machinery of flight;
suspended in the curve of a wave
it suffocates its breath,
greedily sucks up its space
violates its connected order.

Having torn the ties of knotted threads
Having rent the woven cloth
Emptiness wedges itself with power
distancing the mutilated knots,
rejected by one another
in the unstoppable drift.
However, the arcane machinery
still throbbed, suspended
in a moving, brief tremor,
before perishing.

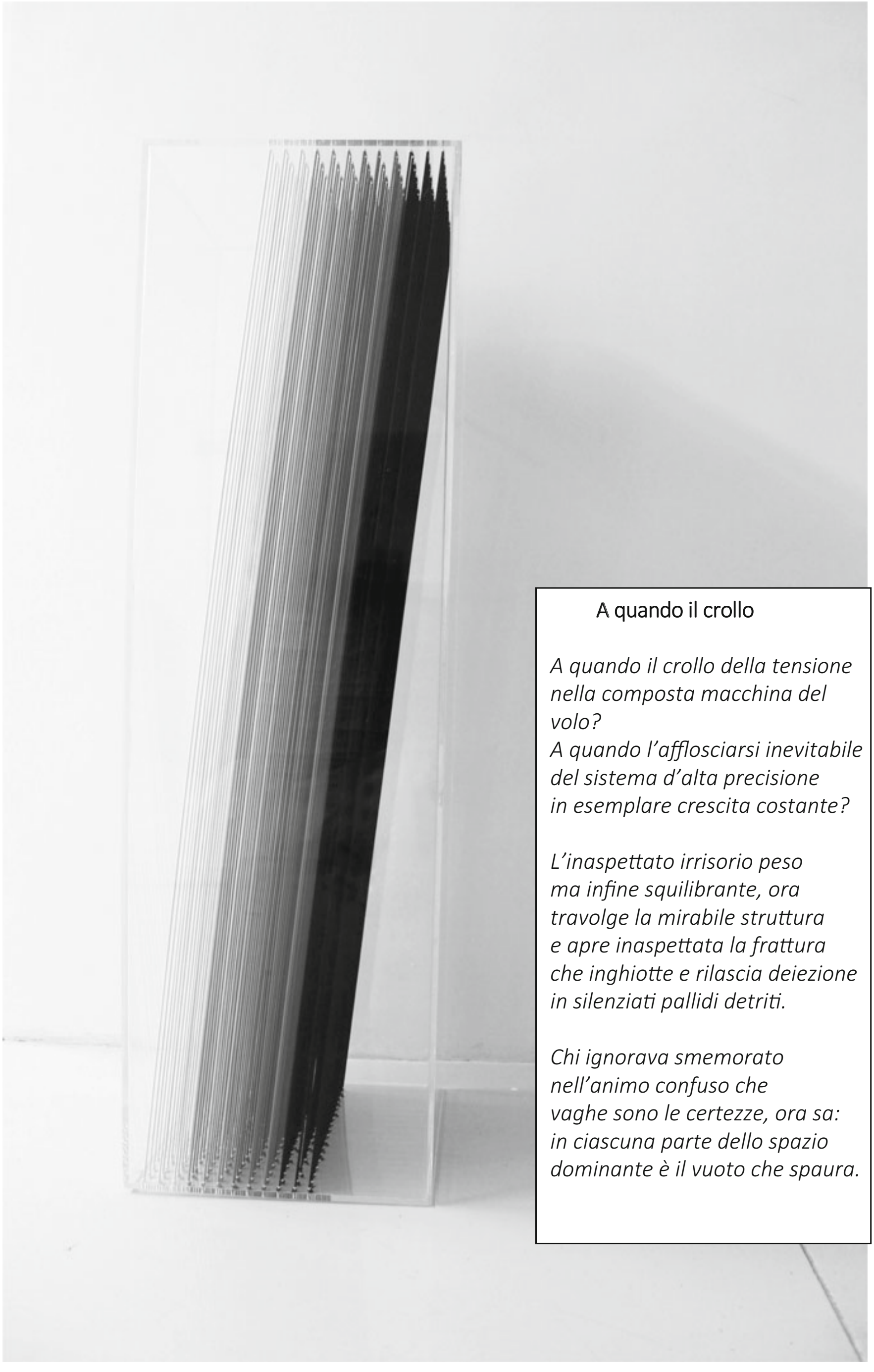


Fig. 5 Emanuela Fiorelli, Sculptor box 2, cm 100 × 30 × 30, plexiglass, elastic thread, 2017

Fig. 6 Emanuela Fiorelli, Sculptor box 2, Covid effect

When the Collapse

When will the tension
of the poised flying machine crumble?
When, the inevitable collapse
of the high-precision system,
exemplary in its constant growth?

The unexpected negligible, but
ultimately unbalancing, weight now
overwhelms the impressive structure,
and unexpectedly opens the fracture
that swallows and releases ejection
in silenced pale detritus.

Those who forgetfully overlooked,
in the confused soul, that
certainties are vague, now know:
in each portion of space,
the frightening emptiness prevails.

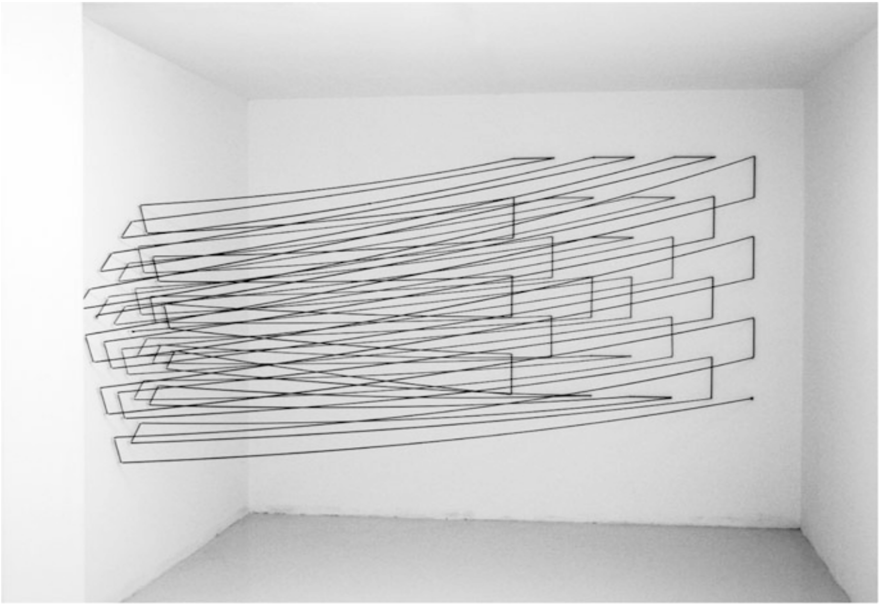


Fig. 7 Emanuela Fiorelli, site specific installation, elastic cord—400 × 150 × 100 cm, 2011

La ferita verticale

Preme la ferita verticale,
divaricando penetra
la macchina del volo.
Recide i magri filamenti
ne sventra l'ordita trama
degli elastici segmenti.

Sembrava mormorasse
quasi un alito sommesso
la macchina sospesa
un impercettibile vibrare
un sottilissimo stormire.

Ora tace nel cavo della faglia
spazio attonito e silente
vuoto collasso verticale.

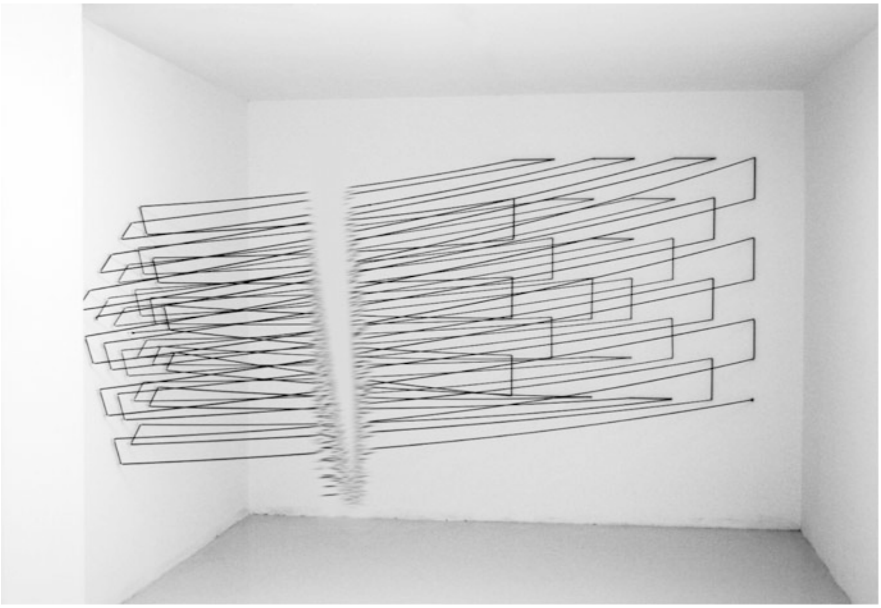


Fig. 8 Emanuela Fiorelli, side specific installation-Covid effect

The Vertical Wound

The vertical wound presses,
spreads, penetrates
the flying machine.
It severs the scant threads,
guts the woven cloth
of the elastic segments.

The suspended machine
seemed to be murmuring
almost a quiet breath,
an imperceptible vibration,
the faintest of rustles.

Now it is silent in the hollow of the fault-line
an astonished and silent space,
a vertical empty collapse.

Lo scarto trascurabile

Era, all'inizio,
impercettibile anomalia,
lo scarto trascurabile
entro l'equilibrato assetto
del segmento pencolante.

Non era ancora
baratro sfondato senz'appiglio,
né la scivolosa china
precipitosa verso il ciglio.
Non era ancora fatale sbando.

Volge ora, tuttavia
nell'asimmetrico traslare
inarrestabile torsione
entro gorgo che l'annienta
nell'inesplicabile sparire.

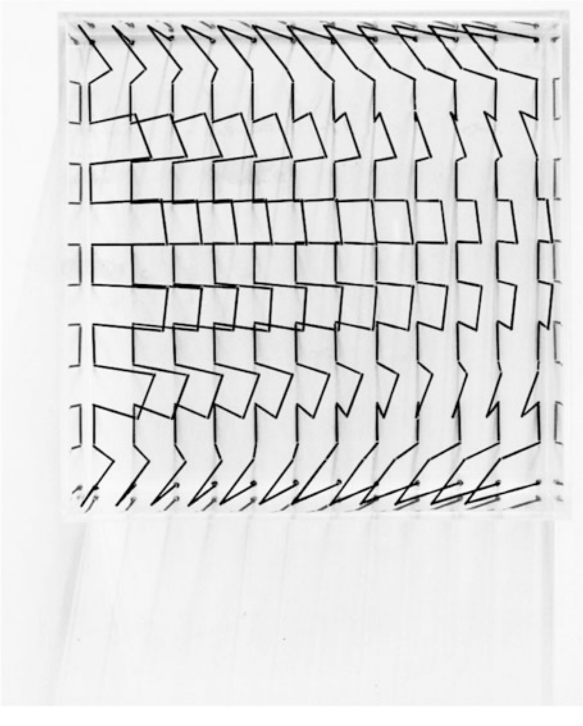


Fig. 9 Emanuela Fiorelli, 1 + 1 = 1, 30 × 30 × 18 cm, plexiglass, elastic thread, 2018

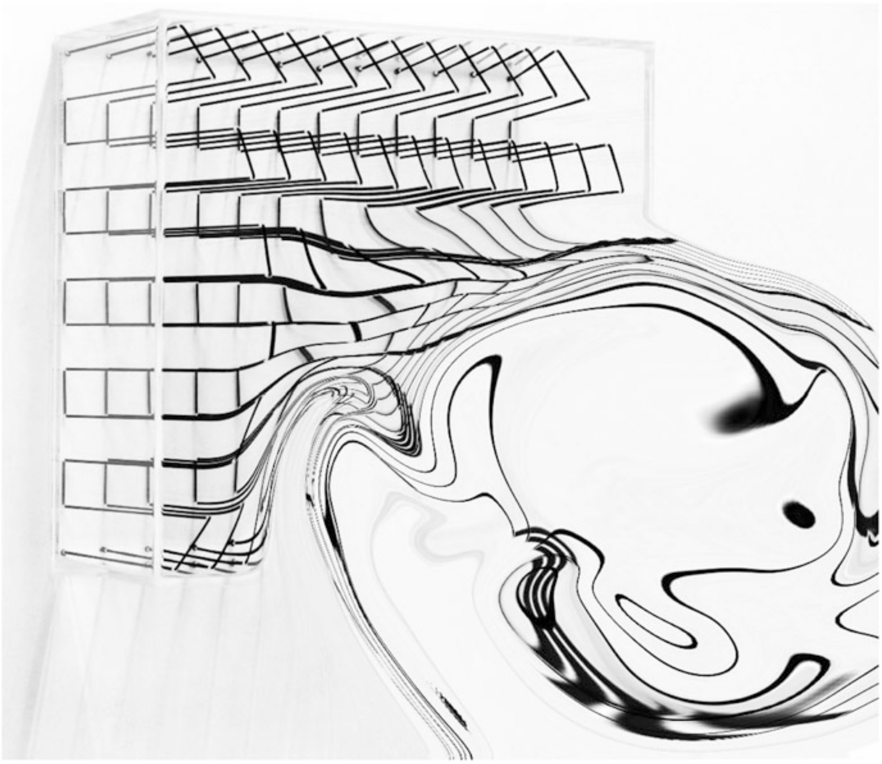


Fig. 10 Emanuela Fiorelli, 1 + 1 = 1 Covid effect

The Negligible Reject

At first, it was
an imperceptible anomaly
the negligible difference
within the balanced structure
of the loose segment.

It still was not
a broken, gripless chasm,
nor the slippery slope
rushing towards the edge.
It still was not fatal disorder.

However, it turns now,
in an asymmetrical movement,
an unstoppable twist
inside an annihilating whirlpool,
an inexplicable disappearance.

In torsione vorticosa

In torsione vorticosa, frulla
la geometria, esplode e sborda
nel suo disperdersi centrifugo,
collassa l'equilibrio
nell'insostenibile espansione.
Tende ad avvitarsi in sé, spaura
quale grumo nero perturbante
magmatica matassa oscura.

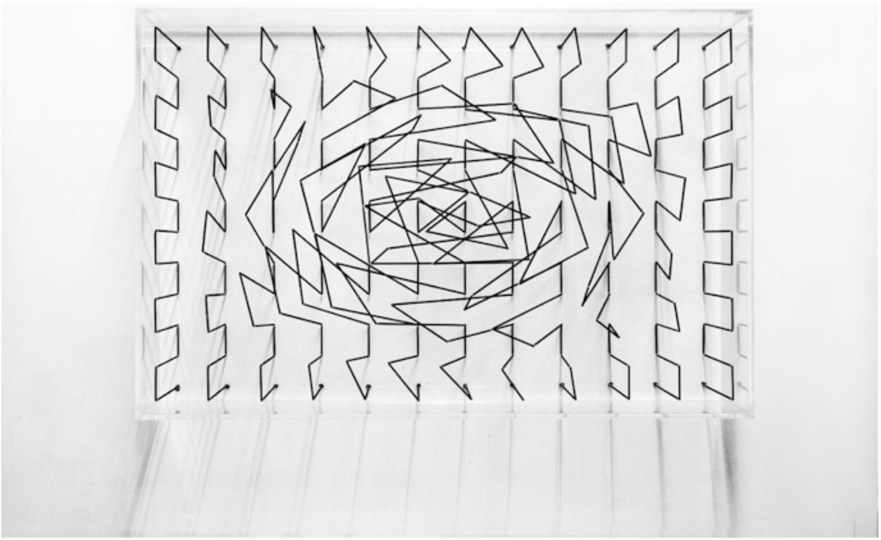


Fig. 11 Emanuela Fiorelli, Basic box 8, plexiglass, elastic thread, 40 × 60 × 14, 2020

In a Twisted Cyclone

In a twisted cyclone, it spins
geometry, explodes and overflows
in its centrifugal scattering,
collapses the balance
in its unsustainable expansion.
It tends to coil itself, frightening
like a perturbing black clot,
a magmatic dark tangle

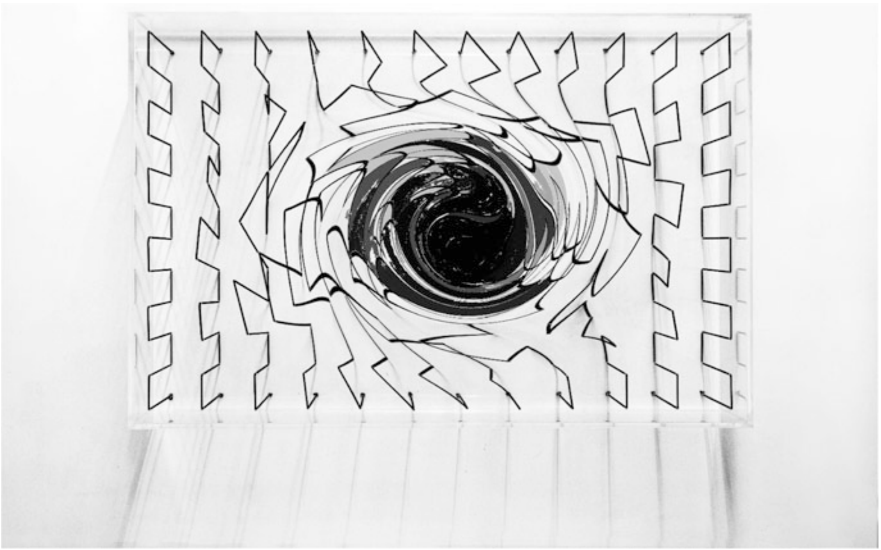


Fig. 12 Emanuela Fiorelli, Basic box 8, Covid effect

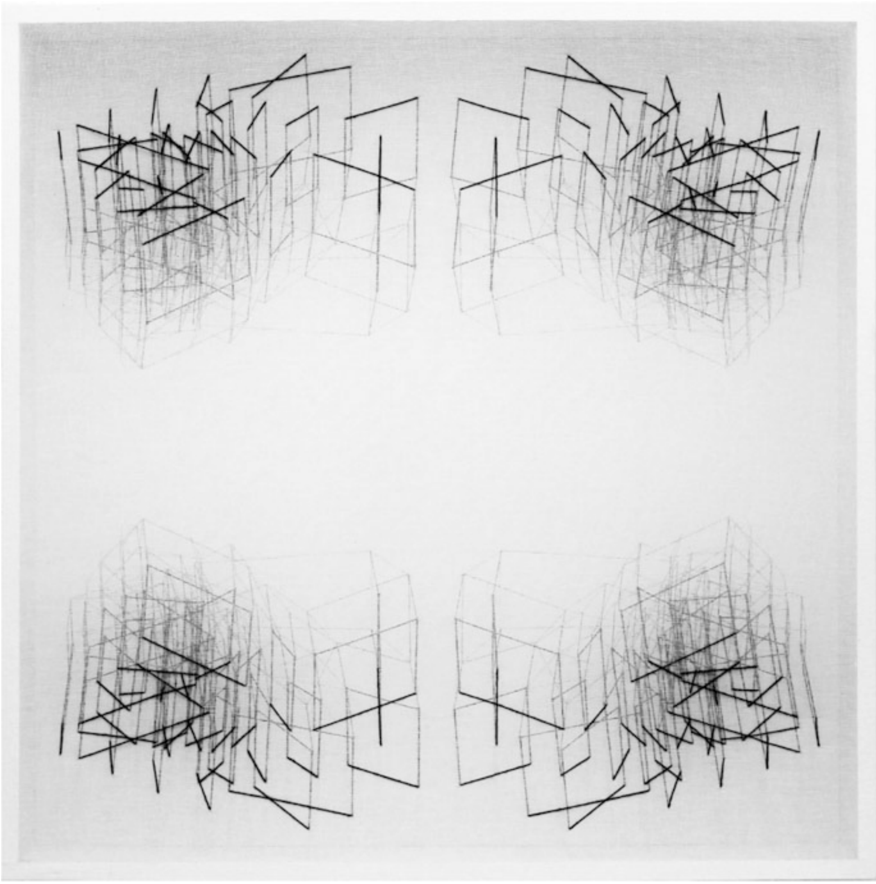


Fig. 13 Emanuela Fiorelli, Dilatato, 94 × 94 × 11 cm—cotton thread, tarlatan, 2016

2 A Thread for Two

Work by Emanuela Fiorelli (Figs. 14–22)
Philosophical Contribution by Lucio Saviani

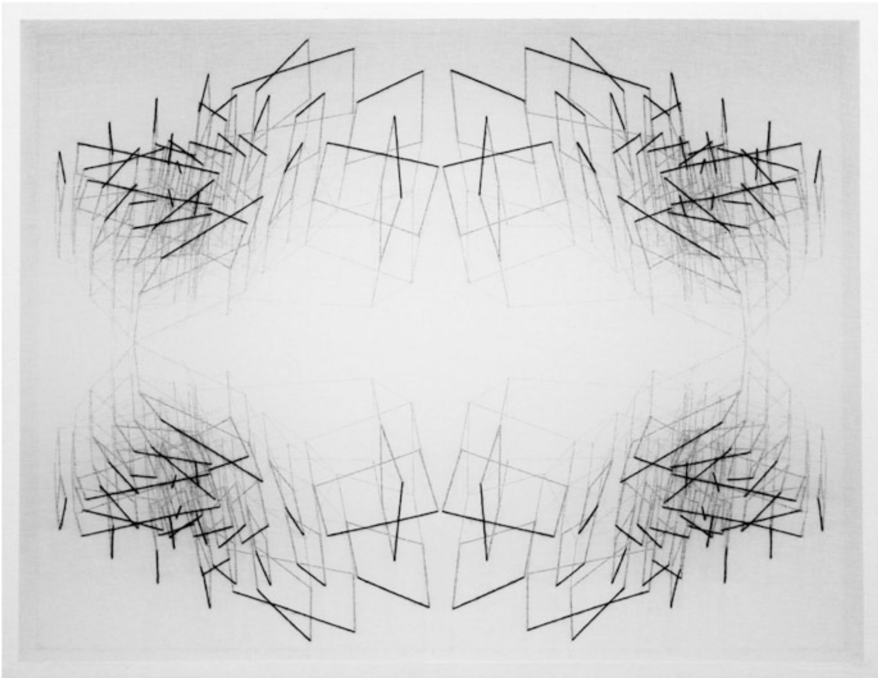


Fig. 14 Emanuela Fiorelli, Coincidente, 70 × 90 × 11, cotton thread, tarlatan, 2015

1
It is displacement, dilation, duplication. It is suspended time. Like swinging suspended from a branch, or like a spider from its thread, which that one time was the same, by the terrible will of the gods.

Solitude, isolation, confinement. Speaking to yourself as if you were speaking to another. Call yourself by your first name and make yourself into a second person singular.

It is Arachne, and it is Minerva.

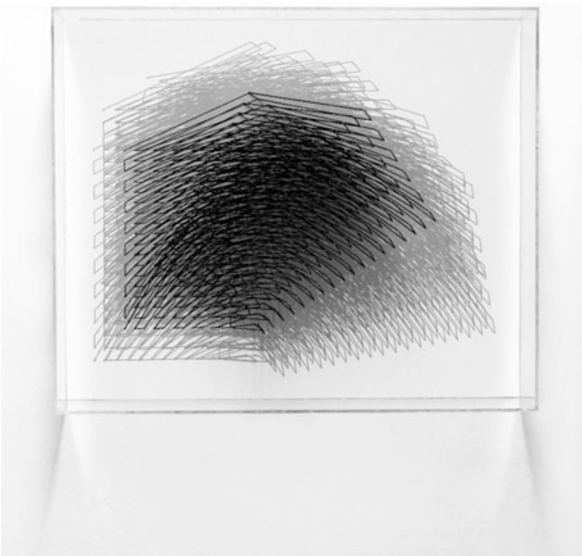


Fig. 15 Emanuela Fiorelli, s.t. 50 × 50 × 14, plexiglass, elastic thread, vinyl, 2016

2

One weaves order, rule, distance. The other bursts and overflows, frays, dilates, deforms. One demands and arranges concordant harmony, the other foreshadows and warns of disorder, and the bewilderment that arises and overflows from it.

Fig. 16 Emanuela Fiorelli, s.t., Covid effect

Fig. 17 Emanuela Fiorelli, Affetto ottico 39.5 × 34 × 19 cm, plexiglass, elastic thread, 2006

3

What is that superhuman, or too human, superb delirium? What threads is its mind weaving? Or will it not be some sort of agreement, between the spider artist and the loom goddess, a secret pact in their minds, as secret as the confrontation will be public? Minerva acknowledges defeat, but then punishes her rival. She is attracted to her opposite but does not accept what she cannot separate herself from.

Fig. 18 Emanuela Fiorelli, Affetto ottico-Covid effect

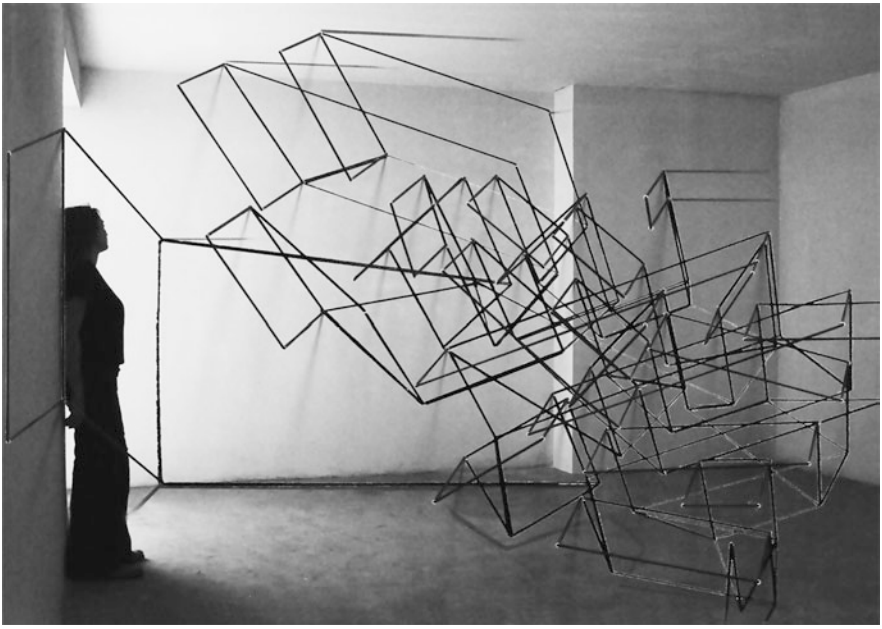


Fig. 19 Emanuela Fiorelli, Archi box 1, 40 × 60 × 14 cm, plexiglass, elastic thread, vinyl, 2017

4

Facing each other, the two works challenge each other in an unprecedented public confrontation. A motionless duel, made up of spikes, crossings, threads, deceptions, hooks, and counterpoints. Was Arachne's hand taken by such sudden disproportion, or was Minerva simply summoned by her rival and turned into her own double?

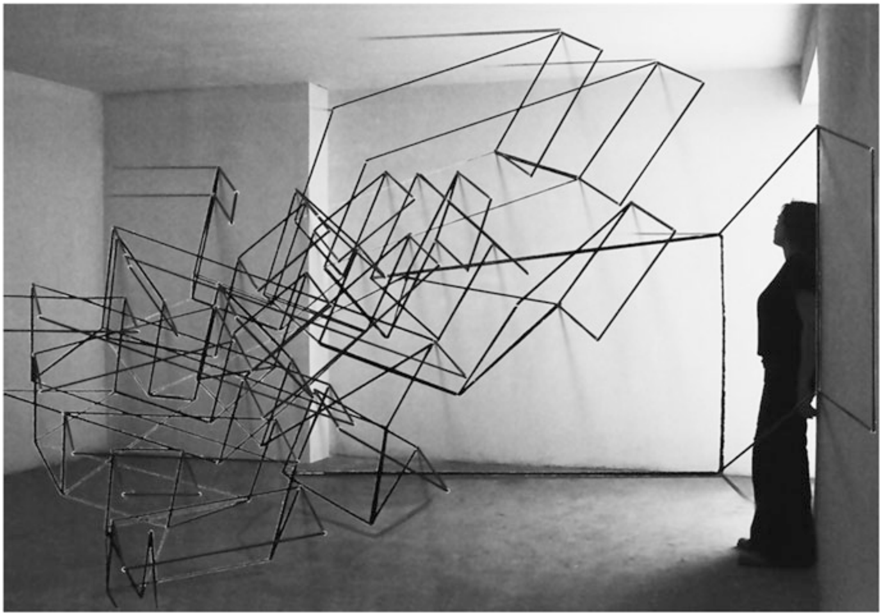
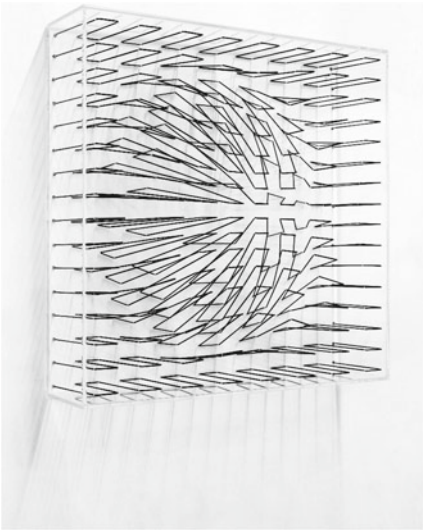


Fig. 20 Emanuela Fiorelli, Archi box 1, 40 × 60 × 14 cm, plexiglass, elastic thread, vinyl, 2017



5

On this side of the line it is a work of exactitude, ingenuity, balance, clarity, and proportion. An enraptured gaze and an expert manoeuvre, the work of the hand and a suspended mind, free hand, and logical speech.
Beyond the line it is toil–faithful to disorder, sensitive to the unexpected, to the coming disruption, to invisible nature. Continuous beginning, and a challenge to the established order, it reveals its clandestine causality.

Fig. 21 Emanuela Fiorelli, Basic box 6, lateral vision, 60 × 60 × 14 cm**,** plexiglass, elastic thread, 2019

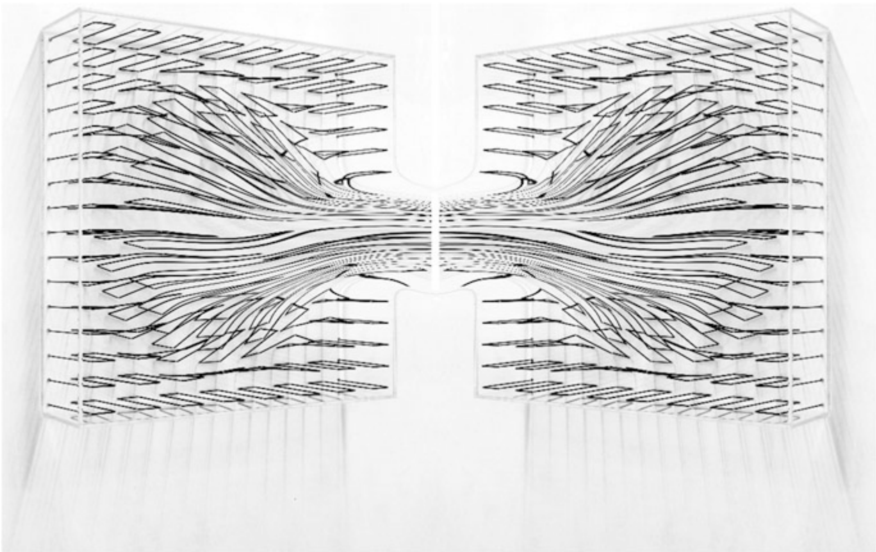


Fig. 22 Emanuela Fiorelli, Basic box 6, Covid effect

The Multivalent Fourth Dimension and the Impact of Claude Bragdon's *A Primer of Higher Space* on Twentieth and Twenty-First Century Art

Linda Dalrymple Henderson

After German Expressionist painter Max Beckmann fled Germany to escape the Nazis in 1937, he gave a lecture in London in 1938, titled "On My Painting." This was an important first appearance of Beckmann outside Nazi Germany, and he sought from the start to emphasize that his paintings did not engage national politics. ". . . I would like to emphasize that I have never been politically active in any way," Beckmann declared at the start [1]. The international concept on which he drew to characterize his painting, instead, was the "fourth dimension:" "To transform height, width, and depth into two dimensions is for me an experience full of magic in which I glimpse for a moment that fourth dimension which my whole being is seeking." [2] A few paragraphs earlier, he had noted his focus on imagination, space, and transcendence: "My dream is the imagination of space—to change the optical impression of the world of objects by a transcendental arithmetic progression of the inner being." "I am seeking the bridge that leads from the visible to the invisible . . . ," he asserted [3].

Although Beckmann's expressive figurative paintings remained rooted in the visual world, transcending the visible had been a central goal of many earlier twentieth-century painters, including the French Cubists and abstract painters such as František Kupka, Kazimir Malevich, and, as we shall see, Hilma af Klint [4]. For these artists, a possible higher dimension of space functioned as a sign of a truer reality, liberating them from allegiance to the visible world. In declaring his interest in the fourth dimension, Beckmann clearly sought to align himself with modern European painting. Yet, in 1938 his allusion also risked misunderstanding, because of the almost overnight rise to prominence of Einstein and Relativity Theory in late 1919. That had occurred in the wake of critical proof of one of Einstein's postulates by an eclipse expedition that year. Since that point, the general public

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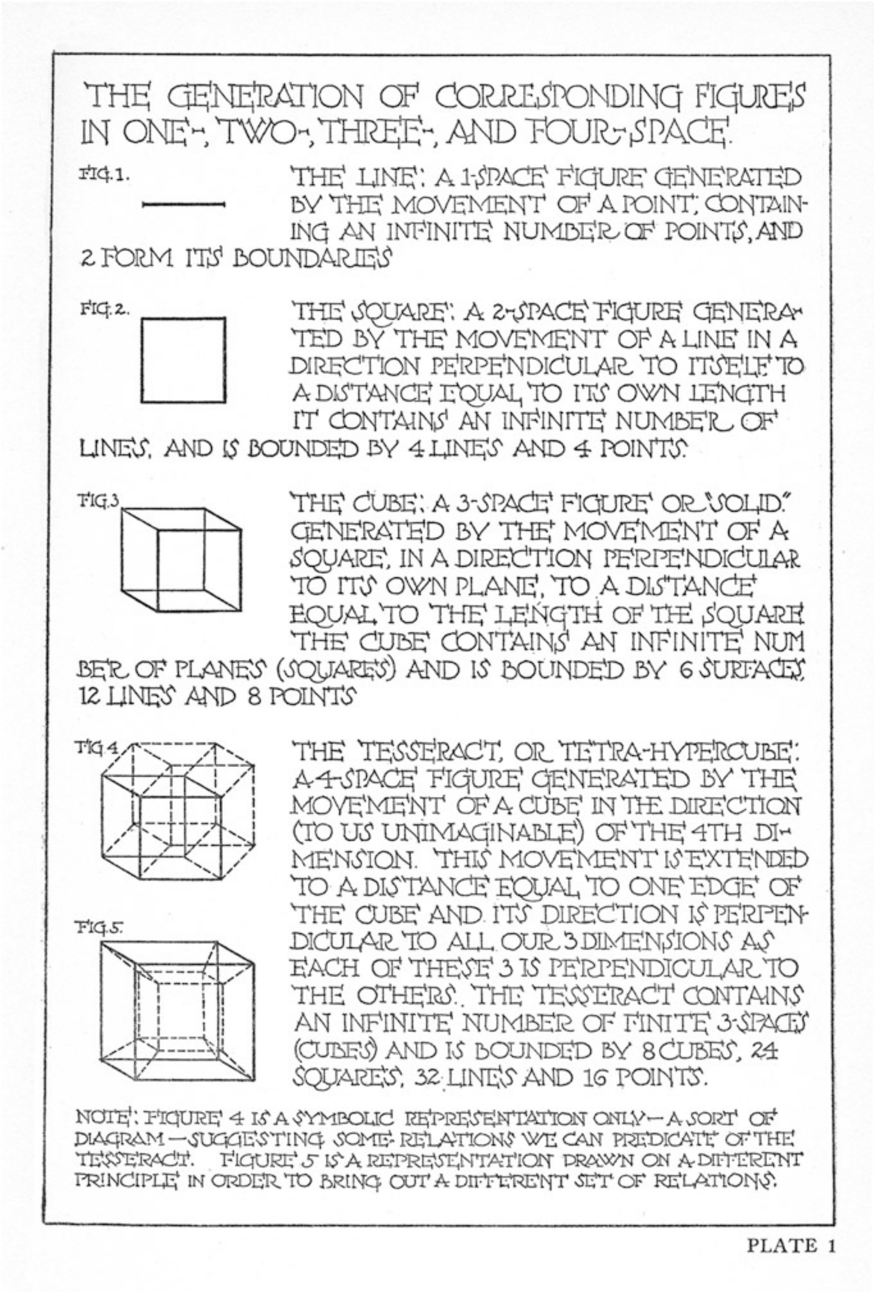
had heard increasingly of a fourth dimension defined not as space but as time in a four-dimensional "space-time continuum." [5] This potential confusion would persist through much of the twentieth century and into the twenty-first: was the term "fourth dimension" referring to the earlier concept of space or to Relativity Theory's time?

It has been 50 years since the first scholarly historical study of the "fourth dimension" and art began, in sources such as my own 1971 essay, "A New Facet of Cubism: 'The Fourth Dimension' and 'Non-Euclidean Geometry' Reinterpreted." [6] My graduate student research had been triggered by my adviser Robert L. Herbert's curiosity about articles from the 1940s and 1950s propounding a supposed connection between Cubism and Relativity or Picasso and Einstein. This "Cubism-Relativity myth," as I subsequently termed it, was widely cited in literature until scholarship beginning in the 1970s began to counter that error; unfortunately, one still finds such ideas in survey textbooks, for example [7]. Surveying the topic in popular journals and books from the 1880s to ca. 1920, however, establishes immediately that the focus of the international fascination with the "Fourth Dimension" (as it was often written), was on a spatial realm that suggested a truer, invisible reality [8].

Of the books, I encountered in 1970 as I began my research, one of the most useful was Claude Bragdon's *A Primer of Higher Space (The Fourth Dimension)* of 1913 (Figs. 1, 2, 4, and 9). Bragdon (1866–1946), an architect and designer in Rochester, New York, had gathered everything he could find on the popular fourth dimension, including from Europe, and published the book through his own Manas Press [9]. In the *Primer* Bragdon created a compendium of 30 beautiful, hand-drawn plates setting forth ways of understanding a higher spatial dimension (e.g., Fig. 1), preceded by an introductory text that argued for the "reasonableness of the higher space hypothesis" and linked it to the idealist philosophical tradition, among other touchstones [10]. At that moment in 1970, however, I could never have imagined what a widespread cultural impact the *Primer* had had and would continue to have beyond the twentieth century. My 1975 dissertation and 1983 book *The Fourth Dimension and Non-Euclidean Geometry in Modern Art* established Bragdon's impact in New York on figures such as painter Francis Picabia (and, undoubtedly, the latter's friend Marcel Duchamp), the inventor of the "Clavilux" Thomas Wilfred, and designers Norman Bel Geddes and Buckminster Fuller [11]. Most importantly, as summarized below, it proposed Bragdon's importance for the Russian mystic philosopher P. D. Ouspensky and the Russian Suprematist painter Malevich [12].

Bragdon's national and international impact, however, was much broader than this, as I have since discovered. His effect upon a number of prominent artists establishes his centrality to the history of art and the fourth dimension in the twentieth century and beyond. With its unprecedented, boldly designed plates, Bragdon's 1913 *Primer of Higher Space* initially introduced the idea to a wide public, spreading it internationally [13]. In subsequent decades, the *Primer* would serve as a conceptual anchor for the spatial fourth dimension in culture—to be rediscovered periodically and stimulate artists to explore what seemed an increasingly historical



THE GENERATION OF CORRESPONDING FIGURES IN ONE-, TWO-, THREE-, AND FOUR-SPACE.

FIG.1. THE LINE: A 1-SPACE FIGURE GENERATED BY THE MOVEMENT OF A POINT; CONTAINING AN INFINITE NUMBER OF POINTS, AND 2 FORM ITS BOUNDARIES

FIG.2. THE SQUARE: A 2-SPACE FIGURE GENERATED BY THE MOVEMENT OF A LINE IN A DIRECTION PERPENDICULAR TO ITSELF TO A DISTANCE EQUAL TO ITS OWN LENGTH IT CONTAINS AN INFINITE NUMBER OF LINES, AND IS BOUNDED BY 4 LINES AND 4 POINTS.

FIG.3. THE CUBE: A 3-SPACE FIGURE OR "SOLID." GENERATED BY THE MOVEMENT OF A SQUARE, IN A DIRECTION PERPENDICULAR TO ITS OWN PLANE, TO A DISTANCE EQUAL TO THE LENGTH OF THE SQUARE THE CUBE CONTAINS AN INFINITE NUMBER OF PLANES (SQUARES) AND IS BOUNDED BY 6 SURFACES 12 LINES AND 8 POINTS

FIG.4. THE TESSERACT, OR TETRA-HYPERCUBE: A 4-SPACE FIGURE GENERATED BY THE MOVEMENT OF A CUBE IN THE DIRECTION (TO US UNIMAGINABLE) OF THE 4TH DIMENSION. THIS MOVEMENT IS EXTENDED TO A DISTANCE EQUAL TO ONE EDGE OF THE CUBE AND ITS DIRECTION IS PERPENDICULAR TO ALL OUR 3 DIMENSIONS AS EACH OF THESE 3 IS PERPENDICULAR TO THE OTHERS. THE TESSERACT CONTAINS AN INFINITE NUMBER OF FINITE 3-SPACES (CUBES) AND IS BOUNDED BY 8 CUBES, 24 SQUARES, 32 LINES AND 16 POINTS.

FIG.5.

NOTE: FIGURE 4 IS A SYMBOLIC REPRESENTATION ONLY—A SORT OF DIAGRAM—SUGGESTING SOME RELATIONS WE CAN PREDICATE OF THE TESSERACT. FIGURE 5 IS A REPRESENTATION DRAWN ON A DIFFERENT PRINCIPLE IN ORDER TO BRING OUT A DIFFERENT SET OF RELATIONS.

PLATE 1

Fig. 1 Claude Bragdon, *A Primer of Higher Space (The Fourth Dimension)* (Rochester, NY: The Manas Press, 1913), pl. 1

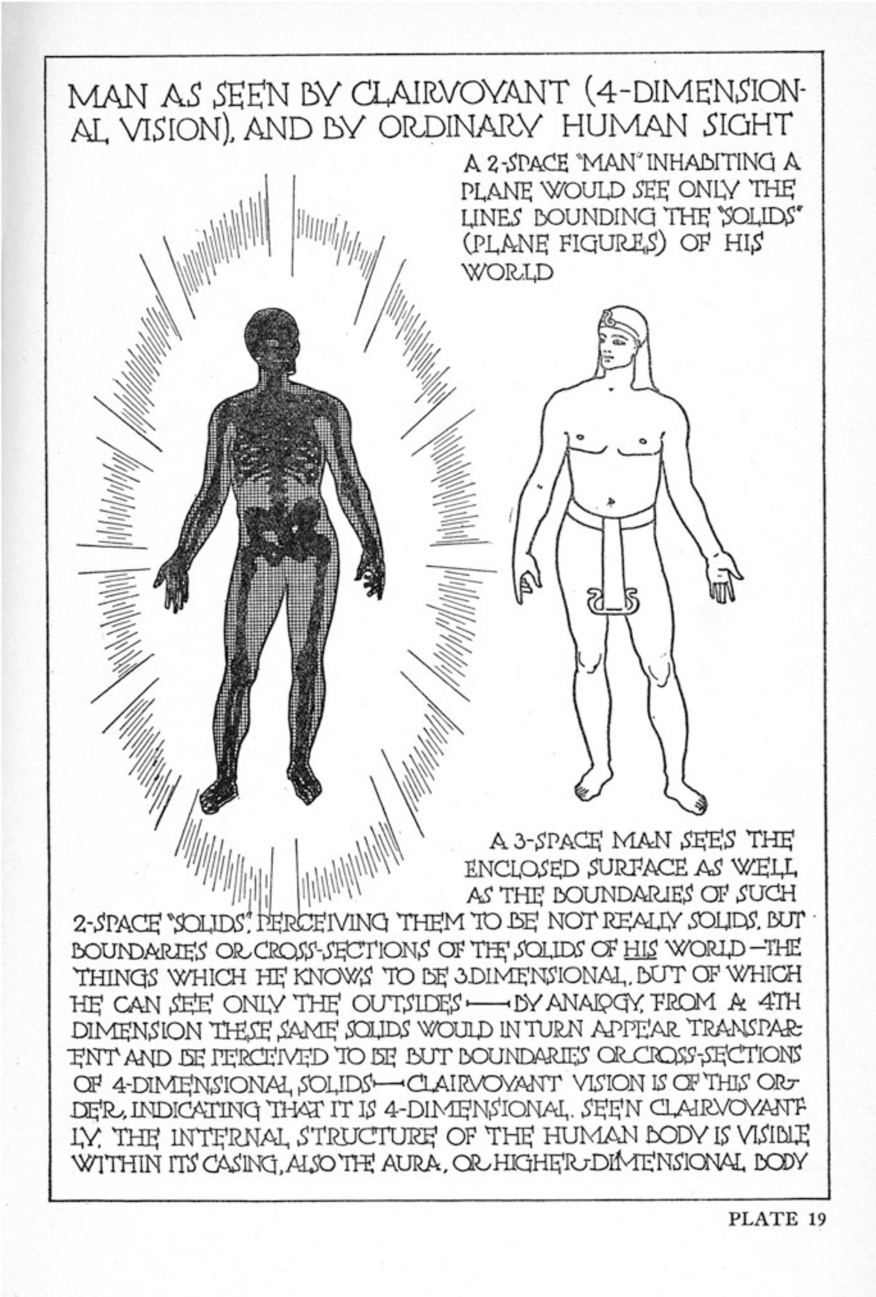


Fig. 2 Bragdon, *A Primer of Higher Space*, pl. 19

idea. With the emergence of computer graphics and string theory in physics in the 1970s and 1980s, however, a gradual cultural revival of higher spatial dimensions commenced and interest in Bragdon's book began to grow [14]. The *Primer* was first reprinted in 1972 in Tucson, Arizona, in an unauthorized edition that did not acknowledge Bragdon's Manas Press as the publisher [15]. Gradually, more reprint editions have become available, culminating in the artist Tauba Auerbach's homages to Bragdon in the 2016 Diagonal Press editions of *A Primer of Higher Space* and his 1915 *Projective Ornament* discussed below.

Bragdon was a member of the Theosophical Society, so prominent in this period, and his collecting of international information for the book as well as its international impact owed a debt to the Society [16]. Following upon the first figure to develop the philosophical implications of the fourth dimension, the Englishman Charles Howard Hinton in the 1880s, Theosophist C. W. Leadbeater had embraced the fourth dimension to argue both for the experience of clairvoyance and for the Theosophical doctrine of the "astral plane" as interpenetrating other levels of existence [17]. Bragdon was an active distributor of his Manas Press publications, and in addition to major bookstores and individuals, he sent his books to Theosophical Society offices in Europe as well as to individual Theosophists such as Leadbeater and Rudolf Steiner [18]. Although a specific Bragdon connection to Beckmann is not known, the German painter, too, was involved in Theosophy and might well have encountered Bragdon's books in that context. However, as we shall see, Bragdon's impact is quite certain now for numerous additional artists, including Johannes Itten (and his friend Wassily Kandinsky) at the German Bauhaus, Swedish abstract pioneer Hilma af Klint in Stockholm, artists Peter Forakis and Robert Smithson in 1960s New York, and, most recently, contemporary New York artist Tauba Auerbach.

As Bragdon's involvement with Theosophy demonstrates, the geometrical fourth dimension had acquired a variety of associations, becoming a truly multivalent term. Beyond its roots in the development of geometries of higher dimensions in the 1870s, the "fourth dimension" was soon linked to idealist philosophy (Plato's world of ideas and Kant "thing-in-itself"), the magical spiritualism of German astronomer J. C. F. Zöllner, the Christian conception of Heaven, the evolution of consciousness, the Romantic pursuit of infinity, and science fiction [19]. As noted above, the most radical change in understanding of the term occurred around 1920 with the redefinition of the fourth dimension as time in the four-dimensional space-continuum of Einstein's General Theory of Relativity. Hermann Minkowski had proposed that structure for Einstein's theory in 1908, but, significantly, laypersons did not hear of that redefinition until after November 1919. Indeed, early popular explications of Relativity Theory drew heavily on the spatial fourth dimension to elucidate the concept of dimensionality [20].

After 1920 and for much of the twentieth century, any scientific associations of the term "fourth dimension" were with Relativity Theory and space-time. Yet in the late nineteenth and early twentieth century, the fourth dimension of space was indeed associated with other aspects of science—specifically the X-ray [21]. Wilhelm Roentgen's discovery of X-rays in 1895 had established the limited nature of human vision, which perceives only the narrow band of visible light in the much vaster spectrum of electromagnetic waves. More than I realized in the 1970s, the discovery of the X-ray provided critical support for continued cultural interest in a fourth spatial dimension, since after that point no one could deny the possible existence of a higher dimension simply because it could not be seen. Bragdon illustrated the X-ray/fourth dimension link in his 1913 *Primer of Higher Space,* suggesting that four-dimensional vision could explain clairvoyance, allowing an individual's vision to penetrate three-dimensional objects just like an X-ray (Fig. 2).

Examining plates from the daybook of German Bauhaus teacher Johannes Itten in 1920 (Fig. 3), it is clear that Itten had a copy of Bragdon's *Primer of Higher Space* in hand by that year. In contrast to Berlin, whose resident Einstein was the talk of the city, Weimar offered Itten, a member of the mystical Mazdaznan sect, a remote place to explore Bragdon's plates, including his illustration of X-ray clairvoyance (Fig. 2). In several drawings, Itten responded to aspects of Bragdon's book, including the planes produced as cross-sections of three-dimensional cubes intersecting a plane (Fig. 3, center; Fig. 4) [22]. The plane within the cube at the center of Fig. 3 derives directly from Bragdon's parable *Man the Square*, first published in 1912 and reprinted in the *Primer* [23]. In that tale a "Christos" cube unfolds down into the plane to illustrate the unity of individual squares who focus on their differences only because they are skewed in their relation to the plane (as in Fig. 4). Bragdon's *Man the Square* source image, labeled "THE ARCHETYPAL WORLD [The Cube] AND THE PHENOMENAL WORLD [The Square]," along with the *Primer*'s introductory text, made clear the effectiveness of dimensional difference as a means to contemplate spiritual or idealist philosophical concerns. In the context of the enthusiasm for Einstein and a temporal fourth dimension during the 1920s, most often expressed by incorporating time in kinetic art, Bragdon's *Primer* now took on a new role as a record of the pre-World War I era and its utopian visions of higher realities, helping to preserve the spatial fourth dimension [24].

Given their similar worldviews, it is highly likely that Itten would have shared his copy of the *Primer* with fellow faculty member Wassily Kandinsky [25]. The Russian-born Kandinsky, working in Munich before World War I, had been deeply interested in Theosophy and especially the Christian-oriented Theosophist (later Anthroposophist) Rudolf Steiner, to whom Bragdon had sent book copies [26]. In the prewar period, Kandinsky had studiously avoided any reference to the fourth dimension in his writings, including *On the Spiritual in Art* (1911), since he had roundly rejected French Cubism with its pursuit of a geometrical fourth dimension [27]. Kandinsky's developing abstraction of the prewar period had been highly organic and used hidden images of the Apocalypse to help sensitize viewers and prepare them for a future spiritual epoch. He hoped to transform individuals by his paintings' creation of vibrations in the souls of his viewers [28]. After returning to Russia during the World War I, he had encountered the new geometric style of Malevich and the Russian avant-garde (e.g., Fig. 5). By the time he joined the Bauhaus faculty in early 1922, he had embraced geometric forms while maintaining

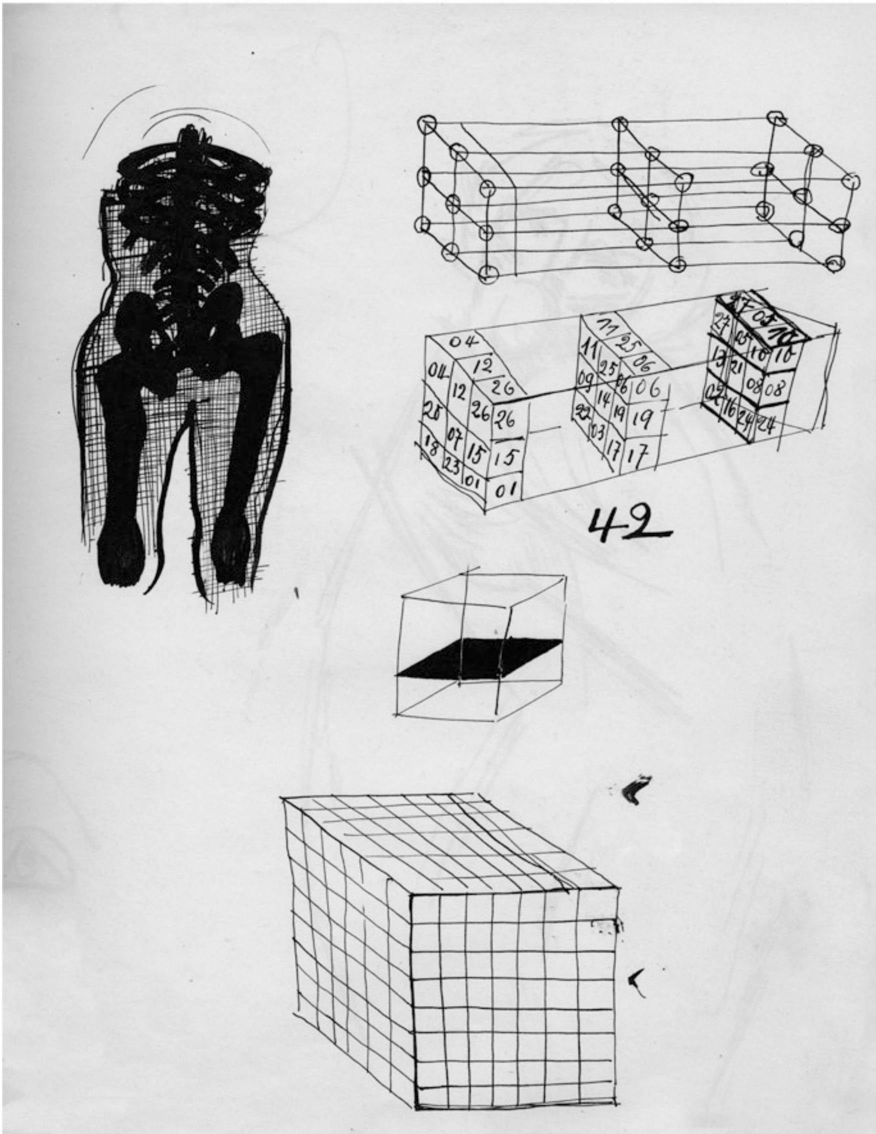


Fig. 3 Johannes Itten, *"Tempelherrenhaus" Tagebuch*, 1920, p. 131. Ink on paper. Kunstmuseum, Bern, Johannes Itten-Stiftung, Bern, Gift of Anneliese Itten, Zurich

his idealistic goal of communicating meaning to a viewer through form and color [29].

Kandinsky's attitude toward the "fourth dimension" changed during the 1920s as well. When the young Hungarian artist László Moholy-Nagy replaced Itten in 1923, the new world of space-time arrived at the Bauhaus, with Moholy championing

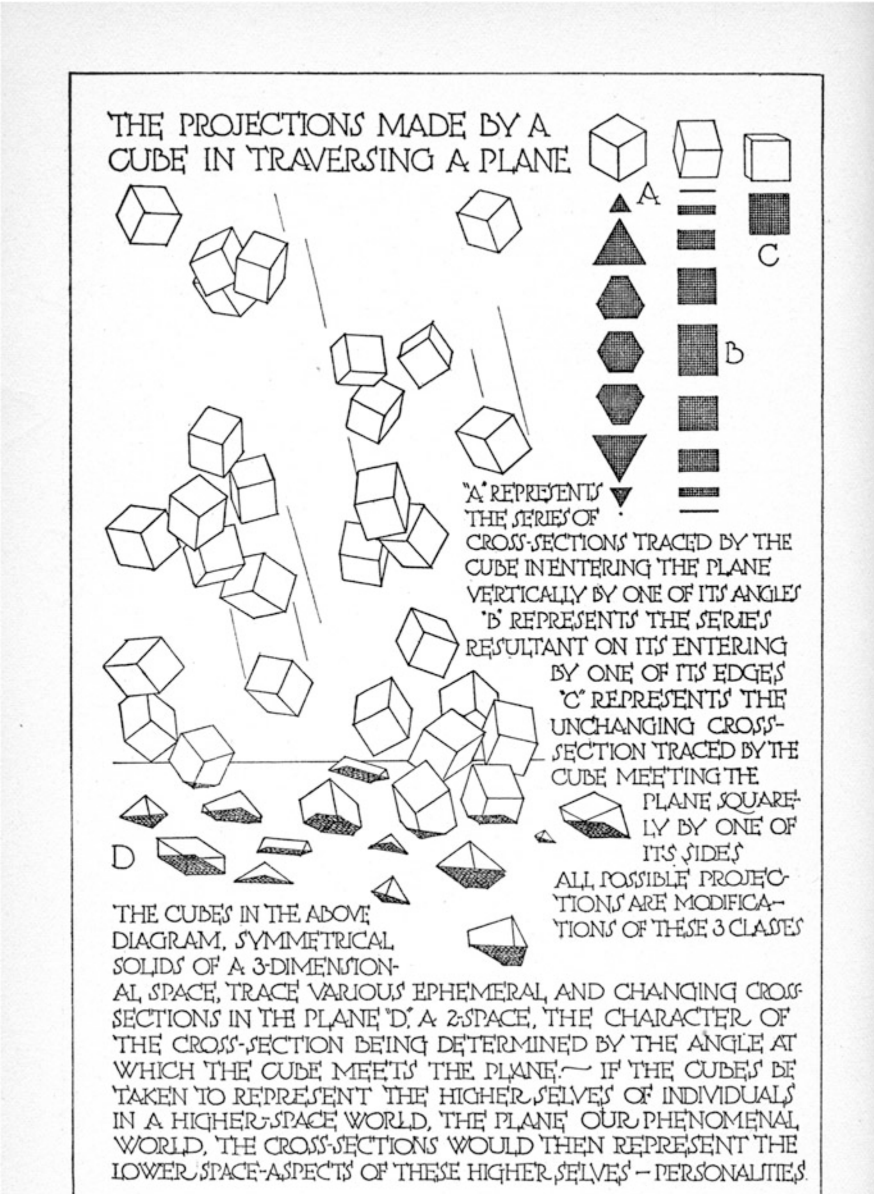


PLATE 30

Fig. 4 Bragdon, *A Primer of Higher Space*, pl. 30

Fig. 5 Kazimir Malevich's Multicolored Suprematist Works at the "0.10" Exhibition, St. Petersburg, December 1915

kinetic art as the appropriate expression of a new temporal fourth dimension [30]. This is the context in which we can better understand Kandinsky's positive 1930 reference to the fourth dimension in a letter to art historian Will Grohmann. There he wrote, "The circle . . . is the synthesis of the greatest oppositions. It combines the concentric and the excentric in a single form, and in equilibrium. Of the three primary forms [triangle, square, circle], it points most clearly to the fourth dimension." [31] It may well have been Bragdon's discussion of the concept's philosophical dimensions—so sympathetic to Kandinsky's own views—that helped transform his thinking on the subject in the face of Relativity Theory and space-time. His 1930 comment to Grohmann suggests that Kandinsky had come to see the earlier spatial fourth dimension as the sign of transcendent meaning it represented for so many artists, including Malevich and the remarkable Swedish painter Hilma af Klint (Fig. 6).

As noted earlier, in the 1970s, I suggested a connection to Bragdon's illustrations, such as Fig. 4, for Malevich, the Russian creator of "Suprematism," first shown in December 1915 (Fig. 5) [32]. Malevich use of subtitles such as "Color Masses in the Second Dimension" and "Color Masses in the Fourth Dimension" for his paintings documents the centrality of the spatial fourth dimension to his style. Russian mystical philosopher P. D. Ouspensky, author of *The Fourth Dimension* (1909) and *Tertium Organum* (1911) and advocate of four-dimensional "cosmic

Fig. 6 Hilma af Klint, *The Swan, No.* 9, Group IX, The SUW/UW Series, Oil on canvas. 1915. By courtesy of the Hilma af Klint Foundation, Stockholm

consciousness," was a crucial source for the Russian avant-garde, including Malevich. Ouspensky extensively summarized Hinton's writings, which had initially set forth the analogy of three-dimensional objects passing through a plane as a means to reason about higher dimensions [33]. Malevich could certainly have derived the idea of geometric forms that appear to leave traces in a plane or float in multicolored spatial configurations simply from his reading of Ouspensky/Hinton. However, Ouspensky later recorded that Bragdon's *Man the Square* had reached him in St. Petersburg several years earlier, and it is possible that members of the avant-garde also saw Bragdon's powerfully geometric plates [34]. Ouspensky had also been associated with Theosophy, and, once again, that network may have played a role in getting the book to St. Petersburg. Bragdon would later assist in the English translation and publication of *Tertium Organum*—first by his Manas Press in 1920 and then by Alfred A. Knopf in 1922. By the 1960s, Ouspensky's book would play a role similar to that of the *Primer* in reigniting artistic interest in the spatial fourth dimension, as discussed below.

It is appropriate to consider Hilma af Klint (Fig. 6) in proximity to both Kandinsky and Malevich, since, beginning with her first major retrospective in 2013, she has been discussed widely as a pioneer of abstract painting missing from the history of art. Critics have been mystified as to how af Klint could have developed her abstract style, working largely in isolation in Stockholm and never having traveled to Paris or encountered avant-garde art theory. Yet, af Klint actually serves as an effective demonstration of the way in which circulating journals and books spread new ideas from science and occultism, such as the fourth dimension, X-rays, spiritualism and Theosophy (she was involved in both), and the ideas of the

Swedish mystic Emanuel Swedenborg [35]. The latter had been a key inspiration for the French Symbolists' and avant-garde painters' interest in "correspondences" between a higher, true reality and its signs or shadows in our world. These were ideas being discussed in various international capitals, but af Klint did not need to travel to find them. As noted earlier, in the textual introduction to Bragdon's *Primer*, in addition to the fourth dimension, she would have encountered idealist philosophy, Swedenborg, and the idea of a "movable threshold of sensibility" of highly influential German writer Carl du Prel, whom both af Klint and Kandinsky read [36]. Af Klint also shared with Kandinsky a deep interest in the ideas of Steiner. (Af Klint, Bragdon, Kandinsky, and Steiner were of the same generation—all born in the 1860s.) Af Klint attended Steiner's Stockholm lectures in April 1912, and he may have introduced her to Bragdon's publications in that context, in addition to Bragdon's having sent copies to international Theosophical Society offices. In his lectures beginning in 1905, Steiner had regularly invoked Hinton and his method for acquiring knowledge of four-dimensional space via the study of blocks of multicolored cubes as a model for Theosophists working to gain knowledge of the astral world. [37]

The af Klint paintings that have been celebrated for their early abstract language are from her series of "Paintings for the Temple," which commenced in 1906, when she believed she had received a commission from one of the "High Masters" of her spiritualist circle [38]. "The Ten Largest" of these paintings (1907) are wall-sized paintings on paper that strongly reflect the Swedish decorative arts and motifs derived from nature [39]. When af Klint resumed work on the Temple paintings in 1912, after a four-year hiatus, her style began to evolve toward the geometric form language she would adopt wholeheartedly in 1915 (Fig. 6). A key stimulus for her new language was undoubtedly Bragdon's *Primer*, which offered a model of beautifully drawn plates using isometrically projected cubes and bold visual contrasts between black and white [40]. Af Klint was deeply committed to overcoming the duality between matter/spirit and male/female, and the concept of a spatial fourth dimension itself—with its association with the astral plane where dualities could be transcended—would have had a particular appeal. Thus, in the course of her series *The Swan*, the paintings progress from the juxtaposition of a female swan above (symbolized by blue) and a male swan below (symbolized by yellow) to her restatement of the theme in her tumbling isometric, color-coded cubes that unite on a circular plane suggestive of ether vibrations, another of her, Bragdon's, and Kandinsky's interests [41].

With the popularization of Einstein and space-time, starting in 1920, Steiner began to receive questions about this change in his lectures discussing the fourth dimension [42]. The same was true for Bragdon, who, in publications of the 1920s, had to acknowledge Einstein and slightly adjust his discussions of the spatial fourth dimension, to which he remained true nonetheless [43]. Artists who had been deeply engaged with the spatial fourth dimension, such as Duchamp, felt this challenge especially keenly. Four-dimensional space had been a central theme in his project *The Bride Stripped Bare by her Bachelors, Even* (1915–23), for which he had made hundreds of notes beginning in 1912. It is little wonder that, while he published a major group of notes in his *Green Box* of 1934, he did not release his notes relating to the fourth dimension until 1967, when he could observe the topic coming back into culture [44].

At the same time in the 1960s that Duchamp sensed a renewal of interest in the spatial fourth dimension, driven in part by Martin Gardner's "Mathematical Games" columns in *Scientific American*, space-time rhetoric in the art world reached a high point in Robert Delevoy's book *Dimensions of the Twentieth Century* of 1965. Delevoy presents a history of the century's art in which any remnant of a spatial fourth dimension is eradicated, and all is transformed into the "space-time dimension" or the "movement dimension." [45] Delevoy was echoing Moholy-Nagy's highly influential book *Vision in Motion* (1947), which also focused on space-time, but also argued for the societal need to incorporate the new integrated relationship of space *and* time into one's thinking as well as into art [46]. Given the dominance of this language, for artists in this period to discover the earlier, largely forgotten spatial fourth dimension was like encountering some kind of ancient wisdom.

That happened for sculptor Peter Forakis in 1957 when he found in a book sale at the California School of Fine Arts copies of Bragdon's *The Frozen Fountain* (1932) and Ouspensky's *Tertium Organum* [47]. If Ouspensky's *Tertium Organum* offered a method for pursuing higher consciousness, it was Bragdon's visual language in the *Frozen Fountain*, echoing the *Primer of Higher Space* (Fig. 7), that was critical for Forakis's subsequent development as a sculptor. In *Hyper-Cube* of 1967 (Fig. 8),

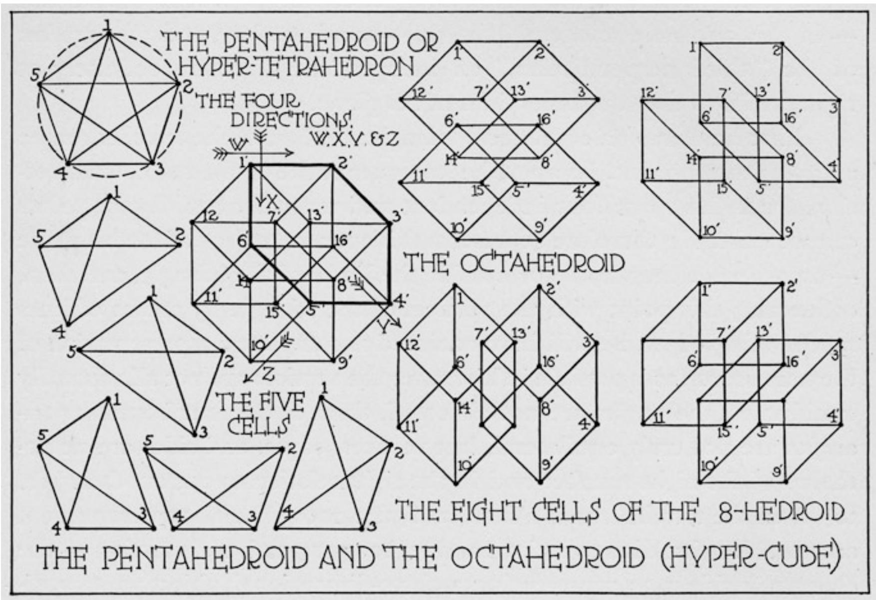


Fig. 7 Claude Bragdon, *The Frozen Fountain* (New York: Alfred A. Knopf, 1932), Fig. 66

Fig. 8 Peter Forakis, *Hyper-Cube*, 1967. Aluminum, 35–7/8 × 36–7/16 × 36–1/8 in. overall. Walker Art Center Minneapolis, donation of Virginia Dwan, 1985

he responded directly to Bragdon's isometric projection of the four-dimensional hypercube in *The Frozen Fountain* [48]. Here Forakis physically modeled the three-dimensional "shadow" of the hypercube, incorporating the four interconnecting pairs of bounding cubes that make up Bragdon's figure and that he illustrates to its right. At the cooperative Park Place Gallery operated by Forakis and nine other artist friends between 1963 and 1967 in New York, the spatial fourth dimension became a central theme. The 10 artists approached the multivalent fourth dimension in different ways, ranging from Forakis's geometry and Dean Fleming's pursuit of higher, four-dimensional consciousness to Mark di Suvero's adulation of Einstein and space-time [49]. Not surprisingly, Park Place audiences were often confused by their discussions of the "4D," as they termed it, since the public was still largely unaware of the spatial fourth dimension so central to the group.

In 1962 Robert Smithson, a friend of the Park Place artists, had strongly critiqued a "Fourth Dimension" he associated Einsteinian space-time and kinetic art [50]. By 1965, however, Smithson had discovered a completely different fourth dimension—involving geometry and space—which now seemed fresh, iconoclastic, and relevant to contemporary art. Crucial sources for Smithson's discovery were Martin Gardner's *Scientific American* columns as well as his 1964 book *The Ambidextrous*

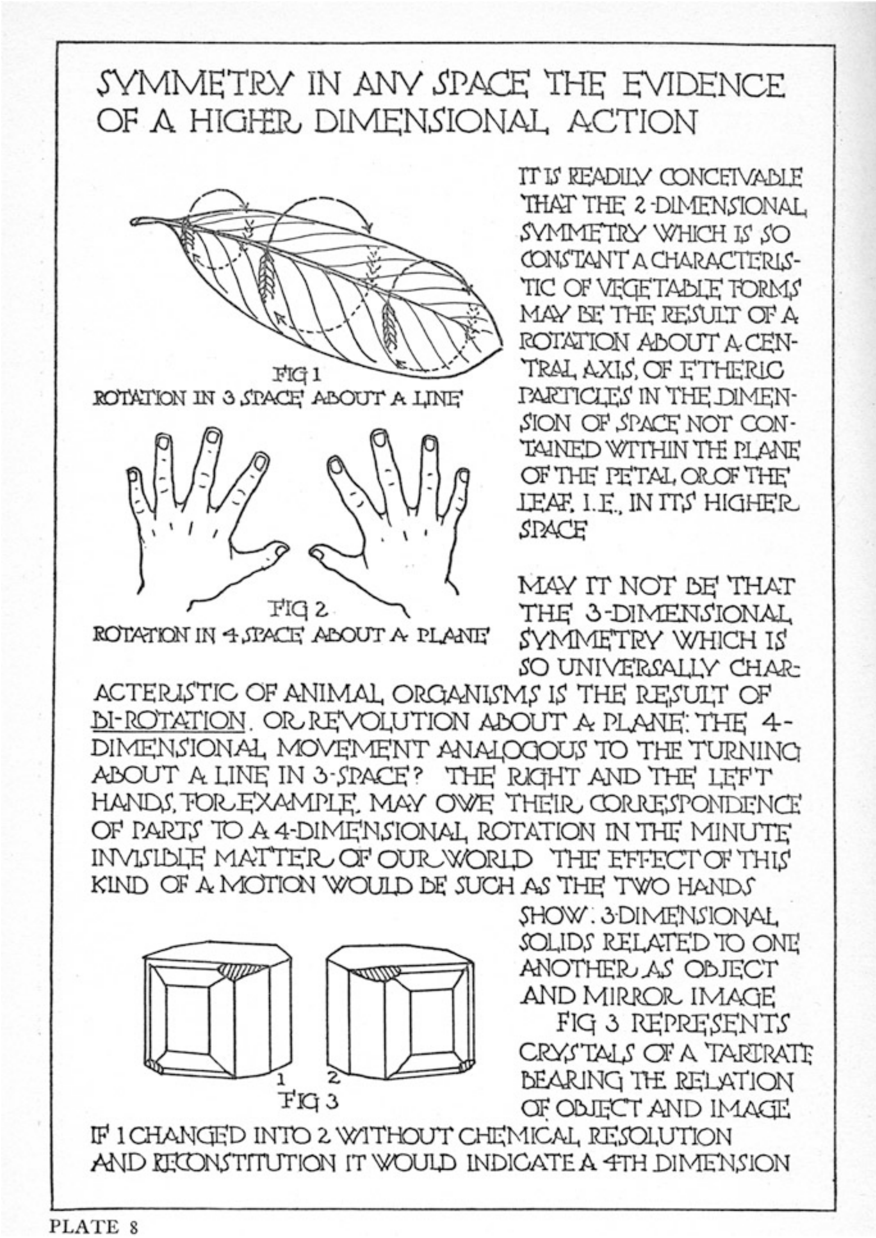
Universe, which centered on the issue of mirror symmetry or chirality. That theme, embodied in left- and right-handed spiral growth in plants and in crystals, had been discussed widely in the early twentieth century to argue for the existence of a higher dimensional space. Rotation in a fourth dimension would be necessary, for example, to turn a right hand into a left hand. Bragdon had treated mirror symmetry in the *Primer of Higher Space* (Fig. 9), and Gardner in his *Ambidextrous Universe* chapter on "The Fourth Dimension" adapted Bragdon's symmetrical crystals at the base of Fig. 9. Smithson actually cut that illustration out of his copy of Gardner's book and used it as the basic schema for his *Enantiomorphic Chambers* of 1965 (Fig. 10; now lost) [51]. Composed of two 34-inch-square mirror images, with blue steel frames holding fluorescent green panels, these structures also contain mirrors within, with which Smithson consciously subverted the normal process of vision and its association with three-dimensional space and perspective.

Smithson also employed complex mirror reflections in his various *Mirror Vortex* sculptures in 1965 [52]. And in his subsequent works juxtaposing mirrors and rocks in the later 1960s, such as his 1969 *Chalk-Mirror Displacement* (Art Institute of Chicago), the artist provided further cogent demonstrations of the theme of right- and left-handed symmetry and its dimensional overtones. Finally, Smithson's devotion to the spiral at various scales is embodied in his 1970 earthwork *Spiral Jetty* and his related film that traverses the scale of spirals from salt crystal growth to spiral nebulae. The notations in his notebook "The Metamorphosis of the Spiral," discovered after his death, make clear that for Smithson the spiral carried strong higher dimensional associations, which Bragdon had also emphasized in the *Primer of Higher Space* [53].

Bragdon's most enthusiastic and discerning advocate at present is the artist and designer Tauba Auerbach, whose commitment to making his ideas and images better known resulted in the Diagonal Press editions of the *Primer* and his 1915 *Projective Ornament* in 2016. Auerbach had responded forcefully to Bragdon's books upon discovering them in 2011:

> I was preoccupied with the notion of higher spatial dimensions and had read, watched, listened to, ingested, and generally exposed myself to whatever I thought might bring me into closer contact with the dimensions(s) running perpendicular to these three. Other texts, drawings, and videos on the matter had long ago won my mind, but Bragdon's book won my heart. He writes about geometry in terms of humanity and spirit, and we share in speculating about a connection between consciousness and higher space [54].

In the last decade, Auerbach has developed an art practice devoted to exploring dimensionality and "incorporating another dimension into my spatial sense." [55] Like Bragdon in his advocacy of four-dimensional "projective ornament," the artist is deeply interested in ornament, which "evoke[s] essential natural structures—the wave, the vortex, the helix; and gestures—oscillation and spin; and . . . exist[s] at every scale, even deeply inside of us." [56] Auerbach has extended Bragdon's study of higher dimensional ornament in a book titled *A Partial Taxonomy of Periodic Linear Ornament—Both Established and Original—Arranged by Shape, Symmetry, Dimension, Projection and Interation* [57]. Figure 11, a "square wave extruded twice, once into the third dimension and again into the fourth," is grounded in



SYMMETRY IN ANY SPACE THE EVIDENCE OF A HIGHER DIMENSIONAL ACTION

IT IS READILY CONCEIVABLE THAT THE 2-DIMENSIONAL SYMMETRY WHICH IS SO CONSTANT A CHARACTERISTIC OF VEGETABLE FORMS MAY BE THE RESULT OF A ROTATION ABOUT A CENTRAL AXIS, OF ETHERIC PARTICLES IN THE DIMENSION OF SPACE NOT CONTAINED WITHIN THE PLANE OF THE PETAL, OR OF THE LEAF. I.E., IN ITS' HIGHER SPACE

FIG 1
ROTATION IN 3 SPACE ABOUT A LINE

FIG 2
ROTATION IN 4 SPACE ABOUT A PLANE

MAY IT NOT BE THAT THE 3-DIMENSIONAL SYMMETRY WHICH IS SO UNIVERSALLY CHARACTERISTIC OF ANIMAL ORGANISMS IS THE RESULT OF BI-ROTATION. OR REVOLUTION ABOUT A PLANE: THE 4-DIMENSIONAL MOVEMENT ANALOGOUS TO THE TURNING ABOUT A LINE IN 3-SPACE? THE RIGHT AND THE LEFT HANDS, FOR EXAMPLE, MAY OWE THEIR CORRESPONDENCE OF PARTS TO A 4-DIMENSIONAL ROTATION IN THE MINUTE INVISIBLE MATTER OF OUR WORLD THE EFFECT OF THIS KIND OF A MOTION WOULD BE SUCH AS THE TWO HANDS SHOW: 3-DIMENSIONAL SOLIDS RELATED TO ONE ANOTHER AS OBJECT AND MIRROR IMAGE

FIG 3 REPRESENTS CRYSTALS OF A TARTRATE BEARING THE RELATION OF OBJECT AND IMAGE

1 2
FIG 3

IF 1 CHANGED INTO 2 WITHOUT CHEMICAL RESOLUTION AND RECONSTITUTION IT WOULD INDICATE A 4TH DIMENSION

PLATE 8

Fig. 9 Bragdon, *A Primer of Higher Space*, pl. 8

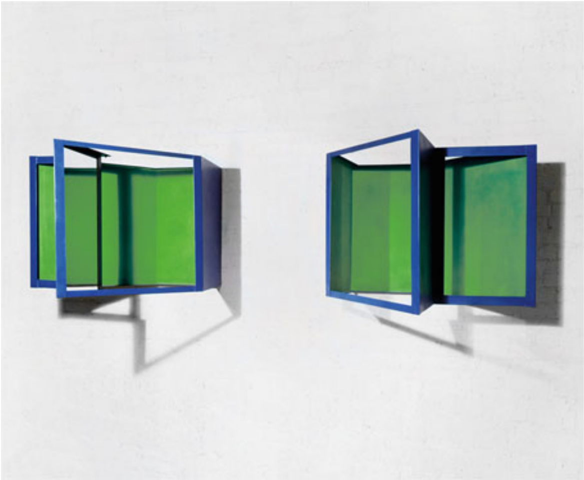


Fig. 10 Robert Smithson, *Enantiomorphic Chambers*, 1965. Painted steel and mirrors, 24 × 30 × 31 in. Now lost. © Holt/Smithson Foundation / Licensed by VAGA at Artists Rights Society (ARS), NY

Fig. 11 Tauba Auerbach, "Square wave extruded twice, once into the third dimension and again into the fourth" from back cover of Bragdon, *A Primer of Higher Space* (1916 Diagonal Press ed.). Photo: Steven Probert

Fig. 12 Tauba Auerbach, *The New Ambidextrous Universe III*, 2014. Aluminum and plywood, 98 × 46 × 1–1/2 in. San Francisco Museum of Modern Art, gift of David Schrader. © Tauba Auerbach. Photo: Paul Knight

Bragdon's isometric projection of the four-dimensional hypercube but developed here as a continuous wave. As *A Partial Taxonomy* explains, "a helix drawn in two dimensions looks like a wave," and thus "all of the ornaments in this book can be seen as flattened, elaborate helixes, if you allow for their varied pitches." [58]

Auerbach's commitment to the helix/spiral and symmetry, essential aspects of Bragdon's *Primer* (Fig. 9), led to the discovery of Martin Gardner's *The New Ambidextrous Universe* of 1990, which is now subtitled "Symmetry and Asymmetry from Mirror Reflections to Superstrings." [59] In response, Auerbach made a series of wooden sculptures titled *The New Ambidextrous Universe* in 2013–14 (Fig. 12). The sculptures were generated by programming an irregularly drawn line into a waterjet and then slicing a plywood sheet into narrow, irregularly curved strips. Reassembling the strips in reverse order produced an object that is a kind of mirror reversal of itself, as if it had been turned through a fourth dimension. Just as for Smithson in the 1960s, Auerbach 50 years later found in Gardner's book stimulation to explore a theme that had first been widely promulgated by Bragdon. In Auerbach's case, with the emergence and development of string theory and new cosmologies involving higher dimensions and symmetry, the artist's explorations parallel scientific developments in contrast to Smithson's "counter-discourse" versus the temporal fourth dimension of Einstein [60].

Claude Bragdon was the first advocate of the spatial fourth dimension to give striking geometrical form to ideas circulating about the highly popular concept during its heyday from the 1880s to ca. 1920. His *Primer of Higher Space* stood in stark contrast to a design world still filled with the organic, curvilinear form languages descended from Art Nouveau. Bragdon's volume was prescient: painting would move toward geometry in the wake of Cubism, and by the 1920s geometry and modernism were practically synonymous in both painting and architecture. In architecture, however, it was not the four-dimensional geometrical "projective ornament" Bragdon had proposed, but the stripped-down pure geometry that would come to be known as the International Style. That shift would sideline Bragdon's architecture career, although happily his work is now being reevaluated. But his *Primer*, more than he could have imagined, would assure the currency of his ideas on the fourth dimension and contribute significantly to its continued relevance nearly 100 and 10 years later.

References

1. See Max Beckmann, "On My Painting," in *Modern Artists on Art*, ed. Robert L. Herbert (Englewood Cliffs, NJ: Prentice-Hall, 1964), p. 131
2. Ibid., p. 135
3. Ibid., pp. 132, 134. In contrast to the other artists discussed here, Beckmann was a figurative painter, whose style is characterized by works such as his 1938 *Self-Portrait with Horn* (Neue Galerie, New York)
4. For the history of the fourth dimension in twentieth-century art, see Linda Dalrymple Henderson, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art* (Princeton University Press, 1983; rev. ed., Cambridge, MA: The MIT Press, 2013). The 2013 edition includes a new 100-page "Reintroduction," which focuses particularly on the period 1950–2000
5. On the delayed impact of Einstein's theories and their content, see ibid. as well as Linda Dalrymple Henderson, "Einstein and 20th-Century Art: A Romance of Many Dimensions," in *Einstein for the 21st Century*, ed. Peter L. Galison, Gerald Holton, and Silvan S. Schweber (Princeton: Princeton University Press, 2007), pp. 101–29. (All essays by the author cited in these notes, are available at Academia.edu under Linda Dalrymple Henderson)
6. Linda Dalrymple Henderson, "A New Facet of Cubism: 'The Fourth Dimension' and 'Non-Euclidean Geometry' reinterpreted," *The Art Quarterly*, vol. 34 (Winter 1971), 410–33.
7. See Linda Dalrymple Henderson, "Four-Dimensional Space or Space-Time: The Emergence of the Cubism-Relativity Myth in New York in the 1940s," in *The Visual Mind II*, ed. Michele Emmer (Cambridge: MIT Press, 2005), pp. 349–97.
8. See Henderson: *Fourth Dimension*, Appendix B for a listing of such articles
9. See Claude Bragdon, A Primer of Higher Space. *The Fourth Dimension*. (Rochester, NY: The Manas Press, 1913)
10. Ibid., p. 3. In his 21-page introduction Bragdon mentions the scientists Helmholtz and Kelvin, philosophers from Plato to Kant, popular figures such as Swedenborg and Carl du Prel (German author of *The Philosophy of Mysticism*), fourth dimension advocate Charles Howard Hinton, and Theosophist C. W. Leadbeater
11. See Henderson, *Fourth Dimension*, chap. 4 as well as Henderson, "Reintroduction," *Fourth Dimension*, (2013), pp. 42–45, for a further discussion of Fuller. For Bel Geddes, see Christina Cogdell, "The Futurama Recontextualized: Norman Bel Geddes's Eugenic 'World of Tomorrow," *American Quarterly*, 52 (June 2000), 193–245; see specifically pp. 214–25 on the Bragdon-influenced "Crystal Lassies" exhibit at the 1939 World's Fair

12. See Henderson, *Fourth Dimension*, chap. 5. For a more recent treatment of Bragdon as a source for artists, including Johannes Itten and Peter Forakis (both discussed below), see Linda Dalrymple Henderson, "Claude Bragdon, the Fourth Dimension, and Modern Art in Cultural Context," in *Claude Bragdon and the Beautiful Necessity*, ed. Eugenia Ellis and Andrea Reithmayr (Rochester, NY: Rochester Institute of Technology, 2010), pp. 73–86
13. The first 2000 copies of the *Primer* sold out and the book had to be reprinted in 1915; see Jonathan Massey, *Crystal and Arabesque: Claude Bragdon, Ornament, and Modern Architecture* (Pittsburgh: Pittsburgh University Press, 2009), p. 140.
14. The publisher Alfred Knopf took over the publication of Bragdon's Manas Press books and printed a revised edition of the *Primer* in 1923 and subsequent editions in 1929 and 1938
15. The Omen Press in Tucson, Arizona, published the facsimile reprint, making it available for the first time in decades
16. Demonstrating his knowledge of international sources, Bragdon's diagram in Plate 15 is borrowed from A. de Noircarme, *Quatrième Dimension* (Paris; Éditions Théosophiques, 1912), fig. 35. On Bragdon's joining the Theosophical Society and its dovetailing with his growing interest in the fourth dimension, see Massey, *Crystal and Arabesque*, chap. 4. Massey also discusses Bragdon's virtuosity as a designer (ibid., pp. 138–40)
17. On Hinton's books, *A New Era of Thought* (1888) and *The Fourth Dimension* (1904), and the role of Leadbeater and Theosophy in popularizing the fourth dimension, see Henderson, *Fourth Dimension*, chap. 1
18. According to Bragdon's records ("Publishing Ventures") in the Bragdon Family Papers, The University of Rochester, he sent one of the first copies of his 1912 *Man the Square* to Steiner and one of the first copies of *A Primer of Higher Space* to Leadbeater
19. See Henderson, *Fourth Dimension*, chap.1; see also Linda Dalrymple Henderson, "Mysticism, Romanticism, and the Fourth Dimension," in *The Spiritual in Art: Abstract Painting 1890-1985* (Los Angeles: Los Angeles County Museum of Art, 1986), pp. 219–37.
20. See, e.g., E. E. Slosson, "That Elusive Fourth Dimension" *The Independent and Weekly Review*, 100 (Dec. 27, 1919), pp. 274–75, 296–97. Following Hinton's model, both Bragdon and the Russian author P. D. Ouspensky discussed time as a means of conceptualizing higher dimensional space, but never treated time itself as the fourth dimension
21. Since Hinton's first book in 1888, the fourth dimension had often been associated with the impalpable "ether of space," a concept still widely accepted by the general public through the 1910s. The ether also surfaces in the *Primer* (Plate 8, fig. 9 herein) and elsewhere in Bragdon's writings. On this subject, see Henderson, "Bragdon, The Fourth Dimension, and Modern Art," pp. 75–76, 82–84
22. These pages from Itten's *Tempelherrenhaus Tagebuch* are reproduced in Christoph Wagner, ed., *Das Bauhaus und die Esoterik: Johannes Itten, Wassily Kandinsky, and Paul Klee* (Bielefeld/Leipzig: Kerbert Verlag, 2005), pp. 12–13. For a fuller discussion of Itten and Bragdon and the possible role of Steiner as the source of Itten's copy of the book, see Henderson, "Bragdon, The Fourth Dimension, and Modern Art," pp. 80–81
23. See Claude Bragdon, *Man the Square: A Higher Space Parable* (Rochester, NY: Manas Press, 1912), p. 11; reprinted in *A Primer of Higher Space*, p. 61. Itten copied the block of cubes at the bottom of the page (fig. 3) from the last page of *Man the* Square; the drawings at upper right derive from Bragdon's *Primer* plates of "magic cubes" and "magic tesseracts" (*Primer*, Plates 25, 27). A simplified version of *Primer* Plate 30 (fig. 4 herein) with this title but no text or diagram at upper right appeared in the 1912 edition of *Man the Square*
24. For the artistic response to Einstein in Berlin and beyond, see Henderson, "Einstein and 20th-Century Art"
25. Kandinsky's close friend Paul Klee, so interested in the form language of art, may well also have seen the book; this is a subject for further research
26. For Bragdon sending books to Steiner, see again n. 18 above
27. On Cubism and the fourth dimension, see Henderson, *Fourth Dimension*, chap. 2
28. Rose-Carol Washton Long established the centrality of Steiner for Kandinsky and his use of hidden imagery; see, e.g. Long, *Kandinsky: The Development of an Ab-*

stract Style (Oxford: Clarendon, 1980). For Kandinsky's rejection of the fourth dimension and his focus on ether vibrations, see Linda Dalrymple Henderson, "Abstraction, the Ether, and the ether vibrations Fourth Dimension: Kandinsky, Mondrian, and Malevich in Context," in *Kandinsky, Malewitsch, Mondrian: Der Weisse Abgrund Unendlichkeit/The Infinite White Abyss*, ed. Marian Ackermann and Isabelle Malz (Düsseldorf: Kunstsammlung Nordrhein-Westfalen, 2014), pp. 37–45 (German), pp. 233–38 (English). This essay is now readily available in the British online journal *Interalia*, no. 61 (Nov. 2020). https://www.interaliamag.org/articles/linda-dalrymple-henderson-abstraction-the-ether-and-the-fourth-dimension-kandinsky-mondrian-and-malevich-in-context/

29. The contrast in style is readily apparent in the juxtaposition of Kandinsky's *Compositions VI* (Hermitage Museum, St. Petersburg) and *Composition VIII* of 1923 (Solomon R. Guggenheim Museum, New York)
30. See, e.g., Moholy's *Light-Space Modulator* (1923–30), as discussed in Henderson, "Einstein and 20th-Century Art," pp. 111–12
31. Kandinsky letter to Will Grohmann, October 12, 1930, quoted in Angelica Zander Rudenstine, *The Guggenheim Museum: Paintings 1880–1945*, vol. 1 (New York: Solomon R. Guggenheim Foundation, 1976), p. 310
32. This interpretation first appeared in my 1975 dissertation, in two articles of 1975/76 (*The Structurist*) and 1978 (*Soviet Union*), and then in my 1983 *Fourth Dimension* book. For a more recent reading of Malevich, see also Henderson, "Abstraction, the Ether, and the Fourth Dimension: Kandinsky, Mondrian, and Malevich in Context"
33. See P. D. Ouspensky, *Tertium Organum: The Third Canon of Thought, a Key to the Enigmas of the World*, trans. from 2nd Russian ed. (1916) by Claude Bragdon and Nicholas Bessaraboff (2nd American ed., rev., New York: Alfred A. Knopf, 1922). Ouspensky also published Russian translations of Hinton's books in 1915
34. See Ouspensky, "Author's Preface," in *Tertium Organum*, p. xv
35. This section is drawn from Linda Dalrymple Henderson, "Hilma af Klint and the Invisible in Her Occult and Scientific Context," in *Visionary: On Hilma af Klint and the Spirit of Her Time*, ed. Kurt Almqvist and Louise Belfrage (Stockholm: Axel and Margaret Ax:son Johnson Foundation, 2019), pp. 71-91, 118-20. For an excellent introduction to af Klint's art, see, e.g., *Hilma af Klint: Paintings for the Future*, ed. Tracey Bashkoff (New York: Solomon R. Guggenheim Museum, 2018)
36. For Bragdon's reference to a "moving threshold of consciousness," see Bragdon, *Primer*, pp. 14–15. Af Klint owned two books by du Prel, and he was a critical resource for her; see Henderson, "Hilma af Klint and the Invisible," for this discussion (pp. 71–72) and for Kandinsky's knowledge of du Prel (ibid., n. 6)
37. For Steiner's lectures, see the compendium of his lecture notes published as Rudolf Steiner, *The Fourth Dimension: Sacred Geometry, Alchemy, and Mathematics*, trans. Catherine E. Creeger (Great Barrington, MA: Anthroposophic Press, 2001), p. 19
38. For af Klint's spiritualist activities, see David Max Horowitz, "'The World Keeps You in Fetters; Cast Them Aside': Hilma af Klint, Spiritualism, and Agency," in *Hilma af Klint: Paintings for the Future*, ed. Bashkoff, pp. 128–33
39. For these paintings, see *Hilma af Klint*, ed. Bashkoff, pp. 105–15
40. See Henderson, "Hilma af Klint and the Invisible," pp. 82–83, for what seems to be af Klint's first response to Bragdon with its use of strongly contrasting black and white planes. Since Bragdon had sent Steiner a copy of *Man the Square* in January 1912, he could well have shared it when he was in Stockholm in April 1912
41. For more of the "Swan" series paintings, see *Hilma af Klint*, ed. Bashkoff, pp. 140–53; and Henderson, "Hilma af Klint and the Invisible," pp. 84, 87–89
42. See, e.g., the series of lectures in Stuttgart in March 1920 (Steiner, *Fourth Dimension*, pp. 98 and after)

43. On Bragdon's 1923 adjustments as he worked to navigate the new "space-time" world of Einstein, including removing references to the ether of space that was so important for him and for others, see Henderson, "Claude Bragdon, The Fourth Dimension, and Modern Art," pp. 82–84
44. For a brief introduction to Duchamp's *Large Glass* project/notes and his navigation of the changes after 1919, see Linda Dalrymple Henderson, "The Image and Imagination of the Fourth Dimension in Twentieth-Century Art and Culture, *Configurations*, 17 (Winter 2009), pp. 131–60. This essay also addresses Cubism, Malevich, and contemporary artist Tony Robbin, who has been the most prominent advocate of the spatial fourth dimension since the 1970s. Robbin recalls *Village Voice* jazz critic Nat Hentoff showing him a copy of Bragdon's *Primer of Higher Space* in Provincetown, RI, in summer 1963 (Robbin e-mail to the author, Jan. 12, 2021); however, computer graphics were the crucial stimulus for him, not Bragdon's *Primer.* For Robbin's art and writings, see http://tonyrobbin.net
45. Robert L. Delevoy, *Dimensions of the 20th Century* (Geneva: Skira, 1965). On the reappearance of the spatial fourth dimension in culture in the later 1950s–1960s, including Gardner's important role, see Henderson. "Reintroduction," to *Fourth Dimension* (2013), pp. 46–57
46. See L. Moholy-Nagy, *Vision in Motion* (Chicago: Paul Theobald, 1947); see also Henderson, "Four-Dimensional Space or Space-Time" on Moholy's theories
47. On this event and Forakis and Park Place, see Henderson, "Reintroduction," to *Fourth Dimension*, pp. 61–65; and Linda Dalrymple Henderson, *Reimagining Space: The Park Place Gallery Group in 1960s New York* (Austin, TX: Blanton Museum of Art, 2008), pp. 22–24. A pdf of this catalog is also available at Academia.edu
48. Claude Bragdon, *The Frozen Fountain: Being an Essay on Architecture and the Art of Design in Space* (New York: Alfred Knopf, 1932), chap. 8 and fig. 66. See also Henderson, "Claude Bragdon, the Fourth Dimension, and Modern Art," pp. 84–86, where the Bragdon diagram is illustrated
49. See Henderson, "Reintroduction" to *Fourth Dimension*, pp. 57–65; and Henderson, *Reimagining Space*, pp. 8–11 and individual artist essays
50. See Henderson, "Reintroduction" to *Fourth Dimension*, pp. 58–61
51. For the crystal illustration, see Martin Gardner, *The Ambidextrous Universe* (New York: Basic Books, 1964), p. 107
52. See Henderson, "Reintroduction" to *Fourth Dimension*, pp. 59–61 and fig. R.20, for his *Four-Sided Vortex* of 1965; for this work and other mirror sculptures, see also https://holtsmithsonfoundation.org
53. In addition to his interest in right- and left-handed spiral growth in nature, Bragdon centered his Plate 16 on an evolutionary spiral interacting with a plane representing the current moment. For a further discussion of Smithson and the spiral, including Smithson's notebook, see Linda Dalrymple Henderson, "Space, Time, and Space-Time: Changing Identities of the Fourth Dimension in Twentieth-Century Art," in *Measure of Time*, ed. Lucinda Barnes (Berkeley, CA: Berkeley Museum of Art and Pacific Film Archive, 2007), 97–99 and n. 28
54. Tauba Auerbach, "Note from the Re-Publisher," in Claude Bragdon, *A Primer of Higher Space (The Fourth Dimension), to which is added Man the Square, a higher space parable*, 2nd rev. ed., 1923 (New York: Diagonal Press, 2019), n.p. This section on auerbach is drawn from my essay in the book accompanying the 2021 retrospective exhibition at the San Francisco Museum Modern Art. See Linda Dalrymple Henderson, "The Fourth Dimension as 'The Beyond Which is Within,'" in *S v Z: Tauba Auerbach* (San Francisco: San Francisco Museum of Modern Art, 2020), pp. 124–128; 128–124 (10-page essay spanning the catalog's middle and its symmetrical rising and falling pagination)
55. Auerbach e-mail to the author, Feb. 17, 2019
56. Auerbach, "Note from the Publisher," in Bragdon *Projective Ornament*. (New York: Diagonal Press, 2016), n.p.
57. See Tauba Auerbach, *A Partial Taxonomy of Periodic Linear Ornament—Both Established and Original—Arranged by Shape, Symmetry, Dimension, Projection and Interation* (New York: Diagonal Press, 2017)

58. See the section titled “Helix,” in Auerbach, *A Partial Taxonomy*, n.p
59. Gardner, M.: *The New Ambidextrous Universe: Symmetry and Asymmetry from Mirror Reflections to Superstrings*, 3rd rev. ed, (New York: W. H. Freeman, 1990)
60. On Auerbach’s deep interest in contemporary science, including string theory and the “Twistor” theory of Roger Penrose, see Henderson, “The Fourth Dimension as ‘The Beyond which is Within,’” in *S v Z*: *Tauba Auerbach*, as well as the other essays in the book. See also Auerbach, *Partial Taxonomy*

"Where Natural Law Holds No Sway": Geometrical Optics and Divine Light in Dante, Michelangelo, and Raphael

Martin Kemp

The revolution of optics in Renaissance art, which set geometrical and physical reality as the goal for the learned artist, brought in its train a severe difficulty. If all the visual resources of pigmentary tone and colour were wholly devoted to the portrayal of naturalistic light and space, what was left for the evocation of divine light [1].[1] How, for instance, was the *Transfiguration of Christ* to be rendered?

> The face of Christ did shine as the sun, and his raiment was white as the light. And, behold, there appeared unto them Moses and Elias talking with him. Then answered Peter, and said unto Jesus, Lord, it is good for us to be here ... While he yet spake, behold, a bright cloud overshadowed them: and behold a voice out of the cloud, which said, this is my beloved Son, in whom I am well pleased; hear ye him.[2] ([2], Matthew 17:2–5).

In Early Christian, Byzantine, and mediaeval art, the passage of light could be described in literal terms as visible rays, and actual gold was used to designate the ineffable radiance of heavenly realms and divine apparitions. The Sanai mosaic is a splendid early example (Fig. 1).

During the course of the fifteenth century, above all in Renaissance Florence, the recourse to actual gold was progressively excluded from the tool-kit of painters who aspired to create a new kind of naturalism founded upon the geometry of light. Alberti's pioneering little book *On Painting* set the tone.

[1] This is the central question in [1].

[2] Mark, 9:3 adds, "And his raiment became shining, exceeding white as snow; so as no fuller on earth can white them" [2].

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Fig. 1 *The Transfiguration of Christ*, mosaic, sixth century, Church of The Monastery of Saint Catherine, Sinai © St. Catherine's Monastery at Mt Sinai, by kind permission

> There are some who make excessive use of gold, because they think it lends a certain majesty to painting. I would not praise them at all. Even if I wanted to paint Virgil's Dido with her quiver of gold, her hair tied up in gold, her gown fastened with a golden clasp, driving her chariot with golden reins, and everything else resplendent with gold, I would try to represent with colour rather than with gold this wealth of rays that strike the eyes of spectators from all angles [3].

A writer has no such problem. A poet can write effectively of the gleam of gold, of the passage of a light ray, or of the kind of blinding effluence that typically accompanies miracles, and we can envisage them in our mind's eye. Virgil and Dante are not limited by the painter's material pigments.

1 Dante and the Limits of Optics

The greatest master of divine light in poetry was Dante. His canticle *Paradiso*, the third great book of his *Divina Commedia* from the early fourteenth century, contains dazzling accounts of divine radiance that lies beyond the normal range of our sense of sight. Yet, Dante expressed a keen adherence to optical science, *perspectiva*. His understanding of optics was superior to Alberti's and only surpassed in the arts by Leonardo's accounts of how light could be seen and should be rendered.

The clearest testimony of Dante's engagement with optical science occurs in the commentaries he provided on the poems in his *Convivio* (Banquet), written shortly after 1300. He was well aware of the highly developed science of light in mediaeval philosophy, which was founded upon ancient authors, most notably Euclid and Ptolemy. He also drew extensively upon sophisticated Islamic optics, exemplified by Ibn al-Haytham (Alhazen) and transmitted by theologians such as Roger Bacon and John Pecham [4, 5]. Dante was by no means aspiring to be an authority on *perspectiva* as such, but he was sufficiently well informed to exploit the prestigious science to full effect in the service of his majestic narrative.

Of the sciences of observable nature, geometrical optics was the most perfect in Dante’s eyes, while at the same time testifying to its own finite limits.

> As Euclid says, the point is the beginning of this [geometry], and, as he says, the circle is its most perfect figure, which must therefore be adopted as its end. As between its beginning and end, geometry moves between the point and the circle and these two are inimical to certainty; for the point in itself cannot be measured because of its indivisibility, and it is impossible to square the circle perfectly because of its curvature, and so it cannot be measured precisely. Geometry is furthermore the most inviolate [most white] in that it is without the stain of error and is most certain both in itself and in its handmaid, which is called *perspettiva* ([6], *Convivio*, II, xii, 26–27).

Even something as perfect and rational as Euclidian geometry has ineffable unknowns at its heart.

In the *Convivio* he was much concerned with the geometrical working of the eye, describing its internal structure in a way that simplifies the complex accounts in the standard literature.

> The passage that the visible ‘form’ makes through this medium [air] is completed in the water within the pupil of the eye, because that water has a boundary—almost like a mirror, which is glass backed by lead—so that it cannot pass any further but is arrested there like a ball that is stopped when it strikes something, so that the ‘form’, which cannot be seen in the transparent medium, now appears lucid and defined. This is why an image is seen in leaded glass, and not in any other glass. The visual spirit, which passes from the pupil to the front of the brain where is the sensitive power, and where it is represented without passage of time, and thus we see it ([6], *Convivio,* III, ix, 7–9).

The “form” that comprises the image passes through the air in an immaterial manner.

Lacking any sense of the lens (crystalline humour) as a focusing device, the philosophers needed to explain how the eye could sort out the diverse bombardment of the eye by a multitude of rays at varied angles. They generally adopted the solution proposed by al-Haytham:

> Only one among all the points confronting the eye, which have simultaneously arrived at that point on the surface of the eye, will have travelled along the perpendicular to that point on the eye’s surface, while the forms of all the other points reach that point along the eye’s surface along inclined lines ... Only the form of a single point among all of them will rectilinearly pass through the transparency of the eye’s coats, namely the point at the extremity of the perpendicular drawn to that point on the eye’s surface ([7], *De Aspectibus* I, 6, 20).

With characteristic ingenuity, Dante identifies the key perpendicular ray (Alberti’s “Prince of rays”) with the direct course of the arrow that Cupid fires into the lovers’ eyes and thence into their heart.

It should be known that although many things can enter the eye at the same time, it is true that whatever enters along a straight line into the centre of the pupil is truly seen and is the only one that imprints itself upon the imagination. This is because the nerve along which the visual spirit runs is orientated to this part. And therefore one eye cannot really look into another eye unless it is seen by it, just as the one who aims their sight receives the form in the pupil along a straight line, so along that same line its own form departs into the one it aims at; and many times along the extension of this line is discharged the bow of him [Cupid] against whom all weapons are feeble ([6], *Convivio,* II, ix, 4–5).

Alongside such optical felicities, Dante's *Convivio* also introduces us to what becomes a dominant visual theme in *Paradiso*, namely the overwhelming of our eyes by light of such brilliance that it becomes unseeable. The ineffable visual delights of his beloved lady, Beatrice, act in a similar manner.

In her countenance appear such things
As exhibit a part of the joy of Paradise.
I mean in her eyes and in her sweet smile,
For here Love bears them as if to his lair.
They overwhelm our intellect,
As a ray of sunlight does weak vision ([6], *Convivio* III, canzone 2).

In his commentary, he explains that

Here we must understand that in a certain way these things dazzle our intellect, in as much as certain things are affirmed to exist which our intellect cannot observe, that is to say God, eternity, and primal matter, things which most certainly are known to exist and are with full faith believed to exist. But given the nature of their essence we cannot understand them ([6], *Convivio,* III, xv, 8).

The mediaeval texts all stress that certain visual phenomena lie outside the reach of our "weak vision" This is true above all of extreme brightness, which causes pain in the eye, an observation that provided one of the main arguments for the theory that the eye operates though intromission of light rays into the eye rather than the extramission of "seeing rays" from the eye.

Repeatedly in *Paradiso*, as the poet journeys miraculously with Beatrice through the heavenly spheres towards the ultimate bliss, his sight is repeatedly dazzled and fails over greater or lesser periods of time. This is foreshadowed more than once in *Purgatorio*:

... I felt weighing on my forehead
A splendour much greater than before
...
So that I raised my hand to the top
Of my eyes and made myself a shield from the sun's rays,
To scrape away the visual superfluity.
As when from water, or from a mirror,
A ray of light jumps to the opposite part.
Rising in a manner equivalent
To that of its descent, just as the line
Of a falling stone deviates at equal angles;
As is demonstrated by experience and mathematics;
So it seemed that the light which struck me
Was reflected from in front of me;

And therefore my sight was quick fly from it ([6], *Purgatorio,* XV, 10–24).

The terrestrial law of reflection is cited but has no explanatory value outside the earthly context, and he cannot ward off the blinding rays by shading his eyes.

Near the beginning of *Paradiso*, he draws picturesquely upon the legend of the eagle in the mediaeval bestiary. Alone of creatures, the eagle could stare unwaveringly at the radiant sun.

Beatrice had to her left flank
Turned round to look at the sun:
An eagle never looked at it more steadily.
And as the second ray always issues
And rebounds from where their first ray struck,
just like a pilgrim who wishes to return,
So from her action, infused into my eyes
By my imagination, my action was enabled,
And I fixed my eyes on the sun
other than as we can do. ([6], *Paradiso,* I, 46–54)

While the celestial voyagers are in the heaven of the moon, discussing the moon’s mottled appearance, Beatrice insists that Dante’s learned optics simply does not apply in heaven. She dismisses the value of earth-bound “experiment [*esperienza*] ... which is the only source for the streams of your learned disciplines [*arti*] ([6], *Paradiso,* II, 94–6).[3]

In Canto XXX, he presents an ecstatic account of celestial light in the vision of the sempiternal rose, the vast luminous flower of petal thrones that accommodate the spirits of the blessed. It is literally like nothing on earth, but Dante has been miraculously granted the privilege to surpass his normal sensory limits.

O splendour of God, through which I saw
The exalted triumph of the true kingdom,
Grant me the power to say how I saw it!
The light up there renders visible
The creator of that creature
Who only finds peace in seeing him.
...
So, rising above the light and all around it,
I saw it mirrored in more than a thousand tiers
the numbers of us who had returned on high.
And if the lowest tier encloses
So great a light, what is the full size
of this rose extending to its furthest petals?
My sight in breadth and height
Was not confounded, but took in
The full extent and nature of the jubilation.
Nearness and distance add nothing, take away nothing,
because where God governs without an intermediary
natural law holds no sway ([6], *Paradiso,* XXX, 25–133).

[3] See [1], p. 40.

As Dante moves towards the climactic end of *Paradiso*, experiencing an elusive image of the Trinity, all his faculties are finally overwhelmed. Characteristically his analogy is scientific. In his futile quest for ultimate understanding, he compares himself to

. . . a geometer who sets himself
To measure the area of a circle, and, pondering,
Is unable to think of the rule he lacks ([6], *Paradiso,* XXXIII, 115–45).

2 Michelangelo and Raphael: Painted Light

A series of Renaissance painters invented various solutions for characterising light in paintings that was other than strictly naturalistic ([1], Chap. 4). It was a Dantesque non-naturalism that endowed the spiritual lights with properties that could be identified as exceptional either in source or in action or both. Here we will look at just two artists who operated in very different ways, Michelangelo and Raphael.

Light for Michelangelo was not for the most part a separate agent but a servant of sculptural form. Its independent performance in real life - in outdoor scenes, townscapes, interiors, and portraits - was not his concern. Yet, he achieved two masterpieces of divine light in narrative contexts. The first and most obvious is the *Separation of Light from Darkness* in the first compartment of the vault in the Sistine Chapel. Here, inspired by *Genesis*, God performs a separation that is both in his mind and a plastic manipulation of light as a tangible substance.

God said, Let there be light: and there was light. And God saw the light, that *it was* good: and God divided the light from the darkness. And God called the light Day, and the darkness he called Night ([2], *Genesis* 1–5).

As if to stress that the light he is creating above his silhouetted head is a spiritual concept rather than a naturalistic phenomenon, God's body is illuminated independently from below to display the expressive motions of his head, body, and loosely draped limbs. It is noticeable in the next compartment in the ceiling that the newly created sun and moon do not obviously radiate on the two hefty figures of God flying across and into the vacant space (Fig. 2).

The other great manifestation of light in Michelangelo's paintings also has a narrative trigger.

On one of the side walls of the Capella Paolina, he shows the dramatic moment when the sudden apparition of Christ with a great ray of light blasts Saul from his horse.

And Saul, yet breathing out threatenings and slaughter against the disciples of the Lord ... came near Damascus: and suddenly there shined round about him a light from heaven: And he fell to the earth, and heard a voice saying unto him, Saul, Saul, why persecutest thou me? And he said, Who art thou, Lord? And the Lord said, I am Jesus whom thou persecutest ... And the men which journeyed with him, stood speechless, hearing a voice, but seeing no man. And Saul arose from the earth, and when his eyes were opened, he saw no man: but they led him by the hand, and brought him into Damascus ([2], Acts, 9:1–8).

Fig. 2 Michelangelo, *The Separation of Light from Darkness* c. 1511–2, fresco, Rome, Vatican, Sistine Chapel. Foto ©Governatorato SCV—Direzione dei musei, Tutti i diritti riservati

Michelangelo's telling of the story is truly Dantesque. Not only has the surge of golden light blinded Saul, but it also refuses to be obstructed physically by St. Paul's raised hand and it penetrates his closed eyelids. It is just as Dante narrated: "I raised my hand to the top of my eyes and made myself a shield from the sun's rays, to scrape away the visual superfluity" - all in vain. Michelangelo was noted as an interpreter of Dante and would have well understood that for divine light "natural law holds no sway" (Fig. 3).

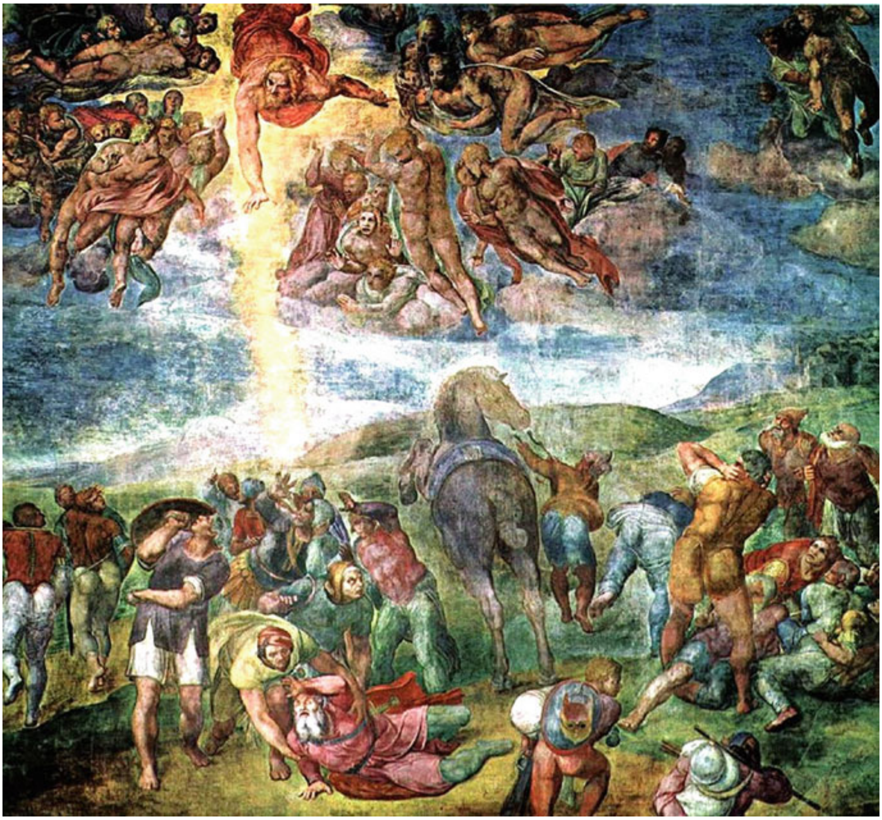

Fig. 3 Michelangelo, *The Conversion of St. Paul*, 1536–41, Rome, Vatican, Pauline Chapel. Foto ©Governatorato SCV—Direzione dei musei, Tutti i diritti riservati

The pairing of Raphael and Dante is more unexpected. But the links are strong. Raphael's father, Giovanni Santi, an accomplished painter in Urbino, was also an author, composing a substantial *Cronaca rimata* (Rhyming Chronicle) in honour of Duke Federico da Montefeltro [8]. A contemporary referred improbably to Giovanni as a second Dante. Giovanni died when his son was 11, but there is little doubt that a seed had been sown. In Rome, Raphael drafted some love poems on his own account (see [9], p. 249). Dante was conspicuously present in Raphael's *Parnassus* (amongst the poets) and in the *Disputa* (amongst the theologians) in the Stanza della Segnatura in the Vatican. Dante is the only person to be honoured by two appearances.

Raphael, who was the supreme master of all-round naturalism in the early sixteenth century, was also the great innovator in the portrayal of divine light. He developed a hugely effective way of transmuting cloud putto-heads into an amorphous celestial glare that emulates the unseeable dazzle that Dante so brilliantly evokes ([1], pp. 114–120). We will see a later example of this.

Fig. 4 Raphael, *The Liberation of St Peter from Prison*, fresco, 1514, Rome, Vatican, Stanza d'Eliordoro, Foto © Governatorato SCV—Direzione dei musei, Tutti i diritti riservati

Here I am going to look at three narratives that present him with an opportunity to juxtapose natural optics and divine radiance. The most varied and brilliant is the *Liberation of St. Peter* in the Stanza del Eliodoro. Raphael responds very directly to the Biblical account (Fig. 4).

> Peter was sleeping between two soldiers, bound with two chains: and the keepers before the door kept the prison. And, behold, the angel of the Lord came upon *him*, and a light shined in the prison: and he smote Peter on the side, and raised him up, saying, Arise up quickly. And his chains fell off from *his* hands. And the angel said unto him, Gird yourself and bind on thy sandals. And so he did. And he saith unto him, Cast thy garment about thee, and follow me. And he went out, and followed him; and wist not that it was true which was done by the angel; but thought he saw a vision ([2], Acts 12:6–9).

To the left, Raphael masterfully describes three different sources of light: a cloudy moon, the first glimmers of sunlight on the distant horizon, and a blazing taper that plays games of reflection on the soldiers' armour. The uppermost soldier has seemingly been struck by a pulse of divine light that has escaped from the central compartment. The severe prison cell above the actual window is a wondrous theatre of light. The glare of the angel's aura, set off by the dark bars of the grill, somehow transcends the limits of material pigments. The light is not seen by the three other protagonists. This is also true of the slumped soldiers to the right, while Peter, although led to safety by the angel, appears bewildered by a visual experience that lies outside his understanding—in keeping with the Biblical account and consistent with Dante's frequent bewilderment in *Paradiso*.

The glare of cloud putto-heads, already exploited by Raphael on large scales in frescoes and altarpieces, is most brilliantly realised in a notably small picture, *The Vision of Ezekiel*. God is transported into our view with the angel the three animals (eagle, lion, and ox) associated with the four Evangelists. The main protagonists, with two smaller angels, are strongly modelled by directional light and silhouetted against a spaceless radiance that again seems to transcend the brightest pigmentary white. In the leafy background to the left a divergent shaft of celestial light strikes a tiny ecstatic figure, presumably Ezekiel. Yet, we see the apparition as close to us not to him. Or do we? The underside of the cloud platform which bears the group aloft is actually *behind* the large tree in the distance. We may recall that the rules of perspectival scale do not pertain in the zones of miraculous vision in the *Divina commedia*. Raphael's ravishing naturalism is in subtle dialogue with spatial ambiguities that emerge when we look hard (Fig. 5).

The final example takes us back to our first Biblical episode, the Transfiguration. Raphael's last painting before his early death in 1520 was undertaken in implicit competition with Michelangelo, who was assisting Sebastiano del Piombo with a matching altarpiece of the *Raising of Lazarus* for the same patron, Giuliano de' Medici [10]. Raphael has amplified the basic narrative with the dramatic tale of what transpired when Christ was absent on the mountain. The remaining disciples failed to cure a mad boy who had been brought to them by his distraught father. In the lower half of the picture, the confused figures mill turbulently in the foreground, systematically illuminated by a rather lurid natural light from the upper left. Above, unseen by the disciples, Christ rises in an explosion of brilliant light. The flanking prophets, Elias and Moses, are already spirits and can bear the blinding radiance, while the three earthly disciples, Peter, James, and John, are forced to the ground and unavailingly endeavour to fend off the unbearable light. There can be no more potent juxtaposition of the logical optics of the terrestrial zone and the transcendent illogicality of any aerial realm that Christ graces with his presence (Fig. 6).

From these Raphaelesque foundations arose the great Parma domes by Correggio of the airborne *Vision of Ezekiel* and the heavenly *Assumption of the Virgin*, and later the rapturous dome and vault illusions of Baroque Rome. By the seventeenth century, Dante's writing was more distantly involved with what the painters were doing. The violation of optical law had become so deeply entrenched in the depiction of heavenly visions that Dante was no longer directly needed.

What we see in Dante and those painters who are in conscious and unconscious dialogue with him is a high regard for optical truth coupled with an acute awareness that something lies beyond the scope of rational analysis. To pick up the title of this volume, mathematics reveals the divine structure of natural law. Imagination stretches out to what is unknown (and perhaps unknowable).

Fig. 5 *The Vision of Ezekiel*, c. 1517, oil on panel, 40 cm × 30 cm (16 in. × 12 in.), Florence, Palazzo Pitti. Foto©Gabinetto Fotografico, Gallerie degli Uffizi, Firenze, by kind permission

Fig. 6 Raphael, *The Transfiguration of Christ*, oil on panel, 410 cm × 279 cm (160 in. × 110 in.), Rome, Vatican, Foto © Governatorato SCV—Direzione dei musei, Tutti i diritti riservati

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On the Classification and Recording of Colours According to the Methods of the Painter Adolfo Ferraris: A Brief Note

Marco Pierini

Little or nothing is known about the painter Adolfo Ferraris. Despite being systematically ignored by all the repertoires and utterly forgotten by art critics and art historians, Ferraris now comes across as a personality deserving of some consideration, if not for the results of his art—which to a great extent are unknown to us—at least for his substantial, original theoretical output [1, 2].

Between 1921 and 1934, in Alessandria (where according to the records in our possession he was a resident), Ferraris brought out a series of books and pamphlets in which he reported on his experiments and illustrated his artistic theories, publications that now constitute our main source of information about him and his work. We learn, for example, that the *Società Promotrice delle Belle Arti* (Fine Arts Promotion Society) in Turin refused some of his paintings in May 1920, while the *Albertina Academy* in the same city hosted an experiment in "paint and colour transmission by wire" on 4 March 1921.

His work *La classificazione dei colori e delle tinte*, written in 1920 and published in the following year, furnishes the foundation for all subsequent speculations. Unlike all previous systems devised for classifying colours—from Newton's circle to Runge's sphere and the solids of modern colorimetry employed by Munsell and by Ostwald—the method expounded by Ferraris does not use a geometric model, but one that is purely mathematical. In fact, the treatise states that it sets out to classify colours precisely by reducing them to numerical expressions, since: "if we say that white is the lightest colour, and give it the number zero (0), and that black is the darkest colour, and give it the number (1000), we have stated two mathematical truths [3]". At the end of a complex sequence of formulae and numerical expressions, Ferraris then created a "mathematical-coloured framework" in which, when analysed for their tonality, their intensity, and their reciprocal

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relationships, colours are allotted a position in a strict systemic structure. So if we mix black with white, we can achieve a thousand different degrees of intensity, all of which can be quantified with a number, each one approximating to zero as it approximates to white and, naturally, vice-versa. While the degree of intensity "marks the limit of darkness and lightness in relation to black and white", the tonality is the other crucial parameter identified by Ferraris to indicate "the exact point between the warmest tonality—Neapolitan yellow—and the coldest—Prussian blue [4]".

Ferraris identified eight basic colours, giving each a letter related to its name in Italian: white (*bianco*: B), black (*nero*: N), yellow (*giallo*: G), red (*rosso*: R), green (*verde*: V), blue (*azzurro*: A), brown (*terra d'ombra*: O), and siena (*terra di Siena*: S). Combinations between these colours generate derived colours, each one identified with a symbol (e.g. Bs stands for *bianco scagliola*, or scagliola white, Nfu for *nero fumo*, or sooty black, and Gcr for *giallo cromo*, or chrome yellow). Once this code had been established, all that remained was to express each blend (M for *miscela*) of several shades in formulae: "If we use lime white Bc (*bianco di calce*), with yellow soil Gt (*terra gialla*) and plumbago black Np (*Nero Piombaggine*), we get a blend or a shade that can also be called plaster shade, or rapid-dry cement shade. We can use the symbol M. Bc+Gt+Np = M for this blend" (Fig. 1).

Every shade would contribute naturally as a percentage to achieving the desired blend and its *weight* in the compound was indicated by a numerical value that may refer both to grams and (as in the case of preparing a cocktail) to the *parts* employed. The resulting formulae could be breathtakingly articulated, also looking graphically very impressive, as in the examples reproduced here, which stand respectively for the "fresh, non-glossy ivory shade" (*tinta avorio a fresco, non lucida*) and its glossy version (Fig. 2).

$$\overset{500}{Bc} + \underset{400}{Gt} + \underset{100}{Np} = M^{\frac{3}{=}} \begin{matrix} 500\ Bc \\ 400\ Gt \\ 100Np \end{matrix}$$

Fig. 1 Formula of a white colour that turns into "stucco" (from La *classificazione dei colori e delle tinte col metodo di Ferraris Adolfo pittore*, p. 13)

$$\overset{740}{Bc} + \underset{257}{Gd} + \underset{3}{A_{\frac{s}{u}}} = M^{\frac{3}{=}} \left\{ \begin{matrix} 740\ Bc \\ 257\ Gt \\ 3A_{\frac{s}{u}} \end{matrix} \right\}^{1/c} = \text{gradi} + 130{,}9\ \text{toni} + 499{,}3825$$

$$\overset{740}{Bz} + \underset{257}{Gd} + \underset{3}{A_{\frac{s}{u}}} = M^{\frac{3}{=}} \left\{ \begin{matrix} 740\ Bz \\ 257Gt \\ 3A_{\frac{s}{u}} \end{matrix} \right\}^{3/c} = \text{gradi} + 130{,}9\ \text{toni} + 499{,}3825$$

Fig. 2 Fresh, non-glossy ivory shade and its glossy version (from La *classificazione dei colori e delle tinte col metodo di Ferraris Adolfo pittore*, p. 75)

A practical application of this classification of colours in the field of art—but Ferraris also believed that his efforts could be extended to apply to industry, to trade, to craftsmanship, and even to gardening—mentioned briefly at the end of the book, was later developed in a pamphlet published in 1924 with the title *Resoconto su alcuni esperimenti fatti riguardanti la Pittura artistica radioelettrica*. Ferraris created an ingenious method for recording painting that set out to isolate every single dot of each work—"the modern painters of *Divisionism* who used patches, like Segantini and Pellizza, [5]"—using a system of coordinates more reminiscent of the game of battleships than of a pair of Cartesian axes, and to identify the shade to use for that single dot. A painting can be replicated anywhere and at any time, using "a box of empty dots" (which we can imagine as tiny upturned cones) destined to act as recipients for the colours, with the aid of a matrix that Ferraris called a *compositoio* [6] with the twin effect of preserving the work thus made and recorded for eternity and being able to set up exhibitions all over the place simultaneously. "For example: the recording of a painting is transmitted by radiotelephony from Rome to London, to Paris and to New York; the receivers in those cities take the recording they have received, then deposit the dots in the *compositoio* exactly as stipulated in the recording. The result will be four works painted identically in the four cities thus named" [7].

From the description provided by the painter, the system would appear to be very sophisticated, since it would take no less than 8720 dots, each measuring only 2.5 millimetres in diameter, to paint a work on the small scale of 15 $\times$ 27 centimetres. Here is an example of the recording of a painting made by Ferraris: Line 78 = 4 Rm 9 A 3 R [...]

This explains that the line in question comprises four dots of red lead, nine of dark blue, three of vermilion [...] [8], and so on until the surface of the painting has been completed (Fig. 3).

Fig. 3 Adolfo Ferraris, Landscape (from *Resoconto su alcuni esperimenti fatti riguardanti la Pittura artistica radioelettrica*, p. 20)

Recording a painting and the subsequent possibility to replicate it on the basis of *user instructions* constitute an extremely innovative idea that would seem to be following in the footsteps of the research being undertaken at the same time by Duchamp (who sent his sister Suzanne, who was in Buenos Aires, the instructions for making the *Unhappy Readymade* in 1919) and Moholy-Nagy, the first to provide indications for making a work by telephone, in 1922, thus pre-empting the celebrated *Art by telephone* exhibition set up in the Chicago Museum of Contemporary Art in 1969.

Radioelectric painting also appears to contain another innovative element, the intention—which may not have developed fully consciously—to eliminate what Pierre Restany, when discussing Duchamp's *ready-made*, called the *taboo of everything handmade* [9]. Ferraris's object was not an industrial product, despite the fact that its creation was totally mechanised: the person executing a radioelectric painting merely followed a set of instructions, without interfering in any way in the process of creating the work of art. Lastly, the possibility to replicate the work anywhere and at any time would appear greatly to exceed the modern concept of the *multiple* and to promote an inexhaustible technical reproducibility of the original (or, even better, of a work whose matrix is the numerical recording in itself). Ferraris actually showed scanty interest in the object and was concerned above all—adopting a clearly idealistic approach—in laying claim to the importance of conceiving a painting and then later recording it, rather than of the painting itself. Recording a painting also enabled its image to be maintained durably stable, since it would always be possible to tackle any ageing, fading colours, or defects continuously with subsequent *restoration* or by redoing the entire painting—something that anyone could do, since the operation would be completely mechanical—safeguarding both the work's aesthetics and its commercial value, as Ferraris was at pains to point out.

Radioelectric painting also touched on an issue that was much frequented in the environs of the twentieth-century avant-gardes: that of the relationship between colour and sound [10]. In fact, in the second part of the pamphlet entitled *Breve descrizione metafisica della pittura sinfonica*, Ferraris established a relationship of correspondence between musical notes and colours arranged according to his method. The note *so* annotated beneath the pentagram was thus the equivalent of black (N), *la* the equivalent of light green (V′), *ti* that of a darker green (Vc), and so on through the whole pentagram. He then developed this equivalence further to express the duration of each note in terms of the amount of colour: "sixty-four dots of the same colour can be done with a whole note. If the note *ti* on the third line is a whole note, it calls for 64 white dots on the painting [...] if the same *ti* is a half note, it calls for 32 white dots. If the same note is a quarter note, it calls for 16 white dots [...]" [11]. In this way, Ferraris believed he could translate pieces of music into colours and create music from paintings. And even, at the same time, see a painting and listen to the music derived from the recording made from it. This is how he described his ingenious system:

"The painting is placed on a prepared frame that can be raised and lowered with a crank as the need arises. A canvas is stretched in front of the painting to cover it. In front of this arrangement, the concert will play the music obtained from the recording and all the notes issuing from the concert will be uncovered on the painting as they are played, by using the crank to raise the painting and another scroll feature that will shuttle back and forth as the need arises. In this way, the *aspettatore* (sic: a neologism that seems to intend to combine the words meaning 'watcher' and 'listener') has the twin sensation of hearing and seeing at the same time" [12]. Nevertheless, however unique the system invented by the painter, Ferraris's synaesthesia does yet seem to be very artificial: the relationship between notes and colours does not in fact have any scientific (or maybe it would be better to say parascientific) basis, as in the case, for example, of Enrico Prampolini's chromophony [13], nor does it have any emotional basis—still with a symbolist matrix—as in the case of Alexander Scriabin's table of musical chromatic equivalences and, in part, of the considerations of Wassily Kandinsky.

Radioelectric painting underwent a further stage of evolution in 1934 with the invention of cyclozonic painting, a more refined recording system capable of defining the confines of minute fields of paint—including fields with very irregular shapes—and even individual brushstrokes or touches. The handful of pages of the pamphlet devoted to this innovation mention a *Universal and analytical dictionary of shades*, of which no trace has been found but which, reading between the lines, seems to have been a repertoire—"the greatest book in the world (...) of 1056 shade samples"—compiled in the wake of the previous volume about colours.

It would be easy to deduce that Ferraris's lively imagination and a certain degree of harmony between his theories and some research developed in the latest contemporary artistic fields might be met by a taste and a stylistic intention in line with the avant-garde character of his thinking. Yet on the contrary, what little we know indicates that the painter's output never ventured away from an academic conformity that was already then so outdated as to look almost the result of the honest dabblings of an amateur.

It could be argued that the elementary landscapes reproduced in his pamphlets are the consequence of the need for him to limit the difficulties encountered during recording—and so of reproduction—of the painting during the experimental phase, but if the oval with San Rocco on the façade of the church dedicated to him in Alessandria is the one that Ferraris proudly claimed as his own work in his report of cyclozonic painting [14]. Then the modesty and backwardness of his art can hardly fail to stand revealed with striking clarity. Once again in this case, however, the painter did not miss the opportunity to surprise us with his originality and inventive manner: this is no ordinary mural, in fact, but an *elastic portable fresco* [15] (Fig. 4).

Fig. 4 Adolfo Ferraris, *St. Rocco*. Alessandria, Chiesa di San Rocco

References

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2. Ferraris, A.: Relazione su la pittura ciclozonica, Alessandria (1934) p. 14
3. Ferraris, A.: La *classificazione dei colori e delle tinte col metodo di Ferraris Adolfo pittore*, Alessandria, p. 7. (1921)
4. in [3] p. 45
5. Ferraris, A.: Resoconto su alcuni esperimenti fatti riguardanti la Pittura artistica radioelettrica, Alessandria, p. 3. (1924)
6. The *compositoio* (literally, a device for making a composition) was then replaced by the "grating (..) a simple metal frame like the one used in a sieve (. . .) made of galvanised iron, of brass or also of a finer metal—without ruling out precious metals—according to the work to be done. It must be absolutely regular, so have vertical lines exactly on the perpendicular of the horizontals" (Ivi, p. 15)
7. in [6] p. 11
8. in [6] p. 12

9. Restany, P.: L'altra faccia dell'arte. In: La Diana, I. (ed.) 1997, pp. 47–55 (1995)
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11. in [5] p. 27
12. in [5] p. 28
13. See Marco Pierini, *La cromofonia e il valore degli spostamenti atmosferici. Affinità e divergenze fra il manifesto di Enrico Prampolini e le coeve teorie delle avanguardie musicali e artistiche*. In: Annali della Facoltà di Lettere e Filosofia di Siena, vol. 16, pp. 123–134. (1995)
14. in [2] p. 12
15. in [2] p. 11. "Despite being heavy with its relative elasticity" as we read on p. 12, "the elastic fresco is still rigid and, since it is a composition of chosen minerals, it will resist longer and, since it has no persistent luminance, it allows for a perfect vision of the painting from all angles. On the other hand, being portable, it can be taken down, cleaned and placed elsewhere"

Colored Figurative Tilings in Pre-Incan Textiles

Anthony Phillips

1 Introduction and Preliminaries

Pre-Columbian Peruvian weavers are renowned for the skill with which, over some two millennia, they produced textiles of a complexity that has never been duplicated, including techniques found in no other cultures [9]. The visual content of their textiles also has unique formal aspects. Until very recent times and the work of artists like Koloman Moser [21] and Maurits Escher [12], theirs were the only cultures that systematically decorated with what John Osborn [15] has called *figurative tilings:* "A figurative tile, as I use the term, is a zoomorphic outline devised in such a way that multiple copies will interlock to tile the plane."

I am using the term "tiling" in a non-standard way: Osborn clearly has *bounded* tiles in mind, whereas in the Peruvian examples considered here each "tile" is (potentially) infinitely long; it is actually a frieze with zoomorphic elements. The decoration of these textiles will be analyzed as two-color two-dimensional patterns, using terminology from [23], but with the motifs, or units, taken as copies of the frieze in order to follow what I believe to be the thinking process of the ancient Peruvians. One of our patterns (Sect. 5) is in fact generated by a finite tile; but as a two-color pattern, its natural units are friezes composed of repetitions of that tile. A more complete discussion is in that section.

Here, I use this perspective to analyze five textiles, including one which was published [8] in 1924 but which to my knowledge has never been explained. All five of these textiles are of a special construction (terms explained below): bands about 10 cm wide with warp-faced plain-weave borders enclose a central area decorated with interlocked snakes or birds, the pattern woven with complementary warps [10]

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of 3 or 4 colors. Characteristic is that the white-colored warps are cotton and much thinner than those of the other colors [9, p. 153], leading to a distinctive texture in the white areas. A similar class of textiles is described in [7], analysis of item 127. Two of this family of textiles appear, about 80 cm long, as the hem decoration on tunics in the Ethnological Museum, Berlin [2, p. 151]. All of them are attributed to the central coast of Peru, and the Late Intermediate period, roughly 1000–1500.

My thanks to Ann Rowe, George Hart, Philip Palmedo, and Cecilia Toro; their comments on a draft of this work led to significant improvements.

Warp-Faced Plain Weave; Complementary Warps Weaving begins with the *warps* (shown horizontal in these diagrams) strung on the loom. There may be one (Fig. 1), two (Fig. 2), or several yarns in each *warp location*. Then, the *wefts* are woven in, one by one, perpendicularly to the warps. *Warp-faced plain weave* is illustrated in Fig. 1. In plain weave, "[e]ach weft passes alternately over and under successive warp units, and each reverses the procedure of the one before it" [10]. The weave is warp-faced if the warps are so thick and tightly packed that they completely hide the wefts.

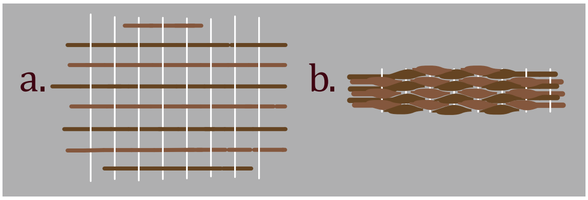

Fig. 1 Warp-faced weave. (**a**) Expanded view. (**b**) In warp-faced weave, the warps are thicker than the wefts and packed together: the wefts disappear. White wefts and two different shades of warp are used for legibility

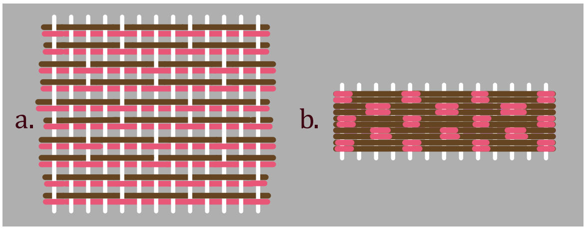

Fig. 2 In 3/1 alternate-pair complementary-warp weave, the warps appear in 3-span floats aligned in alternate pairs [17, p. 70]; diagram adapted from [4], as it appears in the solid brown areas of Textile 1 (Fig. 3). (**a**) Expanded view of weaving, with wefts white for legibility. The top four warps represent the brown and pink yarns in two adjacent warp locations. For this pair, brown goes over 3 under 1, while pink goes under 3 over 1. Subsequent pairs of locations are treated the same, except each time the sequence of overs and unders is staggered by two steps, giving the "alternate alignment." (**b**) In the textile, the warps are compressed together; then, the brown warps slide over and hide the pink, except where the pink surfaces; the wefts are completely hidden

Table 1 The criteria that define the symmetry types of the textiles in this study. Note that Pattern **B** and Pattern **D** have the same symmetry type

Pattern	**A**	**B**	**C**	**D**
A reflection preserves or exchanges colors.	No	No	No	No
A 180° rotation preserves colors.	No	Yes	No	Yes
A 180° rotation exchanges colors.	Yes	Yes	No	Yes
A glide reflection preserves colors.	No	No	Yes	No
A glide reflection exchanges colors.	No	No	Yes	No
Symmetry type	$p2'$	p'_b2	p'_b1g	p'_b2

In weaving with complementary warps, the loom is strung with two or more different-colored yarns in each warp location. Then, at each undercrossing, the weaver runs the weft under one of the warps and over the others, which form *floats* on the back, while the chosen warp contributes to the design. This is how d'Harcourt [8, p. 17 §2], describes the process. In fact, the way warps appear on the surface is quite rigidly organized in all these examples, in part so as to keep the floats on the surface short (length ≤ 3). The speckled texture apparent in "solid color" areas is the result of *3/1 alternate-pair complementary warp weave* ([18], Chapter 7, note 3), sometimes called "pebble weave" [4, p. 22]; see Fig. 2.

Symmetry The decoration in each of these four bands determines a 2-colored tiling of the entire plane. For a planar tiling, a *symmetry* is a rigid motion of the plane (translation, rotation, reflection, or glide reflection), which places the pattern back in congruence with itself. When the tiling is 2-colored, we can ask if a symmetry preserves or exchanges the colors. All the possible two-colored planar tilings have been categorized: Washburn and Crowe [23] give flow charts for identifying the symmetry type of any particular example; for the textiles in this study, the charts can be compressed into Table 1.

Notation I will use capital letters **A, B,** ... for colored patterns, and the corresponding lowercase **a, b,** ... for the underlying monochrome pattern. A combinatorially different colored pattern based on **a** would be **A**$'$, etc.

All textiles are shown with warps horizontal on the page. Photo credits: except for Figs. 10, 17, 21a, b, and 23b, these are my photographs of items I have collected.

2 Pattern A

In Textile 1 (Fig. 3), the central panel is 3-color complementary-warp weave: the yarns are dark brown, (thin) white, and pink. Pink is used for edging and as complementary to the main color in the 3/1 alternate-pair sections. Plate V (p. 92) in [17] shows a textile (Textile Museum 91.593) containing a band very closely related to this one, only with snake heads instead of birds.

Fig. 3 Textile 1. Detail of a band of width 11 cm (the total length of fragment: 76 cm)

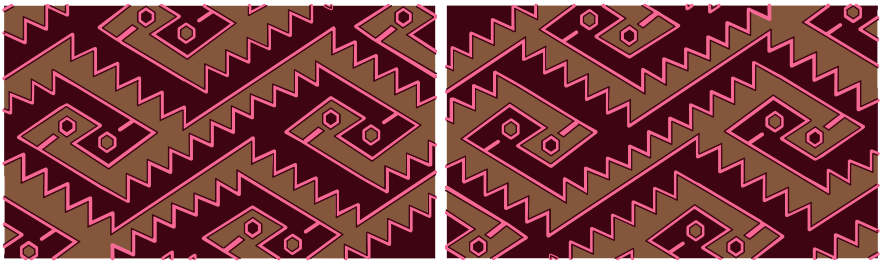

Fig. 4 Pattern **A**. Left: the planar tiling extrapolated from Textile 1; right: that image rotated 180°. The images are the same, with colors exchanged. Symmetry type $p2'$ (Table 1)

The top edge of the patterned band can be seen to match the bottom edge after a translation of about one-fourth of a repeat. Stacking shifted copies of that pattern leads to a tiling of the plane (Fig. 4).

3 Pattern B

The pattern in this strip (Fig. 5) seems at first glance to be loosely nested gray and crimson copies of the ornithomorphic tile in Fig. 6.

Closer examination shows that the background cream-colored space separating the gray and crimson tiles shares their shape exactly and that what we see is an exact figurative tiling with three different colors. There can be no symmetry involving all three colors since cream appears twice as often as either of the others; this type of pattern is discussed in [23, p. 65]. Symmetry is analyzed in Fig. 7. Figure 8 shows how a period-3 coloring would eliminate the background/foreground effect and manifest the equivalence of the tiles. For a period-2 coloring of Pattern **b**, woven in a different technique, see Fig. 21b in Sect. 6.

Figure 9 shows another instance of Pattern **B**, woven with 4-color complementary warps: dark blue, yellow, (thin) white, and red for edging. Where the thin white warps are on the surface, their color blends with the brown of the weft to give a

Fig. 5 Textile 2. One end of a band, 7.5×23 cm

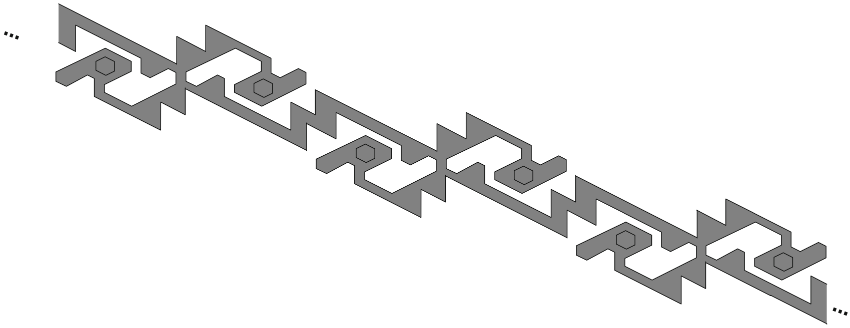

Fig. 6 The infinite ornithomorphic tile suggested by Textile 2

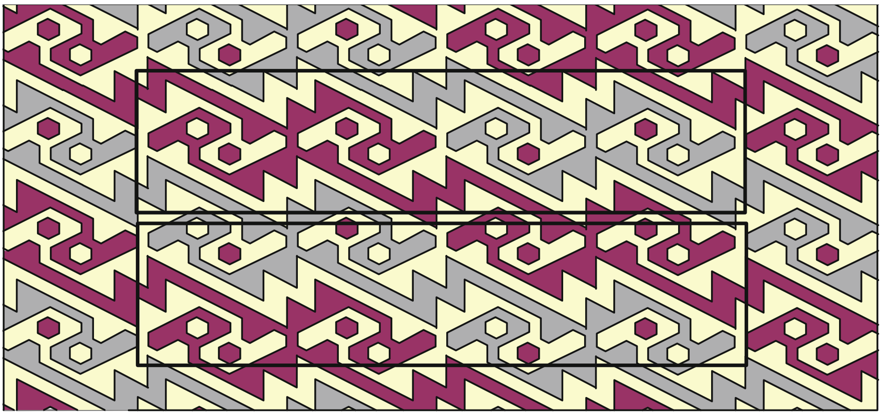

Fig. 7 Pattern **B**, the planar tiling extrapolated from Textile 2. If we consider it as a 2-color tiling (crimson, gray) with cream-colored background, then a 180° rotation about the center of the upper rectangle reverses colors, while a 180° rotation about the center of the lower rectangle preserves them. Symmetry type $p_b'2$ (Table 1)

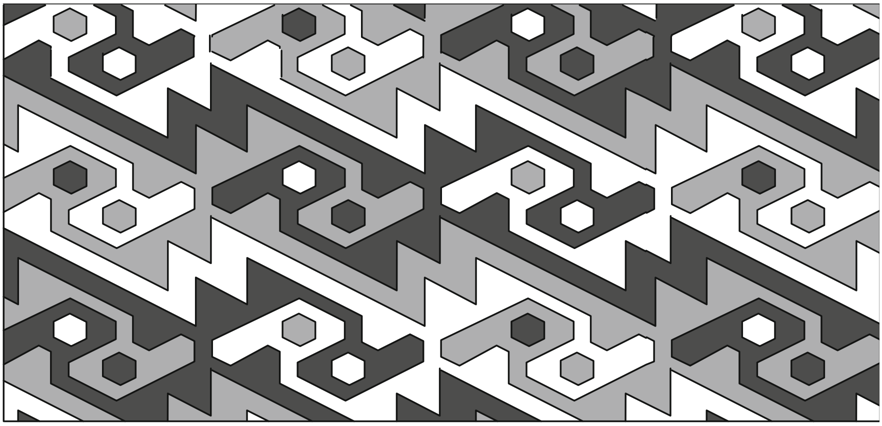

Fig. 8 A period-3 coloring of the pattern would bring out the equivalence of the tiles

Fig. 9 One end of a band fragment 14 cm wide

kind of tan. The symmetries involve the dark blue and tan areas, with the yellow as background.

4 Pattern C

Textile 3 (Fig. 10) is clearly a slice of something, but of what? A clue comes from another textile: the pattern (disregarding colors) in Fig. 11 stacks copies of the pattern in Textile 3 except, as detailed in Fig. 12, that the top copy (yellow circles)

Fig. 10 Textile 3. Band, 9×38 cm, from the central coast of Peru, area of Lima, town of Huacho. Musée du Quai Branly—Jacques Chirac, Paris, Inventory No. 71.1964.86.166. Originally in the collection of Raoul d'Harcourt, illustrated in [8], Plate 33b and in [9], plate 19. Image ©RMN-Grand Palais/Art Resource, NY

Fig. 11 Central band (width 4.5 cm) from a complete textile, 27.5×31 cm, sewn together from various elements. In this band, the white warps have the same thickness as the others

has been altered to allow the particular coloring planned for this exemplar. Undoing that alteration results in part of a complete planar tiling, Pattern **c**; from there we can deduce the tiling (Pattern **C**) of Textile 3: see Fig. 13. Symmetry is analyzed in Fig. 14.

We will probably never know why an inscrutable decoration like that of Textile 3 appealed to those who wore it. Perhaps they were familiar with the whole intricate design (Pattern **C**) of which it is a section? Since, to my knowledge, there are no surviving examples of that whole design (except a monochrome version in double cloth, see Fig. 22), one can only speculate. In fact, *inconsistent* implementations of the pattern do exist: besides the item shown in Fig. 11, there is an example in the Museum of Fine Arts, Boston (Fig. 15) and another (Fig. 16) of a band much like Textile 3; neither of those patterns can be extended to the plane.

The band in Fig. 16, while similar in many respects to Textile 3, does not have a coherent decoration. It contains 5 repeats (2 shown here) of a motif, each glide-reflected from the previous; the motif itself corresponds to $1\frac{1}{2}$ repeats of the motif of Textile 3, so the copies of the motif do not match where they meet.

Fig. 12 One copy of the repeated motif in Textile 3 and a detail from Fig. 11. The white rectangles enclose areas where the patterns on the two textiles match, disregarding colors and details of bordering. The yellow circles show where the third vertical repeat in the Fig. 11 pattern does not match the first two

5 Pattern D

Pattern **D** (Fig. 17) is one of two different two-colored figurative tilings, in bands of the type described in Sect. 1, both derived from a standard planar tiling (Pattern **d**, see Fig. 18) of type $p2$ etc. by grouping copies of the motif into infinite tiles; the other is Pattern **D**′ (Fig. 19). The Peruvian way of analyzing Pattern **D** would seem to be as a union of bicolored tiles, of type Fig. 18a: (black, yellow) and (gray, yellow). In those terms, the symmetry type is $p_b'2$ (Table 1), the same as Pattern **B**.

In Textile 5, the tiles are colored blue, yellow, and white, with yellow serving as a background. The color symmetry occurs between groups of three: three blue tiles interchanging with three white tiles. In those terms, the symmetry type is also $p_b'2$.

6 Patterns a, b, c, and d Woven in Other Techniques

Techniques for a weaver are like media for a painter: each has its own potential and limitations. So far we have examined single-faced textiles woven warp-faced with multicolor warp substitution. But the planar patterns underlying our examples also came to be implemented in plain weave with fine white supplementary warps,

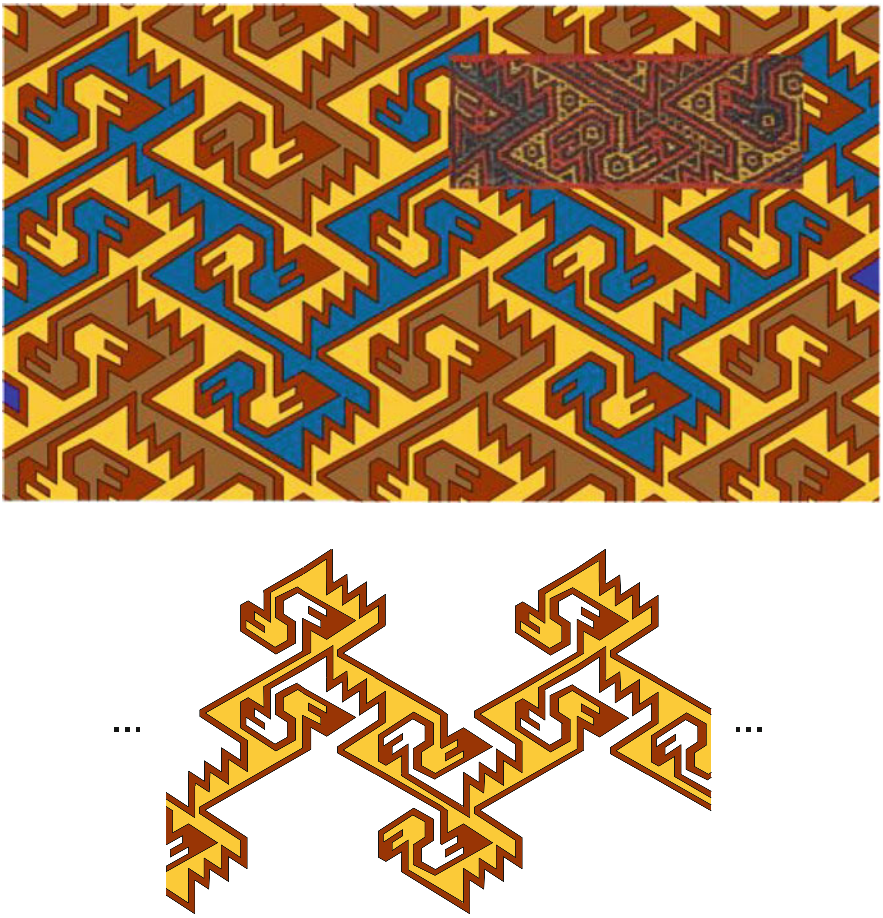

Fig. 13 Top: Pattern **C**, with one of the repeats of Textile 3 superimposed on the image. Symmetry type $p_b'1g$ (Table 1); see Fig. 14. Bottom: one copy of the infinite tile in Pattern **C**, with both of its borders

in double cloth, in brocading, and in double-faced tapestry weave (kilim). The first two of those techniques involve just one color and a background, while for the last two the decorative or the weft colors may be chosen locally during the weaving and may not show the consistency built into warp-patterned schemes, where colors are fixed when the loom is strung.

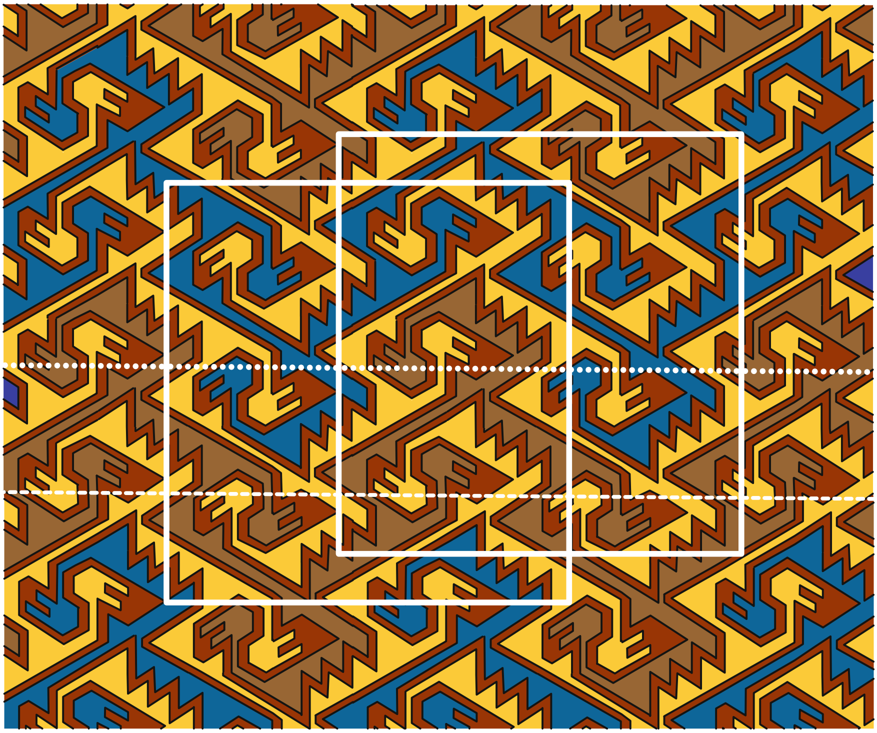

Fig. 14 Symmetries in Pattern **C**: reflection in the dashed white line and horizontal translation by half a repeat brings the pattern in congruence with itself with colors matching. Reflection in the dotted white line and horizontal translation by half a repeat takes one white rectangle to the other: the glide-reflected pattern is congruent with itself, but blue and tan are interchanged

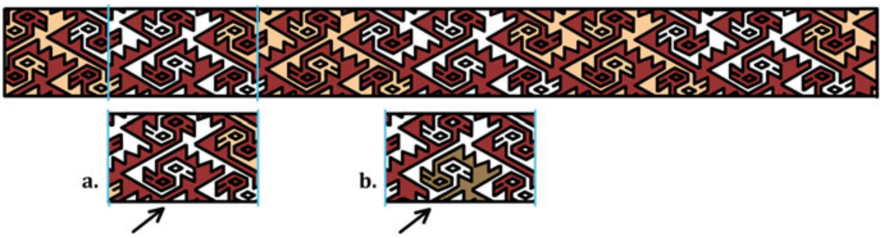

Fig. 15 Another non-systematic extension of the design of Textile 3. Coloring scheme of a band (width 5.5 cm) in a textile in the Museum of Fine Arts, Boston (Arthur Mason Knapp Fund 1942.440), Plate 34 in [20]. (**a**) Detail showing where modification of the design has destroyed vertical periodicity and allowed a coloring different from (**b**), the coloring scheme of Pattern **C**

Fig. 16 Detail of a band of width 14 cm; entire fragment 96 cm long. Central portion executed with 4-color complementary warps: black, yellow, (fine) white, with red and black edging

Fig. 17 Textile 4. Pattern **D**. Fragment of band 17×59 cm, Museum of Fine Arts, Boston 42.452, listed as item 230 in [20]

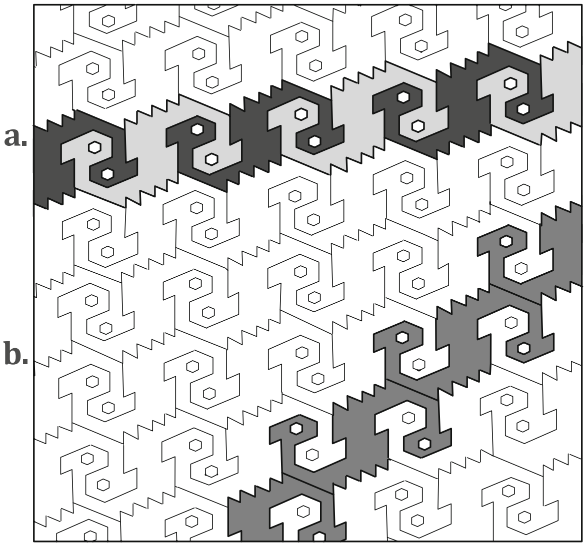

Fig. 18 (**a**) Tile from Textile 4; (**b**) tile from Textile 5 (Fig. 19), as subsets of Pattern **d**

Fig. 19 Textile 5. Pattern **D**′. Detail of a complete band, width 13 cm, woven with dark blue, yellow, and (thin) white warps; edging in pink and dark blue, length 126 cm. Band has a 9-cm wide fringe

Fig. 20 A double cloth fragment, 13 × 18 cm, shows the Greek key frieze along with its adaptation as an interlocked snake motif

6.1 Pattern a

Pattern **a** most likely evolved from the "Greek key" or "running dog" frieze pattern ubiquitous in textile decoration. The ancient Peruvians, besides using the Greek key itself, tweaked it into naturalistic forms in many different ways; so Pattern **a** has many cousins even though its precise geometry, as far as I know, only occurs in bands of our type. One close relative is the interlocked snake motif, which was already at least 1000 years old when our textiles were woven (see [6], Fig. 4a). In Fig. 20, a fragment of double cloth exhibits the Greek key motif along with an elementary version of the interlocked snakes. *Double-weaving* [4, p. 7], [9, p. 44] uses two complete plain woven sets of warp and weft, almost always cream and brown cotton; they trade places on the surface to create the pattern.

a. b.

Fig. 21 (**a**) Fragment, 16×23.5 cm, single-faced supplementary weft patterning or embroidery, AMNH Catalog No. 41.2/8989. (**b**) Pattern **B**′. Double-faced tapestry weave with paired warps, tubular edging, 14×12 cm. (The red flecks point to the design's origin in a complementary-warp implementation). AMNH Catalog No. SAT/84. Both images courtesy of the Division of Anthropology, American Museum of Natural History

6.2 Pattern b

Pattern **b** is a distant and considerably more elaborate cousin of the double "Greek key" pattern exhibited, for example, in Fig. 21a. I know of just one non-warp-faced example, a tapestry-woven triangular corner piece (Fig. 21b). We can call this Pattern **B**′, a two-color implementation (gray, yellow), with red outlining.

6.3 Pattern c

Pattern **c** occurs in a double cloth fragment, Fig. 22. A textile of almost identical fabrication (a 1-pixel difference in the birds' beaks), item 138–752 in the Museu Etnològic, Barcelona, is illustrated as No. 48 in [19, p. 107].

6.4 Pattern d

Figure 23 shows two examples of Pattern **d** itself: one (a) in blue cotton plain weave with fine white supplementary warps. See [7, p. 171] for detailed diagrams of this particular structure. A complete panel (25.8×44.5 cm) with a more finely woven, smaller scale version of this same decoration, RPB1611 in the Rietberg Museum, Zürich, is shown as item 140 in [5]. The other example (b) is double-woven.

Fig. 22 Pattern **c** in a double cloth fragment, 20×33 cm. Along the bottom, a selvedge, there is a couple of weaving errors. I have picked out one of the Pattern-**c** tiles in yellow

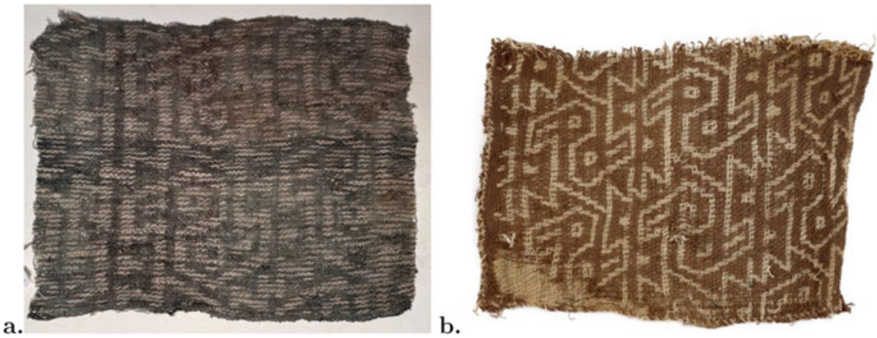

Fig. 23 Pattern **d**. (**a**) Fragment of a textile of width 23 cm, blue cotton plain weave with fine white supplementary warps. (**b**) Double cloth fragment, 21×28 cm, The Textile Museum, Washington, D.C., 1961.30.128, Burton I. Jones

In both the textiles of Fig. 24, the colors are pink, brown, and various shades of yellow. The coloring of the birds' eyes strongly suggests that the grouping is as in Fig. 18a, with tiles colored (dark brown, yellow) and (pink, yellow) alternately. In Fig. 25, the inconsistent color of the birds' eyes suggests a hybrid coloring between Fig. 18a and b; there does not seem to be any global color symmetry.

Figure 26 compares two textiles of essentially identical construction: dark brown plain weave with brocaded double-birds in white, orange, and taupe. But the organization is different. The one on the left shows Pattern **D**, colored alternately (orange, white) and (taupe, white), whereas the Museo Amano example shows

Fig. 24 Pattern **D**. (**a**) A complete textile, dark brown open plain weave with tapestry inserts. (**b**) Schematic coloring scheme for a complete square (32×32 cm) of essentially identical construction, item V 9.600 in the Roemer- and Pelizaeus-Museum [3, p. 67]. The photograph of another example of this pattern and construction, from Lauri in the Chancay Valley (Museo Amano, Lima), appears as [22], Fig. 26

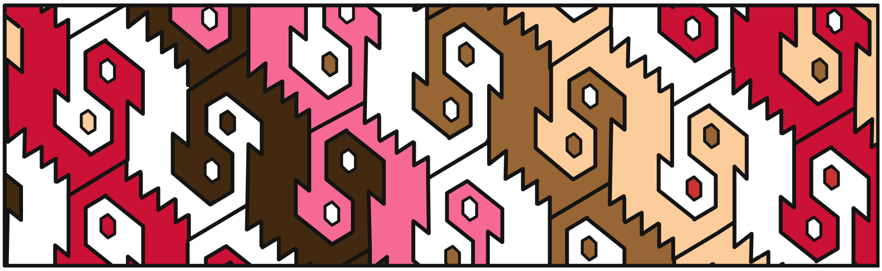

Fig. 25 Schematic coloring scheme for the tapestry woven fragment AS. 1838 [13] (17.2×52.3 cm) in the Textile Museum of the Fondazione Ratti, Como

Fig. 26 Left: Pattern **D** in a brocaded fragment (one selvedge), 16×37 cm. Right: Pattern **D**″. Schematic coloring scheme (detail) of a brocaded textile in the Museo Amano, Lima, No. 128 in [1]

Pattern **D**″: the tiles are of type Fig. 18b, like Textile 5, except that the coloring is *periodic of order 3:* taupe, orange, white.

The brocaded example from the Museo Amano shown in Fig. 26 is unique in the set of textiles we have examined in that there are three colors used equivalently. The relevant 3-colored symmetry class $pgg[3]$ is illustrated in [11], Figure 8.2.3.

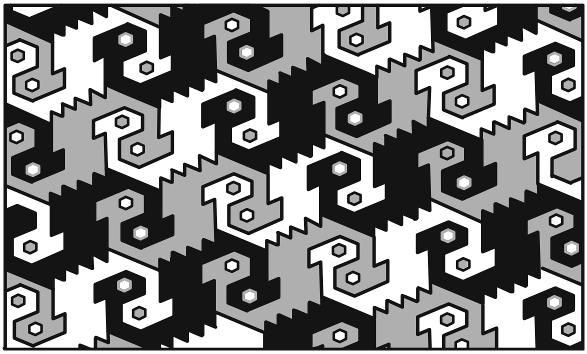

Fig. 27 A non-Peruvian coloring of Pattern **d**

One general feature of all the Peruvian colored tilings based on Pattern **d** is that copies of the double-bird motif are grouped into linear friezes. The Peruvian weavers and brocaders do not seem to have entertained the possibility of a coloring motif by motif. In particular, I do not expect that the coloring that might seem most natural to a modern eye, namely the essentially unique 3-coloring in which same-colored motifs do not share an edge (Fig. 27), will ever be found in a textile from this period. See [14, p. 964] for more general remarks on n-colorings in traditional art, $n > 2$.

7 Conclusion

It is commonly understood, and I have remarked on it elsewhere [16], that we know almost nothing about the intellectual life of the Peruvian cultures that preceded the Inca. These textiles give a hint of how much we are missing. Patterns like these are not found by chance: they must have taken a large, sustained, essentially mathematical program to develop. Additionally, the possibility that a small generating section (as in Textiles 1 and especially 4) could be both a decorative band on a garment and a token of a much larger complex pattern suggests that the general population understood and appreciated the decorative potential of intricate geometrical symmetry. The mathematical genius of these peoples found its expression in these wonderful works of art.

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The Artistic (and Practical) Utility of Hyperspace

Tony Robbin

I have felt and given evidence of the practical utility of handling space of four dimensions as if it were conceivable space. -James Joseph Sylvester, 30 December 1869.

The best way to show how four-dimensional geometry has enriched my artwork is to show how my artwork has enriched four-dimensional geometry. Of course, I had teachers.

1 Hypercubes Tessellated

In the summer of 1979, when I was 35, I traveled to Brown University to meet Tom Banchoff, chair of the mathematics department, and to see his computer representation of a hypercube rotating in four-dimensional space. Banchoff was generous with his time, and with time on his million-dollar VAX computer. Subsequent visits, handwritten letters, which I have cherished and kept, proofs of my conjectures, invitations to conferences, and authentication of my computer programs and of the mathematical content of my work—all this followed.

Do you know the plane if you only know a square? Wouldn't it be better to contemplate a whole page of squares fitted together, a *tessellation* of squares? Likewise, do you know space if you only know a cube? Soon after I visited Banchoff for the first time and learned to replicate his program for the rotating hypercube (at Pratt Institute with Herb Tesser and *his* million-dollar VAX), I programed *9 Tessellated Hypercubes*, linked here—tonyrobbin.net/quasi/TessHyperCubes.mp4, and see drawing at the end of this section. One hypercube above, one below, to the left, to the right, in front of and in back of, and also one fore and one aft in the

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M. Emmer, M. Abate (eds.), *Imagine Math 8*,
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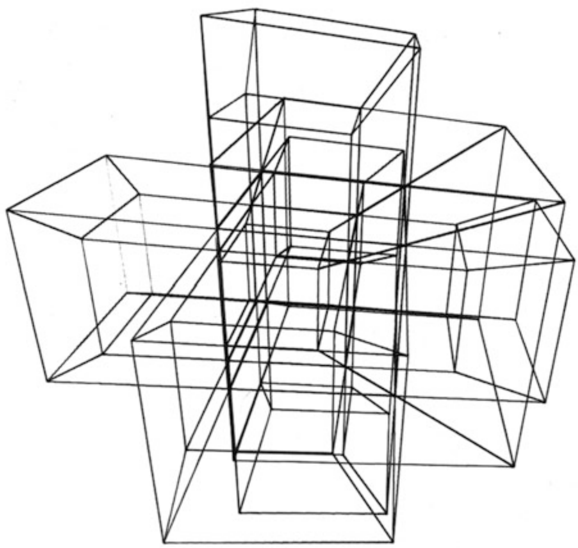
fourth dimension were place around a central hypercube. To my knowledge, this was the first time a program was written (and a resulting video was made) that showed tessellated hypercubes rotating in real time about four mutually perpendicular axes taken two at a time. My program also showed the rotations in both isometric and perspective projection, and with the option of viewing in anaglyphic (red and blue) stereo. Further, the hypercubic rotations also included body-centered rotations such as the pitch, roll, and yaw that a pilot would have to control. Seeing the tessellated hypercubes gives a more vivid understanding of 4space than seeing just one hypercube.

Squares are tessellated if their boundary edges are edges in precisely two squares; cubes are tessellated if their boundary squares are squares in precisely two cubes; and hypercubes are tessellated if their boundary cells are cells in two and only two hypercubes. Said this way, the tessellation rule is clear, but I found it hard to visualize what Donald Coxeter called a "honeycomb" of hypercubes. With an introduction from Linda Henderson, I wrote to Coxeter, and received back one page of diagram and text that made the problem clear, and I quickly wrote the code. (I have cherished that handwritten correspondence too.) I published pictures of the 9 tessellated hypercubes in my 1992 book *Fourfield: Computers, Art & the 4th Dimension* and even included 3D glasses and a print of the tessellation in 3D.

Tony Robbin, *9 Tessellated Hypercubes*, *c.* 1983. Plotter drawing on paper. Collection the artist



2 Planar Rotations

When I visited Banchoff at Brown that first time, I traded a drawing for a print of his pioneering film T*he Hypercube: Projections and Slicing*, 1978 (https://www.youtube.com/watch?v=90olwwLdEYg). Leaving the slicings aside for the moment, consider the four-dimensional rotations and resulting changes in the projections of the hypercube. A paper square can rotate around a pin stuck into it. A cardboard box of safety matches can rotate around a bamboo skewer shoved through it (an axel). But a hypercube can rotate around a plane. This is a special kind of rotation that can only happen in four or more dimensions; it is a rotation that requires the extra degrees of freedom that 4space can offer. This higher-dimensional rotation is called planar rotation, and without planar rotation one does not really have 4D art.

These planar rotations are best understood by looking at the matrix algebra, see Appendix 1.

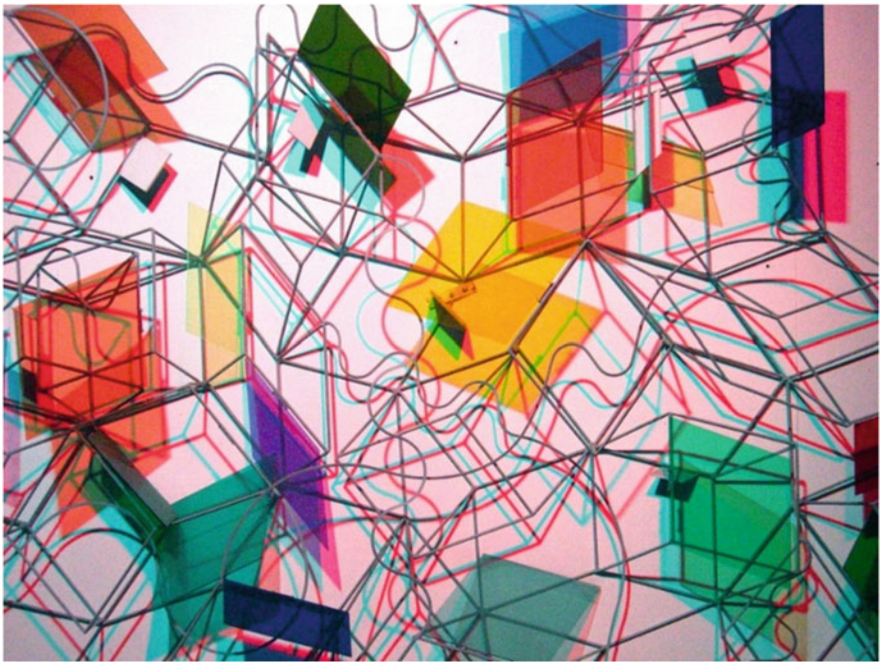
George Gamow's *One, Two, Infinity* was one of the trove of books given to me by John McIlroy when he retired from the math department of Trenton State College where I briefly taught in the 1970s; McIlroy lit a fuse when he challenged me: "Read these books and they will change the way you see!" In that book, Gamow made the odd claim that the Lorentz transformations of special relativity were planar rotations in the fourth dimension. And eventually I did see that the 4x4 matrix of Minkowski's spacetime metric has the form of a planar rotation: push on one thing and another thing changes: make lengths shrink in one of three spatial dimensions and time has to run slower, i.e. the interval between "clicks" has to expand. Two things change but the rest do not. That is, we see a spaceship traveling close to speed of light snubbed in length in the direction it is going, and we also notice its clocks running slow. But the height and width of the spaceship remain the same as it always was. The rubbery, reciprocal nature of spacetime has this formal association with planar rotations in 4space.

All the mathematics of 4D rotation, including those in Banchoff's film and my programs too, are planar rotations about the origin—about the center of the hypercube, or the center of the central hypercube in a tessellation. But for my artworks, I wanted to rotate about a plane that is a face of one of the cells of the hypercube. I thought I knew what that would look like, but I went to Brown to ask Banchoff to confirm my understanding on his computer, which he was able to quickly do. It looks different than rotations about the origin.

My works from the 1980s, especially *Fourfield, Lobofour,* and the light pieces use two-dimensional elements (lines painted on a canvas or fixed shadows from colored lights) and three-dimensional elements (thin steel rods welded to become three-dimensional line drawings) that work together to give the visual information of planar rotation. The fixed two-dimensional elements do not change as you walk around, but the three-dimensional elements do parallax. (Some things change, and some do not.) In the light pieces, red and blue colored light reflect white on the wall, but where the red light is blocked a blue line of light appears, and where blue is blocked a red line appears. These colored lines can be combined by red and blue

3D glasses to make an illusion of three-dimensional elements rotating through the steel rod structures for a convincing display of four-dimensional planar rotation.

The wall becomes a face of the hypercube, and the floor becomes the zw plane. Walking on the floor while wearing the 3D glasses you are causing a four-dimensional, planar rotation—you are walking in the fourth dimension.



Tony Robbin, *1987–3*, 1987. Welded steel, acrylic plates, and colored lights. 84 × 84 × 8 inches. Collection the artist

3 Slices Vs. Projections

Banchoff's 1978 film distinguishes between slices and projections of the hypercube. My 2006 book *Shadows of Reality, the Fourth Dimension in Relativity, Cubism, and Modern Thought* reviews the art and math history of the first few decades of the twentieth century (making some new discoveries) and reaches this conclusion: the projection model of the fourth dimension does all the work and the slicing model gets all the credit.

Shadows discusses the flatland model where three-dimensional objects pass through a two-dimensional world in analogy to four-dimensional objects that pass through our three-dimensional world. This model, popularized by Edwin Abbott Abbott's 1884 book *Flatland*, dominated popular thought until well into the twentieth century. As Henderson has amply demonstrated in many papers and books, the whole of European culture was fascinated by a hidden reality: atomic structure, what was revealed by the new X-rays, radio waves penetrating everywhere, and things, from atoms to ghosts, that are hidden in the fourth dimension. The flatland or slicing model explained this world beyond.

But soon after the turn of the twentieth century, papers by Washington Irving Stringham (1847–1909) and Victor Schlegel (1843–1905), and especially books by Esprit Jouffret (1837–1904)—among others—were passing beyond mathematical circles to enter the general culture as well as artworld. These math papers and books examined the projection model, based on projective geometry, which is intuitively understood as shadows from a higher dimension. My book *Shadows* shows that it is this projection model that Pablo Picasso (1881–1973) used to further his creation of Cubism (perhaps it should be called Hypercubism). Picasso used four-dimensional geometry to free himself from the tyranny of the surface, the skin, to show the psychological reality within. *Shadows* then shows that it is the projection model that is the basis of Hermann Minkowski's (1864–1909) spacetime formalism of Special Relativity. *Shadows* continues by discussing such challenging contemporary physics topics as Twistors, Entanglement, Category Theory, and Quasicrystals. Each of these topics owes far more to the projection model of 4space than to a flatland model; indeed, the flatland model of higher dimensions plunges one into hopeless confusion when thinking about these topics. *Shadows* also rediscovers the remarkable polymath T. P. Hall.[1] I was motivated to explore the implications of the distinction between slices and projection by Banchoff's pioneering 1978 film.

The relentless assault on commonsense reality provided by new technologies in the early twentieth century needed the artist to make the new reality, and the fourth dimension, comprehensible and stable. Heavier-than-air ships could not possibly fly, just think about it! said the famous mathematician Simon Newcomb (1835–1909). But what a mistake it was to ever adopt the more reasonable space+time slicing model of four-dimensional reality over projective spacetime, the less reasonable, more accurate, more giving model of another fungible *geometric* dimension fully inserted into our familiar width, length, and height. Projective geometry is essential to a vision of an invisible, fecund, immaterial primal extra-dimensional soup that makes reality.

[1] My hero, the great polymath Thomas Proctor Hall (1858–1931) studied projective models of four-dimensional geometry to master his image of reality: as a physician, that X-ray could not only diagnose, they could cure; as a mathematician, that dynamic, telescoping, and hinged glass-tube models of hypercubes could show how planar rotation works 75 years before it was seen on the computer screen; and a science fiction writer, that stories of trans-material essential beings could teach us about the divine. He was, by turns, chemist, physicist, mathematician, physician, and writer. His life should be better known, but I think there are some secrets that cancelled his fame. I still want to write his biography.

4 Fat Topology

In 2014, I deconstructed a painting from 2006 for the online journal *Symmetry* in an article called "Topology and the Visualization of Space" https://www.mdpi.com/2073-8994/7/1/32. The journal's special issue "Diagrams, Topology, Categories, and Logic" was guest-edited by topologist Louis Kauffman. I met Kauffman years before when I took a booth at an AAAS meeting to show my quasicrystal sculpture. We had dinner and had a conversation that changed my life. He subsequently invited me to a small conference at the University of Minnesota's Superconducting Computer Center where I met Scott Carter, Jeff Weeks, Charles Gunn, and other topologists. This is some of what I learned from Carter: to braid one-dimensional threads you need access to three-dimensional space to pass the threads over and under each other, and to braid sheets, not ribbons but infinite sheets, you also need to have access to two more dimensions than the two-dimensional sheets themselves.

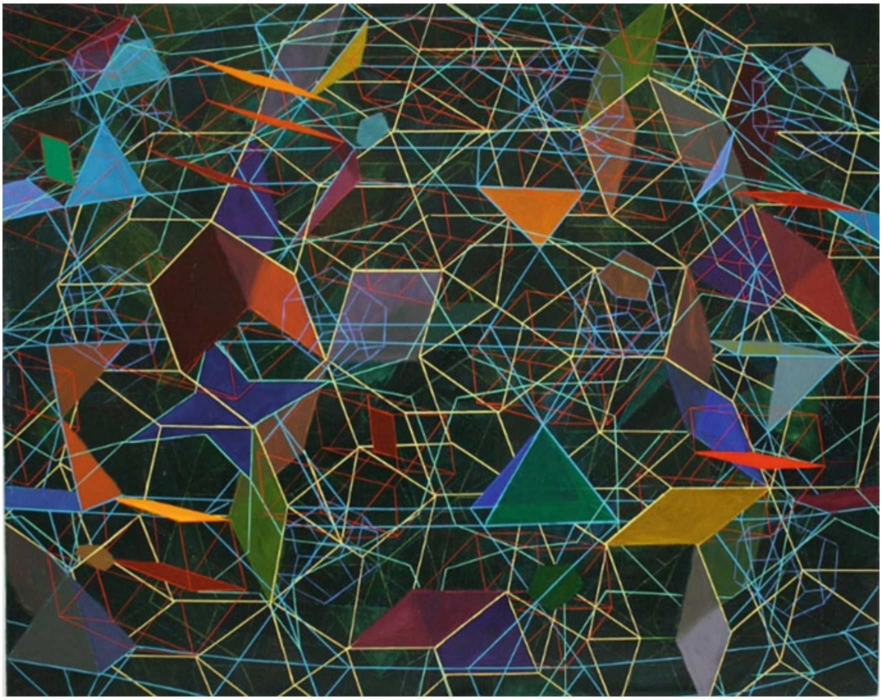
As a semi-pro four-dimensional geometer, unstructured, curving, and braided infinite sheets made me nervous. After renewing my friendship with Carter through invitations to speak at his University beginning in 1998, I took the flat patterns of my previous paintings and swooped them, curved them, interlaced them, and braided them. Like before, each patterned sheet was color-coded with a single color, and like before each sheet was defined by a different geometric pattern. But unlike before, each pattern had a thickness and was defined by three-dimensional polyhedra rather two-dimensional shapes. Fat sheets, fatter and fatter: not planes but hyperplanes, maybe even one could think of them as braided spaces.[2]

Almost 50 years ago, I read the following passage from Albert Einstein's fifth appendix (1952) to his popular 1916 book *Relativity: The Special and the General Theory*.

> When a smaller box *s* is situated, relatively at rest, inside the hollow space of the larger box *S*, then the hollow space of *s* is a part of the hollow space of *S*, and the same "space," which contains both of them, belongs to each of the boxes. When *s* is in motion with respect to *S*, however, the concept is less simple. One is then inclined to think that *s* encloses always the same space, but a variable part of the space *S*. It then becomes necessary to apportion to each box its particular space, *not thought of as bounded*, and to assume that these two spaces are in motion with respect to each other. Before one has become aware of this complication, space appears as an unbounded medium or container in which material objects swim around. But it must now be remembered that there is an infinite number of spaces, which are in motion with respect to each other. [Dover edition of *The Principle of Relativity*, pp. 138-9, emphasis added].

[2] How lucky to have discovered the programming language *Formian* just at this time. Developed by Hoshyar Nooshin at the University of Sussex at Guildford for the civil engineering department, *Formian* is a language of pattern generation, in an arbitrary number of dimensions, with many procedures to curve surfaces and lattices. I was invited to Guilford for a two-week seminar to learn this software and was presented with several versions of the language.

In my paintings from 1998 onward, braided lattices represent the multiple unbound spaces that Einstein wrote about, and that had a special resonance for me given the way I grew to see the world. Many spaces in the same space.



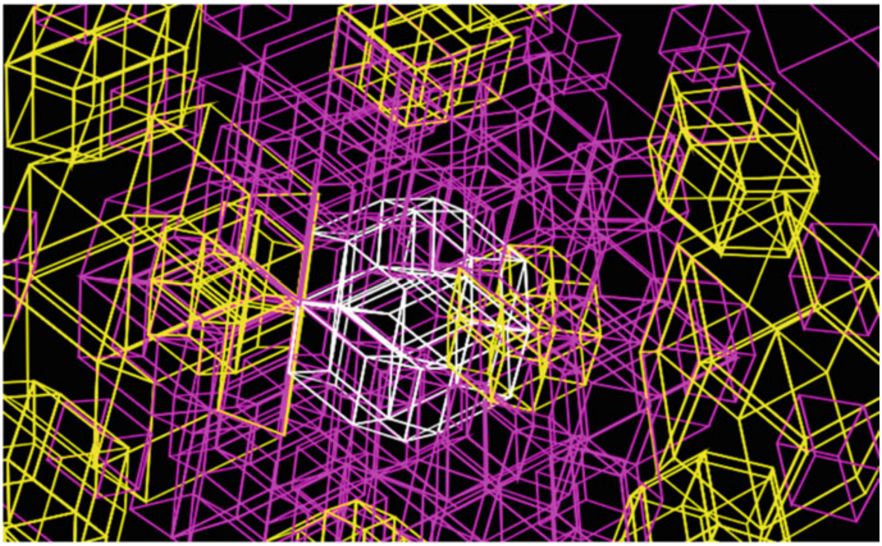
Tony Robbin, *2006–6*, 2006. Acrylic on canvas. 56 × 70 inches. Collection the artist

5 Quasicrystals

Reality is in higher dimensions; our experience is but a projection of that higher-dimensional reality. Such a statement smacks of religion or at least Platonism. But if the statement can be divorced from those associations, a new understanding of modern physics and the richness of projective geometry can be had.

For those interested in four-dimensional geometry, especially my friends Koji Miyazaki and Haresh Lalvani, quasicrystals had an odd familiarity. I went to Philadelphia to visit Paul Steinhardt to learn more. Steinhardt received me cordially, gave me the thorough papers he had written with his collaborators, and answered my letters as he coached me in learning to code the deBruijn algorithm for quasicrystals (Nicolaas Govert deBruijn,1918–2012). Later I met deBruijn himself, was a guest in his home, and we stayed up late talking quasicrystals. Steinhardt sought to discover foolproof local matching rules that obviated Roger Penrose's conjecture that, since no local matching rules existed, real, physical quasicrystals must self-assemble according to some quantum-like method.

Soon after I managed to write my Pascal programs for quasicrystals implementing the deBruijn algorithm, I visited George Francis at the University of Illinois/Champaign-Urbana. Francis threw me, without warning, into his honors class in mathematics, and I started talking about quasicrystals. There followed decades of collaboration, in which Francis and his students translated my archaic Pascal to Python and expanded my code to include features I wanted but did not have. Francis put the enhanced program in the "Cube," a virtual environment where one could wander at will through a quasicrystal, peering at its components, and gaining an intuitive feel for the structures otherwise unobtainable.



The Robbin/Francis program can be linked here: http://new.math.uiuc.edu/Tavares/htr4.html

Originally, I wanted to make quasicrystal architecture and sculpture, and did make large pieces for the Danish Technical University and the Jacksonville Library system, as well as for a number of temporary exhibitions. But then, I started thinking more and more about Penrose's conjecture of an application to quantum physics.

Altogether, I spent 35 years studying and programming the deBruijn algorithm for quasicrystals, and recently I have concluded that the algorithm *is* a geometric model of quantum non-locality. I had been discussing what is known as quantum information theory with Padmanabhan K. Aravind because Aravind was investigating the use of figures from projective geometry to model the fundamental quantum paradox of entangled particles. The deBruijn algorithm is also based on projection: a regular six-dimensional cubic lattice is projected to 3space in just the right way to induce a foolproof potentially infinite quasicrystal. In fact, the algorithm bounces back and forth between these two states: the six-dimensional *before*, and the three-dimensional *after*. If the algorithm represents a physical system, then it passes back and forth through what should be a one-way, entropy gate. But this is precisely the quantum paradox of entangled particles: impossible, spacelike separations of information are overcome!

I wrote this all up: https://www.researchgate.net/publication/343099608_The_deBruijn_Algorithm_for_Quasicrysals_as_a_Model_for_Quantum_non-Locality but the problem with making my argument convincing is that one has to get into the difficult nitty-gritty of the algorithm, to get your hands dirty as the mathematicians say. It is challenging to understand just how and why the algorithm works.

6 Something from Nothing

The conversation with Kauffman that changed my life was on the topic of how do you get from pure abstraction to physical reality: what John Wheeler called "pre-geometry," and Roger Penrose investigated with spin networks, and Carter, Kauffman, and Masahico Saito called Topological Lattice Field Theory. In analogy to the origin of life, where organic compounds spontaneously arise from inorganic ones fitted into a clay template, the origin of physical existence spontaneously arises from something that is nothing-at-all, and from that nothing-at-all to geometry, and from geometry to spacetime, and from spacetime to everything.

That physics could ask these metaphysical questions, that famous mathematicians begged for these questions to be answered and were working on it, and that I could have a considered opinion—all of this ennobled me. Going toe to toe with Kauffman at dinner at the AAAS meeting in Boston (must have been the one in 1988) gave me the impetus to study and speculate further.

For years and years, I had the notion, intuitive and vague, that four-dimensional space must be curved. Now, almost every mathematician will tell you that this is nonsense: four-dimensional and non-Euclidian are two separate geometries that have nothing to do with one another. And a mathematician might also suggest that an artist's fantasies could not possibly be connected to real thought. But as I later read in Felix Klein's (1849–1925) *Development of Mathematics in the 19th Century*, 1926 (who had a similar problem with his fellow mathematicians), if you happen to be *in* space, then infinity to the left and infinity to the right are the same point at infinity because in projective geometry there is only one point at infinity. And as Coxeter concludes: "Thus, if metrical ideas are left out of consideration, elliptical geometry is the same as real projective geometry" (1942, p.15).

There must be some set of logical relationships in the universe or else the universe would not be as consistent as it is. John Wheeler calls that set of logical relationships a "pre-geometry." Penrose's "spin-networks" defines that set as tri-valent diagrams of spins of would-be particles, diagrams that assemble to look like space. Penrose's spin networks are like an immaterial version of a mycelium underground that from place to place erupts in a fruiting mushroom. As John Baez writes in an important paper for the arXiv, that concept was powerful to those working on quantum loop theory because of loop theory's "insistence of a back-ground free approach." In other words, spin networks do not happen in space, they make space.

Carter, Kauffman, and Saito were influenced by spin networks when they collaborated on "Topological Lattice Field Theory." I spent a fair amount of time working to understand the theory for my book *Shadows*. Starting from a simple diagram of associations and noting that the associations change if the diagram is rotated or looked at from a different location, the authors build a multidimensional topological structure on which a universe could be built. "Pancher moves [extended] to 4 manifolds" is an argument made with diagrams that climbs the dimension ladder from two dimensions to four, and from logical associations to quantum gravity.

My thoughts about something from nothing center on the role of projective geometry in describing, no in *making*, reality. I learned this from Penrose's writings, and a couple of conversations with him: a light ray is more like a projective point than a line in space. A projective point is what an artist would call the completely foreshortened edge. If I am reading Penrose correctly, the Minkowski metric for spacetime (and special relativity) is the inevitable consequence of space being projective. And as I have argued, it is something like the N/2 projection of the deBruijn algorithm that permits the paradoxical logic of the quantum world. Say that it is the nature of physical space to be projective and all else fall out.

7 Shape Shifting in the 1970s

An artist's life experiences influence that artist's esthetics; sometimes art history forgets this. (Could it possibly be the same for a mathematician, even a little?) I grew up in Japan, Kansas City, Missouri, Okinawa, Teheran, and spent time in Frankfurt, and Andover before settling in New York. In each place, I was expected to be a different person: a juvenile delinquent, an athlete, a scholar, a hippie, an artist. When I was a professor at Trenton State College, I marveled that many of my students never went anywhere: maybe to the mall in New Brunswick, never to Manhattan. But the world came to them via the movies and television, and they had somewhat of the fluid identity that I needed in order to be welcome in the various societies I grew up in. Just after graduate school at Yale, I met Robert Jay Lifton, who interviewed young Japanese men after the war. He was astonished: where they once worshiped the Emperor, they now embraced Christianity; where they once hated their enemy America, they now loved Americans; where they once cherished their agricultural heritage, they now favored capitalism. Lifton coined the term "proteanism" to describe those young Japanese men, and suggested that their behaviors were admirably adaptive, and further, that identity could be far more fluid than we had been led to believe.

Linda Ronstadt sang country-rock in a boy scout shirt and hot pants, sang Gilbert & Sullivan in a bonnet, 50s torch songs in an evening gown, country-western in jeans, and Mariachi with flowers piled on top of her head. Lee Breur presented Sophocles as an African-American gospel service, and Dante's *Divine Comedy* with Bunraku puppets and Motown music. The Beatles started out as a Mod band from Liverpool playing American Rhythm & Blues, then they transitioned to music influenced by Indian Raga, to the goof of St. Pepper's Lonely Hearts Club Band, and finally to avant garde electronic music.

Thomas Berger's novel of 1964 *Little Big Man*, later a popular film directed by Arthur Penn and starring Dustin Hoffman, 1970, tells the story of a protean man who changes back and forth from White to Native American, usually to avoid threats to his life. At times, he is also a farmer, a gunslinger, a merchant, and a drunk. Hoffman truly inhabits the different identities with differences of costume, speech, and body language. The Dave Brubeck Quartet's biggest hit, and the bestselling jazz single ever, is *Take Five*. The title is not a command to take a short break, but rather is a reference to the non-Western 5/4 time that the quartet heard during a State Department sponsored tour of Turkey in 1958. Paul Desmond, writing for the group, made the most of the quirky rhythm, quirky to Western ears.

I use projective geometry to depict this protean worldview: many spaces in the same space.

And in these worried times, cross-cultural fertilization shapeshifting with its try-on identities is sometimes denigrated as imperialist cultural appropriation. But there is another way to look at it, a way of looking that we need now more than ever: see the commonality of sapiens as exemplified by the universality of pattern. We need to see common biology, not cultural differences. Mathematics is the same for everyone, too.

Finally, in a paraphrase of an adage of mathematical Chaos Theory: a bat shits in WuHan and 900,000 Americans die. The whole world is focused on every point on earth; every point of earth is projected on the whole world.

Tony Robbin, Gilboa 2020, during Covid lockdown.

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I must also praise two private tutors. John Swartz who patiently walked me through *Gravitation* by Misner, Thorne, and Wheeler. And Charles Scheim who helped me understand projective geometry when I was working on *Shadows.*

Appendix 1

To rotate a point in the xy plane around the origin by an arbitrary angle (a), and "multiplying on the left."

$$\begin{matrix} \cos(a) - & \sin(a) \\ \sin(a) + & \cos(a) \end{matrix} \times \begin{matrix} x \\ y \end{matrix} = \begin{matrix} x' \\ y' \end{matrix}$$

If plane xy is in 3-space, then $\begin{matrix} \cos(a) - \sin(a) & 0 \\ \sin(a) + \cos(a) & 0 \\ 0 \quad\quad\quad\quad 0 & 1 \end{matrix} \times \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} x' \\ y' \\ z \end{matrix}$

And this allows for two more rotations:

$$\begin{matrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{matrix} \times \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} x' \\ y \\ z' \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 0 & 0 \\ 0 & \cos(a) & -\sin(a) \\ 0 & \sin(a) & \cos(a) \end{matrix} \times \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} x \\ y' \\ z' \end{matrix}$$

Rotation in x, z rotation in y, z

If plane xy is located in 4-space, then

$$\begin{matrix} \cos(a) & -\sin(a) & 0 & 0 \\ \sin(a) & \cos(a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \times \begin{matrix} x \\ y \\ z \\ w \end{matrix} = \begin{matrix} x' \\ y' \\ z \\ w \end{matrix}$$

And this allows for a total of 6 rotations: *xy*, *xz*, *xw*, *yz*, *yw*, *zw*

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From *Vision* to *Perception*: Chardin's Eighteenth Century Cultural and Scientific Approach to Painting (and Soap Bubbles)

Carla Scagliosi

> *M. Chardin est de nos jours le Peintre, qui peut-être était le plus doué du talent de la couleur; ce sentiment en lui était exquis et ne s'est jamais affaibli. Il semblait qu'il avait les yeux disposés comme le prisme, pour composer les différents tons de tous les objets et les passages imperceptibles de la lumière à l'ombre: personne n'a mieux connu la magie du clair obscur.*
>
> *Antoine Renou, Éloge funèbre de J.-B.-S. Chardin (1779) [1]*

Among Chardin's *oeuvre*, *Soap Bubbles* has always been one of the most admired and commented paintings. This has occurred not only because it was the first (or one of the first) genre scenes that the artist realized,[1] but because its composition and theme, as well as the artist's role in the cultural context of eighteenth-century French painting, have favoured various and sometimes opposite interpretations.

As for many other Chardin's paintings, *Soap Bubbles* was replicated various times by the artist[2]; in his 1933 *catalogue raisonné* Wildenstein includes a dozen works featuring this subject ([4], pp. 166–167).

I am very grateful to Prof. Michele Emmer for having involved me in this publication and introduced me, few years ago, to the fascinating and inspiring world of soap bubbles. I thank him for his esteem and friendship, and for our precious talks and sharing of ideas.

[1] See, among the others, [2], p. 8.

[2] For this aspect, see in particular [3].

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The three known versions[3] (Figs. 1, 2 and 3), are dated by most scholars between 1733 and 1734[4]; they are later versions of a first painting that Chardin realized around 1733 and then exhibited in the Salon of 1739.

Differing in shape, dimensions and for some detail (like the presence of the honeysuckle leaves),[5] the paintings present a pyramidal composition given by the figure of the youth leaning from the sill and the plane of the parapet with its large stones. The colours are a harmonious composition of grey, brown and green, where some white spot (the soapy water in the glass, the shirt of the youth coming out from his waistcoat and detachable sleeves) and little touches of red and blue emerge. The richness and the thickness of the *impasto*, which is peculiar of Chardin's painting, gives an illusionistic three-dimensional shape to the figures and the objects and renders all the different textures and materials present in the picture. It is a plain and not overloaded composition, which leads the beholder's eyes to the main focus: the bubble.

Light is what orchestrates the whole scene and renders it coherent: light and shadows endow space, objects and figures with depth and solidity. The light comes from the left and illuminates the face and the arm of the young man leaning forward, projecting his shadow on the large blocks of the side wall. Light is reflected twice in the bubble, enhancing its spheric and three-dimensional shape while penetrating its iridescent film. "In the *Soap Bubble* the transparent, slightly distended globe at the tip of the young man's blowpipe seems almost to swell and tremble before our eyes" ([9], p. 50). Chardin used white and ochre brushstrokes to render the bubble's transparency and red and blue hints to enhance its variable shades.

The youth is not depicted with his cheeks swollen, though he is blowing air through the straw. He seems very concentrated in what he is doing, that is to control his action in order to reach the maximum bubble expansion without letting it burst. The little boy with the plumed cap in the background, his hands levering on the sill, is trying to lift himself up so as to have a better view. Being his eyes the only thing we can see of his face, our attention is focused on them, which in turn are fixed on the magic arising of the bubble.

As in all Chardin's genre paintings, action has been frozen: it is the very moment of a pause or stasis, a suspension of movements and gestures: "Like the toddler from behind the sill, we watch with bated breath the bubble straining at the end of its straw, at that precise, wobbly instant before it either bursts or breaks free and floats

[3] Now in the National Gallery of Art in Washington (Fig. 1), the Metropolitan Museum of Art in New York (Fig. 2) and the Los Angeles County Museum of Art (Fig. 3).

[4] Recently J. Patrice Marandel has asserted that "it is hardly imaginable that the artist would execute three versions of the same painting in a single year" and assumed that the one in Los Angeles could be later than the one exhibited at the Salon of 1739. In [5], p. 71.

[5] For deeper analysis, comparison, history of the three known paintings, see, among the vast scholarship, [2]; [6]; [7], pp. 205–210; [8], pp. 208–210.

Fig. 1 Jean Siméon Chardin, *Soap Bubbles*, 1733–1734, oil on canvas, Washington, National Gallery of Art, 1942.5.1

away. Chardin's brush has arrested on the canvas a moment of maximum instability and tension, evoking from viewers the wordless absorption of the painted figures".[6]

This feeling of silence and suspension, which is Chardin's signature style, is shared both by his still lifes and genre scenes. Chardin, in fact, treated genre scenes

[6] [10], p. 222. For a deeper analysis on the matter of absorption in Chardin's painting see [9].

Fig. 2 J. S. Chardin, *Soap Bubbles*, 1733–1734, oil on canvas, New York, Metropolitan Museum of Art, 49.24

as if he were painting still lifes, with the same obsessive attention to the palpable rendering of materials, light and composition. Thus, the solemnity which emerges from these artworks contributes to the sense of mystery that only he was able to translate into images.

In the traditional iconography, soap bubbles are intended as a symbol of life transience and vanity of the human things. Bubbles can be the symbol of whatever passes and fades away, "anything that is attractive but evanescent" ([10], p. 219), so they can also be associated with childhood and youth, as well as with the fleeting nature of love. Both references could be found in emblems,[7] still in fashion

[7] See, for instance, the book of amorous emblems *Ambacht van Cupido* by Daniël Heinsius (1613), in which the many "occupations of Cupid" can be read as allegorical interpretations of love features. See also [3, 11, 12].

Fig. 3 J. S. Chardin, *Soap Bubbles*, after 1739, oil on canvas, Los Angeles County Museum of Art, M.79.251

in Europe during the eighteenth century, and in contemporary works of art such as those of Lancret or Boucher, in which soap bubbles appeared as a frivolous amusement in line with the "lightness" of rococo painting.[8] "Soap Bubbles was both an admonitory image, for the youth is wasting his time, and a vanity, reminding us of the transience of human life and endeavour".[9] According to Conisbee, this message was reinforced by the *pendants* of *Soap Bubbles*, that are either scenes with similar games in which young people indulge (*The Game of Knucklebones* or *The House of Cards*, Fig. 4) or, conversely, examples of good behaviour (*The Little Schoolmistress*, Fig. 5).

[8] See for instance Nicolas Lancret, *L'Air*, 1730–1732, now in Waddesdon Rothschild Collections, UK; Jean Daullé, *La Souffleuse de savon* and *Le petit soufleur des bouteilles de savon*, both engravings from paintings by F. Boucher (1758), Stockholm, Staten Kunstmuseum.

[9] [2], p. 20. For an analysis on the interpretation of moral or allegorical meanings in *Soap Bubbles*, its pairs and the engravings taken from Chardin's paintings, see also [3]; [6], p. 225; [13], p. 123; [14], p. 126; [10], p. 222; [11], p. 165.

Fig. 4 J. S. Chardin, *The House of Cards*, probably 1737, oil on canvas, Washington, National Gallery of Art, 1937.1.90

The *Homo Bulla* iconography and its symbolism appeared in the second half of the sixteenth century and then spread widely in Northern Europe. It derived from the *adagio* by Erasmus of Rotterdam *Homo bulla est*, which the scholar took from

Fig. 5 J. S. Chardin, *The Little Schoolmistress*, after 1740, oil on canvas, Washington, National Gallery of Art, 1937.1.91

Latin sources and that was figuratively translated into the image of a putto blowing soap bubbles, symbolizing the transience and vanity of life.[10]

The *boy blowing bubbles* iconography became very popular in seventeenth-century Dutch painting, especially among the *fijnschilders*, the "fine painters" of the Leiden School. Artists like Gerrit Dou and his pupil Frans van Mieris (Fig. 6) influenced generations of artists that gave their own version of the theme, from Dou's pupils such as Pieter Cornelisz. van Slingelandt and Domenicus van Tol to Mieris' son Willem, to Caspar Netscher[11] (Fig. 7). These painters disguised the *vanitas* iconography in a genre scene, depicting children leaning at the window and intent in playing with soap bubbles.[12]

[10] Erasmus of Rotterdam, *Adagia* (Venice, 1508). See [15], pp. 42–44, and p. 62, note 5. See also [12].

[11] Among the others, Chardin must had seen Willem van Mieris' *Soap Bubbles* now at the Louvre, Paris (1710–1720, oil on panel, INV 1550) and a copy of Frans van Mieris' *A Boy Blowing Bubbles* (Fig. 6), but with the attribution to Caspar Netscher, that was in Joseph Aved's collection ([16], p. 38).

[12] [15], pp. 42–50 and [17], pp. 146–150.

Fig. 6 Frans van Mieris, A Boy Blowing Bubbles, 1663, oil on panel, Mauritshuis, The Hague, 106

In the first half of the century, the Flemish and Dutch Golden Age masters were very requested in France, especially from the 1730s, and many well-known collections and art dealers owned paintings and engravings that Chardin could have known.[13] The painter Jacques-André Aved, who was Chardin's close friend, was also an art dealer in Dutch and Flemish paintings and "traveled to the Lowlands regularly to procure 'old paintings' for himself and others".[14]

Chardin had already measured himself with Netherlandish still life painters like Willem Kalf or Jan Weenix, and could count on many sources of inspiration

[13] There was no difference, for eighteenth-century French critics and art *amateurs*, about Flemish and Dutch schools.

[14] [16], p. 37. Radisich's research on Chardin's cultural *milieu* and the relationships between his paintings and his contemporaries' reception of them, proves that he was an artist deeply rooted in his time. Moving from the previous scholarship ([11, 18], etc.), an accurate survey of the seventeenth century Dutch sources in Chardin's paintings gives new light to the artist's use of *pastiche*. This was a very fashionable eighteenth century term to define a modern painter's version, endowed with "*goût moderne*" and innuendo, of subjects and themes derived from the old masters, conducted in a witty and imaginative way, proof of the artist's genius and uniqueness.

Fig. 7 Caspar Netscher, Boy Blowing Soap Bubbles, 1670 c., oil on wood, Mauritshuis, The Hague, L120

among the paintings and prints of artists that were well represented in collections or publications. Chardin decided then to follow the art market demand and realized genre paintings with everyday life scenes, in the fashion of Dutch seventeenth-century masters, investing "these modest scenes [. . .] with some moral or didactic message" ([2], pp. 9–10). Eighteenth-century intellectuals discussing art compared Chardin to Rembrandt and called him "the French Teniers" ([6], p. 224) to remark the affinities with artists such as David Teniers, Gerard Ter Boch, Pieter de Hooch, etc. ([14], pp. 118–119).

Like those of the Dutch painters, Chardin's pictures are characterized by a solemn silence given by the suspension of time and action; all attention is focused on the inanimate still life elements he depicted so mimetically. Even human figures, in their freezing actions, are silent and poetic, thus generating a sense of wonder. It is not only a question of themes or compositions: as we will see, the main affinities with Dutch art lay on a similar scientific approach to painting, considered as a means of knowledge of the physical world.

Fig. 8 J. S. Chardin*, The Washerwoman*, 1733, oil on canvas, Stockholm, Nationalmuseum, NM 780

In the same years of *Soap Bubbles*, or maybe earlier,[15] the artist realized another representation of a child blowing bubbles in *The Washerwoman* (Fig. 8), painted as a pair for *Woman Drawing Water at the Cistern.*[16] As for *Soap Bubbles*, both these pictures show their debts with Dutch Golden Age painting for their iconography and composition.[17] In a humble interior, a woman is depicted while she is doing laundry. She has stopped her activity for a moment to look at her right, maybe taking a pause or because something caught her attention. Besides her, other isolated and silent figures are represented: a crouching cat, a sitting child blowing bubbles, another woman seen from the back, whose silhouette is enclosed in an adjacent space invaded by light, seen from a half-open door. She is immersed in a light that

[15] If these were Chardin's first attempts in genre painting. See note 1.

[16] Both in Stockholm, Nationalmuseum NM 780 and NM 781. Another version of *The Washerwoman* is now at The State Hermitage Museum in St. Petersburg, ГЭ-1185.

[17] See, among the others, [16], p. 40.

is palpable and enveloping thanks to Chardin's extraordinary pictorial rendering of the atmospheric particles.

The little child is painted in the foreground, sitting next to the large tub and is characterized by the same attitude of his older counterpart in *Soap Bubbles*: both figures are represented as absorbed while they are watching the bubble expanding from the straw they are holding.

The bubble is rendered by white, ochre and red brushstrokes. Chardin has succeeded in representing the thickness of the bubble at its bottom, giving it an oblong shape that recalls the presence of a drop, thus producing the effect that the bubble is going to burst within a short time. The artist faithfully reproduces the colours identified by Newton in his *Observations,* in which he stated that, at its thickest point, the bubble appears to be white ([19], p. 195).

Chardin's painting can be associated to the experimental practice of eighteenth-century "natural science" and the theories about optics and human perception.

Moving from the premises of Francis Bacon's empiricism, sixteenth- and eighteenth-century "natural philosophers" thought that mere speculation was not sufficient to gain knowledge. Natural mechanisms and phenomena had to be investigated through experience, by the means of experimentations that have to prove the veracity of the scientific assumptions based on abstract theories. Experiments and demonstrations had therefore a main and active role in knowledge, conceived as the final stage of a process that had to be verified and validated in its progress.

Isaac Newton can be considered as one of the founders of modern sciences and his studies on light and colours became very popular in the eighteenth century. He began his research on optics at the end of the 1660s and during the 1670s; the results of his experiments and observations were finally published in 1704 in his *Opticks, or, a Treatise of the Reflections, Refractions, Inflections and Colours of Light* [20].

Newton's theories were translated into French in 1720, commented and spread by popular scientific writings, such as those of Algarotti[18] and Voltaire.[19] Experimental demonstrations, conferences, public lectures, scientific theatrical performances contributed to their dissemination, not only among the intellectuals who animated the cultural debates, but also among lay people. In line with Enlightenment spirit and beliefs, in fact, science, culture and knowledge in general had to be universally accessible and shared. Performances, demonstrations and experiments were carried out with an everyday language and very simple equipment, allowing the non-specialistic public to understand their scientific contents and to repeat them at home, thus experiencing a direct and active learning.[20]

[18] Francesco Algarotti, *Il Newtonianismo per le Dame, ovvero dialoghi sopra la luce e i colori* (Naples, 1737).

[19] Voltaire, *Éléments de la philosophie de Newton mis à la portée de tout le monde* (Amsterdam, 1738).

[20] As Hosseini remarks, it i's interesting to notice that both words "experience" and "experiment" are translated into French as *expérience*, in [21], p. 19; see also [22], p. 82.

After using prisms, mirrors and lenses, Newton began to study light and colours phenomena by the means of soap bubbles. Unlike the prism, through which it is possible to observe the phenomenon of light refraction, he discovered that the iridescence of soap bubbles depends on the so-called interference. It "occurs when the thickness of the film is comparable to the wave-length of visible light. It is caused by the fact that, in soapy liquid, the different colours that make up sunlight move at different speeds".[21] The light falling on the bubble is reflected from two contiguous surfaces, the outer surface and the inner surface of the soap film: they are not totally mirror-like, so some light penetrates them, thus creating this particular physical phenomenon. As subjected to the force of gravity, which pulls the liquid downwards, the thickness of the soap film is not constant, but it gets thinner and thinner on top till the bubble bursts. In this time interval and in relation with these changings, the full range of colours appears onto the sphere superficies, in a succession of bands expanding and moving downward that create the peculiar iridescence of the bubbles: "Newton showed that the different thickness of soap film reflect different colours. Soap film appears white at its thickest, black at its thinnest, and as it becomes thinner it shows a series of hues" ([19], p. 195). In some conditions, the colour that will be visible on the soap bubble outer surface will be a mixture of red and blue, as Newton wrote in his *Observations* [20], pp. 20–27.

The artist's way of depicting soap bubbles (Figs. 9 and 10) follows Newton's theories: the hues of red and blue, the white brushstrokes at the maximus thickness, the two reflections of the light source on the spheric shape, all are pictorial translations of optical observations. Also, the glass with the soapy liquid on the sill is painted with blue touches on the bottom and hints of red among the shades, while its rim is enlightened by two white strokes that follow the direction of the light ray. As already noticed, the same attention is given to the bubble painted in *The Washerwoman*, with its white drop of liquid at the bottom.

In his essay on Chardin [25], Baxandall puts in relation Newton's research with John Locke's theories about perception and education, accurately illustrating the connections that can be found between the two intellectuals, the role they had in the scientific and cultural debate of the time and the characteristics of Chardin's painting that can derive from his embedding in this culture.

Newton's studies on light and optics, in fact, were part of a wider field of research that concerned perception in general and visual perception, sight in particular. He understood that the individual plays an active role in the process of perception, discovering that "colours [. . .] are a sensory product of the mind and do not exist in the object we see, nor in the light, which nevertheless allows us to appreciate them and which is the immediate object of vision" ([25], p. 116).

In the same years in which Newton was studying optics, the contributions of Philippe de La Hire[22] were published; the French mathematician (and painter!)

[21] [23], p. 17. See also [24], pp. 106–112.

[22] *Dissertation sur les différents accident de la vue* (1685) and *Traité de la Pratique de la Peinture* (posthumously published). See [25], pp. 130–132.

Fig. 9 Detail of Fig. 2

Fig. 10 Detail of Fig. 8

discovered that the colour of an object changes in relation to the different light conditions or to its distance from the eye.

In accordance with Newton's and La Hire's studies on vision, perception intended as a process of knowledge became a main interest also for John Locke; in his writings,[23] he investigated the interrelation of the senses and the ability of the human mind to collect experiences, turn them into comprehension and then into knowledge. Locke considered "the subject of perception, rather than the object perceived, as the central element of the question. [. . .] Man is not born already endowed with the ability to visually perceive intrinsic and primary details such as figure or form, i.e. with the knowledge to see, for instance, a sphere. It is through experience and the comparison of different sensations, in fact, that we learn to associate specific sensory perceptions with specific qualities of the substance" ([25], pp. 116–117). The eye perceptions of colours, shadows, shapes, must be intertwined with the other senses, first of all touch, to experience textures, materials, tactile sensations and to associate them to the visual stimuli. Chardin knew it well, as his painted objects, spaces and figures have those tactile features that make them appear absolutely real. His technique and style, gained by a strenuous workmanship, aim at reaching the most palpable and believable representation of reality. Renou's comparison of the artist's eye to a prism[24] proves that his contemporaries had understood the extraordinary relevance of his scientific, optical and perceptive approach to painting.

The connections between Newtonian theories and Chardin's painting are evident in *Soap Bubbles.*[25] Chardin starts from a double layer priming, obtained by a red-ocher layer and a light grey-brown one, which together produce "an optical gray, the purpose of which is to give vibrancy and depth to the painting" ([26], p. 23). The same careful workmanship is displayed for the colour mixtures used for rendering flesh, objects and garments, the wall and ledge stones. Chardin's colours are always made up of pigments combination, and what is interesting from the optical point of view is that this gives vibrant chromatic effects and tones that interrelate and contribute in creating a homogeneously lightened, believable scene. The quick and large brushworks make "these images appear to be observed from the corner of our eye. [. . .] With the image almost complete the artist scumbled light green—or rather a mixture of yellow, black and white pigments—over the background, blending it with the still-wet outlines of the primary forms to create a fuzzy atmospheric effect [. . .] [and] a very believable atmospheric space with light naturalistically modeling the forms" ([26], p. 24).

Chardin's paintings seem to be pervaded by air: "By depicting natural blurring, the diminution of details and the dissolution of strong contours, Chardin succeeds

[23] John Locke, *An Essay Concerning Human Understanding* (London, 1690) and *Some Thoughts Concerning Education* (London, 1693).

[24] "It seemed that his eyes were set like a prism", see [1].

[25] It is maybe not by coincidence that Chardin exhibited his *Soap Bubbles* in the Salon of 1739, only a year after Voltaire's book on Newtonian optics.

in capturing the natural perception of distant objects in his paintings. Due to the air atmosphere between the viewed object and the viewing subject, details become indistinct, colours diminish and the overall appearance becomes blurred" ([27], p. 203).

These features bring to mind two mechanisms of visual perception related to focusing: "adjustment", which allows to focus on foreground objects placed at different distances, and "optical sharpness", that "concerns the different gradations of the sensitive response at different points on the retina and determines the degree of sharpness within the visual field" ([25], p. 121). This is the reason why the peripheral vision is not as sharp as the central one, and sharpness cannot embrace the whole field of view, but just some selected elements.

Chardin's painted images, which seem to be "observed from the corner of our eye" ([26], p. 24), are built up by using sharpness and blurring, applying the distorting effects of human peripheral vision and focusing on light and its capability to give a credible setting to space and things, being the fundamental medium by which the visible is given. "Chardin's still life paintings seem hazy with few selected items in focus, mimicking the way the eye comfortably scans a tabletop with objects on it; the eye moves easily over those objects familiar to it and rests only on certain objects of interest" ([18], pp. 90–91).

Just as Chardin observed the objects of his representation from various distances so to render them in an overall view on the canvas and strip them of their excessively vivid details, the observer is required to have a similar approach. When in front of the painting, he cannot place himself in a fixed and privileged point of view, but has to "adjust" his vision moving closer and further, as he does while perceiving reality. "Through his movements in front of the picture, he imitates with his body the moving eye, which regulates the visual acuity through pupil and accommodation movements and reacts to proximity and distance" ([28], pp. 542–543).

The painting thus acquires the same value as a scientific experiment, since it becomes the means of recognizing and understanding the mechanisms of optical perception. In this sense, Hosseini finds an epistemic value in Chardin's painting, because images in general can be considered as a form of scientific dissemination that is accessible to all and immediately understandable ([21], p. 20). The experience of observing a painting that has been composed following the rules of perception involves the same physiological mechanisms used for deciphering reality, activating then a real process of knowledge in the beholder. Furthermore, in *Soap Bubbles*, the study of perception and colours through physical experimentation is also the *subject* of the painting and, while painting, Chardin too is experiencing human perception and knowledge.

Painting, scientific experimentation and playing have in common the element of curiosity, which must arouse questions and a thirst for knowledge. This is why, in Lockean theories on education, the component of play, fun and involvement became fundamental to the learning process and the education of children.

If we focus on the representation of the two characters in *Soap Bubble*s, we can notice that the boy is more sharply depicted, and this is not only because he is on the foreground or because Chardin is applying optical theories to the composition. Jasin considers the blurred features of the younger kid to be a reference to the Lockean definition of the child as a *tabula rasa* ([18], p. 100), an empty page that must be filled with life and experience. What Chardin depicts in his many scenes with children and adolescents, then, can be defined as the process of *Bildung* intended "as the production of the self" ([29], p. 258). In *Soap Bubbles,* the little kid is not *on focus*, yet, but he will.[26] Watching the elder boy blowing bubbles with an absorption similar to that of his companion, he is learning how to blow bubbles and the characteristics of the bubble itself. As soon as he learns how to do it, he is going to have an active part in the process, as the second straw in the glass seems to suggest. Parallel to the diagonal line of the one held by the youth, this straw is an element that Chardin includes to refer to the dimension of time. Its presence opens up to a series of thoughts about what will happen, in a succession of "narrative moments or ideas. Liquid is transformed into bubble; bubble is blown; child observes; child will pick up second straw and emulate what he has observed and experienced".[27] Moreover, the suspended time of painting is endowed with a pedagogical value, since it allows to freeze what is mutable in nature, thus enabling to perceive it better and better comprehend it.[28]

In line with the Lockean interpretation of *Soap Bubbles*, its *pendants* can be then read in a different way. *The House of Cards* (Fig. 4), as well as for *The Game of Knucklebones*, can be considered as a visual explanation of Locke's theories about learning through play. The boy playing with cards is experimenting gravity and balance; as Terpak states, "the youthful natural philosophers of Chardin's [. . .] paintings reveal the fascination with science that pervaded everyday life in 18th century France and England. They also subtly advance the Enlightenment idea that nature can be fully discovered by those willing to investigate her secrets" ([19], p. 196). A Lockean interpretation can be applied also to *The Little Schoolmistress* (Fig. 5). In this case, the pair can be read as the transposition in pictures of Locke's idea that the child is learning both by studying and playing. Another element conveys to this reading: the little kids that Chardin painted in the two scenes are so similar that they can even be the same child.

[26] See [29], p. 256: "This fetus-like figure, this Lockean *tabula rasa*, occupies the zero-point of knowledge preceding the maturation of meaning. Meanwhile the adolescent, crossing over the threshold, enters into that broad foreground space into which an ever-greater number of adult viewers will constantly arrive (following after the adolescent, as it were, yet always arriving before him) to inscribe their own, proliferating readings". See also [30], p. 153.

[27] [18], p. 102. See also [22], p. 82 for a different interpretation.

[28] Newton himself had to cover the bubble with a clear glass dome to protect it from air moving and colour mixing, and to let it last longer so to better observe it (see [20], p. 20). See also [22], p. 82.

1 Conclusion: From the Objective Sharp Vision to a Human Perceptive Experience

"*Trompe l'oeil*, with its supernatural form of vision, can take still life from mundane reality to hyper-reality, creating flawless, polished, pure and spectacular representations of objects. This phenomenon is at work in the studied formality of the [. . .] paintings from the seventeenth century Dutch Republic" ([18], p. 90).

What fascinates most in Dutch Golden Age painting is that illusionism reaches its highest level. The *trompe l'oeil* is the final result of a painting conceived through the seventeenth-century scientific approach and optical theories, in which a great impulse was given by Kepler's research.[29] The scientist perfected the knowledge of eye physiology and sight mechanisms, and consequently improved the studies on the functioning of lenses, promoting their use in empirical science. He associated the term *pictura* to the inverted image which forms on the retina, comparing the eye to a perfect *camera obscura* in which light projects the "representation" of the outside world. In this perspective, seeing means producing images (*ut pictura, ita visio*) ([15], p. 41).

The Dutch painters conducted their optical and perspective studies by the means of squared frames, mirrors, lenses and devices such as the optical cameras, peepshows, perspective boxes, and so on. The depicted image, so sharply clear and deceiving in its mimetic aspects, was the result of the idea that a faithful image of the world allows us to know it. Therefore, applying the laws that govern our mechanical way of seeing to painting, amplifying and enhancing the power of vision by the means of lenses, Dutch artists obtained an illusionistic objectiveness (Fig. 11). They reached amazing results in depicting textures and materials, reflections and effects of light on various surfaces, microscopic details and scientific reproductions of nature (from botanical or enthomographic drawings to topographical or chorographic representations) ([15], pp. 41–42). In Dutch painting, the protagonist is not the individual, but the eye: "The strength of his method lies in the deanthropomorphising of vision. [. . .] It is a dead eye, and the model of vision, or if you will of painting, that is proposed to us, is a passive model".[30] The process of vision goes from the outside to the inside, to the object to the subject, through a mechanical process which is passive and does not imply any active involvement.

Eighteenth-century studies discovered and demonstrated that the act of merely *seeing* is not sufficient to gain an experience (and therefore a knowledge) of the perceptible world. To reach knowledge, it is necessary to elaborate on the notions that arrive from all the senses. These *data* coming from the outside through perception must be collected, related to each other and connected with our previous

[29] *Ad Vitellionem paralipomena quibus astronomiae pars optica traditor* (1604, in particular in chap. 5, *De modo visionis*) and *Dioptrice* (1611).

[30] [31], p. 55. See pp. 44–141 for a deeper analysis.

Fig. 11 Willem Claesz. Heda, *Still life with a Gilded Beer Tankard*, 1634, oil on panel, Amsterdam, Rijksmuseum, SK-A-137

experiences, to be recognized, elaborated and treasured. Therefore, we do not simply *see* the world, but we *perceive* it, and we can even perceive it differently according to the physical, physiological or even psychological conditions in which we are at that moment. Perception is an act that involves the individual as a whole and, as such, implies a certain level of subjectivity. It is no more a matter of objects and objectivity, of universal and abstract laws that mechanistically govern a pure transferring of *data* from the outside to the inside. On the contrary, eighteenth-century science recognized the active role of the individual in the process of perception: this goes from inside to outside, from the subject to the object. The object is invested with qualities and features by the "perceiving agent," as he activates perception mechanisms and the consequent "recognition" (or comprehension) of reality. If the world is subjectively perceived and not objectively given, what can be represented is not so much the object in its tangible and external materiality, but rather the perception of the objects and the qualities our experience attributes to them (Fig. 12).

In eighteenth-century painting, vision becomes an experience that, although common to men in their being rational and "sensorial" beings, presumes an individual, personal component. This individual vision of the world, which substitutes what

Fig. 12 J. S. Chardin, *Still Life with a White Mug*, c. 1764, oil on panel, Washington, National Gallery of Art, 1972.9.6

is *seen* with what is *known,*[31] what is *objective* with what is *perceived*, will lay the foundations for the artists' claim for subjectivity in nineteenth century, finally derogating from universal laws in favour of a personal interpretation of the world and, consequently, of art.[32]

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[31] "Chardin's ability to render the objects as we *know* them rather than just as we *see* them, insists on the interconnectedness of touch and sight in the process of perception and of painting", in [18], p. 91.

[32] It is not by chance that Chardin was rediscovered and celebrated as one of the fathers of French art by nineteenth-century art critics and artists; painters like Manet recognised the modernity of his approach to art and reserved him a special place among their sources of inspiration.

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Part IV
Architecture and Mathematics

Andrea Palladio and Zaha Hadid

Michele Emmer and Fulvio Wirz

1 Palladio, Villa *Malcontenta*

> We motored out to tea at *Malcontenta,* by the new road over the lagoons beside the railway. Nine years ago Landsberg found *Malcontenta*, though celebrated in every book on Palladio, at the point of ruin, doorless and windowless, a granary of indeterminate farm-produce. He has made it a habitable dwelling. The proportions of the great hall and staterooms are a *mathematical paean*. Another man would have filled them with so-called Italian furniture, antique-dealers' rubbish, gilt. Landsberg has had the furniture made of plain wood in the local village. Nothing is 'period' except the candles, which are necessary in the absence of electricity.
>
> R. Byron, *The Road to Oxiana*, [1] (Byron, 1937)

Robert Byron was an English travel books writer who died in 1941 during World War II, while serving as a correspondent for a London newspaper on a ship from Scotland to West Africa that was torpedoed by a Nazi U-boat.

Byron is refering to the famous villa *La Malcontenta*, one of the most famous villas created by Andrea Palladio (Andrea di Pietro, son of Pietro della Gondola, born in Padua on November 8th 1508, called Palladio). The villa is located on the banks of the river Brenta, it is not very far away from the lagoon and the center of Venice (Fig. 1).

> If you can build near the river, it will be very comfortable and beautiful; because with little expense at any time it will be possible to transfer with the boats any kind of goods from

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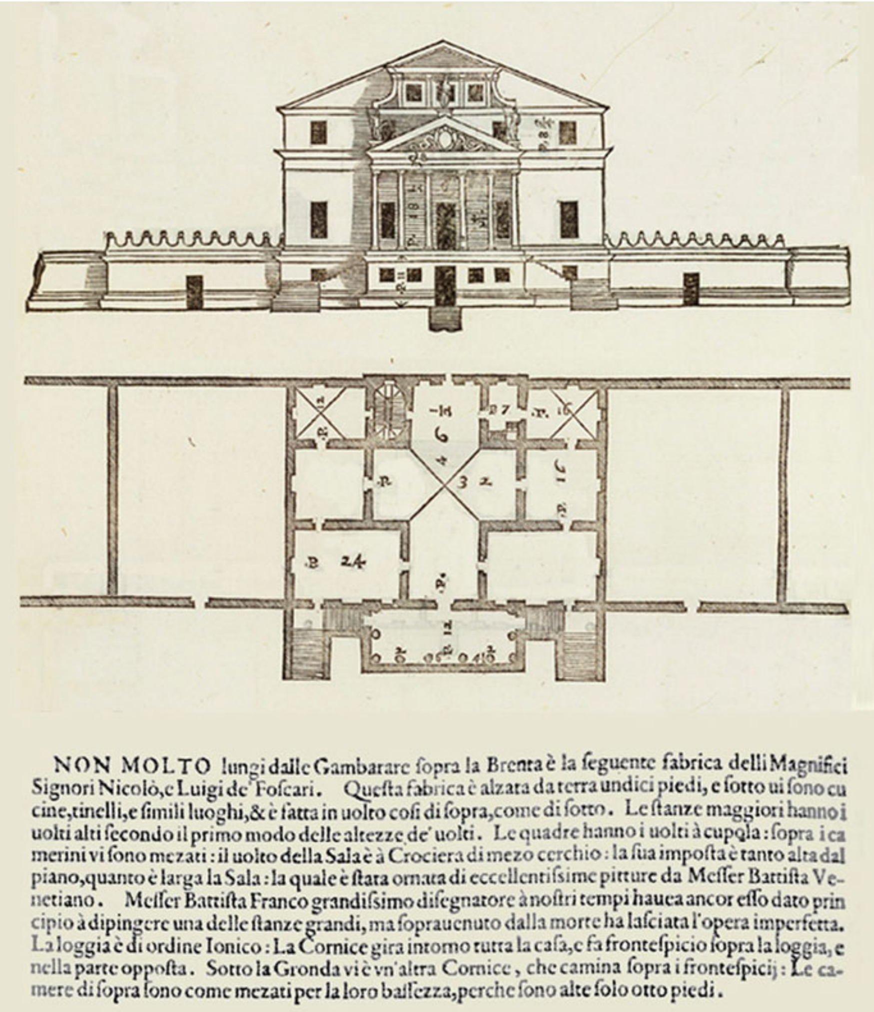
Fig. 1 Andrea Palladio, *Villa La Malcontenta*, 1556–1559 (© photo Matthias Schaller, courtesy Villa La Malcontenta S. r. l.)

the town, and the villa will serve for the uses of the house and the animals, as well as to bring very fresh summer, and it will have a beautiful view, and with great utility and ornamentation it will be possible to water the possessions, the Gardens, and the *Bruoli (vegetable gardens)*, which are the soul and pleasure of the Villa" wrote Palladio in the book II of the *Quattro libri dell'architettura* (Four Books of Architecture) in chapter XIV *Dei Disegni delle Case di Villa di alcuni Nobili Veneziani (The draughts of Several Country-houses built by noble Venetians*) [2, 3].

And so he describes the villa: (Palladio, Andrea & Leoni, Giacomo, 1715)

Near the *Gamberare* (a small village) on the *Brenta*, is the following building, which is the house of the magnificent Lords *Nicolò* e *Luigi* de' Foscari. The house is raised eleven feet from the level of the ground, and below are the kitchen, pantries, and the like places. Everything is arch'd as well above as below. The arches of the great chambers are made after our first manner. Those of the squares are arch'd like a cupola. On the closets are *mezanini*. The hall is arch'd half round grinded: its imposts as high from the floor as the breadth of the hall, which is excellently painted by Messer *Battista Venetiano* Messer *Battista Franco*, one of the best draughts men of our time, did also begin to paint one of the great chambers, but he dy'd before he could finish his work. The portico is of the *Ionic* order. The cornice goes around the whole house, and makes a pediment above the portico, as well as on the opposite part. Under the eves of the roof there is a second cornice, which passes above the pediments. The upper rooms are like *Mezaninos*, because of the little height they have, which is about eight foot (Fig. 2).

Nicolò Foscari, who was also the owner of the famous *Ca 'Foscari* palace on the *Canal Grande*, now home of the University by the same name, wanted a villa to be



NON MOLTO lungi dalle Gambarare ſopra la Brenta è la ſeguente fabrica delli Magnifici Signori Nicolò, e Luigi de' Foſcari. Queſta fabrica è alzata da terra undici piedi, e ſotto ui ſono cucine, tinelli, e ſimili luoghi, & è fatta in uolto coſi di ſopra, come di ſotto. Le ſtanze maggiori hanno i uolti alti ſecondo il primo modo delle altezze de' uolti. Le quadre hanno i uolti à cupola: ſopra i camerini vi ſono mezati: il uolto della Sala è à Crociera di mezo cerchio: la ſua impoſta è tanto alta dal piano, quanto è larga la Sala: la quale è ſtata ornata di eccellentiſsime pitture da Meſſer Battiſta Venetiano. Meſſer Battiſta Franco grandiſsimo diſegnatore à noſtri tempi hauea ancor eſſo dato principio à dipingere una delle ſtanze grandi, ma ſopraucnuto dalla morte ha laſciata l'opera imperfetta. La loggia è di ordine Ionico: La Cornice gira intorno tutta la caſa, e fa frontespicio ſopra la loggia, e nella parte oppoſta. Sotto la Gronda vi è vn'altra Cornice, che camina ſopra i frontespicij: Le camere di ſopra ſono come mezati per la loro baſſezza, perche ſono alte ſolo otto piedi.

Fig. 2 A. Palladio, I quattro libri dell'architettura, chapter XIV Dei Disegni delle Case di Villa di alcuni Nobili Veneziani, table 34. [4]

built along the river Brenta in an isolated place, not a farm-villa but a real residence that was easily reachable by boat from Venice. It is believed that the project was entrusted to Palladio between 1556 and 1559. The building was certainly completed also in internal decorations in 1566 when Giorgio Vasari visited the villa.

The nickname of *Malcontenta* (never satisfied) came later, perhaps because of a noblewoman, Elisabetta Dolfin, of the important Dolfin family. Cardinal Giovanni Dolfin bought the *Ca 'Dolfin* palace in Venice in 1621 where until a few years ago the mathematics department of the Ca' Foscari University was located. And the young Elisabetta, a widow, married Nicolò Foscari, who, given the rumours about his wife's infidelity, closed her up, although she declared herself innocent, in the villa on the Brenta until her death.

Palladio decided to direct the main façade of the villa towards the north, towards the river, transforming the south façade into a sort of wall of light, with large windows that let in natural light.

Among the sources of inspiration of Palladio *Il tempio ch'è sotto Trevi* (The temple under Trevi), as he calls it in the book IV of *I quattro libri di architettura* (The four books of architecture), the temple of the sources of Clitunno which is located in Umbria, just near Trevi [4]. Chapter XXV of the fourth book describes the temple of which Palladio made four tables, including one dedicated to the half of the façade. More interesting is the design that the architect does not include in the fourth book in which the façade is complete. The resemblance to the north face of the *Malcontenta* is undoubtedly significant.

Probably the temple was originally a shrine dedicated to the river god *Clitumnus* that was described by Pliny the Younger in his book [5]: "Adiacet templum priscum et religiosum. Stat Clitumnus ipse amictus ornatusque praetexta; praesens numen atque etiam fatidicum indicant sorties. (Next to it [to the river] stands an ancient and venerable temple in which is placed the river-god *Clitumnus* clothed in the usual state robe; and indeed the prophetic oracles here testify the immediate presence of that divinity".

Considered one of the most interesting early medieval monuments in Umbria, it is among the seven jewels of art and Lombard architecture in Italy included in the list of *UNESCO World Heritage sites*. Recent studies have allowed to limit the chronology of the building to the Lombard period, with an oscillation between the beginning of the seventh and the full eighth century (Fig. 3).

In the monograph *Palladio: tutte le opere* (Palladio: all the works) Paolo Marton, Manfred Wundram and Thomas Pape observe [6]: "If we look at the design of Palladio we see that the temple rests on a mighty basement interrupted in the middle by a portal; on this rises a portico surmounted by a pediment framed with shelves: In the *Malcontenta* on the façade facing the river, we find, even if in a slightly varied form, those same elements, ie the very tall basement, the columned portico and the framed pediment from shelves. "And they add "Palladio's choice of harmonies that follow mathematical rules is again evident: from the basic unit of measurement the volumes increase progressively, in proportion to the previous ones."

And it is the mathematical rules, the classical proportions that Palladio uses reinventing them continuing to fascinate architects all over the world for hundreds

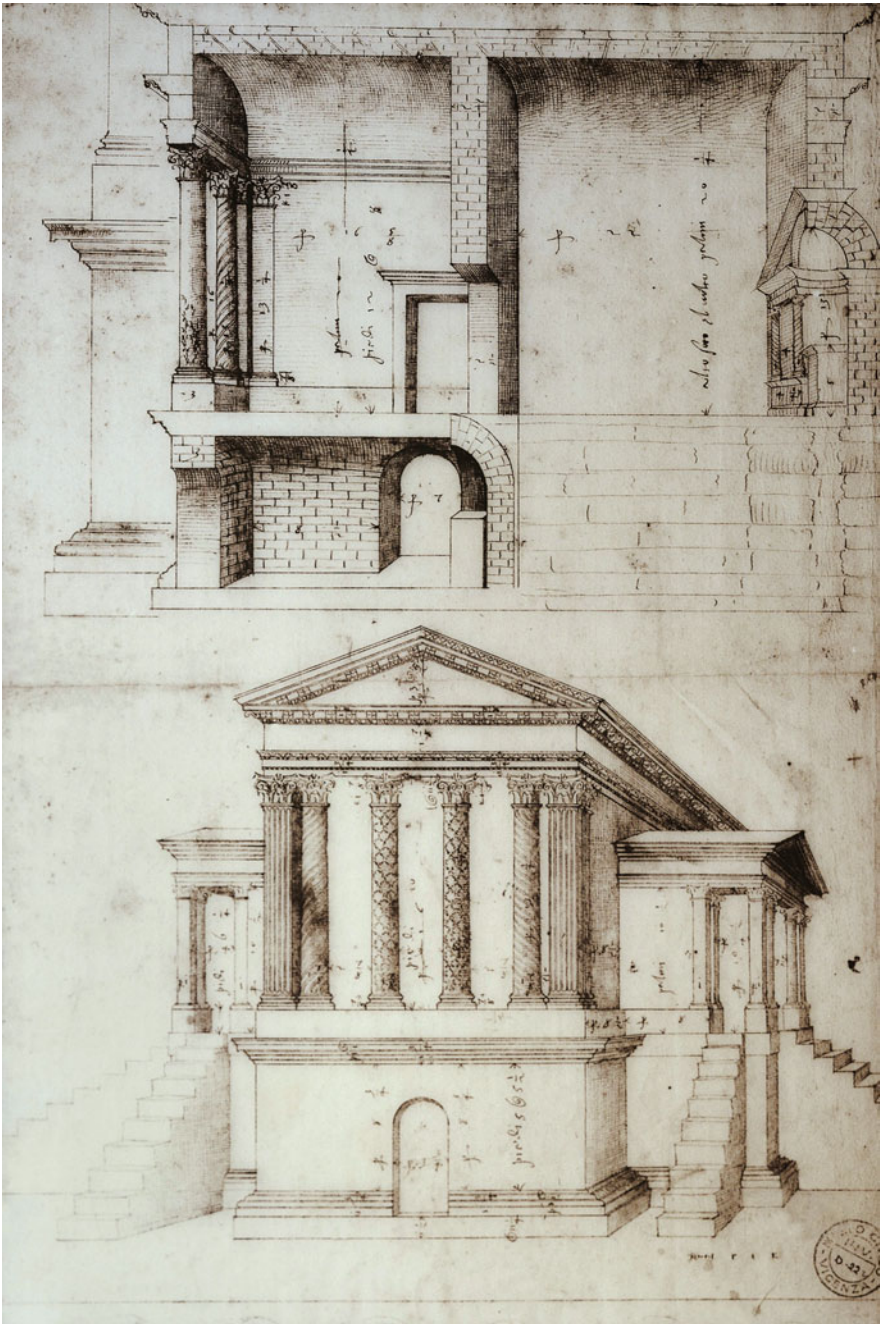


Fig. 3 A. Palladio, *Prospetto e spaccato prospettico del tempietto del Clitunno presso Spoleto*, drawing, Museo Civico Palazzo Chiericati, Vicenza. Photo credit: DeA Picture Library, licensed to Alinari

of years, including one of the most important architects of recent years, who has made fluid architecture and new digital technologies his credo: Zaha Hadid.

2 Venezia, Mostra Internazionale Di Architettura, 2008

The 2008 Biennial had as general title *Architecture Beyond Building* and was organized as always in different sections; one of these, with its own catalogue was called *Experimental Architecture*. *Zaha Hadid Architects* participated. This is what Patrik Schumacher, co-founder of the Zaha Hadid Atelier, wrote in the catalogue on the relationship between innovation and tradition in contemporary architecture [7]:

> Architecture is often associated with longevity, primordial archetypes or eternal values; this is the vision that you have outside of the discipline. Inside originality is the fundamental criterion of self-evaluation in the architectural field ... which implies the necessity of a permanent adaptable innovation ... The experimentation must not be arbitrary; it only makes sense if framed within a paradigm that is a guide to a collective research effort.

Zaha Hadid Architects also participated in the section *Eventi speciali e collaterali* (special and side events) with the project *Andrea Palladio and Contemporary Architects: Zaha Hadid and Patrik Schumacher* [8]. This was a very appropriate theme in the city of Venice where Andrea Palladio realized so many important works. One theme in particular concerned Hadid and Schumacher: that of the villa, which although built for obvious reasons on the mainland, was an important part of the lagoon town. Schumacher wrote a few years earlier on the theme of the Villas of Palladio in the volume *Digital Hadid. Landscape in Motion* [9]:

> The Analogy of building and organism is as old as the self-conscious discipline of architecture itself. Traditionally, the analogy focused on key ordering principles like symmetry and proportion. These principles were seen as integrating the various parts into a whole by means of setting those parts into definitive relations: In this conception, the organism is approximating an ideal type which implies strict rules of arrangement and proportion for all parts. It also assumes a state of completeness and perfection. The organism is a closed form: nothing can be added or subtracted: The Palladian Villa is perhaps the best example of this idea of the organism as an ideal of perfect order.

Our projects remain incomplete compositions, our concept of organic integration does not rely on such fixed ideal types. Neither does it presuppose any proportional system, nor does it privilege symmetry. Instead, integration is achieved via various modes of spatial interlocking, by formulating soft transitions at the boundaries between parts and the means of morphological affiliation."

The 2008 Biennale was an opportunity for a direct confrontation with the Villa par excellence of Palladio, *La Malcontenta*. Giulia Foscari, curator of the exhibition of Zaha Hadid Architects, which took place right in *Malcontenta*'s rooms, wrote in the Biennale catalogue: [10]

> Palladio's architecture is the built manifestation of Palladian utopia, consisting in the synthesis of all humanistic values through the definition of the exact role and relationship of each component of the architectural composition, from the organism as a whole to each

> individual environment. The proportions of each room are determined by a series of specific "harmonic" relations deriving from the Euclidean mathematics practiced in the 16th century ... The *Malcontenta* represents the ideal context for this investigation, as it was conceived and built by Palladio as a manifesto that demonstrated to the *Serenissima* Republic of Venice the perfection of its architectural theories ... The frequency curves generated by the system of harmonic proportions of the villa are subjected, through mathematical algorithms, to a progressive transformation aimed at defining an elementary genotypic form that contains in its DNA the whole set of Palladian rules. The result of this experimentation consists in the generation, through a series of respectful variations of the classical Palladian proportions, of multiple complex environments. The natural equilibrium of *Malcontenta* is *shaken* by the dynamic component introduced by Zaha Hadid and Patrik Schumacher, who have long rejected Euclidean mathematics, at the basis of Palladian theories of proportions—which could only lead to the definition of a single singular relational system, 'perfect'—to explore the potential of advanced digital techniques.

This explanation is not entirely convincing, because visiting the exhibition one had the distinct impression that experimenting with the application of mathematical algorithms to the perfect Palladian proportions wanted rather to claim for the digital experimentation a sort of nobility of origin, to continue in the wake of an idea of architecture that certainly always refers to originality without forgetting its history. And therefore not a *dynamic shake-up of* the harmony of the Villa but rather a reinterpretation in the light of new technologies and topological interpretation of architectural fluidity, of which Zaha Hadid has often spoken, of Palladio's innovative architectural ideas which in turn are rooted in classical architectural culture. So the place could only be within one of the highest expressions of the *mathematical harmonies* of the buildings of Palladio. It should also be noted that Zaha Hadid received a degree in mathematics from the *American University* of Beirut before moving to London.

This is the way in which the project was described by its creators, in which some words take on a somewhat different meaning, especially in the conclusions [11]:

> The Aura installation for the 2008 *Venice Biennale* represents a dialogue between the fluid contemporary language of the Zaha Hadid studio and the mathematical principles of harmonious architectural composition of Andrea Palladio. The work focuses on the *piano nobile* of Palladio's Villa Foscari *La Malcontenta*, which encapsulates his theory of perfect form. Accordingly, the proportions of the sequence of spaces provided the starting point for Zaha Hadid and Patrik Schumacher's study.
>
> The frequency curves generated by the harmonic proportional system of the villa were progressively transformed, through mathematical algorithms, to define an elementary form that contains in its DNA the full set of Palladian rules. As a result of this experimentation, multiple complex spatial environments were generated based on lawful variations of Palladio's classical proportions. The natural equilibrium Palladio achieved in the design of *La Malcontenta* is thus shaken by a dynamic component introduced and made possible by advanced digital design technologies.
>
> With a *mathematical integrity* rooted in Palladian algorithms, the graceful curves of Aura reflect the structure of this ethereal space in a contemporary formal language and materials. Demonstrating the generative potential of Palladio's proportional system, a second installation was developed for the adjacent symmetrical room. The two installations—*Aura L and Aura S*—were presented together, both generated by a contemporary translation of Palladio's harmonic system—at five centuries' remove.

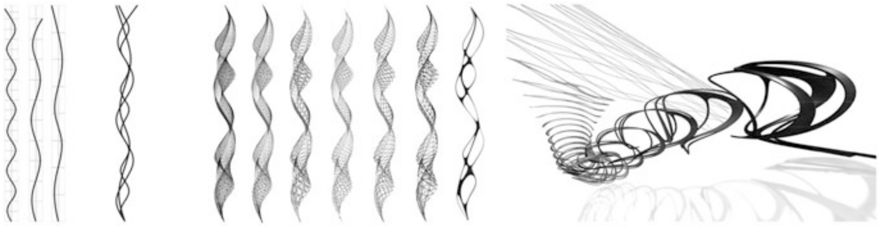


Fig. 4 *Harmonic Projects* for the *Aura Project* (2008) Courtesy of Zaha Hadid Architects

Mathematical integrity is the key to reading through the algorithmic generation of the intervention of Hadid and Schumacher. A mathematical procedure, the algorithm is applies to the initial data which are the *mathematical harmonies* of the Villa *Palladiana*. An algorithmic procedure. Precisely mathematical, allowing to reach at new geometric proportions that start from the initial Palladian data.

The malleability of the digital process is wisely used not only in reinterpreting the Palladian proportional system but also in the subtle interplay of transition and contrast between the perfectly circular envelope of the front elevation and the sinusoidal articulation of the longitudinal profile. The experience of circumnavigating the work thus becomes itself the transient manifestation of the evolution from the geometric perfection of classicism to the apparently chaotic complexity, but in reality perfectly controlled by complex algorithms, typical of computational design.

And the experimentation carried out by the two architects, the various steps that are taken to reach an aesthetically interesting result, are just *harmonic experiments*, as they are called. Which will lead to new forms with their own harmony (Fig. 4).

A few years later, again in Venice, Hadid and Schumacher were asked to organize an anthological exhibition of paintings, projects, objects that had been recently conceived and partly realized until 2016. The best place was Venice again, as part of the *Biennale di Architettura* (Architecture Biennal). In one of the Venetian historical buildings of relevant architectural interest.

From 27 May to 27 November 2016 at Palazzo Franchetti in Venice there was a large exhibition dedicated to Zaha Hadid Architects. Hadid designed the exhibition by choosing works, paintings and projects to be included in the exhibition's itinerary. She died on March 31, 2016.

In the introduction to the exhibition, Schumacher stressed that the publication of the book *Digital Hadid* in 2004 remained an important milestone in his attempt to reflect on the genealogy of the digitally generated style he called *Parametricism* at the *Venice Biennale of Architecture* in 2008. He felt the need to give a name to the new language or style of architecture that had been formed by the strong convergence of an entire generation of young architects since the nineties [12] (Fig. 5):

> My 2004 thesis focused on the pre-digital desire for complexity and fluidity as a motivating force for the introduction of certain digital tools drawn into architecture from the realms of computer graphics, movie animation and scientific simulation . . . These tools are the ever

Fig. 5 Zaha Hadid Architects, ***Project****Aura*, **Client:** Venice Biennale, Fondazione La Malcontenta. 2008. Photographs by Luke Hayes at La Villa Malcontenta

expanding set of algorithms that shape, discipline, and rationalize our design in unexpected and sometimes even counter-intuitive ways. These tools have become truly generative and intelligent, augmenting our design capacity in profound ways.

Schumacher wrote always in 2004 [13]:

There is an unmistakable new style *manifesto* within avant-garde architecture today. Its most striking characteristic is its complex and dynamic curve—linearity. Beyond this obvious surface feature, one can identify a series of new concepts and methods that are so different from the repertoire of both traditional and modern architecture that one might speak of the emergence of a new paradigm for architecture.

Schumacher reports some of the answers that Zaha Hadid gave about the role of design media in general and digital media in particular in an interview with the Chairman of the *Architectural Association of Schools of Architecture Mohsen Mostafavi* [14].

It recalls and praises the practice of design, of the importance even in the digital age, of using design as an expressive method and to formulate one's ideas:

> I still think that even in our later projects, where the computer was already involved, the 2-dimensional plan drawings are still seminal. I still think the plan is critical. The computer shows what you might see from various selected viewpoints. But I think this doesn't give you enough transparency, it's much too opaque. Also, I think it is much nicer on the screen that when it is printed onto paper, because the screen gives you luminosity and the paper does not, unless you do it through a painting. Further, I think if you compare computer renderings with rendering by hand, I must say that you can improvise much more with hand drawing and painting . . . Only 1920s *Modernism* really discovered the full power and potential of drawing as a highly economic trial-error mechanism and an effortless plane of invention—in fact inspired by the compositional liberation achieved by *abstract art* in the first decade of the twentieth century. Drawings accelerate the evolution of architecture. Modern architecture depends upon the revolution within the visual arts that finally shook off the burden of representation. Modern architecture was able to build upon the legacy of modern abstract art as the conquest of a previously unimaginable realm of constructive freedom . . . Abstraction meant the possibility and challenge of creation. Through figures such as Malevich and vanguard groups such as *De Stijl* movement, this exhilarating historical moment was captured and exploited for the world of experimental architecture.

Schumacher comments: "Abstraction implies the avoidance of familiar ready-made typologies. Instead of taking for granted things like houses, rooms, windows, roofs . . . Hadid reconstitutes the functions of territorialisation, enclosure and interfacing by means of boundaries, fields, planes, volumes, cuts, ribbons . . . One of Hadid's most audacious moves was to translate the dynamism and fluidity of her calligraphic hand directly into equally fluid tectonic systems. Another incredible move was from isometric and perspective projection to literal distortions of space and from the exploded axonometry to the literal explosion of space into fragments, from the superimposition of various fisheye perspectives to the literal bending and meltdown of space. All these moves initially appear as plenty illogical, akin to the operation of the surrealists." (Figs. 6 and 7).

Those words of Schumacher became at the 2008 Biennial the project of experimental architectures, where a large part had the paintings which Zaha Hadid created, paintings of architecture as utopias, before starting to realize her ideas as finished architectures, including the project of the link between contemporary architecture and Palladio within the *Malcontenta*. Another project was always presented at the *Venice Biennal*. A project which linked the forms of the Villa of Palladio mediated by mathematical algorithms to the other projects that Zaha Hadid and Patrik Schumacher presented in the *usual* rooms of the *Biennale* at the *Arsenale*, inside the city of Venice: *the Lotus Project* [15].

"Instead of representing a system already domesticated through internal rules, the *Lotus* room seduces through the folds of undulating rhythm, its exclusions, its reconfigurability and its ability to remain outside of categories.

Fig. 6 Zaha Hadid Architects, ***Project****Aura*, **Client** Venice Biennale, Fondazione *La Malcontenta*. 2008. Photographs by Luke Hayes at *La Villa Malcontenta*

Fig. 7 Zaha Hadid Architects, ***Project****Aura*, **Client** Venice Biennale, Fondazione *La Malcontenta*. 2008. Photographs by Luke Hayes at *La Villa Malcontenta*

3 Conclusions

There was no doubt that it was necessary to modify formal language in order to use geometric instruments that were up to graphic and pictorial intuitions: [16]

> Computer technology, i. e. the new digital design tools, have had an important and increasing influence on the work of *Zaha Hadid Architects* over the last years. This concerns primarily the handling of increasingly complex geometries within the designs. However, the desire for such tools to be imported from the animation industry originated in the fact that the tendency towards complexity and fluidity was already manifest in the work before those tools were available.

All this leads to a new conception of space. Schumacher clarifies: [17]

> These techniques lead to a new concept of space which suggests a new orientation, navigation and inhabitation of space . . . The significance and the ambition of these projects is that they might be seen as manifestos of a new type of space. As such, their defining context is the historical progression of such manifestos rather than their concrete spatial and institutional location . . . including the legacy of modern architecture and abstract arts as the conquest of a previously unimaginable realm of constructive freedom.

Schumacher concludes: [18]

> All compositions are seen as tasks for creative organic interarticulation. A refined organic architecture resists easy decomposition, a measure of its complexity.

The comparison with Palladio was inevitable and it was fruitful. The revolutions even in architecture must have a solid foundation from which to start. As the mathematicians of the second half of the nineteenth century have taught, in particular Henri Poincaré, there is no geometry which is better than another, there are only different geometries. Some, like the Euclidean, seem more useful than others. But geometry, the idea of space, architecture, even the so called classic continue to offer benchmarks and challenges even to digital fluid architecture, an architecture that would seem indeed far from the Palladian ideals.

Addendum

An homage to Zaha Hadid was held at the *Imagine Math 6: Mathematics and Culture* conference in Venice in March 2017, in the place of the exhibition of the previous year. The two presentations by Gianluca Racanà, Zaha Hadid Architects, and Michele Emmer, are published in a special session of the Proceedings of the Venice conference, M. Emmer and M. Abate, eds., *Imagine Math 6*, Springer Nature verlag, Switzerland, 2018.

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16. in [9], p. 8
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18. in 9 p, 32

Sergio Musmeci and the Calculation of the Form

Tullia Iori

Sergio Musmeci was one of the four *heroes* of the twentieth-century Italian School of Engineering, along with Pier Luigi Nervi, Riccardo Morandi, and Silvano Zorzi. Musmeci was an eccentric and countercultural engineer, the most visionary of the four.

Today, Musmeci is most famous for being the designer of the Basento River Bridge in Potenza (1967–1975), probably the most beautiful bridge in the world.

The bridge is an equicompressed *nameless* form. Musmeci designed it in accordance with his theory: when an engineer designs, the form must be the only unknown. The shape that responds perfectly to given constraints and loads, according to Musmeci, can be calculated mathematically.

The bridge over the Basento is the last masterpiece of the Italian School of Engineering; it is the final moment of its glorious adventure [1], reaching its apex during the *economic miracle*. In those years, Morandi and Zorzi, together with all the best engineers, designed the *Highway of the Sun*; Nervi built his famous works for the Rome Olympics in 1960 and for the celebrations of the Unification of Italy in 1961 [2]. At that time, Musmeci was still too young.

His most productive years coincided with those of the crisis, which began slowly in 1963 and in the following years worsened without end. He designed many works, but few of his projects were built. Others remained on paper: *missed chances*, testifying to Musmeci's relentless design curiosity.

Musmeci died young, at only 55. At that moment, everyone felt to have lost a genius who could have given so much more to Italian engineering.

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1 The First Experiences

Musmeci was born in 1926 in Rome. In November 1948, he graduated in civil engineering from the University of Rome La Sapienza. He then enrolled in a master's program in aeronautical engineering to postpone his mandatory military service. At the time, this was a typical trick among very good students. In January 1953, he graduated from the master's program and then served in the Army Air Corps. During his military service, he married Zenaide Zanini, who later became his partner in the design studio.

In the meantime, he served his apprenticeship in the technical office of the company Nervi & Bartoli, owned by Pier Luigi Nervi. Thus, he became Nervi's favorite. In 1954, he founded a design studio together with Antonio, Nervi's eldest son, who graduated in architecture in 1950. That studio did not last long. Soon Antonio no longer had time for his friend Sergio because he had to help his father, who had begun to build the works for the 1960 Rome Olympics.

At the same time, Musmeci embarked on an academic career. In 1954, he began as voluntary assistant at the School of Architecture at the University of Rome La Sapienza (Architecture, not Engineering; Nervi also taught all his life at the School of Architecture). He later became a tenured assistant and was appointed to teach *Rational Mechanics* course. Beginning in 1969, he taught *Bridges and Large Structures* course, delivered to lucky students who were allowed to choose the course in their elective exam package [3].

Early in his design career, Musmeci was above all a beloved design accomplice. He was able to solve complex structural problems, helping the greatest Italian architects of his time. Just to mention the most famous ones, he collaborated with mutual satisfaction with Adalberto Libera, Eugenio Montuori and Leo Calini, Giuseppe Vaccaro, Francesco Palpacelli, and Carlo Mollino.

In some cases, those architects left Musmeci complete autonomy and entrusted him with part of the project, asking to design the form from scratch. In other cases, the project was already defined and the architect called for Musmeci's help to solve problems of stability.

The first fun exercises were two roofs, which were never built. In 1954, together with Vaccaro, Musmeci designed the roof of a rural market in Puglia, composing some parabolic vaults. Through a few analytical calculations, he demonstrated with satisfaction that his geometric composition was perfectly funicular to the loads. The architectural form, therefore, responded perfectly to the static functionality. Musmeci exercised his mathematical expertise in another project in 1955: the roof of the *Araldo* movie theater in Rome, which completed a block of houses designed by Carlo Ammannati. The plan of the cinema was almond-shaped; in the roof, Musmeci weaved a large number of polygonal arches. He established that the number of crossings between the arches should be minimal along the axis of symmetry; the *golden number* was used to establish the position of the pillars in the plan; the vertical height of the crossing points between the arches was the unknown; it was determined mathematically with a system of equations, introducing

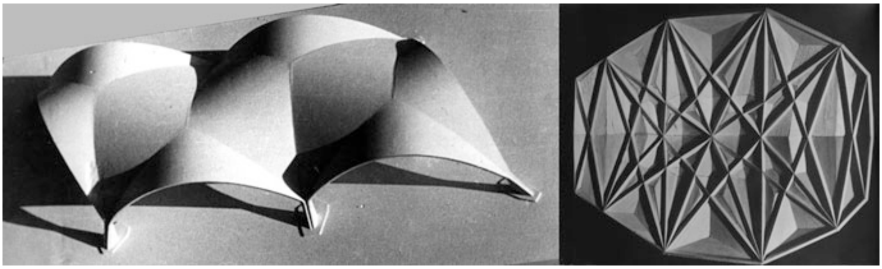

Fig. 1 (left) Rural Market in Tressanti, S. Musmeci, G. Vaccaro, 1954. Courtesy Private Archive G. Vaccaro, Roma (SIXXIdata); (right) The roof of the Araldo movie theater in Rome, S. Musmeci, C. Ammannati, 1955. Courtesy MAXXI Architecture Archive (SIXXIdata)

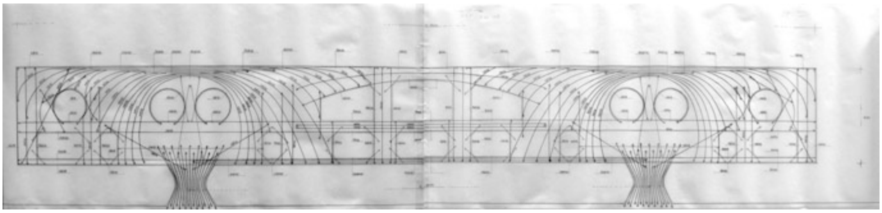

Fig. 2 Palazzo della Regione in Trento, 1956–1962, S. Musmeci, A. Libera. Courtesy MAXXI Architecture Archive (SIXXIdata)

the new condition that all the thrusts of the arches should be equal; maximum static efficiency had been achieved. The result of this apparently overly mathematical way of designing is incredibly beautiful: the roof resembles a *modern* Gothic vault (Fig. 1).

In the project for the *Palazzo della Regione* in Trento (1956–1962), on the other hand, Libera decided to span a large central beam against two enormous pillars, visible at the level of the square. That beam supported the cantilevered office floors, symmetrically on both sides.

Musmeci drew with maniacal care the voids and the solids of that beam and the reinforcement bars accordingly; however, in the end, the concrete casting concealed the beautiful web of rods. Disappointed, Musmeci understood that evidently the shape of the beam did not correspond to the natural flow of the internal forces; therefore, his careful design work had become invisible (Fig. 2).

Musmeci's most independent research began thanks to Annibale Vitellozzi, who in the mid-1950s was the consultant to the Italian National Olympic Committee (CONI). Since the 17th Olympics—to be held in 1960—were assigned to Rome, Vitellozzi had many works to do. He involved Musmeci in several projects; the most important was the roofing of the restaurant of the Swimming Stadium in the sports area called *Foro Italico* in Rome (1958). Musmeci designed a corrugated roof: he repeatedly folded the ceiling, obtaining resistance through the form. A folded sheet of paper resists much better than the same sheet left flat; the natural

Fig. 3 The roofing of the Restaurant for the Swimming Stadium, 1958, S. Musmeci, A. Vitellozzi. Courtesy Coni Archive (SIXXIdata)

pattern of internal forces guided Musmeci in establishing the shape. Thus, he varied the amplitude of the corrugation in tune with the variation of stresses: the greater the stress at a point, the greater the amplitude of the fold. He designed a didactic structure: even a non-expert user could understand the behavior of that structure; the stress of the members and the consequent static solution were clear to everyone (Fig. 3).

In the roof, the result of the calculations is no longer transferred to invisible quantities (the reinforcements inside the concrete casting) but to quantities evident in the form: "the degree of explicitness of the nature of the structure has definitely increased," commented Musmeci.

"Giving visible form to the bending moments": Musmeci did many tests to achieve this goal. In previous years, he experimented with his folding solution in smaller works: the roof of the *Raffo* industrial plant in Pietrasanta, Tuscany, designed with Calini and Montuori (1956); the roof of the *San Pietro* Cinema in Montecchio Maggiore, near Vicenza, designed with Sergio Ortolani (1957); and the roof of the gymnasium in Frosinone, together with Uga de Plaisant (1958). The walkways and the covering of the foyer of the *Regio* Theatre in Turin (1966), the last in order of time, designed for Carlo Mollino, are the most famous (Fig. 4).

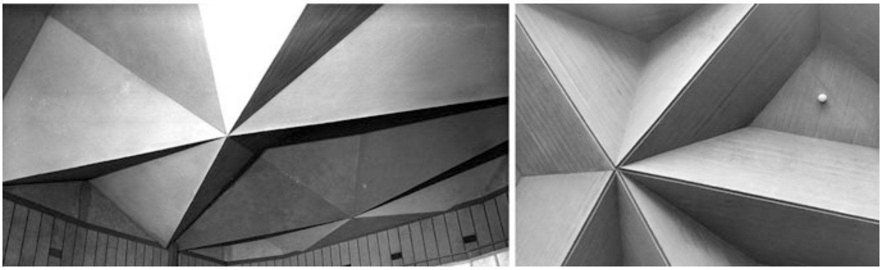

Fig. 4 (left) The roof of the San Pietro Cinema in Montecchio Maggiore, Vicenza, 1957, S. Musmeci, S. Ortolani (SIXXIdata); (right) The foyer of the Regio Theatre in Turin, 1966, S. Musmeci, C. Mollino. Photo Alessia Sisti (SIXXIdata)

2 Networks of Beams

At the end of the 1950s, Musmeci, boosted by these early successes, attempted the synthesis of form and structure by other means.

In the design of his folded surfaces, he noticed that the reinforcement was concentrated almost all along the edges; so he decided to simplify the structure, keeping only the edges, which were the only ones really useful. What remained was a network of beams.

Musmeci conceived many projects according to this criterion. In 1960, he designed with Calini and Montuori the roof of the Auditorium of the *Bhabha Atomic Research Centre*, in India. The roof is a shallow pyramid on a hexagonal plan; the six inclined pitches of the roof are generated by a triangular mesh of ribs (Fig. 5).

Musmeci told an anecdote about the project. He was able to estimate the behavior of that roof with a few calculations made by hand, introducing some simplifications. He submitted his report, not feeling that more complete and lengthy calculations needed to be added.

The Indian evaluating committee had never seen a structure so conceived. So they asked for a comprehensive calculation that took into account all the elements of the structure. Musmeci reluctantly set up the new calculation; he obtained a system of 122 equations, impossible to solve by hand. The commission then asked for a scale model; load testing on the model was very expensive. In the end, the scale model perfectly confirmed Musmeci's initial intuitions, based on his simplified calculations. This experience was very fruitful for the engineer: it proved to him the value of intuition in structural design.

In the same year, Musmeci designed the breathtaking roof of the *San Carlo* Church in the *Villaggio del Sole* district in Vicenza (1960–1962), whose general project had been elaborated by Ortolani.

The roof of the church is a kind of inverted funnel. On the outside it appears perfectly smooth; on the inside, however, three different families of logarithmic spirals, crystallized in the concrete, move toward the center, rising like a whirlwind. The spirals embrace the community of worshippers, favoring the participation of

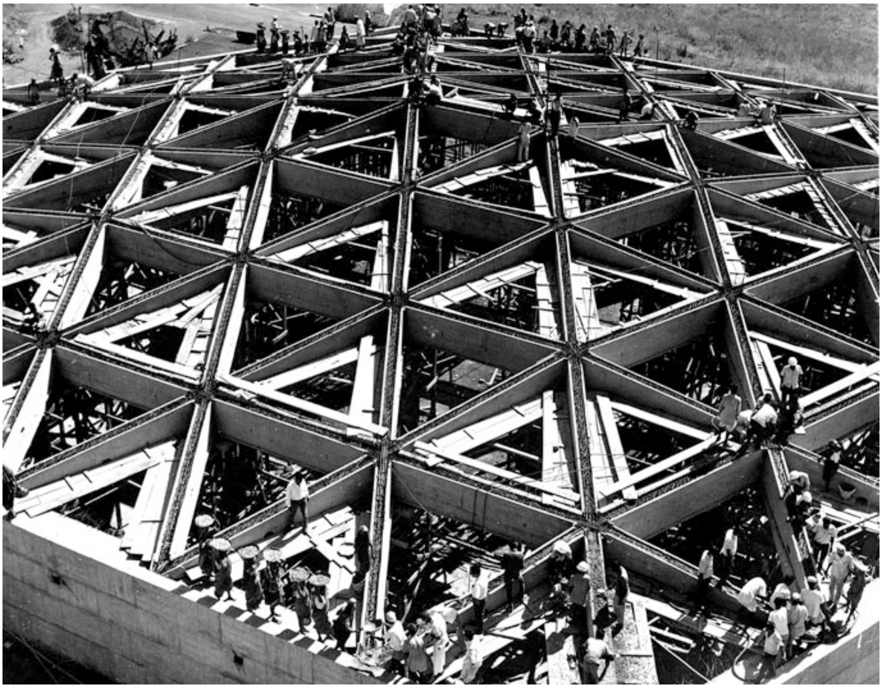

Fig. 5 The roofing of the Auditorium of the Bhabha Atomic Research Centre, India, 1960, S. Musmeci, L. Calini, E. Montuori. Courtesy MAXXI Architecture Archive, Montuori Archive (SIXXIdata)

the assembly around the altar; the center of the spirals, at the top, coincides in plan with the position of the altar. The shape of the structure is adapted to the liturgical requirement.

At each point three spirals intersect, with angles whose tangents are equal to 4, −1, 1/4. The mesh of the spirals is triangular. Pure mathematics: and the effect is spectacular. The stress on the spiral ribs changes gradually from the center to the edge: at the center, the behavior is that of a membrane; then it transforms and gradually bending takes over; on the edges, the ribs behave like cantilever beams. These variations in behavior are reflected in the gradual increase in the thickness of the ribs: small in the center and much larger on the perimeter. The construction site was quite complicated, creating very stormy relations between the designers and the construction company (Figs. 6 and 7).

The Marian Temple overlooking Trieste (1963–1967), designed together with Antonio Guacci, also belongs to the *triangle phase*. In the church, the usual elements of architecture—the pillars, the beams, the roof—disappeared. The exterior facades, the roof slabs, and the interior walls have been transformed into ribs, whose triangular meshes are closed by panels. The striking texture is entirely in reinforced concrete. For verifications to the stress of the *bora* wind, a model is tested at *Ismes*, the Experimental Institute for Materials and Structures of Bergamo (Fig. 8).

Fig. 6 The roofing of the San Carlo Church, Villaggio del Sole, Vicenza, S. Musmeci, S. Ortolani, 1960–1962. Photo Sergio Poretti (SIXXIdata)

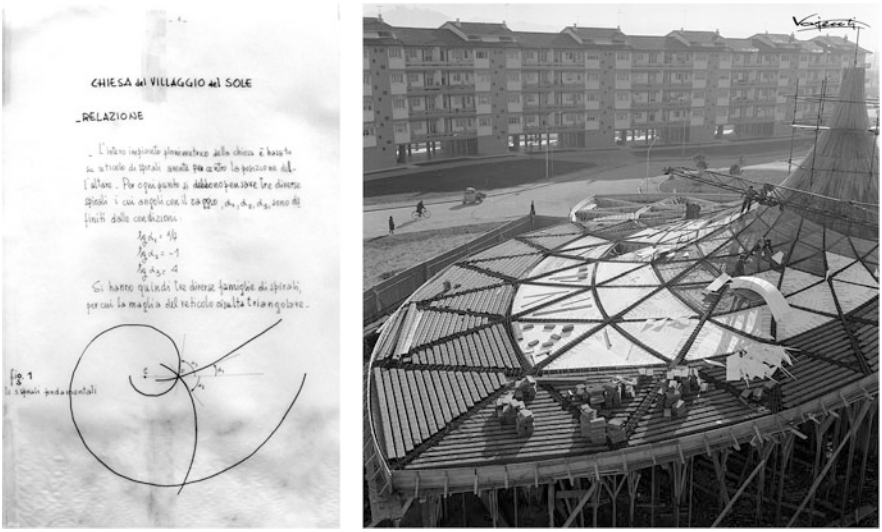

Fig. 7 The roofing of the San Carlo Church, Villaggio del Sole, Vicenza, S. Musmeci, S. Ortolani, 1960–1962: (left) Calculation report, 1960. Courtesy MAXXI Architecture Archive; (right) The construction site, 1961, Vajenti Photo (SIXXIdata)

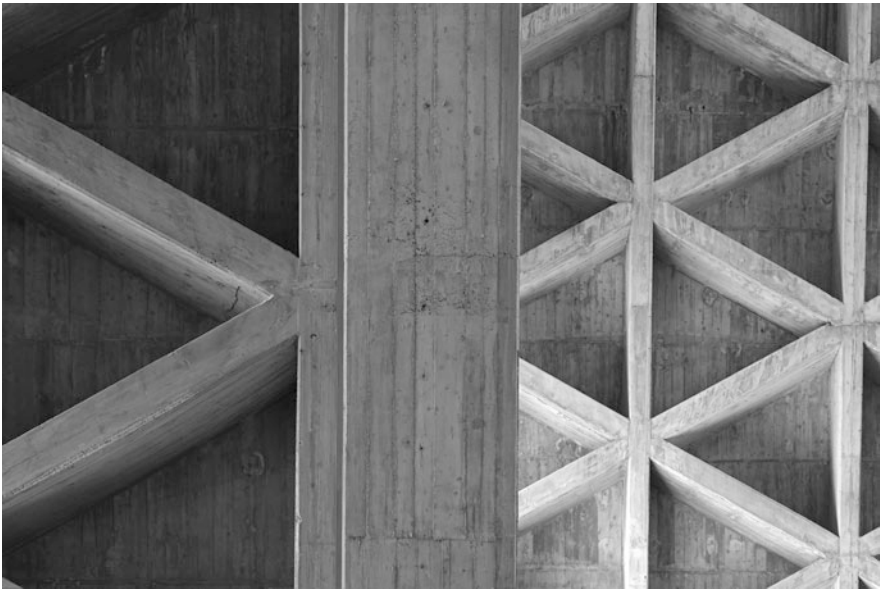

Fig. 8 The roofing of the Marian Temple, Trieste, 1963–1967, S. Musmeci, A. Guacci. Photo Sergio Poretti (SIXXIdata)

Musmeci returned to studying spatial networks of rods even in the last part of his life. But at this stage, the composition of triangles was very difficult to compute. In 1956, Turner, Clough, Martin, and Topp published the famous paper on the finite element method. Between 1954 and 1957, John Argiris systematized tensor notation and applied it to the mechanics of structures. Olgierd Zienkiewicz's textbook did not appear until 1967. To easily compute his reticular textures, Musmeci should have used finite element meshes. He was, however, ahead of the research that later became the future of automatic computation of structures.

3 Equicompressed Minimal Surfaces

Musmeci continued to work on beam networks for a long time. In 1964, he participated in the competition of ideas for the bridge over the Lao River for the Salerno-Reggio Calabria Highway. His project did not win but was later awarded by Inarch, the Italian Institute of Architecture.

The general shape of the bridge is a double-curved surface, uniformly compressed. The shape was suggested by a soap film model, which materialized the uniformly compressed members. Perfectly flexible wires, subjected to simple normal stress, located the edges of the membrane. The soap film placed tension on the wires determining their geometry. The resulting surface was uniformly stretched.

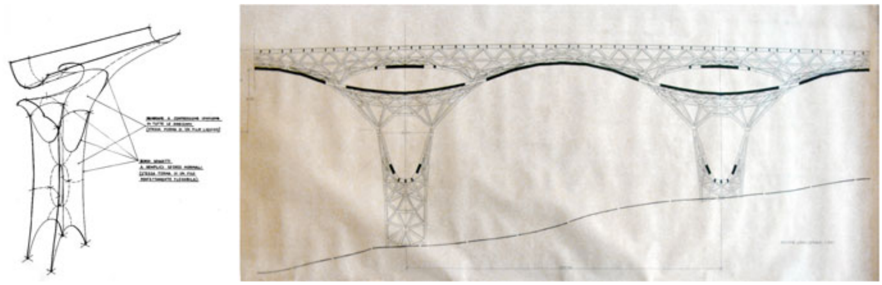

Fig. 9 Bridge over the Lao River for the Salerno-Reggio Calabria Highway, 1964, S. Musmeci: (left) the soap film; (right) the triangular mesh. Courtesy MAXXI Architecture Archive (SIXXI-data)

Musmeci showed that the surface described the minimum area compatible with the fixed contour, achieving the condition of maximum material economy (Fig. 9).

Soap bubbles are *minimal surfaces*: the bubble finds the state of minimum energy and assumes the shape that minimizes its area. In architecture, minimum area means, with the same thickness, minimum volume and therefore minimum weight and minimum consumption of material [4].

In the project submitted to the competition, a spatial reticular structure approximated the shape identified thanks to the soap bubble. Musmeci designed a network of linear elements, with constant section, uniformly compressed, in a triangular mesh.

In that project, Musmeci investigated the minimal surfaces with wisdom, because he had already elaborated several projects with this technique. Musmeci's research, in fact, had changed again, remaining on the frontier between mathematics, geometry, and theory of structures.

After the folded slabs and the networks of discontinuous beams, he returned to continuous surfaces, where the loads flowed more naturally and softly.

The genesis of that new experimentation can be traced back to the project for the bridge over the Astico river in Chiuppano, near Vicenza, whose tender was closed in August 1956. Musmeci designed the bridge following a rudimentary process of optimization; the two-dimensional curve of the arch was schematized with ropes loaded by uniformly distributed weights obtaining a uniformly compressed shape. Then in 1958 Musmeci drew his first *minimal* surface, on the occasion of the competition for the bridge over the Tiber at *Tor di Quinto* district in Rome, together with his colleague Ugo Luccichenti (Fig. 10).

In the calculation report attached to the drawings for the competition, Musmeci wrote: "Once shape and external forces are assigned, the equations of equilibrium allow to calculate the stresses in the membrane. But those equations can also be used to determine the shape once certain conditions are imposed on the stresses." In this particular case, Musmeci imposed the condition that the stress regime be hydrostatic.

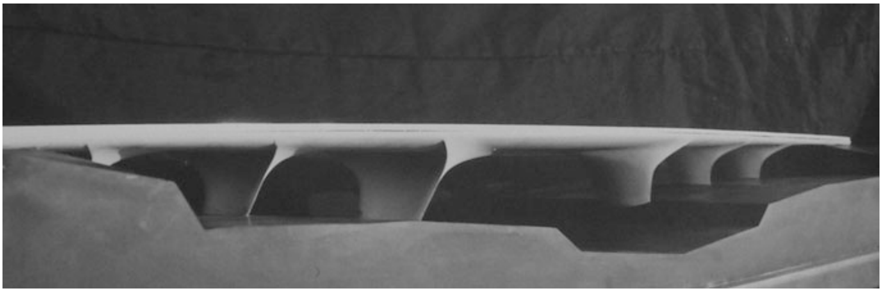

Fig. 10 Bridge over the Tiber at Tor di Quinto district in Rome, 1958, S. Musmeci, U. Luccichenti. Courtesy MAXXI Architecture Archive (SIXXIdata)

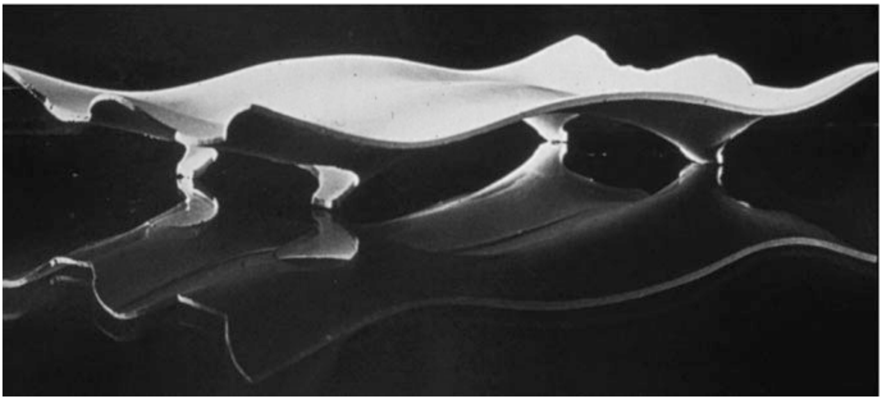

Fig. 11 The New General Market in Rome, project, S. Musmeci. Courtesy MAXXI Architecture Archive (SIXXIdata)

In the calculation reports of all his works, whether actually built or never built, Musmeci never addressed the specific static problem. Normally an engineer's goal is to solve a structure, simplifying so as to quickly apply to the data of the single project, in a given site, with certain specific dimensions. On the contrary, Musmeci always looked for a general mathematical equation, the closed-form solution, in order to be able to apply it in a thousand other occasions, simply by varying the parameters appropriately. In most cases, however, he never used his beautiful closed-form equation again.

When the project for the bridge over the Tiber was ready, none of his colleagues had understood Musmeci's hypotheses and calculations. The last night before the deadline for the submission, Luccichenti, who was a very witty man, improvised a poem to the bridge designed by his friend. He wrote in rhymes that he would never cross the bridge: "Who says it won't fall? You don't fool me. I don't move from the mouth."

Musmeci also created minimal surfaces for the competition project for the New General Market in Rome on Via Prenestina (1957) (Fig. 11) and for the competition for the monument to the *Mille*, in Marsala (1960). Both equicompressed and unreleased surfaces, however, remained only drawings [5].

But that was the road that gave Musmeci international fame.

4 Equi-Stressed Surfaces

In Musmeci's research, there was also room for two prototypes and a project based on equi-stressed surfaces, which certainly felt the influence of contemporary international experimentation.

These works are also related to the initial projects by Musmeci, in which he sought geometric forms uniformly stressed by compression. In these, on the contrary, the prevailing tension is traction.

In 1964–1965, Musmeci designed the chapel of the Pontifical Sanctuary of Pompeii, now called *La Vela*: two conical surfaces and a hypar were juxtaposed to create a soaring roof, whose highest point again coincides with the plan position of the altar (the chapel has some formal analogies with the chapel of St. Vincent de Paul in Coyoacán, Mexico City, designed by Felix Candela in 1959) [6] (Fig. 12).

The *Sant'Alberto* Church at Sarteano (1969–1972) near Siena is more interesting from the point of view of the original formal conception. Musmeci intervened on a project already approved, drawn up by the architect Giancarlo Petrangeli.

The roof is wrapped in a helix around a vertical cylindrical element, which acts as a bell tower. A system of radial cables is anchored, on one side, at different heights to

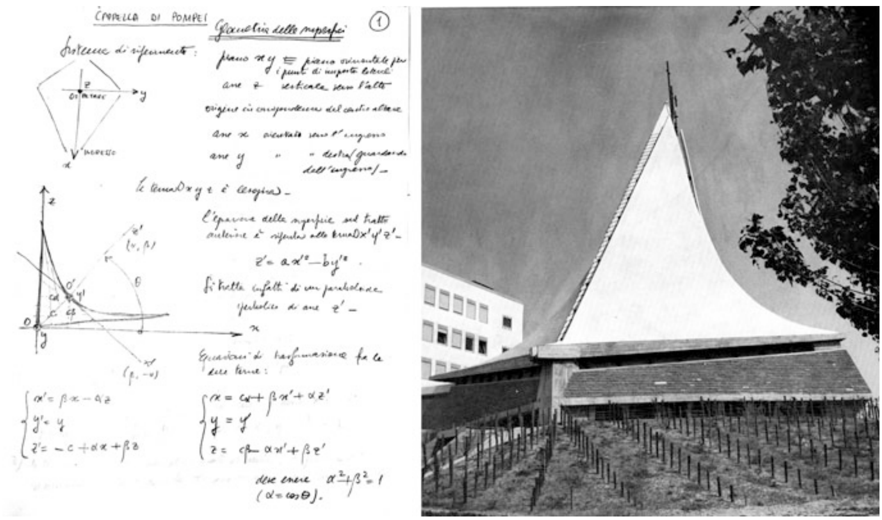

Fig. 12 The chapel La Vela in the Pontifical Sanctuary of Pompeii, 1964–1965, S. Musmeci. Courtesy MAXXI Architecture Archive (SIXXIdata)

Fig. 13 The Sant'Alberto Church at Sarteano, Siena, 1969–1972, S. Musmeci, G. Petrangeli: (left) Inside view; (right) The plan. Courtesy MAXXI Architecture Archive (SIXXIdata)

the cylinder and, on the other side, to the perimeter walls, which follow an irregular geometry. Each cable has a different length and different anchorage height, but the geometry of the cables is identical. From a static point of view, the roof behaves like a suspended shell of reinforced concrete, prestressed by the load-bearing cables. Musmeci was maybe indirectly inspired by some Eero Saarinen's works (David S. Ingalls Rink, 1953–1958; Dulles Airport in Washington, 1958–1962) (Fig. 13).

Then, on May 28, 1969, the Azienda Nazionale Autonoma delle Strade and the Italian State Railways announced a *Competition of ideas for a stable connection between Sicily and the Continent*. Connecting Messina to Reggio Calabria was an old dream, the dream of every engineer. About 150 competitors from all over the world took part in the competition, but only 85 presented solutions respecting the rules of the call. Of those, more than half proposed the solution of a bridge, suspended or cable-stayed, with one or more spans; the others preferred the tunnel or various solutions, including dams, floating islands, and some crazy geometry. On November 25, 1970, the Judging Committee, composed of foreign and Italian experts, awarded 6 first prizes and 6 second prizes ex aequo to the best solutions for each proposed type. Greatest attention was inevitably paid to the most daring project: the suspension bridge that with a single gigantic span crosses in a single hop the three kilometers of water of the Strait. The winner for this solution was Musmeci himself; Nervi was only second (Fig. 14).

The structure proposed by Musmeci was a tensile structure, like those designed in those years by Frei Otto (Fig. 15) [7]. Musmeci's project was a mixture of a suspension bridge and a cable-stayed bridge. According to Musmeci, founding piers in the Strait was impossible. So he designed a suspension bridge with a span of 2 kilometers, which in turn was cable-stayed to 600 meters high and 3 kilometers apart antennas.

Then he invented another trick: in addition to the main suspension cable, he placed a second cable, in the opposite position, under the bridge. That cable looks like an arch but is made of steel strands. A spider web of well-stretched strands connected the main suspension cable and that second one, called *stabilizing* cable according to the typical terminology of tensile structures. The spider web wraps and tightens the two decks, the one for the cars and the one for the train, which travel at

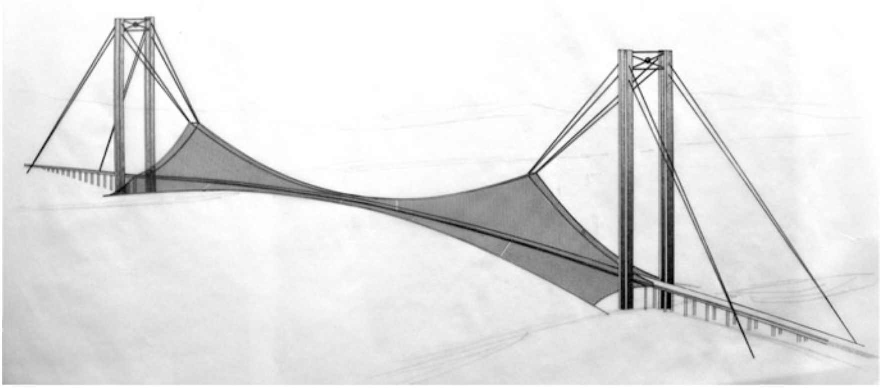

Fig. 14 Suspension/Cable-stayed Bridge over the Messina Strait, 1969, S. Musmeci. Courtesy MAXXI Architecture Archive (SIXXIdata)

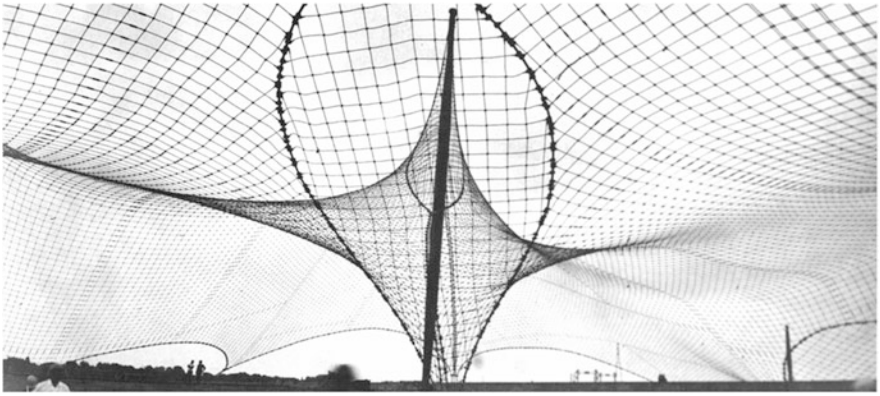

Fig. 15 Frei Otto, German Pavilion, Montreal, 1967, a detail (SIXXIdata)

different heights. The decks are aerodynamic: they are fusiform like the wings of an airplane, so the wind slips better and the bridge sways less.

That adventure was a story without a happy ending.

5 The Nameless Shape Bridge

Meanwhile, in 1967, Aldo Livadiotti involved Musmeci in the project for a bridge at the gates of Potenza. The bridge had to cross the Basento River and connect the city with the state road *Basentana*. The client was the Consortium for the industrial nucleus of Potenza, chaired by Gino Viggiani. In fact, the bridge was meant to encourage the industrial development of the area. The *Cassa per il Mezzogiorno*, which was responsible for promoting the South of Italy through various types of economic investments, financed the work [8].

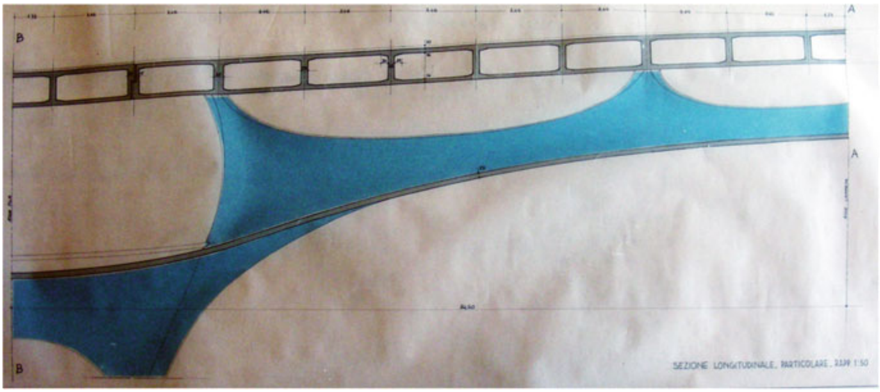

Fig. 16 Bridge over the Basento river in Potenza, 1967–1975, S. Musmeci: partial cross section. Courtesy MAXXI Architecture Archive (SIXXIdata)

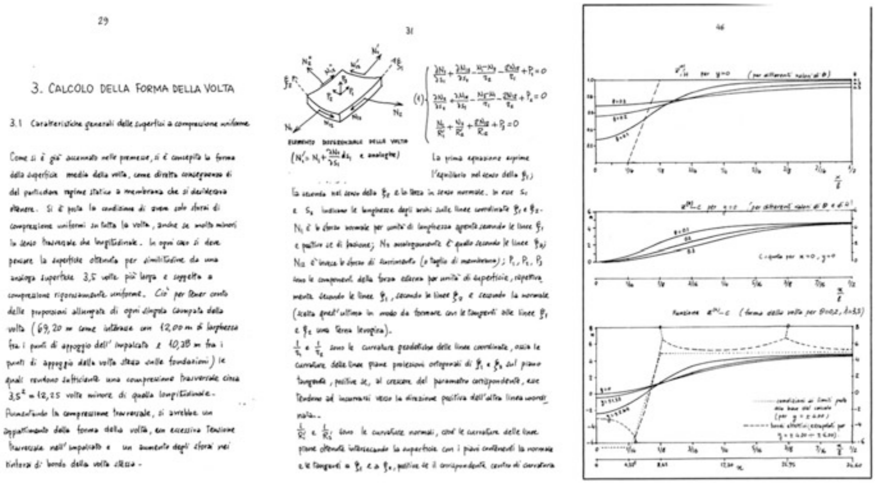

Fig. 17 Bridge over the Basento river in Potenza, 1967–1975, S. Musmeci: Calculating the form of the shell. Courtesy MAXXI Architecture Archive (SIXXIdata)

Musmeci radically changed the very simple first project by Livadiotti and created a *nameless form*. The project was based on this theory: Knowing the constraints and loads, then the unknown is the form.

The designer must *calculate* the shape of the bridge. How do you calculate a shape? Musmeci introduced an additional condition: the shape must be minimal, that is, as light as possible; more, the material must be equally stressed; that is, it must be equicompressed.

The shape of the structure was deduced from its static regime. Musmeci developed a mathematical theory for this, with relative closed formulas (Figs. 16 and 17).

In the technical report dated January 30, 1968, attached to the preliminary design, Musmeci made explicit reference to the methods used by Frei Otto to calculate the tensile structure that covered the German pavilion at the Montreal Expo 67.

The bridge is composed of four *lowered arches* with spacing of about 70 meters and free span of about 60 meters. A double-curved shell generates the arches; in each arch, the shell bends upward, creating four pairs of apophyses. Toward the foundation, the shell bends creating four supports, which behave as virtual hinges (or elastic joints). The deck consists of a hollow beam of lenticular shape. The shell and the deck are made of reinforced concrete; the deck is transversally prestressed at the apophyses. The shell is not isotropically stressed: longitudinal equicompression reaches values 12 times higher than transversal equicompression.

On July 26, 1968, the Cassa per il Mezzogiorno approved the project, but Musmeci was asked, in addition to more complete analytical calculations, to verify the shape on scale models.

The first model was a simple soap film, stretched between iron wires and cotton threads. In November 1968, Musmeci created a model with a stretched rubber membrane; under inverted loads, it ensured an equicompressed shape. The rubber model was accurately measured and became the basis for building another model, at a scale of 1:100, out of transparent acrylic material (*perspex*) (Fig. 18).

In April 1969, that perspex model was subjected to loads in the laboratory of the Faculty of Engineering in Rome, under the direction of Musmeci himself, measuring the deformations with electrical strain gauges. The result was that the stress values in the shell were about half of those indicated by the analytical calculation.

Then, a much larger model, in microbeton, on a scale of 1:10, limited to two spans of the bridge, was commissioned to Ismes in Bergamo.

In February 1970, the Ismes technicians carried out the first test, but it was not positive: the model required important modifications in the drawing of the shell, in particular on the central part of the arch, to increase the two curvatures; in July, they tested the second correct model. Thanks to the results obtained, Musmeci further improved the project and made important variations to the model: for example, he interrupted the continuity of the deck by inserting Gerber joints and beams. He also

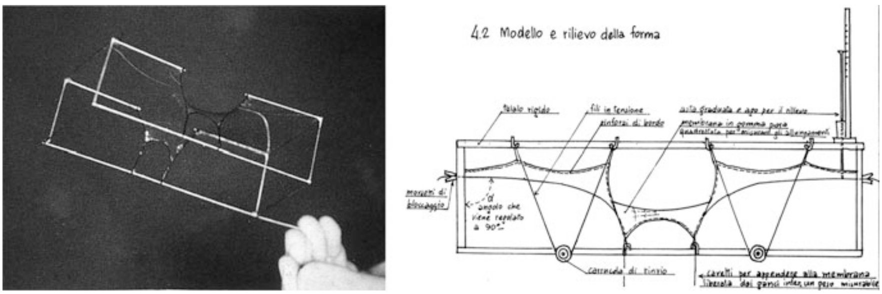

Fig. 18 Bridge over the Basento river in Potenza, 1967–1975, S. Musmeci: Soap film model and rubber model. Courtesy MAXXI Architecture Archive (SIXXIdata)

Fig. 19 Bridge over the Basento river in Potenza, 1967–1975, S. Musmeci: Microbeton model, 1:10, 1970–1971. Courtesy ISMES Archive (SIXXIdata)

Fig. 20 Bridge over the Basento river in Potenza, 1967–1975, S. Musmeci: the complex construction site. Courtesy Edilstrade Archive (SIXXIdata)

thickened all the edges of the shell. Between March 29 and 31, 1971, the new revised model was brought to failure; the test was interrupted because the foundation block broke before the shell. The many tests cost a total of 24 million liras (Fig. 19).

The final form is difficult to draw as well as to describe in words. In order to follow the dream of the *perfect form*, Musmeci eliminated all conditioning: above all, he disregarded construction problems.

On July 1, 1970, the Edilstrade Company of Forlì, directed by Gilberto Flamigni, won the private bidding for the construction of the work. Thirty companies were invited to bid, including Condotte and Salini, which were much more experienced but would probably have modified the project, simplifying it. Musmeci was appointed director of works and the construction site began in September. Mr. Buattini, an elderly worker expert in the construction of wooden boats, was called to design the complex formwork, based on a plaster model provided by *Ismes*. The formwork, supported on tubular scaffolding, changed for each span, with no possibility of reuse. Musmeci thanked Buattini warmly in many articles (Fig. 20).

After the consolidation of the concrete castings, when the wooden formworks were lowered, the shell made very dangerous movements. At that time, Edilstrade consolidated the foundations with inclined steel micropiles, 14 centimeters in diameter, in order to absorb the thrust. The engineer Arrigo Carè, a great designer of bridges, was entrusted with the testing; on May 22, 1975, load tests were carried out on the span above the railway; on September 25, 1981, at the end of all contracted works, Carè issued the test certificate. In 1989, Edilstrade was still waiting for the last payments.

In the transition from design to construction, the form was not distorted, as could be expected (especially after the many recommendations of the *Ismes*). On the contrary, the form became even more mysterious. Today, looking at it, on the one hand the shell seems very light, like a fabric hanging from the deck, descending to the ground; on the other, it looks like a wrinkled rhinoceros, or an extinct pachyderm. The fabric petrified in the concrete appears powerful and robust. The indelible traces left by the wooden formwork on the shell reveal the fatigue and difficulty of construction. In this contradiction, beauty lives on; the bridge is a unique sculpture, visited every year by hundreds of engineering and architecture students from all over the world (Fig. 21).

Fig. 21 Bridge over the Basento river in Potenza, 1967–1975, S. Musmeci: The indelible traces left by the wooden formwork on the shell. Courtesy MAXXI Architecture Archive (SIXXIdata)

The visitor can walk inside the bridge, over the shell, under the deck: this is a very rare case. Children ride bicycles or skateboard. It is forbidden: but if you don't try it, you can't feel the bridge.

The bridge over the Basento River, the highest point of the Italian School of Engineering, coincides with the definitive decline of that School. It is its swan song: when the bridge was completed, the Italian School of Engineering had already lost its way, disappearing along with Pasolini's *fireflies* [9].

That work by Musmeci is visionary; it anticipates by decades the engineering of *form finding* and the parametric engineering of this millennium. Musmeci anticipated our time. Today engineering, like architecture, has lost its strictly functional role; it must rather astonish and it must attract attention, with new and complex forms. In this millennium, pop structures have taken over. Bridges can be repeated identically anywhere, in Buenos Aires, in Dublin, or in Cosenza, like the ones designed by Santiago Calatrava. They are both repeatable and stunning, like the *Pop Art* masterpieces.

The previous generation engineers were faithful to the strict principles of maximum economy, assimilated directly from Gothic construction sites. Today's designers follow new principles, much closer to the astonishing and limitless cost of Baroque construction. Musmeci was the transitional element in this process of transformation. With his exceptional mathematical ability and his pursuit of optimized but uneconomical forms, he symbolizes the transition to the baroque engineering, dedicated to wonder, of the new millennium [10].

After the bridge in Potenza, Musmeci continued to search for minimal forms. But Italy no longer had much to offer him. By the end of the 1970s, the economic crisis, which had begun in the mid-1960s, became very harsh.

On March 5, 1981, Musmeci died, taking with him many good structural ideas and his personal vision of the future of engineering. Musmeci had a dream. He saw the birth of the computer, but he could use the machine only once for small verification calculations. He understood that the computer was the future. He wrote in a manuscript: "When the computer becomes powerful enough and when we really understand how to use it, it will help us not only to verify structures, but to design them." Musmeci dreamed of the powerful computers of today's times, which make it possible to correct form, to optimize structures. Today, computers help when we have to take really important decisions, at the design stage and not just verification step. Perhaps, Musmeci lived too far in advance: if he were reborn today, the Italian School of Structural Engineering would be reborn with him [11].

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This paper is dedicated to Sergio Poretti (1944–2017).

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Twenty Years of *Il Giardino di Archimede*

Enrico Giusti

I cannot remember precisely how and when I got myself involved in the exhibition first, and later in the museum of mathematics. It may have been in the early 1990s when Franco Conti, then professor at the Scuola Normale Superiore in Pisa, and I began to investigate the possibility of a non-formal approach to mathematics that was both rigorous and interesting for the public, in general, especially for students.

We started from the observation of the marginal role reserved for mathematics in the various science museums, and from reflections on the reasons for this situation. We agreed on the belief that its systematic nature could not depend on occasional reasons, but must have causes inherent in the nature of mathematics and its modes of diffusion, and in the differences between mathematics and other sciences.

A first difference immediately catches the eye: unlike the experimental sciences, mathematics has no objects to describe or phenomena to show. Rather, and better, its objects are not immediately perceivable as such and require different supports from which they do not emerge without an action aimed at the purpose.

To clarify this point, let us ask ourselves: how is it possible to describe a parabola to an audience non particularly trained in mathematics? One answer may be the use of explanatory panels, perhaps made more attractive by good graphics and captivating explanations that describe their genesis and main properties. But when you try to imagine a museum in which this solution is applied systematically, it turns out that what you are thinking of is essentially a book applied to the walls. Nor does the situation much improve with the presence of models that are sometimes esthetically valuable, but which do not change the general approach. The mathematics they include does not appear to the visitor: it is buried within the objects and never comes to the surface. It is “dead” mathematics.

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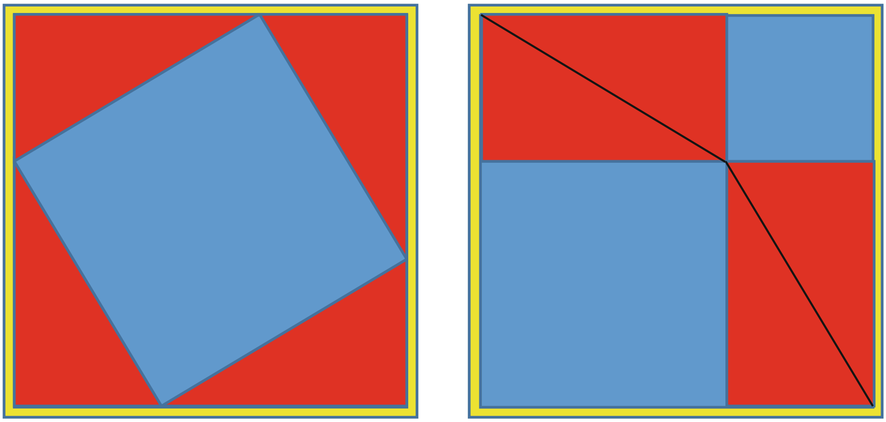

Fig. 1 A proof of Pythagoras' theorem

What we wanted instead was a "living" mathematics; a mathematics caught in the moment it displays its power, determining the behavior of the machines we operate and the phenomena we observe.

The following example can help us understand the difference between living and dead mathematics. It concerns one of the oldest theorems, one that most people are familiar with: good old Pythagoras' theorem.

Several science museums exhibit a "hydraulic" proof of this theorem. The squares of the legs are filled with water, while the one of the hypotenuse is empty. When turning the figure, the square of the hypotenuse goes down and the ones of the legs go up. The water flows through tubes until the square of the hypotenuse is filled and the squares of the legs are empty, which shows that their sum is equal to the first.

Another very simple proof involves a large square, the side of which is equal to the sum of the two legs (Fig. 1).

The relevant right triangle is in one corner, and identical ones are placed on the other corners. The area that remains is the square of the hypotenuse, so the large square is equal to the square of the hypotenuse plus four triangles. Let us now move the triangles in order to form two rectangles; what remain are the two squares of the legs, so in this configuration the large square (which was equal to the square of the hypotenuse plus four triangles) is now equal to the same four triangles plus the squares of the legs. Q.E.D.

The first proof, which is based on filling either the squares of the legs or the square of the hypotenuse, is a perfect example of dead mathematics. In fact, mathematics was sometimes used while proving the theorem, but there is no trace of this proof in the experiment. This hydraulic test is based on a number of assumptions that have nothing to do with mathematics: that the containers are actually square; that their thickness is uniform; that there are no hidden parts where the water could go; that the vessels are in fact precisely filled. Even if these assumptions are all reasonable and natural, we can only trust in a visual impression, since there is no logical reasoning that can assure us that this is in fact the case and that the appearance of areas being the same is not a trick. Moreover, even if the object on display actually shows that the square of the hypotenuse is equal to the sum of the squares of the two legs, nothing proves that a different right-angled triangle would possess the same property. In essence, the hydraulic device gives a verification without mathematics.

The second proof is completely different, because it speaks not to the eye but to the mind. It is not even important that the four triangles are absolutely identical, that their sides be perfectly straight or that their right angle be exactly right. As with the figures we draw in geometry, we just need the objects to be reasonably precise, so that they can guide reasoning while avoiding glaring errors. Once we assume that they are indeed right-angled identical triangles, then the quadrilateral of the first figure is a square, just like those of the second figure, whose sum is equal to the first. What is more, although the specific object by necessity only describes a particular case, it is easy to understand that the same reasoning will be valid for any right-angled triangle. All this is possible because it is in fact a reasoning, and not an empirical verification with no mathematical content. In brief, while the hydraulic device displays dead mathematics, the second proof shows mathematics in its making—living, active mathematics.

Live mathematics has another characteristic that distinguishes it from its dead counterpart: it can be read on more than one level. The lowest level consists in the visual verification that the uncovered parts of the large square are, on the one hand, the square of the hypotenuse, and on the other, the squares of the legs of our right-angled triangle.

A level immediately above would prove, for example, that the first quadrilateral is in fact a square, that is that its angles are right angles.

A third level of interpretation may consist of the remark that the proof of the last assertion depends on the fact that the sum of the angles of a triangle is equal to two right angles. This theorem can only work if the parallel postulate is assumed: in fact, it is equivalent to this postulate. Consequently, Pythagoras' theorem is a theorem within Euclidean geometry and it does not hold in non-Euclidean geometries.

After a short discussion, our approach immediately turned in another direction: the mathematics of the museum should not be superimposed on the objects on display, but should emerge from these very same objects. In some sense, the description of the mathematical objects would have been all the more effective the more it became useless, as the properties that characterized them would spontaneously emerge from the physical objects and phenomena that the visitor would observe.

To remain in the example of the parabola, its generation would have been observed by illuminating a wall with a flashlight or thanks to the surface of the water in a conical container; its focal properties would materialize in burning mirrors and mirages, its applications would have involved the motion of projectiles and the shape of suspension bridges.

This approach immediately seemed to us the most suitable, but it was immediately evident that it required relatively large spaces: the parabola alone would have needed about ten stations! A space that no science museum could offer. If this is the reason for the secondary role of mathematics in science museums, the answer could only be one: a museum dedicated solely to mathematics. Starting from these ideas Conti organized in 1992 a conference at the Scuola Normale in Pisa, on the theme "Experiences, ideas and proposals for a museum of mathematics." The conference was widely attended and allowed the participants to have a clear view of the Italian situation.

At that time, Conti had already begun to build in his garage a series of instruments, that would be the initial nucleus of the "Oltre il compasso" (Beyond compasses) exhibition. More complex objects were entrusted to artisans whom he trusted; still others to technicians working in the workshop of the Scuola Normale. I contributed to the exposition by writing the texts and finding the images for the panels, designed by Salvatore Di Pasquale at the Department of Constructions of the University of Florence.

The exhibition was shown for the first time at Palazzo Lanfranchi, Pisa, from 8 to 31 May, 1992. It was a great success. During the subsequent years it was exhibited in more than 30 cities within and without Italy. In particular, the exhibition was hosted at Palais de la Découverte in Paris for almost a whole year and had more than 500,000 visitors.

In the meantime, Franco Conti and I were looking for a location where the exhibition could find a permanent settlement, being the first nucleus of the projected mathematical museum. For the first time we were to experience the difficulties and subtleties necessary in dealing with political authorities. In particular, we learned that when a politician says "yes" he means "maybe," and when he says "maybe" that means "no."

The city of choice for the museum was obviously Pisa, since the exhibition was planned and realized at the Scuola Normale Superiore, where Conti was a professor. We went therefore to see the mayor of Pisa, to whom we illustrated the project, its novelty (at that time, there was in the whole world no museum completely dedicated to mathematics) and its impact on the mathematical education and culture. The reception was almost enthusiastic, and we were promised of a building where the museum could be hosted. A few months later we learned that the same building had already been destined for other activities.

A second possibility, which was proposed shortly after, had not better luck. It was a building close to the Certosa di Calci, a few kilometers away from Pisa. The Certosa already hosted the Museum of Natural History of the University of Pisa, and the eventuality of a synergy between the two museums seemed promising. On the other hand, the cost of the necessary restauration of the building appeared immediately exorbitant, and even that possibility vanished into thin air.

Several years passed thus since the first exhibition, without any advance toward the establishment of the museum. Then one day I was visiting my parents in my hometown Priverno, a city south of Rome, when I met Mario Renzi, a longtime friend who was then mayor of the city. Together we visited the newly restored

"Castello di San Martino," a magnificent Renaissance villa that originally belonged to Cardinal Tolomeo Gallio, secretary of the Pope Gregorio XIII. To my question "what will you do here?" he answered in rather general terms of art exhibitions and conference site, but it was clear to me that he had no definite plans for the use of the villa. So when I proposed a mathematical museum he became very interested, even more so when the exhibition "Oltre il compasso" was hosted in May 1996 at the old infirmary of the Abbey of Fossanova.

The bureaucracy took its time, but in July 1999, a Consortium was founded between the City of Priverno, the Scuola Normale Superiore and the Universities of Florence, Pisa, and Siena, with the purpose of "constituting and managing a Museum for mathematics." In September of the same year, the museum opened in Priverno. "Il giardino di Archimede" (Archimedes' garden) was born.

For the occasion, a new exhibition was built, centered on Pythagoras' Theorem. It was by far smaller than the previous one, but much simpler to manage. Moreover, it was easily duplicated in reduced form, and could be used outside the museum, in schools and other facilities. This was the first nucleus of what later developed into a budget of laboratories, offered both within the museum and outside it.

In spite of its location in a small town of about 10,000 inhabitants, the museum had a success beyond expectation, receiving annually not less than 8000 visitors.

Meanwhile, the offer of the museum grew of two more exhibitions, this time of historical character. In the year 2000, the Italian Ministry of Education sponsored a series of events under the general title of "1000 anni di scienza in Italia" (A thousand years of science in Italy). The giardino di Archimede's project was among those funded, and with the help of Luigi Pepe we created the exhibition "La Matematica in Italia: 1800–1950," an excursus on mathematics and mathematicians from the Napoleonic era to the years after the second world war. The second exhibition: "Un ponte sul Mediterraneo. Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente" (A bridge over the Mediterranean. Leonardo Pisano, Arab science and the rebirth of mathematics in the West) was the celebration of the 800th anniversary of the publication of Leonardo Fibonacci's *Liber Abbaci*. The catalog released on that occasion is still cited as one of the main reference works on Fibonacci and his legacy.

In the meantime, things were moving.

The original idea, which Conti and I shared, was for a "diffuse museum," a series of small museums in different locations, an alternative to a large concentrated museum, which would have required a large financial and personnel commitment. In principle, the museum could have been the result of a confederation of previous experiences and exhibitions carried out in different cities by different subjects. We made some timid advances in this direction but we immediately realized that this could not work: the diffuse museum could only be a filiation of Archimedes' garden.

An important step was an occasional encounter with Pietro Zecca, a mathematician who at that time was a member of the Council of the Province of Florence. Talking of the museum and why it was situated so far from Florence, I complained of the fact that it was extremely difficult to obtain a definite commitment from a politician. His answer was immediate:

Come with me—he said—I am in the Council, and they can’t make fun of me. They must say yes or no.

A few days later, we met the Vice President of the Province; with him we visited a school building in the western outskirts of Florence, a part of which—about 1000 square meters—was currently unoccupied by the school and could be used for the museum. Within a few months the project was approved by the Provincial Council and a sum of 700 million liras was allocated for the necessary works.

The works took some time, but in 2004, the Giardino opened its headquarters in Florence. Alas, Franco Conti was unable to see the new museum, since he had died the previous year.

The opening of the Florence branch was quite a fortunate event, since the relations with Priverno were rapidly deteriorating. The municipal elections of 2004 brought about a change in the administration, and the new mayor did not reconfirm the director of the museum, on the basis that “he was a political opponent.” The agreement between the Municipality and the Consortium provided that the director was appointed by the Municipality “on the proposal of the Consortium.” None of the alternative proposals managed to win the approval of the municipality, and after a grueling back and forth the mayor appointed a trusted director of him, presumably a political ally. The rupture that followed led to the withdrawal of the Giardino di Archimede. In 2005, a different set-up took the place of the previous one, with a completely different philosophy. Above all, once the new museum was built, there was no maintenance program, so that in less than a decade the new installations practically fell apart.

With the opening of the Florence branch, which soon became the only seat of the museum, we experienced a period of relative tranquility and ordinary management. Despite being located on the outskirts of the city, and therefore outside the main tourist flow, the number of visitors to the museum was constantly increasing, from around 10,000 when it opened to 15,000 in the following years. The visitors were mostly students with their professors, coming not only from Florence and Italy, but also, and significantly, from other countries. We have managed to establish a stable relationship with some of these, and some schools have sent their pupils to visit each year. The 2008 financial crisis resulted in a steep drop in visitors, but we slowly recovered.

In 2009, we produced another exhibition, whose title “Helping the nature” came from Galileo’s remark that the machines could never deceive the nature, but rather help it producing wonderful effects (Fig. 2). The exhibition consisted of a series of simple devices, the operation of which was explained using Galileo’s own words, taken from his book “On Mechanics.”

The following years passed without important novelties and everything seemed to be heading toward a substantial routine, when events apparently far from the reality of the museum came to disturb normal activity, putting the very survival of the Giardino di Archimede at risk.

The relations between the Giardino di Archimede and the Province of Florence were regulated by an agreement, which provided among other things for the free concession of the premises. The agreement had a duration of six years, and was

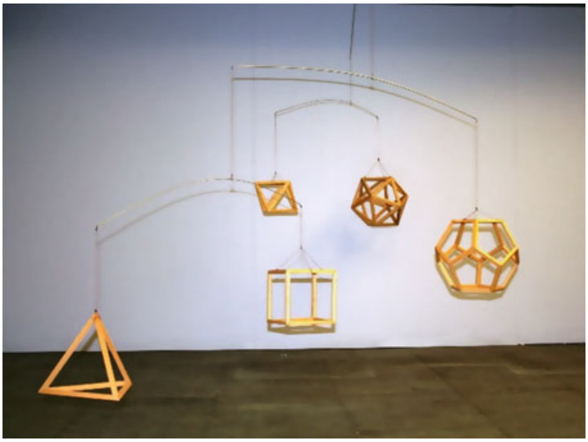

Fig. 2 The law of the lever: a mobile with polyhedra

automatically renewed unless canceled by one of the parties. The first deadline in 2010 passed without problems, and the agreement was renewed until 2016.

In the meantime, in 2014, a law was approved which established among other things the "Metropolitan cities," including that of Florence, which replaced the province. The new law slashed the budget of the metropolitan city drastically, and the following year we were asked more or less covertly to pay a rent, estimated at around 100,000 euros per year, practically our entire annual budget. That this was impossible was clear to everyone; in fact, the request was only the first move by the Metropolitan city to regain possession of the premises.

When the agreement with the Province—now Metropolitan city—of Florence expired in 2016, it was renewed for a single year, with the possibility of an extension for no more than a second year. The decision, which was quite similar to a death sentence, was motivated by "the intention of the Metropolitan city to study for the Museum, as you often suggested, a more suitable and easily accessible location within the city center, as part of an overall project for the creation of a museum dedicated to Science and Technology, in order to attract a greater number of visitors, in addition to school visits" (my translation).

Needless to say, the museum dedicated to Science and Technology never saw the light.

The new expiration date was set for March 9, 2017, and then at our request it was extended by one year.

In the meantime, another exhibition saw the light, "Armi di istruzione di massa," that is "Weapons of mass instruction," whose subtitle was "mathematical games, puzzles and pastimes." The idea was that every game or puzzle should introduce an important mathematical concept or tool: proofs by contradiction and by induction, binary notation, metrics, and the like. At the same time, the didactic offer was enriched with various new laboratories (in addition to those already proposed under the title "All'inizio del conto," and dedicated to calculations and numbers in ancient civilizations) and with the "Mathematical walks in Florence" in which, under the guidance of Giuseppe Conti, the mathematics underlying many Florentine monuments was brought to light.

In March 2018, the new concession expired, this time without the possibility of renewing it. No solution was in sight and we were granted an extension of a few months, until the end of July. In September, we were offered a space of about 300 square meters (about a third of what we had) on the top floor of a school building. Regardless of its size, the space was absolutely unsuitable for a museum: there was neither a separate entrance nor a fire exit, and several rooms on the same floor were occupied by the school and other institutions. We and the administration both knew all too well that the Giardino di Archimede would never be housed there; however, we were forced to accept it under penalty of the immediate closure of the museum. As some works were needed, which involved, among other things, the completion of an auditorium attached to the school, we were allowed to continue our activity in the previous building until the end of 2019.

The year which followed was entirely dedicated to the research of a new location. One after another, several possibilities seemed on the verge of materializing, and then vanished into thin air. The most promising of these was the possibility of occupying a building owned by the University and located on the hill of Arcetri, a stone's throw from the ancient physics department and not far from the villa "Il gioiello" which once belonged to Galileo. The available area, it is true, did not exceed 300 square meters, by far insufficient, but the building had a large terrace which if covered would have added at least another 200 square meters. It was always a much smaller space than that occupied by the museum, but sufficient to continue the activities, even if on a smaller scale. Obviously the building would need some major works, which would have involved considerable costs, which the museum could not afford on its own. Thus, the problem of finding ad hoc financing arose, and various possibilities were being studied when queries addressed to the Municipality revealed that, since the hill of Arcetri is subject to landscape constraints, the Municipality would never have given permission to close the terrace. Other possible solutions—in Pisa I visited an entire completely empty building in the city center—lasted even less.

At the beginning of 2020, the convention was not confirmed, and we were declared "non-contractual occupants," just a small step before eviction. Short of a miracle, the fate of the Giardino di Archimede seemed sealed, and we were prepared to close all activities.

But sometimes miracles do happen. A few years earlier I had been asked to organize a three-day event for high school students, focusing on mathematics. The event was sponsored by a banking foundation in the nearby city of Pistoia. Under the impulse of Ezio Menchi, a professor of physics, the Foundation "Cassa di Risparmio di Pistoia and Pescia" had created a project for the enhancement of scientific culture in schools, which included, among other things, study stages for selected groups of students. The event I had organized was one of them. Subsequently, the project was enriched, including a competition among the schools of the province for the realization of original scientific projects, and later on with the creation of an Academy for the most scientifically active young people.

For the management of the project the Foundation had appointed a Scientific Committee, of which I had been called to be a member since its inception in 2017. One day, during a break in a committee meeting, speaking with Ezio Menchi and Giovanni Palchetti, then Vice President of the Foundation, I happened to ask—mainly to keep the conversation alive—if they were not interested in a possible transfer of the Giardino di Archimede to Pistoia. To my surprise, the response was almost enthusiastic, and in a subsequent meeting with the President of the Foundation, a path began to be outlined that could lead to a positive conclusion.

Of course, there was still a long way to go, also because the approach adopted by the Foundation was very different from that experienced years ago in relations with local administrations. In that case, the concession of the premises for the museum was based exclusively on personal relationships (with me in Priverno, with Zecca in Florence). No one had asked me to document my skills, nor to see a project for the museum; least of all an economic plan that would guarantee the sustainability of the initiative. Here the first request was a visit to the museum and an examination of the economic and management situation (personnel costs, economic accounts of recent years, balance sheet). Only after these investigations were successfully concluded could a possible location for the museum be thought of.

Another striking difference between the local administrators and the Foundation was that the latter wanted us, while the former tolerated us. An obvious symptom of this situation is that when we said that an efficient museum needed at least a surface of 1000 square meters, the response of the local administrators was systematically: "can't you do with less, say with 300?" while that of the Foundation was: "no, 1000 are too few for a satisfactory setting up; at least 1300 are needed."

In July 2020, the Florentine headquarters of the museum was cleared and all the material was transported to Pistoia, where it was stored waiting to be placed in the new premises. In the meantime, some possible locations have been identified, and the Foundation has commissioned Cesare Mari, an architect specializing in museum design, to develop an exhibition plan. The works for the adaptation of the premises should begin shortly, and we hope to be able to open the museum again to the public as soon as the health situation allows.

What lessons can be learned from the events of the Giardino di Archimede? The differences in behaviour between public administrators and politicians on the one side, and private Foundations on the other are striking. But I can be wrong; perhaps not all politicians are alike. Maybe I came across the wrong ones. Maybe there are those who simply say yes or no, and when they say it they mean it. Maybe the fact that I have never met any does not mean anything. Maybe. But I confess that I feel much better now.

Part V
Design and Mathematics

The Multifaceted Abraham Sharp

George W. Hart

1 Introduction

Abraham Sharp (1653–1742) was a mathematician and scientific instrument maker who worked for several years as assistant to England's first royal astronomer, John Flamsteed, receiving great praise for the high quality of his workmanship. But equally noteworthy are the accomplishments that went into his remarkable 1717 book, *Geometry Improv'd* [1]. Figure 1 shows the title page. It is actually two independent books packaged under one cover. We will ignore the first 64 pages, which concern trigonometry tables, and focus on Part 2 of the book, *A Concise TREATISE of POLYEDRA or SOLID BODIES of many BASES*. Sharp presented 12 original "solid bodies" and detailed a unique method by which they could be constructed. These range in complexity from 18 to 120 faces and include examples that were later rediscovered. Although Sharp's polyhedra are significant for being new to the geometry literature, they have been almost entirely ignored for three centuries by the mathematics community. *Geometry Improv'd* is unusual in many ways and I maintain that the only way to make sense of its contents, its style of presentation, and its poor reception is to think of Abraham Sharp as an artist.

There has never been another mathematics text that compares to the obsessive labor of love in which Sharp meticulously presents his 12 original polyhedra. What stands out most saliently is how he gives all the necessary dimensions both as exact formulas (with "surds," i.e., irrationals expressed by radical signs) and as overly precise numerical values. His decimal numbers are painstakingly hand-calculated with 16–30 digits of precision, sometimes via intermediate terms up to 50 digits long. The book often presents multiple detailed constructions starting

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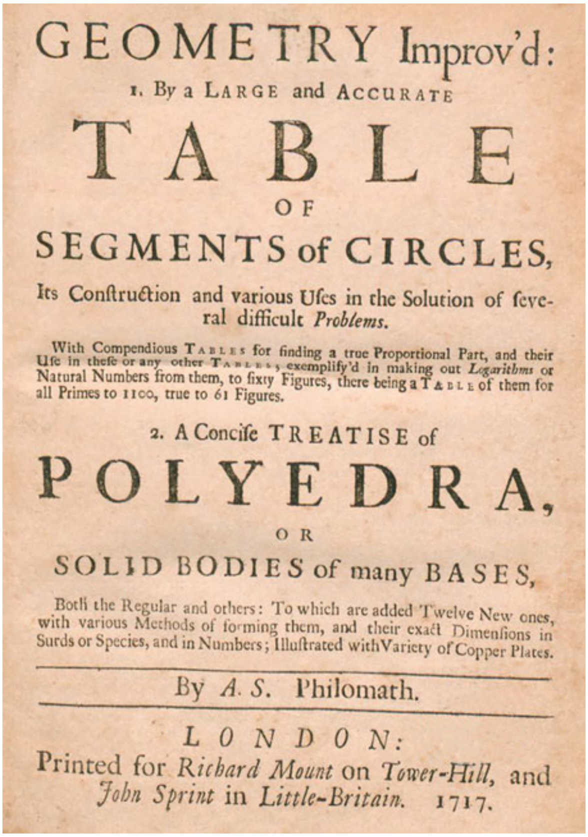

GEOMETRY Improv'd:
1. By a LARGE and ACCURATE
TABLE
OF
SEGMENTS of CIRCLES,
Its Conſtruction and various Uſes in the Solution of ſeveral difficult *Problems*.
With Compendious TABLES for finding a true Proportional Part, and their Uſe in theſe or any other TABLES, exemplify'd in making out *Logarithms* or Natural Numbers from them, to ſixty Figures, there being a TABLE of them for all Primes to 1100, true to 61 Figures.
2. A Conciſe TREATISE of
POLYEDRA,
OR
SOLID BODIES of many BASES,
Both the Regular and others: To which are added Twelve New ones, with various Methods of forming them, and their exact Dimenſions in Surds or Species, and in Numbers; Illuſtrated with Variety of Copper Plates.
By *A. S.* Philomath.
LONDON:
Printed for *Richard Mount* on *Tower-Hill*, and *John Sprint* in *Little-Britain*. 1717.

Fig. 1 Title page of *Geometry Improv'd*

variously from a cube, a rectangular parallelepiped of specified proportions, a sphere, and/or a previously prepared simpler body (dodecahedron, icosahedron, or rhombic triacontahedron) to attain the same intricate result. Furthermore, redundant alternate ways to locate the same point are often included, e.g., the distance to measure in from a corner and the distance to measure out from a center are sometimes both stated.

Figure 2 shows a typical section of Sharp's text. A scanned copy of the full book is available online [1]. The concise treatise of polyhedra is packed into only 32 pages, but one immediately gets the impression that Sharp may have put years of passionate work into the calculations alone. He also made wooden models that must have required a great deal of time. Although they reify his methods and validate some of his numbers, they are not mentioned in the book.

Sharp made three large copperplate engravings, "neatly engraved by his own hands" [2], which illustrate *Geometry Improv'd.* These are works of art in their own right and, like the text, they are overwhelming in their density of information. Figure 3 shows his Plate 1, illustrating just the simpler forms from the book. This one page includes 18 examples of how to start with a solid block (a cube or a parallelepiped)

The ſhorter Diagonal of the Semi-rhomb $= b\sqrt{\frac{55}{2} - \frac{21}{2}\sqrt{5} - \sqrt{50 - 22\sqrt{5}}} =$ $=,327514283511240623860835.$

The longer Diagonal $= \frac{4}{181}b\sqrt{7765 - 482\sqrt{5}} - \frac{449b - 376b\sqrt{5}}{362} = ,338311174575652\text{-}6222550172912.$

The longer Part of it, extending from the acute Angle to its Interſection with the ſhorter Diagonal $= \frac{1}{4}b\sqrt{\frac{185}{2} - \frac{11}{2}\sqrt{5}} - 2b = ,2388840150249961735062634218.$

The ſhorter Part $= \frac{275b - 376b\sqrt{5}}{362} - \frac{b}{1448}\sqrt{438050 - 21158\sqrt{5}} = ,09942715955065\text{-}6448748753869 4.$

The Radius of the Circle paſſing through the three acuter Angles of the Semi-rhomb, which are equidiſtant from the Center of the Body $= b\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{5}} - b = ,175570504\text{-}5849462583374119.$

The remaining Part of the longer Diagonal extending from the Center to the obtuſe Angle $= \frac{3}{181}b\sqrt{\frac{1565}{2} + \frac{599}{2}\sqrt{5}} - \frac{87b - 376b\sqrt{5}}{362} = ,1627406699907066.$

The longer Side $= b\sqrt{\frac{23}{2} - \sqrt{5} - \frac{1}{4}\sqrt{1970 - 278\sqrt{5}}} = ,2896238493467569633 3559.$

The ſhorter Side $= \frac{1}{181}b\sqrt{83770 - \frac{53835}{2}\sqrt{5} - \frac{1}{4}\sqrt{105661319210 - 43669848002\sqrt{5}}}$ $= ,1915780820769485491326.$

The Radius of the Circumſcribing Sphere $= \frac{1}{2}b\sqrt{15 - 2\sqrt{5} - 4\sqrt{10 - 2\sqrt{5}}} =$ $= ,529929242522256508336504793.$

The Radius of the Sphere paſſing through all the obtuſe Angles of the Semi-rhombs $= \frac{1}{181}b\sqrt{\frac{75341}{4} - 4305\sqrt{5} - 6\sqrt{5229965 + 2338682\sqrt{5}}} = ,5258179586786894775\text{-}85957.$

Area of the Semi-rhomb $= \frac{1}{724}b^2\sqrt{7875095 - 2922894\sqrt{5} - 8\sqrt{15443262563 45 - 678139662118\sqrt{5}}}$ $= ,0554008709724955568920352234.$

The Superficies $= \frac{15}{181}b^2\sqrt{7875095 - 2922894\sqrt{5} - 8\sqrt{1544326256345 - 678139662118\sqrt{5}}}$ $= 3,32405225834973341352211 3402.$

The Solidity $= \frac{5}{362}b^3\sqrt{7875095 - 2922894\sqrt{5} - 8\sqrt{1544326256345 - 678139662118\sqrt{5}}}$ $= ,554008709724955568920352234.$

The Radius of the inſcrib'd Sphere $= \frac{1}{2}b = ,5.$

Fig. 2 Sharp's text showing exact values and decimal approximations (up to 28 digits) of dimensions, surface area, and volume of one of his original solids

and mark the surface to construct slicing planes that pass at appropriate angles through the interior. After cutting off all the exterior material (beyond the slicing planes) the shape remaining at the center of the block is the desired polyhedron.

In Fig. 3, Sharp's Plate I, Fig. 16, parts 1 and 2 show two views of the "very elegant" rhombic triacontahedron, which comprises 30 congruent golden rhombi. His Fig. 17 indicates where the surface of a cube is intersected by the 24 slicing planes needed to release the polyhedron from the interior of the cube. (Each of the 24 slices produces one face, plus six faces arise as the central uncut portions of the cube's original square faces, to make the total of 30 rhombic faces.) He notes that it is "so commodiously cut from a Cube as to exclude all other Methods." This "subtractive" method is a very different way to make polyhedra than the tape-together-paper-faces technique which is now standard for introducing students to new 3D forms. It evokes Michelangelo working with his chisel more than the Euclidean construction of Platonic solids in *The Elements*. Figure 4 is my reconstruction—a rhombic triacontahedron cut from a solid block of wood by following Sharp's diagrams and using his calculations.

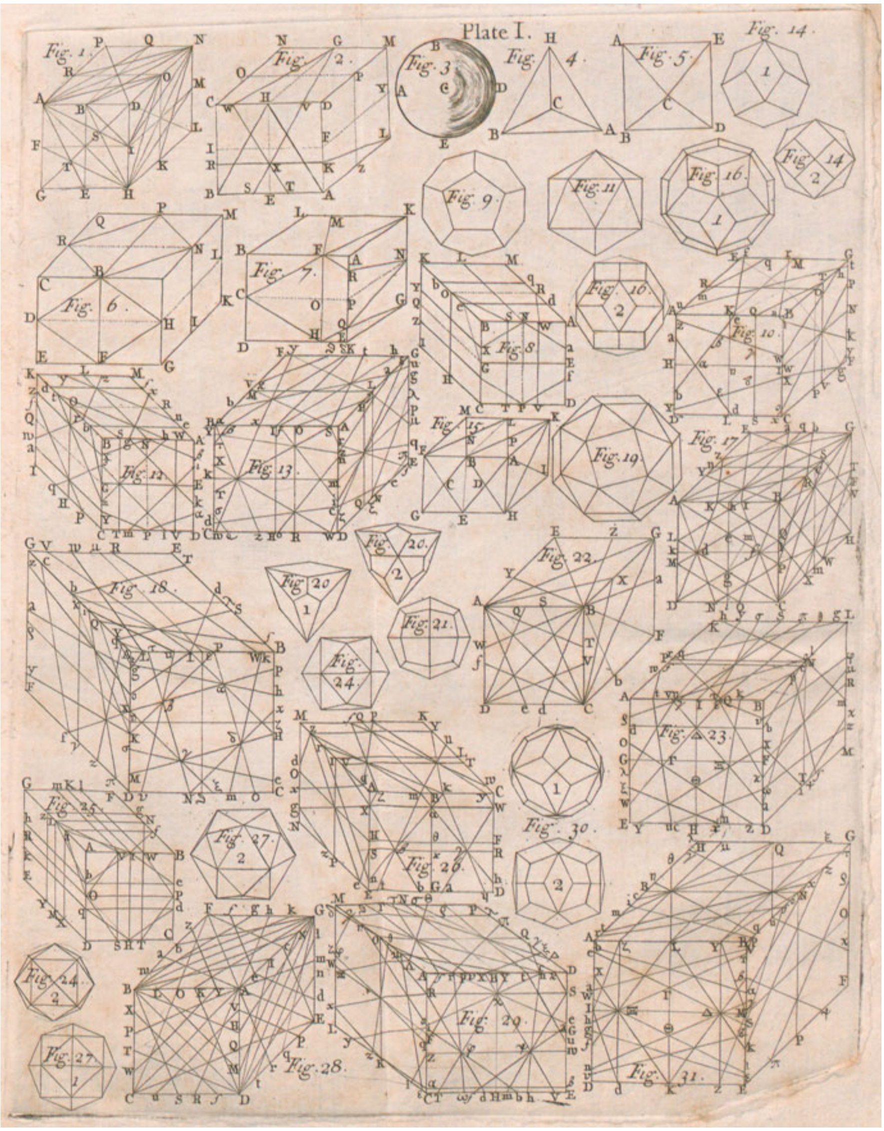

Fig. 3 Plate I of *Geometry Improv'd*

Sharp's unusual methods and laborious presentation undoubtedly contributed to his 12 original polyhedra being long ignored. It is quite an ordeal to carefully read his book and make sense of his geometric intentions. From the dearth of citations to his work in the mathematics literature, I have come to doubt that anyone in the past 300 years other than Sharp and me has ever thoroughly read it all.

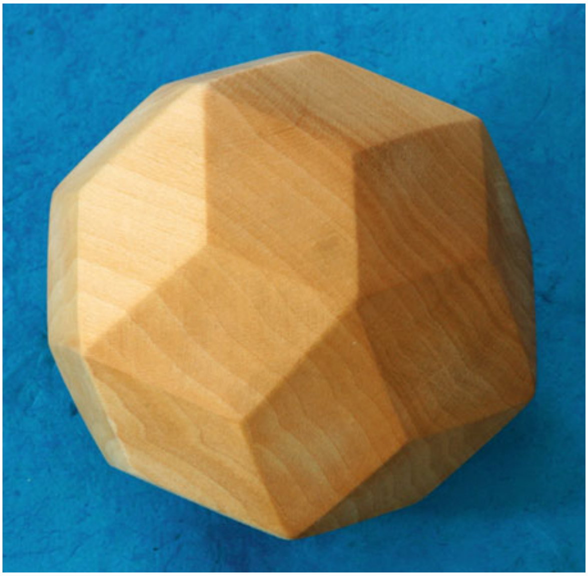

Fig. 4 Wood rhombic triacontahedron, sliced by the author

At least two of Sharp's forms were later rediscovered by others: He describes a body with 24 kite-shaped faces that is now called the "deltoidal icositetrahedron." It is an example of a "Catalan Solid," named after the 1863 paper in which Eugène Catalan described it, the dual to the rhombicuboctahedron, 150 years after Sharp did [3]. And Sharp's body with 60 isosceles triangular faces is exactly what Buckminster Fuller called a "Class II geodesic sphere," an elevated dodecahedron with all vertices the same distance from the center, so that it is inscribable in a sphere.

While the book is certainly troublesome to read, I believe *Geometry Improv'd* has been ignored mostly because Sharp does not discuss the original construction ideas underlying the novel forms. This is not the type of math book that aims to communicate insightful understanding of any underlying structure, nor to explain the relationships between the new discoveries and previous work, nor to teach general methods that might be used in solving future problems. While ostensibly aimed toward a mathematics audience and claiming that the calculations provide "Exercise and Improvement of the Doctrine of Surds," the book does not present the style of organized thought that a mathematician looks for. It is filled mainly with the tedious minutiae of how to reproduce particular examples. The results can only be appreciated after following the laborious steps and attempting to re-imagine the designer's creative process. Sharp also did not explain why one should follow his rules other than to claim that the results will be "Elegant and Beautiful."

The remainder of this chapter is organized as follows: First some background is presented about Abraham Sharp and his unusual book, to place it in historical context. Then the wooden models he created are described, along with the little I have been able to find out about what happened to them after his death. Next, I consider in detail just one of Sharp's 12 original forms, one so notable that I informally call it "the Sharpohedron." My intention is to explain some of what is left out from the book: the mathematical thinking that enriches our understanding of the form and allows us to interpret it in a wider context of geometric knowledge. No doubt some of these broader ideas inspired Sharp's creative process, but we can only speculate. The concluding section is a more subjective exploration of how Sharp's approach to his work has an organic coherence when he is viewed as an artist.

2 Abraham Sharp's Life and Work

> *Twelve* (I presume) *New Geometrical Bodies* are herein to be exhibited to publick Notice, some of which may possibly be as Elegant and Beautiful (if truly form'd according to the Rules herein prescrib'd) as any of that Nature hitherto known.

With these words from *Geometry Improv'd,* Abraham Sharp introduced his 12 novel polyhedra. That apostrophe-D participle form was common and *X Improv'd* was a meme of sorts in the bookstalls of its time. An online search turns up scores of similar British book titles from the late 1600s and early 1700s: *Trigonometry Improv'd, Surveying Improv'd, Farriery Improv'd, Hocus-Pocus Improv'd,* etc. The Enlightenment was underway and a reading audience sought new knowledge.

The Platonic solids were well known from Euclid's *Elements,* and an assortment of other symmetric polyhedra had sporadically appeared in print. As prelude to his 12 original bodies, Sharp first describes how to cut out from blocks seven solids that had been previously described in the mathematics literature: the tetrahedron, octahedron, dodecahedron, icosahedron, rhombic dodecahedron (called simply "the Body of Twelve Rhombs"), the rhombic triacontahedron mentioned above (called "the Body of Thirty Rhombs"), and the snub cube (discussed below).[1]

Though these seven polyhedra existed in the literature of the time, Sharp's method of cutting them from solid blocks is entirely original. I know of no other mathematics reference, before or since, that presents polyhedra by his technique of calculated slicing. The closest analog is in the ornamental woodworking tradition. For example, the nineteenth-century Holtzapffel reference volumes give the compound miter angles to cut simple polyhedra from blocks and describe turning simple polyhedra on a lathe in a spherical chuck, starting from a solid wood or ivory sphere[2] [5]. From the way Sharp describes first marking the surface of the block to identify the cutting plane and then making cuts along the lines, one assumes this is to

[1] The rhombic dodecahedron, rhombic triacontahedron, and Archimedean solids were described in Kepler's 1619 *Harmonices Mundi* [4], which Sharp could have encountered in his astronomical work, but he does not specify any sources other than Euclid.

[2] Sharp also begins with a sphere in some of his instructions, and as a scientific instrument maker was an expert using and making lathes, but nothing in his book discusses using a lathe to make polyhedra.

be done with a hand-saw. But physical processes and materials are never mentioned, only the abstract geometrical steps.

It is not known how Sharp came to be in a position to make scientific instruments or precision polyhedra. He was born in Little Horton, near Bradford, in Yorkshire, but much of his life is undocumented, starting with his year of birth. A valuable early reference, the entry on Sharp in Charles Hutton's 1795 *Philosophical and Mathematical Dictionary,* says "he was born about the year 1651" [2]. But in 1889, almost 150 years after Sharp's death, when a trove of his correspondence was discovered, William Cudworth wrote the only book-length biography of Sharp and claimed he was born in 1653 ([6], p. 200). Recent scholarship is more cautious; e.g., the *Oxford Dictionary of National Biography* only states his documented baptism year of 1653 [7].

Sharp learned his Latin and mathematics at a local grammar school in Bradford, before being apprenticed to a wool merchant in York at the age of 16. But that did not work out. Cudworth states that Sharp "did not take kindly to the yardstick and counter" and that his interests "unfitted him for dealing in dimity and calicoes." He abandoned his apprenticeship and fled to Liverpool where "he opened a day school and taught writing and accounts" and met the astronomer John Flamsteed. Before long he was working for Flamsteed and in that capacity he created one of the most celebrated astronomical instruments of its time, the Mural Arc at the Royal Observatory, Greenwich [8].

Flamsteed later wrote: "The making of this [instrument] was principally the work of Abraham Sharp, my most trusty assistant, a man enriched with gifts and resources of every kind to render him competent to complete a work so intricate and difficult" [6]. Elsewhere, he calls Sharp "a man much experienced in mechanics, and equally skilled in mathematics." Hutton gushed: "Indeed few or none of the mathematical instrument-makers could exceed him in exactly graduating or neatly engraving any mathematical or astronomical instrument, as may be seen in [the Mural Arc], or in his sextant, quadrants and dials of various kinds; also in his curious armillary sphere, . . . , his double sector, with many other instruments, all contrived, graduated, and finished, in a most elegant manner by himself." How exactly Sharp went from the wool counter to world-class expertise in instrument making and the art of engraving is a mystery. Much of his knowledge appears to have been self-taught.

After completing the Mural Arc in 1690, Sharp left Flamsteed's employ. His occupations are not certain for a few years, but likely included teaching and work as an instrument maker. Then Sharp's eldest brother died in 1693 and he moved to the family estate at Little Horton in 1694 to manage it for his widowed sister-in-law. When his nephew died in 1704, Abraham Sharp became heir to the estate and apparently enjoyed the gentlemanly leisure to work on whatever pet projects he wished. He spent the rest of his life as a wealthy recluse at Little Horton, unmarried, traveling hardly at all and receiving very few visitors, but connected to the scholarly scientific world by correspondence.

An undated portrait of Abraham Sharp (by an anonymous painter) on display at Bolling Hall, Bradford, shows him in gentleman's clothing, holding some of his instruments [9]. In 1744, 2 years after his death, this portrait was the model for a

Fig. 5 Engraved portrait of Abraham Sharp by George Vertue

printed engraving of Sharp by the prolific engraver George Vertue [10]. See Fig. 5. In this posthumous engraving, a frame-within-a-frame motif places Sharp in a separate realm from his earthly instruments and a rectangular panel is portrayed that includes a geometrical figure (discussed below). A detailed study of this portrait [11] notes that Vertue sold themed collections of engravings, e.g., of poets or royal houses, and this portrait of Sharp "represents perhaps the earliest attempt by members of the popular culture to 'possess' collections of intelligent and powerful people by buying posters."

As a gentleman of leisure, Sharp continued to make instruments and chose to occupy himself at length with meticulous calculations. He carried out his computational work for no fee, simply for his own entertainment. Sharp volunteered as an astronomical computer for Flamsteed, calculating predicted positions for the planets, the moon, and Jupiter's satellites, plus tables of data that eventually appeared in Flamsteed's posthumous *Historia Coelestis Britannica*. In 1699, he worked out "for his own amusement" (using 150 terms of an infinite series for the arctangent function) the decimal value of π to 71 digits, doubling the world record of the time. He did this in two different ways so he could check for correctness. He also produced a table of 60-digit logarithms, published in 1705 by Henry Sherwin. These are impressively difficult accomplishments, but in addition, at some time in this period he also produced the original polyhedral designs, 20-plus digit calculations, detailed engravings, and wood models of *Geometry Improv'd.*

What motivated Sharp in his polyhedra project? He suggested no practical application for his 12 new bodies.[3] He received no payment for his book from the publisher, only a dozen printed copies. He did not even seek authorial fame, modestly publishing under the pseudonym "A. S. Philomath." He certainly understood that 20 digits of accuracy gave far more precision than any physical construction could ever require.[4] And yet he put an enormous amount of time and work into this project. Sharp gives no indication of intent other than a desire for beauty and elegance. His 12 solid bodies and their peculiar presentation are a gift to the world from an obsessive, passionate creator. For me, they are art.

Apparently, the world was not ready for this type of art, as *Geometry Improv'd* has been little noticed in the ensuing three centuries. Some copies must have sold as there was both a 1717 and a 1718 printing, but I conjecture that the demand was mostly for the practical tables in the first half of the book. There are almost no citations of the geometry portion of the book, and it had zero impact on the subsequent development of polyhedra theory.[5] But I do not think that would have bothered him. Sharp was a truly modest and generous man, known for his peculiar habit of donating to the poor. He would walk through town every week with his hands behind his back holding a pile of coins. People in need could come up behind him and pluck money from his hands without having to reveal themselves. I see a parallel in his artistic generosity: we all are invited to follow in his footsteps and pick out what we wish from what his book offers.

[3] It is not that Sharp was a "pure" mathematician who eschewed all applications. He was very applied in his instrument making and seems enthusiastic in describing how the faces of a snub cube could be used as the planes for a set of sundials. Nowadays, a utilitarian might propose using new polyhedra for dice.

[4] Starting with a block the size of the Earth, 20 decimal places gives a resolution smaller than an atom. I have spot-checked a few dozen of Sharp's calculations and found he is usually correct to as many digits as he lists, though he is occasionally incorrect in the least significant three to five digits. For example, one of the distances to mark off when constructing the icosahedron is (1/2)Sqrt[15]-Sqrt[12], which he gives as 0.2044408655348311488923, but the correct value is 0.2044408655348311490622.

[5] I learned of the book from the epigraph and a footnote in the Polyhedra chapter by Coxeter in [13]. But even Coxeter does not delve into the technical details, merely commenting how Sharp's figure with 90 faces "somewhat resembles" a later polyhedron discovery, the rhombic enneacontahedron. (I have checked that it is topologically equivalent, but geometrically distinct.) The only other reference I know in the mathematics literature is in a discussion of Descartes' understanding of polyhedra [14], where the modern editor cites *Geometry Improv'd* as a general indication of the contemporary state of the art. In the world of fine art, Raphaël Zarka references Sharp in a sculpture series consisting of wood beams marked with burned lines [15]. An interesting paper from the Oxford Master's program in Literature and Arts takes an interdisciplinary look at Vertue's work and is the only printed reference I know that specifically discusses any of Sharp's individual polyhedra, the "Solid of Eighteen Bases" [11].

3 Sharp's Boxwood Polyhedra Models

Sharp's presentation in *Geometry Improv'd* is purely geometric. We are instructed to cut off prisms, pyramids, or segments from a cube, but never to saw a chunk from a block of wood or other concrete material. There is no mention at all of physical cutting tools such as saws. In this purely abstract realm, the book makes no mention that Sharp actually made wooden models. When I first read it 25 years ago, I wrote a web page emphasizing how his thinking was like that of an artist [16]. But I did not know then that he had also made wooden models of his 12 solid bodies, or I might have been tempted to also describe him more specifically as something of a sculptor. I was surprised years later to happen upon Hutton's summary of Sharp's life and to read: "...the models of these polyhedra he cut out in boxwood with amazing neatness and accuracy" [2].

For Sharp to make accurate wood models of his polyhedra was an impressive accomplishment. His more complex bodies have 60, 80, 84, 90, and 120 faces, as can be seen in his Plate II, Fig. 6. It is one thing to calculate precise dimensions on paper, but quite another to be true to them in the physical world. Why did he not mention that accomplishment in his book? This might be another example of his modesty, or perhaps Sharp wanted to position his discoveries in a lofty mathematical realm and not debase them with Earthly matters? I wish he had written at least a little about the woodworking process so we would have more specific information about his methods. I have reproduced all 12 of his original polyhedra by 3D printing, in which the machine is responsible for the fabrication accuracy. But I have made only a few from solid wood blocks by his slicing method, which demands considerable time and care to obtain an exact result.

If Sharp had written about the making of his wooden polyhedra, perhaps they might have been recognized as an important part of his work and so better preserved. I have contacted many museums and historical societies trying to track them down, but with little success so far, mainly finding tantalizing records in a few old collections, since dispersed. His 12 original bodies are not conserved in any major museum as they deserve, but I still hold some hope that they may be sitting as under-appreciated curios on a dusty shelf in some British manor house, waiting for an informed viewer to recognize them.

The earliest evidence I have found of the wood polyhedra is in a letter from Sharp to Flamsteed, dated February 2, 1701-2 [17]. Their correspondence generally concerns astronomical matters, but in this letter Sharp mentions: "I have some time agoe made 12 new Geometricall Bodys, ... some of these you may possibly have seen or heard of per Mr Kirk when he was at London about 2 years agoe, whither at his desire I sent 3 of them to him." On February 6, Flamsteed, who was a Fellow of the Royal Society, responded "I heard your bodyes were before the society but saw them not." It is unclear if Flamsteed is saying the polyhedra were physically displayed at a Royal Society meeting, or if someone gave a presentation talking about them, or what precisely. I have not been able to locate any more concrete information, but this suggests that additional documentation about the wood models

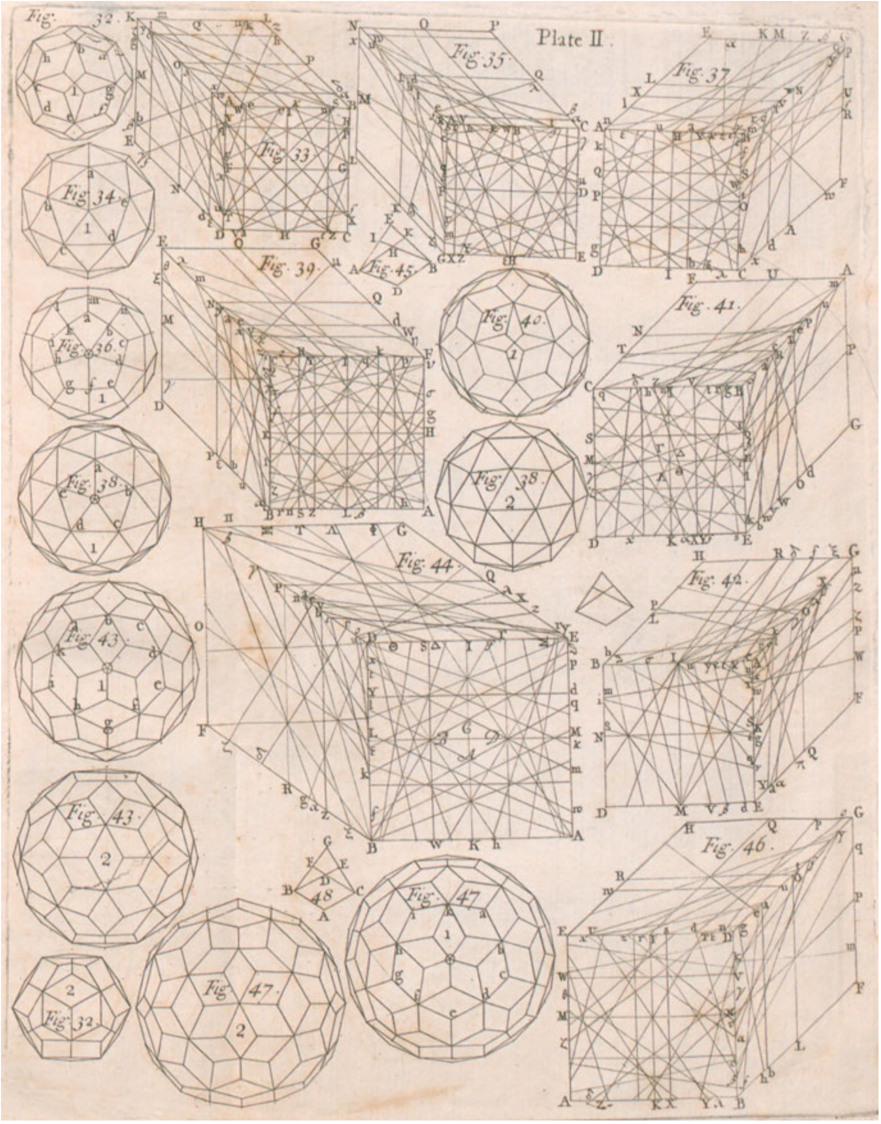

Fig. 6 Plate II of *Geometry Improv'd*

may be preserved somewhere in the minutes of a Royal Society meeting or in notes or letters of someone who attended.

As to the three models that Sharp says he gave "Mr Kirk," they seem to have left more of a trail. This was Thomas Kirke, FRS (1650–1706), who is occasionally mentioned in Sharp's letters and Cudworth's biography. Seven years after Kirke's death, a 1713 catalog of the *Musaeum Thoresbyanum* lists material both from Sharp and from Kirke ([18], pp. 489 and 498). This museum was a substantial "cabinet of curiosities" rich with relics and marvels assembled by the celebrated antiquarian Ralph Thoresby, who was also known to Sharp. Three of the museum catalog entries are intriguing, though frustrating:

- A Body of thirty Rhombs composed by the late ingenious Virtuoso Tho. Kirk Esq; F.R.S.
- Other larger Mathematical Bodies.
- Some Mathematical Bodies by the curious Pen of the incomparable Mr. Sharp.

Was the listed rhombic triacontahedron truly *by* Kirk, or was it perhaps received via Kirk and actually one of the models that Sharp mentions he gave to Kirk? What were the other large bodies and were they by Sharp? And should we interpret "bodies by Sharp's pen" to mean drawings or are they wood models that he first designed? We can only speculate about these questions as Thoresby's collection was dispersed at auction after his death. The 1764 London sales catalog lists many antiquarian items, but no geometrical bodies [12]. I have found no record of what might have happened to them.

Cudworth's biography contains a three-page inventory of Sharp's Instruments that was "probably made after his death." It includes telescopes, quadrants, micrometers, sectors, a microscope, sundials, a large burning glass, a fine large lodestone, a quantity of boxwood, and much else that one might interpret in the act of gauging a man's life by what remains after his death. One specific entry is very encouraging: "Curious Set of Solid Bodies No 24." I expect that these include the 12 original boxwood polyhedra mentioned by Hutton. Unfortunately, these 24 curious models, along with almost everything else Sharp made or owned, are not to be found. Investigating in the 1880s, Cudworth reported that "Many of the instruments with historic associations have been utterly lost, and the materials of others have had narrow escapes of being put to debased uses. A few only are known to be in safe keeping."

Before Cudworth, the Reverend N.S. Heineken had made a search in the 1840s for any "Relics of the Mechanical Productions of Abraham Sharp" [19]. He tells the sad tale of what happened to many of Sharp's hand-written papers after his death: "many years since, when they had been neglected by the owner of the house and left in a closet, the cook was in the habit of supplying herself from the ample store for the purpose of lighting fires and singeing fowls!" It is painful to imagine Sharp's many pages of hand calculations and geometric diagrams being reduced to ashes in this way. More positively, Heineken is able to report on the location of a few instruments and states: "some geometrical solids, turned in the first-mentioned lathe, now belong to my friend J. Waterhouse, Esq., of Well Head near Halifax." I have found that some items from Waterhouse's collection were passed on to the Halifax Literary and Philosophical Society Museum, and from there some went on to other area museums, but I have found no record of the polyhedra John Waterhouse once held.

Fig. 7 Three wood polyhedra made by Abraham Sharp: rhombicuboctahedron, icosidodecahedron, and snub cube. Image courtesy of Bradford Museums and Galleries

Among all those dead ends, I can report one positive result. Three of Abraham Sharp's wooden models are preserved in the collection of Bolling Hall of the Bradford Museums and Galleries. The only record of their provenance is that they came to the museum in 1916, with the donor recorded as F.S. Bardsley-Powell. This is Sir Francis Sharp Powell, a descendant of the Sharp family, baronet, and member of parliament [20]. He died in 1911, so the donation most likely came from his wife, Lady Powell. The Powells resided in the Little Horton Hall where Abraham Sharp had lived and worked most of his life, so it seems likely these three wooden models simply sat there for 200 years. Might there be others? The building was demolished in the 1960s.

Figure 7 shows the three wooden solid bodies made by Sharp, held in the Bolling Hall museum: a rhombicuboctahedron, an icosidodecahedron, and a snub cube. All three are Archimedean solids that appear in the literature predating him, so these are not any of his 12 original forms. The first two are not mentioned in *Geometry Improv'd*, but could have been included among the simple introductory examples. The first is easy to cut from a cube and the second is trivial to cut after first following Sharp's instructions to make a dodecahedron or icosahedron, so I can see that he might have felt no need to include them in his text.

The third body in Fig. 7 is a snub cube—a lovely, chiral, tricky-to-cut form. It is discussed in detail in the book, though not named as such, being described simply as "another Geometrical Solid, comprehended under six equal Squares and thirty-two equal equilateral Triangles." In Fig. 3, Sharp's drawing is his Fig. 19. Some of the 32 slicing planes for cutting it from a solid cube are indicated in his Fig. 18. This polyhedron had appeared in Albrecht Dürer's *Underweysung der Messung* in 1525 [21], but that is not where Sharp learned of it. In the only morsel of human interest in the entire book, Sharp writes: "The Notion hereof was imparted by a Friend, who understood so much of it as enabled him to draw the several Parts upon Paper or Past-board, and fold them up into a due Form: At his Request I undertook to give a more full and exact Account of all its Parts and Dimensions, and to lay down a regular and certain Method of forming or cutting it." This is notable because calculating the reference points for the slices requires formulating and solving a cubic equation, confirming that Sharp was a creative and able geometer. For a modern derivation, see Lines [22]. Sharp's wormholed model in the Bolling Hall museum may be the oldest snub cube extant.

Guided by Sharp's meticulously calculated dimensions, I have sawn a solid snub cube using a version of Sharp's slicing method. I wonder if I am the first in 300 years to try this. My result is shown in Fig. 8. It is a very curious and pleasing object to hold. I tend to continually turn in my hand as if it were necessary to repeatedly verify that one square and four equilateral triangles meet at each vertex. To guide the 32 cuts, instead of marking the cube's surface, I made a custom miter box. Figure 9 shows the laser-cut plywood parts I prepared; note the slots at the proper angles to hold the saw. Figure 10 shows the assembled box, the cut-off corners, the Japanese pull-saw I used, and the resulting sliced body. The block was rotated after each cut to bring the proper region to the saw slots. The initial cuts removed large corners; later cuts removed smaller remaining bumps. Note that my slicing planes were not precise to 20 decimal places and small errors resulted in faces that are visibly off from equilateral triangles and squares. This was easily adjusted by sanding the surface to produce the final result of Fig. 8. I wonder whether Sharp's instrument-making expertise guided him to be more accurate and not require such sanding.

Sharp did not use a custom miter box in this way. His method was to draw all the lines on the surface of the block and cut at the lines. Typical woodworking techniques to remove material up to a marked line include sawing, planing, chiseling, whittling with a small knife, and sanding. (A lathe with a spherical chuck might also be used, if starting from a sphere.) Careful examination of Sharp's existing models might indicate his exact cutting methods. A complication is that

Fig. 8 Wood snub cube, made by the author

each cut removes portions of the marked lines that will be needed to guide future cuts. Sharp is specific about how to deal with this: “Always observing to draw all the Lines upon the Cube or Parallelepipedon before any Segment is cut off; and new Lines must be drawn upon the Plane made by every Section, from the Termination of the Lines which are cut off on every side.”

We can be confident that Sharp used this method of redrawing the lines after each cut because he gives a warning that could only come from experience. After explaining where to draw new lines in his simplest example, the tetrahedron, he adds: “But the Neglect hereof, where there are more Bases, will involve the Work in an inextricable Confusion; so that ‘tis of absolute Necessity it should be observed in cutting of all the other Bodies, where the Lines and Sections will be more numerous: Wherefore this caution is to be taken along through the whole.” It was to avoid the drawing, the redrawing, and the possible confusion that I devised the custom miter-box method.

It is hard to know how Sharp’s models would have been perceived in his time: perhaps as elegant educational models, artisanal curios to display, or finely crafted exemplars of a strange mathematical realm. These are the sorts of categorization that would have put them at home in an eighteenth-century cabinet of curiosities. Certainly, they would not have been considered fine art or sculpture in any modern

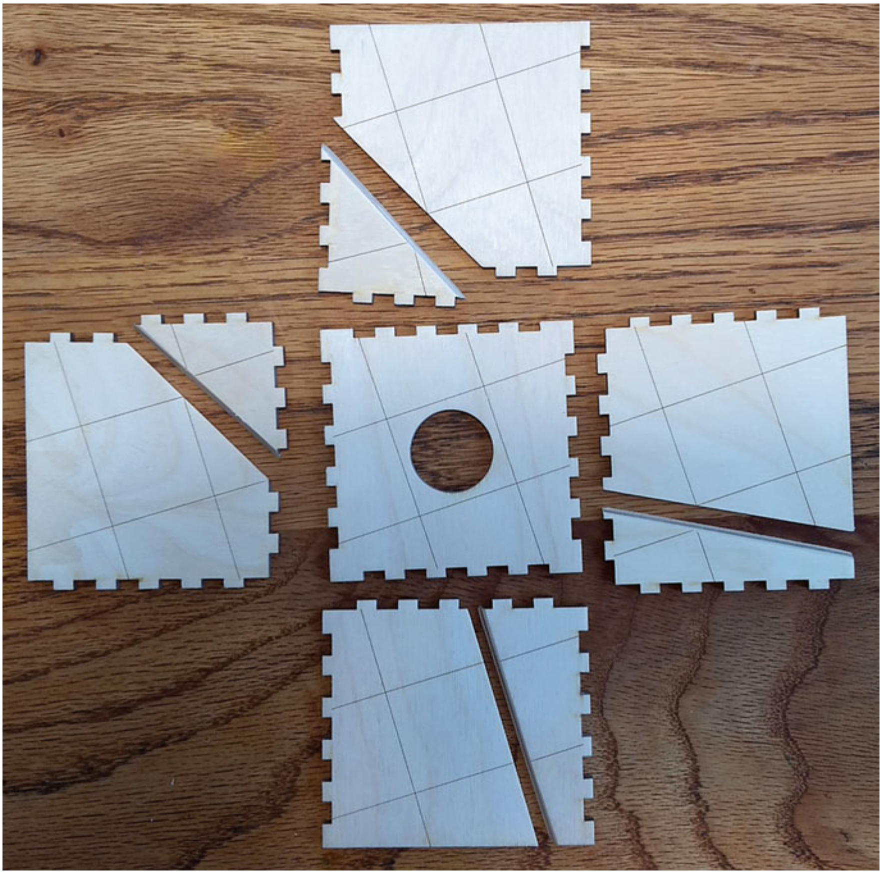

Fig. 9 Laser-cut parts to make miter box to guide slicing

sense, as this was long before the idea of non-representational sculpture. Yet, I am certain Sharp enjoyed the process of creating them in the same way any sculptor must enjoy bringing his or her imaginings to reality. Only a creative passion could lead someone (especially a gentleman of leisure) to invest such focused labor into a block of wood. One can hope that however his models were dispersed, the new owners recognized some of this artistry and kept them safe somewhere. Thus, it could happen, over time, that more will come to light.

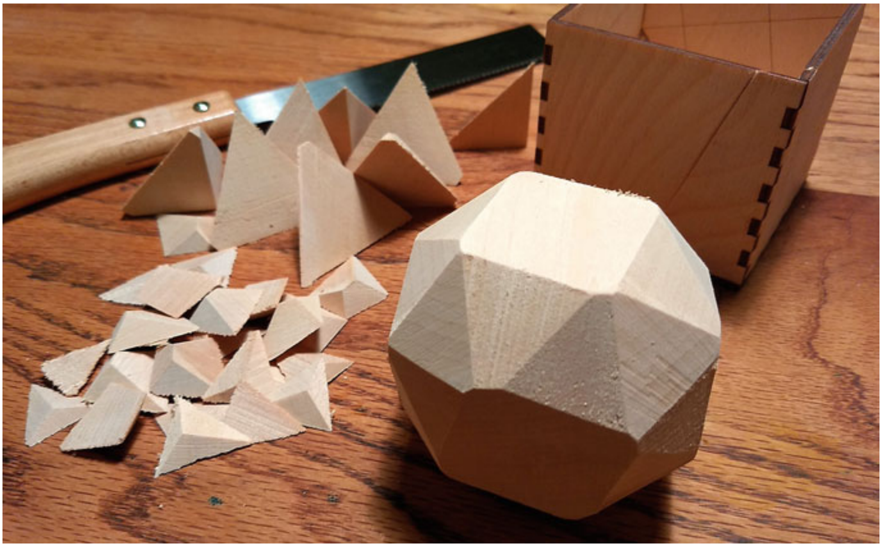

Fig. 10 Wood snub cube sliced by the author (before sanding), with custom miter box, saw, and parts removed from the solid cube

4 The "Sharpohedron"

I have studied and reproduced all of the polyhedra in *Geometry Improv'd*. With some straightforward graphics programming, a patient coder can translate Sharp's descriptions into computer images or 3D-printed models that are faithful to his geometry. Even though Sharp does not describe his creative intentions, his numbers are so precise that one can test any hypothesis and know for certain whether or not one is thinking of exactly the same shape that he had in mind.

Analyzing all 12 of Sharp's original forms is beyond the scope of this chapter, but to do some honor to his vision, we must at least take up one example and see what it is about. Sharp's first and simplest body has 18 faces. In the 300 years since *Geometry Improv'd* appeared, it has not been rediscovered, reproduced, analyzed, or even mentioned once in the mathematics literature. Yet, I find it so worthy that I have dubbed it the "Sharpohedron." Of course, with random slices anyone can create an infinite variety of polyhedra, so why would any particular one be worthy of a name? Indeed, why does "the Mona Lisa" or any human creation warrant a name? Let me try to explain what makes this polyhedron notable and suggest some reasons why, after he imagined it, Sharp might have decided to physically build it and then write about it in such detail for us.

Figure 11 shows four versions of the Sharpohedron. The first I made from a solid block of basswood following Sharp's precise marking-and-slicing method. The second is made with scissors and paper, by cutting and taping together polygons, following the template of Fig. 12. (The tape is hidden on the inside.) The third is

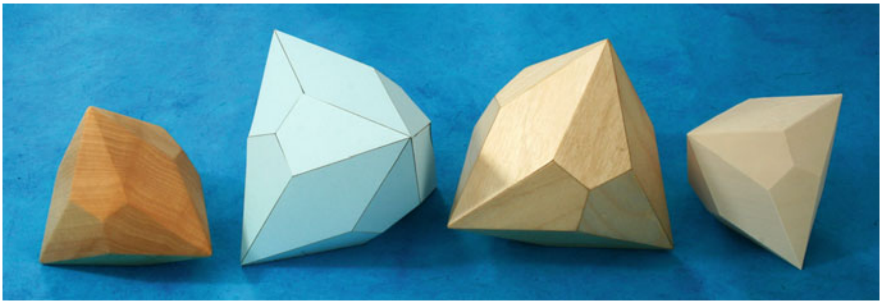

Fig. 11 Four versions of Sharpohedron: solid wood, paper, assembled wood, and 3D printed

made by cutting out 18 individual faces from plywood, beveling the edges to mate at the proper dihedral angles, and gluing them together. This technique allows for a larger, lighter, hollow wooden model. It is akin to the paper version but stronger, and enriched by the luster and feel of wood. The last is made of ABS plastic by 3D printing [23]. I recommend the reader make some kind of physical model to aid in following the analysis below. The fastest method is to tape together a paper model using the template or data in Fig. 12.

With a model in hand, one naturally turns it about to see it from all angles, explores it tactilely to feel its geometry, passes it from hand to hand while trying to understand its structure, and eventually explores the different ways in which it rests on a flat surface. I imagine that Sharp went through such an appreciation process and expected that his followers might come to know each of his 12 original solids from a similar experience with models. A fair test of successful understanding is whether one can close one's eyes and form a mental image clear enough for answering questions like "how many vertices does it have?"

One natural reference point for understanding the Sharpohedron is the regular tetrahedron. As Sharp notes, it "bears some Resemblance to a Pyramid."[6] Like a tetrahedron, the Sharpohedron has four "corners," but these are rounder peaks, less

[6] A few comments on symmetry are appropriate here, as it is central to any modern treatment of polyhedra. The Sharpohedron has "tetrahedral symmetry with reflections," i.e., what is now called orbifold type *332. This means that it has four axes of threefold rotational symmetry, three axes of twofold rotational symmetry, and six mirror planes. Sharp worked long before the nineteenth-century development of group theory formalized symmetry, so he would not see it in these terms, but he certainly had an adequate intuitive notion. All 12 of the bodies he created have some sort of polyhedral point group symmetry, with the Sharpohedron standing alone as his only one in the *332 class. (His later ones fall into four other groups: octahedral or icosahedral rotations, with or without reflections.) In his instructions, Sharp typically details a half-dozen cutting planes and then says "and more like these," relying on the reader's intuitive understanding of the correct symmetry for each example. When coding up his slicing process, I found it convenient to specify just one cutting plane, and then generate the others algorithmically by applying the appropriate group of symmetry transformations. If Sharp had formalized an analogous notion, his book could have been even shorter.

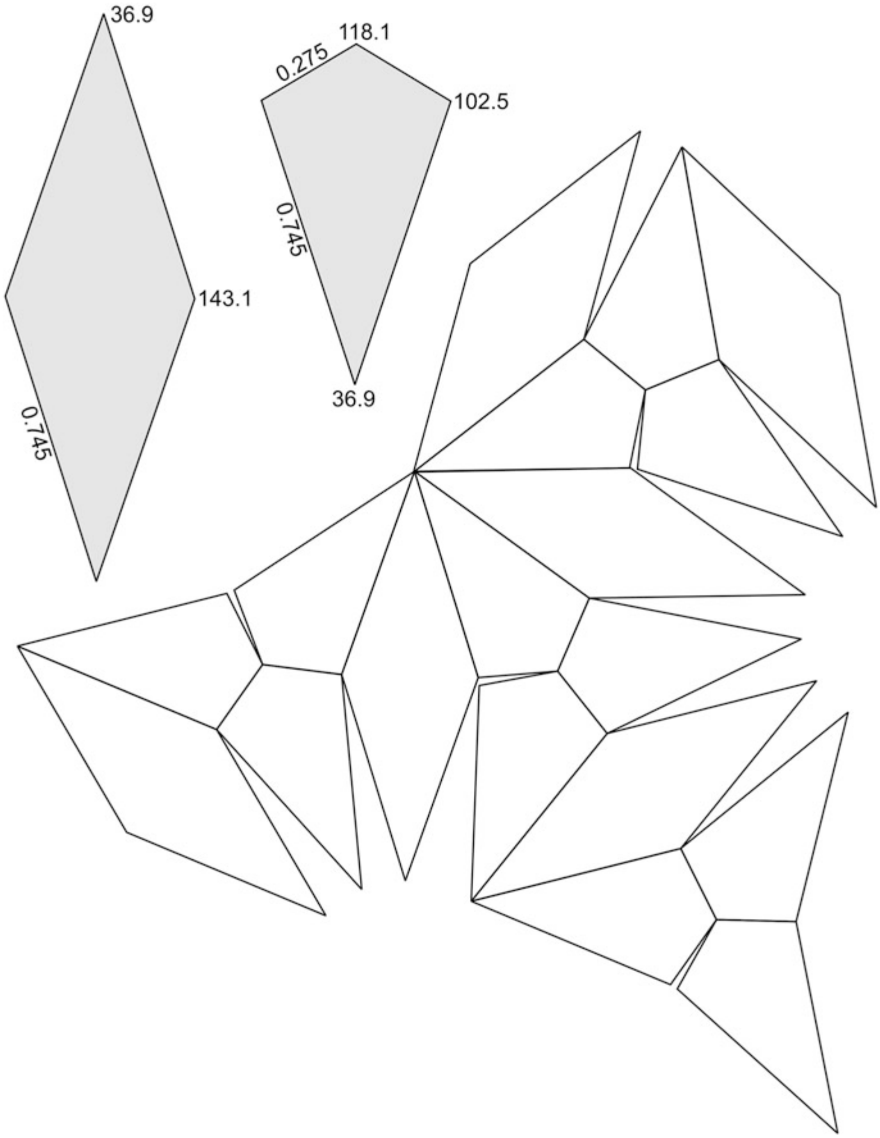

Fig. 12 Net and construction data for Sharpohedron (face angles and edge lengths to fit in unit cube). Print net on cardstock, cut outer lines, crease inner lines, and tape together. The dihedral angles are 48.2° (long edges) and 27.3° (short edges)

penetrating. While the tetrahedron has three sharply knife-like edges incident to each vertex, here we find that six edges with flatter dihedral angles meet at each peak. The overall form is less of a caltrop and more of an overstuffed tetrahedral pillow, rounded as if it is trying its best to impersonate a sphere. In fact one can imagine a perfect sphere inside it that would be tangent to all 18 faces. The technical term is that the polyhedron is "circumscribable about a sphere."

Examining the individual faces, one finds two types: six congruent rhombi and 12 congruent kite-shaped quadrilaterals. Kite-shaped faces on polyhedra were extremely rare in the literature before Sharp[7] and might be viewed as implying a kind of experimental artistic daring. Seven of Sharp's new solid bodies involve kites, so he needed a term for them. Because a kite can be assembled by joining half of two different rhombi, he calls them "double Semi-rhombs," then shortens the term to "Semi-rhombs." It is evident that the kites of the Sharpohedron are arranged in four groups of three, with each group positioned like a face of an imagined tetrahedron. The groups are separated from each other by the six rhombi, which are stretched out along the edges of the imagined tetrahedron. Depending on how we turn it, the Sharpohedron may present us with one of its four six-sided peaks or with a Mercedes-like insignia of three kites.

Examination shows that the Sharpohedron's kites and rhombi are not independent shapes. A deeper structure relates them. They share their common long edge length, and the acute angle of the rhombus face is equal to the acute angle of the kite face, so six equal face angles meet at each peak. Also, the short diagonal of the rhombus shape exactly equals the short diagonal of the kite. If we view the kite as "a double semi-rhomb," one of the two rhombus shapes that it derives from is the one we see sitting next to it. Thus, each peak is a regular hexagonal pyramid. The overall polyhedron has a threefold rotational axis through each vertex, but if you hold it in your hands so just one peak peeks through, what you see has twice as much symmetry: a sixfold rotational axis.

When it is time to place the Sharpohedron down on the table, we discover that it sits in a peculiar manner. With a kite face down, a peak is not quite vertical. This uncooperative personality insists on leaning a bit to the side. Or it can be placed rhombus-down to show a totally different character: Now we feel some tension in the fact that if the lower rhombus points North/South, there is a parallel one on top, but facing East/West. Turning it 45 degrees, so those rhombi point to the NE, SW, NW, and SE, one may suddenly have an aha! experience of how the form fits snugly into a cubical volume. Of course, if you personally sliced it from a wooden cube that will be no shock, but it may be a surprise for those who first understand it as a pillowy tetrahedron. The experience of sliding it in and out of a snug five-sided cubical box is particularly pleasing.

From Sharp's slicing construction, it is natural to see the Sharpohedron as a special case among a continuum of related forms. First, observe in Fig. 13 how one can truncate the vertices of a cube to produce new solids. There is a continuous range of depths possible and particularly interesting forms arise if one stops either (a) at the depth that leaves regular octagons, giving the Archimedean truncated cube, or

[7] I am aware of only one previously published polyhedron with kite-shaped faces, which is found among Wentzel Jamnitzer's 1658 imaginings [24]. In later centuries they became commonplace with the Catalan solids.

Fig. 13 Truncating the vertices of a cube to varying depths

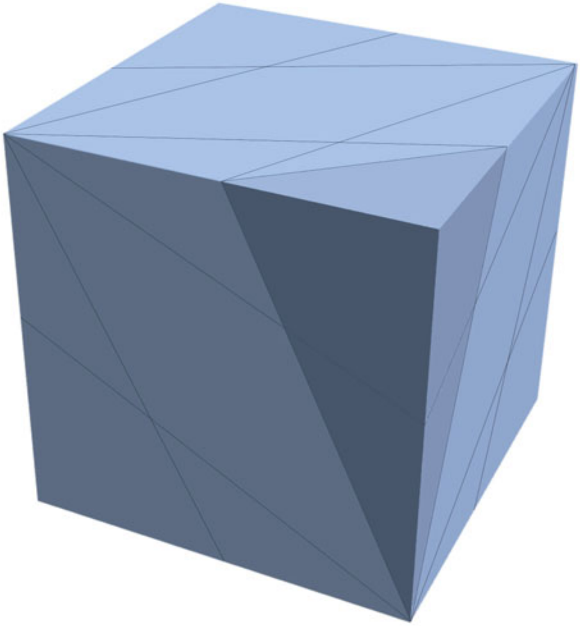

Fig. 14 Cube shaded to indicate one slanted piece (of 12) to remove to make a Sharpohedron

(b) at the cube edges' midpoints, giving the cuboctahedron. Or instead of truncating vertices, a related edge-truncation operation produces the 12 slanted squares of the rhombicuboctahedron (Fig. 7, left). The Sharpohedron construction can be seen as a variation on these familiar cutting processes.

In the text supporting Sharp's Fig. 1 (our Fig. 3), he instructs us to construct the midpoints of the cube's edges and then cut along a plane that goes through one vertex and two midpoints, to remove a corner pyramid like the one shaded in Fig. 14. This is done 12 times, once for each cube edge, to produce the 12 kite faces. The six rhombi remain as the central portion of each original cube face, so 12 cuts suffice to make an 18-sided body. (We have seen this before, as it is also how six of the faces of the rhombic triacontahedron arose.) Figure 15 shows one of the marked cube faces; the shaded portions are removed by four cuts, leaving the central rhombus.

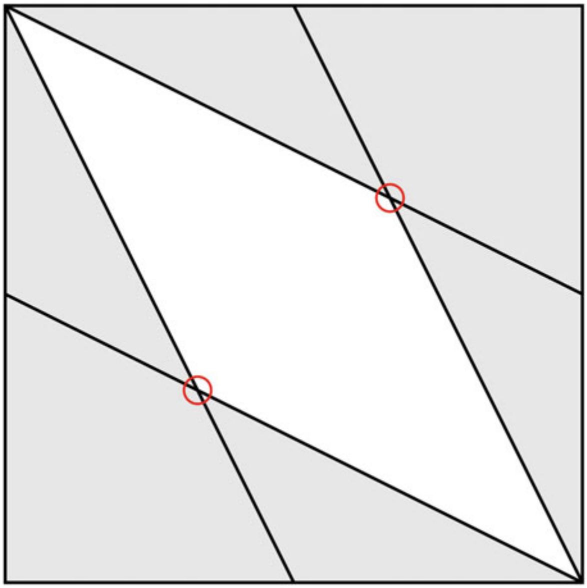

Fig. 15 One face of cube marked to indicate cuts

Fig. 16 Adjusting the depth of the slanted cut through a vertex, varying from a cube to a Sharpohedron to a tetrahedron

A very natural generalization of this process is to imagine varying the angle of the cut by constructing two points an arbitrary fraction of the way along the cube edge, instead of exactly half way. Varying the cut depth produces a family of related forms with different kite and rhombus shapes, shown in Fig. 16. From this continuum, the one Sharp presented is special not just because the slice goes to the edge midpoints. It is also the only one that is circumscribable about a sphere and that creates six equal face angles and six equal dihedral angles at the points. It makes the two circled intersection points in Fig. 15 lie exactly at the one-third and two-third point of the diagonal they lie on, it makes the area of the remaining rhombus exactly one-third the area of the square, and it makes the area of each kite exactly one-fifth the area of the cube face. There is no way to know which properties of this body Sharp found most interesting, but he did specifically state the circumscribability, the rhombus diagonal (Sqrt[2]/3 for a unit-edge cube), and these surface areas.

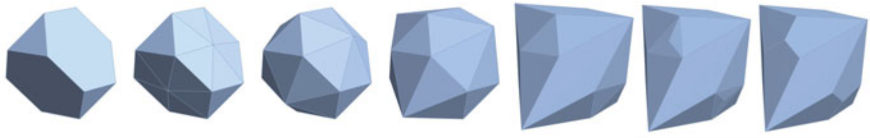

Fig. 17 Starting with a truncated tetrahedron, elevating first the hexagons, then the triangles, to create Sharpohedron

Truncation is just one process for creating new polyhedra from old. Another familiar method is "elevation," i.e., erecting pyramids on the faces of a given polyhedron.[8] Again there is a continuum of possible forms, as we are free to choose the height of the pyramids. Polyhedral elevation has been familiar since 1509 in the images by Leonardo da Vinci for Luca Pacioli's *De Divina Proportione*, in which they elevated polyhedra to a height that created pyramids of equilateral triangles over each face, but there is no record of what Sharp knew. The rhombic dodecahedron can be derived by elevating either the cube or the octahedron; the rhombic triacontahedron can be derived by elevating either the dodecahedron or the icosahedron. In each case, the elevation height is chosen so that pairs of adjacent isosceles triangles become co-planar and merge across an edge to become rhombi. Sharp discusses these relationships without specific mention of elevation or pyramids, so it is not clear if he thought of elevation as a formalized conceptual process. One of his later bodies (consisting of 24 isosceles triangles) is an elevated cube, which he describes as being "compounded of six square Pyramids ... and a Cube." See his Fig. 24, parts 1 and 2, in Fig. 3. Another (consisting of 60 isosceles triangles) can be derived by elevating the dodecahedron and yet another (consisting of 80 triangles) can be derived by elevating the pentagons of an icosidodecahedron. See his Figs. 34 and 38 in Fig. 6. In these three cases, the elevation height puts all vertices on a common circumsphere. But Sharp gives us just the results so we do not know how he came to think of them.

With this background, we note that the Sharpohedron can be simply derived by elevating the Archimedean truncated tetrahedron. Figure 17 illustrates the process. The truncated tetrahedron comprises four regular hexagons and four equilateral triangles. We can first elevate the hexagons into pyramids of a height where three of their isosceles triangles merge with neighbors to become the rhombi, and then elevate the triangles until their isosceles triangles merge with the remaining faces of the hexagonal pyramids to become kites. This is another elegant way to understand the Sharpohedron. If you pick up your model and visualize just the short diagonals of the faces, you can see the truncated tetrahedron hiding within it. This construction explains why the overall shape has threefold rotational symmetry, yet the peaks are regular six-sided pyramids. We can also understand why there are 18 faces:

[8] Some people have misunderstood the term "stellation" and used it for this, but I prefer to use "stellation" as Kepler originally defined it—for a process of extending the face planes [4].

because the truncated tetrahedron has 18 edges. Once elevation is understood, it seems natural to apply it to familiar Archimedean solids, but there is no way to know whether Sharp thought along these lines. He only tells us the final shape.

A different way to specify a polyhedron is to list a finite set of infinite bounding planes, with the understanding that we are interested in the region of 3D space that is interior to all the planes. For example, we can give equations for six planes ($x = 1$, $x = -1$, $y = 1$, $y = -1$, $z = 1$, $z = -1$) to specify the cube that lies interior to them.[9] Abraham Sharp appears to be thinking very much along these lines when he defines cut planes and discards all the exterior regions. For construction purposes, he is very specific in locating points on the surface of the starting block that delimit how the plane intersects the surface, but he also understands more abstractly that any way of characterizing planes determines a unique result. For example, he explains how the rhombic dodecahedron is "apparently deriv'd either from the Cube or Octahedron, by fixing Planes upon every Edge perpendicular to that Axis which passeth from the Center of the Body through the middle of each Edge ...." He similarly finds the face planes of a rhombic triacontahedron as planes through the edge midpoints of a dodecahedron or icosahedron.

With this type of characterization in mind, one might ask what happens with other "natural" sets of planes, for example ones derived from the edge midpoints of other simple polyhedra. (Given a point, a uniquely defined plane to choose, as Sharp notes, is the one that passes through the point and is orthogonal to the line that connects the point to the center of the body.) Once this process is familiar, it turns out that the Sharpohedron can be beautifully and elegantly derived as what is inside the planes defined by the edge midpoints of a truncated tetrahedron. This is my best guess as to how Sharp derived it, simply applying an operation he understood on Platonic solids to the simplest Archimedean solid, but there is no way to know for sure. The conjecture is supported by the fact that *Geometry Improv'd* also includes a body with 36 faces, which can be derived analogously from the edge midpoints of a truncated octahedron. See Fig. 30, parts 1 and 2, in Fig. 3. However, others of his 12 original bodies have separate derivations. Sharp's second body, one with 24 faces, can be derived by applying this process not to the edge midpoints, but to the 24 vertices of a rhombicuboctahedron. Here, Sharp appears to be anticipating the idea behind the Catalan solids, which arise if one defines a plane for each vertex of an Archimedean solid. But he was not consistent in this method; some of his other bodies are approximate Catalan solids with slightly different dimensions.

So there are a number of possible routes by which Sharp might have arrived at his 18-sided body (just as networks of interrelationships arise in any interesting mathematical domain). And there is a variety of distinctive geometric and personality characteristics that could have bolstered his assessment of the Sharpohedron's beauty and elegance. We will never know for sure how Sharp felt

[9] In the twentieth-century formalization of this idea as "Nef polyhedra," one constructs the intersection of a finite set of "half-spaces" defined by the planes. This method can only produce convex polyhedra, but it is suitable because all of Sharp's examples are convex.

about his 12 “children” that he presented in *Geometry Improv’d*, but there is strong evidence that the Sharpohedron was particularly important to him. An indication of its significance can be found in Vertue’s posthumous engraving of Sharp. The geometric figure that appears on the panel within the picture is a portrait of the Sharpohedron, centered on a group of three kite faces and perhaps slightly atilt, as it tends to be. It is unclear how Vertue knew of the Sharpohedron, but for some reason he chose this particular example to embody Sharp’s mathematical work in the 1744 portrait [10]. This was just 2 years after Sharp’s death, so he would have been able to learn from those who knew Sharp which form was most significant.

5 Conclusion: Sharp the Artist

Abraham Sharp’s self-taught geometric genius appears to have come out of nowhere, gone in its own direction, and disappeared, not to be pursued for 300 years. Although biographies label him as a mathematician (or more precisely “an ingenious mathematician, mechanist, and astronomer,” per Hutton), I categorize him first as an under-appreciated artist. Yes, he certainly had the technical skills for calculation and could apply mathematics to solve difficult problems, but there was much more to his creative compulsion. Mathematics was just one of the abilities, along with mechanical skill, craftsmanship, perseverance, and imagination, that he applied toward his artistic aims.

Of course, the term “art” has had differing meanings and associations over the centuries and one cannot claim that Sharp himself saw his polyhedra project as fine art in any modern sense. But certainly he gave us what modern definitions call for: artifacts that express the creator’s imagination, conceptual ideas, or technical skill, intended to be appreciated primarily for their beauty or emotional power. The beauty Sharp saw in his creations undoubtedly included both a visual layer, as one might enjoy the gleaming craftsmanship of a brass scientific instrument, and a deeper mathematical layer, as when one understands how the relationships of truncation, elevation, and other geometric transformations provide a mental graph connecting different structures.

While it is always risky to guess at someone’s intentions, especially one living in times so different from ours, we do have Sharp’s statements that he was interested in creating attractive new works for viewers to appreciate. At the point in *Geometry Improv’d* where he introduces his novel polyhedra, he says: “as an Addition to the Geometrical Store, I shall subjoin Twelve more; none of which (I presume) have yet been expos’d to publick View, and some of them perhaps being more beautiful and elegant than any of the former.” He wants to share his creations, and his exacting calculations might be seen as the extreme perfectionism of an artist who wishes his work to be “just so.”

Mathematicians have ignored *Geometry Improv’d* precisely because it is more art than math. They look to books for logical development and conceptual understanding, not for lists of idiosyncratic instructions. Lacking a definition-theorem-proof arrangement, Sharp's writing does not immediately show a casual reader that he indeed thinks like a mathematician. The underlying structures and relationships that connect his creations to each other and to the existing body of geometric knowledge are not spelled out. A leap of faith is required to invest the time needed to dive in and appreciate his unique contributions.

Sharp worked meticulously and at length both on the calculations and on the woodworking that resulted in his physical models. He saw the visual and conceptual value in his geometric creations and freely presented them to the world, for us to appreciate both in print and as concrete objects. The very human pleasure of visualization must have underlain all this. Both as an instrument maker and as a polyhedron slicer, Sharp would begin with a concept in his mind's eye, one that he felt was worthy of physical existence, and take whatever steps were required to bring it to reality. This is the creative drive which is the engine inside any artist.

The individual character of a work may place it somewhere along a continuum between mathematical creation and fine art. Many beautiful mathematical structures seem to have a certain inevitability and it happens that different mathematicians formulate them independently. (This is why some feel mathematics is *discovered* in a Platonic realm rather than *invented*.) At the other extreme, unequivocal works of fine art typically show the hand of one particular artist. We feel that if Beethoven had never lived, no one else would ever have produced any particular piano sonata of his. Within this spectrum, where can we place Sharp's polyhedra? I see the fact that no one else came up with his Mona Lisa, the Sharpohedron, in over 300 years as a testament to Sharp's personal artistry, but it is the presentation in *Geometry Improv'd* that is absolutely unique. The hand of the artist is evident throughout the book in every square inch of the densely packed engravings and in every 20-digit measurement.

By thinking of Sharp as an artist and his polyhedra project as art, we can make some sense of the enormous number of digits he presents. The unnecessary over-precision of the dimensions might incorrectly suggest (to a reader unfamiliar with his instrument-making fame) that this was the writing of a very abstract mathematician, one out of touch with the practicalities of real-world fabrication. No metalsmith could work to even four digits of precision, so why waste time doing 20-digit calculations and why waste space in the book printing such exact values? One might propose that he was merely showing off his mastery of calculation, of trigonometry, and of logarithms, but that does not accord with his modest nature. I have come to view his over-precision as a form of artistic style. The digits are a kind of decoration he adopted, a brush stroke of sorts, akin to Van Gogh's swirls, Seurat's dots, or Beethoven's tremolos. Sharp is decorating his document with digits in an expression of personal taste, thereby creating a unique impression on the reader.

I see an (admittedly anachronistic) analogy in all this to Sol Lewitt's twentieth-century conceptual artwork. Lewitt emphasized the role of the artist as a thinker rather than craftsperson and gave constructive instructions for wall drawings to be painstakingly executed by assistants, allowing for them to be reproduced over time in different locations. He thereby separated the essence of a visual artwork from its repeatable process, just as a composer's musical score provides a representation that is of a distinct nature from its many possible performances. In an analogous manner, Sharp's presentation is very much an instruction sheet that merely commands us in how to proceed mechanically. Lewitt was happy for different practitioners to adapt his wall drawing instructions to their own particular rooms, wall sizes, wall materials, etc. Similarly, I believe Sharp would be pleased to see his creations rediscovered and newly instantiated not just in boxwood, but in 3D printing, computer graphics, and any other medium yet to be invented.

It took some decades for Lewitt's notion of conceptual art to be broadly understood by the public, but now one finds giant wall drawings of his at large museums all over the world. Similarly, many works of Beethoven and other innovative artists were not immediately popular, but came to be understood over time. Perhaps after 300 years, the moment has come for a wider appreciation of Abraham Sharp's remarkable artistry. It would be wonderful to see the polyhedra of this multifaceted genius wrought large. A wider recognition of his uniquely mathematical artwork would certainly be deserv'd.

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Learning by Metadesigning

Giordano Bruno, Massimo Ciafrei, Claudia Iannilli, Giacomo Fabbri, and Marzia Lupi

Encomium of Metadesign

1 Introduction by Giordano Bruno

My interest and work in the Design area started many years ago. I consider myself a mathematician adopted by Design since I am a Professor in Mathematical subjects at ISIA Roma Design (Istituto Superiore per le Industrie Artistiche) for almost 40 years. I have discovered that Mathematics and Design could share many shining facets in Geometry, Topology, Combinatorics, etc.. In Mathematics of Complexity, such as the theory of complex systems and the *uncertain* logic, we find even more fertile ground.

This paper came to life to celebrate my friend and colleague Michele Emmer 75th birthday.

With my colleague Massimo Ciafrei, ISIA's Professor of Metadesign, we will present here a selection of students' projects originated from the strong interdisciplinarity that intertwined our subjects. As we already made in 2017, in *Imagine Math 6* [1].

Professor Claudia Iannilli, and Doctors in System Design Marzia Lupi and Giacomo Fabbri, have provided their wise and useful help.

I proposed "Encomium of Metadesign" as the title for the essay. There are two main reasons.

First, I believe that *metadesign* is the more fruitful cultural area to develop a project like the space to experience complexity.

It promotes the interweaving of ideas, notations, links, analogies, transpositions, building bridges with other matters. Mathematics, in particular, could supply a whole common language, general and coherent. This concept is central in the

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metaprojects we are going to present; they are carriers of fascinating open thoughts. A gallery of concepts and terms which belongs to systemics: incompleteness, chance, uncertainty and many more that will be readers' pleasure to explore. These projects have a peculiarity that every project, in my opinion [2], should have: to be *systetics* (Systetics is the concept that I coined, merging Systemics, Esthetics and Ethics).

Second, I think that the whole of Michele Emmer's work has been an elegant and refined metaproject. Able of place in touch and comparison mathematics and the culture, expressed in every form. He held innovative and original seminars at ISIA Roma Design about the relationship between Mathematics, Art and Design for more than 30 years.

The conferences he organized in Venice, named "*Matematica e cultura*", and the related proceedings, allowed many people and young students to be part of a very *systetic* event. A magnificent and sparkling blend of ideas that join several subjects, highlighting the cultural, aesthetic and ethic features they share. Therefore, owing him an *encomium* is dutiful for the whole of us.

I want to thank him for all that I learned from him: he has allowed me to travel with my mind and my esprit in the endless space of *beauty*.

2 The Projects

2.1 Raphaël

A becoming project

Designer: Barbara Muratori

A huge wall. Movable, soft, fluid and shining. This image is the lead concept of *Raphaël*. A surface with no defined dimension made up of small modules that let the light filter and sparkle, the air to flow and the time to show off. The modules' shape allows them, once fixed, to move and readjust. The entropy of the system will increase with time. Every touch, every blow of air will change the configuration. Copper is primary for his aesthetic features: his shininess, his oxidation, his warm colour. *Raphaël* is a pondered *non finito*: human interactions will modify its copper surface. Every module will be characterized distinctively by oxidation formed through years. The history of the object and its life will be readable out of its surface *imperfection* (Fig. 1).

2.2 Paròla

Intimate union

Designer: Chiara Simpson

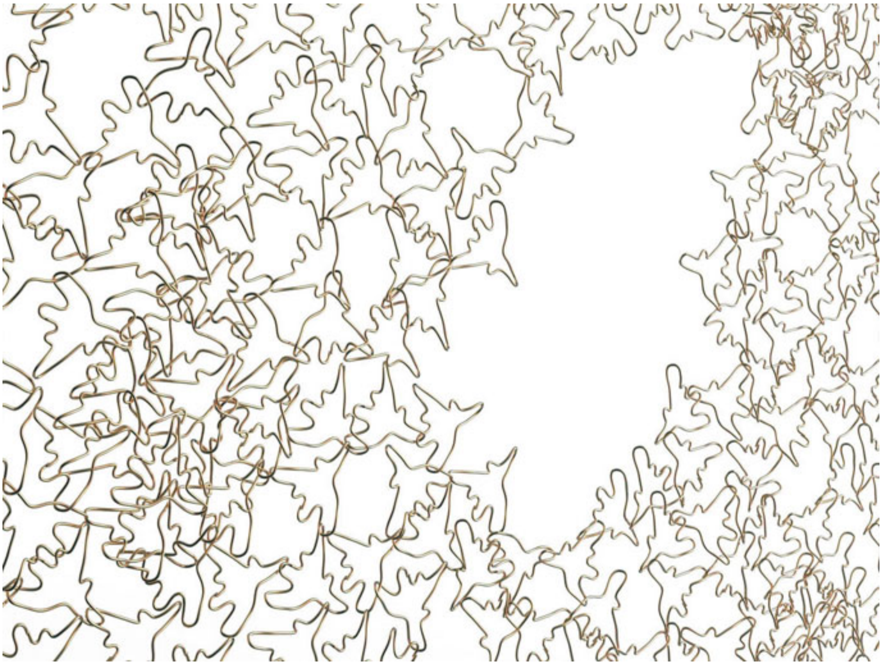

Fig. 1 *Raphaël* by Barbara Muratori

Paròla is an intimate union of parts, spaces and ideas. *Paròla*, which means *word*, is a complex of phonemes through which man expresses a notion in the context of a sentence.

We may consider the term as an abstract engine since it consists of a specific quantity of parts that, all together, if well arranged, produce a meaning.

The project *Paròla* takes form in a series of circular, transparent crowns with a scalar centre. Some rays, traced from the central point towards the perimeter of the object, emphasize the conceptual growth of the figure. Some letters have been arranged on the rays, to outline the *connections* between the crowns. The letters make sense when read from the outside to the inside of the object.

By using *Paròla*, the user has the possibility to construct the visualization of a word with a concrete sense or even of a pseudo-word, which is a meaningless ensemble of letters; enabling the user to work, touch and see something that usually vibrates in the air and has no dimension (Fig. 2).

2.3 Ápeiron - ἄπειρον

Permutation e indeterminacy

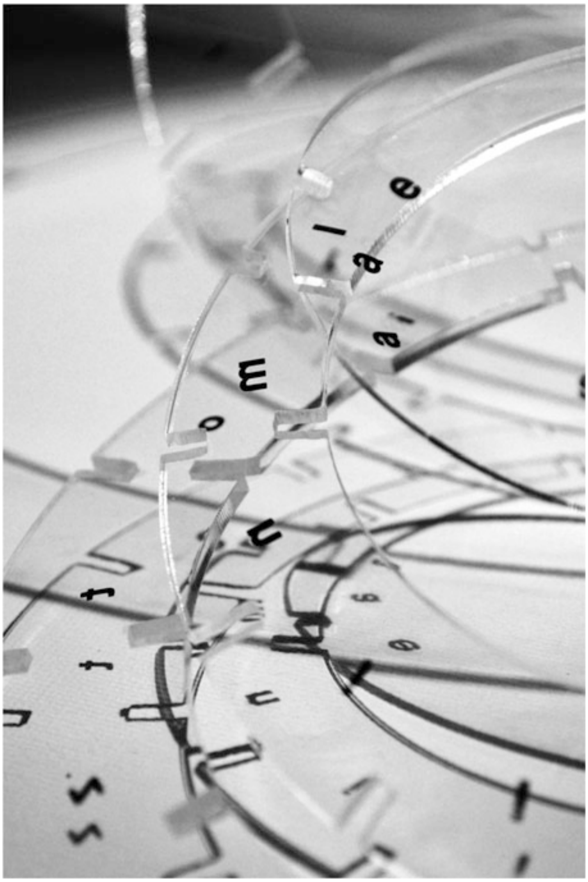

Fig. 2 *Paròla* by Chiara Simpson

Designers: Alessandro Olinteo, Enrico Sciardi

Ápeiron is an ancient Greek term meaning "anything with a not clear definition, with no shape or precise determination". A primordial disposition of reality in which objects are indistinct and share a condition of *uncertainty* and *undefinedness*. *Ápeiron* is the Anaximander concept of *indeterminacy*. The first principle of everything, the primordial state of elements: ἀρχή, the *Arche*. It is the specular peculiarity of sensible world definiteness. Unlimited in both his acceptions. Positive as infinite endlessness that transcends every limit and moves towards perfection. Negative as a substance without boundary and therefore without shape.

The project takes life from the will to replicate the movement in the space to capture his trail (Fig. 3).

The final object is the natural outcome of research and investigation of its shapes and components. Although it is limited in parts number, still capable of endless permutations, formalizations and paths definition (Fig. 4).

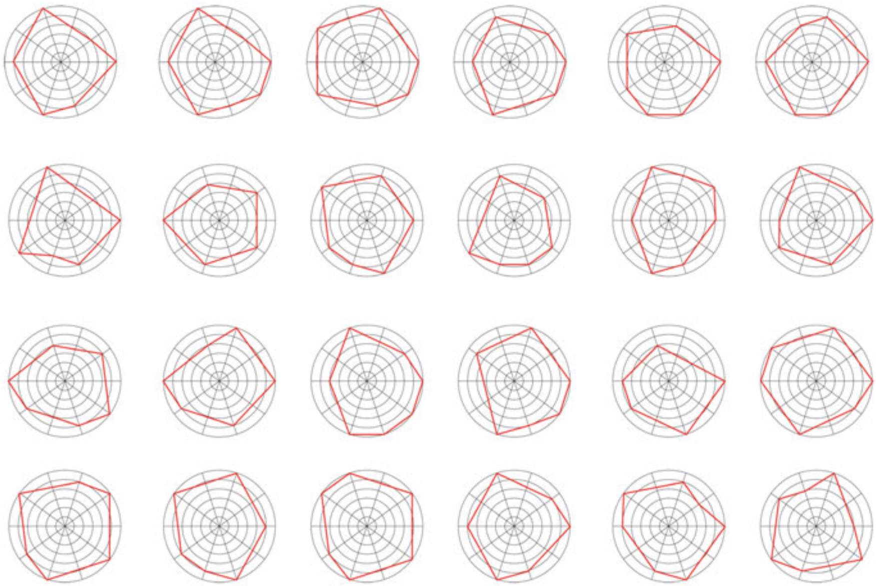

Fig. 3 *Ápeiron* by Alessandro Olinteo, Enrico Sciardi

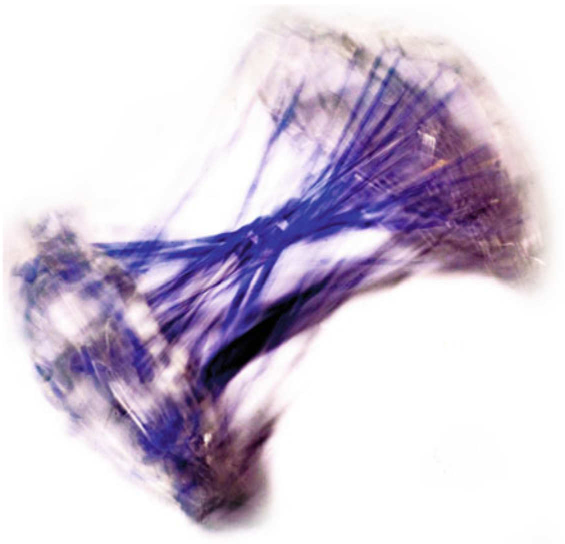

Fig. 4 *Ápeiron* by Alessandro Olinteo, Enrico Sciardi

2.4 Koi

Just a gesture

Designers: Biagio De Vecchi, Marco Di Donè, Noah Gabriel Zandonà

Koi springs from the will to build a spontaneous relation between elements of a system in water. The project has clear analogies with the Japanese iconographic imagery of waterlilies and their unusual behaviour on the water's surface. Despite the appearances, the way it groups is not accidental; it has an impressive *systemic* print that reveals their natural laws.

Koi system's primary components are constructed from geometric and regular shapes by processes of cut and removal. Every module's face is divided into a grid, defining three typologies of sign and asportation. Each *injured* face is marked with colour to underline its cut, its lack of matter. The *Koi* family comes to life when thrown in the water following their buoyancy and their magnetic field generated by an internal magnet (Fig. 5).

2.5 Caustics

Searching relationship

Designers: Alessio D'Angeli, Giulia Celeste Lanzafame, Marco Porpora

The analysis was oriented, since its origin, towards *caustics* behaviour focusing on the relation between water and other materials. The aim was to explore and comprehend how water, interacting with substances and light, creates moods and atmospheres with unexpected, elegant and unique results.

Applying an empirical and graphic approach, from an extensive iconographic investigation, we inferred a *geometric system* that emules what is usually seen in nature.

This transposition underlined the huge utilization possibilities brought from real elements to visual communication.

We did not aim to define use in a finite product. We strived, instead, to define a methodology to research and experience nearby the design project (Fig. 6).

2.6 Tangle

Line, knot, link

Designer: Giorgia Grippa

Tangle is a research stemmed from three key elements: a polypropylene sheet, the cutting process and the creation of a modular *system*.

The final shape in his planar version is deliberately simple. Inside an equilateral triangle, a broken line wraps a triangular hole. A single gesture could reveal the module in his tridimensionality in space as a ribbon, like a helix. The module locks in a three-dimensional conformation, connecting it on itself using a little cut as a slot, gains tension becoming an essential curve in space. Every module could be

Fig. 5 *Koi* by Biagio De Vecchi, Marco Di Donè, Noah Gabriel Zandonà

either a graph's node or part of an edge forming an unlimited gallery of possible aggregations of topological links (Fig. 7).

2.7 Mutaforma

The instability of the balanced poles

Fig. 6 *Caustics* by Alessio D'Angeli, Giulia Celeste Lanzafame, Marco Porpora

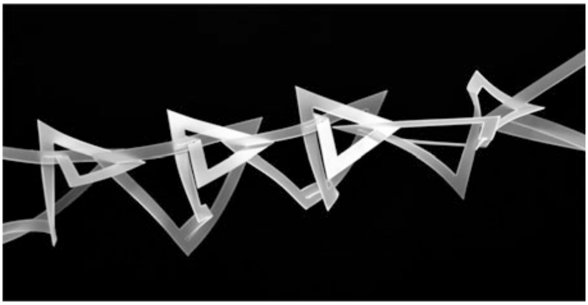

Fig. 7 *Tangle* by Giorgia Grippa

Designer: Angela D'Alonzo

Mutaforma is an unstable body, poised, in extreme precariousness. The maximum degree of expressiveness is given by the casual event, that is the need to contrast the generative rules, through a dynamism of opposing forces.

A metal wire becomes the shape of a body. The rationality of thought is expressed through generative rules to achieve a situation of balance, harmony and coherence.

The need for a demolition of rigour through the destruction of a metal wire that gives birth of an evanescent reality and leads to a world of abstract forms (Fig. 8).

2.8 / Dis-còrde /

Flight casuality

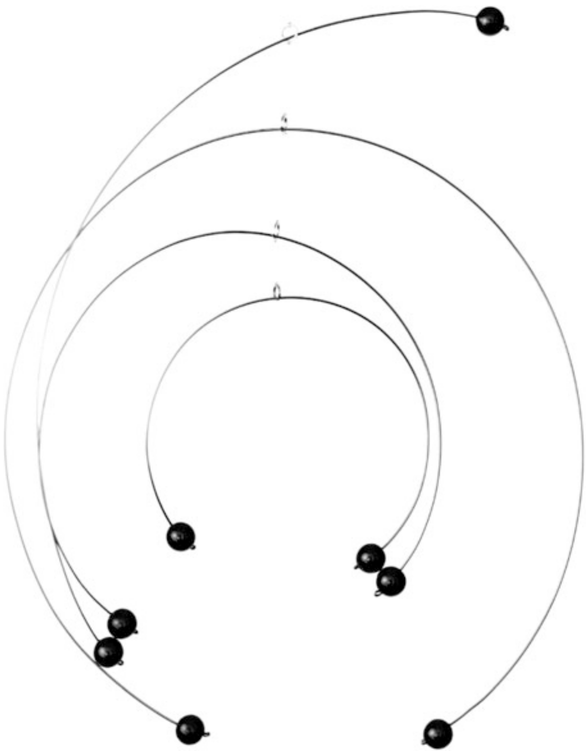

Fig. 8 *Mutaforma* by Angela D'Alonzo

Designers: Giorgia Bortone, Giorgia Malizia

The interest in movement generated by the interaction between parts linked by a cause–effect relationship arises from the observation of nature. From this concept comes the desire to investigate flight with its causes, its effects and its evolution.

/ Dis-còrde / is a system capable of making a helical element fly through the thrust given by the release of tension of two rubber bands wrapped around an axis. The movement will change according to the charge given to the bands.

Through photography it was also possible to study the signs that the propeller creates when moving in the air (Fig. 9).

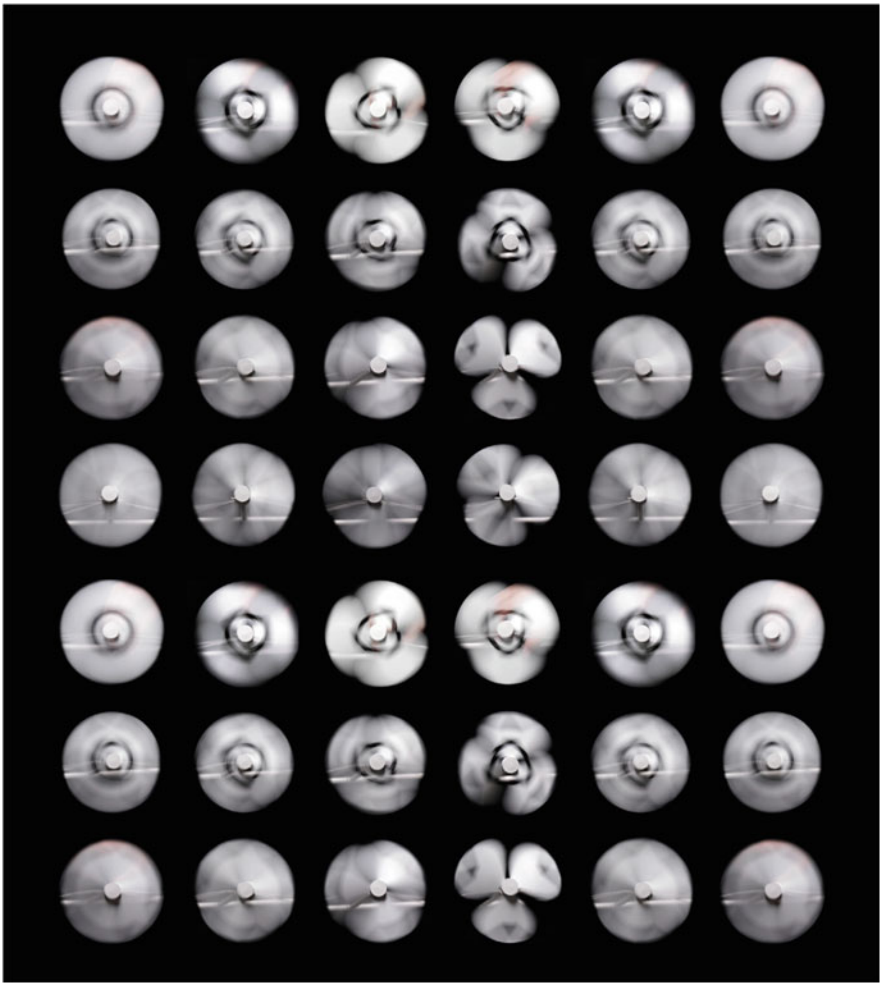

Fig. 9 / *Dis-còrde* / by Giorgia Bortone, Giorgia Malizia

2.9 Tensioni

Taut forms in space

Designers: Elena Berardi, Lidia Catena

The area in which the research is developed concerns the behaviour of matter subjected to tension. The aim is to explore the possible configurations generated in three-dimensional space through the relationship between solids and voids. With *Tensioni,* which means *tensions,* we investigated how static structures that expand

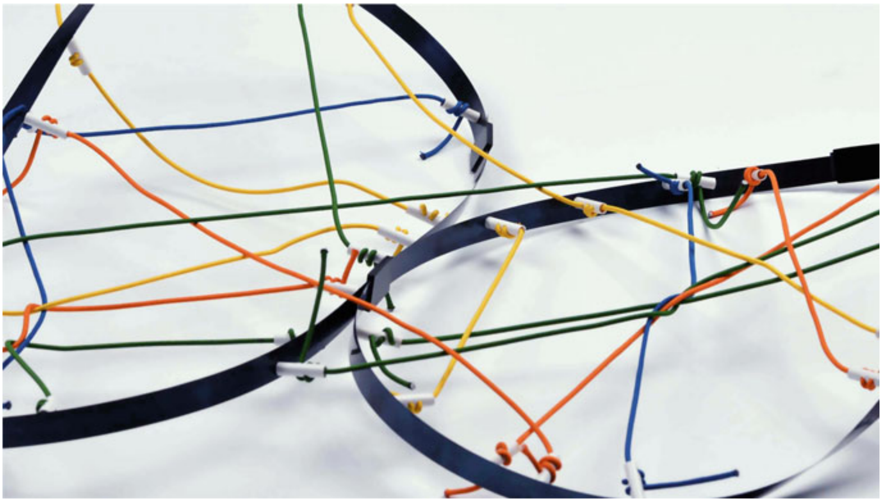

Fig. 10 *Tensioni* by Elena Berardi, Lidia Catena

and articulate in several directions can offer a variety of visual effects. Another starting point for analysis is provided by the suggestions observed by inserting a dynamic component, that is, allowing the material to flex in space and then return to the initial configuration (Fig. 10).

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Part VI
Homage to Roger Penrose

A Little Homage to Roger Penrose

Michele Emmer

Sir Roger Penrose was awarded the 2020 Physics Nobel Prize at the age of 90 (he was born on August 8, 1931) for his research on the black holes of the universe, which he had initiated many decades earlier. He worked for many years with Stephen Hawking, who died in 2018. In Hawking's life film *The Theory of Everything* [1] for which Eddie Redmayne received the Oscar for Best Actor, Roger Penrose appeared briefly, played by actor Christian McKay. Penrose's insights and creativity span many fields. Among his many interests are Escher's works and quasi crystals. Penrose is a kind, helpful person who from the first moment fascinated not only me but all my physics students who spent unforgettable days with him during the Escher conference I organized in Rome in 1985 [2].

1 M. C. Escher and Penrose

The meeting between Penrose and Escher was inevitable. It took place at the world congress of mathematics in Amsterdam in 1954. At the time, Penrose was a student of mathematics. The conference was the site of the first major exhibition dedicated to the Dutch graphic designer. In the film *Escher: Geometries and Impossible Worlds* [3] shot in Rome, Penrose recalls:

> I was told that Escher's prints and drawings would be of particular interest to mathematicians. It was known that I was interested in mathematical curiosities, especially geometric ones. In fact, when I went to see it, I found it particularly fascinating. When I returned to

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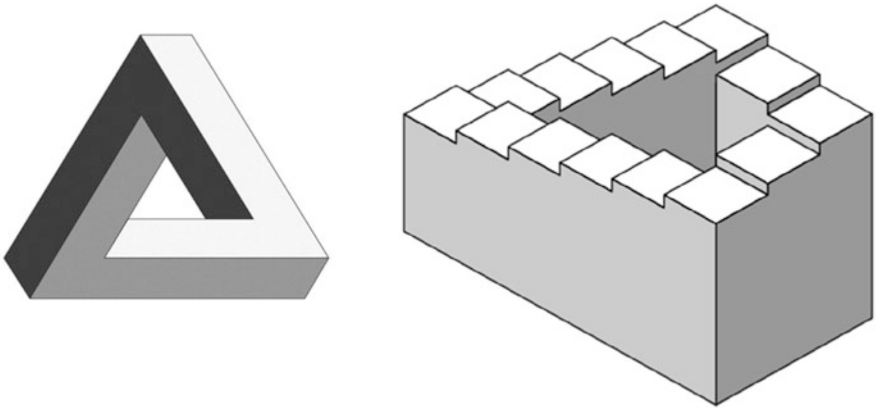

Fig. 1 Penrose' impossible triangle and staircases

England I began to wonder if I would be able to do something geometrically bizarre too, but not quite the same kind of things I had seen at the Escher exhibition.

So I started making drawings that were somewhat impossible. I gradually simplified them until I drew the triangle, which is a kind of thing, how could I say, an impossible object [today it is known as the Penrose triangle] (Fig. 1). What is the idea? It is that in the triangle every part of the figure could exist as a three-dimensional object, but the whole configuration is something that could not exist in space. I showed the drawing to my father Lionel (psychiatrist and geneticist) and he drew a number of impossible figures and came to draw the stairs that always go down in an impossible way. Sometime later we sent an article containing a number of drawings to the *British Journal of Psychology* [4], the Penrose triangle and scales have become a classic not only in visual perception theory but also in psychology]. When it was published we sent a copy to Escher, and he incorporated the designs into some of his works.

Penrose later confirmed in his work *Escher and the Visual Representation of Mathematical Ideas* [5, 6] that Escher had created his engraving Belvedere in the same period but in a completely independent way. In the section of his first book, dedicated to impossible objects, Escher recalls how it was the Penrose drawings that inspired his works *Ascending and descending* and *Waterfall* [7].

It is worth remembering that neither Penrose nor Escher were the first to use impossible objects graphically, especially the triangle. Aside from situations due to more or less conscious errors on the part of artists, it seems that the first was the Swedish artist Oscar Reutersvärd since 1934. Neither Escher nor Penrose were aware of his work.

I met Penrose while shooting the Escher film. If I remember correctly it was the Toronto-based English mathematician Harold Scott MacDonald (Donald) Coxeter who sent me his contact details. Coxeter also participated in the film. Penrose agreed to come to Rome for the shoot. I think it was 1980 or 1981. For simplicity, his two interventions, one for the film on Escher of about 13 min and another on aperiodic symmetries of about 5 min, were made in my living room and in my library studio respectively. Penrose had brought with him the original models of the triangle and stairs. The film was shot on 16 mm with live sound. The details of the two objects were filmed separately and then inserted during the assembly phase. I cut only my questions and a couple of hesitations from the shoot.

2 Penrose and Quasi-Crystals

One of the things that always fascinated Escher were the symmetries of the plane coverings, the so-called tessellations [8], motifs that fill the entire plane in a symmetrical and endlessly repetitive way. Fish, birds, there are many periodic drawings invented by Escher. Only after his death in 1972 were Escher's notebooks with more than a hundred watercolors and periodic drawings found by a collector who had purchased the material left by the artist. The drawings and watercolors of the notebooks were sold one by one by the collector but fortunately, before the destruction of the notebooks, Doris Schattschneider was allowed to have them reproduced along with many other preparatory drawings in the book *M. C. Escher: Vision of Symmetry* [9]. Most of the symmetries present in many works by the Dutch graphic artist are called crystallographic because they are typical of crystals. These symmetries are all periodic. Crystals in nature have the same symmetry properties as those found in the decorations of flat surfaces. For this reason their group of symmetries is called crystallographic; crystals cannot have a symmetry which is not compatible with the structure of these groups. So, for example there can be no pentagonal symmetry. A floor cannot be covered by pentagonal tiles, as there would remain gaps, or it would not be possible to avoid overlaps. Classical crystallography is based on the so called crystallographic restriction, which requires that there are only symmetries of the permitted type.

Can crystalline forms be in the form of an icosahedron and have pentagonal symmetry? Until 1984, the answer was very simple: no, it is not possible. Indeed, the negative response to the question was and is one of the foundations of classical crystallography. In 1984, a new event disrupted some deeply held beliefs: the discovery of a Platonic form where no one thought one could be found: 1984 marks the year of the discovery of quasicrystals.

Roger Penrose wrote:

> Mathematical objects are just concepts; they are the mental idealizations that mathematicians make, often stimulated by the appearance and seeming order of aspects of the world around us, but mental idealizations nevertheless. Can they be other than mere arbitrary constructions of the human mind? At the same time there often does appear to be some profound reality about these mathematical concepts, going quite beyond mental deliberations of any particular mathematicians. It is as though human thought is, instead, being guided towards some eternal external truth – a truth which has a reality of its own, and which is revealed only partially to any one of us.

One of the most interesting examples of how mathematics is *unreasonably* tied to physical reality regards some mathematical results by Roger Penrose. In November 1984 a scientific paper was published by the title *Metallic Phase with Long-Range Orientational Order and No Translational Symmetry* [10]. The authors were the physicists Dany Schechtman, Ilan Blech, Denis Gratias and John Cahn. The publication of this work gave rise to a huge debate among mathematicians, physicists, chemists and crystallographers. The reason is that the paper put under discussion one of the bases of classical crystallography. If a homogeneous material is in the crystalline state it cannot occur, as already mentioned, in the icosahedral form. Although the icosahedral packing is the energetically most favorable way to put together a large number of polyhedra, no crystal can have the symmetry of the icosahedron because a structure based only on icosahedra cannot fill all the space; the reason is that between the symmetries of the icosahedron there is the pentagonal one, and with a pentagon the plane cannot be tessellated without leaving gaps. With an icosahedral symmetry it is impossible to have extensive structures, the disorder that would be created to fill the gaps between the polyhedra would destroy the structure itself.

Moreover, in crystals the presence of a long-range order is synonymous with periodicity, and each periodic structure has an elementary cell that if indefinitely repeated by translations can generate the all structure. In the work published in 1984 even the title was in contradiction with the basic ideas of crystallography: long-range order and no translational symmetry, two properties that seemed in stark contrast to each other. In this sort of quasi crystals, investigated with electron microscopy or techniques of diffraction, a large scale accommodation of pentagonal symmetry was observed, and there was no elementary cell infinitely repeated by translations to form the final structure. In short, as Marjorie Senechal and Jean Taylor, both mathematicians, wrote "The impossible occurred ".

Since 1984, scientists have tried to change their models to take account of these results. In fact they had already been discussing for a long time, in theory, if there could be a new area of crystallography in which it was possible to obtain the pentagonal and icosahedral symmetry. This question was often asked in the late seventies and early eighties of the last century. Roger Penrose had discovered in those years a number of tessellations of the plane which had symmetries that were not permitted in classical crystallography and that were not globally periodic, in the sense that there were regions that clearly repeated themselves here and there in the structure but without a global periodicity. The tessellations discovered by Penrose had a new property, quasi periodicity. The name quasicrystals came from the contraction of quasiperiodic crystals [11]. When asked if his results could be the basis for an entirely new area of crystallography, usually Penrose replied: "In principle, yes; but how could Nature be doing this?" Penrose also made a children's version of some of these tessellations; he changed the two geometric figures that were used to generate the non periodic tessellations, named for their shape *Kite* and *Dart*, into two types of birds. Unfortunately, apart from a few prototypes, the toy version of the tessellation found by Penrose was never built on a large scale: it was considered too complicated.

Fig. 2 Frames from the movie *M. C. Escher* © M. Emmer

In the film *Symmetry and tessellation* [12], Penrose illustrated this sort of mosaic-puzzle with the two types of birds The sequence ended with the composition in animation of the original tessellation. The animation was made in a animation studio with a vertical camera perpendicular to the table with the objects. In the animation you always start from the end and taking one piece at the time you reach the starting point. Than the film is printed in the opposite direction (Fig. 2).

The pieces used, *Kite* and *Dart*, are generated by a rhombus whose angles are respectively 72° and 108°; they are quadrilaterals with angles multiple of 36° and sides of length 1 and φ, where $\varphi = (1 + v5)/2$ is the golden mean. The irrationality of φ is the key to the construction. The fact that the ratio is an irrational number is the fundamental principle that made the tessellation aperiodic. The proportion is rational in a regular tessellation.

The *unreasonable* thing is that if the diffraction pattern generated by the distribution of points at the vertices of rhombs in Penrose's tessellation is generated with a computer, it is possible to obtain the same symmetry of the diffraction images obtained with the quasicrystals! This is a striking example of how mathematical research which is a priori useless, or completely internal to mathematics, became a key point for research in physics and crystallography.

On October 5, 2011 Dan Schechtman was awarded the Nobel Prize in Chemistry for the discovery of quasicrystals. This how the discovery is introduced in the official web site for the Nobel Prize, the article is entitled *A remarkable Mosaic of Atoms* [13].

> In quasicrystals, we find the fascinating mosaics of the Arabic world reproduced at the level of atoms: regular patterns that never repeat themselves. However, the configuration found in quasicrystals was considered impossible, and Dan Shechtman had to fight a fierce battle against established science. The Nobel Prize in Chemistry 2011 has fundamentally altered how chemists conceive of solid matter. On the morning of 8 April 1982, an image counter to the laws of nature appeared in Dan Shechtman's electron microscope. In all solid matter, atoms were believed to be packed inside crystals in symmetrical patterns that were repeated periodically over and over again. For scientists, this repetition was required in order to obtain a crystal. Shechtman's image, however, showed that the atoms in his crystal were packed in a pattern that could not be repeated. Such a pattern was considered just as impossible as creating a football using only six-cornered polygons, when a sphere needs both five- and six-cornered polygons. His discovery was extremely controversial. In the course of defending his findings, he was asked to leave his research group. However, his battle eventually forced scientists to reconsider their conception of the very nature of matter. (Fig. 3)

To illustrate the story of the discovery the Nobel prize's site included five tables that make up a kind of comic book that tells the whole story of quasi-crystals. After the result of Shechtman different types of quasi-crystals have been made in the laboratory, such as alloys of nickel-chromium, vanadium-nickel-silicon, and quasi-crystals have been discovered also in nature. In 2009 a mineral in the Khatyrka

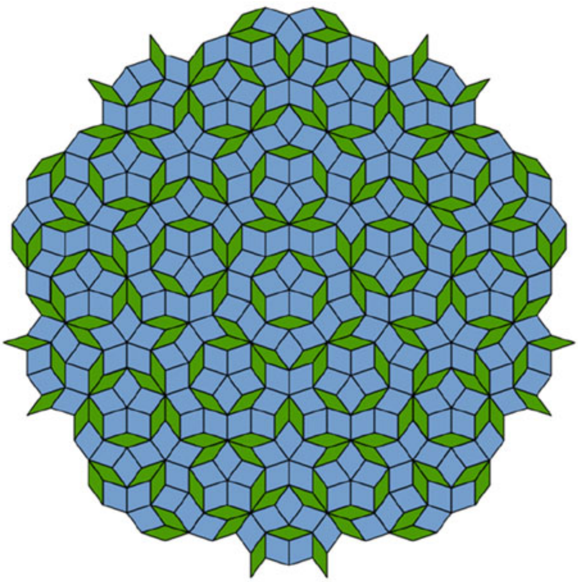

Fig. 3 A Penrose tiling

river in the South East of Russia was discovered. It has been called Icosahedrite, $Al_{63}Cu_{24}Fe_{13}$, (aluminum, copper, iron and other materials).

The specific atomic structure of quasi-crystals makes the corresponding materials robust materials, with a good capacity of conduction and therefore efficient for the transmission of heat and electricity, able to well absorb deformations compared to the crystalline materials. These properties have opened up many prospects for applications.

A few years ago a new surprise, quasiperiodic structures were discovered in some Arab mosaics. They are dated from the tenth to the fifteenth century AD. Examples are found in the Darb-i Imam Mosque (1453) in Isfahan, Iran. In an article of 2007 [14], Lu and Steinhardt thoroughly studied the structures of the decorations found in different buildings of the Islamic world of the past looking for patterns that might be non-periodic, in short, Penrose type. They found plenty of examples on the walls of many mosques in different regions of the world. They managed to reconstruct the way in which craftsmen and artists of that time arrived to assemble the decorative motifs that cover the walls in order to maintain the complicated non periodic symmetry of quasicrystals. This was the case obviously for a relatively small surface. In short, the Penrose tiling, the almost symmetrical structure of quasi-crystals, was known hundreds of years ago! Both contemporary Arab and western architecture have incorporated Penrose patterns in recent years in some of their new building.

Roger Penrose attended the Escher conference in 1985 both as a member of the scientific committee and as editor of the Proceedings. In his article Penrose talks about impossible objects, almost periodic coverings and obviously cites the article by Schechtman and others on quasi crystals that had been published in 1984. He also talks about the origin of Escher's last work, made a few months before his death. In particular the watercolor entitled *Ghosts* inspired by the drawing Penrose sent him. It is the only example of a non-isohedral tiling of the plane by Escher. Penrose explains: "The single shape, out of which the entire pattern is constructed and which covers the whole plane without gaps or overlaps, is placed in more than one distinct way in relation to the pattern as a whole. That is to say, not all occurrences of the shape are on an equal footing: one can find two instances of it such that there is no Euclidean symmetry, taking the pattern into itself, which carries the one instance of the shape into the other. Moreover, there is no alternative way of tiling the plane with this particular shape. It will tile only non-isohedrally."

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Part VII
Mathematics and Physics

Identity and Difference: How Topology Helps to Understand Quantum Indiscernibility

Amaury Mouchet

Soy esos otros,
también. La eternidad está en las cosas
del tiempo, que son formas presurosas. (I am those others too.
The eternity is in the things of time, which are precipitate forms
(trad. AM*).)*

Jorge Luis Borges. *Al hijo* (1967) *in El otro, el mismo.*

1 Introduction

This contribution, to be published in Imagine Math 8 to celebrate Michele Emmer's 75th birthday, can be seen as the second part of my previous considerations on the relationships between topology and physics [17]. Nevertheless, the present work can be read independently. The following mainly focusses on the connection between topology and quantum statistics. I will try to explain to the non-specialist how Feynman's interpretation of quantum processes through interference of classical paths (path integrals formulation) makes the dichotomy between bosons and fermions quite natural in three spatial dimensions. In (effective) two dimensions, the recent experimental evidence of intermediate statistics (anyons) [3] comforts that topology (of the braids) provides a fertile soil for our understanding of quantum particles.

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2 Old Puzzles

Among the most common primary concepts that are jeopardised by quantum physics, the related notions of identity, individuality or discernability are not the least.[1] The question of identifying a material object, even if it is immediately accessible to the common sense, has always raised many philosophical issues once we consider it as the set of its replaceable constituents. Heraclite's thoughts on the dynamical changes and, in particular, the paradox of talking about the "same river" while "waters flows" [12, chap. 5, § 2] has never ceased to nurture Western philosophy (and Borges in particular). Another variation on these questions goes back to an even more remote era when some Greek founding myths were forged: is it justified to talk about the ship of Theseus after some or even all of her original parts have been replaced? However, one can nevertheless suspect that all these issues can be reduced to a matter of semantics (what is meant by "same"), keeping in mind the danger of the inevitable tautology that plagues ontology while trying to explain what "existence" signifies.

An attempt to give some empirical flesh to the question of indiscernibility can be found in the writings of Leibniz who reports an observational test of the *principle of the identity of the indiscernibles* which is now attached to his name[2] [19]:

> PHILALETHES. A relative idea of the greatest importance is that of identity or of diversity. We never find, nor can we conceive it 'possible, that two things of the same kind should exist in the same place at the same time[. That is why, when] we demand, whether any thing be the same or no, it refers always to something that existed such a time in such a place; from whence it follows, that one thing cannot have two beginnings of existence, nor two things one beginning... in time and place'.
>
> THEOPHILUS. In addition to the difference of time or of place there must always be an internal *principle of distinction*: although there can be many things of the same kind, it is still the case that none of them is ever exactly alike. Thus, although time and place (i.e., the relations to what lies outside) do distinguish for us things which we could not easily tell apart by reference to themselves alone, things are nevertheless distinguishable in themselves. [...]
>
> If two individuals were perfectly similar and equal and, in short, *indistinguishable* in themselves, there would be no principle of individuation. I would even venture to say that in such a case there would be no individual distinctness, no separate individuals. That is why the notion of atoms is chimerical and arises only from men's incomplete conceptions. For if there were atoms, i.e., perfectly hard and perfectly unalterable bodies which were incapable of internal change and could differ from one another only in size and in shape, it is obvious that since they could have the same size and shape they would then be indistinguishable in themselves and discernible only by means of external denominations with no internal foundation; which is contrary to the greatest principles of reason. In fact, however, every body is changeable and indeed is actually changing all the time, so that it differs in itself from every other. I remember a great princess, of lofty intelligence, saying one day while walking in her garden that she did not believe there were two leaves perfectly alike. A clever gentleman who was walking with her believed that it would be easy to find some, but search

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[1] See, for instance, the contributions in [5].

[2] plato.stanford.edu/entries/identity-indiscernible.

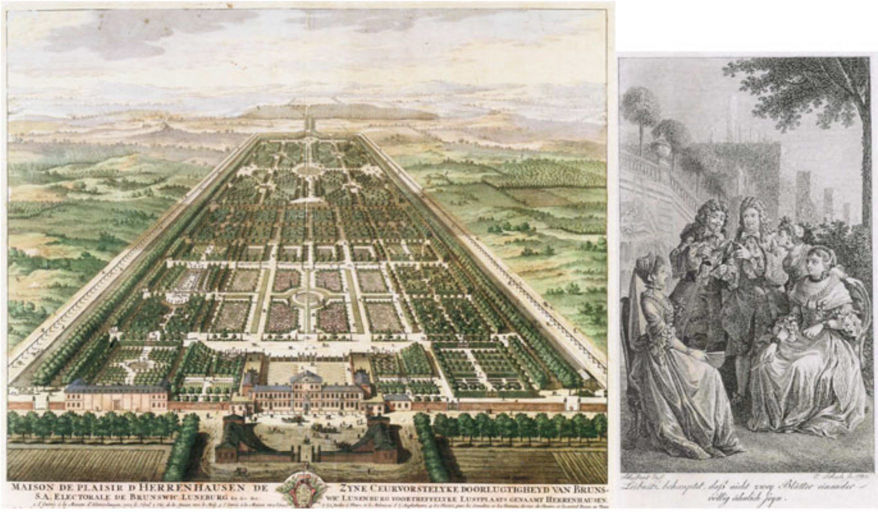

Fig. 1 On the left: The Herrenhäusen Gardens around 1708 (Wikipedia from Gottfried-Wilhelm-Leibniz-Bibliothek, Kartensammlung C Mappe 18 Nr. 178 b). On the right: Leibniz with Duchess Sophie, Carl August von Alvensleben and two ladies-in-waiting in the Herrenhäusen Gardens (wikipedia from a biography of Leibniz by Johann August Eberhard published in 1795)

> as he might he became convinced by his own eyes that a difference could always be found. One can see from these considerations, which have until now been overlooked, how far people have strayed in philosophy from the most natural notions, and at what a distance from the great principles of true metaphysics they have come to be. [13]

We learn from a letter written by Leibniz to Sophie, Electress of Hanover (dated October, 31st, 1705), that the challenge took place in the Herrenhäusen gardens of Hanover between princess Sophie and M. Carl August von Alvensleben, the "clever gentleman". Sure the conclusion of the "experiment" would have been less straightforward if instead of leaves, the bet had concerned bees or ants (since clones are ubiquitous in one hive or in one anthill). However, in the same letter, maybe remembering the geometric patterns of the garden itself (Fig. 1), Leibniz writes

> There are actual varieties everywhere and never a perfect uniformity in anything, nor two pieces of matter completely similar to each other, in the great as in the small. [. . .] Therefore there is always actual division and variation in the masses of existing bodies, however, small we go. Perfect uniformity and continuity exist only in ideal or abstract things, as are time, space, and lines, and other mathematical beings in which the divisions are not conceived as all done, but as indeterminate and still feasible in an infinity of ways. [14, § 68. pp. 327–328, notes 669, 670]

Retrospectively, and somehow ironically, the above argument where perspires Leibniz' aversion against atomism was founded. The existence of atoms does not only undermine the *principle the identity of the indiscernibles* but the refutation is much stronger than Leibniz could have thought: even the difference in position between atoms—a classical notion, on which, as Leibniz explains it, one may always rely to make a distinction between similar material objects—becomes irrelevant at the quantum level as long as it is not measured.

3 Quantum Abandon of Individuality

After a long maturation, mainly done in the first quarter of twentieth century [20, 21], starting with Planck's 1900 work on the blackbody radiation that can be seen as the foundation stone of quantum physics, our concept of indistinguishability of quantum particles proceeds from the quantum theory of fields. Quantum particles, whether considered as elementary or composite, appear as elementary excitations with respect to a ground reference state (the vacuum of the considered particles) that are characterised by a handful of well-determined values which are the only observable quantities that can be attributed *simultaneously* to each of them: the mass, the electric charge, the spin and few other "flavours". For instance, an electron is the particle whose mass is $9.109\cdots\times 10^{-31}$ kg, whose charge is $e = -1.602\cdots\times 10^{19}$ C, whose spin is $1/2$, etc. Other individual observable quantities, even though they remain reasonably stable because of some conservation laws, for instance, the linear momentum, may be affected by an individual measurement of a non-compatible quantity (the two observables do not commute in some precise algebraic sense) including, notably, the position of the particle. In fact, the orthodox interpretation leads to quantum properties that cannot be attributed simultaneously without raising contradictions with observations. According to the famous Heisenberg's inequalities, after an arbitrarily precise measurement of its momentum, it is not that we do not know the position of the particle, it is just that it does not have a precise position at all. In other words, a measurement (say, the component J_z of the angular momentum along one direction z) does not affect the value of the previously measured non-compatible quantity (say the component J_x of the angular momentum along an orthogonal direction x), it completely erases its existence: once J_z is measured, one cannot attribute an even unknown value of J_x to the particle anymore. Therefore any quantum particle cannot have any history nor accidental properties nor contingent secondary qualities that would still allow to individualise it, including its position.[3]

These completely counter-intuitive properties reflect all the more the strangeness of the quantum world that the number of particles is itself a quantity that maybe

[3] Some experiments have succeeded the technical challenge of keeping for several weeks one particle sufficiently localised away from the others. However, on the other hand, the correlations of entangled pairs over long distances show that the individuality cannot be based on spatial separation. Connected to the subject of the present text, the quantum contribution to the Western philosophical analytic-reductionnist/holistic-emergentist dialectics concerning the relationships between the part and the whole is fascinating [16, for instance].

incompatible with other observables.[4] There are quantum *pure* states (that is on which we have the maximal possible amount of information we can conceivably get) where the number of its constituents cannot be attributed.

Even in the cases where the number N of the particles can be attributed and maintained constant because the available energy is not sufficient to create or destroy some of them, the particles of the same species (electrons, neutral alkaline atoms, etc.) still cannot be numbered, even in principle, for such a numbering is nothing but the attribution of a discriminative quantity (essentially based on a position in a given configuration). The consequences are considerable if one wants to study the statistical properties of a set of a such identical particles. We all know that the odds (and the gains) to win at a trifecta horse race are significantly different if we decide to take into account or not the finishing order of the top three horses. In statistical physics, it is energy (not money) that is distributed according to the odds of the configurations and the distinguishability of the particles has observable consequences even at the macroscopical level. The organisation of nucleons in the nuclei, of the electrons on the atoms which explain the chemical Mendeleiev classification as well as the stability of matter, could not be explained if the particles were distinguishable. The conducting properties of materials, notably the superconducting ones, their thermal response, the superfluidity, the behaviour of photons in a laser beam, or the existence of states of matter like a Bose-Einstein condensate are emergent properties coming out from purely collective effect of some set of particles where only the number of its constituent has a physical meaning.

In fact, all quantum particles we know up to now fall into two families according to their collective behaviour. The fermions, whose spin is half an integer, design particles that cannot share two identical states (the Pauli exclusion principle) whereas the bosons, of integer spin, can condensate into the same individual state. All the particles that constitute ordinary matter that we consider to be fundamental are fermions (mainly quarks, electrons). An assemblage of an odd number of fermions remains a fermion whereas any particle made of an even number of fermions (a Cooper pair of electrons, a Helium 4 atom, for instance) follows a bosonic statistics.

One remarkable thing is that this dichotomy can be understood with some topological arguments and the following tries to give some hints about how this works. As a bonus, I will try to explain that for models in condensed matter where we can consider that the dynamics lies in a layer whose effective dimension is $D = 2$, the same topological arguments offer more possibilities. In two dimensions, one may consider some identical particles whose statistical behaviour is characterised by a continuous parameter θ that allows to interpolate continuously between bosons

[4] Some other interpretations try to fix what happens to be, as a last resort, a question of interpretation of quantum probabilities: alike what occurs in the classical world, do the latter reflect a lack of information or not? But as far as I know, these de Broglie-Bohmian points of view[4, for a particularly interesting plea] concern a fixed number of particles only (generally one) and do not venture in the quantum field arena where even elementary particles can be created or annihilated.

(say for $\theta = 0$) to fermions (say for $\theta = \pi$). These existence of these *anyons* (anyons)—the word was coined by Frank Wilczek [22]—has been proposed theoretically by Jon Magne Leinaas and Jan Myrheim [15] forty-five years ago but it is only last year that they received a first experimental confirmation [3].

4 Superpositions, Interferences and Phases

To understand better, the reason why we must give up the systematic attribution of some properties to quantum objects is that their pure states are described in terms of a (linear) superposition of states having definite properties. There is an experimentally accessible manifestation of these superpositions: the interferences they can produce. Think of the most important Young experiment on light diffracted by two holes on an opaque screen that produces an interference pattern. Keeping a pure wave interpretation, the interference pattern is due to the superposition of two waves, each one being diffracted by one hole (the other being closed). But once the two holes are opened it is meaningless to say that the resulting wave has passed though one hole rather than the other. Rather than getting a fuzzy or an unknown path, we actually completely loose the possibility of attribution of a path. These Young-like configurations, as well as others interference experiments, have been set up for individual quantum particles (photons but also electrons, atoms and even organic molecule made of hundreds of atoms). One crucial quantity that is measured in all these interference experiments is the *relative phase* φ between the states being superposed, that is essentially an angle given by the time delay between two periodic oscillations expressed in unit of their common period (like an angle defined modulo one turn, only a delay modulo a period unit can be measured, see Fig. 2).

These relative phases have been identified by Chen Ning Yang as one of the three melodies of theoretical physics in the twentieth century [23] and Dirac, looking back on the development of quantum physics that he contributed to shape, writes

> The question arises whether the noncommutation is really the main new idea of quantum mechanics. Previously I always thought it was but recently I have begun to doubt it and to think that maybe from the physical point of view, the noncommutation is not the only important idea and there is perhaps some deeper idea, some deeper change in our ordinary concepts which is brought by quantum mechanics. [...] [Following Heisenberg and Schrödinger], the probabilities which we have in atomic theory appear as the square of the modulus of some [complex] number which is a fundamental quantity. [...] I believe that *this concept of probability amplitude is perhaps the most fundamental concept of quantum theory.*
>
> [...] The immediate effect of the existence of these probability amplitudes is to give rise to interference phenomena. If some process can take place in various ways, by various channels, as people say, what we must do is to calculate the probability amplitude for each of these channels. Then add all the probability amplitudes, and only after we have done this addition do we form the square of the modulus and get the total result for the probability of this process taking place. You see that the result is quite different from what we should have if we had taken the square of the modulus of the individual terms referring to various

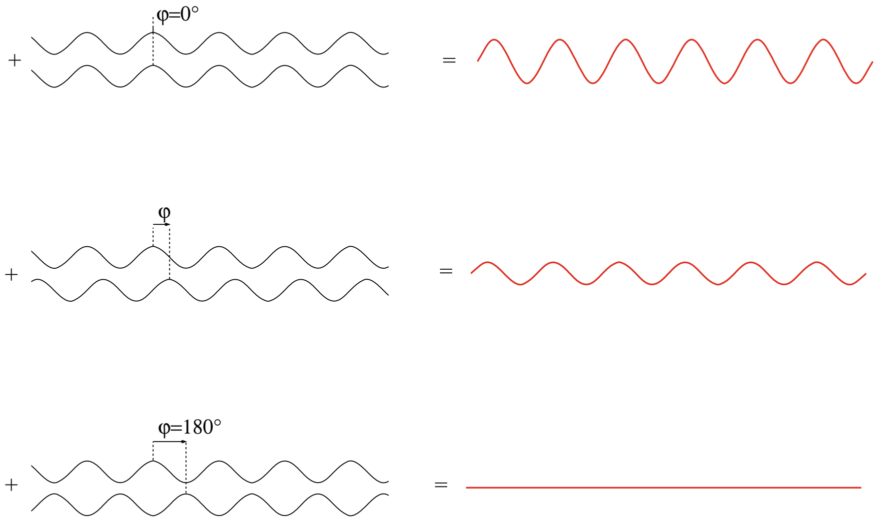

Fig. 2 The resulting superposition of two waves with the same amplitude and frequency $1/T$ is governed by their relative phase φ. When there is no dephasing (upper case with $\varphi = 0$), two maxima (or minima) of the superposed waves coincide and they add constructively into a wave of maximal amplitude. Conversely, when the two waves have opposed phase (lower case with φ is half a turn), each "bump" is compensated by a "hollow"; the two cancel one with the other and the resulting amplitude is almost zero. The relative phase is very much like an angle defined modulo one turn (or 360°), if the horizontal axis stands for the time, and Δt the time delay between two maxima, then $\varphi = (\Delta t/T)\,360°$ and one cannot distinguish between two delays differing by an integer multiple of T. To follow Dirac's argument in mathematical terms, the square modulus of the sum of two complex numbers $z_1 = |z_1|e^{i\varphi_1}$ and $z_2 = |z_2|e^{i\varphi_2}$ differs from $|z_1|^2 + |z_2|^2$ because of the relative phase $\varphi = \varphi_1 - \varphi_2$: $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos\varphi$. The last term in the right-hand side is precisely the interference term

channels. It is this difference which gives rise to the phenomenon of interference, which is all pervading in the atomic world [. . .].

So if one asks what is the main feature of quantum mechanics, I feel inclined now to say that it is not noncommutative algebra. It is the existence of probability amplitudes which underlie all atomic processes. Now a probability amplitude is related to experiment but only partially. The square of its modulus is something that we can observe. That is the probability which the experimental people get. But besides that there is a phase, a number of modulus unity which can modify without affecting the square of the modulus. And this phase is all important because it is the source of all interference phenomena but its physical significance is obscure. So the real genius of Heisenberg and Schrödinger, you might say, was to discover the existence of probability amplitudes containing this phase quantity which is very well hidden in nature and it is because it was so well hidden that people had not thought of quantum mechanics much earlier. [8, pp. 154–158]

5 Feynman's Paths

Half a century after Dirac showed the equivalence of Schrodinger's "wave mechanics" and the Heisenberg's "matrix mechanics" in a unified formalism, Feynman proposed in his thesis of 1942 [11] a third way of computing quantum predictions. Also equivalent to the first ones, Feynman's formalism, at the price of introducing a subtly new mathematical concept of functional integration, gives to the quantum interferences the first role:[5] the probability amplitude $Z_{i\to f}$ for a system to evolves from an initial state i to a final state f are explicitly written as the result of the interference between all the possible histories the system may follow between the two states: up to a normalisation factor we can write[6]

$$Z_{i\to f} = \sum_{\substack{\text{all possible histories } h \\ \text{connecting } i \text{ to } f}} |z_h|\, \mathrm{e}^{\mathrm{i}\varphi_h} \tag{1}$$

Each virtual history h is weighted by a complex number z_h whose phase is φ_h. Only for a subset of histories that interfere constructively, we can think of a quasi-classical evolution—but still fuzzy at Planck's constant scales. Most of virtual histories, even when they have the same amplitude $|z_h|$, have a phase that differ too much from the average of the classical bunch and their contribution is mostly destroyed by their neighbours (Fig. 3). One possible history of one particle is just given by one continuous path made of all possible positions in ordinary space (its trajectory) and the constructive interference occurs precisely when it satisfies Newton's (or Euler-Lagrange's, or Hamilton's) classical equations.

However, as soon as two or more particles evolve, even if non-interacting, the appropriate place to describe histories is not the ordinary space anymore but a more abstract space, the *configuration space* (the same space on which the Schrödinger wavefunctions are defined). To simplify the discussion, we will assume that the initial state has been prepared with a determined number N of particles of the same species and this number will be maintained all along the evolution. In that case,[7]

[5] The Dirac's quotation given at the end of Sect. 4 is the transcription of a conference he gave in April 1970 for a general audience. Though he talks about the recent development in quantum electrodynamics, and though he defends the idea that *this concept of probability amplitude is perhaps the most fundamental concept of quantum theory*, surprisingly enough, Dirac does not mention Feynman at all in his text. The only very vague allusion I could find lies, perhaps, in the sentence immediately following the quotation above: *If you go over the present day theory to see what people are doing you find that they are retaining this idea of probability amplitude* [8, p. 158].

[6] The writing is simplified (it hides that the sum covers an infinite functional continuum) but captures the spirit of the famous Feynman path integrals.

[7] When particles can be created or annihilated, one cannot avoid working with fields whose configuration space is infinitely dimensional. This is almost unavoidable when dealing with quantum electrodynamics since photons are massless particles and thus can have an arbitrary low individual energy; therefore are cheap to create and easy to absorb. It requires a tremendous experimental skill to preserve a fixed number of photons for a while (in a superconducting cavity, for instance).

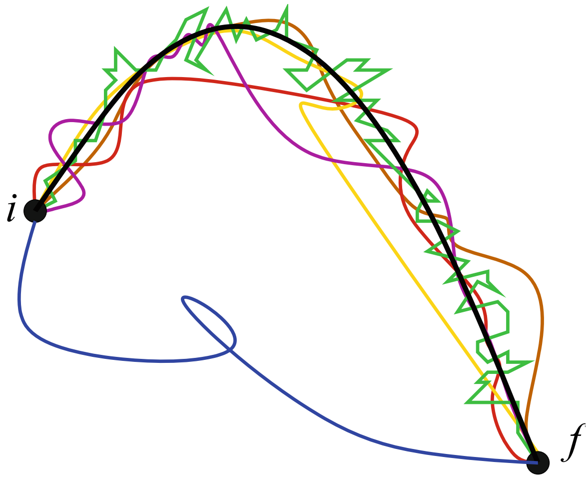

Fig. 3 For one particle starting at initial position i, its probability amplitude to find it in the final position f is given by the sum (1) over all histories, i.e., all the continuous (non-necessarily differentiable) possible trajectories connecting i to f. Only a bunch of trajectories in a neighbourhood—whose size is governed by Planck's constant—of the classical Newtonian trajectory (here the black parabola for a particle in a uniform constant force field) will contribute with a constructive interference (the phases are proportional to the classical action which is stationary). Any other bunch of trajectories far from the classical one (say around the blue lower trajectory) will bring a negligible contribution because their phases φ_h vary extremely rapidly and provoque a destructive interference

the configuration space has ND dimensions where D is the dimension of ordinary space (most of the time, obviously, it is 3 but in some condensed matter models, we shall see that D can be lowered to 2 or even 1). To try to visualise the evolution of such a system, one may come back to the ordinary D-dimensional space where inevitably each particle can be individualised by a numbering that is continuously followed as they evolve from a given initial state to a given final state. But as we explained in the previous section, such a numbering has no physical basis at the quantum level and any continuous permutation among them not only can but must be considered among the Feynman's histories. It is also crucial to keep in mind in the following that the inevitable interactions between particles (the photon being aside, see the last footnote) exclude the histories where two of them overlap (for the corresponding energy of such a configuration diverges which make the phase oscillating infinitely quickly which destroy the superposition).

6 Where the Topological Properties Come From

It is to Dirac [6] that we owe the first identification of a quantum property of topological origin, namely a universal constraint on the electric charges if there would exist magnetic monopole.[8] By Feynman's own admission, Dirac [7] was also an inspiring source for the implementation of the ideas of the sum over histories briefly sketched in the previous section. Indeed, the sum over histories in (1) probes directly the topology of the space where the histories take place, namely the configuration space. In particular, since the histories are necessarily continuous they can be classified according to the so-called first homotopy group of the configuration space, each homotopy class being made of all the histories that can be continuously deformed one into another.

For one particle in ordinary space, most often, the homotopy properties are trivial in the sense that only one class generally exists. Like in Fig. 3, all the paths can be continuously deformed one into another.[9] But as soon as at least two identical particles are involved more than one homotopy class should be considered.

Whenever several homotopy classes are present, some new possibilities are offered in the Feynman's approach [18, and its references for a rigorous justification]. Because of the well-behaved composition law of histories together with some conservation of probabilities, one can attribute to each class $\mathfrak{c}$ a phase $\chi_{\mathfrak{c}}$ that depends only on $\mathfrak{c}$ and not on any specific choice of one of its member. This is why they are called *topological phases* and then (1) can be extended to

$$Z_{i\to f} = \sum_{\substack{\text{all homotopy classes } \mathfrak{c} \\ \text{of histories connecting } i \text{ to } f}} \mathrm{e}^{\mathrm{i}\chi_{\mathfrak{c}}} \sum_{\text{all histories } h \text{ in } \mathfrak{c}} |z_h|\,\mathrm{e}^{\mathrm{i}\varphi_h} \tag{2}$$

Of course when only one class is present or if we take $\chi_{\mathfrak{c}} = 0$ for all classes one recovers (1) but it happens that other choices are actually realised.

7 Permutations and Braids

Consider, for instance, $N = 3$ identical particles. Figure 4 provides two examples of histories represented in ordinary space with the time being given by the vertical axis.

As explained above, because the particles cannot overlap, we can follow each individual trajectory by a thread that allows to keep their individualisation (by a numbering or, more visually, a colour). In the two examples drawn, the two histories differ by a permutation of particles: in the final states the end position of the "green" thread has been exchanged with the end position of the "red" one. And precisely because of this permutation, one cannot deform one history into another without

[8] Monopoles would come with a violation of the Maxwell equation $\mathrm{div}B = 0$.

[9] One can manufacture on purpose some holes in space by creating zones that are forbidden to the paths by a magnetic field. This is the celebrated geometry proposed by Ehrenberg-Siday-Aharonov-Bohm [1, 2, 9] where quantum topological phases are involved even for just one particle.

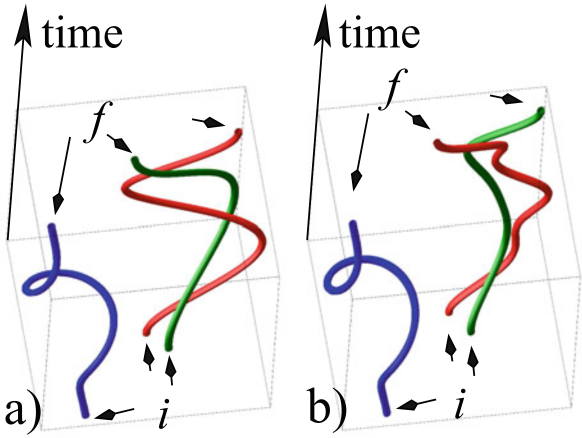

Fig. 4 Representation of two Feynman histories (or paths in the configuration space) for $N = 3$ particles. Time axis is chosen to be vertical while ordinary D-dimensional space is perpendicular to it. Since on these examples, the red and green threads do not connect the same initial and final points, these histories cannot be continuously deformed one into the other and therefore paths (**a**) and (**b**) belongs to two different homotopy classes

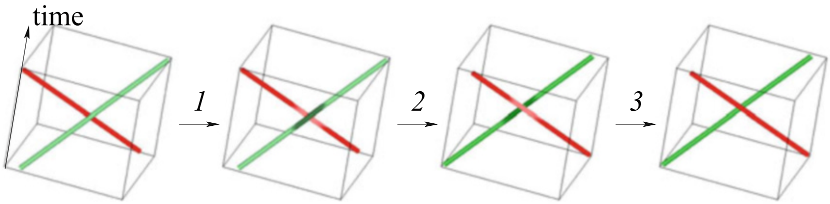

Fig. 5 In these graphical representations of a path for $N = 2$ particles, the time axis is plotted vertically while the ordinary space is perpendicular. When $D = 3$, moving in the third spatial dimension can be thought as changing the darkness of the threads. Because no particle can be at the same place at the same time, in $D = 2$ the threads cannot cross. In $D = 3$, one can always continuously separate them in the third dimension without any cut: in step 1 a darkening of a portion of one thread (here the green one) and a lightening of a portion of the other (the red one). Then, a crossing of these two portions in the two other dimensions is possible and, eventually, in step 3 one can restore the initial value of the third spatial coordinate

cutting two threads before gluing the pieces appropriately which would break the continuity. The representation chosen here can involve two spatial dimensions only (the horizontal plane). When $D = 3$ one can imagine that the motion in the third spatial dimension is represented by a change of darkness in the colour of the thread. With this image in mind, one can understand that when $D = 3$ one can have the thread crosses one with another even though two particles still cannot be at the same place at the same moment (Fig. 5). This latter constraint forbids two threads to cross at some point where their darkness is the same (all the three coordinates would coincide) but one can always bypass this restriction by changing the the darkness of one thread at a point of crossing (that is moving it in the third spatial dimension) then cross the threads at this point and restore the original darkness after this operation. In summary, for $D = 3$, when using the graphical representation of an history given in Fig. 4 or 6, the thread can be crossed but not in $D = 2$. This latter statement makes all the topological difference between $D = 2$ and $D = 3$. In the latter case, on can see that each homotopy class is in fact an ordinary permutation, whereas the structure of the classes in $D = 2$ is much richer. By concatenating the threads in order to represent the composition of two evolutions, we introduce naturally an algebraic internal law that makes the set of classes a group (the so-called fundamental homotopy group of the configuration space). Then, by a straightforward generalisation to N identical particles, in $D = 3$ the topological group is simply the *permutation group* of N elements whereas in $D = 2$, the group is called the *braid group* of N strands. Moreover, it can be shown that the topological phases must naturally compose accordingly: if $\mathfrak{c} \cdot \mathfrak{c}'$ stands for the class obtained by concatenating the N threads in $\mathfrak{c}$ with the N threads in $\mathfrak{c}'$, then, it can be shown that we must have

$$e^{i\chi_{\mathfrak{c}\cdot\mathfrak{c}'}} = e^{i\chi_{\mathfrak{c}}} e^{i\chi_{\mathfrak{c}'}} \tag{3}$$

For $D = 3$, the latter relation leaves us with a simple alternative since the $(2p+1)^{\text{th}}$ iteration of a transposition of two threads, p being an integer, leaves us with the

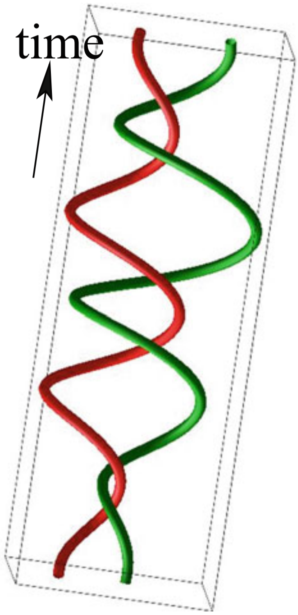

Fig. 6 In $D = 3$, the crossing of two threads being possible, one can always entangle a succession of an arbitrary number of exchanges of two particles, leaving us with two homotopy classes only (the identity and the transposition). In $D = 2$, the braid such obtained is always different from the previous ones and the homotopy classes can be labeled by a unbounded integer

transposition itself, that is $\mathrm{e}^{\mathrm{i}(2p+1)\chi_{\mathfrak{c}}} = \mathrm{e}^{\mathrm{i}\chi_{\mathfrak{c}}}$ whose solution can only be

$$\mathrm{e}^{\mathrm{i}\chi_{\mathfrak{c}}} = 1 \qquad \text{or} \qquad \mathrm{e}^{\mathrm{i}\chi_{\mathfrak{c}}} = -1 \tag{4}$$

for all the classes $\mathfrak{c}$ associated with a transposition of two identical particles. The first choice corresponds to bosons and the second to the fermions. For $D = 2$, if $\mathfrak{c}$ is a bread with two threads associated with a transposition one can iterate the concatenation of the same bread made of two threads an arbitrary number of times and the result is always a different braid because of the interdiction of crossing the threads (Fig. 6). The algebraic constraints on the bread group are much less restrictive than for the permutation group and, in particular, we can take consistently

$$\mathrm{e}^{\mathrm{i}\chi_{\mathfrak{c}}} = \mathrm{e}^{\mathrm{i}\theta} \tag{5}$$

where θ is an angle that characterises the species of the identical particles under consideration: it continuously interpolates between the bosons ($\theta = 0$) and the fermions ($\theta = \pi$). This is precisely this quantity that characterises an anyon. This phase being relative to some terms in a quantum amplitude of probability, they have observational consequences through interferences.

This is the way we can connect the statistical physical properties of N identical quantum particles (the bosonic, fermionic or even anyonic character under permutations) and some topological properties (the first homotopy group of their configuration space).

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Physics in a Small Bedroom

Denis Weaire, Stefan Hutzler, Ali Irannezhad, and Kym Cox

1 Physics Teaching and Research in the Time of the Covid Pandemic

In 2020 the corona virus sent universities and schools into prolonged hibernation, forcing us to ask: how are we to adapt to our confinement? Available space for study might be restricted to a small bedroom. How were we to use it?

An obvious response has been to use remote teaching and learning, with online lectures and assignments. So far so good, and it will be very interesting to see what lasting effect this has on traditional teaching methods. One of us (DW) once wrote a humorous piece, "Rough World" [1], about the horrors of university life, including the torture of the ill-prepared lecturer confronted by a smart student . . . We have not always been honest with ourselves about the quality of our lecturing: we could well have adopted a better mix of online and personal teaching, long ago.

But in physics we insist on hands-on experimental work. In Trinity College Dublin this is imposed even on the theoretical physicists. Simulated experiments on a screen are hardly an adequate substitute for the real thing. So what could we do in such remote confinement?

Some of the best experiments in teaching and research are very simple and safe. They can be done at home, with materials that are readily at hand. Mostly they relate to classical mechanics or elementary properties of materials.

Many have argued that classical physics is the best training for rigorous thought. Even apparently trivial everyday phenomena can throw up teasing challenges to

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analysis. See, for example, the excellent work of Eric Mazur at Harvard [2]. Indeed, “Rough World” centred on a very elementary question that cropped up in a lecture: What happens if you modify the traditional “ladder-against-a-frictionless-wall” problem by allowing friction on the wall? [1]. This could indeed be a candidate for analysis in the bedroom.

What other experiments suggest themselves for this restricted space? Let us start with one that we have recently published in the *American Journal of Physics*: it made it on to the cover of the May 2020 issue [3].

2 Toying with Hard Spheres

Take any convenient number of ball bearings or similar hard spheres. (In what follows, you will find that it matters whether the number of spheres is odd or even, inviting thoughts and analysis on symmetry properties.) Place them in a tube with stoppers at both ends, lay it horizontally, and agitate slightly to encourage the system to come into equilibrium. (Immersion of the balls in oil will help.)

If sufficiently compressed by adjusting the stoppers, the chain of hard spheres buckles in a zig-zag pattern, as in Fig. 1. (Study problem: is this “Euler buckling”?) But the buckling is not uniform, as is evident from Fig. 1, and there are alternative sphere arrangements for higher values of compression. The phenomenon of localized buckling calls to mind the subject of “kinks” and “solitons” found in many nonlinear systems, so thoughts about nonlinearity in general are provoked.

The great John von Neumann said long ago that the computer (which he helped to invent) would release mathematics from the narrow confines of linear problems. So why not try to replicate your results by computer simulation?

There are much more sophisticated laboratory systems that are analogous to the humble set-up described above; they often demand the kind of expensive apparatus (e.g., for ion trapping) which we were determined to avoid. There are lots of possible variations without undue complication, for example, using soft bubbles instead of hard spheres.

Another hard-sphere experiment is associated with Isaac Newton, some of whose greatest achievements were made in seclusion: “All this was in the two plague years of 1665–1666. For in those days I was in the prime of my age for invention and minded Mathematicks and Philosophy more then than at any time since.” (quoted in [5])

Newton’s Cradle (Fig. 2), which appears to have been discussed first in 1662 by the mathematicians/natural philosophers John Wallis, Christiaan Huygens and Christopher Wren (the latter being also the architect of St. Paul’s Cathedral in London), is a popular executive desk ornament, illustrating the principles of classical Newtonian mechanics. It is not hard to understand its obvious properties, but physicists always like to look more closely. John Hinch (one of many leading mathematicians who have followed in Newton’s footsteps at Trinity College Cambridge) did so [6]. So have we [7], looking ever more closely: the experiment and

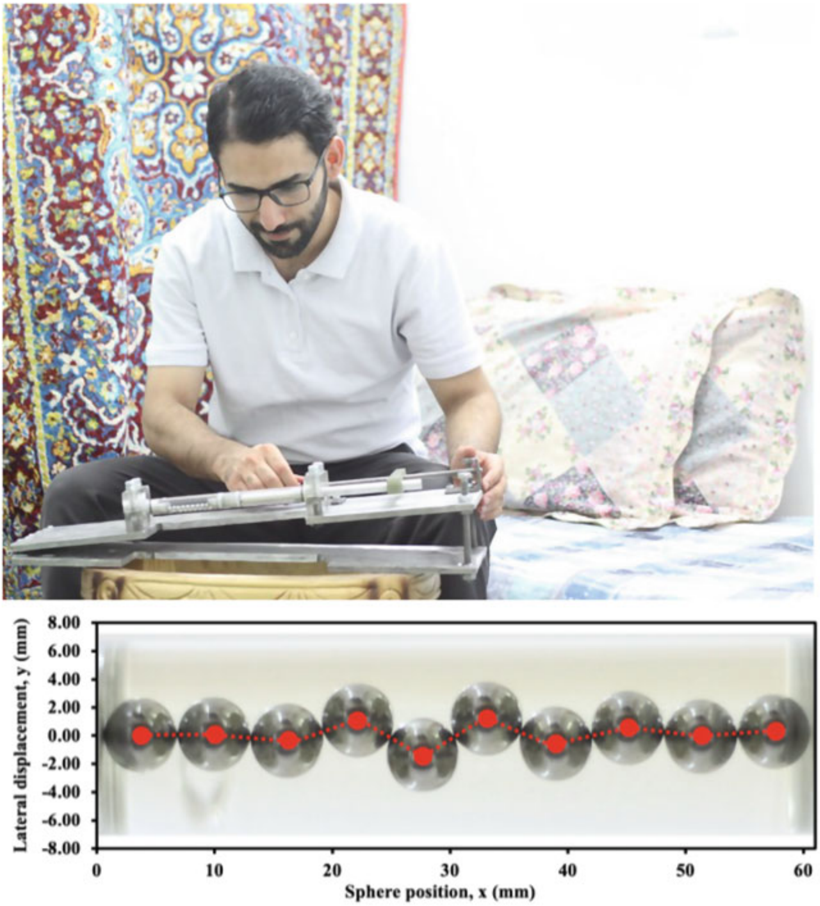

Fig. 1 One of the authors (AI) performing experiments in his bedroom in Tehran, Iran, during a Covid lock-down. His apparatus consists of a cylinder containing metal spheres. Pushing in the sides of the cylinder induces buckling of the initially linear chain. The effect becomes more and more localized as the compression is increased. In a variation of the experiment we study the position of the localisation peak as the cylinder is tilted, in order to determine the so-called Peierls-Nabarro potential [4]. This potential, originating in the theory of crystal dislocations, can tell us how the localized buckling moves when we tilt the apparatus

its theoretical counterpart continue to be fascinating, with a never-ending pursuit of variations and complications [8].

We recently spotted an attractive large scale Newton's Cradle in a local hospital, Fig. 2. Alas, its spheres were bolted together, so both art and science must have been frustrated by other considerations.

Fig. 2 In the standard textbook description of Newton's Cradle there is always only one sphere in motion. However, careful observation of the experiment shows that, already after the first impact of a displaced sphere on one side, the entire chain begins to break up, with all spheres in motion. Theory and computer simulations confirm this, and attribute it to the finite elastic modulus of the spheres [7]. The photograph shows the stainless-steel sculpture *For every action ...* (2005) by Fiona Mulholland permanently sited at Beacon Court South Quarter, Sandyford, Co Dublin. (Photo reproduced with kind permission by the artist)

3 Playing with Soft Bubbles

Soap bubbles can provide another homely source of inspiration, as they have done for centuries, in art, for which the definitive review is that of Emmer [9], and science, whose elementary aspects were described by Isenberg [10].

Staring into a foam with the naked eye, or with the help of a magnifying glass, will reveal local order amongst the randomness. Three soap films always meet in a line under an angle of 120°, and four such lines meet under the tetrahedral angle of 109.4°. The lines are called Plateau borders in honour of the Belgian scientist Joseph Plateau who was the first to describe them. His experiments were undertaken after he became blind (rashly staring at the sun, in the interests of science), so some of them may well have been performed in a domestic environment, with the help of his wife [11].

Rather than scooping up bubbles from the kitchen sink one may prefer to blow air through a straw into some soapy water. Blowing carefully (or using an aquarium pump borrowed from your fish) results in the generation of bubbles of equal size. These crystallize spontaneously to form a hexagonal pattern (triangular lattice) on top of the soap solution, with some defects, such as dislocations, amongst them (Fig. 3).

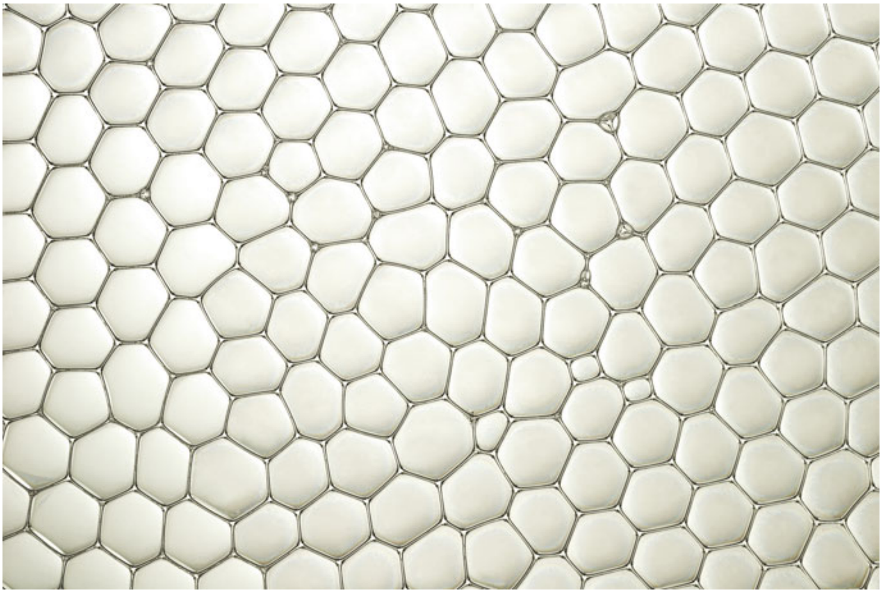

Fig. 3 Example of a (mostly) single layer of bubbles on top of a surfactant solution. The slight variation in size of the bubbles prevents the perfect crystallization seen in Bragg rafts of identical bubbles. ©Kym Cox

In 1947 Nobel Laureate William Bragg and John F. Nye studied such a 2d bubble raft as a source of inspiration for the study of crystalline defects [12]. Why did such prominent physicists experiment on such simple classical physics, working in the Cavendish Laboratory? Probable answer: there was no plague at the time but a World War must have left even Cambridge University impoverished.

Three-dimensional structures formed by monodisperse bubbles can provide more entertainment and questions for physics. Photographer and co-author Kym Cox recently brought them to great prominence via the *New Scientist* (Aug. 24, 2019) and even *The New York Times* (April 9, 2019). Figure 4 shows a particularly amusing example of her work. It was one of us (SH) who taught her the tricks of making such foams in her own kitchen.

In turn, she surprised and challenged us by a further simplification of the experiment which was perfected by our Dublin research group: letting bubbles of equal size flow out of a cylinder is sufficient for generating a variety of regular bubble chains of various degrees of complexity, such as the one shown in Fig. 5.

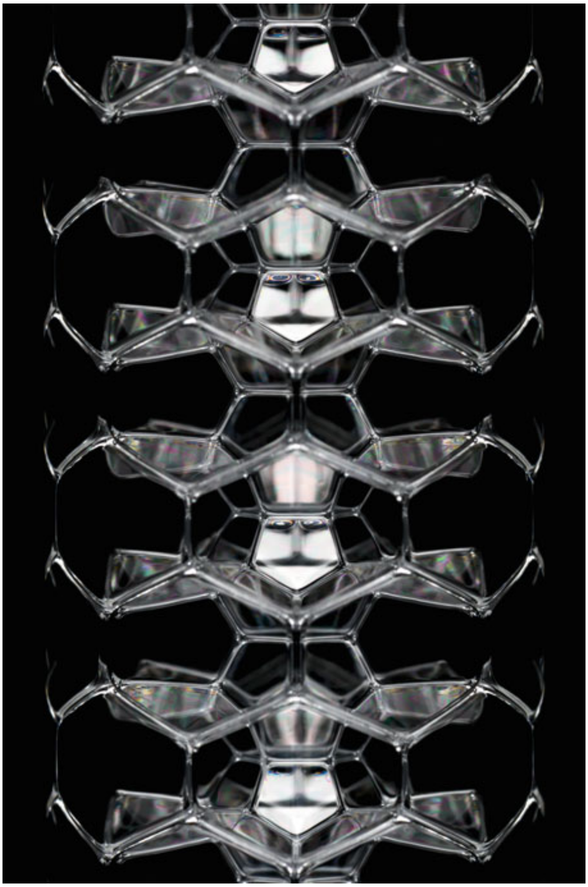

Fig. 4 "Columnar crystal" made of soap bubbles [13, 14], as observed by artist and photographer Kym Cox https://www.kymcox.com/©Kym Cox

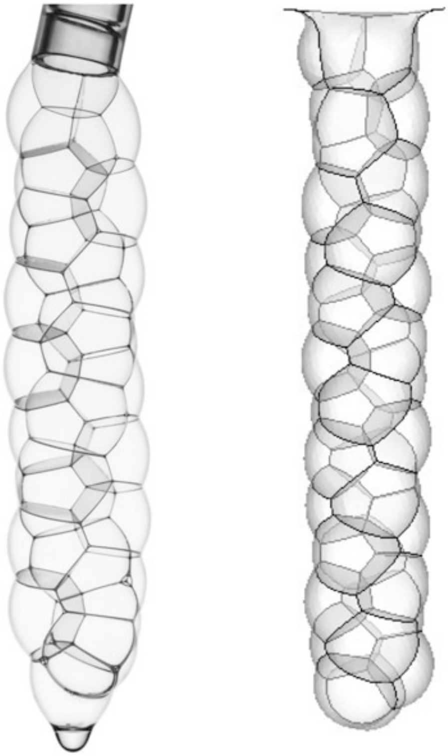

Fig. 5 Photograph of a freely hanging chain of soap bubbles (left, ©Kym Cox), recreated using computer simulations (right)[15]

The figure also shows our response: a computer simulation, which reproduces the key features of the structure.

4 Colours Brighten up a Dull Day

When soap films are viewed under appropriate optical conditions they display a rich spectacle of colours which fascinated both Robert Hooke and Isaac Newton. Figure 6 shows a sequence of four photographs of the interference patterns produced in a crystalline foam similar to the one shown in Fig. 4. As time progresses, the films thin as liquid drain away, leading to ever-changing interference patterns. Eventually, black spots appear on the film when viewed from reflection, as was already described by Hooke and Newton. This is due to destructive interference which occurs when the local film thickness is much less than the wave-length of light. (The phase shift of π due to reflection from the film is responsible for this.) Soap films may be as thin as 5 nm, making nanoscale effects visible to the naked eye!

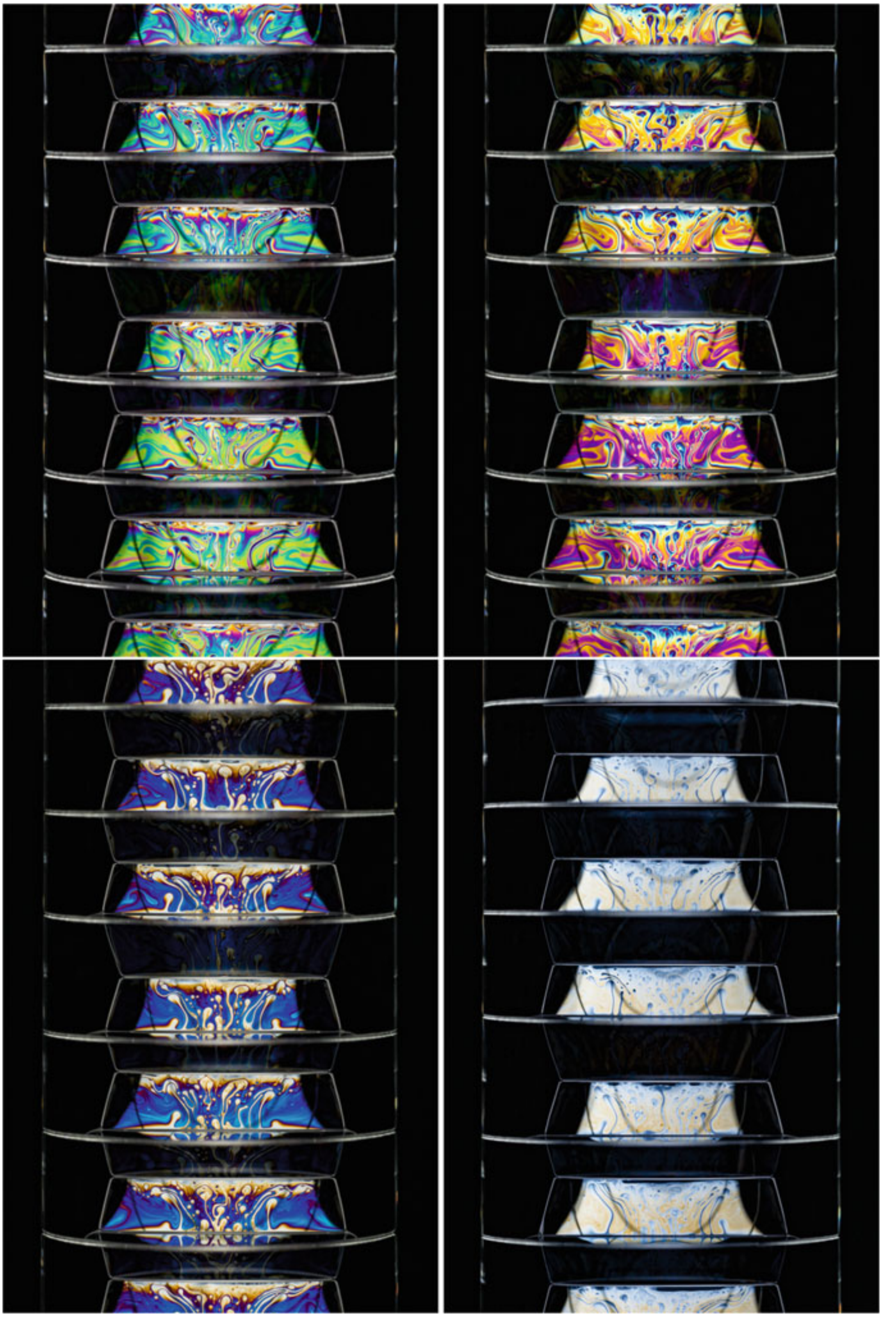

Fig. 6 Four photographs in a time sequence (top left to bottom right) of draining soap films in a "crystalline foam", formed by soap bubbles, confined in a cylinder (©Kym Cox). The interference colours are indicative of the local thickness of the soap films. Since the foam structure is periodic in space [14], also the interference patterns show this repetition, with minor random fluctuations

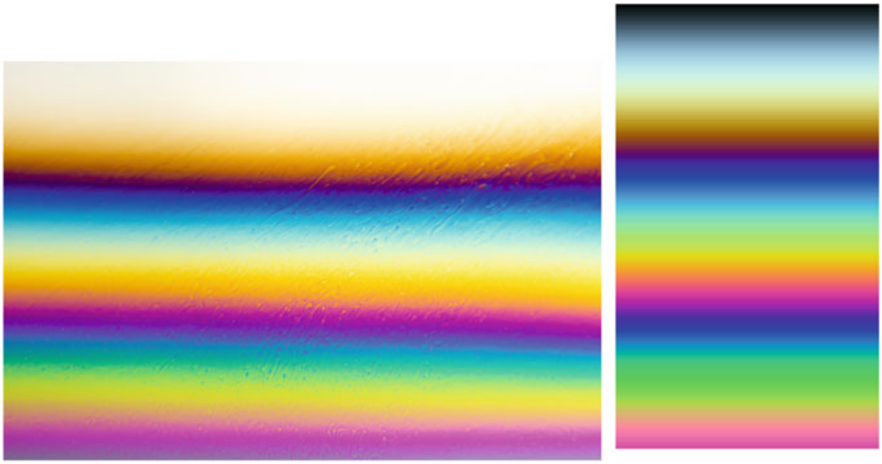

Fig. 7 Interference colours in a free-standing vertical film in photography (left, ©Kym Cox) and a computer simulation (final year TCD undergraduate project by Thomas Greg Corcoran, 2006)

The simple interference pattern of a single vertical wedge shaped soap film is readily computed and displayed using, for example, the Mathematica software. Figure 7 shows the result, together with a corresponding photograph of a real soap film.

Newton's experiments on the refraction of light, producing the colours of the rainbow by a prism, were either accomplished in his Cambridge college rooms or at his home in Woolthorpe Manor, using the light from his window. In our own College, Humphrey Lloyd performed another ground-breaking experiment on light in a similar way, demonstrating conical refraction [16]. Purpose-built and properly furnished physics laboratories are a relatively modern innovation, dating from the second half of the nineteenth century.

Nowadays, laser pointers are ideal for optics demonstrations, and the ever present smart phone offers apps for all kinds of data taking and analysis to conduct physics at home.

On a more philosophical level, we may recall that artists have often painted bubbles to symbolize youthful joy, but also fragility and mortality [9], and many poets have done the same. In a further bedroom-type experiment we have measured the lifetime of thousands of soap films [17, 18]. The time variation of failure rates offers interpretations for those of technical devices and also human mortality.

We should not leave this wonderland of films and bubbles without reference to Cyril Stanley Smith [19]. Despite the great sophistications of many of his ideas, he rejoiced in the beauty of simple bubble structures and their significance. He directly encouraged one of us (DW) to find time to play with them.

5 Putting a Spin on the Lock-in

Finally we have cause to again bring in the prolific mathematician Leonhard Euler. Among his many accomplishments he established the theory of rotating bodies, such as the spinning top. Accordingly, his name is honoured in the Euler disk, although there is no evidence that he originated it. It is a flat disk, preferably a heavy one, although a large coin will do for a start. Search the house. When spun on a firm flat surface it gradually subsides, as its energy is dissipated, but not in the manner of most things. Instead of a dignified gradual exponential approach to equilibrium, it heads dramatically towards a crisis, emitting a sound of ever increasing frequency. It suddenly settles—not with a whimper but a bang. (In mathematical terms this is an "essential singularity".) Can you find a way to measure that intriguing sound? Again, your smartphone might be very valuable.

As for its explanation, enter Keith Moffatt, yet another Trinity Cambridge don (this kind of thing must be infectious …). He has written with erudition on the subject [20]. A previous paper by Moffatt led one of the present authors (DW), while still an undergraduate, to throw eggs out of his second-storey student room on to an adjacent lawn. Moffatt claimed that they would not break, and he was (mostly) right!

Just one warning: if you do find a really heavy Euler disk (a manhole cover?), please try not to wake the neighbours!

6 Return to the Lab

At the time of writing we look forward to a return to our schools and laboratories. Truly remarkable (and very expensive) equipment awaits us. It is a pity that it consists largely of large grey boxes! We might bring back to the teaching and research laboratory an admiration of phenomena that have striking immediate visual or aural impact, and can be brought to your local Science Gallery, bridging science and art. They may look trivial but can still be challenging, teaching us to speculate and analyse until we get to the heart of the matter. And then, as so often in physics, we are tempted to go deeper still!

Note of Precaution Of course, safety is a paramount consideration. We trust that none of our suggestions will pose any significant hazards within the domestic environment, but caution is advised in every case.

Kym Cox is an artist and professional photographer. For a selection of her work see https://www.kymcox.com/.

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Part VIII
Mathematics and Applications

The Train of Artificial Intelligence

Maurizio Falcone

1 AI Is Everywhere

In the last decade, there has been a macroscopic increase in the number of activities in the area of machine learning and Artificial Intelligence (AI) to the point that their algorithms are often cited in newspaper articles and TV shows as a possible solution to many practical problems in everyday life. AI appears to offer easy solutions to many complicated problems like finding an object in a video sequence or in a large archive of pictures, recognizing the pattern in a huge data base, analyzing the trends of consumers, exploiting their selections on WEB pages, or optimizing the configuration of a network. Behind the success of these techniques, there are several mathematical hidden tools. Even more, the understanding of these methods is far from being complete, and AI has become a very active research topic for many scientists working in different areas. As a result, many universities have started new programs in Artificial Intelligence and Data Science attracting a huge number of students, see, e.g., [1–3].

AI is at the intersection of computer science, mathematics, robotics, statistics, and scientific computing. Its success is due to the mixing of different ingredients. The first is the huge computing power now available on main frames, workstations, and also personal computers that has drastically reduced the CPU time necessary to obtain accurate results to some complex problems. The second ingredient is the intense improvement of numerical methods used to deal with huge amounts of processed data. Classical methods, when available, suffer from the so-called "curse of dimensionality", so they have to be adapted to deal with high-dimensional problems. It is interesting to note that many of the mathematical problems and

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techniques are not new and usually involve numerical linear algebra, probability, and optimization. However, they have recently attracted new attention, since developing new and more efficient solutions for high-dimensional problems became a challenging research topic.

This explains why Artificial Intelligence has entered a new era, with a remarkable impact on technology and economy. This progress has been tied to the recent success of machine learning where few rigorous mathematical principles explain how and why methods work (or do not work). Topics include curses and blessings of dimensionality, randomized algorithms, linear and nonlinear dimension reduction methods, graphs and clustering, community detection, sparsity and massive data, diffusion maps and intrinsic geometry of high-dimensional data, as well as convex and non-convex optimization. Some of these techniques are also used to prove new results in mathematics. AI is thus closing the circle, since it is applying a lot of mathematics to mimic the human behavior and to simulate the brain functioning at a high level finally reaching the capabilities of proof typically owned by mathematicians. So far AI is producing new mathematical tools starting from a small number of premises and hypotheses. However, its evolution is very fast, and we do not really know what will come next. Perhaps we are going to reach a new world where the robots will not be distinguishable from humans as in the robot cycle of Asimov's novels [4] or in the "Blade runner" movie [5] by R. Scott (1982).

2 AI Origins and Developments

The idea to formalize the human reasoning is very old and dates back to the Ancient Greece and to the foundations of logic reasoning (that is not always aligned to the human behavior). The interested reader can easily find the history of Artificial Intelligence [6], but let us sketch some recent steps to better understand the modern evolution. The "Church-Turing thesis" implied that a mechanical device, by shuffling symbols as simple as 0 and 1, could imitate any conceivable process of mathematical deduction. The key insight was the Turing machine—a simple theoretical construct that captured the essence of abstract symbol manipulation. This invention inspired a handful of scientists to begin discussing the possibility of "thinking machines."

Game theory, which would prove invaluable in the progress of AI, was introduced in 1944 with a celebrated book [7] by mathematician John von Neumann [8] and economist Oskar Morgenstern.

In 1950, Alan Turing published a seminal article in which he speculated about the possibility of creating machines that think. However, he noted that "thinking" is difficult to define and devised his famous Turing Test. If a machine could carry on a conversation (over a teleprinter) that was indistinguishable from a conversation with a human being, then it was reasonable to say that the machine was "thinking." This simplified version of the problem allowed Turing to argue that a "thinking machine" was at least plausible and the article answered all the most common objections to the proposition. The Turing Test was the first serious proposal in the philosophy of Artificial Intelligence. In 1950, Claude Shannon published a detailed analysis of chess playing, and in the same year Isaac Asimov published his “Three Laws of Robotics” [9].

An important step came in 1955 when Allen Newell and Herbert A. Simon created the program “Logic Theorist” and showed that the program would eventually prove 38 of the first 52 theorems in Russell and Whitehead’s “Principia Mathematica” [10] and find new and more elegant proofs for some of them. This proved that a machine could have the properties of a real mind and answered some of the philosophical objections against thinking machines. Just one year later, the famous Dartmouth Workshop was organized by Marvin Minsky, John McCarthy, and two senior scientists: Claude Shannon and Nathan Rochester of IBM. The goal of the workshop was “to show that every aspect of learning or any other feature of intelligence can be so precisely described that a machine can be made to simulate it.” This can also be used as the definition of Artificial Intelligence and clearly indicates the relevance of learning skills as a part of it. All the most influential scientists working at research programs for AI took part in the workshop, and this was where John McCarthy proposed the name “Artificial Intelligence” for this research area. For this reason, 1956 is considered the birth year for AI.

In 1959, John McCarthy (the inventor of the LISP programming language [11]) and Marvin Minsky founded the MIT AI Lab.

The programs developed in the years after the Dartmouth Workshop were, to most people, simply “astonishing”: computers started to solve algebra problems, prove theorems in geometry, and learn to speak English. Few at that time would have believed that such “intelligent” behavior by machines was possible at all. Researchers expressed an intense optimism in private and in print, predicting that a fully intelligent machine would be built in less than 20 years. Government agencies like DARPA in the USA poured money into the new field expecting a very rapid and successful evolution. However, this vision was too optimistic, since in the early seventies, the capabilities of AI programs were very limited to solving “toy problems.”

One of the major obstacles was associated with the computer architectures of that time. This explains why the history of AI is intertwined with the evolution of hardware and software. In the 60s, there was not enough memory or processing speed to accomplish anything truly useful. For example, Ross Quillian’s successful work on natural language was demonstrated with a vocabulary of only twenty words, because that was all that would fit in memory. We should mention that the celebrated ENIAC (Electronic Numerical Integrator and Calculator) [12], which is considered to be the first general purpose computer, was built in 1943 by J. Mauchly e J. Eckert occupying a surface of 180 square meters for a weight of 30 tons. ENIAC was the first electronic computer without mechanical parts, since its circuits exploited the thermionic valve invented in 1906. It used the decimal system and its memory could contain about 20 numbers with 10 digits. In fact, the “computer” used by Turing [13] at Bletchley Park in 1939 to decrypt Enigma [14, 15] was based on a series of electromechanical devices. Note that its name, the Bombe [16], was following the name of its Polish ancestor (bomba kryptologiczna). ENIAC was rather complicated to program via punched cards and very expensive in terms of energy consumption requiring 1500 kw of power to operate. The first time it was switched on at the Moore School of Electrical Engineering, University of Pennsylvania, it caused a blackout of the west areas of the city. After WWII, the computing facilities were improved but still very limited when compared to our personal computers. As of 2011, practical computer vision applications require 10,000 to 1,000,000 MIPS (Millions of Instructions Per Second). By comparison, the fastest supercomputer in 1976, Cray-1 (retailing at 5 million to 8 million), was only capable of around 80 to 130 MIPS, and a typical desktop computer at that time achieved less than 1 MIPS. Another important limitation was due to the fact that many problems can only be solved in exponential time (exponential in the size of outputs) and this would require an enormous amount of CPU time, too much to solve real problems. The basic knowledge necessary to make decisions and to find commonsense solutions requires a lot of information since the program needs to have some idea of what it might be looking at or what it is talking about. No one in 1970 could build a database so large and no one knew how a program might learn so much information. Moreover, some tasks that we usually accomplish very easily become very difficult for an “intelligent machine.” Proving theorems and solving geometry problems are comparatively easy for computers, but a supposedly simple task like recognizing a face or crossing a room without bumping into anything is extremely difficult. This helps explain why research into vision and robotics had made so little progress by the middle 1970s (and these fields have a lot of intersections with AI). All the above difficulties and limitations convinced many research funding agencies that investing in AI was not very useful, so they started to cut off their fundings by the early 70s. However, the new boom arrived in the 80s with the development of “expert systems.” An expert system is a program that answers questions or solves problems about a specific domain of knowledge, using logical rules that are derived from the knowledge of experts. Expert systems restricted themselves to a small domain of specific knowledge (thus avoiding the commonsense knowledge problem) and their simple design made it relatively easy for programs to be built and then modified once they were in place. All in all, the programs proved to be useful: something that AI had not been able to achieve up to that point. In those same years, the Japanese government aggressively funded AI with its fifth generation computer project.

The enthusiasm for expert systems did not last for many years, and in 1987 there was the sudden collapse of the market for specialized AI hardware. Desktop computers from Apple and IBM had been steadily gaining speed and power, and in 1987 they became more powerful than the more expensive LISP machines made by Symbolics and other companies. There was no longer a good reason to buy them. At the end of the 80s, there was another breakdown of AI fundings as DARPA decided that AI was not “the next wave” and directed funds toward projects that seemed more likely to produce immediate results. Over 300 AI companies were shutdown, gone bankrupt, or been acquired by the end of 1993, effectively ending the first commercial wave of AI

Starting from the middle 90s, the field of AI knew a resurrection finally achieving some of its oldest goals. It began to be used successfully throughout the technology industry, although somewhat behind the scenes. Part of this success was due to increasing computer power, while part was achieved by focusing on specific isolated problems and pursuing them with the highest standards of scientific accountability. Within the computer science community, there was little agreement on the reasons for AI's failure to fulfill the dream of human-level intelligence that had captured the imagination of the world in the 1960s. Together, all these factors helped to fragment AI into competing subfields focused on particular problems or approaches, sometimes even under new names that disguised the role of "Artificial Intelligence." In the late 1980s, several researchers advocated a completely new approach to Artificial Intelligence, based on robotics. They believed that, to show real intelligence, a machine needs to have a body—it needs to perceive, move, survive, and deal with the world. They argued that these sensorimotor skills are essential to higher level skills like commonsense reasoning and that abstract reasoning was actually the least interesting or important human skill. They advocated building intelligence "from the bottom up." However, some new events brought AI to the attention of the public. On May 11, 1997, Deep Blue became the first computer chess playing system to beat a reigning world chess champion, Garry Kasparov [17]. The supercomputer was a specialized version of a framework produced by IBM and was capable of processing twice as many moves per second as it had during the first match (which Deep Blue had lost), reportedly 200,000,000 moves per second. The event was broadcast live over the Internet and received over 74 million hits and showed an incredible skill in abstract reasoning.

The same happened for 1997 First official RoboCup football (soccer) match featuring table-top matches with 40 teams of interacting robots and over 5000 spectators. In 2004, NASA's robotic exploration rovers Spirit and Opportunity autonomously navigated the surface of Mars. In 2005, a Stanford robot won the DARPA Grand Challenge by driving autonomously for 131 miles along an unrehearsed desert trail. Two years later, a team from Carnegie Mellon University won the DARPA Urban Challenge by autonomously navigating 55 miles in an urban environment while adhering to traffic hazards and all traffic laws. In February 2011, in a Jeopardy! quiz show exhibition match, IBM's question answering system, Watson, defeated the two greatest Jeopardy! champions, Brad Rutter and Ken Jennings, by a significant margin.

These successes were not due to some revolutionary new paradigm, but mostly on the tedious application of engineering skills and on the tremendous increase in the speed and capacity of computers by the 90s. In fact, Deep Blue's computer was 10 million times faster than the Ferranti Mark 1 that Christopher Strachey taught to play chess in 1951. This dramatic increase is measured by Moore's law, which predicts that the speed and memory capacity of computers double every two years, as a result of metal-oxide-semiconductor (MOS) transistor counts doubling every two years. The fundamental problem of "raw computer power" was slowly being overcome.

Algorithms originally developed by AI researchers began to appear as parts of larger systems. AI had solved a lot of very difficult problems and their solutions finally proved to be useful throughout the technology industry, such as data mining, industrial robotics, logistics, speech recognition, banking software, medical diagnosis, and Google's search engine.

The field of AI received little or no credit for these successes in the 1990s and early 2000s. Many of AI's greatest innovations have been reduced to the status of just another item in the tool chest of computer science. Nick Bostrom explains "A lot of cutting edge AI has filtered into general applications, often without being called AI because once something becomes useful enough and common enough it's not labeled AI anymore." In 1968, Arthur C. Clarke and Stanley Kubrick imagined that, by the year 2001, a machine would exist with an intelligence that matched or exceeded the capability of human beings [18]. The character they created, HAL 9000, was based on a belief shared by many leading AI researchers that such a machine would exist by the year 2001. It is interesting to note that in the movie, the onboard computer is the only one knowing the final destination of the mission (Jupiter) and does not reveal it to the crew. This conflict with the first rule of robots ("A robot may not injure a human being or, through inaction, allow a human being to come to harm.") creates a contradiction that blows up in the famous scene [19] where the pilot, after a rough discussion with HAL, finally decides to shut the computer down.

In the first decades of the twenty-first century, the access to large amounts of data (known as "big data"), cheaper and faster computers, and advanced machine learning techniques were successfully applied to many problems throughout economy. Indeed, McKinsey Global Institute estimated in their famous 2011 report [20] that "by 2009, nearly all sectors in the USA economy had at least an average of 200 terabytes of stored data." Note that the access to big data archives finally gave the opportunity to train a machine and to develop its new skills simply by a trial-and-error procedure, and we will examine one of this techniques in the following section. By 2016, the market for AI-related products, hardware, and software reached more than 8 billion dollars. The applications of big data began to reach into other fields as well, such as training models in ecology and for various applications in economics. Advances in deep learning (particularly deep Convolutional Neural Networks (CNNs) and recurrent neural networks) have driven progress and research in image and video processing, text analysis, and even speech recognition.

3 Machine Learning and Training

As we said, machine learning indicates the research area dedicated to the development of new automatic skills. These skills are typically used to solve problems related to recognition and classification. The machine learns first by a training process on a large number of known examples (the training set) and then applies this "knowledge" to analyze and classify a new set of object. If the training is supervised by a human, the rate of success increases as the number of processed data increases. Then, a machine learning system is "trained" rather than explicitly "programmed." It is presented with many examples relevant to an activity and finds a statistical structure in these examples which ultimately allows the system to work out rules to automate the activity. To this end, three elements are needed:

1. Examples of data: for example, if the activity is object recognition, they could be images containing data objects
2. Examples of expected output: for example, a "label," indicating the class of belonging of the recognized object, and its position in the image
3. A measurement system to be able to determine the distance between the current output of the algorithm and the expected output, by adjusting the functioning of the algorithm. This adaptation step is the artificial learning process.

Deep learning (DL) is a subfield of machine learning, in which data is represented through learning at successive levels (*layers*), which refers to the adjective "*deep*." The number of levels that contribute to a model is called the *depth* of the model. Typically, these layered representations are learned through models called *neural networks* (NNs). DL models high-level abstractions in data by using a deep graph with many processing layers (this will be clarified later in this section). Deep neural networks (DNNs) are able to realistically generate much more complex models as compared to their shallow counterparts. However, deep learning has problems of its own. State-of-the-art deep neural network architectures can sometimes even rival human accuracy in fields like computer vision, specifically on things like the MNIST database [21], and traffic sign recognition. Language processing engines powered by smart search engines can easily beat humans at answering general trivia questions (such as IBM Watson), and recent developments in deep learning have produced astounding results in competing with humans in Go. *Faster R-CNN Inception-v2* is a particular network that performs object detection. As the name suggests, it is faster than its predecessors R-CNN and Fast R-CNN. It can be divided into three main subnets (Fig. 1):

1. *Feature Network*: a CNN for extracting a map of the image features.
2. *Region Proposal Network* (RPN): it selects bounding boxes from the image, called regions of interest (*Region of Interests*, RoI), which have a high probability of containing any object.

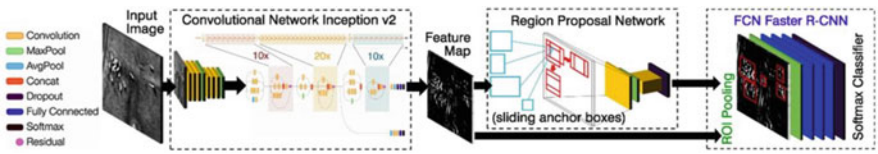

Fig. 1 Faster R-CNN Inception-v2 architecture

3. *Detection Network*: the detection network, sometimes also called the R-CNN network, takes the results of the *Feature Network* and the RPN and generates the final class and the delimiting rectangle (*bounding box*).

A CNN is a deep learning algorithm that takes an input image, assigns values (learnable weights) to the various parts, that is, to the various objects that compose it, and recognizes them, differentiating them from each other. The architecture of a CNN is similar to that of the connectivity model of neurons in the human brain. Individual neurons respond to stimuli only in a narrow region of the visual field, known as the receptive field. These fields overlap to cover the entire visual area. CNN differs from non-convolution neural networks, as the use of convolution allows to successfully capture the spatial and temporal dependencies in an image, by applying filters on each pixel that also involve neighboring pixels. A CNN architecture can be formed by the succession of various levels that we briefly describe below.

Input Level It represents the set of numbers which, for the computer, is the image to be analyzed. Its size is usually $N \times M \times 3$, where N indicates the width in pixels of the image, M is its height, and 3 is the number of colors in the RGB format.

Convolution Level, Conv It forms the main layer of the network. Its goal is to identify with high precision *features*, such as curves, angles, circumferences, or squares, depicted in the image. In a network, there are usually several levels of convolution, and each of them focuses on the search for one of these characteristics in the initial image. Higher is their number, the greater is the complexity of the object they can detect.

Every feature is represented by an array, called a filter (*filter*, *kernel*, or *weights*).

Level *Rectified Linear Units*, ReLU When passing through another level of convolution, the output of the first level becomes the input of the second. Typically a ReLU level follows a convolution level, so the output of the convolution level becomes the input of the ReLU level. It represents a nonlinear level, the purpose of which is that of introducing nonlinearity to a system that is essentially computing linear operations during convolution levels.

Pool Level This level is performed periodically for the sole purpose of reducing the number of trainable parameters. Pooling is performed independently on each depth dimension, and therefore the depth remains unchanged.

***Full Connection* Level, FC** Usually, this is the last level of a CNN. This level takes the output from the previous level, which should represent the activation maps of the high-level features, and determines which ones are most related to a particular class. Moreover, it calculates the products between the weights of the previous level to obtain, for each class, the probability that the object belongs to that class.

4 Looking for a Clapperboard via AI

The clapperboard (ciak in Italian) is a device used in the production of films and videos to aid in the synchronization of images and sounds and to mark the various scenes and shots in a unique way. It is composed of a tablet, on which all the data of the scene being shot are shown, and of a movable rod (*clachette*). The clapperboard is placed in the frame a few moments before the action, and the bar is beaten by a machinist, producing the characteristic "clapperboard" sound. The data written on the tablet will later be used by the editor to accurately identify the shot, and in fact, several shots are often made for the same scene.

The information transmitted on the clapperboard (Fig. 2) may differ, but the more common are:

- *production*: the name or title of the film
- *roll*: the number of the roll on which you are turning, sometimes preceded by a letter that identifies the camera
- *scene*: a reference to the scene, often represented by a number; sometimes, it is followed by a letter that identifies the shot (*slate*)
- *take*: the number of the current take of the shot
- *director*: the name of the director of the production
- *camera*: the name of the director of photography or production
- *date*: the month, day, and year in which you are shooting

Fig. 2 Some frames containing a clapperboard

- Day/Night, Int/Ext, and MOS (shooting without audio recording) are other details to further classify a scene, but they are not always present.

One of the tasks within the CIAK project, developed in collaboration with the Italian company Digital Video, was the detection of the clapperboard in the video shot and the recognition of the superimposed text, to allow meta-dating, as well as an automatic identification of the shot (see the report [22]). The problem can be divided into the following subproblems:

1. Recognition of the clapperboard in each of the frames that make up the video
2. Choice of the "best" clapper among those previously identified and rotation to align it horizontally
3. Recognition of the text on the chosen clapperboard

In the proposed solution, the recognition of the clapperboard in a sequence of frames is carried out using an algorithm based on convolution neural networks. For this purpose, a new initialized network was built using the pretrained model *Faster R-CNN Inception-v2*. Each frame that makes up the video is processed to identify those that contain the clapperboard; the number of such frames depends on their frequency and on the duration of the shot of the clapperboard, varying from a few tens to a few hundred frames. Usually, the take appears for a few seconds at the beginning of the shot, so it is not necessary to process all the frames of the shot and just stop after the "disappearance" of the take identified, i.e., after a certain number of frames does not contain any take.

On the other hand, among all those that contain it, the least blurred one is chosen and preprocessed for the next phase, that is, it is rotated so as to align the text horizontally. To do this, use the horizontal lines on the clapper itself. These lines are identified through the *Randomized Hough Transform* [23] and also used to subdivide the fields of the clapperboard, so that, during the localization phase and the text recognition, each information can be associated correctly. Also, the localization of the words, carried out through Tesseract [24] (a public domain Optical Character Recognition application), is more accurate when the text is less sparse.

Object detection is a general term that groups the artificial vision techniques to locate and identify objects in images, producing in output a label that identifies the class to which the object belongs and its position within the image. Modern object detectors (see [25]) rely largely on the use of CNN. For the take detection in this work, a new network has been built, initializing it using the pretrained model *Faster R-CNN Inception-v2* (see [26]). One hundred images containing the clapperboard and the relative coordinates inside the image were fed to the network, and 80 of these were used to extract the clapper's characteristics and define the net weights. The remaining 20, instead, were used to validate the model by adjusting the net weights by minimizing the loss function. Let us try to give an idea of the training for the CNN. The first step is based on a Region Proposal Network (RPN) to locate the areas containing the information we are looking for. For the training of the RPN, first, a number of boundary regions are generated by a mechanism called *anchor boxes*. After obtaining a series of feature maps from the last convolution level, a "spatially

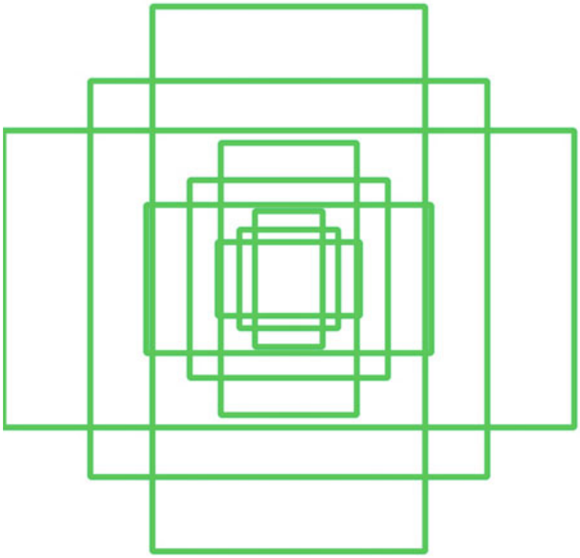

Fig. 3 Anchor boxes in 3 different ratios (1 : 1, 1 : 2, 2 : 1) and 3 different scales, for a total of 9

sliding window" is performed on these feature maps. Each window constitutes an anchor, each of which is typically 3 × 3 in size. A set of anchor boxes is generated for each anchor, usually 9 boxes with the same center, but with 3 different aspect ratios and 3 different scales, as shown in Fig. 3.

These are typically rectangular regions (boxes), and for every box, we are going to compute the value IoU indicating how much these boxes intersect with the real boxes (*ground truth box, GTbox*)

$$IoU = \frac{area(AnchorBox) \cap area(GTBox)}{area(AnchorBox) \cup area(GTBox)}.$$

This value will be used to compute distances in the network and to adjust the weights during the training.

The detection network generates the class to which the object belongs and the rectangle that delimits it. Normally, it is composed of 4 fully connected layers. The Faster R-CNN network differs from its predecessors R-CNN and Fast R-CNN by the addition of the previously described RPN.

R-CNN 2000 regions (region proposals), are scaled to quadrangular regions and fed into a CNN giving in output a vector of features of size 4096 for each proposal. It will be used to calculate the probability of the presence of the object within the region. In addition to predict the presence of an object within region proposals, the algorithm also produces four values, the offset values, which are needed to increase the accuracy of the bounding box. For example, given a region proposal, the algorithm predicted the presence of a person, but his face within the region proposal could be halved. Therefore, the offset values help to adjust the bounding box of the region proposal.

Faster R-CNN Instead of using the selective search algorithm on the feature map to identify region proposals, a separate network is used, namely the RPN described above. After obtaining an image characteristics map, using it to obtain proposed regions with the RPN, and extracting the characteristics of each proposed region, this data is finally used for classification. R-CNN attempts to mimic the final stages of CNN classification, where a fully connected (FC) layer is used to generate a score for each possible class of objects. R-CNN has two different objectives: to classify proposals into one of the classes and to better adjust the selection rectangle for the proposal, based on the expected class.

The last feature characterizing the *Faster R-CNN Inception-v2* is the Inception model. Before their more popular release, CNNs were developed in depth. With the introduction of the Inception model, new techniques have been adopted to increase the performance of CNN, both in terms of speed and accuracy. Its constant evolution has led to the creation of several versions of the network.

The first version (see [27]) starts from the following premises:

- The salient parts in the image may have extremely large variations.
- Due to this huge variation in the position of information, choosing the right filter size for the convolution operation becomes difficult; in fact, a greater nucleus is preferred for information that is more globally distributed and a smaller one for the information that is distributed locally.
- Very deep networks are prone to overfitting and are difficult to update weights across the entire network.
- To have large convolution operations is computationally expensive.

The Inception network applies filters with multiple dimensions to the same level, making it more "wide" than "deep." As stated earlier, deep neural networks are costly from a computational point of view, and to make operations cheaper, the authors limit the number of input channels by adding convolutions progressively.

Inception-v2 and Inception-v3 were presented in the same article [28]. The authors have proposed a number of updates that have increased accuracy and reduced computational complexity.

To implement the clapperboard recognition model, we used `Tensorflow` (see [29]), an open-source software library originally developed by researchers and engineers working on the *Google Brain* team within the *Machine Intelligence research organization* of Google, for the purpose of conducting machine learning and deep neural network research, but the system is general enough to be applicable to a variety of other sectors.

`TensorFlow` provides also API (*Application Programming Interface*) for numerical computation, using *data flow graphs*, where the nodes in the graph represent mathematical operations and its arcs represent multidimensional data arrays, called tensors. A tensor is hence the unity data center in `TensorFlow`.

Any TensorFlow program can be divided into two sections: creation of the computational graph, that is, a series of TensorFlow operations arranged in a graph of nodes, and execution of the graph itself. To actually evaluate the nodes, we need to run the computational graph within a session, which encapsulates the control and state of the TensorFlow . Each graph contains variable data, used to store and update the parameters of a training model.

To judge the performance of the model, *evaluation metrics* are calculated. The following quantities are defined:

- TP (*true positive*), indicating the number of clapperboard frames correctly detected;
- TN (*true negative*), indicating the number of non -clapperboard frames correctly detected
- FP (*false positive*), indicating the number of non-clapperboard frames recognized as clapperboards
- FN (*false negative*), indicating the number of clapperboard frames not recognized clapperboards

According to the metric used by TensorFlow, a detection is considered TP if the *Intersection over Union* (IoU), defined as

$$IoU = \frac{area(GTBox \cap PredictedBox)}{area(GTBox \cup PredictedBox)},$$

is above a certain threshold.

We have considered two parameters to measure the accuracy of the method:

Precision: it describes the skill of a classifier not to label a negative instance as positive (FP). It is defined as

$$precision = \frac{TP}{TP + FP}.$$

Recall: it describes the skill of a classifier to find all positive instances (TP). It is defined as

$$recall = \frac{TP}{TP + FN}.$$

The training process has been stopped after 2000 iterations and has lasted three hours, twenty-one minutes, and ten seconds. Figures 4 and 5 show the graphs of the precision and recall metrics, respectively, obtained by considering as TP the detections such that $IoU > 0.75$. As can be seen from the figures, it is not true that as the iterations increase, the metrics of the model improve. At the 2000th iteration, we have got the following values:

$$precision = 0.808 \quad \text{and} \quad recall = 0.74.$$

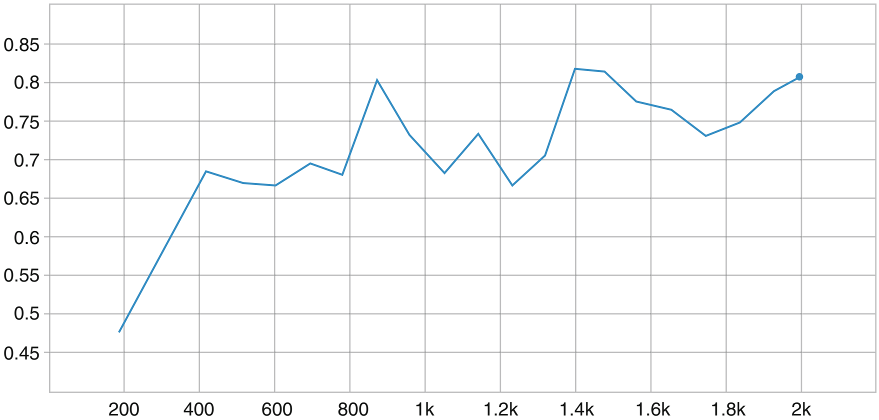

Fig. 4 Precision graph

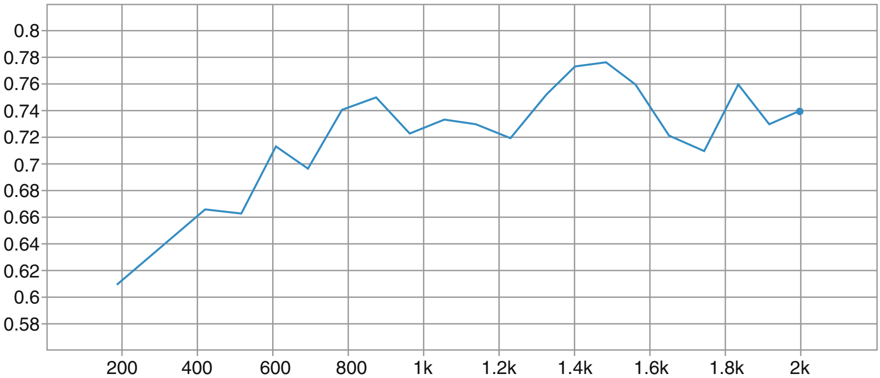

Fig. 5 Recall graph

Actually, the maximum values were obtained at the 1400th iteration with a value of 0.819 for the precision and at the 1480th iteration with a value of 0.777 for the recall. Finally, Figs. 6 and 7 show the performances for the object location and recognition phase, respectively. The performances are measured in terms of some network loss functions that we want to minimize. The values obtained at the 2000th iteration are 0.040 and 0.083, although the minimums were obtained at the 148th iteration with a value 0.037 for the location and at the 1920th iteration with a value 0.071 for the recall object recognition.

4.1 Experiments

To verify the performance of the method, tests were carried out using a video provided by the company 64BIZ and 15 videos from YouTube, containing cuts of

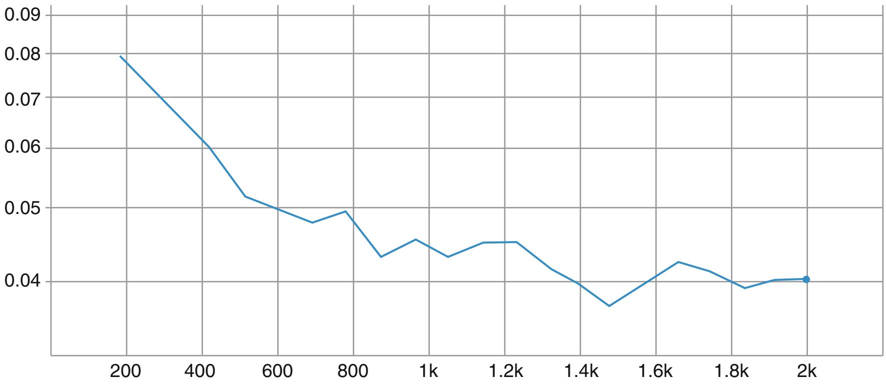

Fig. 6 Loss function for object detection

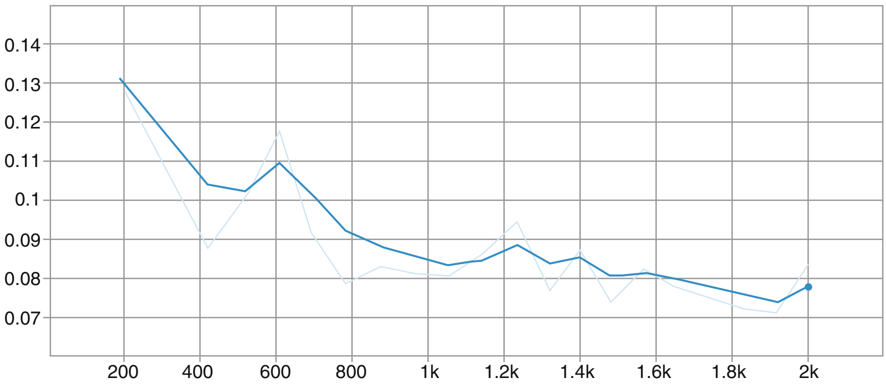

Fig. 7 Loss function for object recognition

footage showing the clapperboard, but the quality of the latter is typically lower than video produced on movie sets.

The search for frames containing the clapperboard is stopped if at least 5 frames that do not contain it follow the last clapper. Only the boxes that have a probability greater than 90% to contain a clapperboard are selected and are saved in memory together with the value of the variance of the Laplacian operator (the variation of the Laplacian is typically used to detect the blur in the image). At the end of the search, all the boxes are sorted by their variance of the Laplacian. The first of them is selected, and its edges are detected through a particular technique (the Canny Edge Detection algorithm [30]), with the parameters and the lines of the image obtained through the Randomized Hough Transform. The lines that, after rotating the clapperboard, turn out to be horizontal are used to divide it into sub-regions corresponding to the different information areas in the clapperboard. If two horizontal lines have a normalized distance to the total height of the box less than 0.2, they are considered as a single line with a thickness of more pixels. Each

Table 1 Results for the tests

Frames	CPU	GPU	Word	L-Word	R-Word	Tot. GPU
126	1.23 s/f	0.13 s/f	20	7.45 s	5.74 s	42 s
9	1.44 s/f	0.17 s/f	19	4.07 s	5.36 s	19 s
12	1.51 s/f	0.15 s/f	17	2.71 s	4.89 s	18 s
15	1.48 s/f	0.15 s/f	12	2.70 s	3.82 s	17 s
22	1.37 s/f	0.13 s/f	10	3.23 s	2.99 s	17 s
29	1.31 s/f	0.13 s/f	25	4.34 s	7.39 s	25 s
31	1.32 s/f	0.33 s/f	10	2.04 s	3.03 s	18 s
63	1.24 s/f	0.33 s/f	18	2.48 s	4.99 s	21 s
75	1.28 s/f	0.13 s/f	24	3.63 s	7.11 s	31 s
75	1.27 s/f	0.13 s/f	15	1.84 s	4.13 s	25 s
100	1.24 s/f	0.21 s/f	12	3.82 s	3.41 s	30 s
100	1.25 s/f	0.21 s/f	21	5.67 s	5.98 s	34 s
120	1.23 s/f	0.21 s/f	20	5.26 s	5.71 s	38 s
140	1.27 s/f	0.22 s/f	22	3.95 s	6.25 s	37 s
180	1.26 s/f	0.17 s/f	9	3.69 s	5.44 s	42 s
290	1.23 s/f	0.16 s/f	8	3.31 s	2.37 s	57 s
87	1.31 s/f	0.18 s/f	16.4	3.77 s	4.91 s	29 s

clapperboard fragment obtained is inserted in a black frame of suitable size, since the word locator network requires the image size to be multiples of 32.

Text recognition is performed on each box obtained with `Tesseract`, to which three configuration parameters are passed: the language, the OCR (Optical Character Recognition) engine, and the segmentation mode of the pages.

The tests have been performed on Windows 10 Pro 64-bit machines using two different processors: a CPU Intel Core i5-7500, 3.40 GHz, 8.00 GB RAM and a GPU NVIDIA GeForce GTX 1050 Ti, to compare the performances: the GPU is faster by a factor 7 (Table 1). The methods were implemented in Python, an object-oriented language, using `TensorFlow v. 1.5` on the CPU and `TensorFlow v. 2.1` for the GPU.

Table 1 shows the results. Every row corresponds to a single test, but the last row contains the averages. From left to right, its columns correspond to:

- Frames: the number of analyzed tests, that is, the number of frames containing the clapperboard plus (at most) 5 frames (to recognize that the clapperboard has disappeared)
- CPU: the time used to search the clapperboard (in seconds per frame)
- GPU: the time used to search the clapperboard (in seconds per frame)
- Word: the number of detected words
- L-Word: the total time used to localize the words on the clapboard
- R-Word: the total time used to recognize the words on the clapboard
- Total GPU: the total run time

Some results are illustrated in Figs. 8, 9, 10, and 11, showing how the system has located and extracted the information from the frames.

Figure 8 shows that the clapperboard has been correctly divided into 4 parts, and however the word locator has cut some letters, resulting in misrecognition of the words “ARRESTI” (“\RRESTI”) and “Martinelli” (“Martinalli”).

Fig. 8 Test 1

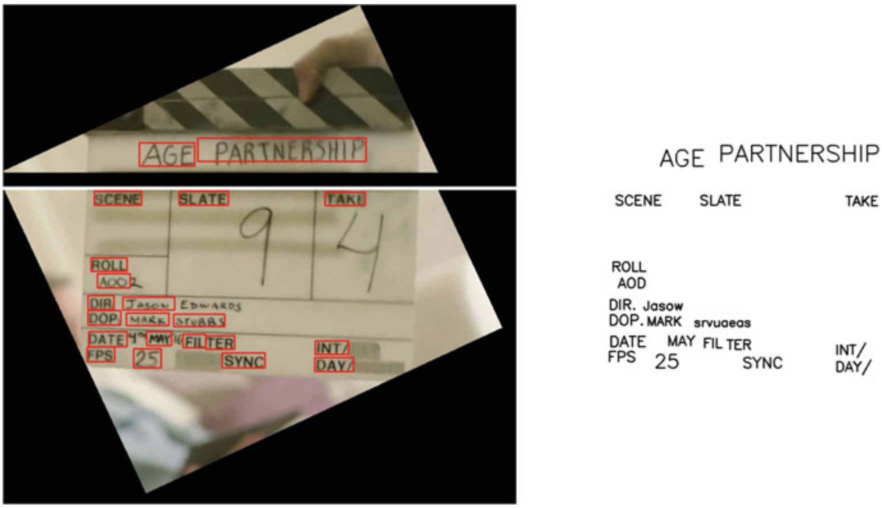

Fig. 9 Test 2

Fig. 10 Test 3

In Fig. 9, we note that the take has not been divided into all its parts, but the text was recognized almost correctly (the locator did not find "Edwards," the manager's surname).

We note that the numeric characters have not been localized, since the method is not well trained for identifying single numeric characters; in fact, in Fig. 10, we note that the scene number (1301) has been located, and the numbers concerning the fields "TAKE" and "ROLL" have not been located and the recognition has interpreted a "3" as a "5." In general, the method is very accurate in locating the clapperboard but is often wrong in the character detection that is a more difficult task, in particular when the character is isolated.

The frames selected from all those identified by the clapperboard recognizer in Tests 1, 2, and 3 actually constitute the best choices. However, in Fig. 11, we note that the selected take is not complete, and this implies the wrong or the absence of localization and recognition of the cut text. This is due to the fact that sometimes the video shot does not start with the take already framed, since this enters the frame slowly. The risk is to choose the less blurred take even if it is one of those in which the clapperboard is still not entirely framed. After beating the bar, on the other hand, the take is quickly released from the frame, and therefore the relative takes are generally blurred, and there is no risk that one of them will be selected. This problem can be avoided by also taking into account the dimensions of the various boxes when selecting the best take.

Fig. 11 Test 4

5 Conclusions

AI has made many steps forward thanks to the hardware and software advances and is now able to give practical answers to a number of problems where it can even outperform humans. However, it still may fail on rather simple tasks where human behavior is much more complex. Rules and skills may depend on many senses and feelings and cannot be reduced to simple logical rules, so they are difficult to describe. In the movie [5], as a tool for identifying replicants, Blade Runners have a mental test consisting of a series of emotion evoking questions to help distinguish a replicant from a human. The mixture of rationality and emotions was very clear to R. Bellman, one of the fathers of modern control theory that describes the optimal behavior of a dynamical system, when in 1957 he wrote the dedication of his famous book “Dynamic Programming” [31] to his wife Betty Jo: “To Betty-Jo whose decision processes defy analysis.”

Despite all the efforts of AI, after more than 50 years, there is still a long way to go.

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Origami and Fractal Solutions of Differential Systems

Paolo Marcellini and Emanuele Paolini

1 A Mathematical Origami from the Analytic Point of View

We consider an open set $\Omega \subset \mathbb{R}^2$ which represents a *sheet of paper*, usually a rectangle in $\mathbb{R}^2$. The *origami* is a *folded paper* and lives in the three dimensional space $\mathbb{R}^3$. We identify the origami with the image of a map $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$.

A sheet of paper is rigid in tangential directions. Indeed, it cannot be stretched, compressed, or sheared. If a sheet of paper is constrained on a plane, it would only be possible to achieve *rigid motions*, i.e., *rotations* and *translations* of the whole sheet. Since origami is a *folded paper*, the map u cannot be everywhere smooth; it is only *piecewise smooth*. Folding creates discontinuities in the gradient. Since we do not allow to cut the sheet of paper, u is however a continuous map. The singular set $\Sigma_u \subset \Omega$, which is the set of discontinuities of the gradient Du, is called *crease pattern* in the origami context. Usually (but not necessarily in the general three-dimensional case), this set is composed of straight segments. Of special interest is the case of the so-called *flat origami*, which is a map u whose image is contained in a plane, and which, up to a change of coordinates, can be represented as a map $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$.

In collaboration with Bernard Dacorogna, we investigated this analytic approach to origami in a series of papers [7–13] and [24]; we refer to these references for further details, proofs, and descriptions of related aspects about origami from a mathematical point of view. In particular, the article [11] in the Notices of the American Mathematical Society contains a less technical description of our analytic approach.

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We also mention some other mathematical approaches to origami, not necessarily of analytic nature: we quote the recent article by Abate [1] and, for instance, Alperin [2], Arkin-Bender-Demaine [3], Bern-Hayes [4], Haga [14], Heller [15], Huffman [16], Hull [17], Kawahata-Nishikawa [18], Kawasaki [19], Kilian et al. [20], Lang [21–23], and Robertson [25].

2 The Fractal Nature of the Solutions of the Dirichlet Problem

In the general three-dimensional case, with $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ being *piecewise smooth* in Ω, the tangential rigidity can be expressed by requiring that the gradient $Du(x, y)$ of the map u is an *orthogonal* 3×2 matrix. That is, in any subdomain where u is smooth, the matrix product satisfies the condition

$$Du^t\,(x, y) \cdot Du\,(x, y) = I$$

where I is the identity matrix. In the special two-dimensional case, with $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, under the notation

$$u\,(x, y) = \begin{pmatrix} u^1\,(x, y) \\ u^2\,(x, y) \end{pmatrix}$$

the same condition $Du^t\,(x) \cdot Du\,(x) = I$ equivalently gives $|Du|^2 = 2\,|\det Du| = 2$ and also $\det Du = \pm 1$. On the regions where the gradient is continuous, the determinant $\det Du$ must be constant and hence has a fixed sign. If we consider a subdomain of Ω where $\det Du = 1$, the equation $|Du|^2 = 2\det Du$ can be easily transformed into the system

$$\begin{cases} u^1_x - u^2_y = 0 \\ u^1_y + u^2_x = 0 \\ \left|Du^1\,(x, y)\right| = 1 \\ \left|Du^2\,(x, y)\right| = 1 \end{cases}$$

where the nonlinear nature of the differential equations is apparent.

The study of this kind of systems of partial differential equations is motivated by the study of elasticity and rigidity properties of materials. If we assume, as it is natural, that the elastic energy vanishes for rigid deformations, then any map with orthogonal gradient must be a minimum for the elastic energy. It is hence interesting to investigate when such maps exist.

In particular, tension and compression of a material are achieved by constraining the boundary of such material in a given position. This is why we are interested in solving the Dirichlet problem:

$$\begin{cases} Du = \begin{pmatrix} u_x^1 & u_y^1 \\ u_x^2 & u_y^2 \end{pmatrix} & \text{orthogonal matrix a.e. in } \Omega \\ u(x, y) = \varphi(x, y) & \text{on the boundary } \partial\Omega \end{cases} \tag{1}$$

It is not difficult to convince oneself that if φ is a dilation problem (1) has no solution. On the other hand, when φ is a strict contraction, there are general abstract results [5, 6] which guarantee the existence of infinitely many solutions.

In the particular case when φ is constant, we are able to find explicit solutions to this problem. From the point of view of origami, we are looking for a crease pattern on a square sheet of paper (for example) such that the whole boundary of the square is sent on a single point. The set of points where the map assumes a fixed value cannot have interior; otherwise, the gradient would be zero and hence not orthogonal. On the other hand, in a region where the gradient is constant and orthogonal, the map is locally invertible, and hence there cannot be two points with the same value.

This forces the crease pattern to accumulate and become dense while approaching the boundary of the domain and explains the *fractal nature* of the solutions of our differential problems.

More precisely, by denoting by τ and ν, respectively, the tangent and normal unit vectors on $\partial\Omega$, up to a sign, we have $\left(Du^1, \tau\right) = \left(Du^2, \nu\right)$ and $\left(Du^2, \tau\right) = \left(Du^1, \nu\right)$. Since $u^1(x, y) = u^2(x, y) = 0$ on $\partial\Omega$, we also obtain $Du^1 = Du^2 = 0$, which contradicts the fact that $\left|Du^1\right| = \left|Du^2\right| = 1$. Thus, any solution to the differential problem (1), with $\varphi = 0$, is Lipschitz-continuous but not of class C^1 near the boundary; therefore, it assumes in a *fractal way* the homogenous boundary datum $\varphi = 0$. The map u will be explicitly defined at every $(x, y) \in \Omega$, and it will be piecewise-affine, with infinitely many pieces, in accord with its fractal nature near the boundary of Ω.

3 A Strategy to Solve the Differential Problem

As usual we denote by $O(n)$ the set of n-dimensional orthogonal matrices; this in particular $O(2)$ is the set of 2×2 orthogonal matrices. Under this notation, the Dirichlet problem (1) with $\varphi = 0$ becomes

$$\begin{cases} Du(x, y) \in O(2) & \text{a.e. } (x, y) \in \Omega \\ u(x, y) = 0 & (x, y) \in \partial\Omega \end{cases} \tag{2}$$

with Ω rectangle in $\mathbb{R}^2$. As we already pointed out in the previous section, only a fractal construction can ensure the boundary condition $u = 0$. When Ω is a rectangle, we can divide it into infinitely many homothetic rectangles which are smaller and smaller while we approach to the boundary of Ω. Then, it is enough to consider a *base map* u_0 defined on one of these tiles. This map will be translated, rotated, and rescaled to fit any other rectangles. To assure that the gluing of the rectangles gives a continuous map, we need the base map to have prescribed *recursive* boundary conditions. That is, we require that on the right-hand side of the *base rectangle* (say a square of side 1), the map is defined so that it reproduces twice the values of the left-hand side, rescaled by half; i.e.,

$$u_0(1, y) = u_0(0, 2y) \quad \text{for } y \in [0, 1/2]$$
$$u_0(1, y) = u_0(0, 2y - 1) \text{ for } y \in [1/2, 1]$$

while on the upper and lower sides we only need periodic boundary conditions $u_0(x, 0) = u_0(x, 1)$ for $x \in [0, 1]$. If the map assumes at least once the value 0 on every rectangle in the net, then by its Lipschitz continuity (every rigid map is 1-Lipschitz continuous) it can be extended to the boundary $\partial\Omega$ with the 0 value.

4 The Dirichlet Problem with Non-homogeneous Boundary Condition

In this section, we propose some new ideas to solve the Dirichlet problem (2) when the homogeneous boundary condition $u(x, y) = 0$ on $\partial\Omega$ is replaced by a non-homogeneous one. From the applicative point of view, the $\varphi = 0$ boundary datum is not really applicable because we are usually interested in finding solutions when a small compression is applied to the boundary of our body. The problem of finding explicit solutions becomes more difficult, and for simplicity we only consider a linear datum such as

$$\varphi(x, y) = (1 - 2\lambda)(x, y) \qquad \forall(x, y) \in \partial\Omega \tag{3}$$

with $\Omega = [0, 1]^2$ and $0 < \lambda < 1$. When $\lambda = 0$, the only solution is the identity $u(x, y) = (x, y)$, while for $\lambda = 1$ the only solution is $u(x, y) = (-x, -y)$.

We build a solution to the Dirichlet problem with a recursive construction, as explained in the previous section.

In particular, we start by defining the mesh of the cells as in Fig. 1. Note that we approach the boundary recursively by splitting each cell into two cells of half the size.

In Fig. 2, we represent the singular set of the map, i.e., the discontinuity set of the gradient. In fact, our solution is Lipschitz-continuous: only the gradient can have discontinuities.

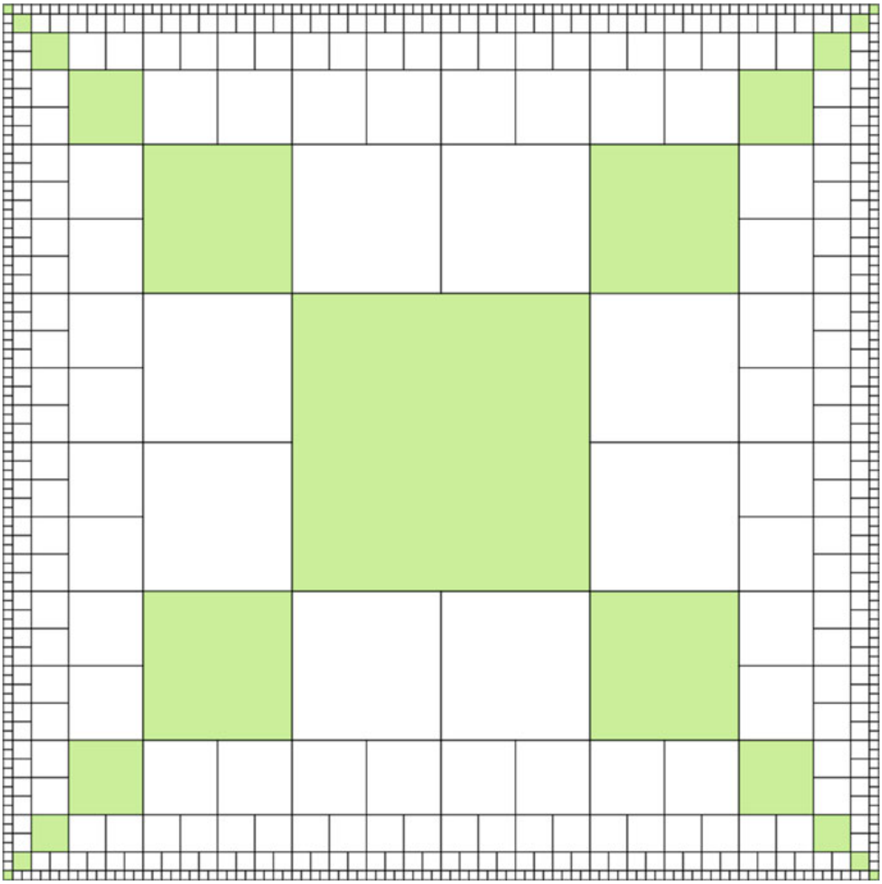

Fig. 1 The scheme of the cells that approach the boundary in a fractal way. The green cells are described in Fig. 3, while all other cells are obtained by a rotation of the construction with the singular set and the solution described in Figs. 4 and 6

We describe the construction of the solution in each cell. We start by the diagonal cells, see Fig. 3, where we emphasize the discontinuity lines of the gradient of the solution.

The main construction is described in Fig. 4 where we have inserted the analytic expression of the solution in each subcell. We also inserted the Cartesian equation of the discontinuity lines. Note that the solution matches continuously on each discontinuity line of the gradient. We invite the reader to check this property.

In Fig. 5, we give the values of the gradient on a base cell. Note that the map has only diagonal gradient matrices. In this figure, we use precisely *six* different gradient matrices.

Finally, a similar detailed analytical description is proposed in Fig. 6 where we show four adjacent cells around a point in the diagonal of the square.

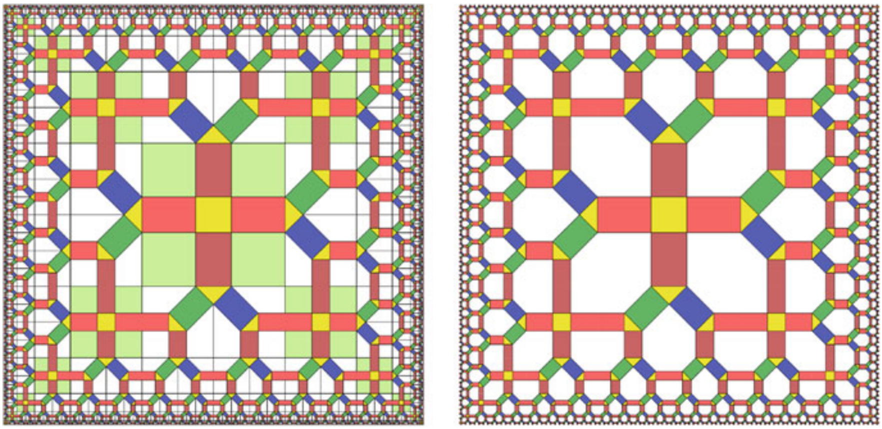

Fig. 2 The singular set, i.e., the discontinuity lines of the gradient. In the picture in the left-hand side, the singular set is superimposed to the grid of the cells, while the picture on the right-hand side shows the singular set alone. Each different value of the gradient of the solution corresponds to each color in the picture. The identity gradient matrix $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is identified with white color, while the yellow color denotes the gradient matrix $-I$

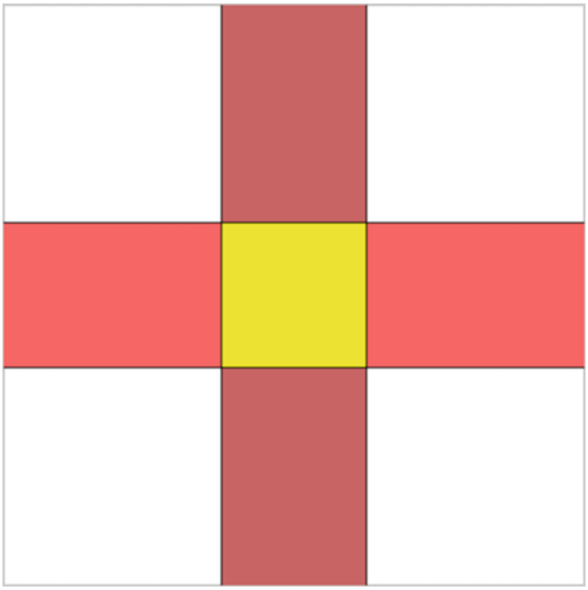

Fig. 3 A detail of each green square in Fig. 1. The set of discontinuity lines of the gradient of the map is represented here. Some analytic details are shown in Fig. 6

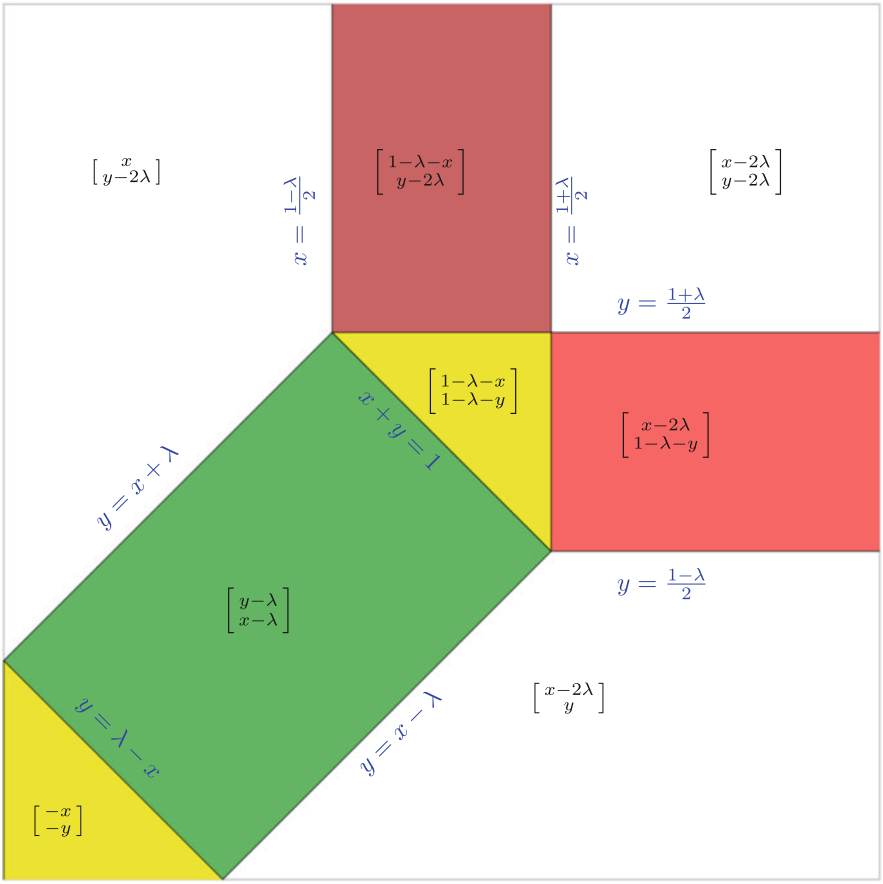

Fig. 4 This is the main construction of the solution in a basic cell. Recall that our solution is a map $\mathbb{R}^2 \to \mathbb{R}^2$. We wrote the analytic expression of the two components of the solution on each subregion where the gradient is constant. The solution matches continuously on the discontinuity lines of the gradient

Up to now, we have a map u whose gradient is orthogonal. To check that this map solves the boundary datum (3), we consider again Fig. 4 (similarly, we could consider Fig. 6). In this basic cell, the map assumes the same values of the boundary datum (3) on the four vertices of the unit square. From a basic cell to another basic cell, the map is rescaled, rotated, and translated so that this property is preserved on all the vertices of the grid in Fig. 1.

Since the cells have diameter which goes to zero as we approach the boundary $\partial\Omega$, and since the map is Lipschitz-continuous, we can extend it to the boundary so that the map assumes exactly the linear datum $(1 - 2\lambda)(x, y)$.

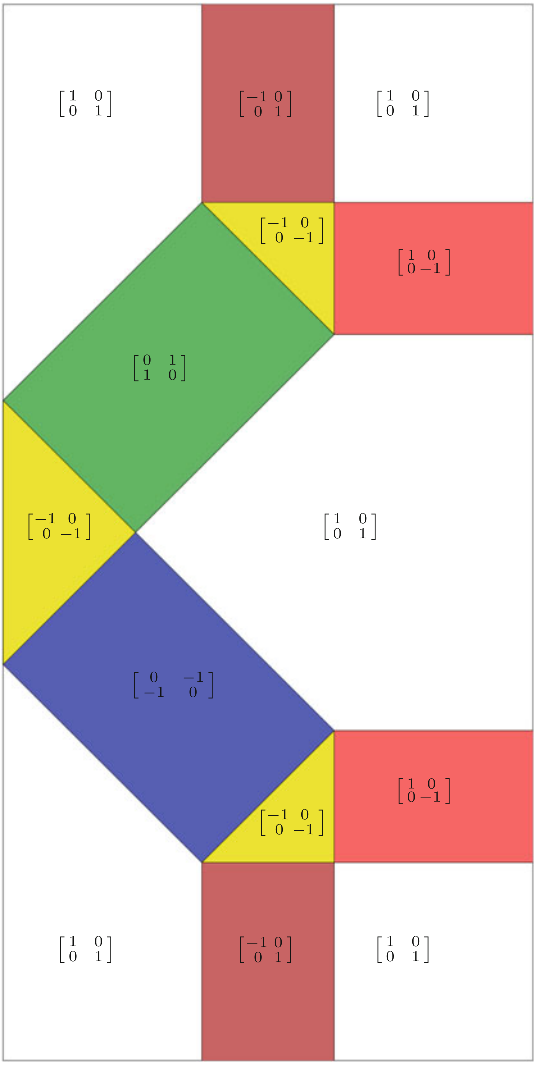

Fig. 5 The gradient of the solution on two contiguous basic cells. Note that the map only uses diagonal unitary matrices. This depends on the fact that the discontinuity lines of the gradient are either parallel to the coordinate axes or rotated by ±45°. In this picture, six different gradient matrices can be seen

In conclusion, our map u solves the Dirichlet problem

$$\begin{cases} Du(x, y) \in \left\{\pm\begin{bmatrix}1 & 0\\ 0 & \pm 1\end{bmatrix}, \pm\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right\} \subset O(2) & \text{a.e. } (x, y) \in [0, 1]^2 \\ u(x, y) = (1 - 2\lambda)(x, y) & \text{for all } (x, y) \in \partial[0, 1]^2 \end{cases}$$

the map being orthogonal almost everywhere on $[0, 1]^2$ and assuming only six gradient values.

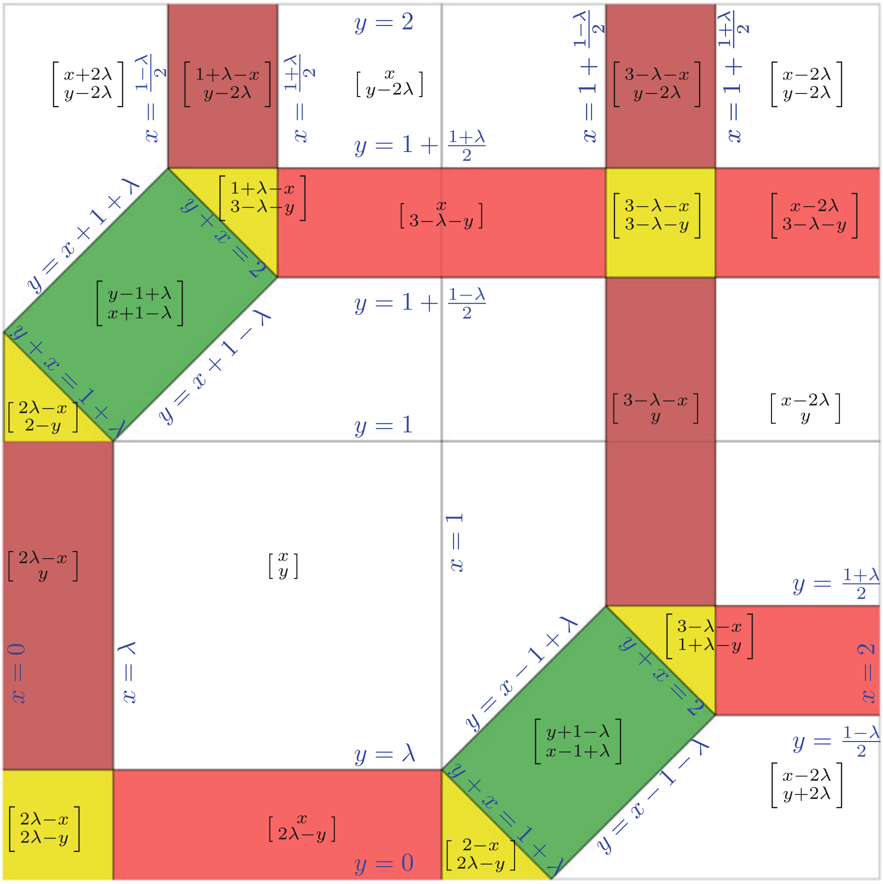

Fig. 6 We represent here the analytic expression of the vector-valued solution, up to an additive constant and a rescaling, specifically around a corner cell (as in Fig. 3). Of course also here the solution matches continuously on each discontinuity line of the gradient. The difference of the values of the solution at the vertex points $(2, 2)$ and $(0, 0)$ is equal to $2–4\lambda$; i.e., it is the same value computed similarly for the boundary value $(1 - 2\lambda)(x, y)$. The same is true for the other two vertices $(2, 0)$ and $(0, 2)$

When λ varies in $(0, 1)$, the singular set of the solution that we have built varies in consequence. In Fig. 7, we present two singular sets: the first one with a small value of λ and the second one with a value of λ close to 1.

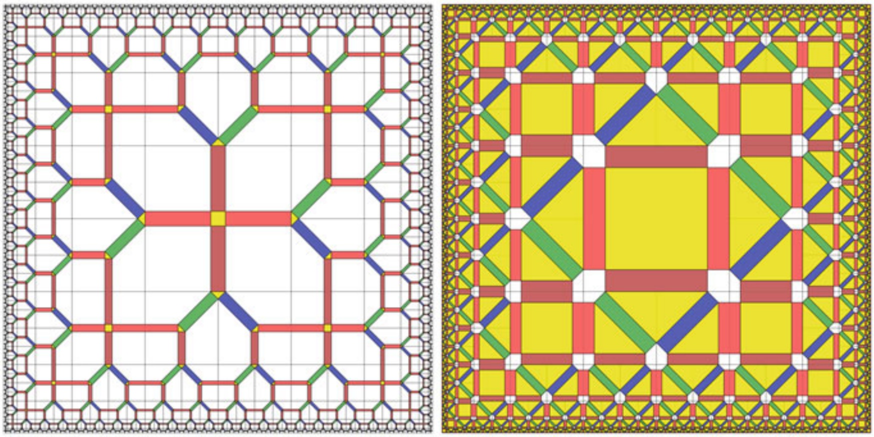

Fig. 7 Two singular sets of the solution in dependence on λ: on the left-hand side $\lambda = \frac{1}{10}$ on the right-hand side $\lambda = \frac{7}{10}$. Again, the gradient matrix of the solution is equal to the identity matrix I in the white regions, while it is equal to $-I$ in the yellow regions

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The Tangled Allure of Recursion

Gian Marco Todesco

One of the oldest algorithms we know, which is still in use, allows computing the greatest common divisor (GCD) of two numbers; it is attributed to Euclid, who describes it in his "*Elements*" (c. 300 BC). It can be expressed in this way:

- If the two numbers are equal, then their GCD is the shared value.
- If the numbers are different, then their GCD is equal to the GCD of the smallest number and their difference.

A noticeable feature of this method is its apparent circularity: to calculate the GCD of two numbers, you must calculate the GCD of two other numbers.

The procedure works (and is very effective) because the numbers shrink at each step while remaining positive, so sooner or later, they will become equal, and the procedure will eventually terminate.

The algorithm (at least in its definition) uses *recursion*, an extraordinary and effective conceptual tool used in mathematics, computer science, and logic. It is a difficult concept to master, and it continues to fascinate and torment generations of students. The following few pages are meant to be a sort of exploratory walk into the recursion world, with particular attention to its applications in the field of graphics.

Recursion is a form of self-reference that is central in human thought when reflecting on its mental mechanisms. It is not surprising that recursion catches the imagination and is often used playfully. For example, a search of "recursion" on Google returns the initial suggestion "*Did you mean: 'recursion'*": winking at the most attentive readers.

Other playful uses of recursion are recursive acronyms. They are acronyms whose first word is the acronym itself. The most notable example is Richard Stallman's *GNU* project. *GNU* is a free software operating system, and the acronym means "GNU is Not Unix" (Unix is a different operating system).

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A book that torments and delights the reader with recursion is *Gödel, Escher, Bach: an Eternal Golden Braid* (1979), by Douglas Hofstadter [1]. The book contains the first example I have met of a recursive acronym and one of the best. Achilles, a character in the book, asks a Genie for a special wish: he wants one hundred desires instead of only three. This kind of wish is a "meta-wish", and the Genie cannot grant it alone. It intends to help Achilles, so it forwards the wish to its *meta-Genie* that in turn has to ask its *meta-meta-Genie*, and so on. Each Genie refers to the whole infinite sequence of all the other Genies as *GOD*. The name is a recursive acronym meaning "GOD Over Djinn." Expanding the first word of the sentence, one gets "GOD Over Djinn Over Djinn" and so on, generating a new Djinn at each step. A very effective name for the infinite sequence of Djins!

Another extraordinary creation in the book is the concept of "quine program" or simply "quine." The name is a homage to the philosopher and logician Willard Van Orman Quine (1908–2000). For Hofstadter, a "quine" is a program that outputs its source code when run. The program must not have access to its code, but it has to "compute" it by itself.

Creating such a program is a fun challenge that requires a fair amount of programming skills. Such a program could be considered a metaphor for living creatures that can reproduce themselves together with the code (DNA) that describes them.

1 Tower of Hanoi

Besides the playful aspects, recursion proves to be a potent and effective tool. Recursive algorithms are often more compact and concise than the non-recursive equivalent, and sometimes the recursive approach can solve problems that seemed too hard with other methods. The GCD and even the factorial (i.e., the product of the first consecutive integers: a typical example in the introductive courses about recursion), while defined with recursion, are usually computed differently, in a more effective way. For many other problems, the recursive solution is almost not avoidable.

A classic example of this kind of problem is the "*Tower of Hanoi*" puzzle. The puzzle, also known as "*Tower of Brahma*" or "*Lucas' Tower*," was invented in the nineteenth century by the French mathematician Édouard Lucas. It features several disks of different radiuses stacked on one of three posts. The game's goal is to move the stack to another post by following two rules: one can move only one disk at a time and not put a larger disk on top of a smaller one. The game does not seem too complex. Indeed, the solution with 3–4 disks can be found quite easily by trial and error. If we consider a higher number of disks, then the number of possible disk arrangements grows dramatically. Even the number of moves needed by a perfect player to solve the puzzle becomes quickly very large. Four disks require 15 moves (see Fig. 1), while 20 disks require more than one million.

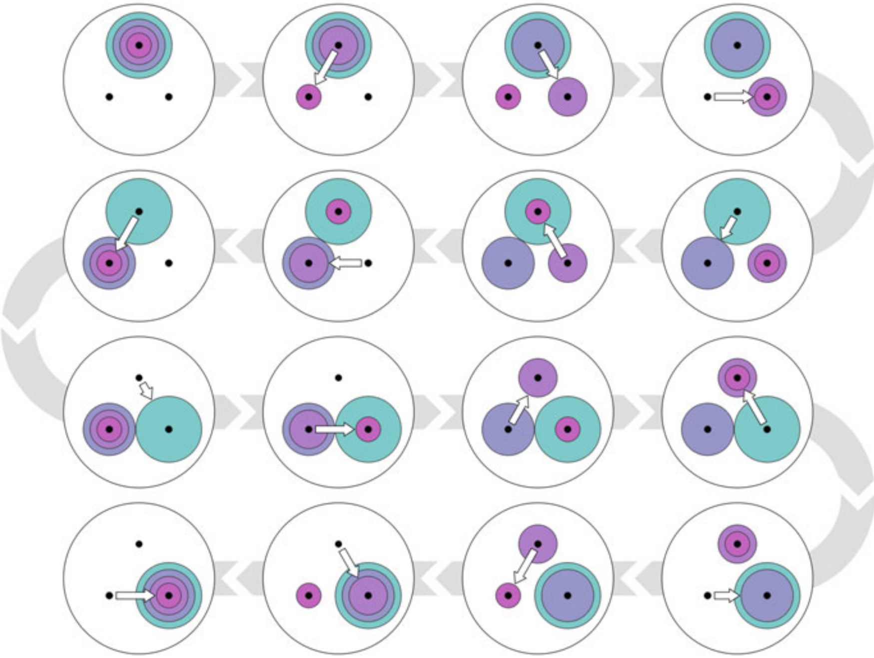

Fig. 1 The shortest solution (15 steps) for the Tower of Hanoi puzzle with 4 disks

When trying to solve the puzzle, there are (almost) always three legal moves at each step. One option "goes back," undoing the previous move, and can be discarded. The other two options require a nontrivial choice. Random choices make us wander in a labyrinth of different states from which it is difficult to escape. The correct solution requires many moves, but suboptimal solutions can require many, many more. Figure 2 shows an alternative solution for four disks. This alternative has no useless cycle: it visits each configuration of disks only once, but it requires 53 moves.

Writing a program that finds the shortest solution is not a trivial task, and the recursive approach is very convenient. To tackle the problem recursively, we must define a straightforward case when the recursion stops (this is crucial to avoid a never-ending program). For the Hanoi Tower, the easy case is when we have just one disk. In this case, the solution is obvious, and it requires just one move.

To solve the general case (N disks, with N larger than one), we must pretend that we know how to solve the simpler configuration with one less disk. With this knowledge (that we assume to have), we move N-1 disks from the first post to the third one: the largest disk on the bottom cannot interfere because the rules allow moving any other disk on top of the largest one as we wish. At this point, we are halfway: we transfer the largest disk from the first post to the second one, which is empty. Finally, we use the N-1 disks solution again and move the stack from

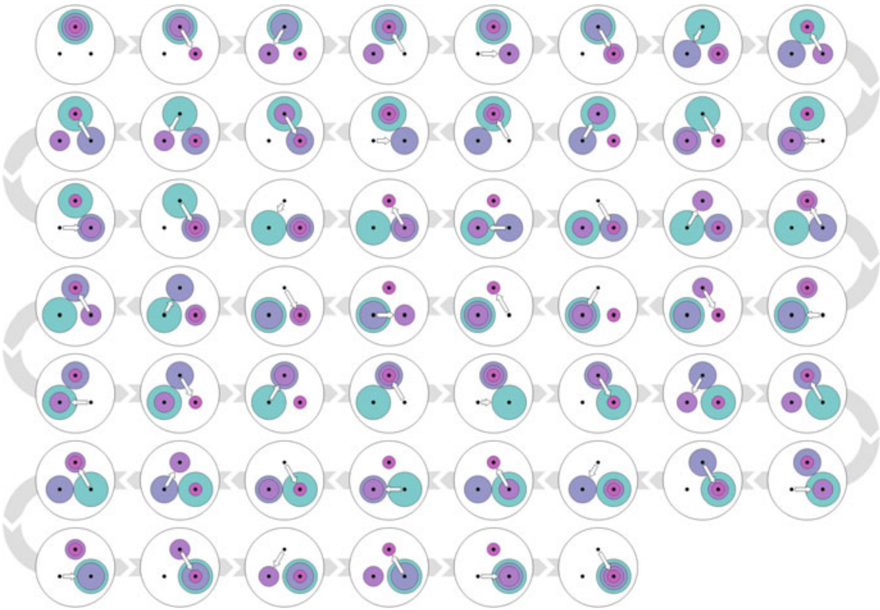

Fig. 2 A nonoptimal solution (53 steps). The solution, although inefficient, contains no repetitions

the third post to the second, on top of the largest disk. This action completes the challenge.

During the actual solution, the N-1 subproblems are solved using the N-2 solution and so on, until we must solve the problem with $N = 1$ that we solve directly, without the recursion.

2 Anagrams

The Hanoi Towers Problem seems (and probably is) meant for recursion, but the recursion is very useful even in more common problems. For example, we can use the recursion effectively to generate all the anagrams of a word. To make things easy, we assume that the word contains all different letters. Moreover, we do not check the generated anagrams in the dictionary, i.e., we accept meaningless anagrams.

In the novel "*Il Pendolo di Foucault*" by Umberto Eco [2], Jacopo Belbo, one of the main characters, owns a personal computer that he loves very much, named Abulafia. He found on the computer manual a small program (written in the *BASIC*

```python
def anagrams(word):
        m = len(word)
        if m == 1:
                yield word
        else:
                for s in anagrams(word[1:]):
                        for i in range(m+1):
                                yield s[:i] + word[:1] + s[i:]

for s in anagrams('image'): print(s)
```

Fig. 3 A Python program that generates all the anagrams of the word "image"

programming language) that can generate all the permutations of four letters. He realizes that he can use the program to generate all the God name permutations (the God name he means is *IHVH*; Apparently, Belbo does not note or does not care that the pair of H's will generate many duplicates).

The novel contains a factual listing of the program (15 lines). Inserting a computer program source code in a fictional book is an intriguing idea. There is a nice contrast between Belbo with his computer program and Diotallevi, Belbo's friend who studies the Kabbalah. He cannot understand the logic of the program and finds it "kabbalistic."

From a programming point of view, the code is not too lousy. It even uses a nice trick to place the last letter, but it has some serious flaws. The worst one is that it can work only with four-letter words. It can generate all the anagrams of "math," but to generate all the anagrams of "image," we should completely rewrite the program.

Writing a program that generates all the permutations of any number of letters is not so easy if we want to start from scratch without using any predefined function (today, many programming languages have predefined library functions to generate the permutations). If we are going to try, then recursion is a perfect approach.

The seed of recursion (the "straightforward case") is when we consider a word with a single letter. In this case, there is only one anagram: the letter itself.

Let us consider a word with more than one letter. We can take out the first letter and generate all the anagrams of the rest (the rest has one less character, and we pretend to know how to solve this easier problem). Then, we insert the first letter of the original word in each possible position into each generated anagram.

This procedure will generate all the anagrams of a word, regardless of the number of letters. Figure 3 shows a program (written in the Python programming language) that produces all the anagrams of "image" recursively (One of the anagrams is, with some punctuation added: "*A gem: I!*". It is a decent anagram that is referring enthusiastically and recursively to itself!).

3 Visual Recursion

The world of images, from paintings to computer graphics, is very suitable for the recursive approach.

The basic idea is to create an image that contains one or more replicas of itself, possibly with some changes. In Western art, this technique is called *Mise en abyme*. It is also known as the *Droste effect*. This last name comes from a Dutch brand of cocoa (see Fig. 4). The image contains a smaller replica that includes an even smaller one and so on. That implies an infinite sequence.

The Dutch artist Maurits Cornelis Escher used this technique with a fascinating twist. In its *Print Gallery* (1956) [3], he creates a stunning vortex that merges the larger image seamlessly with its smaller replica.

Bart de Smit of Leiden University led a group that analyzed the image thoroughly from a mathematical point of view [4, 5]. Today, the effect is readily available in many image editing software. The image in Fig. 5 has been created with *GIMP* (GNU Image Manipulation Program) [6] and the *MathMap* plugin [7].

Fig. 4 The original 1904 Droste cacao tin, designed by Jan Misset (1861–1931)

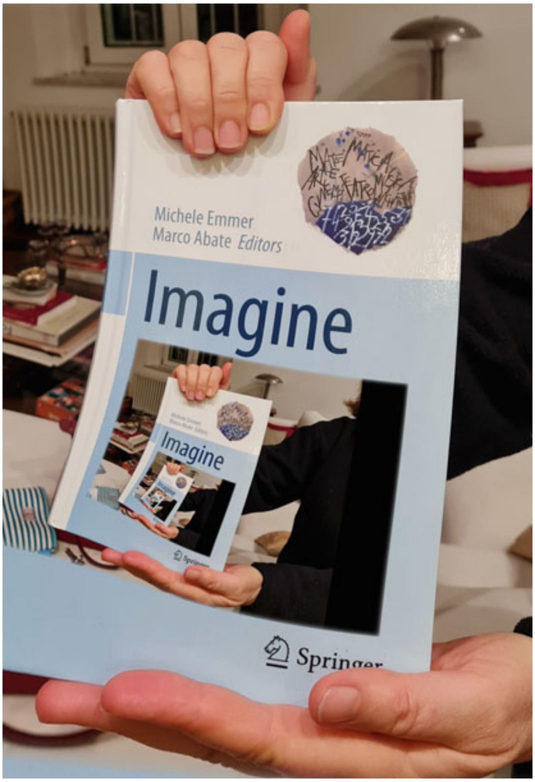

Fig. 5 The "Droste Effect" in action

Recursive drawings can be fascinating even when made of simple graphics primitives: e.g., simple straight segments. The dragon curve (*Heighway dragon*) is a remarkable geometric object with many interesting properties [8]. It can be defined with the following recursive procedure: start with a simple segment connecting two points. Then replace the segment with two shorter segments of equal length, meeting at a right angle. The old segment must be the hypotenuse of an isosceles right triangle, while the two new segments must be the catheti. At each recursive step, apply this procedure to each segment. The new segments must lie alternatively on the left and the right of the old segments.

Figure 6 shows the first steps. Figure 7 is the result after 20 iterations.

The dragon curve is the limit approached as the above steps are repeated indefinitely.

The curve contains infinite replicas of itself, rotated 45° respect to each other, and with a scale factor of $\sqrt{2}$.

It is possible to create a model of the dragon curve by folding a strip of paper. We cut a long and narrow strip of thin cardboard, e.g., a strip one cm wide and 20 cm long. We fold the strip in half, bringing the right edge to the left edge. Then we repeat the folding four times, always folding in the same direction. We then unfold

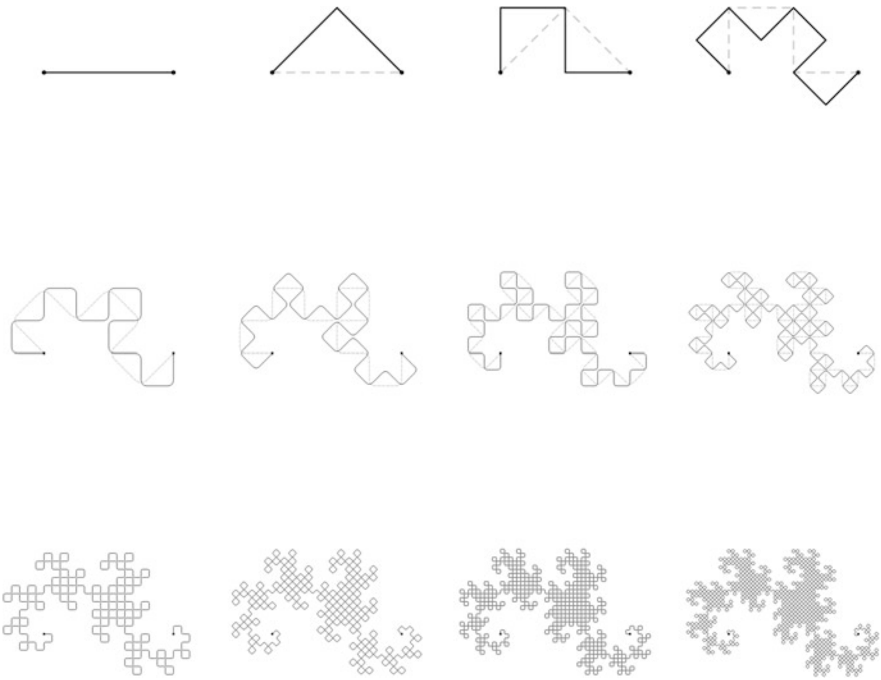

Fig. 6 First steps of the generation of the Heighway Dragon Curve

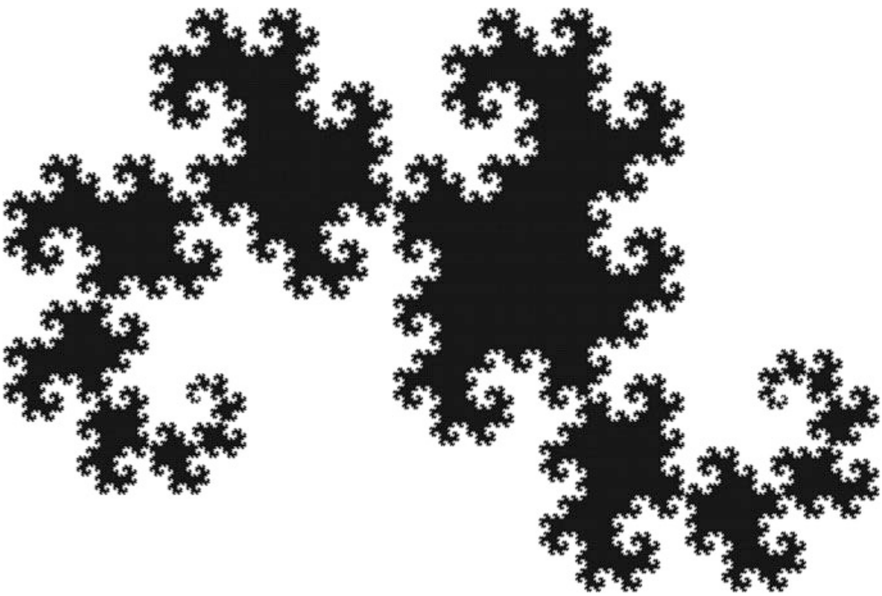

Fig. 7 Level-20 Dragon Curve

the strip carefully so that each fold is at a right angle. To build a larger model is better to fold different strips and tape them together.

It is easy to draw the curve with coding. Figure 8 shows an example that runs in *Google Chrome*, the Internet Browser. Run the browser and then open the console: activate the Chrome Menu in the upper-right-hand corner of the browser window and select *More Tools* > *Developer Tools*. You can also use the shortcut Option + ⌘ + J (on macOS), or Shift + CTRL + J (on Windows/Linux). Copy very carefully the text in Fig. 8 into the console.

Press *return* and the browser will show the 13th iteration of the dragon curve.

Write *dragon(7)* and then press *return* to generate the 7th iteration and so on.

At the limit, the curve fills a whole region in the plane: it touches every point included in its boundary. A bizarre behavior for a one-dimensional entity such as a curve!

It is possible to put two identical curves next to each other: they will share part of their boundary with a perfect match. The dragon curve can even tessellate the whole plane (see Fig. 9).

The first curve that passes through every point of a 2-dimensional region was discovered in 1890 by Italian mathematician Giuseppe Peano. The Peano curve, as the dragon curve and most of the many other space-filling curves, is defined with a recursive process: the actual curve is the limit of an infinite sequence of increasingly detailed curves.

In Fig. 10, there are the first steps of another example, created by David Hilbert in 1891.

This last curve is well suited for creative use: the image sequence shows that the more refined the grid, the darker the image. We can modulate the darkness locally, just changing the depth of the recursion depending on the position [9]. Figure 11 shows the result of this experiment.

```
function dragon(L) {
    let draw = (L,a,b,c,d) => {
        let e=(a+c-d+b)/2, f=(b+d+c-a)/2; return L ?
            draw(L-1,a,b,e,f)+draw(L-1,c,d,e,f) :
            "<line stroke='teal' x1='"+a+"' y1='"+b+"' x2='"+c+"' y2='"+d+"'/>";
    }
    document.body.innerHTML = "<svg width='800' height='500'>"+draw(L,200,180,650,180)+"</svg>";
} dragon(13)
```

Fig. 8 A JavaScript code to draw the Dragon Curve in an Internet Browser

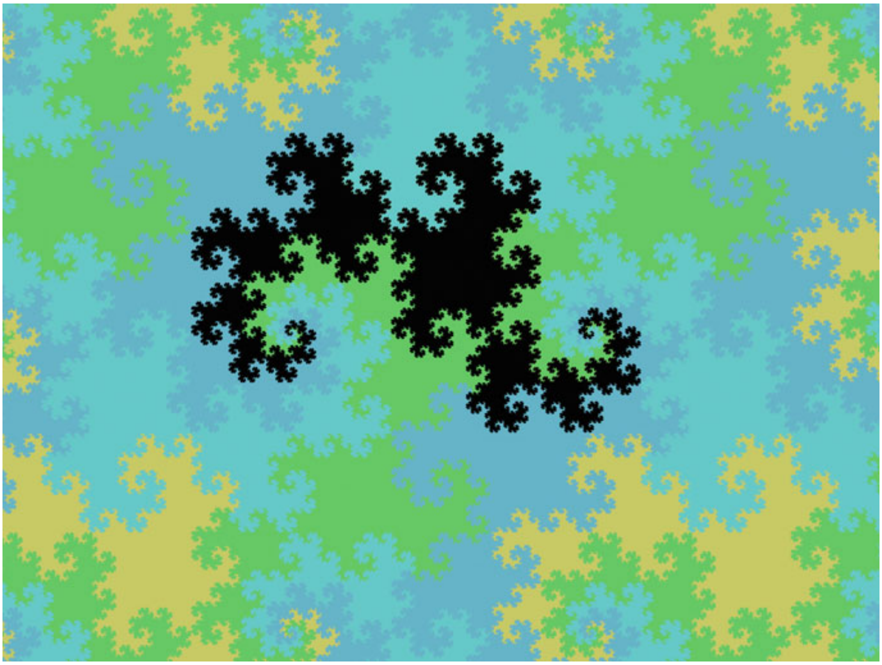

Fig. 9 Many copies of the Dragon Curve can tessellate the whole plane

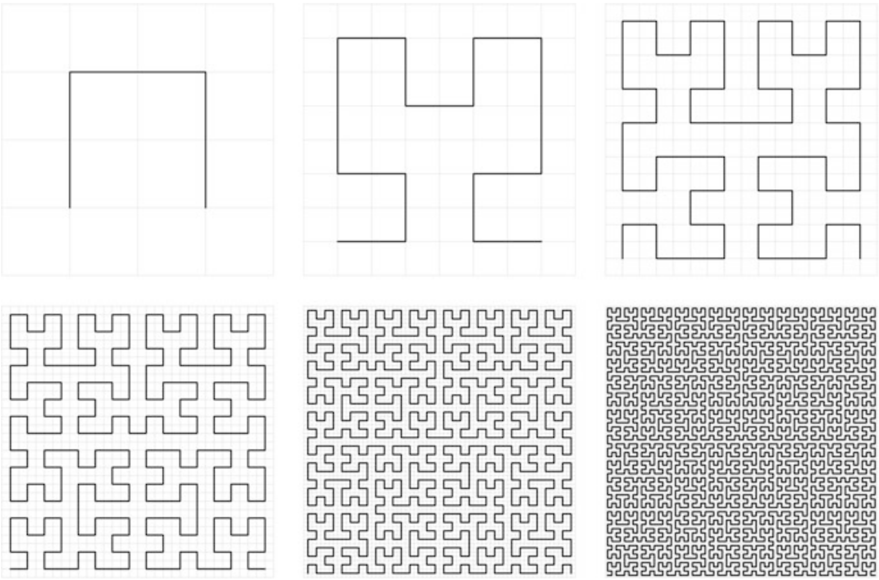

Fig. 10 The first steps of the Hilbert Curve

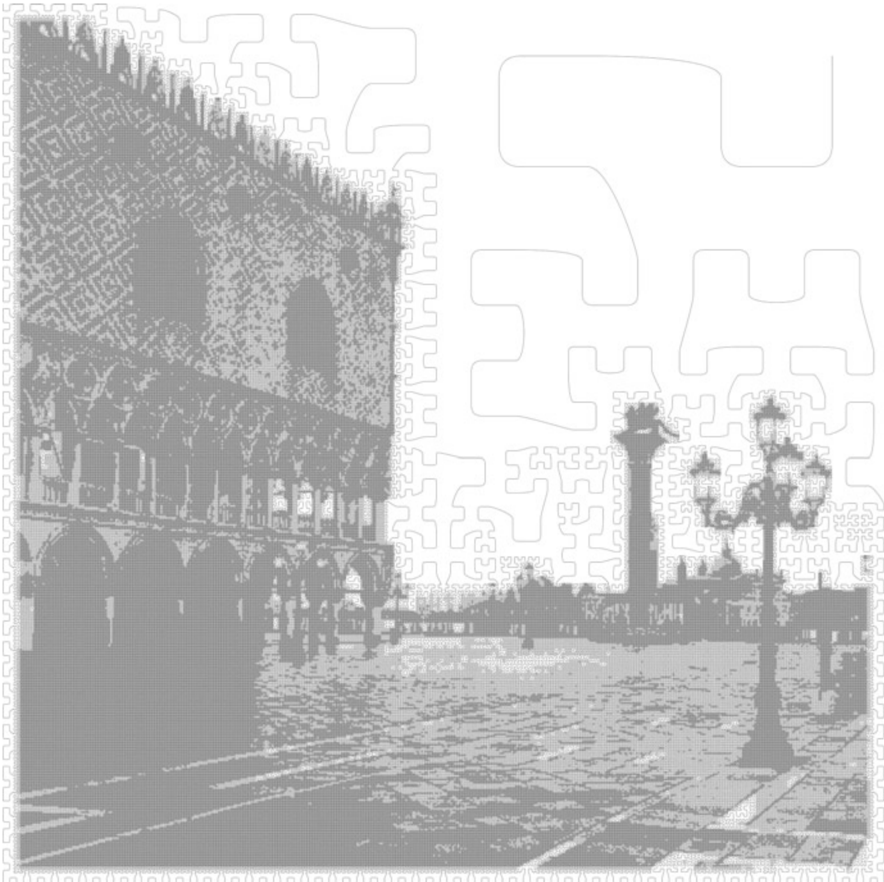

Fig. 11 Hilbert curve with a recursion level depending on the point. The whole drawing is made of a single stroke with same color and thickness

4 Sierpiński Triangle

The Sierpiński triangle is a well-known geometric pattern closely related to recursion.

It comes out in different contexts and, as we will see, shows many deep interconnections among various fields. As in the previous examples, we can generate it with a simple recursive procedure. We start with a triangle (any triangle works well, but we usually select an equilateral triangle). We replace the triangle with three smaller congruent copies, touching at the vertices and arranged in the outline of the original triangle. We repeat the process for each triangle infinitely.

Figure 12 shows the process. The bottom triangle has eight steps of recursion and contains 6561 tiny triangles.

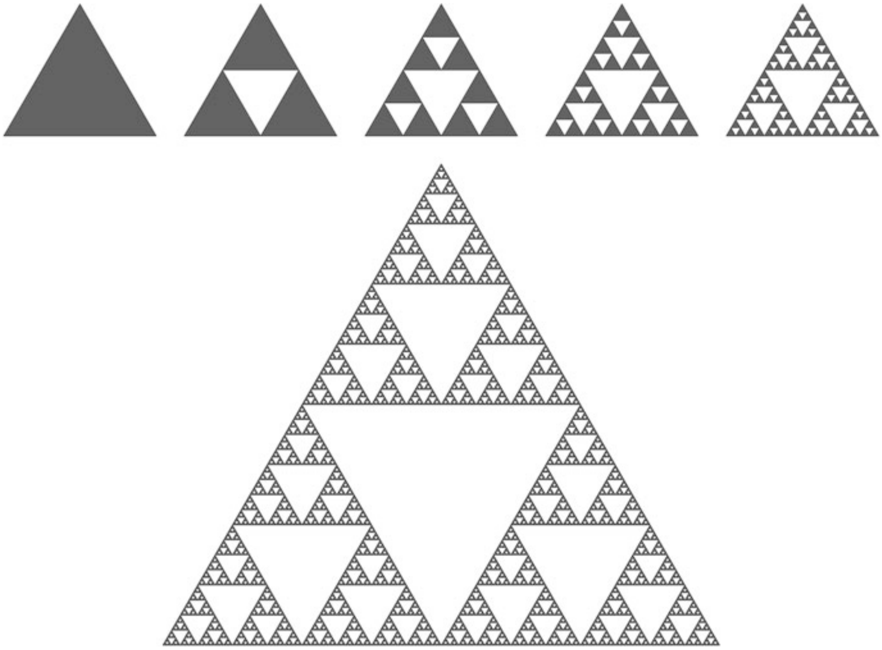

Fig. 12 The first steps of the Sierpiński triangle

The pattern is beautiful, and it has been used in art long before its mathematical formalization. For example, there is a level-4 triangle on the floor of the basilica of San Clemente in Rome, dating late 11th century [10, 11].

One interesting geometrical feature of the Sierpiński triangle (in common with very many other similar shapes) is its behavior when we change the size. If we double the size of the first triangle, we obtain three copies of the original pattern, so the area must be three times larger. This outcome is unusual. We intuitively expect that the effect of a scale transformation depends on the dimension of the shape: if we draw a circle twice the size, then the length of the circumference should double as well ($2^1 = 2$), but the area should be four times larger ($2^2 = 4$). According to this reasoning, we should conclude that the dimension of the Sierpiński triangle is between one and two: the shape is more than a unidimensional curve but less than a plane figure. We can say that it has a "fractal dimension" close to 1.585 ($2^{1.585}$ is close to 3).

To emphasize that the Sierpiński triangle is halfway between one- and bi-dimensional, we will show how to describe it as a curve. The procedure is similar to the one we have already used in the previous paragraphs (the curve is not a plane filler in this case).

Figure 13 shows the first steps. At each stage, each segment is substituted by three smaller pieces forming 120° angles among them. After eleven steps, the result is indistinguishable from that of the first procedure (the one with triangles).

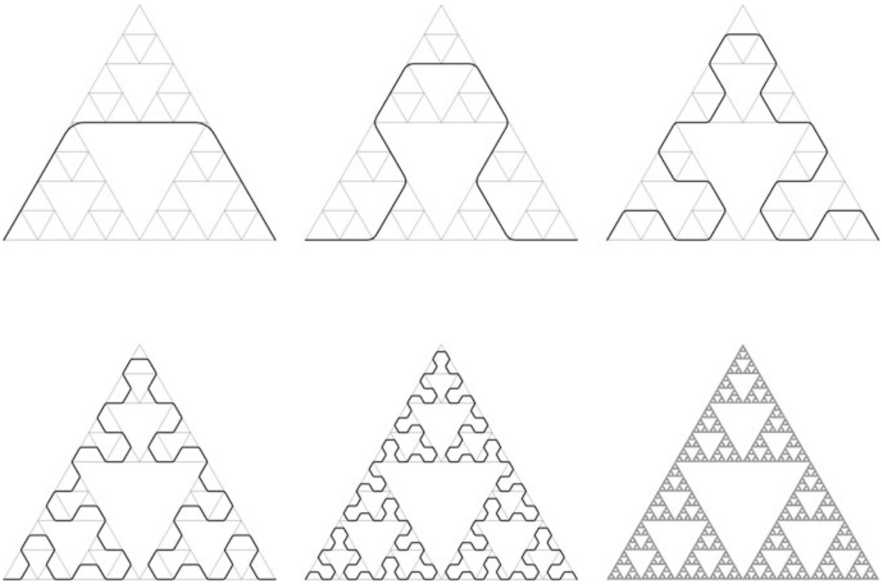

Fig. 13 The Sierpiński triangle as a curve

We inadvertently touched the Sierpiński triangle in a previous paragraph when we played with the Hanoi Tower. The set of all the possible states of a Hanoi puzzle with N disks makes a graph with the arcs representing the allowed moves. The graphs are usually depicted with a diagram, with circles for the vertices and lines for the arcs. The positioning of the circles is not determined by the graph and follows aesthetic criteria. In this case, there is a very natural arrangement. In the game, the smaller disk can always move on each post, and therefore the whole graph must be made of tiny triangles connected by the vertices. The other connections (legal moves) assemble these triangles in a predetermined shape. Figure 14 shows the graph of the puzzle with four disks: it is a Sierpiński triangle!

The link between the Sierpiński triangle and the Hanoi Tower is surprising, but there are many other unexpected connections. An example is Pascal's Triangle, which is made of rows of integer numbers. The first row contains only the number 1. Each number in the following rows is the sum of the two other numbers above it in the previous row (treating missing numbers as zero). Let us draw the Pascal's Triangle using different colors for odd and even numbers: odd numbers form the Sierpiński pattern again!

An even stranger connection comes from the *Chaos Game* invented by Michael Barnsley in 1988 [12]. The game is an algorithm that generates patterns with a random walk. The algorithm depends on some parameters and can create different shapes (in 2D, 3D, and even in higher dimensions). Let us start with the simplest one. We start with a triangle (as in the previous example, we will consider an equilateral triangle, but any triangle will work as well). Then we take a random

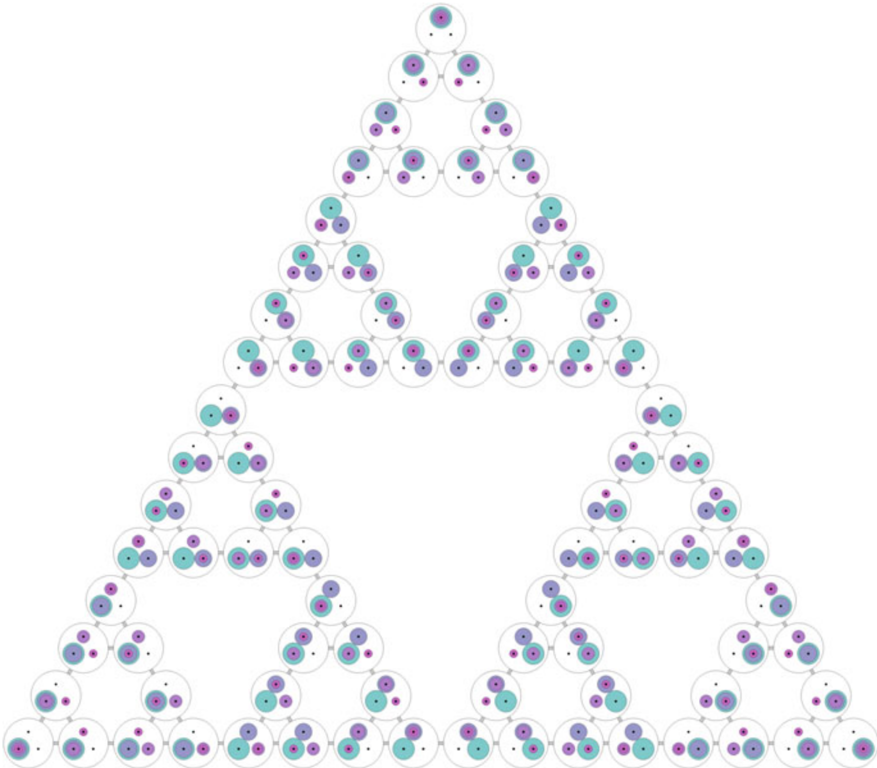

Fig. 14 The complete graph of moves for the Tower of Hanoi puzzle with 4 disks

point somewhere in the plane. It could be the center of the triangle, but the initial position is not very important.

At each step, we select one of the three triangle vertices at random and move the point halfway toward the chosen vertex. Because of the random choices at each step, it is impossible to predict where the point will be after a given number of iterations, but apparently, some spots are more likely than others. If we draw many points, a pattern slowly emerges. Figure 15 shows the first 10,000 points: they look like the Sierpiński triangle!

A connection between our fractal set and the Chaos Game is easy to spot. Each game's move is a scale transformation with the center at the chosen vertex and a scale factor equal to ½. It transforms the whole Sierpiński triangle into one of its three smaller replicas. Therefore if a point belongs to the fractal set, then it will remain on the set forever. Because of the shrink, points outside of the set become closer and closer to the set at each iteration.

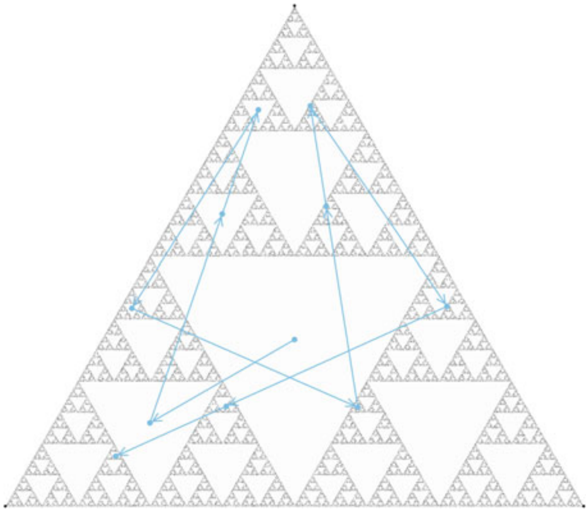

Fig. 15 The Chaos Game. 100,000 points. The first 10 are highlighted

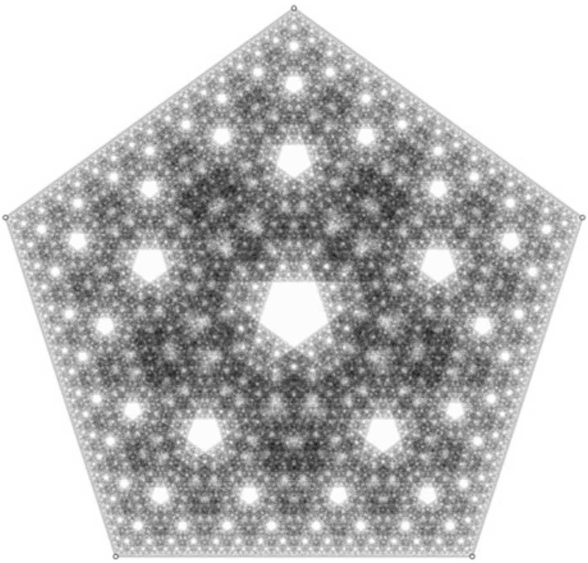

Fig. 16 The Chaos Game with 5 vertices. 5,000,000 points

Playing the Chaos Game with different parameters is also enjoyable. We can change the number of vertices (and their placement), the scale factor (we used ½), and add constraints on the random choice of the vertex (e.g., avoid choosing the same vertex twice in a row). Different parameters will create different patterns (albeit not always fractal).

A remarkable but straightforward variation shown in Fig. 16 uses a regular pentagon instead of a triangle (and no other changes).

5 Natural Shapes

Michael Barnsley extended the Chaos Game using a slightly more complex set of rules. The original Chaos Game uses three scale transformations selected at random at each step (the scale factor is 0.5 and the scale centers are the three vertices of the triangle). Barnsley decided to use a more general type of transformation: the affine transformation. An affine transformation can include a rotation, a translation, a nonuniform scaling, and any combination. It preserves lines and parallelism, but not necessarily angles and distances.

One of Barnsley's best known (and beautiful) models, the *Barnsley fern*, uses four carefully crafted affine transformations selected at random with different weights at each step. The idea is very similar to the original Chaos Game. We start with a point (in a random position); at each stage, we select one of the four affine transformations and use it to move the point to a new position. The selection is made at random but with given probabilities for the four different outcomes.

We draw a dot at each point position. Figure 17 shows the result with three million points. In Fig. 18, you can also see the effects of the four transformations on a given quadrilateral.

Fig. 17 The Barnsley Fern

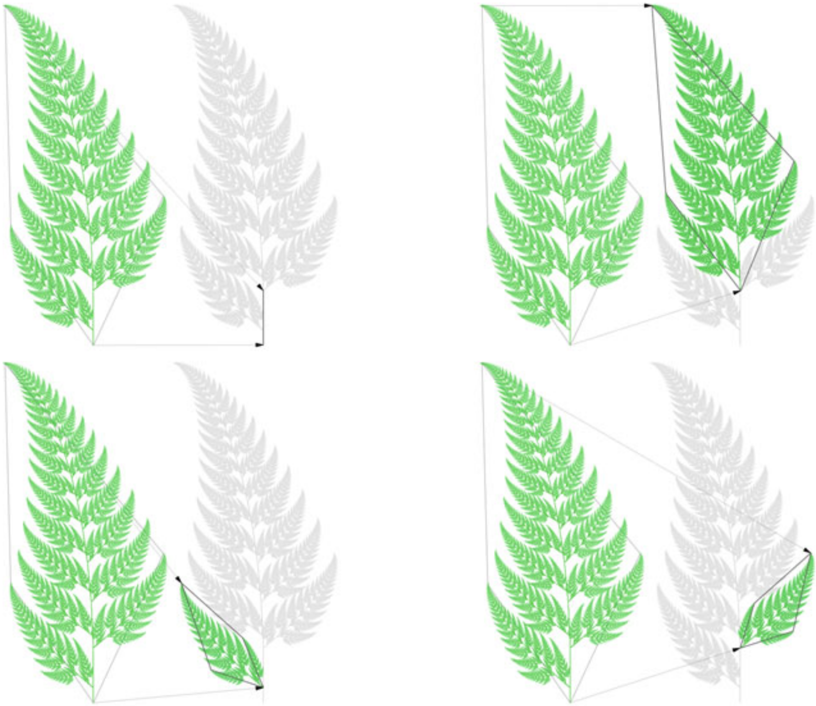

Fig. 18 The four transformations used to create the Barnsley fern

Barnsley selected the four transformations' parameters and weights carefully so that the generated pattern resembles a leaf of an actual plant: the Black Spleenwort. It is interesting that a simple algorithm can obtain such a faithful representation of a natural object. With Barnsley's words: "[. . .] provide models for certain plants, leaves, and ferns, by virtue of the self-similarity which often occurs in branching structures in nature" [13].

Fractals and self-similar, recursive patterns are indeed excellent models for natural shapes.

Even with a deterministic algorithm (one without the random choices of the Chaos Game), we can create forms with some sort of organic features (see Fig. 19).

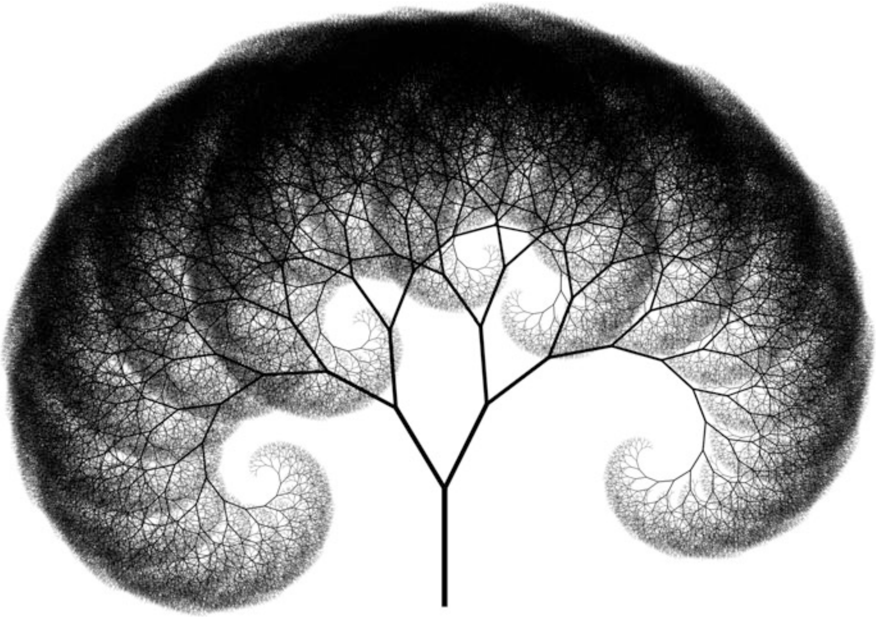

Fig. 19 A deterministic recursive tree (no random parameters); 18 levels of recursion

6 3D

Most of the self-similar shapes we have presented have a natural extension in 3D geometry. For instance, you can build a Sierpiński tetrahedron using the same logic of the Sierpiński triangle. The 3D models maintain the charm of their flat counterpart. Building self-similar 3D models with a computer program, glue and paper, or a 3D printer is a rewarding activity.

The Menger Sponge is a particularly tempting model. It starts with a simple cube. We divide the cube into 27 smaller cubes arranged as in the Rubik's cube. We then remove the inner cube in the center and the six cubes in the middle of the original cube faces. The twenty remaining cubes make a level-1 Menger sponge. We repeat the process on the remaining cubes to create the next-level sponge. Figure 20 shows a level-5 Menger Sponge.

The Menger Sponge is a three-dimensional generalization of the Cantor set: a fascinating uni-dimensional pattern presented by Georg Cantor in 1883 but discovered in 1874 by Henry John Stephen Smith. The pattern starts with a segment. At each step, we delete the middle third of every remaining segment. The process must continue ad infinitum. It leaves a set of points aligned along the initial segment with a lot of interesting properties.

We can extend the process to any dimension. In two dimensions, the pattern is called the Sierpiński carpet. We start with a square, divide it into nine smaller

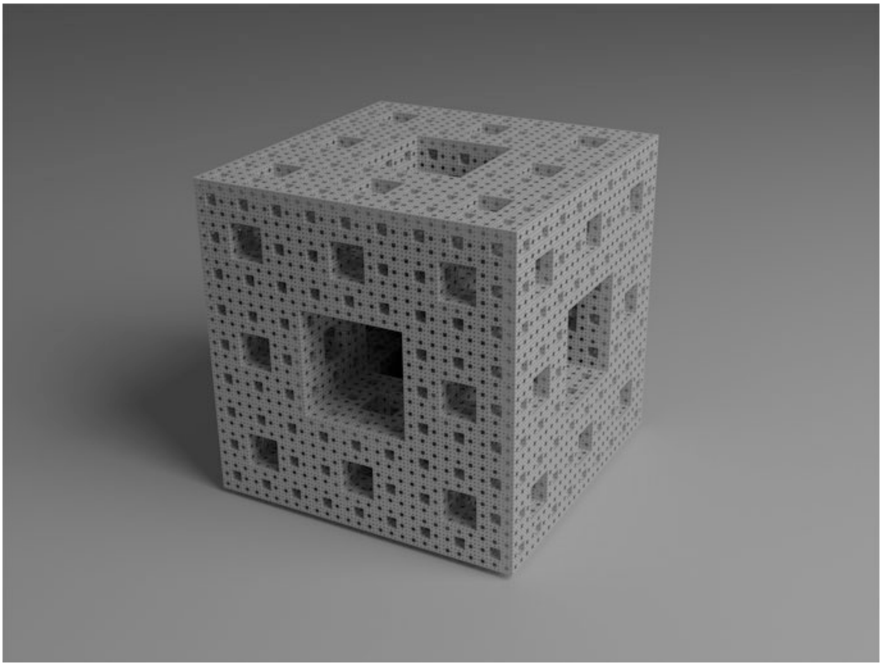

Fig. 20 A level-5 Menger Sponge

congruent squares arranged in a 3 × 3 grid and remove the central square. Then repeat for each remaining square ad infinitum.

The straightforward geometry of the cube makes it easy to build the Menger Sponge (at least for the first few recursive levels). You can even use Origami to construct small cubes and assemble the sponge.

The problem with 3D self-similar shapes is the tremendous growth of the number of elements depending on the recursion level. For the Menger Sponge, each successive level of recursion requires twenty times more elemental cubes. A simple level-2 Sponge is made of 400 cubes.

Creating a paper model of a level-3 (or the astounding level-4) Menger Sponge is a formidable task better suited for teamwork. In 2014, the *MegaMenger* project, conceived by Matt Parker and Laura Taalman, coordinated many worldwide groups to create a level-4 Menger Sponge [14]. The groups eventually built twenty level-3 Menger Sponges, made out of over a million folded business cards.

In 2016 MUSE, the Science Museum in Trento, Italy, launched *MUSEMenger*: a project for the wider public to assemble a large level-3 Menger Sponge. The project lasted 155 days and attracted more than 5000 people.

In November 2016, Serena Cicalò created a stunning level-4 Menger Sponge on her own, achieving an extraordinary result. She used an improved folding technique, quicker to use and capable to support much greater weight. In 15 months, she assembled more than 21 km of paper strips, 1.2 cm wide, to build a model slightly larger than one meter wide and weighing 25 kg [15, 16].

Even with computer graphics, creating shapes with such a large number of elements can be problematic. The problems begin gradually. The 8000 cubes that make a level-3 Sponge are manageable: it is possible (and easy) to create an interactive model that runs on a regular Internet browser in real time. Level-4 requires 160,000 cubes, thus almost two million triangles: the animation becomes slow and jerky. The next level is out of reach with this technique.

We, therefore, must use a different approach to get to higher levels. The crucial point is noticing that, while the number of faces to draw is exceptionally large, the number of distinct planes to which these faces belong is much smaller. The idea is to draw all the faces belonging to the same plane at once.

Let us consider one of the distinct planes. We create an image representing the section of the Menger Sponge relative to that plane. We leave the parts of the image not covered by Sponge faces as transparent. Then we draw a single square face, large as the whole cube, using that image as a texture.

Finally, we repeat this operation for each distinct plane, and we obtain a lovely model using relatively few faces.

Unfortunately, the section images are not identical. Some (at least the first and the last along each cube axis) are just the Sierpinski carpet. The level-1 sponge (4 slices along each axis) requires only that image. Level-2 requires two different slices, and the number doubles for each subsequent level. Figure 21 shows the 16 sections required for a level-4 Sponge; the next level requires 32 different sections. The number is not so small, but that sponge has 244 slices along each axis and 3.2 million elemental cubes! (see Fig. 20 again).

With higher levels, the number of different images becomes problematic. Moreover, the required resolution of each image must be more significant because of the ever more minute details.

To climb to even higher levels, we need a mixed approach: we create a level-5 (or level-6) sponge using the section strategy, and then we assemble 400 (or 8000) copies of this model.

Writing a computer program that visualizes a level-7 Menger Sponge spinning in real-time on the computer screen is a rewarding achievement.

Another challenge is trying to get not-orthogonal sections. This task requires a different and more technical strategy (programming the GPU); the results are lovely images. Figure 22 shows a cross-section passing for the sponge center forming 45° angles with the cube axes.

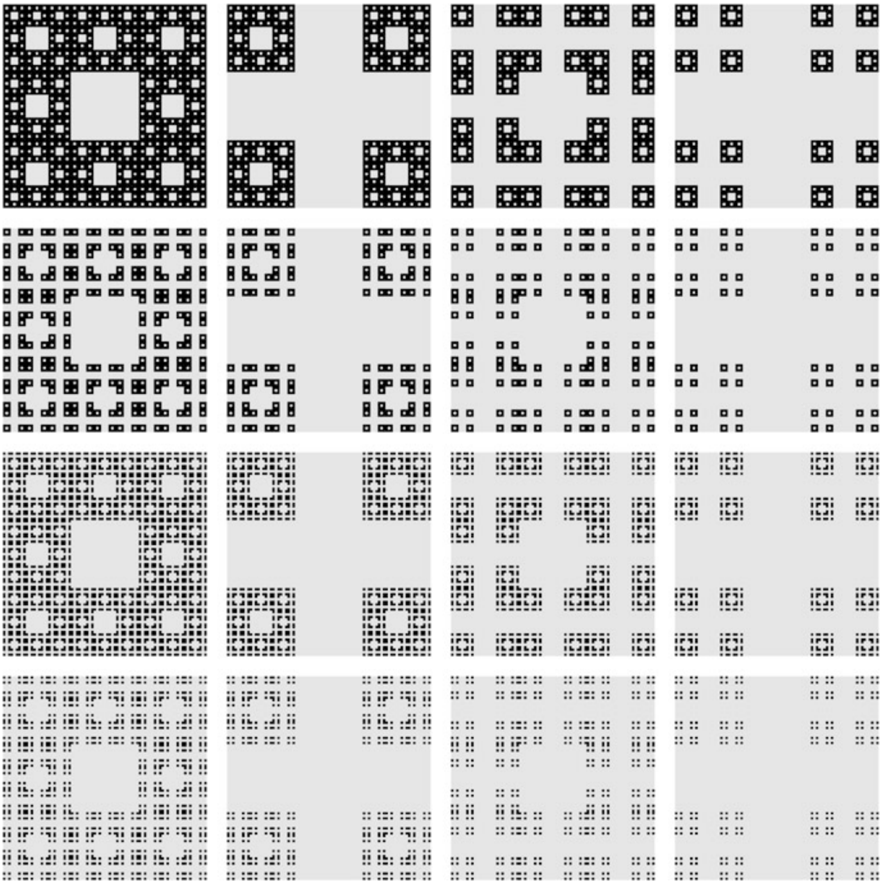

Fig. 21 The 16 different sections for a level-4 Menger Sponge

7 Alexander's Horned Sphere

The last model that we meet in our walk in the world of recursion is another weird shape.

It is a paradoxical figure discovered in 1924 by J. W. Alexander to disprove a very reasonable and intuitive assumption [17].

Let us consider a solid sphere. A loop outside the sphere can be shrunk into a point without touching it. The torus has different properties. A loop linked to the torus cannot be shrunk into a point without passing through the torus.

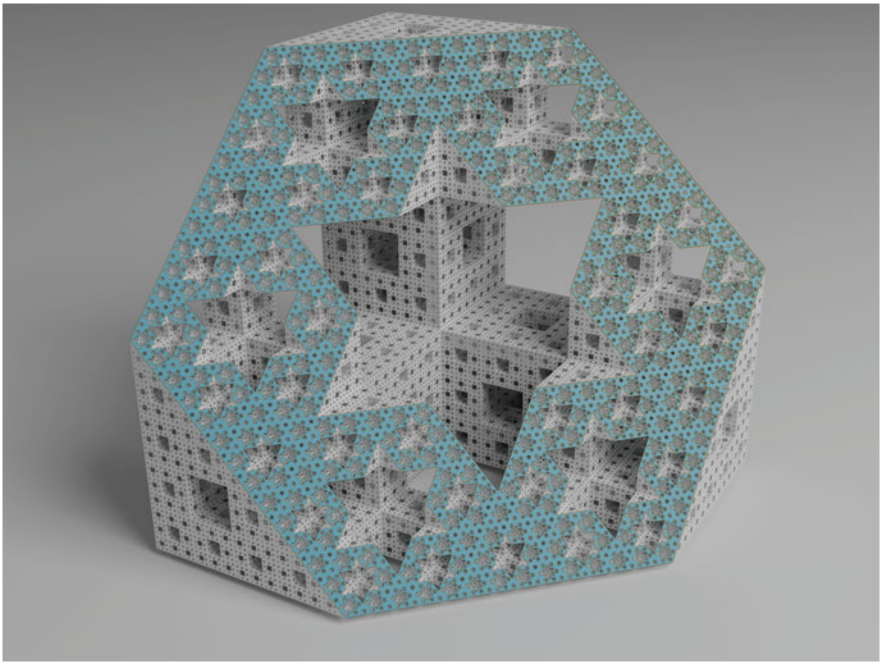

Fig. 22 A level-5 Menger Sponge cut with a non-orthogonal plane

That is a very well-known topological property of the ball and the torus (more precisely, it is a property of the space outside these shapes).

It is clear that the geometrical properties of the solids do not matter: the cube behaves like the ball, and a cup of coffee (with a handle) acts like the torus.

The very reasonable assumption is that the space outside any shape topologically equivalent to the ball is like the space outside the ball: any loop can shrink into a point without touching the surface.

In two dimensions, this is a fact: it is the Jordan–Schonflies theorem. In 1921, Alexander declared it was able to extend this result to the three-dimensional case. Before publishing a paper, he found an error in his demonstration. Eventually, he discovered a counterexample, showing the claim is false indeed! The weird, discovered shape is called the Alexander Horned Sphere. It is topologically equivalent to a ball, but the outside is not equivalent to the ball outside: it contains loops that cannot shrink into points without touching the shape.

To build the Horned Sphere, we must follow a recursive procedure.

We start with a torus with a small missing section. It is topologically equivalent to a sphere; geometrically, it reminds the letter C.

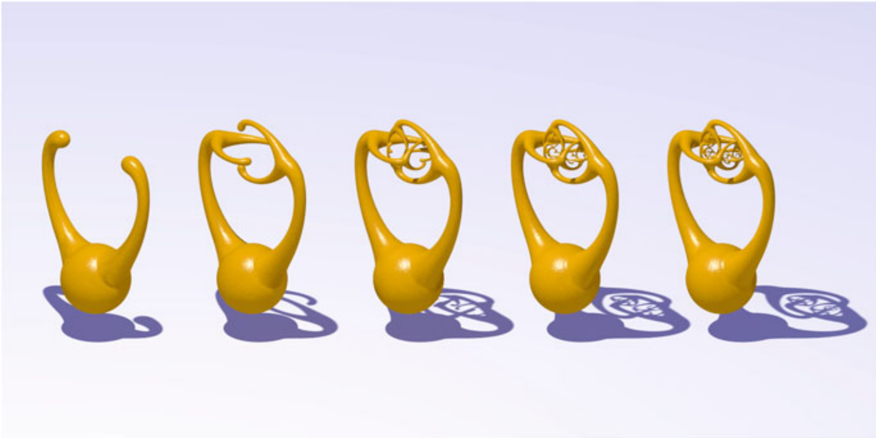

Fig. 23 First steps in building the Alexander's Horned Sphere

We take two smaller copies and glue them to the opposite end of the C. We arrange the position to chain the two smaller toruses together.

Then we repeat the process on the smaller C's and so on, ad infinitum.

At each stage, we get a better approximation of the final shape.

Alexander's horned sphere is the union of all the infinite stages.

It is possible to demonstrate that the resulting weird solid is still equivalent to a ball, like all the successive approximations. On the other hand, the space outside is so deeply tangled and knotted that it is pretty different from the space outside the ball. A loop around the first C is hopelessly linked and cannot shrink to a point without crossing the surface.

The scheme invented by Alexander and described in the previous paragraphs features many "corners" and creases. Still, they can be rounded to obtain a smooth surface without affecting the topological properties.

I have created a small animation that visualizes the growth of the horns up to a certain recursion level [18]. Figures 23 and 24 show the first steps.

The model is lovely even beyond its mathematical meaning.

It seems a couple of hands that want to hug. In the beginning, there are only two, but after a while, there are many, many more.

This image acquired new meanings in these extraordinary and complicated pandemic years.

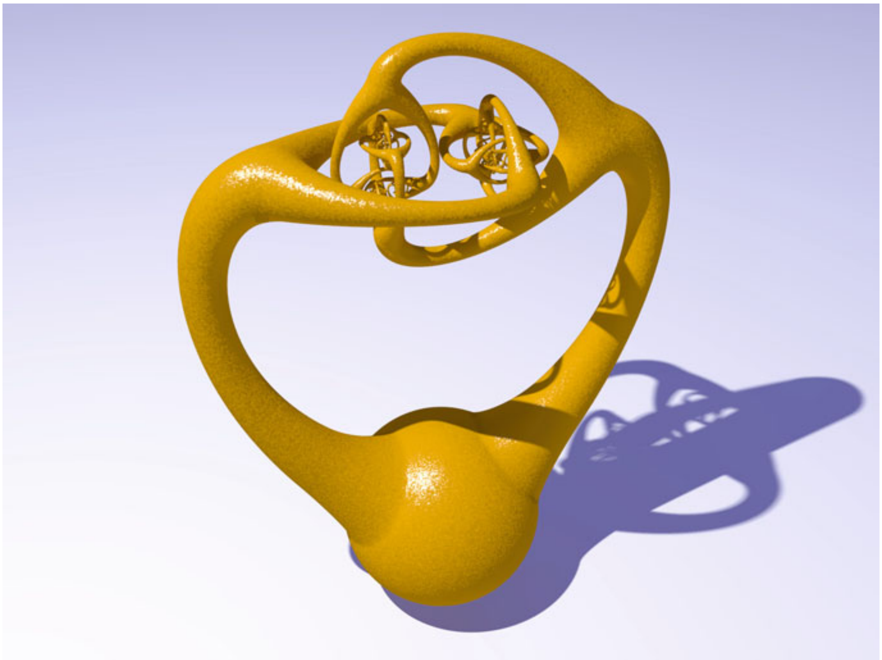

Fig. 24 The Alexander's Horned Sphere at level-8 of recursion

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Desert Locusts: Can Mathematical Models Help to Control Them?

Marcela Villarreal

The ravages created by desert locusts (*Schistocerca gregaria*) on livelihoods and food security have been known and feared for thousands of years. They are known to have been present during the times of the Egyptian pharaohs, around 3200 B.C., and they are identified as a plague in the *Bible* as well as in the *Torah*. By some accounts, they are the world's most devastating pest. Locusts are also interesting from a biological point of view, as they can undergo surprising transformations that enable them to unleash their destructive potential.

Desert locusts are usually present in their solitary form in deserts between Mauritania and India. With rains and the consequent development of vegetation, they can rapidly reproduce and within a few generations (one or two months) transform into their gregarious form, forming small groups or bands of wingless hoppers and small groups or swarms of winged adults. Widespread rains in adjacent areas can cause further reproduction and swarm formation, creating an upsurge and eventually a full-blown plague if conditions remain favorable for breeding.

During this process, locusts' behavior fundamentally changes from solitary to gregarious: "Instead of repelling one another, they become attracted to one another" [1]. They have a remarkable ability to change dramatically in response to environmental conditions, phenotypic plasticity. Their physiology suffers important transformations, including color (from light green to yellow-brown), brain, and increased body size, as well as ability to eat.

With specific climatic conditions, they are able to multiply 20-fold in three months and reach densities of 80 million per square kilometer (Figs. 1 and 2). One swarm in a 1954 plague in Kenya is estimated to have contained ten billion locusts, and it was one of only fifty swarms in the country at that time.

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Fig. 1 © FAO, photo Sven Torfinn, https://doi.org/10.4060/cb3673en

Fig. 2 © FAO, photo Sven Torfinn, https://doi.org/10.4060/cb3673en

In their gregarious form, locusts can eat about the equivalent of their body weight (2 g) per day. FAO calculates that a swarm of just 1 km^2 can consume as much food as would be eaten by 35,000 people (or six elephants) in a single day [2]. One million locusts can eat about one tonne of food each day, and the largest swarms can consume over 100,000 tonnes each day, or enough to feed tens of thousands of people for 1 year. In extreme conditions, desert locusts' voracity can result in cannibalism when other food sources become unavailable.

Locusts have a highly developed migratory capacity, with the ability to travel up to 150 km per day (they are carried by winds), easily going across borders and even continents. In 1988, swarms originating in North Africa crossed the Atlantic Ocean and made it successfully to the Caribbean and South America. They routinely traverse the Red Sea—a distance of 186 miles [1].

Containing locust ravages cannot be done without international action, and is best done through the multilateral system. Indeed, the UN has set up a monitoring and control system including an early warning mechanism that relies to some extent on mathematical models.

1 From Outbreaks to Plagues

Locust invasions differ in intensity, extension, and duration, resulting in a variety of impacts. Outbreaks can evolve into upsurges and into full-blown plagues. According to FAO [2], while outbreaks usually occur with an area of about 5000 km^2, upsurges affect entire regions and plagues develop when two or more regions are infested.

Outbreaks are frequent and only a few lead to upsurges. Similarly, few upsurges lead to plagues. Plagues are defined as periods of one or more years during which there are widespread and heavy locust infestations, the majority of which occur as bands or swarms. The last major plague was in 1987–1989 and the last major upsurge in 2003–2005. In the 1900s there were six major plagues, one of which lasted almost 13 years. The area in which plagues occur covers about 29 million sq. km and can extend across 65 countries in Africa, the Middle-East, and Southwest Asia. This area is extensively cultivated and populated by more than one billion people.

2 Impacts

Locusts are ravenous eaters. Their passage leaves devastation in agricultural fields as well as in most vegetation creating massive loss not only to crop and fodder production, but also to grazing land, affecting livestock too. All types of crops can be severely damaged, including annual rain-fed crops, perennial crops, tree cultivation, and irrigated crops which are even more sensitive since they are exposed throughout the year, with crop losses occurring in just a few hours [3]. Under certain conditions, locust invasions can result in local desertification and rural outmigration. Locust crises have been associated with several famine episodes such as those of Ethiopia and Sudan in the 1950s.

In most of the affected countries, the agriculture sector is the backbone of the economy. For example, in Kenya the agricultural sector accounts for 33% of GDP, employs 40% of total population and 70% of rural population. Rural areas, which are the most affected by the pest, concentrate poverty with 80% of the extremely poor [4].

According to FAO, in the 2003–2005 Sahel upsurge, crop loss ranged from 80 to 100 percent in Burkina Faso, Mali, and Mauritania. Nearly 8.4 million people across six countries (Burkina Faso, Chad, Mali, Mauritania, Niger, and Senegal) were affected, with many households requiring food aid (FAO, 2006). The crop damage was estimated at USD 2.5 billion. The most affected were subsistence farmers. Spillover effects include a significant reduction in children's schooling, especially that of girls. Other documented effects include the destruction of one million vine plants in Libya in 1941, 55,000 tons of cereals in 1954 in Sudan, and 16,000 tons of millet in 1951 in Senegal. The invasion of 1987–1989 left losses of 60% of grazing land and crops in Mauritania, 50% of grazing land in Niger, and close to 100% of market gardening crops in Mali [3].

Desert locusts pose threats to the food chain, food security, livelihoods, and national economies. Many of the affected countries are under a situation of protracted crisis and have undergone successive years of drought followed by heavy rains and floods. Desert locusts are currently considered a potential threat to the livelihoods of one-tenth of the world's population [5].

Making matters worse, many of the countries hit by the worst infestations are already hobbling from protracted crises—recovering from recessions, fighting natural disasters, racked by conflict, and now suffering the consequences of the coronavirus pandemic.

3 Current Situation

At the beginning of 2020, after several seasons of heavy rains and exceptionally wet cyclones, the conditions were ripe for one of the worst desert locust crises in decades. The combination of heat and humidity, made more frequent by climate change, allowed for soaring reproduction rates in breeding areas. Thus, the Horn of Africa became the hotspot of the worst desert locust crisis in over 25 years, and the most serious in 70 years for Kenya and Uganda [6]. Within the first few months of the year, huge swarms of desert locusts began to ravage multiple countries across the Greater Horn of Africa, the Arabian Peninsula, and Southwest Asia. As swarms spread across these regions, the situation quickly spiraled into an unprecedented threat to the food security and livelihoods of affected communities—raising the risk of further suffering, displacement, and potential conflict on top of that already imposed by extended droughts, floods, and geopolitical fragility. Locusts are worsening the conditions for more than 42 million people already facing acute food insecurity [6]. The current situation is set to become a regional plague, as several regions are now being affected simultaneously.

During the current invasion, locusts have swarmed in large numbers in dozens of countries, including Kenya, Ethiopia, Uganda, Somalia, Eritrea, India, Pakistan, Iran, Yemen, Oman, and Saudi Arabia. The situation remains alarming, particularly in Ethiopia, Kenya, and Somalia, where widespread breeding is in progress and new swarms are forming, representing an unprecedented threat to food security and livelihoods at the beginning of the cropping season. Desert locust swarms formed in the spring-breeding areas of Southwest Asia and are moving across India, Pakistan, and the Islamic Republic of Iran. In India, swarms that arrived in Rajasthan in May 2020 continued to move and have reached several central states in the country—this has not occurred since 1961 [6].

The situation in Yemen is particularly worrisome, with the protracted crisis creating conditions of extreme fragility for most of the population. In 2019, 24.9 m (83% of the population) were food insecure, out of which 15.9 m were in situation of crisis, emergency, or famine [7]. This situation is expected to become even worse, with the potential development of locust swarms. Hopper bands are reported to be forming on the southern coast, but the conflict situation makes it extremely difficult to obtain data for the rest of the country.

4 Controls/Programmes

Responding to a progressive weakening of national capacities and regional locust control organizations at the time of the 1987–1989 invasion [3], the *Food and Agriculture Organization of the United Nations* (FAO) launched the *Emergency Prevention System for Transboundary Animal and Plant Pests and Diseases* (EMPRES) in 1994, with a special focus on the desert locust. The EMPRES programme aims to reinforce locust control capacities in the countries with outbreak areas as well as to strengthen regional and international cooperation on this problem. The programme relies on donor funding and expertise at FAO.

An FAO *Commission for Controlling the Desert Locust in the Western Region* (CLCPRO) together with three regional locust commissions were set up for regional coordination. After a decade of strengthening national capacities and regional coordination, a major desert locust threat in the Sahel in 2012 was controlled.

FAO's *Desert Locust Information Service* (DLIS) monitors the locust situation and provides early warning to countries and donors on an ongoing basis. The locust situation is monitored 24/7, producing forecasts, early warnings and alerts on the timing, scale, and location of invasions and breeding. EMPRES and the commissions strengthen national capacities in early warning, early reaction, and contingency planning (Figs. 3, 4 and 5).

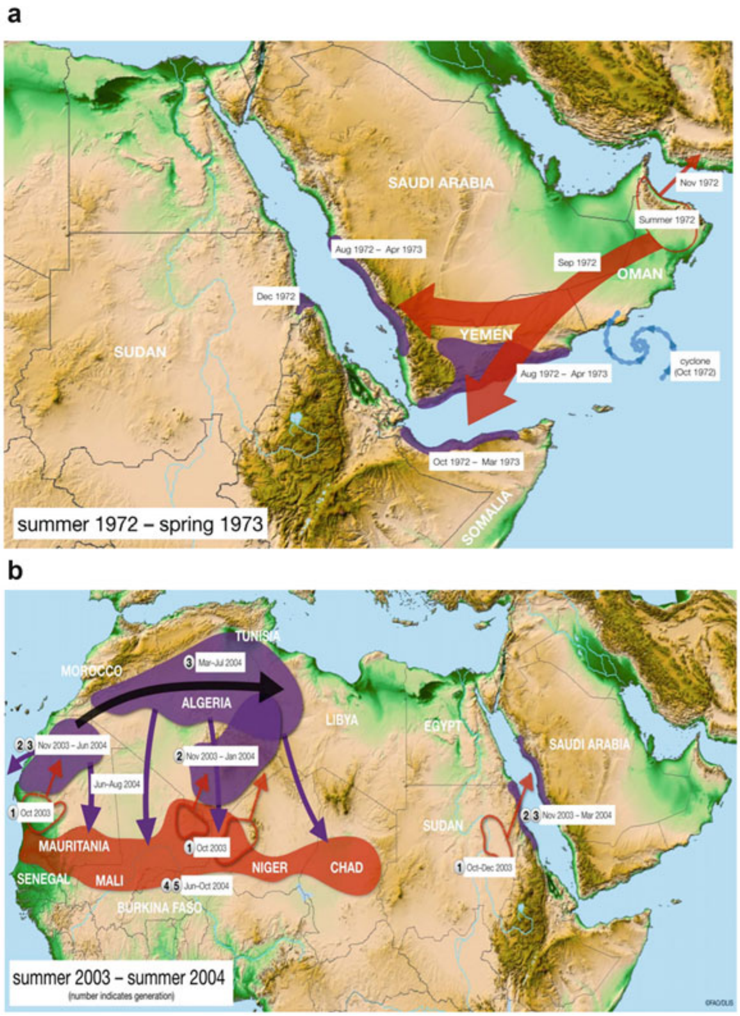

Fig. 3 (**a**, **b**) Monitoring and predicting the direction and size of locust swarm displacements. http://www.fao.org/ag/locusts/common/ecg/1146/en/UpsurgeMaps.pdf

Using data provided by all affected countries, as well as weather and habitat data, satellite imagery, and model inputs, FAO provides forecasts up to six weeks in advance and issues warnings on an ad hoc basis. Locust situations and forecasts on breeding and migration are provided to each country. Furthermore, FAO undertakes field assessment missions, coordinates survey, and control operations as well as emergency assistance during locust upsurges and plagues.

With support from donors, FAO has treated 2 m ha with biopesticides throughout the 2020–2021 invasion.

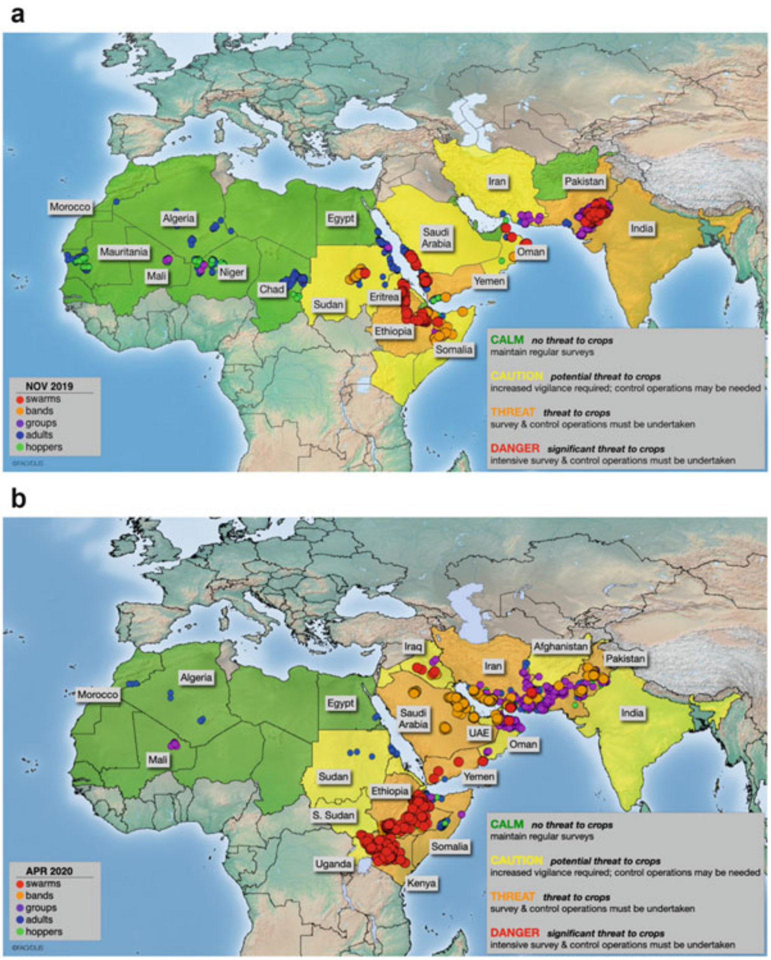

Fig. 4 (**a**, **b**) Monitoring hoppers, adults, groups, band and swarms of locusts to determine levels of risk in a territory (significant threat to crops in red, threat to crops in orange, potential threats to crops in yellow and no threats to crops in green) to alert early warning systems. http://www.fao.org/ag/locusts/common/ecg/1146/en/UpsurgeMaps.pdf

5 Models and Data

Mathematical models have been developed to model different aspects of desert locust dynamics, from demographics to the prediction of breeding grounds. Given the central importance of climatic conditions for desert locusts' breeding and reproduction, temperature and humidity predictions are key variables. Likewise, given that locusts rely on wind for their transportation, wind velocity, direction,

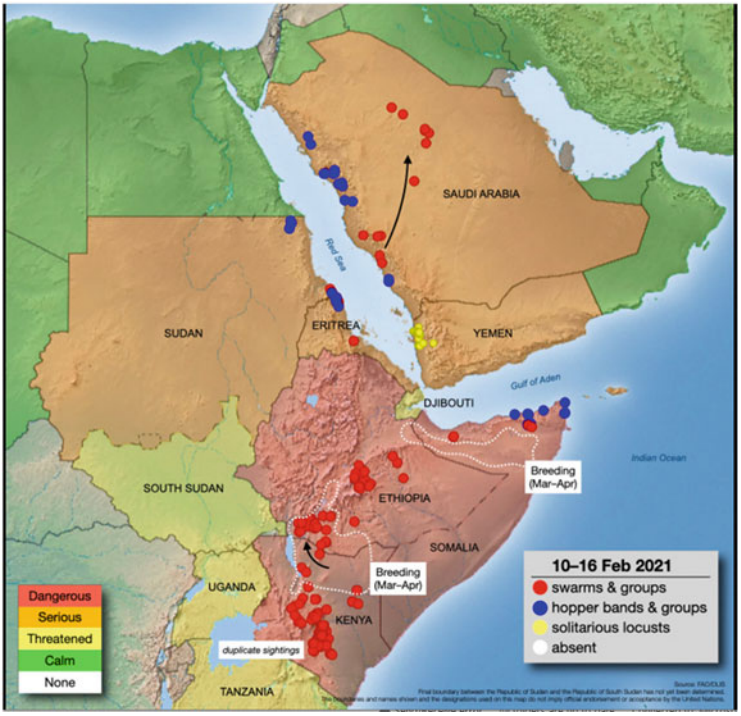

Fig. 5 http://www.fao.org/ag/locusts/common/ecg/1146/en/UpsurgeMaps.pdf

and temperature are also key variables in prediction models. This section illustrates some of these models.

Akimenko et al. [8] study the conditions and particularities of bidirectional phase transitions between solitarious and gregarious phases of *Schistocerca gregaria* using a nonlinear age-structured competitive model with time delays of locust population dynamics. The model uses a variable time of egg incubation that describes the phase polyphenism and behavior of desert locusts. The model is based on the competitive system of linear transport equations with nonlinear density-dependent fertility rates and variable time delay in boundary conditions. The model is able to adequately describe the dynamics of the density of two subpopulations or two phases of locust's polyphenism—solitarious and gregarious. Analyzing the asymptotical stability of trivial and nontrivial equilibriums of the autonomous systems, the conditions and particularities of bidirectional phase transitions between solitarious and gregarious are derived. The model proves useful to predict outbreak dynamics, as corroborated by documented outbreaks.

Kimathi et al. [9] developed a model to predict locusts' breeding grounds. The model relies on the facts that locusts' behavior, physiology, and ecology, including breeding, depend very much on climatic conditions and that females lay eggs at 10–15 cms below ground level, preferring loose/sandy soils. It uses an ecological niche model that applies machine learning algorithms that correlate a set of environmental conditions to records of the species' presence and absence, in order to predict suitable habitats. In particular, the ecological niche maximum entropy (MaxEnt) genetic algorithm predicts suitability using presence-only data. Relying on geo-referenced data from FAO, in addition to weather sources, it uses moisture, surface air temperature, soil quality down to −15 cm, rainfall and prevalence of vegetation (as proxy for locust presence). The application of the model produced accurate predictions of breeding grounds in Saudi Arabia and Morocco. It showed that temperature and soil moisture have the strongest predictive power. The model shows that the onset of greening is a key variable for the start of operations, preparedness, prioritization, and early warning alerts.

Wilson et al. [10] use a *Cox Proportional Hazard Model* to demonstrate that locusts in their gregarious phase are more protected than solitary ones to parasites and pathogens through "density-dependent prophylaxis," as crowding induces higher levels of investment in disease-resistance mechanisms. The model included mean body weight as a covariant and concluded that infected gregaria locusts survived significantly longer than solitaria locusts. The study showed that desert locusts reared under crowded conditions are significantly more resistant than solitary locusts to the entomopathogenic fungus *Metarhizium anisopliae* var. *acridum*, a key natural disease of acridids and an important agent in locust and grasshopper biocontrol. These results have implications for understanding the development and biocontrol of locust plagues (Fig. 6).

FAO operates one of the oldest, largest, and best-known migratory pest monitoring systems in the world. Within this system, remote sensing plays an important role in detecting rainfall and green vegetation.

The incorporation of geographic information systems into the desert locust early warning system at the FAO in the mid-1990s and at the national locust centers a few years later has had a great impact on data management and analysis [11]. GIS technology allows for the integration of data from a wide variety of sources, consisting of different formats, scales, and resolutions, on one platform and displayed as a series of layers on a single map. Vector data such as past and

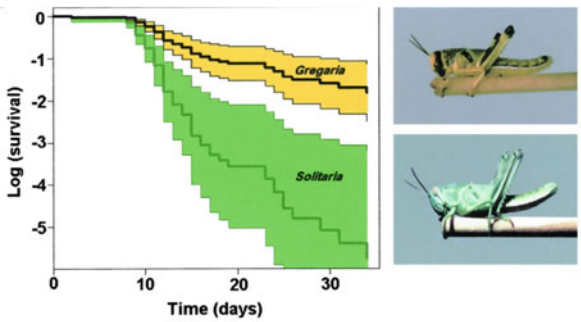

Fig. 6 Source: Wilson et al. (2002) [10]

present field observations on ecological conditions and locust infestations, survey and control results can be combined with raster data such as daily, decadal, or monthly satellite-derived rainfall estimates, periodical vegetation imagery, and six-month seasonal predictions of rainfall and temperature anomalies. This allows users at the national locust centers and the FAO to visualize the past and present situations as a means to assess current infestations and predict changes in population numbers and spatial distribution. FAO uses The Desert Locust Egg and Hopper Development Model that uses 30-year surface air temperature averages at the nearest meteorological station (up to about 250 km) to estimate the developmental rates of locust eggs and hoppers. In addition, The Desert Locust Trajectory Model estimates adult and swarm migration trajectories forward and backward in time (up to 10 days) using temperature, pressure, and wind direction and speed data at 16 atmospheric levels every 6 h [11]. Furthermore, FAO made a massive investment in georeferencing historical data, so that the entire data set from the late 1920s can be accessed using GIS.

Pekel et al. [12] developed an innovative multi-temporal and multi-spectral image analysis method adapted to the detection of vegetation in arid and semiarid areas using satellite imagery. This kind of imagery is widely used in desert locusts and other pest monitoring, as it can provide a continuous overview of ecological conditions (i.e., vegetation and soil moisture) suitable at the continental scale and in near real time. Widely used vegetation indices such as the *Normalized Difference Vegetation Index* (NDVI), on which most remote sensing techniques to monitor green vegetation, do not always provide reliable estimates for sparsely vegetated areas. The analysis method uses a transformation of the color space that decouples chromaticity and luminescence. A complete automatic processing chain combining the daily satellite observations was designed to provide user-friendly vegetation dynamic maps at 250 m resolution over the entire locust area every 10 days. This product informs users about the location of green vegetation and its temporal evolution. The methodology provides vegetation dynamic maps to the *Desert Locust Information Service* at FAO.

More recently, FAO is using trajectory and dispersal models to guide its field operations as well as its country advice and early warning systems [13]. *The National Oceanic and Atmospheric Administration* (NOAA) modified its *Hybrid Single-Particle Lagrangian Integrated Trajectory* (HYSPLIT) model, developed by NOAA's *Air Resources Laboratory*, one of the most widely used models for atmospheric trajectory and dispersion calculations, to predict future locust displacements. HYSPLIT is a complete system to compute simple air parcel trajectories, as well as complex transport, dispersion, chemical transformation, and deposition simulations. In the case of locusts, it inputs observational data from the field, taking the point where swarms are observed, back-computes trajectories (7 days), inputs data on wind, temperature, and precipitation (e.g., Fig. 7) to determine the origin of air masses and forward-track them for 14 days [14]. The model is usually used for tracking and forecasting the release of radioactive material, wildfire smoke, windblown dust, pollutants from various stationary and mobile emission sources, allergens, and volcanic ash. Its calculation method is a hybrid between

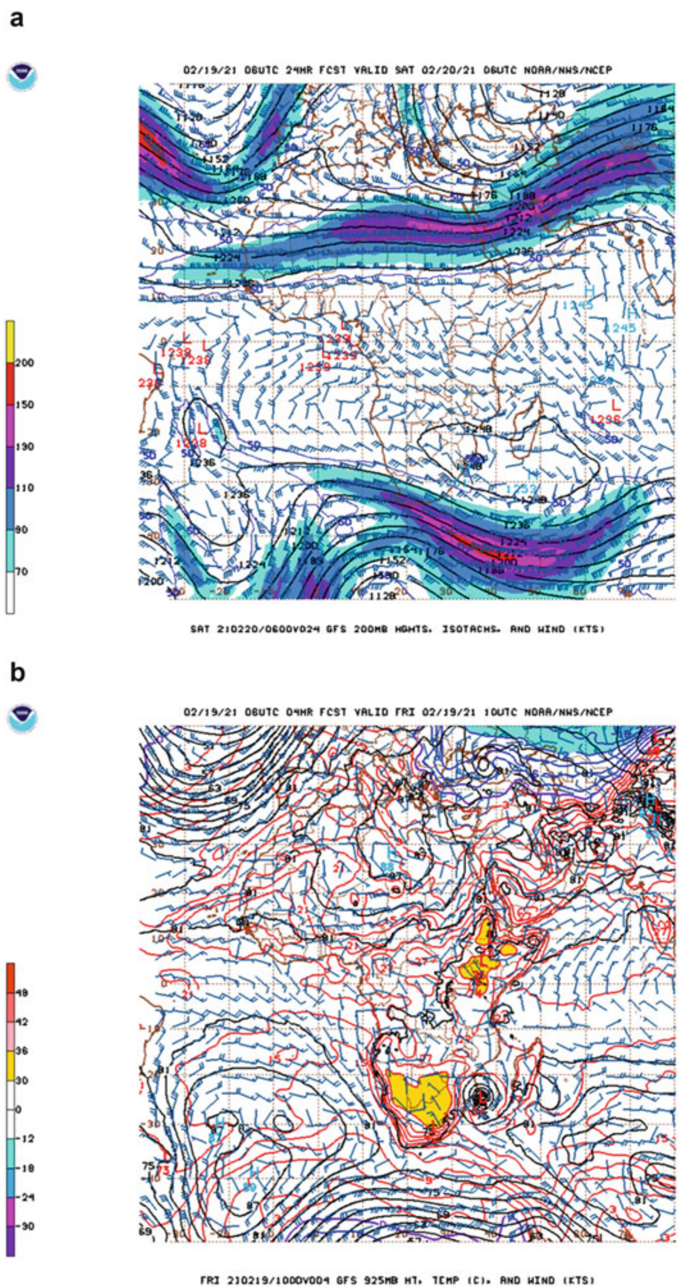

Fig. 7 **a**–**c**)Examples of monitoring of variables used in HYSPLIT (height, wind, temperature and precipitation). https://www.cpc.ncep.noaa.gov/products/international/africa/africa.shtml

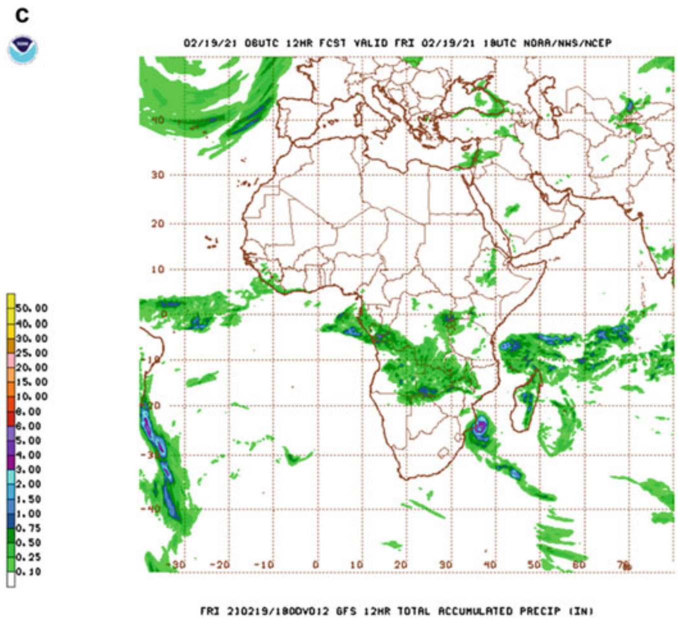

Fig. 7 (continued)

the Lagrangian approach for diffusion and trajectory calculations and the Eulerian methodology's fixed three-dimensional grid as a frame of reference to calculate concentrations [15].

6 Conclusions

Desert locusts have ravaged peoples' livelihoods, generating hunger and even famine for thousands of years. The threat they pose is now being compounded with the effects of climate change, which may create the conditions for increased frequency, intensity, and duration of swarm formation. Early warning and control programmes, such as those led by the FAO require significant donor funding and long-term commitment. In spite of international efforts and local ingenuity, such as an initiative to capture locusts at night time, when they pose on vegetation, in order to convert them into high-protein animal feed, the sheer magnitude of the infestations dwarf the ability of these interventions to effectively control them.

Several mathematical models have demonstrated predictive power regarding desert locust swarm formation and are currently used in control programmes. Technology has improved the quality of imagery data over the past decades, contributing significantly to monitoring and prediction. However, the weaknesses of the system for early warning and early control continue to center around national capacities, availability of broadband for Internet services, and availability of long-term donor funding. In addition, efforts to control transboundary pests such as the desert locust, will be hampered by the situation of protracted conflict in some affected countries, limiting the possibility not only of curbing the spread within the country, but also of providing data to feed into trajectory models at a regional scale.

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Part IX
Literature and Mathematics

Soul Searchin'

Marco Abate

It's late, I'm drinking a bloody Mary and I'm listening to Solomon Burke. I have a paper to write, places to go, theorems to prove. The paper is, as the hour but more, late. I cannot go to the places I would like to (or even have to) go. The theorems are on strike, they refuse to prove themselves. Do not believe mathematicians saying they are working; they are not. They are waiting for theorems kind enough to prove themselves. If not a theorem, at least a proposition. In nights like this one, even a lemma would be welcome. Not a corollary, no, a corollary would not be welcome, not ever. Corollaries ambush you behind the corner, after you've been distracted by a beautiful theorem passing by winking at you. You turn your head expecting—or at least hoping—that it will slowly stop, that it will wait a little meditating about the meaning of the universe, and that then it will quietly turn around, with a gentle smile, a welcoming gesture and it will be yours forever and ever—but it won't. No slow stopping, no turning around, no smiles or gestures, welcoming or else; it won't be yours. And you get stuck with that obnoxious corollary, completely useless without the theorem, but still nagging at your elbow, not even clean enough to be sold out as a conjecture. A useless little brat. A pitiful reminder of the theorem that could—should!—have been mine and it didn't, and won't ever be because I'm not good enough, I cannot be good enough, only dirty corollaries are what I deserve, I'm not a real mathematician, not even an applied one, I must leave right now, in shame, I will go and live in a shanty town, surviving collecting garbage from waste dumps, with dirty corollaries as only company, to constantly remind me that I am a failure, no shining theorems for me, ever.

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Darn, I forgot. I cannot leave. We are in lockdown. The only place I can go to is the kitchen, to fill up this bloody Mary sorely needing filling up. Done. I prepare bloody killing bloody Marys, if you don't mind me saying so. The right amount of vodka. And the right amount is a lot: this is a true theorem, Banach *docet.* Not a dirty corollary. A true bright theorem in its shining armor. A sip—even better two, perchance three—and its light will dispel the darkness in my mind where all those propositions and lemmas are trying to hide, are trying to hide themselves and the true path to the true theorem, the greatest theorem, the theorem to end all theorems… and. It. Will. Be. Mine! Another sip and I'm sure I will start glimpsing it! Another sip and now that I think about it, really think about it—ah, bloody Marys really help you think, yes they do—you know, this dirty corollary is not that dirty after all. A little cleansing here, a bit of scraping there and it might pass for a half-decent conjecture. Why bother to chase ungrateful theorems when you can sell your colleagues a nice enticing conjecture, promising a garden of delights that nobody will enter, but it won't be your fault, you just suggested the way and were so generous to leave it to others to follow, it will be their fault if the conjecture will remain unproved and undeflowered! And yet I have the feeling that something is off… another sip of bloody Mary will surely clear my mind… Mmm…

What? What did you say? That I got carried away? That I promised to write something meaningful? Or at least more meaningful? And that the Solomon Burke album has ended? Decisions are called for. We must decide. Uplifting music: Sergio Cammariere. I do not need a waiter to uplift my glass to toast my deciding strength, but it helps anyways. Lift the music and lift the glass, let's have a musical toast to meaningful mathematics, to sound mathematics! Or at least to mathematics sounding sound! Or to… Whatever, let's start.

1 The Algebraic Playground

Everything will take place in a big large wonderfully smooth complex manifold called M for "mom". Inside it (her?) lies a slightly mischievous possibly singular subvariety called S for—you guessed it—"son". It (he?) might possibly be singular, but he promised his mom to stay reduced, connected (he'll try to be irreducible too, but no promises on that count) and, of course, pure (dimensional).

Our S comes with a whole box full of toys that he mischievously drops on the floor. And, lo and behold, this is the pattern the toys drew on the floor:

$$\begin{array}{ccccccccc}
 & & & \mathscr{I}_S & \xrightarrow{\iota_0} & \mathscr{O}_M & \xrightarrow{d} & \Omega_M & \\
 & & \swarrow \pi & \swarrow D & \swarrow \pi_2 & \downarrow \pi_1 & & & \\
O \longrightarrow & \mathscr{I}_S/\mathscr{I}_S^2 & \underset{\tilde{\rho}}{\overset{\iota_1}{\rightleftarrows}} & \mathscr{O}_M/\mathscr{I}_S^2 & \underset{\rho}{\overset{\theta}{\rightleftarrows}} & \mathscr{O}_M/\mathscr{I}_S & \longrightarrow & O & \\
 & \tau^* \uparrow\downarrow d_{\mathscr{I}} & \swarrow d_2 & & & & & & \\
 & \Omega_M \otimes_{\mathscr{O}_S} \mathscr{O}_S & & & & & & & \\
 & \downarrow & & & & & & & \\
 & \Omega_S & & & & & & & \\
 & \downarrow & & & & & & & \\
 & O & & & & & & &
\end{array}$$

(Ok, ok, the arrow with a D attached to it should be more slanted toward the left, because it goes from $\mathscr{O}_M$ to $\mathscr{I}_S/\mathscr{I}_S^2$, but even my LaTeX-pertise has its limits…). What are all these horrible typographical misfits, I hear you muttering. They are not horrible, the son answers, they actually were showcased in a real art show a few years ago… Hush son, says the mom, let me explain to our honored guests.

Of course, $\mathscr{O}_M$ is the sheaf of germs of holomorphic functions on myself. Very important, the origin of the world. Instead, $\mathscr{I}_S$ is the sheaf of ideals of my son (full of ideals, my son, at least one at each point, I'm very proud of him), ideals of germs of holomorphic functions vanishing on him, and you have no idea how much scrubbing and cleaning is needed to keep those germs vanishing, we don't want any dirty subvariety, oh no. Ω_M is the sheaf of holomorphic differentials that are wonderful things when you have them, but for the life of me don't ask what they are. Ω_S is my son's sheaf of holomorphic differentials. Since he is so singular sometimes, the mad French mathematician told me that I should never mention it in his presence; it is defined by the diagram, so he said to me. I'm only a standard smooth mom, if the mad French mathematician says so it is so.

And then there are all those funny little letters… ι_0 and ι_1 are just inclusions: yes, son, send your ideals into me and call me Giocasta I'll be your victim. And then those annoying three little pigs… I mean, pi's: π, π_1 and π_2 are just projections, sort of canonical if you are of the religious type. θ too is a projection, but it lives at a lower level, and the pigs didn't want it to mix with them. And look to that forlorn arrow with no name, almost at the base of the social echelon: it is a projection too, but of such a lower class that the pi's didn't want it to be named. And the mad French mathematician, nodding madly, confirmed that there is no need to name it; it is defined by the diagram. Yes, sir.

The d's… The first d is the usual d, sending f into df. Ok, it is not much as explanations go, but that's something you already know, don't you? Derivatives and such. To explain d_2 and $d_{\mathscr{I}}$ let me tell you that I don't like that much those pi's, always bringing mud, bricks and wolves into the house… never mind that, but anyway the fact is that sometimes I like to trick them a bit, and I've devised a little notation of my own, writing $[f]_j$ instead of $\pi_j(f)$ just to confuse them. Neat, don't you think? Where f is a little germ of mine, of course, but a very clean one, I assure you, not even a snooze will come out of it. Well, having said that, d_2 is defined as follows:

$$d_2[f]_2 = df \otimes [1]_1,$$

and $d_{\mathscr{J}} = d_2 \circ \iota_1$, of course. A little thought will show, honored guests, that the definition is well-posed. In case of doubt, my son will be glad to explain everything up to the last little microlocal detail. His father was an analytic space, you know.

The capital D... well, let's say that it is a derivation and leave it at that. It is not always there, you know; it'd have to be dashed, but there is no way to find a decent dashed oblique arrow these days. Same thing for those radical arrows going contrarily-wise, ρ, $\tilde{\rho}$ and $\tau^\star$ (why wasting such a good-looking star when there is no τ around is beyond me): they're not always there, thanks God in this country we are still writing left-to-right-top-to-bottom, but when they are around they'll make themselves noticed, no doubt, those rascals.

I think that's all... You asking...? Yes of course, the diagram is commutative with exact rows and columns, I like to keep my home tidy and clean, thank you very much. Oh well, except the first row, yes you're right, but tell me where should I put that Ω_M, just tell me, what a poor smooth manifold has to do? And now, if you excuse me, I'm beginning to have a tiny little headache, I've better go to rest. Where did I leave that covering, it was right here last time I checked... (fade)

Disclaimer

No definition has been harmed in the preparation of this paper. The co-authors of [1] strongly deny any responsibility for this rant and sternly advise young mathematicians to never drink and write.

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Geometric Metaphors and Linguistic Genealogy

Francesca M. Dovetto

1 Similarities and Differences Among Languages

In placing the study of linguistic change as one of their central interests, the nineteenth-century linguists mostly addressed the question of the difference between languages from a historical perspective, feeding nationalist ideals and was attentive to identifying elements and particulars of the differentiation between languages. In opposition to this viewpoint, some models with a different orientation were proposed. For example, the philosophy of mixture proposed by William Dwight Whitney was an original attempt to overturn the dominant linguistic perspective, as it levered on elements of sharing and mixing among populations and languages rather than on differentiation. It was probably inspired by continuist models that started to appear, with prudence, in the scientific debate of the different models for representing the genealogy of languages ([1]; cf. [2]).

At the same time, research about universal traits in the relationships of similarity and difference among languages led to models that existed in nature: natural forms and types could be adapted to assimilating languages and their mutations over time. These images worked long in the history of linguistics, some of them are still very well-known and alive, others less so. Of these two, the first has given impetus to the genealogical representation of languages through the image of the tree trunk from which branches, twigs, and leaves split off; the other, more problematic image, imagines the life of languages as the rippled surface of a pond, in this way, yielding a decidedly more complex and realistic vision of mutations.

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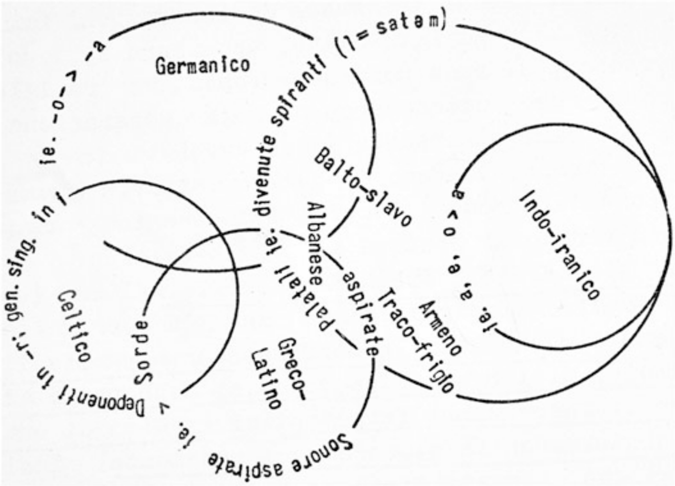

Fig. 1 The Schmidt *Wellentheorie* [4], p. 10

2 Trees and Waves in the Representation of the Relationships Between Languages: From the Discrete to the Continuous

Timewise, the first image that we encounter in the history of the classification of language is, without doubt, the best-known and most widespread, that is, the metaphor of the genealogical tree (*Stammbaumtheorie*), which is the base for the famous genealogical classification of the Indo-European languages proposed by August Schleicher [3], in imitation of the Darwinian model of the origins and transformations of the species. This was immediately followed by the other, just as well-known metaphoric image, contrasting the former, the image of ripples on the surface of a pond (*Wellentheorie*, see Fig. 1), used contemporaneously but independently by Johannes Schmidt [5] and by Hugo Schuchardt [6].[1]

Of the two models, it is certainly the latter that produced the more metaphorical variants, as Schmidt associated two other images to it, that of the inclined plane (*geneigte Ebene*) and that of the stairway (*Treppe*), and Schuchardt extended it to images of the fan (*Fächer*), of the cone (*Kegel*) and of the rainbow (*Regenbogen*).

According to Schmidt, in representations of the kinship relations between languages, the image of the plane (*Ebene*) corresponds to an inclined surface, crossed

[1] Schuchardt had already presented this theory in the third volume of *Vokalismus* [7], p. 34, stressing the independence of his formulation in the preface to the publication of the report for conferring a teaching position made in Leipzig in 1900 [1870]: 4. On this cf. Dovetto [8], pp. 40–41; more recently cf. also Tani [9] and Venier ([10], pp. 56–58, 60.

by lines that track and represent the linguistic varieties present in the dominion. Originally, there would be no clear linguistic borders inside this ideal geographical space. When one of the linguistic varieties of the dominion acquires dominance over the others for political, religious, social, or other reasons, this causes the disappearance of the immediately contiguous varieties. Therefore, distinct linguistic borders begin to appear at this point. According to Schmidt, a step (*eine Stufe*) appears and later the image of the inclined plane transforms into a stairway (*eine Treppe*). The sharing of a certain number of phenomena determined by phonetic mutation breaks the original continuity and, with the disappearance of the intermediate varieties, the languages, corresponding to the steps of a stairway, acquire finally identities and distinct borders. Like the image of the inclined plane, the image of the stairway also implies a well-defined, directionally predetermined hierarchy: in fact, its constant incline runs from Sanskrit to Celtic, following to an uninterrupted succession of mutations.

Therefore, in the representation proposed by Schmidt, inside the inclined plane a series of reciprocally intersecting isoglosses represent, metaphorically, the kinship relations among the Indo-European languages. Every language or dialect constitutes a ring of conjunction with the contiguous languages, in such a way that no language is isolated from the others.[2]

As further specification of the same image, Schmidt uses another metaphor, that of the chain: *das organische mittelglid,* where the term *Glied* (in Schmidt's script *glid*) may indicate the link of a chain or even the element of a generation chain (*Geschlechterfolge*).[3] However, as it is composed of links, the image of the chain evoked by Schmidt through reference to intersecting circles also takes us back to discreteness, suggesting, though in a hazy way compared to the stairway metaphor, the existence of "qualitative leaps" in differentiating of languages over time and, as a consequence, in constituting their respective, single individual histories.

At the same time, the monodimensional model of the family tree, which projects the languages along the dimension of time, opens toward the horizontal dimension, represented by the geometric figures—rings in Schmidt, triangles, and quadrilaterals in other contemporary representations[4]—that reproduce the areas occupied by the

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[2] "Verständlich wird er [der ganze charakter des slawolettischen] nur, wenn wir anerkennen, dass das slawolettische weder vom arischen noch vom deutschen losgerissen werden kann, sondern die organische vermittelung beider ist" [5], pp. 17–18. Eìlsewhere we find: "dass unser sprachgebiet keinen kreis bildet, sondern höchstens einen *kreissector* [. . .], tut nichts zur sache" (ivi: 27, [emphasis added).

[3] All the successive representations of the *Wellentheorie* recur to the image of the chain (e.g., [4], p. 10 and [11], p. 67).

[4] Schleicher's image of the genealogical tree is reproduced (see Fig. 2) in the unpublished notes of the glottologist Giacomo Lignana who, trained in the German school, imported principles and linguistic research methods (*Sprachwissenschaft*) into Italy and was the first Italian professor of this science. In Lignana's image the lines/branches of the tree tend to join with other branches. In a space like a tress of hair the branches intertwine and touch, tracing out a set of geometric figures, among which there is demarcation but not detachment. This is both a singular and an original representation of Schleicher's discrete model, probably due to suggestions deriving from

Fig. 2 The Schleicher *Stammbaumtheorie* according to Lignana

different historical languages and the reciprocal intersections metaphorically on the plane.

With the inclusion of the spatial dimension, the monodimensional model becomes bidimensional.

In the same year that Schmidt outlined the famous image of waves, Schuchardt also used the same metaphor to represent the kinship relations among languages. In this case, however, the result is diametrically different and the very same image suggests continuities rather than differences among languages.

As Schuchardt also asserts, in its unity a language can be compared to a peaceful pond put into movement by the formation of ripples that cross over each

reading Schmidt (cf. [12]). Against Schuchardt's radical observation, according to which uniting the branches of the Stammbaum would have it cease to exist ("Wir verbinden die Äste und Zweige des Stammbaums durch zahllose horizontale Linien, und er hört auf eine Stammbaum zu sein," Schuchardt [6], Lignana observed that: "sarebbe molto facile, anche scostandosi in qualche punto, *concentrare* le linee dello schema di Schleicher" ([13], p. 19, emphasis added).

other according to their energies. Within this model, no language is isolated, no linguistic border is well-defined. In this case, the differentiation among languages is transformed from an amount measured with relative certainty to a variable of continuous transformation.

Between the two analogous images of the lives and the differentiation among languages, both inspired by the rippled surface of a pond, it is the second, Schuchardt's proposal, that decisively counters the metaphor of the tree, a metaphor pre-imprinted with discreteness, with the image of the *continuum*, calling to mind models of liquids.

To represent the relationships among languages in space and time, Schuchardt also proposes some other images besides the image of ripples in a pond. These are the images of a fan, a cone, and a rainbow, all useful metaphors for showing the complexity and the dynamic intertwining of transformations and contaminations, both historical and geographical, of languages. In addition to the temporal dimension, all these models include the spatial, mentioned in Schmidt but irrenounceable, according to Schuchardt, in their representations of relationships among languages and dialects. The models are seen as bidimensional.

The metaphor of the cone (*Kegel*; see Fig. 3) projected onto the lives of languages is particularly complex: the height of the cone represents evolution over time, the different strata into which the figure is sectionable represent the spatial relationships, while the continuity of the surface of the solid represents the continuity of the transitions from one language/dialect to another [14], p. 191.

In this context the transformation of the plane geometric image (the surface of the pond or the fan) into a solid is also of particular interest: then, the different strata or sections of the uninterrupted conic surface correspond to planar images of concentric circles.

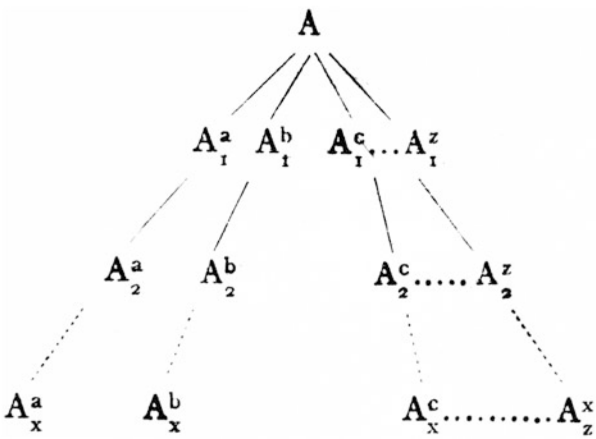

Fig. 3 The metaphor of the cone (*Kegel*)

Among the various metaphoric representations that Schuchardt used to illustrate his model, clearly projected towards the *continuum*, this is the only image that proposes the figure of a solid, thereby—as Schuchardt writes—giving substance to the image of the fan (*Fächer*), of languages that descend from a common ancestor.[5] In this case, the bidimensional model also gains the third dimension of depth, the very essence of the notion of languages being related to nearness/distances over time and space.[6]

The transitions of languages, projected onto the solid form of the cone and which Schuchardt recognizes as being of varied evidence and depth, are, therefore, compared to waterfalls (*Wasserfälle*) or to other steep climbs (*Steilen*). In this way, the image of the solid immediately reconverts to the aquatic metaphor, where fluidity represents the distinctive trait and the guarantee of the indemonstrability of the existence of precise borders between languages: there is mixing (*Mischung*) everywhere, between languages, between adjacent dialects, between related languages, and also between languages without any spatial or temporal connection.[7] Schuchardt comments harshly against the pyramidal system of the genealogical tree, which is completely unrealistic in the case of kinship among languages (*wie es sich nirgends findet*): to us (linguists), this is not allowed (*dies ist uns nicht vergönnt*).[8] Even the present, in fact, cannot be disjoined from the past (*Heutiges mit Vergangenem vermengt werden*).

In the end, Schuchardt [6] compares the image of the genealogical tree with yet another image, also in this case a fluid one with nuanced borders. This is the image of the chromatic continuum (*Färbungen*) or of the rainbow (*Regenbogen*). As is clear from the colors (*die Farben des Regenbogens*) that imperceptibly merge into each other (*unmerklich ineinander überfliessend*) this image is also, basically, a liquid metaphor. With this image, the scholar represents the simultaneous emergence of dialectic coloration[9] in the same area, at the same time confirming the intrinsic mixing (*Mischung*) that allows all language change. This is the same metaphor that the philologist and linguist Gaston Paris [15] recurs to, in the same period, to represent the subject of structural dialectology: a vast tapestry, where "les coleurs variées se fondent sur tous les points en nuances insensiblement dégradées." The scholars of Linguistic science would soon have to face such complexity in their study matter and rethink their own methods and working tools.

[5] "Nur müssen wir uns das Flachbild verkörperlichen, nämlich den ausgespreizten Fächer als Kegel denken mit einer Grundfläche gekreuzter Linien" ([14], p. 191).

[6] "[. . .] das heisst je entfernter die Glieder voneinander in Zeit und Raum sind, desto entfernter auch ihrem innern Wesen nach" ([14], p. 191).

[7] "Mischung durchsetzt überhaupt alle Sprachentwicklung; sie tritt ein zwischen Einzelsprachen, zwischen nahen Mundarten, zwischen verwandten und selbst zwischen ganz unverwandten Sprachen" ([14], p. 193).

[8] "Wir würden eine Pyramidalisystem von Sprachen erhalten, wie es sich nirgends findet. Darwin konnte sich in seinem Falle durch die Theorie vom Kampfe um's dasein und dem Aussterben der Zwischenformen retten. Dies ist uns nicht vergönnt" (Schuchartd [7], p. 79).

[9] "das Entstehen dialektischer Färbungen in den einzelnen Gegenden als gleichzeitige" [6], p. 21.

3 Stairways, Chains, and Ropes: From the Continuous to the Discrete

As it is clear, these are all cases of metaphoric images that mirror generalizations and that themselves constitute indicators of co-occurrence, which perhaps deserves more attention in the history of linguistics.

On the other hand, while it is true that the very possibility of a taxonomy is guaranteed by the fact of being projected onto a continuous object, at the same time the taxonomy, in identifying discrete recurrent elements, induces the belief that these same artificial elements are aligned to the subject matter by the observer and are not intrinsic to the object itself (languages).

In any case, the wave model should be recognized for the merit of having introduced, even with some uncertainties, the *continuum* into what, up to then, had been considered the reign of the discrete. In fact, Schmidt still adhered to the latter when he imagined waves that transmute into steps. Besides, in Schmidt's model, the step has very little in common with the stairway of continuity used in taxonomies of the realm of nature, as it is rather a step that marks, above all, discontinuity in the classification of languages and that seeks to mark specific identities.

The image of the chain, also implicitly invoked by Schmidt (see Fig. 4) in reference to intersecting on interlinking circles, takes us to discreteness, which uniquely distinguishes itself from analogous taxonomic representations of nature, where instead the chain constitutes a particular variant of the stairway, which it

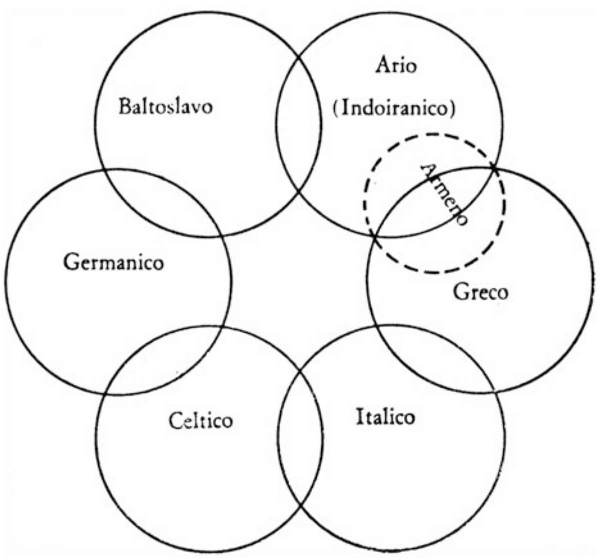

Fig. 4 The Schmidt image of chain [11], p. 67

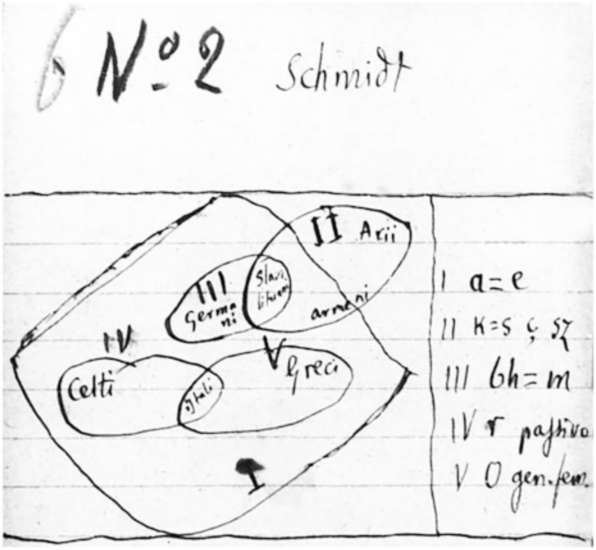

Fig. 5 The Schmidt ropes according to Lignana

would correct of the possibility of recurring to discreteness (the steps of the stairway could, in fact, suggest the existence of "qualitative leaps").

Actually, in its applications to the realm of nature the chain served to support the idea that there was no fracture between the elements, which would, instead, all be connected, "ring to ring." In turn, the chain as a variant of the stairway had a particular variant, the rope, in which every link connected to two others.

Curiously, in the unpublished notes that the glottologist Giacomo Lignana dedicated to the representation of the kinship relations among the Indo-European languages, we find exactly this image of circles/intersections of rings. However, in his papers, Lignana does not draw a chain of links, following a (proto)continuist model, but some ropes (two; see Fig. 5), that reinforce the historically and linguistically determined identities and differentiation among languages: Slavic-Lithuanian link to German on the one hand and to Arians and Armenians on the other; instead, Italic acts as the link between Greek and Celtic ([12], p. 35; cf. also [8], pp. 40–45, in partic. 44–45).[10]

[10] Cf. also what Lignana wrote to Pietro Merlo in 1884, in which, among other things, his use of the metaphorical model of the chain clearly emerges: "due punti di più stretta affinità ariana sono certi, cioè l'Indo-Iranico e l'Italo-Celtico. E quando i due anelli estremi della catena sono saldi, il resto più e meno si può determinare" [16], p. 184). In the image that Lignana traces (cf. [12]), the only model we have that is contemporary to Schmidt's, the apex of the inclined plane is represented by the vocalism of the Arian, believed older, with respect to the unity of the European languages, at the time defended by Lottner and by Fick and represented by the weakening of *a* to *e*, while Arian would have conserved the *a*. Clearly this reflects the period in which Schmidt wrote, when the precise, and in fact opposite, direction of the mutation was still unknown. However, for

The chain of the continuity of nature is broken at birth, though, as Schuchardt remarks, where there is no separation of the child from the mother, there can be neither brothers nor sisters.[11]

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Lignana, who knew about the more recent discoveries on protoArian vocalism, though remaining unmutated the image, like the isogloss II of the model ($a = e$), changes the direction of the mutation (the original European e mutates into a in the Arian languages).

[11] "Wo es keine Abtrennung des Kindes von der Mutter gibt, es auch keine Geschwister geben kann" [14], p. 191.

A Mathematical Physicist in Hell

Galileo on the Geometry of Dante's *Inferno*

Jean-Marc Lévy-Leblond

A modest contribution to the celebrations of the 700th anniversary of Dante's passing.

Among the earliest publications of Galileo are his *Two Lessons on the Shape, Location, and Size of Dante's Inferno*, hereafter abbreviated as *Lessons on Hell* [1]. This work is little known except to experts, and is often considered at best as an exercise in mathematical expertise as well as a work of circumstance intended to make its author known. The relegation to the background of these *Lessons on Hell* was, moreover, very early. Galileo seems to have been reluctant to mention it afterwards and to communicate this text [2]. Viviani, his pupil and first biographer, does not even mention it. However, far from being a negligible and lateral element of Galileo's work, the *Lessons on Hell* contain elements that prefigure several essential themes in Galileo's major contributions to mechanics. This text announces, in a sometimes paradoxical way, one of the "new sciences", which Galileo will make public at the end of his life, more than forty years after the *Lessons on Hell*, and which constitute his essential contribution to modern physics [3].

The young Galileo, after beginning his medical studies in Pisa in 1580, discovered mathematics in 1583 (he was not yet 20 years old), abandoned the university where this discipline was not held in high esteem, and devoted himself to the personal study of the ancient mathematicians, Euclid and Archimedes in the first place [4]. He quickly acquired a considerable mastery of the subject, which enabled him to teach it on occasion to earn a living. As early as 1586, Galileo circulated a small pamphlet, *La bilancetta*, of typical Archimedean inspiration, on a weighing instrument capable of measuring the densities of objects and therefore the quality

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of alloys—in the tradition of the famous anecdote about Archimedes detecting the fraud of the jeweller who had manufactured Hieron's crown. At the same time, he wrote some theorems, again in a very Archimedean style, on the centres of gravity of solids of revolution. Noticed by leading scholars such as Guidobaldo del Monte, he was invited in 1587 by the Florentine Academy to shed light on the controversy that had for decades opposed two interpretations of Dante's *Inferno*. In 1506 the Florentine Antonio Manetti had published a description of the geography and geometry of Dante's Hell as it had been described by Dante, nonobstanding his rather obscure poetical description [5]. To be given particular attention was the evaluation of the reliability of the figurative representations given by Botticelli in the nineties of the fifteenth century in a luxurious illustrated edition which followed the first sketches of Giuliano da Sangallo [6] (Fig. 1).

The illustrations were drawn based on measurements established through complicated calculations taken from references in Dante's text and which, for the intellectual circles of an era for which *The Divine Comedy* was a fundamental reference point, needed to be accurately established. But in 1544 Alessandro Velutello from Lucca, Florence's rival city, published a severe critique of Manetti's work and proposed a very different description of Hell [7]. Galileo was called on the resolve the debated question which he did, predictably, in favour of the Florentine, Manetti. This work of literary exegesis allowed the young and ambitious Galileo to have his mathematical talents recognised as well as his pedagogical qualities,

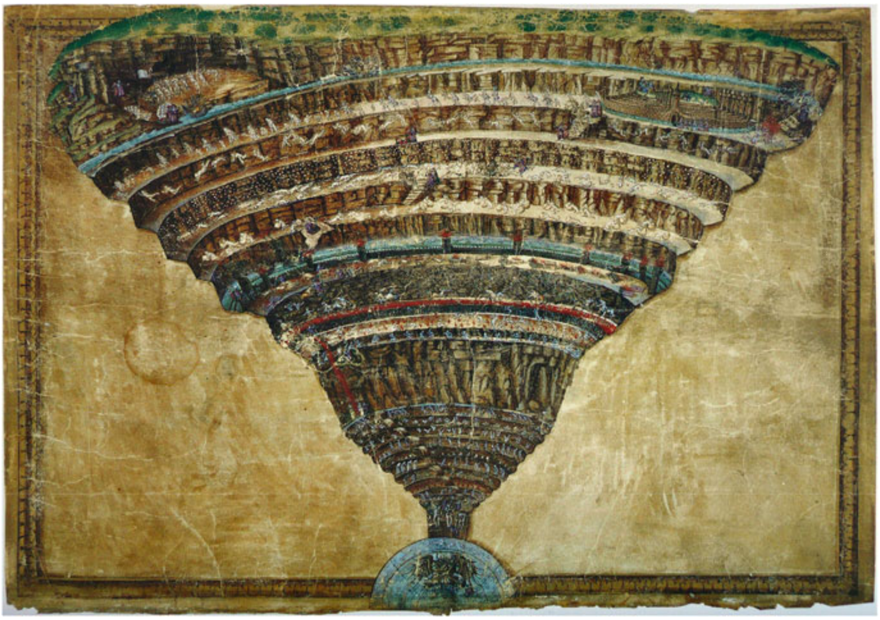

Fig. 1 Dante's Inferno according to Botticelli [https://upload.wikimedia.org/wikipedia/commons/3/3e/Sandro_Botticelli_-_La_Carte_de_l%27Enfer.jpg]

and constituted a fruitful promotional operation. As his reputation grew, in 1589 he finally obtained a chair of mathematics at the University of Pisa, a position that inaugurated Galileo's institutional career.

But what is the place of these *Lessons* in Galileo's work, or rather, in the general project he developed in his youth?

1 *The Pure Tuscan Language*[1]

Of course, it was by no means the case, neither at the beginning of the sixteenth century for Manetti or Vellutello, nor at the end of that same century for Galileo, that the description of Dante was to be taken seriously from a theological point of view. Quite simply, the importance of the *Divine Comedy* in Tuscan culture made it obvious that it had to be understood in all its aspects—including topographical—in order to make it easier to read. Indeed, the geography of Hell and its iconography are a classic theme of Dantesque exegesis, already in the sixteenth century [5, 7–9] up to recent times [10–13]. Beyond the demonstration of his personal mathematical skills, Galileo's *Lessons* had a higher cultural ambition. His training among the Pisan elite had given him an excellent literary, artistic, and musical education (his father, Vincenzo Galilei, was one of the first musicians of his time, a friend of Monteverdi's, and his brother was also an instrumentalist and composer). Galileo was also deeply involved in the literary and artistic debates of his time: in the years following his *Lessons on Hell*, he took part in the fierce controversies between the supporters of Ariosto and those of Tasso, as well as in the very fashionable discussions on the comparative merits of painting and sculpture [14, 15]. But cultivated Tuscans, such as those who formed the Florentine Academy, were obviously far from all possessing the scientific knowledge, especially mathematical, of the young Galileo. If the new science was born, as Galileo offers the emblematic example, from the depths of the culture of his time, the bearers of this culture were far from recognising and assimilating this new twig. And in fact, the following centuries would see the gradual divorce between science and culture of which our time is a victim [16]. Perhaps aware of this risk, Galileo wanted, in any case, to show in his *Lessons on Hell* that mathematical physics is not simply a source of technically efficient calculations but can make its contribution to the noblest cultural debates, and thus acquire an intellectual status comparable to that of the classical humanities—without, however, pretending to replace them, contrarily to a recent claim [17, 18].

It is in this context that one must understand Galileo's use of the Italian language, and not Latin, for most of his major works—as is the case for the *Dialogo* [19] and the *Discorsi* [3], not to mention the *Saggiatore* [20]. The importance of this decision has often and rightly been stressed. But it should not be reduced to a purely political

[1] All of the section titles are taken fom Galileo's *Lessons*, hence the italics.

choice, as is usually argued, by which Galileo would target a readership broader than that of scholars alone in order to gain wider support in his intellectual battles. In truth, Galileo in no way considered Italian (which, at the time, was in fact Tuscan) to be the vulgar language, which it would be necessary, willy-nilly, to use in order to be heard by all. For him, it was the very language of the high culture of his time, the one that both demanded and allowed the greatest clarity and subtlety of expression. And it is Latin, on the contrary, that he considers without complacency as a technical jargon, no doubt among people in the trade, but unsuitable for an intellectual discussion of the meaning and value of ideas. The *Lessons on Hell* are in this respect perfectly revealing and inaugural. Forced to use, in his scientific explanations, certain scholarly terms (geometric in particular), Galileo justifies this and even apologises to the members of the Florentine Academy:

> ... let us hope that your ears, accustomed to hearing this place always resound with the chosen and distinguished words that the pure Tuscan language offers us, may forgive us when at times they feel offended by some word or term peculiar to the field we are dealing with, and taken from the Greek or Latin language, since the subject we are dealing with obliges us to do so.

It becomes clearer then that, even in his great scientific works, Galileo's decision to write in Italian is much more than a tactical manoeuvre of "communication", as we would say today, it expresses the firm will to inscribe his work within the culture of his society and his time. Galileo is not an isolated case here. Contrary to a widespread but simplistic conception, the Scientific Revolution of the seventeenth century is in no way linked to the existence in Europe of a single language of scientific communication, supposedly Latin. On the contrary, this period coincided with the development, within the most demanding intellectual activities, of national languages, henceforth considered as vectors of modern culture. Already in Italy, Guidobaldo del Monte, mentioned above, had in 1585, barely two years before the *Lessons on Hell*, published a work on mechanics in both Latin and Tuscan. During the first half of the seventeenth century, which saw the completion of the Scientific Revolution, Descartes for French, Harvey for English, and up to Leeuwenhoek for Dutch, offer compelling examples of the scientific legitimation of national languages [21].

2 *The Intervals Between the Skies*

There is no question of making these modest *Lessons on Hell* the prolegomena of all Galileo's future work. It is not the future astronomer and cosmologist, the author of the *Dialogues on the two great systems of the world* [19], that we can glimpse in these *Lessons*. On the contrary, they open with an apology of archaic cosmology, praising the results obtained in the measurement of the "intervals between the skies" and their movements, which clearly refers to the ancient and medieval representation of a universe composed of several geocentric celestial spheres. Galileo was far from being the militant Copernican he would become. From his pulpit in Pisa, he taught the Ptolemaic system of spheres without reservation. It was only in 1597, ten years after the *Lessons on Hell*, that he privately (in a letter to Kepler) stated his adherence to the Copernican system, and much later still, after 1610, that he began to defend it publicly, in particular in his famous 1615 *Letter to Christina of Lorraine* [22]. Galileo will no doubt have to distance himself from a somewhat naïve epistemology which led him to write, at the beginning of the *Lessons*, that the celestial survey he celebrates, however, "difficult and admirable" it may be, concerns "things which, totally or in large part, fall within the realm of senses". It is by abandoning the common illusion of a world immediately given to observation that Galileo will be able to find a new vision of the cosmos: whether through instrumentation (the telescope) or theorisation (mathematics), science only understands the world in a mediate way.

However, at this beginning of the *Lessons*, let us note a typical example of one of Galileo's constant rhetorical resources, which he made great use of in his major works—that is irony. It is in this vein that he takes the liberty of explaining the difficulty of assessing the dimensions of Hell, since "that place where it is so easy to get down and yet so difficult to get out" is "buried in the bowels of the Earth, hidden from all our senses"—unlike the "skies" which "fall under the senses"— and that "the difficulty of such a description is considerably increased by the absence of any study from other people".

3 *For a Few Reasons of Our Own*

Galileo's *Lessons on Hell* are above all an exercise in geometry. It is in order to describe and evaluate the underlying spatial structures in Dante's work that he first sets about. The mathematisation of physics, of which Galileo is rightly considered to be one of the initiators, takes place with him under the aegis of geometry, in a directly Archimedean filiation. Nothing yet, in Galileo's case, of the algebraisation that Descartes, so soon after him, would begin to implement. And in Galileo's very famous quotation that the great book of Nature "is written in mathematical language" [20], we must be careful not to forget what follows, namely that "the characters [in which this book is written] are triangles, circles, and other geometrical figures", that is by no means the literal formulae of modern algebra and analysis. In any case, Galileo, from his youth, was an accomplished geometrician, as the *Lessons* show from the outset.

Carefully choosing and commenting on the appropriate verses of *The Divine Comedy*, Galileo begins by confirming Manetti's description: Hell is an approximately conical cavity whose apex is at the centre of the Earth, and whose axis pierces the surface of the Earth in Jerusalem (of all places . . .). The base of the infernal cone, on the surface of the Earth, is a circle with a diameter equal to the radius of the globe; in cross-section, the chord corresponding to a diameter of this circle is, therefore, the base of an equilateral triangle having the centre of the Earth

Fig. 2 Sketchy cross-section of Dante's Inferno according to Manetti [5]

as its vertex, which means that the angle at the vertex of the cone is 60°. And this is where Galileo applies his mathematical knowledge, first of all to invalidate an erroneous opinion:

> If we want to know [the] greatness [of Hell] in relation to the whole aggregate of water and earth, we must not follow the opinion of those who wrote about Hell, believing that it occupied the sixth part of the aggregate.

Indeed, on a central section of the Earth passing through the axis of the cone, the infernal sector occupies one-sixth of the area of the disc—temporarily neglecting the vault of Hell (Fig. 2).

Some people, profane in three-dimensional geometry, might therefore think that the same proportion applies to the volumes. But Galileo continues:

> If we do our calculations according to what Archimedes demonstrates in his books *On the Sphere and the Cylinder*, we will find that the space of Hell occupies a little less than the fourteenth part of the [volume] of the aggregate. I say this even if this space were to reach the surface of the Earth, which it does not; for its mouth remains covered by a very large earthen vault, at the top of which is Jerusalem, and whose thickness is the eighth part of the half-diameter.

Archimedes treatises [23] were then part of the most erudite mathematics, which the previous commentators of Dante, pure literary scholars, certainly did not master. Galileo's contribution calls for very particular expertise, which he can legitimately boast of. This is undoubtedly how his claim to enlighten the controversy "for a few reasons of our own" should be understood. It is not without interest to verify Galileo's estimates using both Archimedean formulations and algebraic expressions that are now commonplace (although at university level—see Appendix 1). It should be noted however that if the vault of Hell is re-established, with its thickness of one-eighth of the earth's radius, the total volume of Hell is considerably reduced, since it is now only slightly less than 1/22 of the Earth's volume (instead of 1/14).

4 *A Line That Leads Naturally Towards the Centre*

Galileo, in his commentary, does not only claim to be a mathematician, but also calls upon his expertise as a novice physicist. It is in this capacity that he is going to deliver a severe criticism of Vellutello's comments. The latter, in fact, conceived the successive tiers of the Inferno as portions of a cylinder with walls parallel to their common axis, like the tiers of an ancient amphitheatre. Galileo contests this interpretation, arguing that such walls are by no means vertical, since they should then be generated by rays drawn from the centre of the Earth, since at two distant points the directions of the verticals are not parallel, but convergent. Thus, according to Galileo, the cliffs bordering the cylindrical steps of Velutello would in fact be oblique in relation to the local vertical(s); the outer edges of these steps would be markedly overhanging, and therefore absolutely unstable:

> If [Vellutello] supposes that the gulch rises between equidistant banks, we will have upper parts without supports to hold them, and therefore, inevitably, they will collapse. We know in fact that the heavy bodies follow a line that leads them directly towards the centre, and if on this line they find nothing to stop and support them, they continue to descend and fall.

In Manetti's architecture, on the other hand, the walls of the tiers are truncated cones, segments of nested cones with the centre of the Earth as their apex, so that these walls, oblique to the axis of Hell, are directed towards the geometric centre of the globe, considered by Galileo to determine the direction of the earth's attraction. The edition of the Crusca, which comes very shortly after Galileo's *Lessons*, offers a representation that is undoubtedly based on these *Lessons*, or at least confirmed by them, which is perfectly clear in this respect. Galileo's endorsement of this point of view seems at first glance to be based on a convincing physical reasoning which reinforces the scientific relevance of his discourse (Fig. 3).

On reflection, however, the physicist today is obliged to distance himself from Galileo's argumentation. First of all, Vellutello's Inferno is very small: both its depth and its maximum diameter are no more than a tenth of the values taken by these dimensions in Manetti's version (its bottom is about a tenth of the Earth's radius from the surface, far from the centre of the Earth), and its volume is therefore

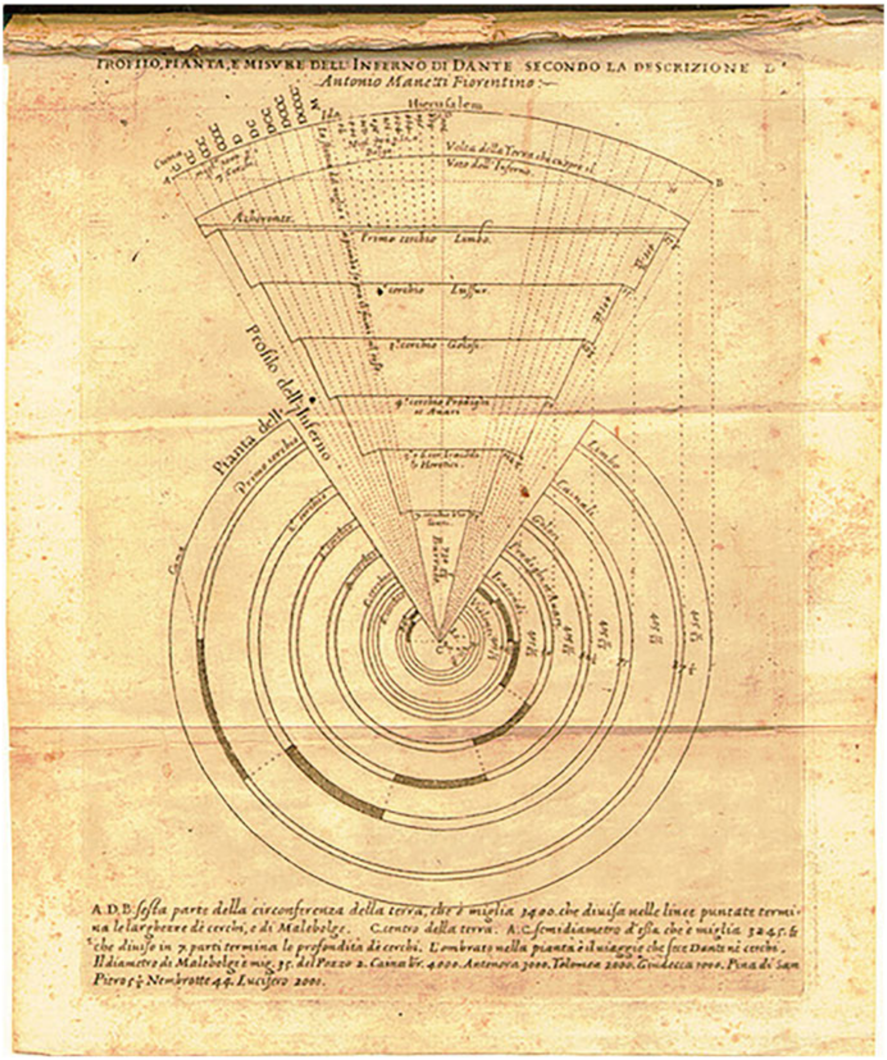

Fig. 3 Dante's Inferno as shown in the edition la Crusca (1595)

a thousand times smaller. Under these conditions, the variation in the direction of gravity from one point to another of Hell would be small, a few degrees at most, and could be practically assimilated to the direction of the axis of Hell. It is therefore a reasonable approximation to consider the tiers as portions of cylinders, with parallel walls, as Vellutello proposes. But if one wants to be more precise, which is necessary since Hell, according to Manetti supported by Galileo, has dimensions of the order of the earth's radius, Galileo's reasoning comes up against a serious modern objection. Indeed, the verticals (meaning: the directions of the force of gravity) at each point are directed towards the centre of the terrestrial globe only if the latter is a complete sphere, uniformly full. However, as soon as the globe is partially emptied by removing the vast conical space required by Hell, the inner gravity field of this incomplete sphere is affected and the directions of the local verticals are disturbed. Galileo certainly could not grasp this point and even less had the means to evaluate the necessary modifications, which can only be calculated using Newtonian gravitation theory. It is then somewhat ironic to note that, according to this theory, the situation is rather similar to the one described by Vellutello! Indeed, the attraction due to the hollowed part being removed, the force of gravity at any place on the edges of Hell would be directed not towards the geometrical centre of the sphere, but towards a point situated lower on the axis of the infernal cone. Thus, the vertical walls of the tiers should not be cone trunks with the centre of the Earth as a common vertex, but much more tightened cone trunks, akin to cylindrical segments. Seen from the centre of the Earth, the walls of the tiers would therefore appear to be more or less steeply overhanging. In fact, the calculation according to the Newtonian theory of gravitation shows that in Manetti's Hell, the direction of gravity would in fact be almost parallel to the axis of Hell everywhere (Appendix 2). Thus, and very curiously, from the point of view of modern physics, the situation would finally be intermediate between the one described by Manetti, and the one proposed by Velutello and retoqued by Galileo. Galileo's criticism, in that case, is therefore much less devastating than he thought and then a first reading would lead one to believe.

The nodal point of the argument developed above consists of distinguishing the geometrical centre of the globe from the attractive gravitational centre. From an Aristotelian perspective, where geocentrism is absolute and where the centre of the Earth is the intrinsic centre of the Universe, the natural place of gravity, this distinction is obviously unthinkable. Galileo, in his *Lessons on Hell*, misses the point, but one can legitimately wonder whether a later reflection on the situation was not one of the sources of his remarkable anti-Aristotelian discussion of the first Day of the *Dialogue*. Here we think of the passage where, long before the development of Newtonian theory, the spherical shape of the Earth (and other stars) is explained, not by the attractive power of the centre of the Earth considered as an intrinsic property of a single privileged point in the universe, but as the result of the mutual forces of attraction between the parts of the globe:

> . . . the parts of the Earth move not because they tend towards the centre of the world, but in order to reunite with their whole, and that is why they have a natural inclination towards the centre of the globe, by virtue of which they conspire to form and preserve this globe.

In any case, there is no doubt that the Galileo of 1632, if he had taken up the question of the verticality of the tiers of the Inferno, would have understood that on and in an Earth deprived of a large volume, falling bodies would not follow "a line that leads them directly to the centre" of the globe.

5 *In Search of the Size of a Giant*

Finally, Galileo summons the theory of proportions to clarify certain aspects of Dante's description. First of all, he tackles the question of assessing the depth of the icy well where Lucifer is sunk up to the waist at the bottom of Hell, his navel coinciding exactly with the centre of the world. Galileo begins by evaluating the size of the "giants" described by Dante as having a face as high as the famous enormous terracotta *pigna* (pine cone) that decorates a Vatican courtyard (where it is still visible today, with its three metres and a few feet high). Considering that the ratio of head to body is the same in giants as in humans (i.e. 1 to 8), Galileo, by a simple rule of three, attributes to the former a height of about 25 m. As for Lucifer, he is so tall, according to Dante, that the ratio between the length of one of his arms and the size of a giant is greater than the ratio between the size of a giant and that of a human. Hence, by two new rules of three, the length of Lucifer's arm, is about 340 m at least, and the height of Lucifer himself, not far from 1200 m.

But the problem is that Galileo here thinks like a pure geometrician, interested only in the shapes of objects and beings, and not at all in their physical constitution. Now, the resistance of materials follows laws of scale which are not those of simple geometric proportions. This is a phenomenon that is empirically well known in craft practice: if, starting from an object of modest dimensions—i.e. boat, framework, cart—one increases all the dimensions in the same ratio to make a similar but larger object, one realises that its fragility increases rapidly with the enlargement factor. The first to have drawn the attention of physicists to this crucial point, laying the foundations of the modern theory of the resistance of materials, was none other than Galileo himself! This is the essential result of one of the two "new sciences" he develops in the *Discorsi* [3]. The very beginning of the work forcefully announces this conception:

> SALVIATI: (. . .) Do not therefore believe any longer, Lord Sagredo, (. . .) that machines and constructions made of the same materials, scrupulously reproducing the same proportions between their parts, must be equally or, better said, proportionally capable of resisting or yielding to shocks coming from outside, because it can be geometrically demonstrated that the largest are always less resistant than the smallest; so that ultimately all machines and constructions, whether artificial or natural, have a necessary and prescribed limit which neither art nor nature can exceed, - it being understood, of course, that proportions and materials always remain the same.

Galileo applied these ideas to the case of living beings, clearly demonstrating that spatial homothety does not respect physical constraints, and that a large animal needs limbs thicker in relation to its size than a small one in order to support its weight (compare, for example an elephant, a dog, and a mouse):

> (. . .) it would be impossible, whether in the case of men, horses or other animals, to make skeletons capable of lasting and regularly fulfilling their functions, at the same time as these animals would grow immensely in height – unless (. . .) their bones were deformed by enlarging them excessively, which would result in making them monstrous in form and appearance.

Echoing perhaps and in any case rectifying his considerations in the *Lessons on Hell*, he further considers in the *Discorsi* the case of the giants:

> (. . .) if one wished to keep in a particularly large giant the same proportion as the limbs have in an ordinary man, one would have to either find a much harder and more resistant material to constitute its bones, or admit that its strength would be proportionally much lower than that of men of mediocre size; otherwise, to increase its height without measure, one would see it bend under its own weight and collapse.

Thus, Lucifer could not have the proportions of a human being, with dimensions simply multiplied by a single scale factor. Either he must be singularly disproportionate, with limbs of monstrous relative thickness, or he must be very fragile. The latter conclusion, moreover, could be defended since Lucifer is apparently immobile and in a zone of practically zero gravity, thus hardly risking a fatal fall. Give or take a few decades, this is the thesis that Galileo could have defended . . .

In the *Lessons*, Galileo again uses the reasoning of geometric proportionality to develop an argument that is even more crucial than that of Lucifer's size, since it concerns the resistance of the earth's cap serving as the vault of Hell. He writes:

> [According to some], it does not seem possible that the vault covering Hell, as thin as it must be with a Hell so high, can hold without collapsing and fall to the bottom of the infernal abyss, (. . .) if it is not thicker than one-eighth of half a diameter (. . .). One can easily answer that this size is quite sufficient: indeed, if one considers a small vault, made according to this reasoning, which would have an arch of 30 fathoms, it would be about 4 fathoms thick (. . .); [but if] one had even one fathom, or ½, instead of 4, it could already be maintained.

The comparison used by Galileo between the skullcap of Hell and a masonry vault undoubtedly refers to the relationship between the structure of Dante's Inferno and the architecture of the famous dome of the Duomo of Florence designed by Brunelleschi, which played an emblematic role in the Italian Renaissance [24]; incidentally, let us note that Manetti, whose views Galileo defended in his *Lessons on Hell*, was also Brunelleschi's biographer. But if the analogy between the cupolas of Hell and the Duomo had a deep cultural meaning, its scientific value is nil, for the same reasons that Galileo develops in the *Discorsi*: a vault as gigantic as that of the Inferno, if it had the same geometric proportions as a small masonry vault, would certainly not have the same solidity. In the light of modern conceptions about gravity and the resistance of materials, the cover of Hell would inevitably collapse, according to the very arguments that Galileo initiated. Indeed, the resistance of a vault, like that of a beam or a bone, grows like the area of its section, whereas its weight varies like its volume. If all dimensions are multiplied by the same scale factor, 10, for example the weight will be multiplied by 1000 but the resistance to collapse by only 100; it will be proportionally 10 times more fragile. There is therefore necessarily a limit to the strength of a structure obtained by simply changing the scale from a smaller solid structure. And in the case of the vault of Hell compared to the small masonry vault envisaged by Galileo Galilei, where the scale factor is several hundred thousand, this limit is more than obviously exceeded and by a great deal.

We touch here on a crucial point concerning the *Lessons* and their role in the development of Galileo's thought. It is indeed very likely that he quickly understood his error of reasoning, resulting from a purely geometrical conception that does not take into account the laws of scale concerning the physical properties of matter. And it is the realisation of this misunderstanding that would have been at the origin of his work on the resistance of materials, which the *Discorsi* exposes. This thesis, put forward by Mark Peterson, is based on serious arguments [25]. The fact that Galileo soon realised the fallacious nature of the changes of scale implemented in his *Lessons on Hell* would explain in particular the discretion, even reticence, which he showed almost immediately with regard to this work, as we have pointed out. He certainly devoted intense reflection to these issues in the 1590s and 1600s. One can even speculate that Galileo, understanding his error, suffered a real psychological shock, which is echoed in the *Discorsi*, when immediately after Salviati's founding statement recalled above, his interlocutor reacts with a surprising emotionality:

> SAGREDO: Already my head is spinning, and my mind, like a cloud suddenly torn apart by lightning, fills for a moment with an unusual light that from afar lets me glimpse strange and disordered ideas, only to fade and hide them immediately. For it seems to me that one should conclude from your words that it is impossible to execute two constructions, at once similar and unequal, with the same material, and whose resistance would be proportionally identical.

These lines constitute an explicit refutation of the argument in the *Lessons* assimilating the dome of Hell to an enlarged small masonry vault. There is every reason to believe that Galileo quickly understood his error. In fact, although the *Discorsi* were not published until 1638, the material was already ready before 1610, when Galileo devoted himself to astronomical observations and published his first major work, the *Sidereus Nuncius*. Thus, in a letter of 1609 to Antonio de Medici, he sets out what is, almost 30 years ahead of time, an explicit summary of the *Discorsi*, at least of their first Day:

> I have recently managed to obtain all the results, with their demonstrations, concerning the strengths and resistances of pieces of wood of various lengths, sizes and shapes (. . .), a science which is absolutely necessary for making machines and all kinds of constructions, and which has never before been treated by anyone. (quoted in [25], note 15).

It is, therefore, possible to consider the *Lessons on Hell* as the crucible in which Galileo's fundamental work in the *Discorsi* was initiated.

Another physical difficulty in Dante's description can be highlighted, linked as the previous one to the break-in symmetry that the infernal cavity would impose on the distribution of the land masses. Galileo, if he had, a few decades later, raised the question of the physical coherence of the Dantean model, would not have failed to perceive the problem. The problem is that an Earth largely hollowed out by the conical Hell would see its centre of inertia shifted: it would no longer coincide with the geometrical centre of the globe, but would find itself offset on the axis of the cone, in the opposite direction to that of Jerusalem. A quick calculation shows that this shift would be of the order of 3% of the Earth's radius, i.e. about 200 km. The Earth would rotate on itself around this centre of inertia; the distribution of masses would no longer be isotropic (spherically symmetrical), but axially symmetrical, so it would behave like a top. Since its axis of symmetry (passing through Jerusalem) would not coincide with the axis of the poles, the diurnal rotational movement of the Earth on itself would be considerably disturbed by a simultaneous precession of the hyerosolomitan axis, which is contrary to the most banal observation. Thus, mechanics alone would suffice to invalidate the idea of an infernal cavity within the Earth. But, of course, placing the apex of Hell, Lucifer's home, at the centre of the Earth, only makes sense if this place is also the centre of the world, which supposes the validity of geocentrism (which, let us recall, the young Galileo did not yet reject). In the system of the Copernican world, it would be more natural to transport Hell into the Sun—which would ensure the functioning of its furnaces. Such a theory was seriously put forward in 1727 by Tobias Swinden [26].

6 *By Inviting Him to Press the Pace*

Remains an enigma in the Galilean commentary on Dante's Inferno. Indeed, it is strange that there is hardly any mention of the temporality of the journey of Dante and Virgil. However, given the distances so precisely established by Galileo, it is clear that the poet and his guide had to cover thousands of kilometres—on foot, and even at the cost of quite arduous steps and climbs. But the entire journey of the two men into Hell lasts less than three days—from the night of Maundy Thursday 7 April 1300 to the evening of Holy Saturday! Let us note that in Vellutello's little Hell, where the distances amount to hundreds of kilometres rather than thousands, the difficulty would be (a little) less serious . . . The paradox is all the greater as we still find in Galileo's text an allusion to the time of the journey, precisely used as an argument against Vellutello's geometry. Virgil, having brought Dante to the first circle, urges him: "Let us go on, for a long road pushes us on". Galileo concludes that the distance they still have to travel is much longer than that already covered, and that therefore Hell is certainly much deeper than a tenth of the earth's radius proposed by Vellutello. How then can we understand why Galileo, being so precise in his numerical measurements of distances, did not make a quantitative estimate of travel times? This is all the more strange since Galileo was already concerned with the movement of bodies at the time. It is most likely that he did make these calculations, and having realised that they were incompatible with a realistic interpretation of Dantean narrative, he decided to ignore them—without, however, being able to prevent himself from using them to cast a qualitative argument, highly dubious in any case, against Vellutello. No doubt we must resign ourselves, with Galileo, to admitting that Dante's description, while it allows for a coherent geographical interpretation of Hell, leaves the chronology of the journey to the poetic license.

7 *. . . to Give Others the Opportunity to Interfere so Much More . . .*

Of course, there can be no question of making the *Lessons on Hell* the key to the whole of Galileo's work. For example, some have tried to see in the passage where the monster Geryon carries the poet and his guide into the air, without them perceiving the movement other than by the caress of the air as it passes, a premonition of the principle of relativity, from which Galileo could have drawn inspiration [27]. A similar idea was put forward by Primo Levi in one of his very last texts, where he interprets this same passage as a foreboding of the sensation of weightlessness experienced by astronauts in inertial orbits [28]. But these suggestions do not really stand up to scrutiny, either of the text of Dante's poem or of its possible scientific significance.

Perhaps more interesting from a literary point of view is the visit that the young Milton says he paid in 1638 to the old Galileo, who was then confined to his forced residence in Arcetri—an encounter that is not attested to, but immortalised by a statuary group representing the poet leaning on the physicist's shoulder, which can be seen on the ground floor of the physics department of the University "La Sapienza" in Rome. In any case, in his great poem *Paradise Lost*, Milton explicitly mentions Galileo on several occasions. One can only dream of the dialogue the two men may have had on Dante's work, a hypothetical source of Milton's vision of Hell [29, 30].

We would be remiss if we did not mention an interesting recent work, of an inspiration very close to that of Galileo's *Lessons on Hell*, in which a subtle modern geometrical interpretation of Dante's rather obscure description of Paradise, and more generally of his entire spatial universe, is proposed. The physicist Mark A. Peterson, already mentioned, shows that it is quite coherent to understand the Dantean universe as intrinsically curved and closed, presenting the topology of a three-dimensional sphere, which is unrepresentable within an infinite and flat Euclidean three-dimensional space [31]. We would like to know how Galileo could have felt about this analysis!

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These hasty comments on a little-known episode in the history of the beginnings of modern science allow to renew somewhat the old debate on the criteria of scientificity that we apply to such and such a statement in order to award or deny it the label "quality science". Neither the logic of the argumentation—contradictory, according to the orthodox methodological rules—nor the rigour of the calculations or the accuracy of the observational facts is lacking in Galileo's *Lessons on Hell*, nor in the serious infernal studies of natural theology or in the (more or less funny . . .) hoaxes of modern physics on the same subject [32]. Where we see that what characterises the admissibility of a statement in the scientific corpus is less an assessment of its validity than a judgement on its relevance. It is, moreover, easy to crosscheck this thesis, since many of the assertions of science as it is made—the majority, no doubt—prove to be erroneous without disqualifying the research that produced them, as long as their interest is recognised by the scholarly community. From this point of view, the classical opposition between an internalist history of science, favouring the intrinsic dynamics of disciplinary work, and an externalist history, emphasising the effects of their social environment, loses much of its vigour. Indeed the question of the recognised relevance of such or such a research programme precisely allows us to establish a link between the conceptual organisation of a field of research and its cultural, ideological, economic or political determinations.

Thus, the classical question of the validation or refutation of scientific ideas gives way to considerations on their qualification or disqualification. In fact, many works are abandoned without ever falling under explicit and redhibitory criticism; more often than not, it is insidiously that the paths of research take a different direction, leaving half-explored land lying fallow by the side of the abandoned road. If few scientific works reach the Paradise of definitive recognition, few too are condemned to the Hell of absolute oblivion or rejection. Most of them end up in Purgatory—just like Galileo's *Lessons from Hell*.

Annex 1 The Volume of Hell

According to Dante read by Galileo Galilei, Hell is a conical abyss whose apex is at the centre of the Earth, the half-angle at the apex being equal to $\theta = 30°$, bounded by a spherical cap and covered by a vault whose thickness O is equal to 1/8 of the Earth's radius (Fig. 4).

If Hell "reached the surface of the Earth", and the volume of the vault covering it is not taken into account, the volume of the cavity (delimited in section by the OAJBO line) would be:

$$V' = \frac{2}{3}\pi R^3 (1 - \cos\theta), \tag{1}$$

compared to the total volume of the Earth:

$$V_{\text{Terrc}} = \frac{4}{3}\pi R^3$$

that is, in proportion:

$$\frac{V'}{V_{\text{Terrc}}} = \frac{1}{2}(1 - \cos\theta);$$

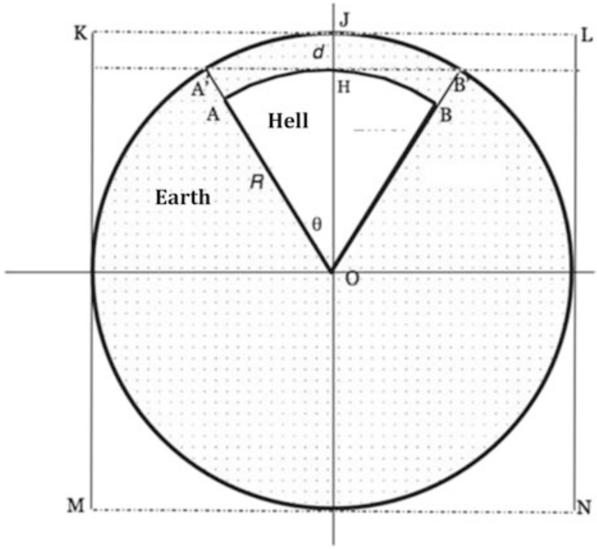

Fig. 4 Earth and Hell (section). The disc with centre O (and radius OA' = R), and the curved triangle OAB (with half-angle 30° at its apex O) respectively represent the spherical Earth and the conical Hell in a sectional view. The arcs AHB and A'JB' delimit the vault covering Hell, with a width HJ = d = R/8. KM and LN show the section of a cylinder circumscribed to the Earth sphere. The area of the cap with section A'JB' is equal to the area of the cylinder with height HJ (Archimedes)

numerically,

$$\frac{V'}{V_{\text{Terrc}}} = \frac{\left(2 - \sqrt{3}\right)}{4} = \frac{1}{14,92} \ldots,$$

a little less than the fourteenth part, as Galileo wrote.

But taking into account the vault, the height of Hell is reduced by a factor of 7/8, and its volume by a factor of $(7/8)^3$, hence now:

$$\frac{V_{\text{Enfer}}}{V_{\text{Terrc}}} = \frac{2 - \sqrt{3}}{4}\left(\frac{7}{8}\right)^3 = \frac{1}{22,3} \ldots$$

Galileo certainly did not have the modern notations, let alone the resources of integral calculus on which the above expressions are based. But Archimedes' results, in his *Treatise on the Sphere and the Cylinder* (Archimedes c. 250 BC), allowed him to reach the same conclusions without difficulty.

Annex 2 Gravity in Hell

When the terrestrial globe is emptied to make room for Hell in the mode proposed by Dante and commented on by Galileo, the distribution of masses loses its spherical symmetry. Now it is this symmetry, as Newton later showed, that causes the attractive forces exerted on any object by the different parts of the Earth to combine into a total force directed towards the centre of the Earth and of an intensity equal to that which would be exerted by a mass located at that point equal to the sum of the masses of the parts of the Earth closer to the centre than the object under consideration. This result depends crucially on the fact that the gravitational force between two masses varies as the inverse of the square of their distance, which obviously put it beyond Galileo's reach. In the case of a hollowed-out sphere, the calculation of the gravitational force at any point is far from immediate, even for a hollow of elementary geometric form such as the infernal cone, and fairly heavy numerical calculations are required. However, a simple model will provide us with an interesting exact result, which confirms our conclusions on the perturbation of the verticals.

Let us suppose a Hell no longer conical, but spherical. In all the interior space of the infernal sphere as well as on its edges, the force of gravity would then be uniform, having in any point the direction of the line joining the centres of Hell and Earth and the same value (this is a small theorem of the Newtonian theory of gravitation which can be demonstrated quite simply).

Thus, in this spherical model of Hell, whatever its size, it is paradoxically the design of Vellutello (cylindrical steps with parallel edges), which would prevail over that of Manetti (steps in truncated cones)!

In the Dantesque case of a conical Hell, the situation is certainly more complicated. The numerical resolution of the equations which determine the potential and the gravitational field, in this case, confirms, however, that the direction of gravity within the Hell and up to its edges varies very little. Figure 5c shows the result of such a calculation.

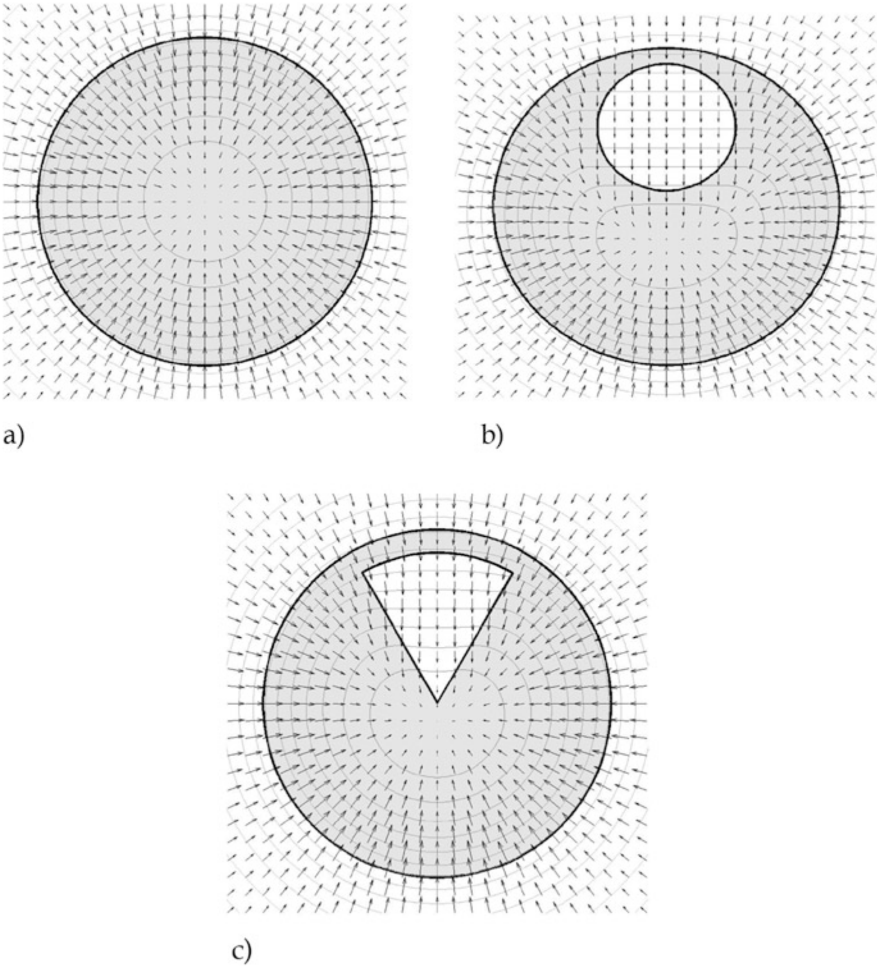

Fig. 5 Gravity in Hell. The arrows show the direction and magnitude of the gravity field; (**a**) in a filled Earth; (**b**) in an Earth hollowed with a spherical Hell; (**c**) in an Earth hollowed with a conical Hell. For the two shapes of Hell, one sees that the field shows a constant direction—exactly for (**b**), approximately for (**c**) [Calculations and figures by Jean-Paul Marmorat.]

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Don’t Tell Me the Cybersecurity Moon Is Shining…

Cybersecurity Show and Tell

Luca Viganò

1 Show, Don’t Tell!

In May 1886, the Russian playwright and short-story writer Anton Chekhov wrote a letter to his brother Alexander, who too had literary ambitions, providing him with the following advice:

> In descriptions of Nature one must seize on small details, grouping them so that when the reader closes his eyes he gets a picture. For instance, you’ll have a moonlit night if you write that on the mill dam a piece of glass from a broken bottle glittered like a bright little star, and that the black shadow of a dog or a wolf rolled past like a ball. [1]

This is often misquoted as

> Don’t tell me the moon is shining; show me the glint of light on broken glass.

but the misquote is understandable: it is crisper and, essentially, preserves the intended meaning. It is a concise injunction that has become the literary commandment for any writer: *show, don’t tell*!

The distinction between telling and showing was popularized by the literary scholar Percy Lubbock in his 1921 book “The Craft of Fiction” [2]. *Show, don’t tell* is a writing technique in which story and characters are related to the reader through action, words, dialogues, thoughts, senses, and feelings rather than through the author’s exposition and description. In a nutshell, telling states, showing illustrates.

Several other literary scholars and writers have since then discussed the *show, don’t tell* style of writing, including Ernest Hemingway and Stephen King, two of the style’s most prominent proponents. For instance, in his memoir “On Writing: A

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Memoir of the Craft," Stephen King writes:

> Description is what makes the reader a sensory participant in the story. [. . .] Description begins with visualization of what it is you want the reader to experience. It ends with your translating what you see in your mind into words on the page. It's far from easy. [. . .] Thin description leaves the reader feeling bewildered and nearsighted. Overdescription buries him or her in details and images. The trick is to find a happy medium. [3, pp. 173–174]

Show, don't tell applies also to all forms of fiction (including poetry, scriptwriting, and playwriting) and to non-fiction (including speech writing and blogging). Does it apply also to scientific writing? Of course, it does. In fact, even though I am not aware of any explicit theoretical (or practical, for that matter) investigation of the use of *show, don't tell* in scientific writing, the *show, don't tell* spirit lies at the heart of many successful scientific communication and storytelling approaches, such as those discussed in John Brockman's "The Third Culture: Beyond the Scientific Revolution" [4].[1] Storytelling has been used widely, and very successfully, as a pedagogical device in textbooks and science outreach endeavors, e.g., [5–9] to name a few. Rina Zazkis and Peter Liljedahl, in particular, have been instrumental in promoting storytelling in the mathematics classroom. They begin their book "Teaching Mathematics as Storytelling" by writing:

> We like to tell stories. We tell stories about mathematics, about mathematicians, and about doing mathematics. We do this firstly because we enjoy it. We do it secondly because the students like it. And we do it thirdly because we believe that it is an effective instructional tool in the teaching of mathematics. We are not alone in this. There is ample literature to support the enjoyment of storytelling on the part of both the story teller and the story listener. There is also an abundance of anecdotal data that suggest "telling a story creates more vivid, powerful and memorable images in a listener's mind than does any other means of delivery of the same material" [10, p. xvii]. Aside from the educational value, however, there is also beauty. There is beauty in a story well told, and there is beauty of a story that can move a listener to think, to imagine, and to learn. [11, p. ix]

I find this remark about beauty particularly fascinating, especially since my colleague Giampaolo Bella and I have been reflecting about beauty in security [12], which has led us to work with Karen Renaud and Diego Sempreboni to investigate the beautification of security ceremonies (i.e., protocols that are executed by machines and human users) [13]. We are currently working on a deeper analysis of the role that beauty plays in security, but I am digressing from the main topic of this paper, so let me return to Zazkis and Liljedahl, who, a few lines after the quote above, discuss the purpose of telling stories in the classroom:

> We tell stories in the mathematics classroom to achieve an environment of imagination, emotion, and thinking. We tell stories in the mathematics classroom to make mathematics more enjoyable and more memorable. We tell stories in the mathematics classroom to engage students in a mathematical activity, to make them think and explore, and to help them understand concepts and ideas. [11, p. ix]

[1] See also Edge.org, the website of the Edge Foundation, Inc., which was launched in 1996 as the online version of "The Reality Club" to display the activities of "The Third Culture."

They quote Egan:

> Telling a story is a way of establishing meaning. [14, p. 37]

and then talk about the power of images:

> One result of the development of language was the discovery that words can be used to evoke images in the minds of their hearers, and that these images can have as powerful emotional effects as reality might, and in some cases even more. [11, p. 15]

In fact, when Egan, Haven, Zazkis, and Liljedahl, as well as many others, speak of telling, and then, more concretely, of storytelling, they invoke images and imagery. One would be tempted to cite the old adage "A picture is worth a thousand words."[2] So, if storytelling is powerful, and images and pictures even more so, why not combine them? Why not tell *and* show? Or, better, *show and tell*?

2 Show and Tell

Dan Roam has written a number of books, including "The Back of the Napkin" [15] and "Show and Tell" [16], in which he has been proposing visual thinking and storytelling for problem solving.[3] Roam is also an engaging speaker and some of his presentations are available online. In particular, when he presented "The Back of the Napkin" at Google [18], he said:

> Any problem can be clarified significantly, if not outright solved, through the use of a picture.

Drawing on his collaborations with visual scientists and neurobiologists, he added:

> If we can take the time to use these simple pictures to help us figure out what we're talking about, we now have this incredibly powerful tool to use to share with other people when we meet them, and the beauty of it is that they're not going to forget what we told them. [...] When we draw in front of someone at the same time that we're talking, [...] we are actually activating processing centers in the brain that are really, really excited. Our brain wants to get information visually as well as verbally [...] when we draw the picture at the same time that we're talking, they get it and it's like manna for the person's brain. This is the way the

[2] In 1921, the advertising trade journal "Printer's Ink" published an article by Frederick R. Barnard titled "One Look is Worth a Thousand Words" in which Barnard claims that the phrase has Japanese origin. But in 1927, "Printer's Ink" published an advert by Barnard with the phrase "One Picture Worth Ten Thousand Words," where it is labeled a Chinese proverb. The Japanese and Chinese attributions were meant to give it more credibility, a sense of gravitas and a touch of mystery and philosophy, so much so that the proverb is nowadays commonly, and wrongly, attributed to the Chinese philosopher and politician Confucius.

[3] "Show and tell" is also the name of a common classroom activity in elementary schools, especially in English-speaking countries, in which a child brings an item from home and explains to the class why he/she chose that item and other relevant information. This activity is useful also for adults [17], but it is quite different from the *show and tell* that Roam champions and the one that I discuss here.

> brain wants to process information. [. . .] The picture is something that is archival and can be taken along, and we essentially guarantee that the person we gave it to, that we drew this picture for, really does understand what we were talking about [. . .] almost invariably we can guarantee that they understood it in exactly the way we meant because we created the picture with them.

In this talk, and in his books, Roam proposes to draw pictures in real time when presenting an idea, when addressing a problem and pitching its possible solution. This is one of the possible ways in which one can realize *show and tell*. In this chapter, I will mainly explore another way, namely the use of existing artworks (films, in particular, but not only). The idea is that while *show, don't tell* is the commandment for fiction, in the case of non-fiction, *show and tell* is often the best approach when one wants to present, teach, or explain complicated ideas such as those underlying notions and results in mathematics and science, or in cryptography and cybersecurity, which is my own discipline.

In my paper "Explaining Cybersecurity with Films and the Arts" in "Imagine Math 7" [19], I discussed how, in the context of research in *Explainable Security (XSec)* [20], popular movies and other artworks can be used to explain a number of basic and advanced cybersecurity notions, ranging from security properties (such as anonymity, pseudonymity, and authentication) to the algorithms, protocols, and systems that have been developed to achieve such properties, and to the vulnerabilities and attacks that they suffer from. As this paper is a natural continuation of [19], let me repeat some text that I wrote there and then expand on it:

> In [20], we discussed the "Six Ws" of XSec (Who? What? Where? When? Why? and How?, as summarized in Fig. 1) and argued that XSec has unique and complex characteristics: XSec involves several different stakeholders (i.e., system's developers, analysts, users and attackers) and is multi-faceted by nature, as it requires reasoning about system model, threat model and properties of security, privacy and trust as well as concrete attacks, vulnerabilities and countermeasures.

This paper, like [19], is mainly about the "Who?" and the "How?". As pointed out in [20], the recipients of the explanations might be so varied, ranging from experts to laypersons, that they require quite radically different explanations, formulated using different languages. Experts typically only accept detailed technical explanations, whereas laypersons are often scared off by explanations of cybersecurity (say, how to interact with a system or an app) that are detailed but too technical. Such an explanation might even repulse the laypersons and make them lose all trust in the explanation and, ultimately, in the cybersecurity of the system that is being explained. This repulsion and lack of trust might lead to users interacting with systems in ways that, unbeknownst to the users and possibly even to the developers and administrators of the systems, are vulnerable to attacks (to the systems and to the users themselves). In practice, however, laypersons are rarely given explanations that are tailored to their needs and their ability to understand.

As discussed in [19], clear and simple explanations with popular films and the arts allow experts to target the laypersons, reducing the mental and temporal effort required of them and increasing their understanding and ultimately their willingness

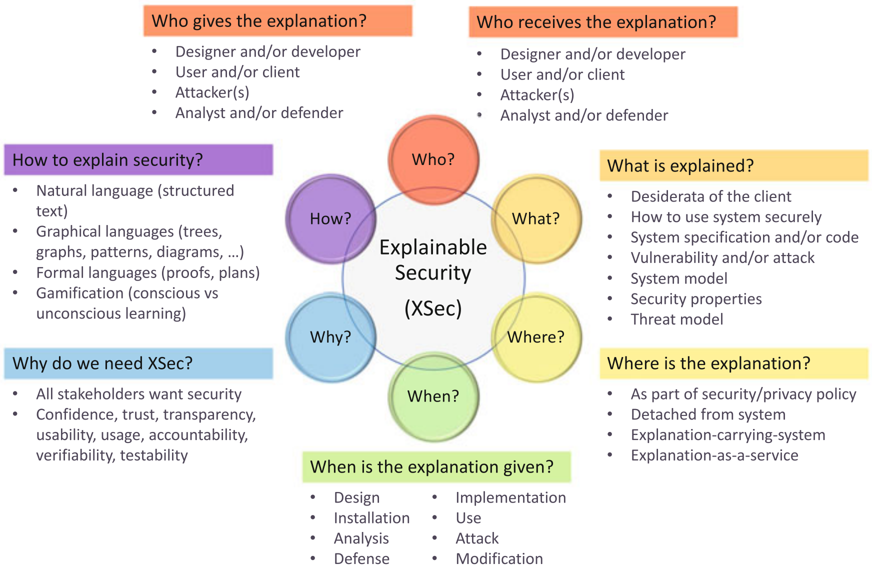

Fig. 1 The Six Ws of Explainable Security (from [20])

to engage with cybersecurity systems. In other words, with reference to the research roadmap for XSec that was laid out in [20], this chapter and [19] focus on:

- **Who?** The experts (the system developers but possibly also independent third parties) provide explanations to the laypersons.
- **How?** Using popular films and the arts. (It would also be interesting to consider "inventing" new artworks, possibly in real time, like Roam's drawings, and I will return to this in the concluding remarks.)

The other "Ws" should, of course, be considered too:

- **What?** Everything that pertains to cybersecurity, such as properties, algorithms and protocols, threats, vulnerabilities, and attacks. Paraphrasing Roam's assertion quoted above, the problem that we want to tackle, and in some cases hopefully outright solve, is the clarification of cybersecurity notions through the use of static or moving pictures (i.e., films) and other artworks.
- **Where?** These explanations could actually be anywhere, either co-located with the system or detached from it.
- **When?** As these explanations mainly target the system users, the explanations will be given when (or after) the system is deployed, but it will also be useful for the system developers to start working on the explanations at system design time (unless the explanation is provided by a third party, independent of the developers). Such explanations are also an effective instructional tool in the teaching of mathematics, as advocated by Zazkis and Liljedahl [11] and as I

have been doing for 20 odd years, using films and artworks during lectures and in public engagement talks on cybersecurity.

- **Why?** To increase the laypersons' understanding, confidence, trust (and more) in cybersecurity.

There is also a second "Why?", namely why use popular films and artworks to explain cybersecurity? Because the added power of telling (i.e., explaining notions in a technical way) *and* showing (via visual storytelling or other forms of storytelling) can help experts to convey the intuition in addition to the technical definition.

I have given some examples in [19] considering security properties such as authentication, anonymity, unobservability, and untraceability, and some algorithms and mechanisms to achieve them (or to attack them). Let me give here another example about anonymous communication or, more specifically, unobservability of message exchanges and untraceability of messages.

Example 1 Assume that Alice wants to send a message to Bob, but she does not want the attacker Charlie to trace the communication, i.e., Alice (and possibly Bob too) does not want Charlie to observe that Alice sends the message and that the message is received by Bob. The point here is not to keep the contents of the message confidential (that can be achieved by encryption, if needed) but rather that the communication Alice—Bob is confidential, i.e., that Charlie is not able to trace the message and observe that the communication is taking place. There might be plenty of reasons to keep a communication confidential. For instance, Alice might be an employee of Charlie's company and would not want Charlie to know that she has applied for a job at Bob's company, or Charlie might be a crime lord, Alice a snitch, and Bob the Police. In such cases and many others, it is therefore in Charlie's interest to monitor the network over which messages are sent and received (say it is the Internet) and trace messages as they move from one machine to the other in the network, from sender to receiver. And it is in Alice's interest (and possibly Bob's too) to find a security solution to impede Charlie to carry out such tracing, even when it is assumed that Charlie is able to monitor the whole network, as is typically assumed in security analysis, where one considers the most powerful attacker possible, e.g., one who is able to monitor the whole Internet. If the adopted security solution is able to withstand the attacks of such most powerful attacker, then it will for sure be able to withstand also the attacks of a real, and thus less powerful, attacker.

Some security solutions for anonymous communication over a network (and unobservability and untraceability of messages) have been implemented, such as *Mix Networks* [21] and *Onion Routing* [22], but they are among the security solutions that are most difficult to explain from a technical point of view. Similarly, one can give a natural language definition of untraceability,

> Untraceability of an object during a process under observation of an attacker is the property that the attacker cannot follow the trace of the object as it moves from one participant or location to another. [23]

and then refine this initial definition by providing a more formal one, a mathematical one, but I am fairly certain that some readers have already found the natural language definition to be quite difficult to digest, and they would find the technical definition and the solutions even more impenetrable.

The intuition behind the property and the solution is, however, quite simple. So, rather than *telling* you that the Mix Network moon is shining, let me first *show* you the glint of light on untraceable communications. Once you get the intuition, the technical explanation will hopefully be much more understandable (if you are, say, a cybersecurity student learning how Mix Networks work) or maybe not needed at all (if you are a layperson interested in understanding why you should trust Mix Networks but not so much in their inner workings). In fact, I will not tell you the technical explanation in this chapter, but only show how and why it works.

Consider the network delimited by the dotted line in Fig. 2, where the squares represent machines that distribute messages in the network, and meet Alice  and Bob . I do not show the attacker Charlie explicitly, but you can picture him at bird's-eye view, observing the whole network. If Charlie is able to ensure Alice's message is the only one in the network, as in Fig. 3, then tracing the communication is a trivial task. It is as if the police were chasing a car on a highway, and that car is the only one on that highway.

In fact, to *show* this, consider the photo in Fig. 4. On June 17, 1994, former NFL player O.J. Simpson was formally charged with the murders of his ex-wife, Nicole Brown Simpson, and her friend Ron Goldman. Instead of turning himself in, Simpson drove off into a white 1993 Ford Bronco SUV and became a fugitive of the law. The low-speed car chase that ensued was watched live by an estimated 95

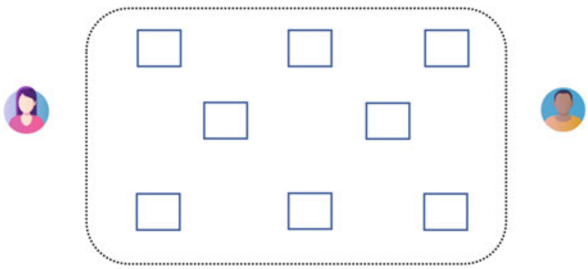

Fig. 2 Alice, Bob, and the network

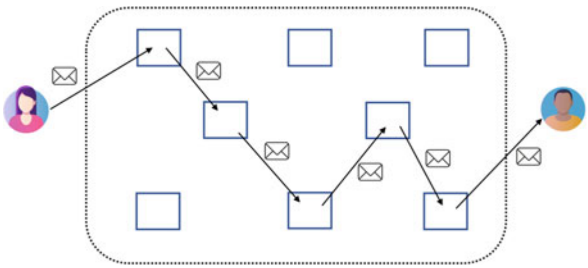

Fig. 3 Alice sends a message to Bob..., but it is the only message in the network and thus can be easily traced

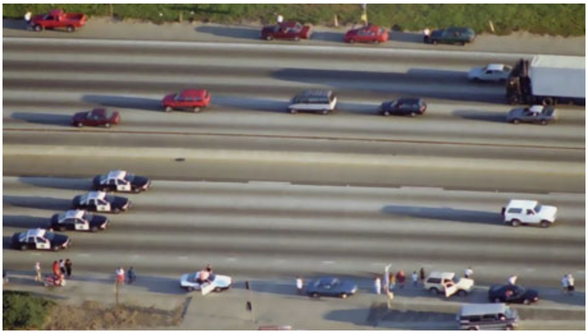

Fig. 4 The police chasing O.J. Simpson's white Ford Bronco (it was impossible to trace the author of this photo even though many newspapers online have used it)

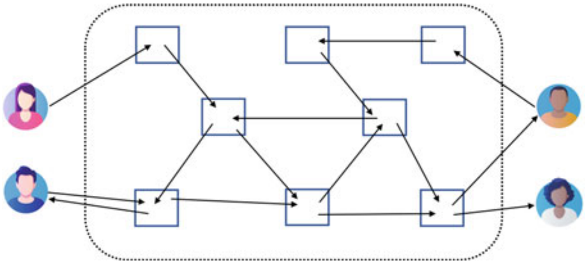

Fig. 5 Some more agents, some more messages

million people and indeed the police closed the highways so that they could easily follow the car for about 60 miles until Simpson's home.

The car chase would have been more challenging had there been other cars on the highway (instead of having been forced to pull over as shown in the photo). So, let us add some more agents who send and receive messages alongside Alice and Bob (the machines in the network are also allowed to send messages), as shown in Fig. 5. Charlie's task is now more complex, but still feasible: if he wishes to find out who Alice is communicating with, Charlie just needs to follow the messages that are sent by Alice to the first machines in the network, then follow the messages that are sent by these machines, and so on, until he has identified all possible traces from Alice to the possible recipients. Basically, this means dispatching one police car for every suspect car. A powerful attacker would certainly have enough police cars at his disposal. Hence, to achieve untraceability, we need suspect cars, lots of suspect cars. That is, we need many more agents sending and receiving many more messages as shown in Fig. 6. Charlie would now need to follow all of the messages, and the more there are, the harder Charlie's task as Alice's message could be any one of the messages that are circulating in the network. In technical terms, this set of messages is called the *anonymity set*: Alice's communication with Bob is anonymous as Alice's message is not identifiable within the set of messages. Of course, the larger the set, the higher the level of anonymity.

I have already discussed anonymity sets in [19] using the "I'm Spartacus!" scene in Stanley Kubrick's "Spartacus" [24] and the climatic museum scene in "The Thomas Crown Affair" [25] (and more). As I observed in [19], the main idea

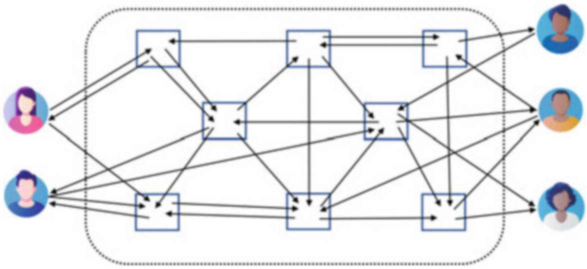

Fig. 6 Toward untraceability

underlying the way in which Mix Networks realize untraceability of messages is that there are plenty of messages circulating in a network and that such messages all look alike (in the sense that they are all plausible messages). But Mix Networks do more than that. The machines in the network are called "Mixes" since they receive a batch of messages in input, mix them, and then output them in different order, so that an attacker who is observing the input and output of each Mix, but cannot open the Mixes to look at the inner working, cannot associate output messages with input messages. There is no first-in/first-out or last-in/last-out association (nor any other association such as first-in/last-out). The first message that is output by a Mix could correspond to any of the messages that the Mix received in input. Moreover, Mixes also ensure that there is always enough traffic in the network by sending "dummy messages" (i.e., fake messages that are then discarded) and they require that all messages have the same size.

How can we "show" that? For instance, by using a scene from the movie "Baby Driver" [26]. The movie begins with a bank heist and the ensuing car chase: three bank robbers are escaping in a red sedan car (a 2006 Subaru WRX STI in San Remo Red) chased by several police cars and a helicopter. Baby, the robbers' getaway driver, has a brilliant idea: he enters a highway by going the wrong way into oncoming traffic[4] and then performs a u-turn to merge into normal traffic when he spots two other similar red sedan cars proceeding side by side. Aerial shot from the helicopter: three red cars side by side, with Baby's car in the middle. But when the cars enter a short tunnel, Baby performs a mix: he drives in front of the car on his left and breaks suddenly, forcing that car to take Baby's place in the middle of three to avoid a collision, so that, when they exit the tunnel, the helicopter continues to chase the car in the middle and Baby's car can safely escape by taking the next exit. Traffic, an anonymity set consisting of similar, if not identical, cars and a mix changing the order of the "messages": Baby gets away using a small Mix Network. □

This lengthy example has hopefully demonstrated that even challenging concepts such as untraceability and Mix Networks can benefit from the *show and tell* treatment. I urge you to watch the movie scenes that I have referenced above (and the ones I will mention below) as doing so will bear witness to the fact that the synergy

[4] This is an homage to another great car chase, the one in "To Live and Die in L.A." [27].

between showing and telling adds another dimension to what is told, a sensory one based on visual storytelling.[5]

As discussed in [19] and above, a clear and simple explanation, with something that they are already familiar with, such as a non-security-related movie or novel, will help make laypersons less irritated, stressed, and annoyed, and thus more receptive. Different from [19], in the remainder of this chapter, I take a more systematic approach, mapping different approaches to cybersecurity *show and tell*.

Zazkis and Liljedahl consider stories that frame or provide the background for a mathematical activity, and they distinguish between stories that introduce and stories that accompany and intertwine with mathematical activity. In particular, they discuss the following categories of stories for the teaching of mathematics:

- *Stories that set a frame or a background*, i.e., stories in which hero(in)es have to overcome obstacles to reach their goal (e.g., Oedipus solving the riddle of the Sphinx), stories of secret codes (e.g., stories in which decoding a message can save lives, or point to a treasure, win a princess' heart, or ensure fame and glory), and stories of treaties or contracts (e.g., the "contract" that Multiplication and Division shall be performed before Addition and Subtraction, but in the order in which they appear in any calculation).
- *Stories that accompany the content* (e.g., derived from the history of mathematics and the lives of mathematicians) and *stories that intertwine with the content* in which mathematical content emerges through the story, at times leaving the story behind and at times staying with the story. Zazkis and Liljedahl point out that the latter are harder to find, but still they are able to provide some

[5] This is reminiscent of the way in which a musical score adds an emotional layer to the images of a film, thus contributing in a fundamental way to the storytelling. This has been explained brilliantly by Stewart Copeland in the second episode, aptly titled "Telling Tales," of the documentary [28], in which Copeland discusses music in films with composer Danny Elfman:

Copeland: Why do the directors need this? They're telling a perfectly good story, with a perfectly terrifying antagonist, a handsome protagonist, a beautiful love interest. Why do they need music?

Elfman: Because music does something they learned very early on, that the pictures couldn't do.

Copeland: Take the decidedly lukewarm chills of early horror movies, for example.

Elfman: In the very first Frankenstein and the first Dracula, no music. All music was, in the first films, was opening and closing, like a play, and then they figured out a few years later, 1933 and 1935, "Why don't we take it up a level?" If you put this dramatic music, it really raises the stakes.

Copeland: As shown in in the pioneering movie King Kong.

Elfman: And if you put something heartbreaking when, you know, your hero or heroine is going to die, it really raises the stakes. [. . .] It goes straight to the heart.

In addition to "Frankenstein" [29], "Dracula," [30] and "King Kong" [31], Copeland and Elfman then also discuss on how Bernard Herrmann's score punctuates and amplifies Alfred Hitchcock's images in the movie "Vertigo."

interesting examples (e.g., using true and apocryphal stories about the Syracusan mathematician Archimedes).[6]

- *Stories that introduce*, i.e., stories that serve well to introduce concepts, ideas, or a mathematical activity (e.g., introducing exponential growth through the classical story of grains of rice and the chessboard).
- *Stories that explain*, e.g., riddles such as the "missing dollar" or "If a hen-and-a-half lays an egg-and-a-half in a day-and-a-half, how many days does it take one hen to lay one egg?" Zaskis and Liljedahl support this kind of stories by writing:

  > Mathematics is often perceived by learners as a collection of facts and skills; facts and skills that are sometimes seen as counterintuitive. When this happens a common reaction is to seek refuge in the meaningless memorization of rules. Experienced teachers can easily point to such places, places in which encounters with mathematics are most puzzling and rules are most prevalent. Instead of reciting rules, however, we suggest explaining these rules with stories. [11, p. 51]

  which reinforces the point that I tried to make above when discussing the benefits of explaining cybersecurity through artworks and to which I will return below.
- *Stories that ask a question* and encourage the students to engage with the story to arrive at the answer.
- *Stories that tell a joke*, since humor can enhance both the telling and the hearing of a story, and thereby indirectly influence learning. Two notable examples about arithmetic and logic are "Mathematics is made of 50 percent formulas, 50 percent proofs, and 50 percent imagination." and "Hofstadter's Law: It always takes longer than you expect, even when you take into account Hofstadter's Law." [37, p. 152]

Zazkis and Liljedahl then also discuss how teachers can create a story and they provide a "planning framework" demonstrating how instruction of specific mathematical topics or concepts can be planned and implemented.

3 Cybersecurity Show and Tell

So, what about cybersecurity? Without limiting the discussion to teaching, but considering all kinds of learning experiences, including scientific communications and public engagement or outreach activities, we can first of all divide the use of artworks to explain cybersecurity into two broad categories:

- *Using existing artworks*
- *Using new artworks that have been created on purpose*

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[6] I have also some experience with this: in the early Noughties, I wrote a play about the French mathematician Évariste Galois, who was killed in a duel at age 20 in 1832 [33–35]. The Teatro Stabile di Genova, which produced the play, had the brilliant idea to schedule morning performances for middle and high school students, and I have been told by many of them that they had never thought that mathematics could be thrilling and moving.

My work so far has focused on the first category, but together with colleagues in computer science and psychology and with artists and curators we have begun tackling also the second category, so let me focus on the first and conclude by providing a few more details about the second. The existing artworks can be divided into the following 4 sub-categories, for which I provide lists of examples that are by no means exhaustive:

- *Artworks about hackers, codebreakers, and cybersecurity experts*
- *Artworks about detectives or spies who use or are confronted with cybersecurity problems and solutions*
- *Artworks about ordinary people confronted with cybersecurity or artworks with references to cybersecurity*
- *Artworks that are not explicitly about cybersecurity but can be used to explain cybersecurity notions*

3.1 "Yes, I Am a Criminal. My Crime Is that of Curiosity. I Am a Hacker, and This Is My Manifesto." [38]

This category includes *artworks about hackers, codebreakers, and cybersecurity experts*, who make for very interesting hero(ine) or anti-hero(in)es. In May 2021, a search on the Internet Movie Database (www.imdb.com) with the keyword "hacker" returned 558 titles of films and TV series, many of which were adapted from novels. Notable examples range from faithful or apocryphal biographies of famous codebreakers and cryptologists like Alan Turing [39, 40] and John Nash [41] or less famous or even made-up ones [42, 43], to stories of bad "black-hat" hackers who carry out cyberattacks or of good "white-hat" hackers who save the day such as in the film "Hackers" [38] quoted in the title of this category[7] as well as in [45–56]. Some are quite realistic in their depiction of cybersecurity, few are real, most are

[7] That quote was inspired by the article "The Conscience of a Hacker" written by the real-life hacker "The Mentor" shortly after his arrest [44]. The article ends with the following words:

> This is our world now. . . the world of the electron and the switch, the beauty of the baud. We make use of a service already existing without paying for what could be dirt-cheap if it wasn't run by profiteering gluttons, and you call us criminals. We explore. . . and you call us criminals. We seek after knowledge. . . and you call us criminals. We exist without skin color, without nationality, without religious bias. . . and you call us criminals. You build atomic bombs, you wage wars, you murder, cheat, and lie to us and try to make us believe it's for our own good, yet we're the criminals.
> Yes, I am a criminal. My crime is that of curiosity. My crime is that of judging people by what they say and think, not what they look like. My crime is that of outsmarting you, something that you will never forgive me for.
> I am a hacker, and this is my manifesto. You may stop this individual, but you can't stop us all. . . after all, we're all alike.

inventive, giving black/white-hat hackers and codebreakers almost superhero-like abilities, but all of them make for a good *show* companion, as a speaker can discuss how faithfully or not cybersecurity has been portrayed. In fact, bad portrayals can be particularly useful to discuss misconceptions and correct possible prejudices.

In addition to films and TV series, there are also a few plays about cybersecurity, such as "Teh Internet Is Serious Business" (sic!), which provides a fictional account of the hacktivism of the collectives Anonymous and LulzSec and in which coding is ingeniously and amusingly symbolized by means of interpretive dance [57], as well as "Hackers" [58] and "The Nether" [59]. There are also many novels, such as, to name just the works of some of the most influential authors, the "Sprawl trilogy" [60] by William Gibson, the cyberpunk writer who coined the word "cyberspace" in the short story "Burning Chrome," Neal Stephenson's "Snow Crash" [61] and "Cryptonomicon" [62], Dan Brown's "Digital Fortress" [63], as well as Stieg Larson's "Millennium" trilogy [64] (and the film adaptations that have been filmed in Sweden and the USA [65–68]), and the sequel by David Lagercrantz [69] (which too has been filmed [70]). *Showing* with plays and novels might be less immediate and thus more challenging, but that does not mean that it will be less effective.

3.2 "I Have Said Enough to Convince You that Ciphers of This Nature Are Readily Soluble." [71]

This category includes *artworks about detectives or spies who use or are confronted with cybersecurity problems and solutions*. This category is closely connected to the previous one, and in many cases there are some overlaps, e.g., serial killers writing in code such as the "Zodiac Killer" [72], criminal masterminds using steganography to hide their messages [73], or the mathematics prodigy who helps his FBI-agent brother solve crimes in the TV series "NUM3ERS" [74].

As pointed out by John F. Dooley, codes and ciphers have a long history in fiction. In addition to his publications [75, 76], Dooley has been collecting a list of pieces of fiction (short stories, novels, chapters in novels, etc.) that contain cryptographic riddles of one sort or another as part of the story line.[8] In fact, several fictional detectives have had to solve cryptographic riddles, starting with the short story "The Gold Bug" by Edgar Allan Poe [71], who is generally considered the inventor of the detective fiction genre, but who also had a keen interest in cryptography and in 1841 wrote an essay about secret writing [78]. The most famous private detective of all, Arthur Conan Doyle's Sherlock Holmes, also had to solve several cryptographic riddles, in particular in "The Adventure of the Dancing Men" and "The Valley of Fear" [79, 80], and so had his doubly fictional sister Enola in Nancy Springer's "The Enola Holmes Mysteries" series [81] (all of these have also received film

[8] See also the collection of stories of code and ciphers edited by Raymond T. Bond [77].

and TV adaptations, e.g., [82–84]). Dilettante detectives have also been tackling cryptographic riddles, e.g., Lord Peter Wimsey in [85], the historian and amateur cryptologist Benjamin Franklin Gates [86, 87], and Robert Langdon, the Harvard professor of history of art and religious symbology[9] created by Dan Brown [89–92].

Also these professional and dilettante detectives are typically given abilities that border on the fantastic as they "see" the solutions of the cryptographic riddles in a matter of minutes, if not seconds. They picture them in their head before anyone can and often when nobody else does. This is "showing" in a way that is exclusive rather than inclusive. Similar to the characters of the first category, this facility for cryptography and cybersecurity creates a clear distinction from ordinary people and turns them into true hero(in)es and anti-hero(in)es. Portrayals of spies and secret agents (such as Jason Bourne, the former CIA agent who is the hero of a series of novels, written by Robert Ludlum [93] and then inherited by Eric Van Lustbader and then by Brian Freeman, and of the subsequent film adaptations directed by Doug Liman, Paul Greengrass and Tony Gilroy) are instead sometimes slightly more realistic, at least in terms of their cryptographic and cybersecurity abilities although maybe not so much in terms of their physical powers, but still almost inevitably go beyond portraying them as simply experts in their field. This is, however, quite understandable, as the story would then otherwise risk being quite dull. Still, using such characters and their stories for *showing* can be very effective also thanks to their worldwide popularity.

3.3 "I'm in a Secret Club." [94]

This category includes *artworks about ordinary people confronted with cybersecurity and artworks with references to cybersecurity*. There are less examples in this category than the previous two, but still several interesting ones, such as: two U.S. Marines in World War II assigned to protect Navajo Marines as their native language was used as an unbreakable radio code [95], a businessman whose identity is stolen after he gets a nice call confirming his name and other identifying information [96], an FBI agent undergoing facial transplant surgery to assume the identity of the criminal mastermind who murdered his only son [97],[10] a security specialist forced into robbing a bank to protect his family [99], a computer geek inadvertently downloads critical government secrets into his brain and is then

[9] There is no such thing as a professor of symbology in real life, but it is tightly connected to the actual discipline of semiotics, which in turn has been investigated also in the context of cryptography [88].

[10] "Con Air," "National Treasure," "Windtalkers," "Face/Off", …, Nicolas Cage has starred in so many cybersecurity-related movies that he would deserve a dedicated paper, perhaps titled "Explaining Cybersecurity with Nicolas Cage" or even better "Nicolas Cage is the Center of the Cybersecurity Universe." In fact, since writing the first draft of this chapter, I have published precisely such a paper [98].

recruited by CIA and NSA to help thwart assassins and international terrorists [100] (the latter two partially belong also in the first category). There are also several animated movies in which cybersecurity features prominently, which suggests that kids, even small children such as those watching "Peppa Pig," are probably much less scared by it than adults; for example, multi-factor authentication through biometrics[11] is used both in "Incredibles 2" [102] and "Monsters vs. Aliens" [103], a guessing/dictionary attack against a password is attempted by Winnie the Pooh's friend Tigger in [104] by singing the "Password Song" with lyrics consisting of every word that Tigger knows, Peppa Pig and their friends use one-time passwords in an episode titled "The Secret Club" [94].

Codes and ciphers (such as the hash function SHA-256, cryptocurrencies, and several different ciphers) feature also in animated series for more adult viewers, in particular in Matt Groening's "The Simpsons" [105] and "Futurama" [106]; many of these examples are discussed by Simon Singh in his book [107] in which he unravels the mathematical secrets of these two series up until 2013.

But it is not just films, TV series and novels. Although examples are more difficult to find, there are also some songs that relate to cybersecurity, such as the song "Secret" by The Pierces [108],

> Got a secret
> Can you keep it?
> Swear, this one you'll save
> Better lock it in your pocket
> Takin' this one to the grave
> If I show you, then I know you
> Won't tell what I said
> 'Cause two can keep a secret
> If one of them is dead ...

which references Benjamin Franklyn's famous saying "Three may keep a secret if two of them are dead" (which, I believe, in turn references the line "Two may keep counsel when the third's away," which is uttered by the villainous Aaron before murdering a nurse to preserve his secret, in William Shakespeare's "Titus Andronicus," Act IV, Scene 2):

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[11] *Multi-factor authentication (MFA)* is an authentication solution that aims to augment the security of the basic username–password authentication by exploiting two or more authentication factors. In [101], MFA is for instance defined as:

> a procedure based on the use of two or more of the following elements—categorised as knowledge, ownership and inherence: (i) something only the user knows, e.g., static password, code, personal identification number; (ii) something only the user possesses, e.g., token, smart card, mobile phone; (iii) something the user is, e.g. a biometric characteristic, such as a fingerprint. In addition, the elements selected must be mutually independent [...] at least one of the elements should be non-reusable and non-replicable.

The underlying idea is that the more factors are used during the authentication process, the more confidence a service has that the user is correctly identified.

This category requires substantial work of the presenter to make the connections between *show* and *tell*, but, trust me, it can be lots of fun, especially if one shows excerpts of the animated films.

3.4 "I'm Spartacus!" [24]

This category includes *artworks that are not explicitly about cybersecurity but can be used to explain cybersecurity notions*. This is, in my view, the most interesting category. I have already mentioned above how "Baby Driver" [26], "Spartacus" [24], and "The Thomas Crown Affair" [25] as well as Alan Moore's graphic novel "V for Vendetta" [109, 110] can be used for explaining anonymity and untraceability. Other popular films and artworks can be used as metaphors for cybersecurity, and in [19] I discussed how a man-in-the-middle attack can be explained by analogy with the *carte blanche* issued to Milady De Winter by Cardinal Richelieu in "The Three Musketeers" by Alexandre Dumas père [111].

Let me give here some more examples by considering authentication and multi-factor authentication (cf. Footnote 11). Questions of authentication and identity occur extensively in the mythology and fairy-tale literature of most cultures, from Greece to India to China. In some cases, authentication occurs by means of passwords, such as the "Open Sesame" that opens the mouth of a cave in which Ali Baba and the forty thieves have hidden their treasure in the story "Ali Baba and the Forty Thieves," which first appeared in Antoine Galland's version of "One Thousand and One Nights." In most cases, however, authentication occurs by biometric traits and also by some form of multi-factor authentication, even in a pre-technology world. For instance, when the hero is (or proves to be or reveals to be) the only one able to do something or to have something:

- When the disguised Odysseus returns home to Ithaca after 20 long years, his faithful dog Argos recognizes him by his smell and his old wet-nurse Eurycleia by a scar he received during a boar hunt, and he finally proves his identity to his wife Penelope by being the only one able to string Odysseus' rigid bow and shoot an arrow through twelve axe shafts. Similar stories exist in Hindu mythology.
- Thor is the only one able to lift and wield the Mjọllnir hammer in Norse mythology (as well as in Marvel Comics and in Marvel Cinematic Universe).
- Arthur is the only one able to pull out the sword Excalibur from a stone, thereby proving to be the rightful king of Britain.

Similarly, mythologies and fairy tales also contain examples of *masquerading attacks*, in which the attacker poses as an authorized user. In the Internet, this would occur by the attacker using stolen logon ids or passwords, in mythology it occurs by the attacker magically or divinely shape-shifting into somebody or something else (e.g., in the "Epic of Gilgamesh," in the "Iliad," or in Ovid's "Metamorphoses"), whereas in fairy tales it often occurs via simpler means, e.g., the wolf pretends to be mother goat by eating honey to soften his voice and by smearing flour over his feet to turn them white in "The Wolf and the Seven Young Goats," a story published by the

Brothers Grimm in the first edition of "Kinder- und Hausmärchen" in 1812 [112]. Similar tales have been told in other parts of Europe and in the Middle East.

Finally, there are also examples of cases in which authentication fails not because the attacker is actively trying to deceive his victims, but rather because the victims (consciously or not) want to be deceived, as in the story of Martin Guerre, a French peasant who, in the 1540s, was at the center of a famous case of imposture: several years after Martin Guerre had left his wife and village, a man claiming to be him appeared in the village and lived with Guerre's wife and son for three years, before eventually being accused of the impersonation. The case of Martin Guerre has been popularized in literature [113, 114], film [115, 116], musical [117], as well as in plays and operas, and has also been the subject of scholar investigations [118]. Many of these works portray how, for different reasons, some people, including his wife, authenticated the stranger to be Martin Guerre even though they suspected him to be an impostor or even knew him to be one. A similar case happened in Italy in the late 1920s: the Bruneri-Canella case concerned an amnesiac patient of the Mental Hospital of Collegno,[12] who was identified first by the Canella family as the professor Giulio Canella, who had gone missing in action during World War I, and then by the Bruneri family as the fugitive petty criminal Mario Bruneri. After several inquiries and trials, the court found that he was indeed Bruneri, but the Canella family kept claiming he was Giulio Canella and he lived with Canella's wife, Giulia Canella, in exile in Brazil until his death in 1941. Also in this case, there were people who wanted to (wrongly) authenticate the amnesiac: some newspapers stated that actually Giulia Canella herself was ultimately convinced that the amnesiac was not her husband, but she had to keep pretending otherwise to avoid a major scandal. And this case too inspired literature [119], plays [120], and films [121, 122], which can be used to explain authentication and related attacks.

Such "popular" explanations are not meant to replace the mathematical definitions and explanations, nor the facts and skills mentioned in the above quote of Zazkis and Liljedahl. Although finding such popular artwork examples is challenging, the synergy of telling and showing via these examples, which laypersons will likely be already familiar with, can help go beyond the mere facts and skills by making them more intuitive, more accessible, more interesting, and more rewarding.

4 Conclusions

Films have been used to explain and teach different disciplines such as philosophy [123–125], history [126], social sciences [127], management and organizational behavior [128, 129], international relations and politics [130, 131], and mental health [132]. In addition to my own research, initial investigations have also been

[12] Collegno is a small town in the North-West of Italy and the case is known in Italy more colloquially as the "Smemorato di Collegno," i.e., the amnesiac of Collegno.

carried out for cybersecurity in [133], but more work is needed to explore the full potential of popular films and artworks for cybersecurity.

I have described four categories of such artworks and provided some examples for each of them. I have been collecting a database that includes many more examples and has benefited not only from the help of colleagues and friends (such as Diego Sempreboni, Sally Marlow, and Gabriele Costa) but also from the lists in [76, 77] as well the entries in the Internet Movie Database IMDB. I am keen to include examples from less considered artworks such as plays and music and to investigate the use of new artworks that have been created on purpose or that are created live during a presentation. The collaboration that I have initiated with artists and curators (such as Hannah Redler Hawes and Alistair Gentry) will be very fruitful to that end.

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Part X
Music and Mathematics

Sounds, Numbers and Other Fancies

Claudio Ambrosini

Where did the idea of a deep connection between music and mathematics come from? Is it due to the famous portrait of Pythagoras measuring the intervals with his monochord? Or to the fact that throughout the Middle Ages music was grouped into the *quadrivium* (alongside arithmetic, astronomy and geometry) as opposed to the *trivium*, where those subjects connected with human expression (grammar, logic and rhetoric) resided? It must be recognized that some issues, such as the different scales developed from Pythagoras onwards (Pythagorean, natural, mesotonic, temperate and others), actually require precise calculations to define their intervals. Even the development of the musical notation itself, from neumes to pentagram, is somehow connected with mathematics: being based on two axes—it is actually a Cartesian coordinate system some nine centuries before Descartes!

Since then, music has gone through a continuous process of innovation and refinement, but probably no other period has been as rich, and variously connected with science, as the twentieth century. Apart from Electronic and Computer Music, many new directions have been tried; from the Futurist *Art of Noise* to *Musique Concrète*, the rigid grids of post-Webernian Structuralism to open Randomness (John Cage), Stochastics (Iannis Xenakis) to the harmonic content of Spectralism (Grisey, Murail), the latter approach in a way recalling the investigations of Pythagoras.

Although as a composer trying to pursue his own path I am not following any of these trends, in my catalogue there are quite a few titles with scientific allusions: *Tecniche per la misurazione dell'infinito* (Techniques for Measuring Infinity, for three pianos, 2014), *Classifying the Thousand Shortest Sounds in the World* (for solo flute, 2012), *Etymon n.6 (Fearful Symmetry)* (for any group of instruments, 2018), *Three Holograms* (for guitar, 1978), *Orienteering* (for one or more electronically

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M. Emmer, M. Abate (eds.), *Imagine Math 8*,
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revealed pianos, 2019), *Big Bang Circus*, an opera conceived as the history of the universe acted out in a strange, surreal circus. A couple of these works even have a humorously "negative" title, as *A Sound a Day Keeps Time Away* (1977), or *Tic-tac (ossia come ammazzare il Tempo)* (Tic-Tac or How to Kill Time, 2013).

In this chapter, I will focus on pieces based on my researches; specifically, those carried out concerning the nature of sound, of perception, the possibilities offered by new instrumental techniques and the problems connected with the notation of "liquid", continuously evolving rhythms. I am not a mathematician in the strict sense, but I think that composing can be both the expression of one's imagination and at the same time an opportunity for knowledge and discovery.[1]

1 Invisible Polyphony

As an introduction I would like to recall a passage from Lucretius's poem *De Rerum Natura*, where the author describes our astonishment as the air in a dimly lit room, apparently empty, is crossed by a beam of sunlight suddenly revealing a myriad of dust particles, tiny floating elements—an unknown world. Something like this may be found in works I wrote starting from the idea of a "double" sound, which may surprisingly reveal other sounds.

Nell'orecchio di Van Gogh, una pulce (In Van Gogh's Ear, a Flea) is a piece for grand piano, upright piano—with mute pedal—and ensemble, written in 1983. Like several other works I composed in the early 1980s, it focuses on the idea of "musical perspective". In particular, the *perspective cone* created by the piano as a front, leading role and the rest of the ensemble pouring a sort of *tinged light* on it, as if it were a character on a stage.

The title refers to the well-known story of Van Gogh's cutting off his left ear. In my somewhat surreal report of this dramatic mutilation, special sounds called *difference tones* (also called *Tartini tone* after his discovery in 1713) are introduced. A difference tone is a frequency audible as a third tone when two other notes are played loudly and steadily. With certain intervals, this resultant frequency can be consonant and in harmonic relationship with the two main notes. For instance, playing a perfect fifth with pure tones: E (660 Hz) over middle A (440 Hz) will produce an audible difference frequency of 220 Hz, that is the A one octave below the middle one.

Using these combination tones in several ways I found that it was possible to develop a special form of polyphony, in which more notes than the number of musicians actually playing are heard: something like an *acoustic mirage*.

At a certain point of *Nell'orecchio di Van Gogh, una pulce* an instrument begins to play solo; when a second one enters 3 notes instead of 2 are heard. Then the

[1] To define this attitude towards the nature of sounds and musical instruments I coined the term *Acuology*, synthesis of acoustics and (sound) ecology [1].

process is continued introducing a third instrument (now producing 4 pitches). Progressively more notes are heard, often mixed in cluster-like groups. It is an *invisible*, or only partially visible, *polyphony* since fewer notes than are actually perceived appear in the score, as Luigi Nono commented on first hearing this piece.[2]

One last detail should be pointed out: difference notes are also called *subjective tones*, both because their listening depends on various subjective factors and because they produce a real, sometimes annoying tingling in the ear. Hence the Van Gogh reference in the title.

2 Natural Counterpoint

Ziggurat, a piece for ensemble written in 1975, produces another type of polyphony, or counterpoint, this time both audible and visible, but not controllable. The score is designed for instruments such as percussion, keyboards, plucked strings and others that, after the attack, unless stopped, continue their decay until the vibrations are naturally extinguished (Fig. 1).

All notes of each instrument must be played from lowest to highest, waiting for the extinction of each note before playing the next semitone. The instrument that has the widest range is the "leader" and starts first, from its lowest note. All other instruments enter one by one when their own lowest note is in unison with the note reached at that moment by the leader. To finish: each instrument stops playing after reaching its highest note.

Though the performers are allowed to choose dynamics, articulations, and the use of plectrums or other objects to modify the sound colour, the result is an unpredictable and almost uncontrollable counterpoint, produced by the *natural* durations of the notes and nicely animated by the "characters" of the instruments themselves: the slow and majestic progress of the piano, the petulant hurry of the harpsichord, the soft, floating "drops" of the vibraphone, the icy shimmer of the bell tree . . . A *theatre of sounds*, all acting in their *natural* habitat. What makes the difference, from a scientific point of view, is of course the material that the single instrument is made of and how long it can vibrate. This depends on further factors, such as the particular range, within the extension of an instrument, to which the single note belongs, the resultant length and thickness of the strings, for a piano or a harp, of the metal bars for a vibraphone and so on. All rather difficult to calculate or control.

[2] ". . . in the wake of the alchemists of Rudolf II's Prague: the *invisible acoustics* or sounds that you hear but don't see the composition mechanics". Luigi Nono in [2].

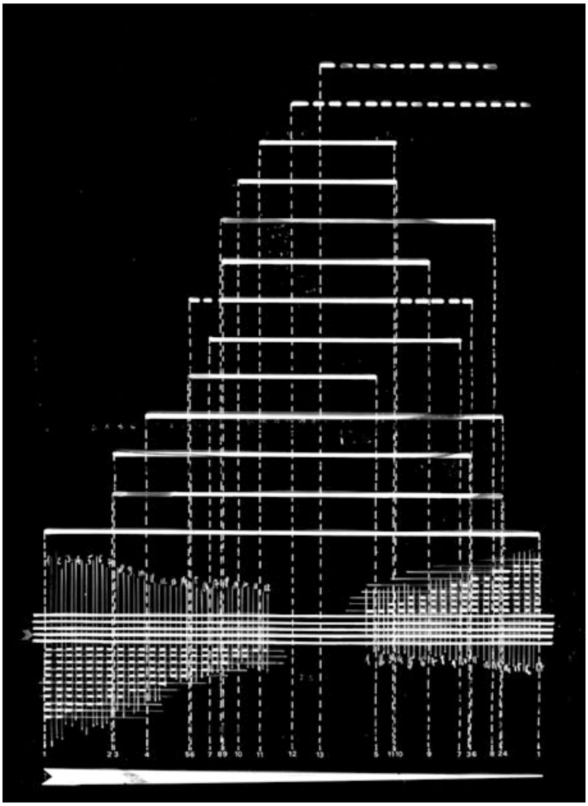

Fig. 1 Claudio Ambrosini, *Ziggurat* (1975), general plan showing the instruments' ranges, whose overlapping recalls a pyramid (from the author's autograph score). © C. Ambrosini, 1975

3 Ghost Notes

Canzone molle (Supple Song, for guitar, written approx. 1973) contains a special kind of "double" sounds, which at the beginning of the seventies, not managing to find a scientific or at least official definition in guitar method books, I chose to call *suono fantasma* ("ghost" sounds), or "non-identical twin notes", and nowadays are called *bi-tones*. These dual sounds are produced by the simultaneous vibration of both parts of a string—from the stopped fret to the bridge and from the same fret back to the nut—that has just been plucked or tapped upon. The two pitches do not have the same intensity, the main note being (much) louder. While the main notes ascend as usual from the nut to the end of the fretboard, the "ghost" notes descend (logarithmically), and vice versa.

Figure 2 shows the first page of the piece and my attempts to notate these softer, mirroring sounds.

In another guitar piece, *Entartete Musik* (Degenerate Music, 2017), the careful choice of the notes made it possible to have a *glissando* almost perfectly mirrored

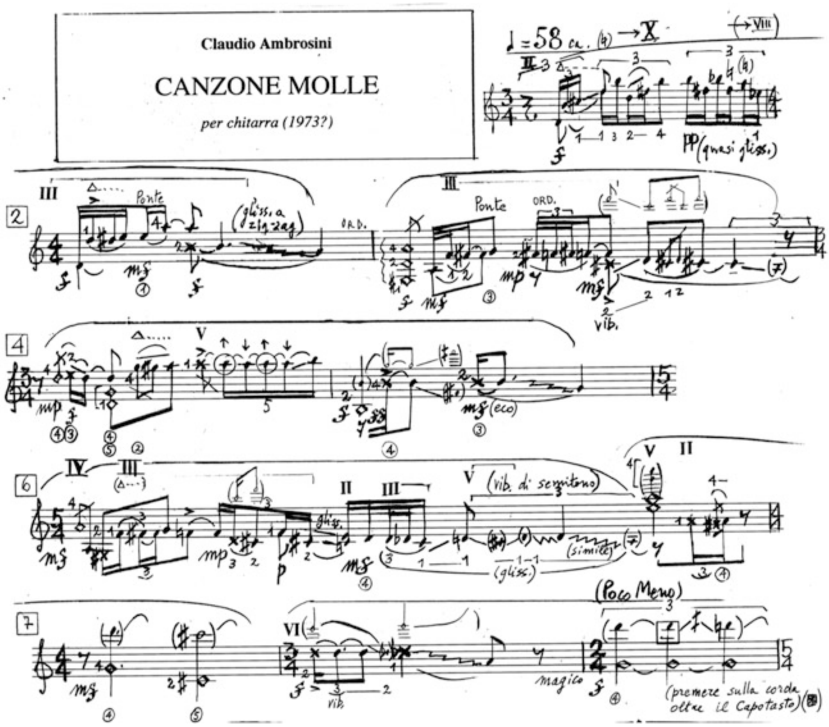

Fig. 2 Claudio Ambrosini, *Canzone molle*, for guitar (undated, early 1970s). The tiny white notes in brackets, written above the staves (in the measures 2, 3, 5, 8), show the *mirror behaviour* of the ghost notes with respect to the main ones below them (from the author's autograph score). © C. Ambrosini

by its "ghost". A bit like a person walking from left to right and his shadow from right to left (Fig. 3).

A piece for solo cello written in 2004 bears a title that intentionally recalls Antonin Artaud's *Le theatre et son double*: *Il suono e il suo doppio* (Sound and Its Double). Among the several moments in which different kinds of double sounds are produced, there is a sort of "anti-cadenza": rather soft and slow, instead of loud and virtuosic. But the difficulty is still high: the cellist must visibly raise the left hand and keep it away from the fingerboard, while the right hand holds the bow on the strings. Then the cellist, while playing on an open string, must be able to simultaneously produce its normal pitch *together* with some of its harmonics, just by precisely controlling the bow. In other words, both the fundamental vibration of the string and some of its partials must be excited. Here again one sees a single action, one bow playing on a single string, but two and sometimes more sounds being heard.

Fig. 3 Claudio Ambrosini, *Entartete Musik* (2017). The two *glissandos* move in opposite directions, using nearly the same notes, only a semitone apart (from the author's autograph score). © C. Ambrosini, 2017

4 Converging and Diverging Forms

Canons are a form of imitation used in music since the Middle Ages. An instrument, the "leader", starts playing and one or more "followers" join afterwards in turn, all playing the same (sometimes transposed) part but each with a certain delay, which is maintained throughout the piece so that the temporal distance of each participant at the end is the same as at the beginning. *Frère Jacques* and *Row, Row, Row Your Boat* are popular canons.

In the early 1980s, I conceived a new type of canon called a *Convergent Canon*, and its opposite a *Divergent Canon*. Based on the idea that each instrument imitates, as usual, the notes of the leader and enters after a delay, but this time plays at a slightly faster tempo (and even faster must be the tempo of the subsequent entry, and so on) so that all instruments ultimately converge on the same final note.

This idea is somewhat related to *Ziggurat* as in both cases a dense counterpoint is produced, each time different and unpredictable. The difference is that in *Ziggurat* the durations are "objective"—as they are determined by the very nature of the instruments—while in the Convergent Canon they depend on the *performance speed gradient*, or the increasing metronomic tempo subjectively chosen by each subsequent follower.

I dare not think how such processes could be transformed into mathematical calculations. Perhaps a possible analogy could be with an unusual funnel, somehow scaled as an inverted ziggurat. The upper diameter could represent the starting

Fig. 4 Claudio Ambrosini, *Dai* Filò *di Zanzotto*, for four female voices and piano, p. 23 (2003). A very short converging canon for two sopranos, mezzo and alto, who enter in sequence to sing "De fien 'pena segà dal faldin"("Of hay freshly cut by the scythe". Andrea Zanzotto, *Filò*, Mondadori Editore, Milano, 1988) and then meet on the last syllable "din" (from the author's autograph score). © C. Ambrosini, 2003

tempo, set by the leader instrument as the "liquid"—that is the music—starts to flow. The speed of the performance then increases step by step as the funnel narrows, until finally everything is reunited in the uniformity represented by the cylindrical neck—the general unison note—with which it ends.

Beyond calculations, the need to notate this process in a score presents some difficulties. *Dai* Filò *di Zanzotto*, for female voice quartet and piano, written in 2003 and inspired by the poems of Andrea Zanzotto, shows a simple intuitive rendering in order to have the four voices enter one after the other, sing the same part but each at a slightly faster tempo and then converge all together on the "meeting" note, corresponding in this case to the syllable "din" (Fig. 4).

Perhaps *Il satellite sereno* (The Serene Satellite, 1989), a piece for ensemble conceived as an "answer" to Bruno Maderna's *Serenata per un satellite*, presents a slightly more effective graphic solution, though this remains only an approximate rendering, since it is impossible to fix on paper what actually happens in each performance, how instruments mix when playing each at a subjective speed (Figs. 5, 6 and 7).

Fig. 5 Claudio Ambrosini, *Il satellite sereno* (for instruments, fixed media and live electronics, 1989), p. 40. The upper part of the page is a graphic explanation, showing when all other instruments must enter after the oboe—the leader in this case—has started. The lower part of the page is the actual beginning of the convergent canon. The marimba, which will be the last to enter, is still silent. (from the author's autograph score). © C. Ambrosini, 1989

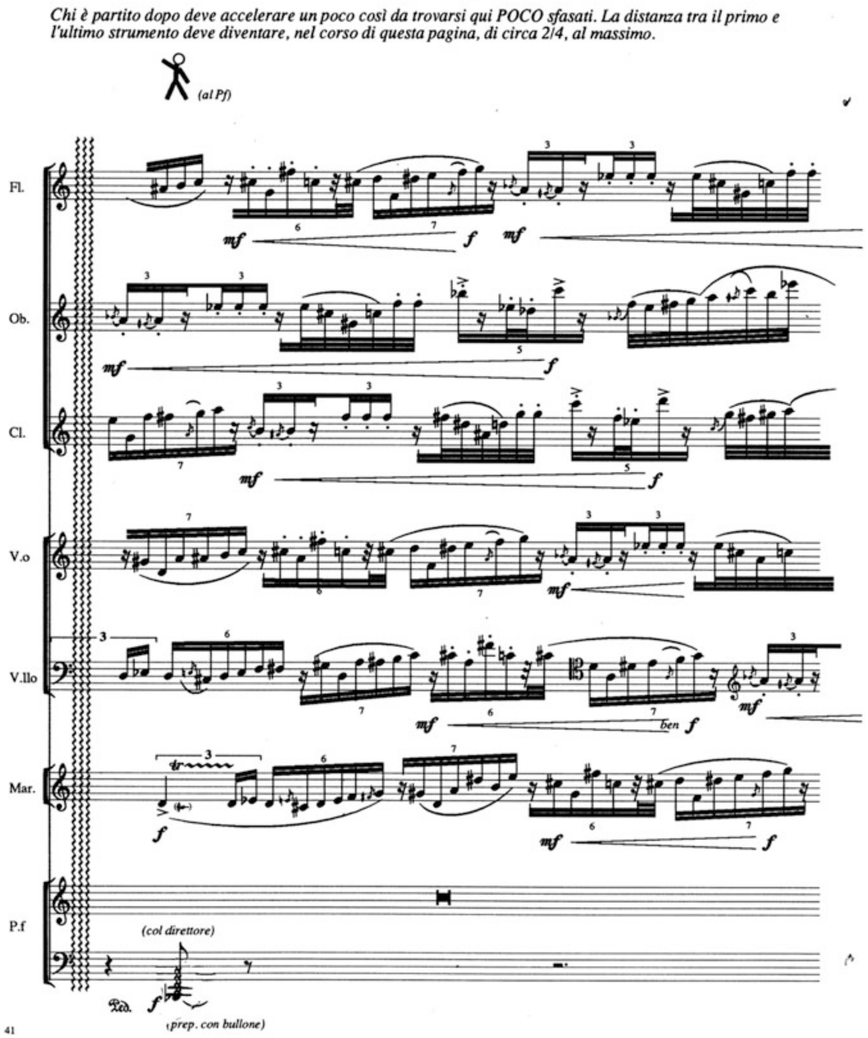

Fig. 6 Claudio Ambrosini, *Il satellite sereno,* p. 41. The marimba now enters playing the same notes as the leader (the oboe, see p. 40, Fig. 5) while the other instruments keep moving forward, each at its own progressively faster tempo. The three wavy vertical lines on the left are a signal to make it clear that the synchrony on this page is only indicative (from the author's autograph score). © C. Ambrosini, 1989

Another example, in this case of an expanding *and* contracting form, could be *Pandora librante*, a lyric-symphonic ballet for soprano, mezzo and orchestra, written in 1997.

It was inspired by Italo Calvino's *Six Memos for the Next Millennium,* a series of lectures on the following topics: lightness, quickness, exactitude, visibility and

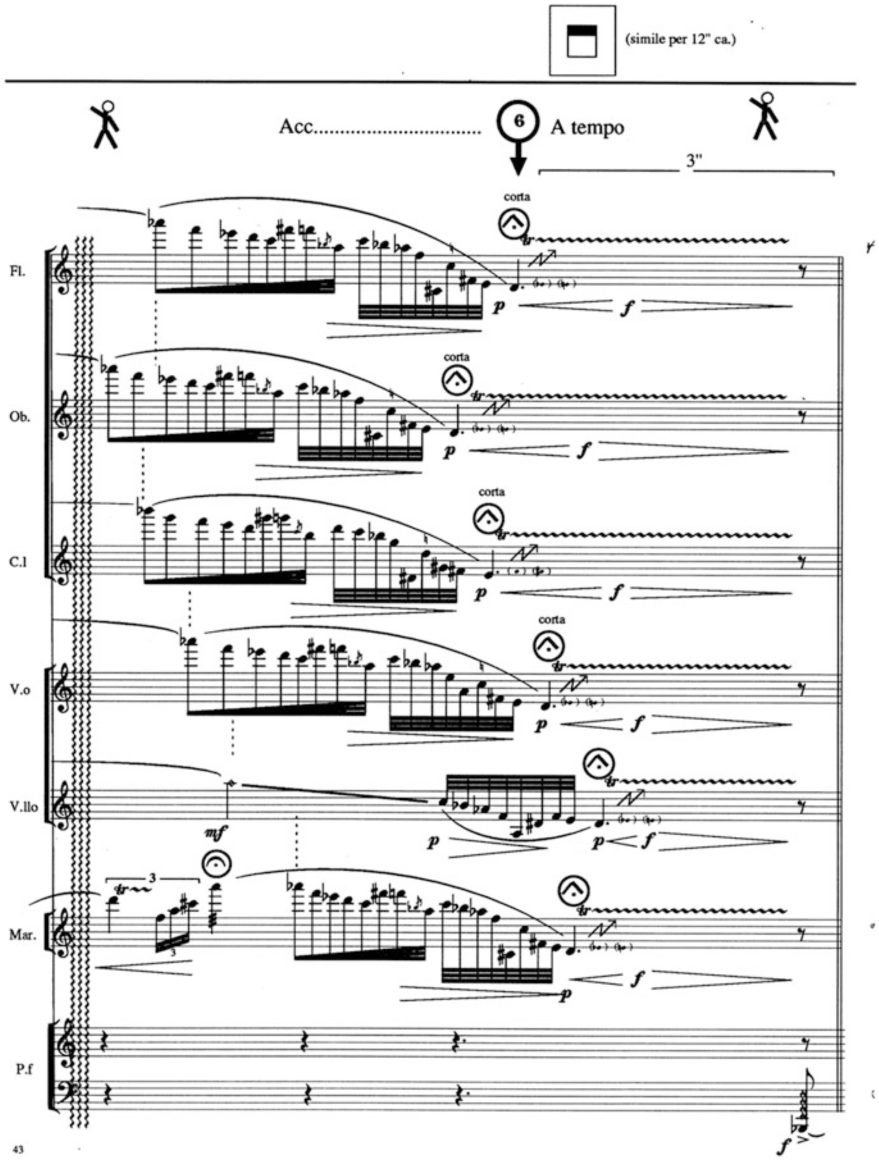

Fig. 7 Claudio Ambrosini, *Il satellite sereno,* p. 43. Here the vertical distance among the instruments is minimal, to show that the delay between them is very small and that they are in a moment converging on the trilled note of the second half of the page (from the author's autograph score). © C. Ambrosini, 1989

multiplicity (and consistency).[3] Calvino related them to literature but, after having read the book, I thought that these subjects could perfectly be applied to dance as well: should not dancers be light, quick, precise, visible, versatile and consistent?

Sketching the ballet I decided that the five large symphonic episodes (one for each topic) should not be separated by short silences, as usual, but by four short songs for the two female singers and just a handful of instruments. Or, in other words, I decided to alternate five large instrumental "frescoes" with four vocal "miniatures".[4] This generated the following expanding and contracting shape, something like a *breathing form*:

FRESCO 1		FRESCO 2		FRESCO 3		FRESCO 4		FRESCO 5
	miniature1		miniature 2		miniature 3		Miniature 4	

The repeated alternation of orchestra and chamber group—some 60 musicians and (maximum) 6—produces an effect that has to do not only with form but also with acoustics and perception.

Simplifying, adjectives normally related to a full orchestra could generally be: dense, powerful, colourful, massive, aggressive etc.; while to a small group like a string quartet they could be: clear, transparent, soloistic, light, attractive etc. Therefore, from the point of view of three-dimensionality, the full orchestral sound could also be considered perspectively extroverted, protruding, *convex*; and the chamber group sounds delicate, introverted and *concave*.[5]

What happens in this continuous alternation of thicker and thinner polyphonies is something like watching a large tapestry from a distance and then approaching for a close look at the single threads. On stage the space occupied by an orchestra or a chamber group is completely different: the sound reaches the audience from all directions when the former plays; from a few isolated points, or from a concentrated area, when the latter does.

In *Pandora librante*[6] after being reached, and sometimes overwhelmed by the sound of the orchestra, the audience subconsciously tends to pay a different, closer

[3] Calvino died before he wrote the sixth chapter, so his *Memos* are actually five.

[4] The first miniature actually sets to music Lucretius' verses quoted above, in the song *Il sole, la polvere* (The Sun, the Dust, for soprano, mezzo and 6 instruments). I have also recently composed a large piece, called *De Rerum Natura*, for three percussionists and electronically revealed environment (2019–2020).

[5] I have already covered this concept in: Escher-Like Perspectives and Music Composition (2002), in [3].

[6] *Pandora librante*. It is not easy to translate the title of this ballet conceived, besides the connection with Six *Memos for the Next Millennium*, also as a tribute to Calvino's literary output. It should not be difficult to recognize some of his famous books in the titles of the five orchestral episodes:

– *Andante leggero, quasi inesistente* (Light, Almost Nonexistent Andante)
– *Prestissimo dimezzato* (Cloven Prestissimo)
– *Misurato kaosmicomico* (Kaosmicomical Misurato)
– *Notturno, dei sentieri* (Paths, Nocturne)
– *Ostinato rampante* (Rampant Ostinato)

attention every time the small group begins to sing and play, transforming the ear into a sort of *acoustic zoom*.

5 Space Translation

The relationship between music and space is another aspect of the connection between music and mathematics.

In Venice, my hometown, there are several places that have been important for both architectural and musical experiments. One could be the "stereophony prototype" developed in St Mark's (and other churches) during the Renaissance by using the *cori spezzati*, or *cori battenti* technique, that is spatially separate vocal and instrumental groups performing from the left and right sides of the basilica. Another example could be the church of San Francesco della Vigna. In order to get a more "harmonically proportioned" building, it was partly designed adhering to proportions connected with the number three—thereby reflecting the Trinity—and natural musical intervals, such as the octave, the perfect fifth and fourth. For example, the best ratio between width and length of the nave would be: 9 to 27, which corresponds to the ratio between octave (9:18, or 1:2) and fifth (18:27, or 2:3).

In 1976, while I was a student of Electronic Music Composition at the Venice Conservatory, I thought to try to *translate* the whole classroom where our lessons were held, *into music*, according to an arbitrary but as coherent as possible code and, above all, fully derived from the architectural features of the space itself. Since the room (*aula*, in Italian) bore the number 104, the piece would be titled *Aula 104*.

As soon as I set to work on this idea, I was surprised by the encouraging dimensions of the classroom. It was a trapezoidal space with square or rectangular walls, and it was characterized, among other things, also by a curious *comb filtering* of the sound.[7]

Its measurements were as follows:

– Right wall: m. 6.28 × 3 (or 3.14 × 6)
– Wall with window: m. 3.14 × 3.14
– Left wall: m. 7.20 × 3.14
– Wall with door: m. 3.14 × 3.14
– Ceiling and floor: m. 7.20 × 6.28 × 3 (again 3.14 × 6).

The recurring presence of π added an unexpected reference to circularity, which could be connected with other acoustic features, such as the concentric waves of sound propagation.

So "Pandora librante" is a word pun as well: it could be translated as Soaring Pandora, but in Italian the word *librante*—actually a neologism—recalls both the verb *librarsi* (to soar) and the word *libro*: book.

[7] A comb filter is an electronic audio device that produces a series of regularly spaced notches within the shape of a sound wave, recalling a comb. In certain conditions (and spatial proportions) something similar can be produced by architecture as well.

Apart from the shape and size, it was built with different materials: herringbone parquet floor, two walls covered by plaster only, two walls and ceiling covered with irregularly perforated insulation panels, a door, a window. I thought that the function expressed by these materials (absorption, filtering, reflection etc.) could be recalled by the use of the electronic equipment of the studio, employing generators, filters, modulators, reverbs, faders etc.

It would take too long to go into detail but, just to give an idea, the 1920 slats of the herringbone parquet have been "transposed" using triangular electronic waves variously filtered (to match the various shades of the wood) and then mixed down into the multi-layered sound which represented the *Floor Episode*. Ultimately, the six components of the room's architecture corresponded to six musical episodes, whose duration was proportional to their physical area. The corners, as elements connecting two different walls, have also been taken into consideration, and several other aspects such as the white colour of the walls, which could be connected with White Noise.[8]

The whole translation aimed at giving life to an "objective", lifeless representation of the environment, which was then to be contrasted by a "human presence", represented by some subjective, emotional solos, which I played on a crumhorn. I thought that the choice of such an ancient instrument could represent the opposite dimension, the totally contrasting compositional and sound element to balance the whole project. And it would also work as a symbol of the wonderful Renaissance palace that houses the Conservatory in Venice.

In 1976, I tended to consider *Aula 104* mainly as a school essay, but in 2017 I had the great pleasure of seeing it included in the selection of works from the 1970s presented by the Boston MIT publication *Leonardo Music Journal Audio Series* n.27 (Fig. 8).

Finally, one more experiment: the use of some mathematical symbols to indicate the function that a certain instrument may have within an orchestration.

I consider the score not only as a "grid" for reading, synchronizing and controlling all the sound elements of a piece, but also as a *laboratory* in which each instrument has one or more functions: as mentioned above for *Aula 104*, it may work as amplifier, equalizer, reverb and more. Graphically highlighting this functional approach might be particularly useful with the large score pages of some productions (Fig. 9).

Therefore, line by line and page by page the composer can indicate, underlining, or circling, or adding himself the most suitable symbol, which is the momentary predominant function(s), making the study of large scores easier for conductors or musicologists. Pieces like *Proverbs of Hell* are dense, complex works, which may have some 50 staves on each page.

[8] White Noise, which draws its name from White Light, is produced by the random sum of theoretically all audio frequencies at equal intensity. The sound is very similar to that of a large waterfall.

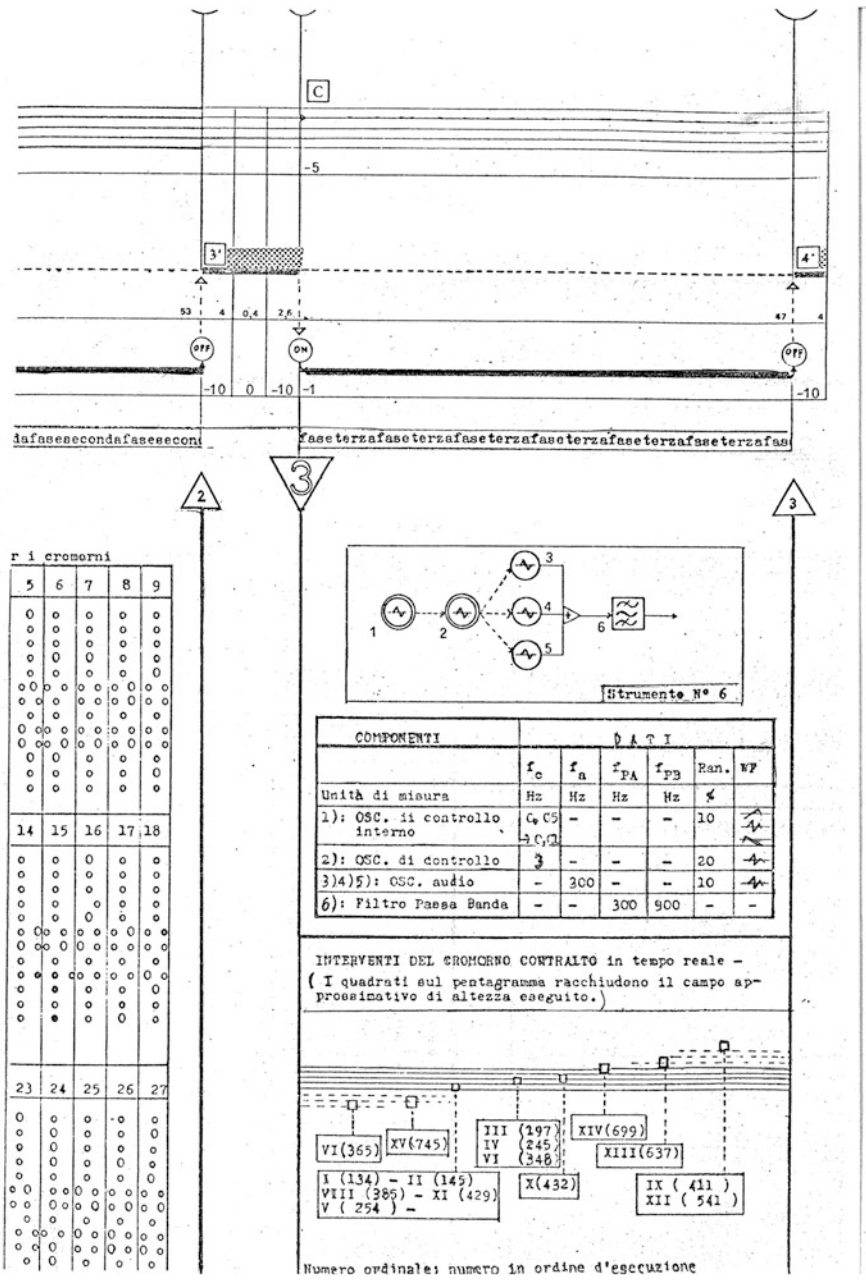

Fig. 8 Claudio Ambrosini, *Aula 104* (Classroom 104, for fixed media: crumhorns, wooden recorders and electronics (1976). The table of vertical tiny circles on the left is a "translation" of the perforated insulation panel holes. It worked as a "tablature" for the fingerings to be used by the recorder and crumhorn players. (detail from the author's autograph score). © C. Ambrosini, 1976

Fig. 9 Claudio Ambrosini, *Proverbs of Hell*, cantata for soprano, alto, tenor, bass, piano, percussion, mixed choir and large orchestra (1990–1991), text by William Blake. String section, page 36. The symbols to the left of each staff aim to intuitively confirm (besides the notes written in those measures) what the composer's intention is at that particular moment: %, <, ≤, =, ≈, ≠, ≥, >, (), &, $,! etc. (from the author's autograph score). © C. Ambrosini, 1991

6 Time

Time is a composer's worst enemy and, as I said before, I have even written a couple of surreal pieces suggesting how to kill it! "Time is unredeemable" T. S. Eliot wrote. Time keeps running inexorably, second by second and what a composer absolutely needs is to manage to catch it, and tame it. It is amazing how much a music fragment can change if one of its notes lasts a fourth, or a sixth, or an eighth of a second. Or if a pause lasts a third instead of a fourth of a second. Minimal yet audible differences. So, it is crucial for a composer to be able to accurately transform the music heard in the mind into precise symbols on the score.

And this is perhaps another thing that musicians and mathematicians have in common: the need to write, to fix solutions and results on paper. And there is probably as much satisfaction in seeing a complex sound event forever "caught" in a score as it is for mathematicians to see their hypotheses and conjectures transformed into successful calculations, theorems, proofs, formulas and discoveries. How delighted I was when, in Einstein's notebook, I saw the pages full of math symbols and operations preceding his universally known formula. Totally incomprehensible for me, but "music" for a mathematician's eye.

Well, it seems to be true that music and mathematics are connected and that composing is also a more or less conscious computation. Completely unconscious at times, since I could not really tell where ideas come from. Anyway—sounds and numbers: how well they can dance together!

Acknowledgement I would like to thank Michele Emmer for the invitation and wish him all the best for his very "musical" 75th birthday (7 as the white keys on a piano and 5 as the black ones . . .).

A detailed list (by date) of the works discussed in the paper:

Canzone molle, for guitar (undated, early 1970s).

Ziggurat, for freely decaying instruments' ensemble (1975).

Aula 104, for fixed media (crumhorns, recorders and electronics, 1976).

A Sound a Day Keeps Time Away, a calendar to compose (for any instrument) (1977).

Three Holograms, for guitar (1978).

Nell'orecchio di Van Gogh, una pulce, for grand piano, upright piano (one pianist) and 7 instruments (1983).

Il satellite sereno, for 7 instruments, fixed media and live electronics (1989).

Proverbs of Hell, cantata for soprano, alto, tenor, bass, piano, percussion, mixed choir and large orchestra, text by William Blake (1990–1991).

Pandora librante, lyric-symphonic ballet for soprano, mezzo and orchestra (1997).

Il sole, la polvere, for soprano, mezzo and 6 instruments (1997).

Big Bang Circus (Piccola storia dell'universo), circus-opera for soprano, mezzo, tenor, bass, actor and 16 instruments (2002).

Dai Filò *di Zanzotto*, for female voice quartet and piano (2003).

Il suono e il suo doppio, for solo cello (2004).

*Classifying the Thousand Shortest Sounds in the World***,** for solo flute (2012).

Tic-tac (ossia come ammazzare il Tempo), for a metronome and a percussionist (2013).

Tecniche per la misurazione dell'infinito, for three pianos (2014).

Entartete Musik, for guitar (2017).

Etymon n.6 (Fearful Symmetry), for any group of instruments (2018).

Orienteering, for one or more electronically revealed pianos (2019).

De Rerum Natura, for three percussionists and electronically revealed environment (2019–2020).

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Euler and Music
Musing Euler's Identity

Davide Amodio

1 Introduction

We would like to explain the reasons that led us to this project and, through it, how the functioning of music can interact with mathematics. It has always seemed reductive to us to act only in the context of our discipline. Already from the entry of music into the Quadrivium[1] it was understood that music could be studied and understood as a branch of mathematics. Through an interdisciplinary approach each study deepens further, naturally with the help and collaboration of experts in the various fields of study. Today, specialization takes us further and further away from this fertile and creative vision. Today it would be unusual for a mathematician like Euler to write a treatise on music from a mathematical point of view and, at the same time, to give indications to musicians on how to compose, how to tune the

Musica est exercitium arithmeticae occultum nescientis se numerare animi (*Music is an occult calculation of the soul that does not know how to number*, Leibniz [9]. The passage quoted is contained in a letter to Christian Goldbach dated April 17, 1712. For details on the relationship between Leibniz and music, cf. Sguben [13], pp. 83–88.)

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[1] *Quadrivium*, in Latin literally four ways, in medieval times, indicated, together with the *Trivium*, the scholastic training of the liberal arts, preparatory to the teaching of theology and philosophy. It included four disciplines attributed to the mathematical sphere: 1. arithmetic 2. geometry 3. astronomy 4. music. This subdivision is due to Marziano Capella, a late Latin philosopher (4th–5th century AD) who took care, among other things, of dividing all human knowledge into categories.

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Fig. 1 Euler, watercolor by Edoardo Amodio

instruments and, above all, to establish which are the intervals that produce pleasure for listeners (Fig. 1).

In a scene from the film *Il Giovane Favoloso*[2] about Leopardi's life, the poet sits on the hill and recites the famous poem *L'Infinito*, as if he was receiving his inspiration right at that moment. For obvious screenwriting needs, the hard work of poetic composition does not appear in the film. The inspiration aspect certainly exists and is important, but it is only a small part of any artistic endeavour. The construction of an organism, be it musical, poetic, or pictorial, requires coherence, balance, inertia, proportions, and many other things. As far as the musical field is concerned, we can observe that the methods and procedures used by the composers adhere to the needs and reflect the taste of their times. The word *composition* itself comes from the Latin *cum ponere*, to put together. Throughout the eighteenth century and a good part of the nineteenth century, composing meant putting together "building blocks" (*Satzmodelle*) that had different origins and provenance, united by the fact that they were free from authorial constraints and, therefore, in the public domain.[3] But also in other times[4] something similar happened: for the inauguration

[2] Film of Mario Martone, 2014.

[3] See Giorgio Sanguinetti, Preface to the Italian edition of: *La Musica nello stile Galante*, R. O. Gjerdingen [12], p.11.

[4] The 25 March 1436 consecration of the Florence Cathedral, on the occasion of the completion of the dome built under the instructions of Filippo Brunelleschi.

of the Dome of Santa Maria del Fiore in Florence, by Filippo Brunelleschi, we saw the close collaboration between Brunelleschi himself and the music composer Guillaume Dufay, who knew how to interact so well with the architect that he was able to harmonize the geometric proportions of the Dome with the isorhythmic motet he composed for the occasion: *Nuper Rosarum Flores*. Motet that contained both elements of the ancient Gregorian repertoire, as well as new elements of counterpoint techniques, which Dufay later continued to develop. The composer has always used schemes, building blocks, and relationships with realities that do not necessarily belong to the musical field, creating organism made up of elements which can be found in different spheres of knowledge.

Parallel to the craft of composing, the practice of tuning and the pleasure of listening have always accompanied musicians of all times. These two aspect are closely intertwined. There is a problem with acoustic physics, the nature of sound, and its resulting harmonics. To simplify, we will say that an interval obtained with the sum of fifths is greater than the same interval obtained with the sum of octaves. This has forced musicians, over the centuries, to continually find solutions that harmonize the notes with each other in a continuous and coherent way. This represented, and still represents, the real problem not just in tuning the instruments but also in the art of intoning and that of playing different instruments, for example, keyboards and string instruments. Between the end of the sixteenth century and the beginning of the eighteenth century, numerous illustrious mathematicians devoted themselves to these problems, writing entire treatises on these subjects. We can think, for example, of Kepler,[5] Mersenne,[6] Descartes,[7] Kircher,[8] the multifaceted character of Caramuel,[9] and Euler.[10]

2 Euler and Music

Leonhard Euler was born in Basel on April 15th, 1707, and died in Petersburg on September 18th, 1783. He completed his studies in his hometown, soon combining his humanistic training with scientific training, by attending the private lessons with Johann I Bernoulli. In 1727 he was invited to join the newly formed Petersburg Academy of Sciences, thanks to the intercession of Bernoulli's sons, Nikolaus II

[5] I. Harmonices Mundi Libri V, Lincii Austriae, sumptibus Godofredi Tampachii excudebat Ioannes Plancvs, 1619.

[6] *Harmonie universelle, contenant la théorie et la pratique de la musique*. Par F. Marin Mersenne [2], pp. 1636–1637.

[7] *Abrégé de musique*, par Monsieur Descartes, 1668.

[8] *Musurgia universalis sive Ars magna consoni et dissoni in X. libros digesta*. Par Athanasius Kircher [4], 1650.

[9] Caramuel Lobkovwitz [5], Juan 1606-1682, *Primus calamus secundam partem metametricae exhibens.*

[10] De Piero [8].

and Daniel, who had reached the capital in 1725. In 1741 he accepted the invitation of Frederick II, recently crowned King of Prussia, to go to Berlin to collaborate in the foundation of the Academy of Sciences. Unfortunately though the second war with Silesia, which lasted until 1745, was an obstacle to this project, which eventually took place only in 1746. In 1762 the accession to the throne by Catherine II also favoured the return of Euler to Russia, which happened only in 1766, with the definitive return of the Swiss mathetatician to Petersburg, the city where he remained until his death.

Although he avoided the worldly events that were organized, at least during the peaceful times, at the court of Frederick the Great, he certainly had contacts with Carl Philipp Emanuel Bach, then court harpsichordist, and with Johann Joachim Quantz, great flutist and theorist, who in those years was the king's teacher and a central figure in court music *genre*. Even in Petersburg, the enlightened government of Catherine II made the court employ some of the most famous composers of the time, from Baldassare Galuppi known as Buranello, "Master of the Ducal Chapel of San Marco" in Venice, who stayed there from 1765 to 1768, to the musicians belonging to the Neapolitan school: Tommaso Traetta who after Galuppi, stayed from 1768 to 1774, and Giovanni Paisiello, who remained from 1776 to 1783, having staged in the previous year his famous *Il Barbiere di Siviglia*, taken from the play by Beaumarchais. Also, in this case, Euler must certainly have known these personalities and it is unlikely that he did not feel the need to engage with them. He also had some correspondence with two of the greatest composers who at the time also dedicated themselves to music theory, namely Jean-Philippe Rameau and Giuseppe Tartini. However, the impression one gets from reading *Tentamen novae theoriae musicae* is that of a treatise written by a scientist who elaborates his own theoretical system aimed at justifying the pleasure of listening to music; but who, at the same time, while quoting an essay by the famous theorist and composer Johann Mattheson, does not seem too interested in arguing with the so-called "practical musicians" or performer-musicians, whom in this view were not equipped enough to understtand or appeciate his theories.

3 Euler Seen by an Eighteenth-Century Musician

Francesco Galeazzi (Turin 1758-Rome 1819), violinist, conductor, and musicologist, as well as a true lover of mathematics, knew Euler's treatise well; he preferred it above all the others mentioned so far. In his brilliant book on the art of playing the violin[11] and composing music, he quotes a paragraph from Euler's treatise, but not before having expressed his opinion on many others:

[11] Galeazzi Francesco [10], p. XXV. Euler's quote refers to *Tentamen*... (op. Cit.) Chap. II paragraph 4.

Sembrerà certamente strano, che dopo tanti egregj, e sublimi trattati di Contrappunto, voglia anch'io correre coraggiosamente una sì battuta, ma sempre difficile carriera. S'abbonda è vero di ottimi, e di assaissimi libri di questa Scienza: ma dicamisi di grazia, dopo che un Principiante, Suonatore o Cantante che egli sia, avrà avuta la pazienza di leggere il Zarlino, il Gaffurio, il Doni, il Fux, l'Artusi, il Penna, il Tevo, il Mersenno[12], il Tartini, il Rameau, ed infiniti altri egregi Autori, cosa mai ha egli imparato? Al più al più qualche ben leggera cognizione Teorica, senza affatto sapere, come adattarla alla Pratica: la maggior parte di questi eccellenti libri riechieggo delle cognizioni matematiche pur troppo poco familiari a molti Professori di Musica (. . .).

It will certainly seem strange that after so many excellent and sublime treatises on counterpoint, I too wish to bravely undertaken such an endeavor, which always remains an arduous one, despite being a path tried by many. It is true that there are plenty of excellent books on this Science: but please tell me, after a Beginner, Player or Singer may he be has had the patience to read Zarlino, Gaffurio, Doni, Fux , the Artusi, the Penna, the Tevo, the Mersenno, the Tartini, the Rameau, and countless other distinguished authors, what has he ever learned? At the most some very superficial Theoretical knowledge, without knowing at all, how to adapt it into Practice: most of these excellent books reflect some mathematical knowledge which is regrettably too unfamiliar to many Music Professors[13] *(. . .).*

It is strange that Galeazzi complains about the need to know mathematics very well for the previous treatises and has no difficulty in reading and studying Euler's one which, on the other hand, is a treatise that requires in-depth knowledge in the field of mathematics and music. Here is a quotation from him:

(. . .) ho inteso più volte co' miei orecchi vilipendere, e disprezzare le stupende composizioni di Haydn, di Paisiello, di Cimarosa ec. Ma che perciò? Toglie forse questo uno zero all'illustre merito di questi inimitabili genj? Odasi il grande Eulero e colla di lui scorta si disprezzino questi Aristarchi, e solo si badi all'osservanza dei precetti dell'arte, da cui sicuramente tutto l'esito dipende di ogni buona Musical composizione**:**

(Quote by Galeazzi from the Euler Treaty). Ma è necessario che il musicista si comporti come l'architetto che non curandosi dei giudizi stravaganti dei più sugli edifici, edifica l'abitazione secondo leggi certe e fondate nella stessa natura; che anche se non piacciono a chi ignora queste cose, tuttavia, purché siano apprezzate dai competenti, è contento. Infatti, come nella musica, così anche nell'architettura il gusto delle diverse genti è tanto diverso che le cose che ad alcuni piacciono, altri le respingono. Perciò come in tutte le altre cose, così anche nella musica è necessario seguire quelli il cui gusto è perfetto e il cui giudizio sulle cose percepite dal senso è libero da ogni pregiudizio. Sono di tal sorta coloro che non solo ricevettero dalla natura un udito acuto e puro, ma anche coloro che percepiscono esattamente le cose che sono rappresentate nell'organo dell'udito e, esaminandole tra sé, ne riportano un giudizio completo.[14]

(. . .) I have often heard with my ears someone vilifying and despising the wonderful compositions of Haydn, Paisiello, Cimarosa and so on. But so what? Does this take any credit away from the illustrious merit of these inimitable

[12] Here he means F. Marin Mersenne [2].

[13] Galeazzi [10], p.VI (translation by the author).

[14] See note 12.

geniuses? Hear the great Euler and despise these Aristarchs, and only pay attention to the observance of the precepts of art, on which surely the whole outcome of any good musical composition depends upon.

(Quote by Galeazzi from the Euler Treaty). But it is necessary for the musician to behave like the architect who, ignoring the extravagant judgments of many on the buildings, builds the house according to settled laws which are governed by the principles of nature; and he shall be satisfied, regardless of those who are ignorant of these things and don't like his creations, as long as they are appreciated by the competent ones. In fact, just as in music, in architecture too the taste of different people may vary so greatly that the things that are liked by some, are rejected by others. Similarly in music too it is necessary to follow those whose taste is perfect and whose judgment on the things perceived by the senses is free from any prejudice. Falling under this category are not only those who are born with acute and pure hearing capabilities, but also those who have the ability to perceive sounds exactly as they are detected by the hearing organ and, by examining them, can make a complete judgment on them.

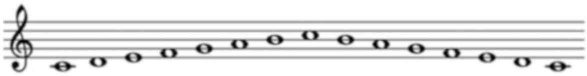
4 Musing the Euler Equation: From Equation to Counterpoint

Before describing in detail my research on the interconnections between Euler's identity and my own musical composition, I should like to give some information on those musical features which, though may be less apparent, music possesses in its own nature, both as an artistic-aesthetic language and a geometric-sound proportion.

The musical scale, as can be seen in Fig. 2, is a succession of ascending and descending notes like the steps that allow to go up and down. However, this is not the case, even if it is called a scale and all the notes seem the same. All major and minor scales have established distances which consist of five tones and two semitones. However, here we are not dealing with the distances between one note and another, (also because the same tones, that is the second interval, actually have different magnitudes from note to note), but we are in fact investigating the latent functions, that each note has and which all notes can take from time to time within a given scale. The notes, therefore, look more like atoms than steps; the degrees of the scale (in music they are so called) have not only different functions but properties that can vary, up to the opposite degree, depending on the given harmonic context. The names of the degrees of the scale, listed in order from first to seventh, are: the *tonic*, the *supertonic*, the *mediant*, the *subdominant*, the *dominant*,[15] the *submediant*

[15] Schönberg writes about the dominant term: To tell the truth, the expression *dominant* for the fifth degree is not entirely correct, because this name suggests that this agreement "dominates" one or more others. (. . .) The name of "dominant" is usually justified by the affirmation that the first degree is introduced by the fifth, so that would be a consequence of this. (. . .) to follow means

Fig. 2 The musical scale of C major



and the *leading tone*. The first degree of the scale is called *tonic* because it gives its name to the scale, it defines the tonality but, above all, it is the starting and ending note and, therefore, it is a static, resting note. Let us consider, for example, of the first and eleventh syllables of a hendecasyllable in Italian poetry, syllables are never stressed (except rarely for the first one). The second note, the *supertonic*, would seem transient and of little interest, but actually thanks to its latent function it can easily become a secondary dominant (dominant of the dominant) and thus acquire a quality of strong tension, so long as it announces (as Schönberg states in his treatise on Harmony) the next tonic. So below the third (the *mediant*) establishes whether the scale is major or minor. The first, third and fifth are the minimum and necessary number of notes to establish a chord.

These are the basic functions but, as mentioned earlier, there can be numerous latent functions that are set in motion by the harmonic context. The most striking one is the transformation of the first degree of the scale into the fifth degree of its fourth. In this process the static *tonic* becomes the *dominant* note, which represents the movement and attraction of tensions, and thus resulting in the relaxation of the new tonic. The third degree note, which defines whether the scale is major or minor, can turn into a dominant chord of the sixth degree, which is nothing else than relative minor of the *tonic*.

All these tensions and release, which come into being as a result of small transformations, are just a layer under the surface of the musical *discourse*. In fact in musical language there are many other components that come into play both upstream in the composer's mind and downstream in the performance by the musician who is familiar with all these components.

These properties, which we have only touched here, are employed in the tonal language and can be found across the centuries, particularly in the classical era. However, even in other musical styles, the music discourse always follows norm and rules that establish a precise relation between the different parts of a complex organism and that realize the same potential, latent or manifest modalities, of which we spoke earlier. The notes are grouped into coherent systems which, in turn, are interconnected within simple or complex structures. In this way, it is possible to interact with the "sound matter," adapting existing ideas and norms which do not even belong to the musical discourse/grammar (Fig. 3).

to obey but also to align, to come later: and if the tonic "follows" the dominant it is like when a king lets himself be preceded by his vassal, the master of ceremony and the quartermaster, so that they make the necessary preparations for the entry of the king who follows them: but the vassal is there for the king, and not vice versa. (Schönberg [11], pages 41–42, (translation from Italian by the author).

Fig. 3 A. Schönberg, example from preparatory series for the violin concerto, with relative connections (Example by the Author)

The groups of notes, about which we spoke earlier, are called *figuration* and are the construction material that rests on some pre-constituted *bricks* or in a contemporary style with the author or by the author himself. The figurations are small or medium groups of notes which, being recognizable, weave the musical discourse and can be changed, producing both the development of the discourse and variety and richness in the musical organism. The possible development of a musical figure can take place in many ways. In this case too, the important thing is to understand that the figures are not mere notes having a predetermined order, but have their own internal structure, more or less simple, and have precise and variable functions. Just as the poet does not limit himself to following the grammar but also builds the sentences according to the rules of metric, seeking linguistic homogeneity and fluidity in his *discourse*, so the music composer chooses the notes bearing in mind the musical structures and functions combined with a sense fluidity that enables him to be both inventive and creative, and, above all, coherent. Poetic language is, in fact, much closer to music than to the language from which it comes.[16] The poetic technique that has been used in Italy for centuries, in addition to the metric and rhythmic properties of each poetic form, involved the use of symmetries, anagrams and transformations of groups of letters to change meanings while maintaining the same sounds. Just to quote an example: … …*a l'ultimo lavoro/fammi del tuo valor* (to the last job / tell me about your value)[17]: here we have an anagram. The intent is to express different meanings by employing homogenous sounds; in the first canto of the Paradise we find symmetries from the first verse: *La gLOria di cOLui che tutto move*: here we have the syllable LO of *gloria* with

[16] Euler wrote: *It is also absolutely necessary that a musical work resembles a prayer or a poem. As, in fact, in these it is not enough to join elegant words and phrases, but there must also be an orderly arrangement of the same things, and an appropriate distribution of the topics; so also in music there must be a similar principle. In fact, it is not very nice to have several consonances placed in series, even if individually they are quite pleasant, but it is necessary that the order be distinguished in these, just as if they were to express some prayer. In this problem it is especially useful to pay attention to the degree of ease or difficulty with which the order is perceived; and depending on how the established object requires, the joy and sadness will have to be changed, or now this, now that, will have to be increased or decreased.* L. Euler, op. cit. p.87, (translation by the author).

[17] Dante: Paradiso Canto 1 versi 13/14.

OL of *colui* (The glory of Him who moves everything); than again the vowels O-U of *cOlUi* mirror the opposite vowels U-O of *tUttO*. Furthermore, in the first verse there are two letter "e" while in the second verse there are seven 'e'. Examples like these can be found from Petrarch to Pascoli, passing through Tasso or Leopardi . . . all the lines are rich in homophonies, echolalia, contrasts, symmetries and many other para-textual elements. Leaving poetry aside, which is in any case a privileged meeting point for music and mathematics (see D. Amodio [14]), let us now look more closely at the mathematical and musical systems and explore the connections they share.

Notes are written on one or more staves, which function in a similar way to the Cartesian plane. Notes are graphically represented by points; their pitch and duration are determined respectively by their position in the stave and their shape; the order in which they are executed follow the reading from left to right; however what is most important here is their relationship with the vertical dimension which establishes their harmonic context. The vertical function is to be considered as a third dimension with respect to pitch and duration. On many istances the horizontal function by changing its nature becomes the same as the vertical one. The direction of the notes, taken as a group and not individually, can be reversed or mirrored or, again, transposed in height, up or down. These movements therefore are functional to the representation of what Euler's equation expresses.

5 Euler's Equation

The Euler equation has not only a scientific value but also an esthetic and cultural one in a broad sense. In music, on the other hand, there are also very short and effective themes, motifs, melodies that not only have an esthetic and cultural value, but also a broad scientific value; we could call them musical formulas, which have a very high communicative impact. For example, the two notes (the first repeated three times and the second long ad libitum) at the beginning of Beethoven's Fifth Symphony convey in a few seconds the substance, style, and overall language of the whole symphony, but also of the author himself and his time (Figs. 4 and 5).

Let us now consider Euler's equation.

From a mathematical point of view, the elements that appear in Euler's identity are 5: e, π, i, -1, 0.

The first thing we notice is that many of the fundamental entities of mathematics appear, one after the other, as in review: the Neperus constant ($e \sim 2.7182818 \ldots$), the value of π ($\sim 3.14159265 \ldots$), the imaginary unit i (square root of -1),

$$e^{i\pi} + 1 = 0$$

Fig. 4 Euler's Identity as we know it at the present moment

Fig. 5 Beethoven V Symphony First Edition, from Petrucci website

the number 1 and the number 0. Even from the historical point of view, these mathematical concepts emerged across different cultures and developed over the centuries making the history of mathematics: the golden period of Greek geometry (constant π), the influences of Indian mathematics, which first introduced the concept of zero, the Italian Renaissance debate between Tartaglia and Cardano on the resolution of the third degree equations (imaginary unit i), the birth of logarithms in the time of Napier (constant e) and finally the number 1, omnipresent in all cultures and in all times.

- 1: it is the neutral element of the product in real numbers; its opposite -1 is a negative integer.
- i: the characterization of i,[18] called imaginary unit, is given by the fact that its square is equal to -1. Thanks to the properties of ordering on real numbers, it can be proved that there are no real numbers with negative squares; for this reason, i is called imaginary, while the term unit recalls the fact that its module is 1 (or, if you prefer, that its fourth power coincides with 1). Since $i^2 + 1 = 0$, the imaginary unit is an algebraic number, that is the root of a polynomial with integer coefficients. An interesting curiosity in the history of mathematics is the fact that already in the mid-1500s Cardano and Tartaglia had the concept of imaginary unit (seen as a number whose square was equal to -1, or rather as a number whose square added to 1 gave 0). Let us recall that in those times in Italy the so-called "syncopated" notation expressed through symbols was not yet in use in the writing of equations, but on the contrary was used the notation known as "rhetoric", in which the equations were described by sentences written in plain language.
- e: The Napier number[19] (in Europe also known as Euler's number) is introduced, in a rather cryptic way, in 1618 in a series of logarithmic tables calculated by the English mathematician and physicist John Napier. Its importance from the point of view of abstract mathematics is due to the fact that the derivative of the

[18] The first use of the symbol i to denote the imaginary unit is in a text of 1777, which Euler addressed to the Academy of Sciences of St. Petersburg and which was published posthumously in 1794 in one of the volumes of the *Institutionum calculi integralis*.

[19] Euler has been the first to use the letter e to denote Napier's number in a short treatise, *Meditatio in Experimenta explosion tormentorum nuper istituta* (Reflection on experiments recently carried out on shooting with cannons) he wrote towards the end of 1727 or the beginning of 1728 (when he was 21 years old).

exponential function with base e coincides with itself; this uniquely identifies it within the set of real numbers. The limit of the sequence as n approaches $+\infty$ is the way in which it is usually introduced in study courses (both high school and university courses). Since this sequence is increasing and is bounded above by 3, its limit exists and is finite; an approximate value is given by 2.7182. The major flaw of this definition is the fact that convergence is extremely slow: it is necessary to get to $n = 135$ to find only 2 decimal digits, to $n \approx 1400$ to find 3 and to $n \approx 14{,}000$ to find 4. A different viewpoint that allows a quicker convergence is obtained via power series. Indeed, we can see e as the sum of the power series $\sum_{n=0}^{+\infty} \frac{1}{n!}$ whose rate of convergence is much faster than the one given by the above-mentioned limit ($n = 6$ gives 3 decimal digits, $n = 10$ gives 7 digits and $n = 15$ so much as 13 digits!). This formula can be generalized by setting $e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$, which defines the exponential of base e as a holomorphic function defined on the complex plane. Since $i^2 = -1$ we can then write $e^{iz} = \sum_{m=0}^{+\infty} (-1)^m \frac{z^{2m}}{(2m)!} + i \sum_{m=0}^{+\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!}$. We then set $\cos z = \sum_{m=0}^{+\infty} (-1)^m \frac{z^{2m}}{(2m)!}$ and $\sin z = \sum_{m=0}^{+\infty} (-1)^m \frac{z^{2m}}{(2m)!}$, thus obtaining

$$e^{iz} = \cos z + i \sin z.$$

- π:[20] A qualitative study of the sine function (defined via power series as described above) proves that it assumes positive values on reals in the range between 0 and 3, while sin4 < 0; this allows us to define π as the first positive zero of the sine function. The calculation of the values of the sine and cosine function is simplified thanks to the fact that for real positive x, the power series that defines them is alternating and that the Lagrange remainder allows a very accurate estimate of the error. Though this definition is not the one more commonly adopted (which sees π as half the length of a circumference of radius 1), nonetheless it is the correct one from the foundational point of view. The fact that the cosine function is the derivative of the sine allows us to prove that $\sin(x + \pi) = -\sin x$ and $\cos(x + \pi) = -\cos x$ for every real x (and therefore also on the complex field); this implies that $\cos \pi = -1$, $\sin \pi = 0$ and that the period of the sine function (and also of the cosine) is 2π. By setting $z = \pi$ in the above-displayed formula we obtain therefore $e^{i\pi} = \cos \pi = -1$, that is, Euler's identity.[21]

[20] Euler popularized this symbol by using it in the *Introductio in Analysin Infinitorum* of 1748 (previously he often used the letter p).

[21] Many thanks to Chiara de Fabritiis for the pleasant explanatory conversations.

6 The Number of Napier as a Musical Theme and Sine and Cosine as a Mutation of a Musical Canon

For what has been said so far, the five symbols represent an extended mathematical world: how might we represent it musically? The first correspondence we can establish is based on Napier's number. The simplest way to musically represent an irrational number that contains numerical sequences seemed like this: composing groups of noteswhich share a system of internal relationship that imitates the system of the number *e* (Fig. 6).

As can be seen in Fig. 5, we start from an octave interval, which Euler in his treatise on music represents with the number 2. The successive pairs of notes are the result of two contrasting forces that generate a sequence of chromatic ascents and descents, with ever smaller intervals. The sequence represents the number *e*, intended as a limit, obtained in twelve subdivisions. The intervals inside the curly brackets symbolize Napier's number. By adding the last couple of notes that share the same sound (therefore the symbol of the number 1) we arrive at silence (the number zero). The sequence of notes inside the brackets as seen in Fig. 7 was eventually elaborated through a series of changes produced by the imaginary number and pi within a circle.

The indications given by the sine and the cosine shorten the theme, raising the pitch of the notes, until arriving, in the first quarter of a circle, at a single note, at

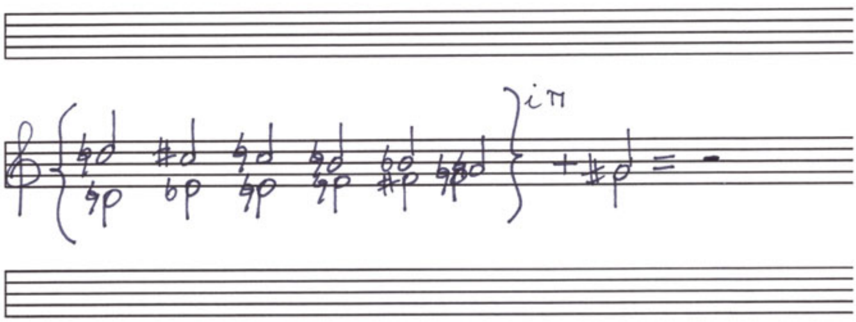


Fig. 6 Euler's equation realized in musical notes

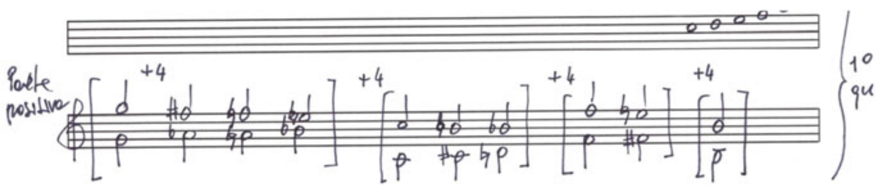


Fig. 7 Euler's theme transposed a fourth above with shortening

Fig. 8 Napier's number realized in a linear way

Fig. 9 Number of Napier made in polyphony (in two dimensions)

the end of a succession of five rises of fourth interval. In the second quarter of the circle, proceeding counterclockwise, from a single note the theme is lengthening and decreasing in pitch until it reaches the same original length but with the arrangement of the notes in reverse order. In the third quarter of a circle we see the same mechanism found in the first one but with the difference that the theme is lowered by a fourth with each passage. Finally, in the last quarter of the circle, the theme goes up by fourths and regains its original height and length. The number *e* was considered in a linear, polyphonic way and, moreover, as part of a canon counterpoint (Figs. 8). Given below are the exemplary (Figs. 9, 10, 11). This work is still being studied and perfected. In conclusion, music and mathematics share elective affinities that reveal deep connections both in form and substance. The author's intervention of the transposition of Euler's identity into music reveals a range of choices which are not always definitive; in fact there could be many other possible solutions. However, an attempt has been made to express a coherence, as close as possible to Euler's identity, by establishing multiple connections between numbers and notes, as well as between the symbolic meaning of the number itself and the corresponding sonic meaning that is represented in the musical figuration.

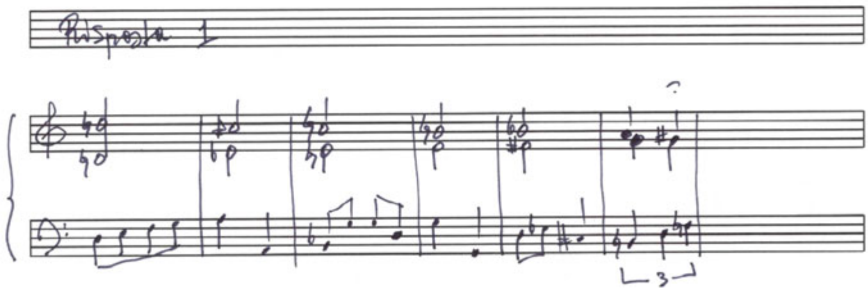


Fig. 10 Two-dimensional Napier number in response form

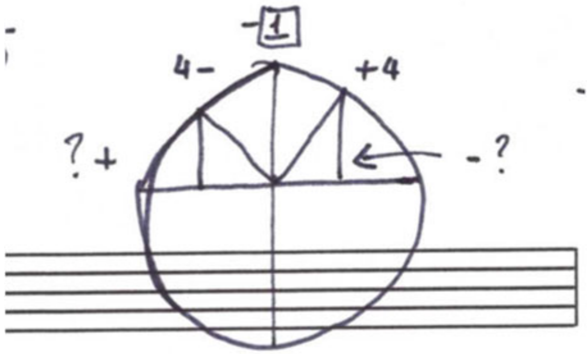


Fig. 11 Preparatory sketch on the movement of the Euler theme with the movement of the sine and cosine

Acknowledgement Co-editor and Supervisor of the English translation Marianna Biadene.

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The Shapes of Violin

Francesco Ciccone

1 From Fidula to Violin

Ed al principio sonar la ribeca
mi dilettai perch'avea fantasia
cantar di Troia e d'Ettorre e d'Achille
non una volta già, ma mille e mille[1]
Luigi Pulci, Morgante (cantare decimottavo)

In the world of classical Greece, the system of sounds and rhythms (always ordered by numbers) was conceived as an exemplification of cosmic harmony: music—from Greek $\mu o \upsilon \sigma \iota \kappa \hat{\eta}$, the art of muses or of combining sounds—was a common practice for those who wanted to pursue beauty and truth.

Although the legend concerning the acoustic experiments of the Pythagoreans on the sound emitted by a taut string is not very reliable (the Greeks tended to backdate all acquired knowledge to Pythagoras), it is a fact that the chordophones—whose first archaeological evidence dates back to the early Bronze Age in Western Asia—developed along the arc of the classical world, often associated with ode and epic, calm and spiritual elevation, the Apollonian side of Greek dichotomous culture [5]; the *monochord*, a very ancient instrument consisting of a fixed bridge and a mobile bridge, was used for the experimental verification of the laws of harmony (whose fundamental intervals are expressed by rational relationships) and as harmonic base for singers up to Late Middle Ages, so much so that it was included in important

[1] And at first I enjoyed playing my "ribeca" (rebec) / Because I really wanted / To sing about Troy, Hector and Achilles / And not once, but thousand and thousand times (translation by Maria Teresa Lambiase).

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M. Emmer, M. Abate (eds.), *Imagine Math 8*,
https://doi.org/10.1007/978-3-030-92690-8_33

dissertations, such as *Musica Enchiriadis* by Odo from Cluny and *Micrologus de disciplina artis musicae* by Guido Monaco.

In an Arabic treatise of tenth century, compiled by Al-Farabi (described by the Dominican friar Hyeronimus of Moravia as “one of the five highest authorities in the field of music”), we find for the first time the use of bow as means to spread more widely the sound of the vibrating string; one of the oldest iconographic attestations of this practice dates back to the first decades of the second millennium, in a nativity scene in the crypt of Sant’Urbano alla Caffarella (Rome), while written documents refer to the bowed instrument through different words, dependent on geographical areas:

- *fidula* in the Latin world (from *fides, fidis*: chord; still today fiddle is used in English to designate the violin with traditional performance practice);
- *giga* in Germanic countries (note that *geige* is a current German term for the violin);
- *crwht* or *rotta* in the Celtic area (from which a famous dance of the late Middle Ages will originate).

In the thirteenth century the aforementioned Hyeronimus da Moravia wrote a treatise on theoretical music which also deals with organology[8], classifying the strings in the two great families of *viella* and *rebeca* (or *rubeba*) (Figs. 1 and 15): the first instrument (*vielle* in English) has between 4 and 5 strings at a distance of the fourth or fifth (the two fundamental intervals of the medieval period) and is mostly used in order to accompany the singing of minstrels and troubadours or support the performances of actors-celebrants in the sacred representations of the para-liturgical functions; the latter instrument (*rebec*), which has between 2 and 3 strings tuned for fifths, has a more acute register and a purely melodic performance practice. However, the naive mixture of sacred and profane elements in many spheres of late medieval culture suggests that there wasn’t a clear distinction between their functions.

As for the construction, the professional figure of the luthier is not documented until the end of the thirteenth century: the player was, in most cases, the creator of his own instruments, inventing and modeling the technical and timbral characteristics to his own playing skills (see [10]); it follows that the types of wood used were mainly of local origin and the decorations drew inspiration from surrounding architectural elements (rosettes, scrolls of capitals, painting of stone floor). Dante Alighieri (Purgatory, Canto IV) mentions the figure of Belacqua, his Florentine contemporary and probably instruments-maker.[2]

Very few instruments from the late medieval era have survived intact up to the present: for example, the so-called *violet of Santa Caterina de ’Vigri* (see [15]), preserved in Bologna in the Corpus Domini church or the *carved citole* of British Museum, transformed in the seventeenth century into a bowed instrument.

[2] “Faciebat citharas et alia instrumenta musica, unde com magna cura sculpebat et incidebat colla et capita cithararum” [11].

Fig. 1 Rebec's player—Santiago's Cathedral; courtesy of Vincenzo Cipriani

Therefore, modern medieval lutherie is mainly based on the description of these instruments from ancient sources, literary or iconographic ones [7], setting the reconstruction on the proportions between instrument, hands, and faces present in the various contexts.

However, for our purposes we can classify the evolutions of ancient models, trying to understand the shape of modern instruments (Fig. 2), reasoning about differences of various nature:

- FUNCTIONAL EVOLUTIONS—the shape is modeled according to the executive needs; for example, the evolution of the instrument's function itself, from accompaniment to melodic, causes a more arched bridge and grooves are created (the so-called *C-bouts*), so that the bow can "cut" the sound box;
- STRUCTURAL EVOLUTIONS—the instrument is meant to stand the test of time; the case takes on a curvature that is not zero as it must counteract the pressure of the strings on the table, thus avoiding basing internal frames that stop the vibration of the wood;
- ACOUSTIC EVOLUTIONS—the need to perform music in increasingly spacious places leads to an optimization of the holes present on the string instruments (the *f-holes)*, up to the current form;

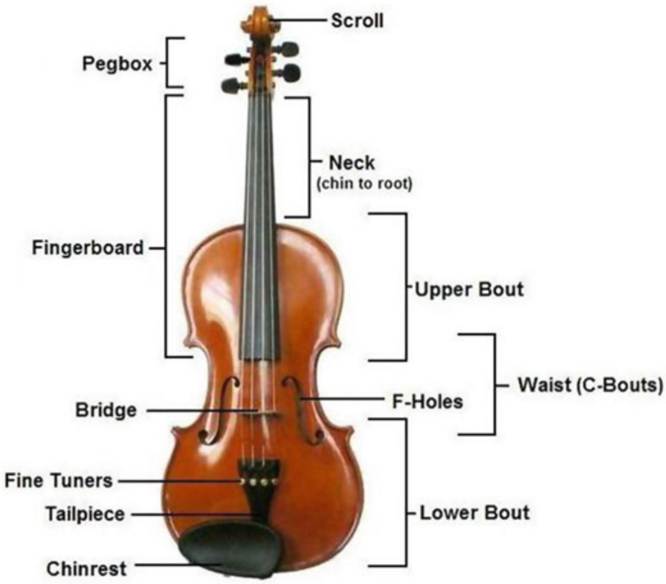

Fig. 2 Depiction of the parts of the violin

- AESTHETIC EVOLUTIONS—the sixteenth century luthiers, children of the Renaissance culture centered on the diffusion of beauty, try to shape their aesthetic taste to the functionality of the instrument; this is particularly evident in the wood decorations (carvings, tessellations, engravings) or in the figures present at the end of the anklet (wooden sculptures, scrolls of the hedgehog, inlays).

2 Echoes from Ancient World

Geometry has two great treasures; one is the Theorem of Pythagoras;
the other, the division of a line into extreme and mean ratio.
The first we may compare to a measure of gold;
the second we may name a precious jewel.
Johannes Kepler

The luthier Simone Ferdinando Sacconi, exegete and great scholar of Antonio Stradivari's work, asserts that the construction of the shape of the violin is based on several golden proportions [13]; before going into this dispute, we will shortly retrace the history of this golden ratio, a true "muse" of thinkers of all disciplines, more than any other number in the history of maths.

Quantitatively, the golden ratio is given by the irrational number

$$\varphi = \frac{1+\sqrt{5}}{2} \quad (1)$$

Although the name *golden section* is a very late attribution (it appears for the first time in 1835 [12]), on a historical plan this number is linked to a geometric problem present in Euclid's Elements (second century BC):

Proposition 2.1 (Elements, II.11) *To divide a given finite line into two segments, so that the rectangle contained by the whole line and one segment may be equal to the square on the other segment.*

This proposition is demonstrated using two previous propositions (I.43 and II.6) and brought back to an area equivalence problem (Fig. 3).

In the sixth book, after having developed the theory of proportions, φ appears like a definition:

Definition 2.1 (Elements, VI.3) A line *(segment)* is said to result in *extreme and mean ratio* (Fig. 4) when it is like the total line with respect to the major segment, so the greater than the minor.

In this way, the definition satisfies problem II.11, since $(\overline{AB})(\overline{BC}) = (\overline{AC})^2$.

A first interesting property is the following: copying the segment $\overline{CB}$ to $\overline{AC}$ (construction with compass), we obtain a new subdivision in the extreme and mean ratio of the latter! By setting the point obtained in centering the compass in A with

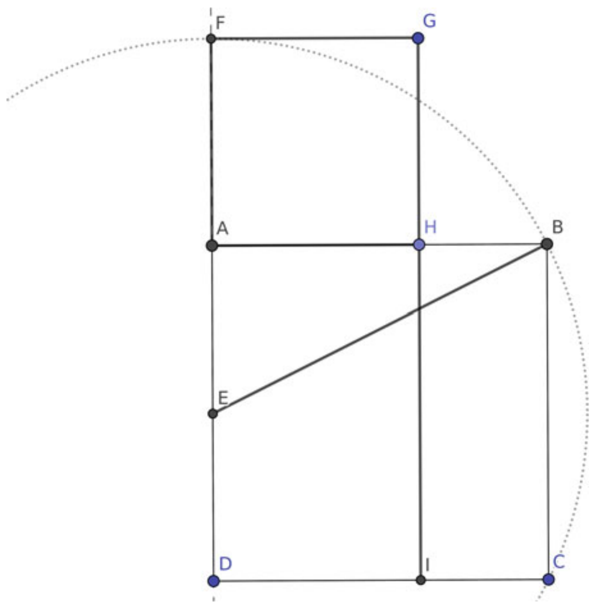

Fig. 3 Constructive scheme of the Proposition 2.1

Fig. 4 Extreme and mean ratio

the opening $\overline{BC}$ equal to D, we actually obtain

$$\frac{AB}{AC} = \frac{AC}{BC} \Rightarrow \frac{AB - AC}{AC} = \frac{AC - BC}{BC} \Rightarrow \frac{CB}{AC} = \frac{AD}{CB} \Rightarrow \frac{AC}{DC} = \frac{DC}{AD}$$

The construction can be repeated an arbitrary number of times.

If we want to give an algebraic connotation to the Euclidean definition (considering the $\overline{AC}$ segment as unitary and setting $\overline{AB} = x$ and $\overline{CB} = x - 1$), we have $\frac{x}{1} = \frac{1}{x-1}$, that is

$$x^2 - x - 1 = 0 \tag{2}$$

By solving the second degree equation, we obtain the two solutions

$$x_1 = \frac{1+\sqrt{5}}{2} = \varphi \quad (\textit{golden ratio}); \quad x_2 = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi} \quad (\frac{1}{\varphi} \text{ is } \textit{golden section})$$

These values are also obtainable through geometric constructions: always taking the Elements as a reference point, a constructive method for a particular triangle is suggested:

Theorem 2.1 (Elements, IV.10) *To construct an isosceles triangle having each base angle double the vertical angle.*

In this way, we obtain a triangle with angles $2\pi/5$, $2\pi/5$ and $\pi/5$. Furthermore, in the triangle ABD (Fig. 5), if $AB = AD = 1$, then $BD = AC = CD = 1/\varphi$, or rather the side of the regular decagon is the golden section of the circumscribed circumference's radius and its interior angles has width $\alpha = 4\pi/5$, that is the double of the angle to the base recalled in 2.1.

From the decagon you can easily build the pentagon, joining the vertices alternately; in the Elements, however, a different path is followed: bisecting the angles of the base side, we obtain five arcs on the circumference that subtend the same angle of amplitude $2\pi/5$ (therefore equal to each other), and the subtended chords are equal (i.e., the sides of the regular pentagon). This construction leads to the graphic realization of the *pentagram* (Fig. 6), the five-pointed star obtained through the diagonals of the pentagon, which generates another regular pentagon within it, from which a new pentagram can be created ...

The pentagram was well known within Pythagorean circles of Magna Graecia in the sixth century BC: Lucian of Samosata quotes that this figure, *a triangle with a triple intersection*, was used as a symbol by their sect; some others argue that it could have offered itself a visual basis for the discovery of the incommensurability of two quantities through the process of *antiferesis*, or subsequent subtractions: $AC - AB = CH$, $AB - CH = GH$, ...

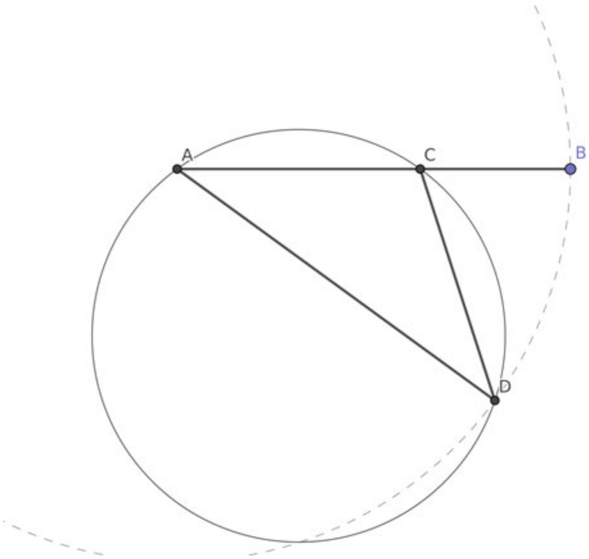

Fig. 5 Constructive scheme of the isosceles triangle required by Theorem 2.1

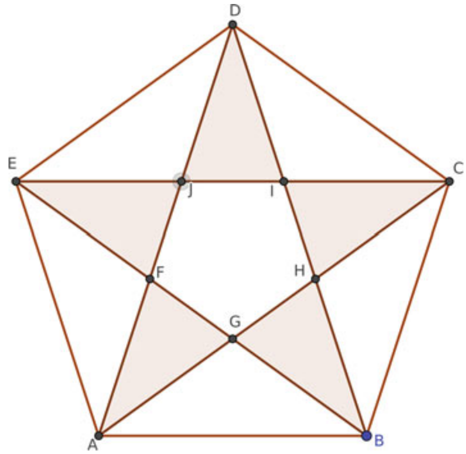

Fig. 6 A pentagram

The iterative process and the reference to φ are analyzed in Elements, XIII.8, where it is demonstrated that in a regular pentagon the diagonals are cut in extreme and medium ratio and the longer segments of this subdivision have length equal to the side of the pentagon.

The *golden rectangle* (i.e., the rectangle that has sides in the same proportion as $\overline{AB}$ and $\overline{AC}$ of Fig. 4) has exerted fascination on the history of arts and architecture, to the point that many scholars and enthusiasts have identified references to φ in every area of human knowledge, sometimes "hazarding" hypotheses linked to periods in which there was no certain attestation of a conscious use (for a detailed analysis of the question see [6]).

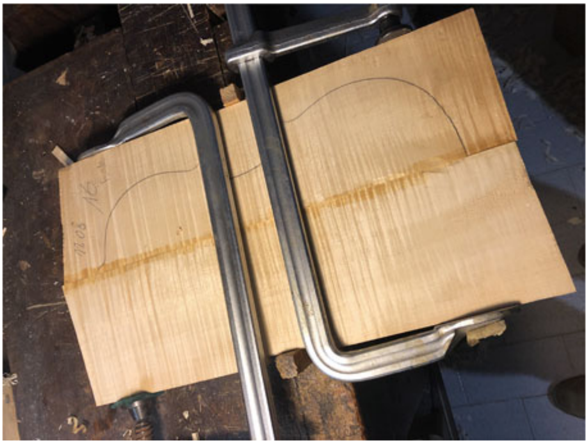

Fig. 7 Back's shape, during the splice; courtesy of luthier Federico Mari

The discourse is quite different with regards to violin making, which sees its *golden century* flourishing soon after in times following the publication of the treatise by the Franciscan friar Luca Pacioli, *De divina proportione* (Venice, 1509): because of this book, probably inspired by a previous work by Piero Della Francesca, the knowledge of φ spreads among painters, sculptors, and architects, who see a real possibility of connecting mathematics to physical universe that surrounded them, in the perspective of Renaissance Neoplatonism. The formal perfection and the acoustic performance achieved by the instruments of the Cremona's school has meant that, for a long time, subsequent later luthiers limited themselves to copy the sinuosities and trends of the various Amati, Guarneri, Stradivari, Guadagnini, Maggini (Fig. 7)... Reading the heterogeneous literature on this subject, there are many allusions to the value of the golden ratio in the construction of the violin, an essential practice both for the acoustic rendering and for the aesthetic balance; despite this, no documents have been found that testify the actual use of φ in the original project, and often the attributions are based only on *a posteriori* measurements and approximations of ratios that approach the value of $\varphi \simeq 1,6180339887\ldots$ although φ, due to its irrational nature, cannot be expressed as a rational relationship! However, φ can be approximated, with increasing precision, by taking the ratio between consecutive terms of the Fibonacci sequence.

Definition 2.2 The recursive sequence $\{F_n\}$, called the *Fibonacci numbers*, is defined as

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1}$$

It is known that $lim_{n\to\infty}(\frac{F_n}{F_{n-1}}) = \varphi$ (for the proof, see [3]).

Among the most followed golden suppositions, we are going to mention the following:

1. the ratio of violin height to lower maximum width would be a good approximation of φ;
2. some portions of the form would be obtained from golden ratios with a lower and higher maximum width;
3. in Stradivari violins, the tracing of the *ff* would be obtained by means of a golden rectangle having a base coinciding with the lower maximum width.

Concerning point (1), as previously mentioned, there is no standard measure for violins, even if the oscillations are minimal; considering that the height h is around 356 mm, and the lower maximum width b around 208 mm, we have that the ratio $h/b \simeq 1,711$ is not a very good approximation; even worse if we consider the unit of measurement used in Stradivari's time, the *Cremonese arm* (about 48.36 cm), divided into 12 *ounces*. Comparing the forms of Stradivari, we usually have $h = 9$ ounces and $b = 5$ ounces, with a ratio of $h/b = 1.8$. As for the other points, since we have not received tables of the original projects but only wooden artifacts, they are almost always with hindsight measurements, which do not prove a real construction of the number φ but, in the limit, various approximations of what it was presumably an aesthetic canon that was inspired by painting, architecture, philosophy.

For the sake of completeness, we report the geometric design made by the luthier Sacconi, in which the stradivarian form G is recreated starting from the maximum lower width (5 ounces) and applying subsequent constructions with ruler and compass (Fig. 8).

3 The Elegance of Scroll

Quando ammiriamo lo svolgersi del viluppo del riccio, così proporzionato, forte ed agile insieme nello slancio, aggraziato e morbido nella sua plasticità, sempre diverso nelle misure e sempre uguale nei rapporti, rimasto unico e inconfondibile fra le migliaia scolpiti dagli altri maestri liutai, ricordiamo ch'esso è stato disegnato con l'applicazione di due regole geometriche quali la spirale di Archimede, per lo sviluppo iniziale della chiocciola, e del Vignola per il suo completamento fino al dorso del riccio. [13][3]

Simone Ferdinando Sacconi

Our purpose is to describe the plane curves that are recalled, on a historical level, in the design of the terminal part of the *pegbox* (the end of the fingerboard where the *pegs* are recessed, by which the strings are tensioned); in modern violins it often has the shape of a spiral, the so-called *scroll*.

[3] When we admire the development of the scroll, so proportionate, strong and agile at the same time in its momentum, graceful and soft in its plasticity, always different in size and always the same in relationships, remained unique and unmistakable among the thousands sculpted by the other master luthiers, we remember that it was designed with the application of two geometric rules such as the spiral of Archimedes, for the initial development of the scroll, and spiral of Vignola for its completion up to the back.

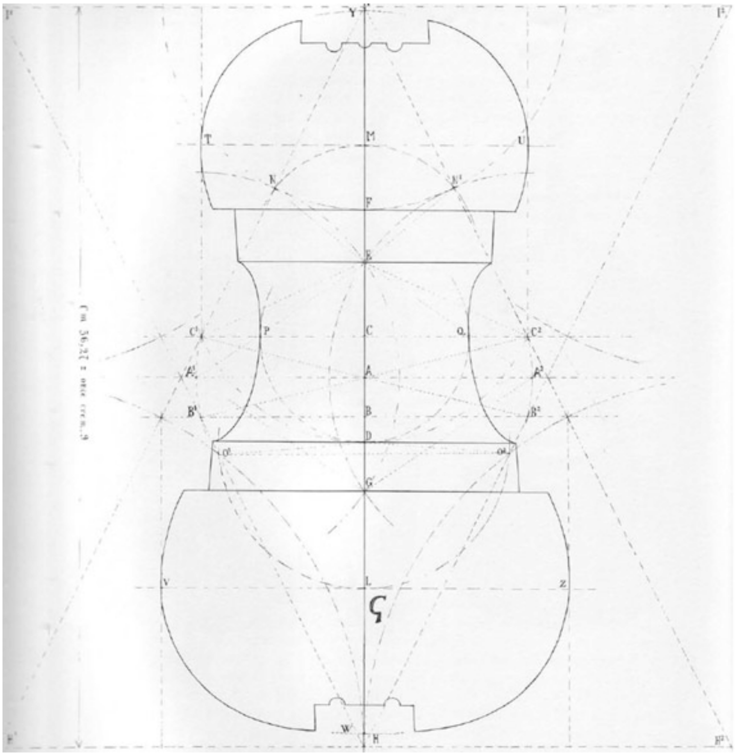

Fig. 8 Geometric constructions obtained from an original Antonio Stradivari's violin form, by Simone Ferdinando Sacconi; courtesy of Eric Blot Edizioni

One of the primitive concepts of our spatial intuition is that of *line*, or *plane curve*; different approaches have been undertaken, throughout history, to be able to describe this entity: from geometric places to the aggregation of small corpuscles, from the dynamic definition of a point moving in a plane to the intuitive definition of a line that can be drawn with a stroke of pen or pencil.

The most suitable idea for the different purposes of modern science is the one describing a curve as an image of an application, a dynamic idea of a point that moves continuously (or differentially) in the plane or in space (see [1]).

A *spiral* is a curve that winds around a certain central point, progressively moving toward or away from it, depending on the orientation in which the curve is traversed.

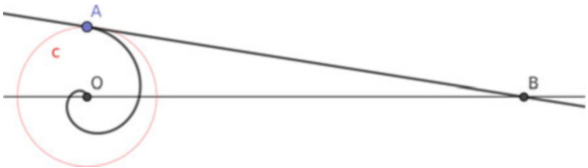

Fig. 9 Archimedean method of construction of the rectification of the circumference

Definition 3.1 Assigned $a, b \in \mathbb{R}$, the *Archimedean spiral* is a curve having parametric equations

$$\begin{cases} x = at \cos t \\ y = bt \sin t \end{cases}$$

This curve can also be described in polar coordinates (r, θ), through the equation $r = a + b\theta$ (in this case $b > 0$).

In Archimedean spiral, the arms are placed at an equal distance, equal to $2\pi b$ (if θ is measured in radians). Archimedes introduced his spiral to solve one of the classical problems of ancient times, namely the rectification of the circumference through a constructive process (Fig. 9):

- An Archimedes spiral with initial point O is constructed in the Cartesian plane.
- After a rotation of 2π it meets the y-axis at the point A.
- A circle of radius OA is constructed.
- We draw the tangent to the spiral at the point A. It intersects the abscissa axis at the B point.
- The length of the circle with radius OA is equal to the length of the segment AB.

This geometric construction does not solve the original problem, as the starting spiral itself cannot be constructed with ruler and compass. However, the method cited by Sacconi in the initial quotation refers to the approximate construction, on the Cartesian plane, of this curve (Fig. 10):

- The Cartesian axes and the two bisectors of the quadrants are traced.
- Concentric circles are drawn with the origin of the axes in the center.
- The circumferences meet the straight lines in numerous points: starting from the origin, we move each time by $\pi/4$ and we radially move away on the outermost circumference, finding the points through which the approximate Archimedes spiral passes (indicated in the figure with A, B, C, D, E, F, G, H).

We also report the construction method (evoked by Sacconi in the initial quotation) by Jacopo Barozzi da Vignola, architect of the seventeenth century, applied to the volute's construction of the Ionic capitals (Fig. 11).

Let us now analyze another type of spiral, more present in nature than the Archimedean one:

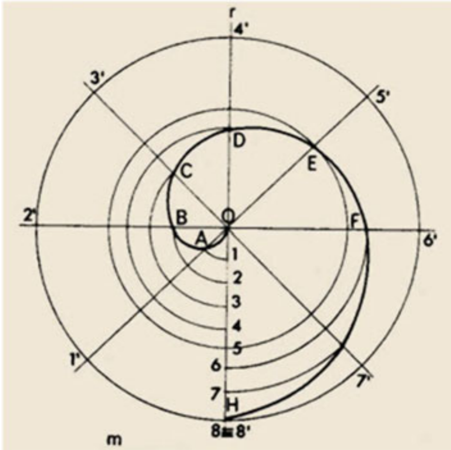

Fig. 10 Constructive Method of the approximate Archimedean spiral; courtesy of Antonietta Zanatta

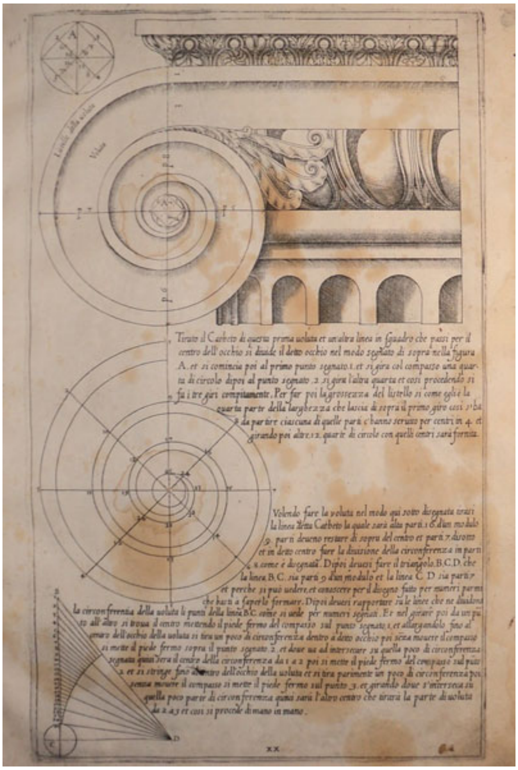

Fig. 11 Constructive method applied to the volute of the Ionic capitals; opus by Jacopo Barozzi da Vignola (1507–1563)

Definition 3.2 Assigned $a, b \in \mathbb{R}$, with $a > 0, b < 0$, the *Logarithmic spiral* is a plane curve $\sigma : \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$.

Recalling that $r^2 = x^2 + y^2$, we see that this curve satisfies $r = ae^{b\theta}$ in a polar coordinate system (r, θ).

The name is due to the fact that, applying the natural logarithm to the polar form, we obtain $\ln r = \ln a + b\theta$, which allows us to express the radius as a function of the argument: moreover, for $\theta = 0$, we get the point $(a, 0)$; $lim_{\theta\to\infty} r(\theta) = \infty$ and when $\theta \to -\infty$ the spiral approaches the origin O, winding up. Also, t coincides with the θ argument of $\sigma(t)$, modulo multiples of 2π, which makes it surjective and periodic of period 2π. The logarithmic spiral was first described by Descartes in a letter sent to Marsenne on 12 September 1638 and subsequently studied by Evangelista Torricelli (*De infinitis spiralibus*, 1645) and by Jacques Bernoulli, who found many properties and defined it *Spira mirabilis*, because moving away or approaching the origin even if its dimensions increase or decrease, it is always similar to itself. Moreover, starting from any point of the spiral and making complete turns, the vector radius varies according to a geometric progression of ratio e^{2bt}—hence the alternative name of *Proportional spiral*, sometimes attributed to the curve. In other texts it is found as *Equiangular spiral*, since in all points of the spiral the angle formed by the vector radius and the tangent line is constant.

4 Straightedge and Compass Construction

Qual è 'l geometra che tutto s'affige
per misurar lo cerchio, e non ritrova,
pensando, quel principio ond'elli indige,
tal era io a quella vista nova...[4]
Dante Alighieri, Paradiso, XXXIII, 133–136

The classical instruments of Greek geometry are still used today by luthiers to trace the nodal points on the soundboards (see [14]). In this section, we will try to introduce, from the algebraic point of view, which points in a real plane are obtainable with straightedge and compass. The possible operations, once assigned a set of $\mathcal{P}$ points, are:

1. draw the segment joining two points of $\mathcal{P}$;
2. draw the line through arbitrary two points of $\mathcal{P}$;
3. draw a circle having as center a point of $\mathcal{P}$ and as radius a segment identified by two points of $\mathcal{P}$.

[4] As the geometrician, who endeavors / To square the circle, and discovers not, / By taking thought, the principle he wants, / Even such was I at that new apparition... (translation by Henry Wadsworth Longfellow).

Definition 4.1 A point is said to be *constructible in one step starting from* $\mathcal{P}$ if it is given by the intersection of any two straight lines, of a circle with a straight line, of two circles, all figures obtained through the possible operations listed above.

A point is constructible *in a finite succession* of steps starting from $\mathcal{P}$, if it can be obtained iterating this construction finitely many times adding to $\mathcal{P}$ the points constructed in the previous steps.

A *straightedge and compass construction* is a finite succession of basic operations.

Some simple synthetic examples of this procedure are the construction of the midpoint of a segment, the construction of a line perpendicular (or parallel) to a given line and passing through a point P (the Italian Lorenzo Mascheroni will demonstrate in the eighteenth century that all constructions with ruler and compass can actually be performed with only the compass, if we consider a straight line as a "assigned" once two of its points are assigned).

By introducing a Euclidean metric, we can associate each segment with the number that expresses its length.

Definition 4.2 A *real number* α is said to be constructible if with straightedge and compass it is possible to construct a segment of length $|\alpha|$. A point is *constructible* if its coordinates are constructible as well with respect to a fixed system of Cartesian axes.

It can be proved that both $\mathbb{Z}$ and $\mathbb{Q}$ are contained in the set of numbers that can be constructed starting from only two points (the extremes of the unit of measurement); called C the set of such constructible points, it is shown that C is a field.

Let now $\mathcal{F}$ be any sub-field of the real numbers.

The *plane* of $\mathcal{F}$ is defined as the subset $\mathcal{F}^2 \subseteq \mathbb{R}^2$.

We define *line* of $\mathcal{F}$ any line joining two points of the plane, with equation $ax + by + c = 0$, $(a, b, c \in \mathcal{F})$. We define *circumference* of $\mathcal{F}$ any circumference that has its center at a point of $\mathcal{F}$ and has a radius belonging to $\mathcal{F}$, with equation $x^2 + y^2 + ax + by + c = 0$ (considering $a, b, c \in \mathcal{F}$).

Given a constructible α number, it is possible to construct the number $\sqrt{\alpha}$.

As a matter of fact, draw the perpendicular for the point 1 and calling P the intersection point with the circumference. By Euclid's Theorem, the y coordinate of the point P is the required root; in fact $1 : x = x : \alpha \Rightarrow x = \sqrt{\alpha}$.

It is concluded that all the points of the real plane constructible in one step starting from the plane of $\mathcal{F}$ are the points having coordinates in the form $\mathcal{F}(\alpha)$, with $\alpha \in \mathbb{R}, \alpha^2 \in \mathcal{F}$. Analyzing the shape of the equations of line and circumference in the plane $\mathcal{F}^2$, and making them intersect algebraically through a system, we note that the solution equations have a degree that is a power of 2.

Proposition 4.1 *A real number c is constructible if and only if there are a finite number of real numbers* $\alpha_1, \alpha_2 \dots \alpha_n$ *such that*

$$\alpha_1{}^2 \in \mathbb{Q}, \quad \alpha_i{}^2 \in \mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_{i-1})$$

so that $c \in \mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_{i-1})$.

We can restrict ourselves to $\mathbb{Q}$ because all constructible numbers contain this field. Considering the field $\mathcal{F}_1 = \mathcal{F}(\alpha)$, with $\alpha \in \mathbb{Q}, \alpha^2 \in \mathcal{F}$, the points of the plane that can be constructed in a step starting from $\mathcal{F}_1$ are all and only those with coordinates belonging to $\mathcal{F}_1(\beta)$, with $\beta \in \mathbb{Q}, \beta^2 \in \mathcal{F}_1$. A point γ is therefore constructible starting from the plane of a field $\mathcal{F}$ if and only if there is a finite succession of subfields

$$\mathcal{F}_0 = \mathcal{F} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_n$$

such that $\mathcal{F}_i = \mathcal{F}_{i-1}(\alpha_i)$ e $\alpha_i \in \mathbb{Q}, {\alpha_i}^2 \in \mathcal{F}_{i-1}$.

Corollary 4.1 *The number φ is constructible (although there are no attestations of its real constructions in the luthier's craft).*

(Compare the constructive process described in the 2.1 Theorem or see the second degree polynomial (2).)

As far as the creation of arching is concerned, the methods are the most varied: the boards that constitute table and back, before being excavated, have a uniform thickness and can be approximated with a portion of the plan; after an initial roughing with planer, the so-called sixth line is traced, or rather the longitudinal profile that coincides with the centerline and which will have to contrast, over the years, the tensions caused by the neck, bridge, soundpost, strings; the difference in the woods used (in almost all cases maple for the back, fir for the table) means that the support that this curve assumes is slightly different between the upper sixth and the lower sixth. We then proceed with the "purfling's insertion"[5] (Figs. 12 and 13) (i.e., the decoration of the external edges, using strips of wood inserted in previously dug grooves) and the "gorge"[6] (channel dug deeper that follows the edge of the shape of the instrument); the gorge must be connected with the rest of the arching, which will result in a transition from a concave area to a convex area.

From this moment, to create the curvatures mechanically, different methods are followed (Fig. 14): some luthiers draw an orthogonal grid, over the table, reporting with a pencil the final height that must be reached by that portion of the top and that will be checked periodically with a caliber; others trace level curves, to be connected continuously; still others make use of models for the cross sections, the so-called fifths of curvature. A Swedish luthier, Bjorn Zethelius, uses a completely different procedure from that of the fifths: taking a catenary line as a reference line, he prepares to excavate the two boards from the inside, establishing maximum depth, obviously different between bottom and top. Following the longitudinal and transverse axis and two geometrically established diagonals, he thus excavates the interiors of the two tables, verifying the various concavities with a chain. Let us explore the characteristics of this particular curve:

[5] Filettatura, in Italian.

[6] Sguscia, in Italian.

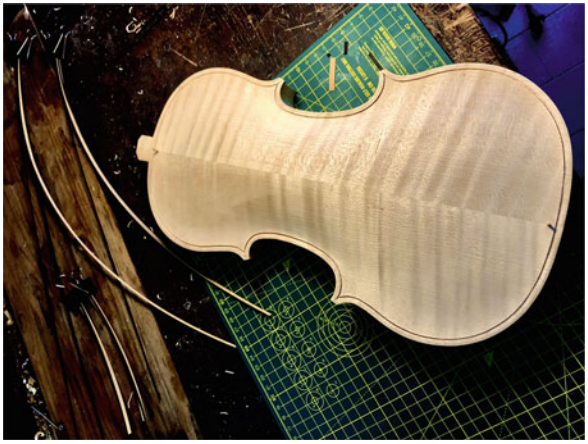

Fig. 12 Excavation of the purfling's channel; courtesy of luthier Federico Mari

Fig. 13 Purfling's detail; courtesy of luthier Federico Mari

Fig. 14 Intermediate phase of the internal sculpture of the bottom; courtesy of luthier Federico Mari

Definition 4.3 The *catenary* is the curve that a homogeneous, flexible, and inextensible thread, subject only to its own weight, forms when its ends are fixed. One possible parameterization is $\sigma : \mathbb{R} \to \mathbb{R}^2$ is given by $\sigma(t) = (t, \cosh t)$.

The first to deal with the catenary was Galileo Galilei, in 1638, mistakenly thinking that the shape of an "ideal" rope (perfectly flexible, inextensible, without thickness and with uniform density) hung by its ends and under the force of gravity were a parabola. Joachim Jungius later proved that the parabola was not the resolution curve.

The catenary is one of the few curves for which we can express the parameterization with respect to the arc length through elementary functions. Since the catenary has the property of having a uniform distribution of its total weight in each of its points, it has often been used to create artifacts and architectural structures and this could be the reason for its use in some descriptive methods of historical lutherie. Finally, the American violin-maker Michael Darnton, after having carried out studies on various ancient instruments (especially by Amati) with Moiré photogrammetry, supported the thesis that the external surfaces strictly followed the support of cycloid curves:

Definition 4.4 The *cycloid* is the curve $\sigma(t) : \mathbb{R} \to \mathbb{R}^2$ with parameterization $\sigma(t) = (t - \sin t, 1 - \cos t)$.

The cycloid arc is the curve that allows a mechanical particle to go frictionlessly from a point A to a point B, not directly situated below A, under the influence of a uniform gravitational field in the shortest possible time and is a solution to the *brachistocrone* problem (i.e., "shortest time curve"). This curve is regular (its tangent vector $\sigma'(t) = [1 - \cos t, \sin t]$ never vanishes) and its support is given by the path followed by a point of a circumference that rolls without crawling on a straight line; this mechanical correspondence probably allowed the construction of wooden models with whom it is possible to compare the progress of the excavation during the sculpture of the table (Fig. 15).

5 Acoustic Surfaces

All religions, arts, and sciences are branches of the same tree.
All these aspirations are directed toward ennobling man's life,
lifting it from the sphere of mere physical existence and leading the individual toward freedom.
Albert Einstein

Analyzing qualitatively the surface of the violin, we note that it is an oriented surface, connected by arcs; the back is simply connected while the table is not, since the closed curves enclosing the *f-holes* cannot be deformed to a single point. By assimilating the table to a two-dimensional surface, we now ask ourselves what the proportionality factors are between the measures of a standard violin and those of its *reduced*; furthermore, there are scaled versions to allow children to approach the instrument in the various stages of physical development. Historically, the popular

Fig. 15 Rebec and organistrum—San Miguel de Estella's church (Navarra, Spain); courtesy of Vincenzo Cipriani

formats for violins (and cellos) are identified in the following ratios

$$4/4 \text{ (standard violin)} \quad 7/8 \text{ (so-called } \textit{women's violin}) \quad 3/4 \quad 2/4 \quad 1/4$$

However, these ratios seem to have no apparent connection with the linear or surface dimensions of the resonance box; in reality, the method used alludes to the reduction in scale factors; for example, the $1/4$ violin is the one that has the length of the resonance box reduced by a quarter compared to the standard one; considering as *standard* the Berthier violin, made in 1716 by Antonio Stradivari (case length equal to 356 mm), we obtain that the $1/4$ violin has length $356 * (3/4) = 267$ mm (length which probably corresponds to another Stradivarian violin, the Aiglon of 1734).

By calculating the difference in surface area S_1 and $S_{\frac{1}{4}}$ between the two models (standard and $1/4$ reduction), we obtain the proportional value that varies in the other violin reductions .

$$S_{\frac{n+2}{8}} = S_{\frac{1}{4}} + \frac{n}{6}(S_1 - S_{\frac{1}{4}}) \quad , n = 0, 1, \ldots, 6$$

This method, apparently anti-intuitive, is the one used to obtain the measures of the violin in scale; what happens, instead, with the other members of the string family? The rational harmonic relationships, already known in the time of the Greeks, relate the length of a string (with the same tension and thickness) with the sound it produces; some of the fundamental intervals (called *perfect*, in musical language) have the following relationships:

2 : 1 Octave 3 : 2 Fifth 4 : 3 Fourth

The three lowest-pitched instruments (viola, cello and double bass) should maintain the same scale ratios as the violin, in proportion to the lowering of the frequency of the tessitura.

Placing l as the length of the violin, the following situation would arise:

- the viola (tuned a fifth below the violin) should be $\frac{3}{2}l$ long;
- the cello (tuned one octave below the viola) $3l$;
- the double bass (tuned a sixth under the cello) about $5l$.

Obviously, these instruments would not have characteristics compatible with the physical structure of man. To remedy this, the luthiers create some tricks to increase the inertia of the case, including the variation of the thicknesses, the addition of the strings' dimensions, the deformation of the sides; despite the compromises adopted have led to well-balanced dimensions and frequencies, the three instruments are not exact copies of the violin [4], as each of them has a particular timbre that characterizes it and distinguishes it from the others, making it unique from an artistic and musical point of view.

Stringed instruments produce a sound through bow or by pizzicato; from the strings, the vibration is transmitted first to the *bridge* and then to the *soundboard*; from this, finally, to the *back* by means of the *soundpost* (a small fir cylinder placed inside the case).

The vibrating ends of the string are the bridge and the *upper nut*; in the case of the violin, the 4 strings are G_3, D_4, A_4, E_5, with respective frequencies 197, 298, 440, 660 Hz (this agrees with the rational ratio 3:2 relating to the fifth interval).

When a string is put into vibration, its motion is described by the so-called *normal modes of resonance*, while the configurations of the vibrations consist of the *nodes* (the fixed points of the string) and the *antinodes* (the points of maximum oscillation) [2].

Like the strings, also the resonance box has specific resonance frequencies (the *resonant vibration modes*), due to the mixture of mechanical properties of the wood of the case and those of the air enclosed in it.

Summarizing the *body*, in addition to "amplifying" the resonance of the strings, has its own resonance which helps to determine the *timbre* of an instrument; this is also one of the reasons why there are no universal thicknesses and standard curvatures for violins, as each wood has certain physical characteristics (elasticity, cut, seasoning, density) which can be combined in various ways ([9]). Wood is, in fact, a "living" material, subject to changes over time.

Fig. 16 Vielle and Rebec, opus of luthier Vincenzo Cipriani; courtesy of Selene Chiozzi

6 Conclusions

La matematica è la più umanistica delle scienze esatte.[7]
Luigi Amerio

The etymological root of the term *mathematics* is closely linked to the concept of *learning* (from $\mu\hat{\alpha}\theta\eta\mu\alpha$ = science, knowledge, what you learn through the technique).

Sometimes music is able to transport us to unexpected worlds, making the physical aspects of sound intersect with the temporal dimensions of memory (Fig. 16); when progresses improve your musical technique, the same enchantment may disappear, for the more you understand something, the less you will be able to observe it with the typical astonishment of a child; the same sensations can be found in mathematics when it is considered as a creation of the human mind.

But, just as "poetry" finds its fulfillment in "doing", in the same way mathematics and music manage to give unexpected worlds that reveal themselves as the level of knowledge and awareness of their own limits increases. Finding an answer makes us happy, but just as the journey gives us more joy than the destination, sometimes the questions are the real fulcrum of knowledge.

Iconographic References

Figure 11: website http://www.faredecorazione.it;

[7] Mathematics is the most humanistic of the exact sciences.

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Part XI
Women and Mathematics

Women, Academia, Math: An Ephemeral Golden Braid

Chiara de Fabritiis

1 Introduction

The relation among women, academia, and mathematics is a long-term romance which, as many love affairs, had its ups and downs (unfortunately, till nowadays more downs than ups, as we shall see).

The aim of this work is to investigate the figures of the presence of female mathematicians in Italian academia, to give an interpretation of their trends, and to suggest possible good practices and affirmative actions, with a special focus on young people, to reduce the gender inequality in this area. In particular, we will report on a new tendency in recruitment that arose in recent years, the so-called Glass Door phenomenon, i.e., the obstacles women face in entering the first levels of the academic career: while the recruiting of "ricercatori" was almost gender-balanced till 2010, in the last decade the presence of women dropped down also in this role, as it was turned to a temporary one.

One may ask why a paper dealing with such an issue appears as a chapter of a book whose title is "Imagine Math": both as a mathematician and a woman, I see gender issues as central ones in the transformations Italian society has to undergo in order to reach a better exploitation of its human potential; this is the reason why the mathematics I imagine for future generations is a gender-balanced one. I must confess this is not an original opinion: the fourth and fifth goals of the UN Sustainable Development Agenda are "Quality Education" (Ensure inclusive and equitable quality education and promote lifelong learning opportunities for all) and "Gender Balance" (Achieve gender equality and empower women and girls), respectively.

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2 Two Faces of an Old Problem: The Leaky Pipeline and the Glass Ceiling

The global underrepresentation of women in academic jobs a well-known phenomenon that the many modifications which took place in the structure of our society in last times scarcely mitigated, in particular in STEM (Science, Technology, Engineering, Mathematics) areas. This paragraph is devoted to a concise analysis of two aspects of an important issue in female academic careers: the decreasing presence of women holding positions at the highest levels of the job ladder. This phenomenon is chiefly due to two concurrent causes: the Leaky Pipeline and the Glass Ceiling. The first expression deals with the fact that women are more likely to leave their academic employments for different sorts of professions (in Italy mainly school teaching) or for staying at home, while the second one concerns the difficulties women experience in reaching the highest levels of the job ladder. For a generalist approach to the reduced participation of women to academia, see, e.g., the She-Figures report 2018 [1], while an investigation more focused on early stages of the career is contained in the publications of the Garcia Project [2] (in particular, the leaky pipeline is discussed in [3–6]) and [7]; a useful source of references for the Glass Ceiling effect is [8].

The percentage of women and men at the different levels of academic career shows that moving from Ph.D. students to full professors a part of the female population "disappears." This evidence is clearly visible in the graph in the following page which displays the percentage of women (orange) and men (green) at the different levels of POST-DOC (assegnista di ricerca), RTD-A (temporary research assistant), RTD-B (tenured temporary research assistant), RTI (permanent research assistant), PA (associate professor), PO (full professor) for mathematicians in the year 2016 (dotted lines) and 2021 (continuous lines). The scissors-shaped curve marks the "evaporation" of a part of the female scholars as the level of the job increases (Fig. 1).

The surveys carried out in Italy for the Garcia project showed once more that

> "the uncertainties connected to these job positions, the lack of long-term perspectives [. . .] seem foster the decision to leave research"; moreover "men and women do not hold the pressure put by the greedy institution between personal and working lives the same way. From this sight, parenthood seems to hold a major role."[1]

[1] Garcia working paper n. 5, p. 6.

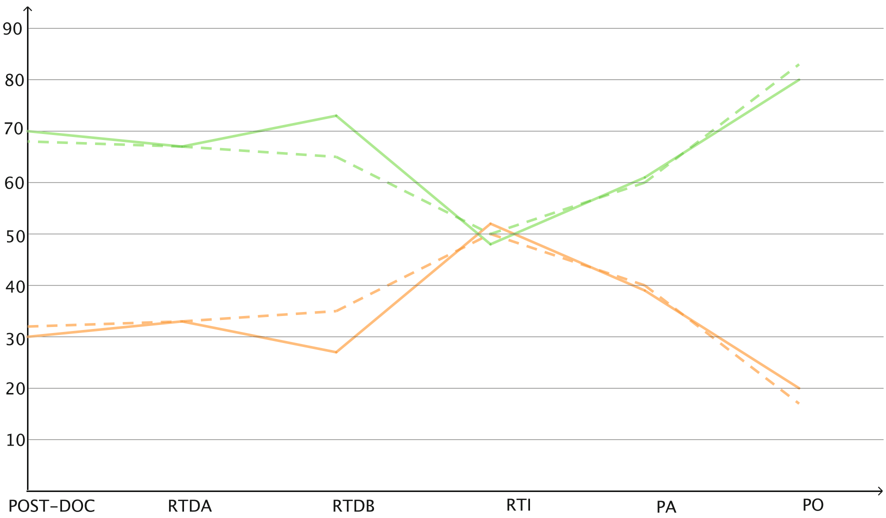

Fig. 1 Percentage of men (green) and women (orange) in academic positions in mathematics from post-doc to full professor; 2016 (dotted lines) and 2021 (continuous lines); elaboration on data taken from the database of the Italian Ministry for University

Another motivation for the small fraction of female full professors is the so called vertical segregation, that is the fact that women experience a greater difficulty than men when applying or competing for the highest level positions (a stunning evidence is given by the fraction of women rectors in Italy which in 2021 was equal to $6/84 = 7\%$); this fact is called the Glass Ceiling phenomenon, where the expression refers to an invisible ceiling which prevents women to go beyond a given level.

In particular, the Glass Ceiling Index (GCI) is a relative index that compares the proportion of women in academia with the proportion of women in top academic positions (full professor level). If the GCI is equal to 1, then the fraction of women in all grades is equal to the fraction of women in the highest level while a GCI greater than 1 denotes a Glass Ceiling phenomenon.

The table in the following page contains the trend for GCI in all mathematics disciplines in the last ten years: the first line displays the percentage of women in academia, that is the quotient of the number of women in all grades (W) and the total number of academics (T), the second one displays the percentage of women in top positions, that is the quotient of the number of female full professors (Wf) and the total number of full professors (Tf). Comparing the GCI for the mathematical area with the general GCI for Italy shows that our field is not an exception to the harder times women experience in STEM, since the global GCI was 1.73 in 2013 and 1.68 in 2016.

An optimistic interpretation of the trend of GCI for mathematical disciplines would underline the fast decrease of this indicator in the last 5 years, implying that it should reach 1 around 2032 (estimate obtained with a linear regression method).

	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
W/T	36.1	36.4	36.6	36.8	36.2	36	35.5	35.3	34.7	34.2
Wf/Tf	16.9	16.6	16.6	17.1	17.3	18.3	19.2	19.3	19.8	20.1
GCI	2.14	2.2	2.21	2.14	2.09	1.97	1.85	1.83	1.75	1.66
GCI_m	0.77	0.76	0.76	0.76	0.77	0.78	0.8	0.8	0.81	0.83
GCI/GCI_m	2.78	2.88	2.91	2.8	2.72	2.52	2.32	2.29	2.15	2

Fig. 2 Percentage of women in all grades (W/T) and women full professor (Wf/Tf), Glass Ceiling Index (GCI), male Glass Ceiling Index (GCI_m), and ratio between GCI and GCI_m; figures obtained from data taken from the database of Italian Ministry for University

Unfortunately, this hopeful analysis is spoiled by the content of the next paragraph, where a new obstacle in the direction of gender balance (the Glass Door) is outlined.

A more revealing indicator for the measurement of the obstacles women find in reaching full professorship is the ratio between GCI (which is computed for the female population) and the same index computed for male population (GCI_m); this number evaluates the difference in the arduousness in becoming a full professor for women and men: even though this quantity has been rapidly decreasing in the last decade, still in 2020 the hardship women undergo for this goal is twice as big as men do (Fig. 2).

Again, linear regression optimistically predicts parity in this parameter around 2031; nonetheless, we must be aware that this index gives a necessary condition for gender equality which is not sufficient at all. Indeed, a trivial algebraic manipulation shows that GCI is equal to 1 if GCI_m is equal to 1 and this is equivalent to the fact that percentage of women at all levels is equal to the percentage of women who are full professors: so a hypothetical academic system in which women hold 1% of the positions of assistant professors, 1% of the positions of associate professors and 1% of the positions of full professors would result in a GCI equal to 1, while being strongly gender unbalanced.

The reasons of the difficulties women experience have been widely investigated and cannot be explained only on a lower tendency to apply for higher rank or on a smaller scientific productivity, as shown in [8], where a detailed study on the cohort of scholars who already obtained Abilitazione Scientifica Nazionale is performed. Authors use years of seniority, macrodisciplinary area, university of affiliation and a parameter that measures individual scientific productivity (standardized h-index, standardized number of citations, standardized number of publications and an overall measure of productivity) as control variables of five different statistical models. In all cases, no matter how scientific productivity is measured, they find that the probability of career advancement for women is significantly lower than for men. In particular, on average female assistant professors have a probability to advance to associate professor which is 8% lower than their male colleagues; this percentage increases to 17% when they consider associate professors looking for a promotion to full professorship.

3 A New Problem: The Glass Door

While the Leaky Pipeline and the Glass Ceiling phenomena have always been a problem in Italian mathematical academia, the access to the lowest degrees of temporary (Ph.D. and post-doc) or permanent (ricercatore/ricercatrice, i.e., research assistant) positions has been almost equal for both men and women for a long time (see [9] for a detailed analysis of the data).

Nonetheless, the modifications of the recruitement rules due to Legge 230/05 and Legge 240/10, which abrogated the permanent position of "ricercatore a tempo indeterminato" and introduced the temporary positions of "ricercatore a tempo determinato di tipo A/B," caused a new phenomenon, which in analogy with the "Glass Ceiling" has been designated as the "Glass Door."

This expression means that access to academic positions is harder for women than for men; this difficulty is measured by an indicator, called Glass Door Index (GDI), which was introduced by Picardi in [10] (see also [11, 12]); GDI is given by the quotient of the percentage of women who work in positions equal or below the first step of academic ladder (that is, post-doc and assistant professor, both temporary or permanent) by the percentage of women who work in positions at the first step of academic ladder (that is, assistant professor, both temporary or permanent). Unfortunately, the computation of this number is more complicated than one could expect, since some of the data are not easily accessible: while the database of the Italian Ministry of University for the academic staff displays many different parameters (level, sex, year, scientific discipline, generic area of research) which allow a simple selection, the database for post-doc positions is very rigid (in particular, there is no sorting for sex) and it contains only current post-docs (Fig. 3).

The GDI for 2011, 2016, and 2021 are easily computed from the numbers in the table and are equal to 0.96, 0.89, and 0.91, respectively.

Since a GDI smaller than 1 denotes that there is no bottleneck for women in the transition from post-docs to more "stable" positions ("personale strutturato" in the

	2011 total	2011 females	2011 ratio	2016 total	2016 females	2016 ratio	2021 total	2021 females	2021 ratio
RTI	887	407	46%	560	283	50%	294	154	52%
RTDA	1	0	0%	87	29	33%	141	46	33%
RTDB	14[2]	5	36%	79	28	35%	200	54	27%
RTD(A+B)	15	5	33%	161	54	34%	341	100	29%
Total	902	412	46%	721	337	47%	635	254	40%
Post-doc	291	108	37%	278	90	32%	354	106	30%

Fig. 3 Figures of "Ricercatori a tempo indeterminato" RTI (total and females), "Ricercatori a tempo determinato di tipo A/B" (total and females), post-docs in mathematics in 2011, 2016, 2021; data come from the databases of Italian Ministry for University. ([2]The figures of 2011 and 2016 RTDB include also a different temporary position, namely ricercatore a tempo determinato L. 230/05, legge Moratti.)

jargon of Italian bureaucracy), these results would give a favorable account on the situation. Nonetheless, the use of raw data introduces a distortion that increases with time and must be taken into account: while in 2011 RTI where the almost totality of assistant professors, in 2021 people covering this role were recruited more than 10 years ago and we are comparing their cohort with present post-docs which will never become RTI, since enrollment in this position was canceled by Law 240/10.

In my opinion, a more significant index for 2016 and 2021 (the number of RTD is too small to make this computation meaningful for 2011) is therefore given by using only RTD-A and RTD-B as "stable" positions and comparing them with post-docs; with this restriction we find that the modified Glass Door Index (GDI_m) is equal to 0.98 for 2016 and to 1.01 for 2021, thus showing a trend which is closer to reality: in recent years the obstruction to female entrance in academic staff has the same strength at post-doc and at assistant professor level.

The figures contained in the above table allow a more detailed analysis of the modifications at the first level of recruitement in academia. In particular, two important trends can be underlined: the first is the fact that the fraction of women who are now "ricercatrici a tempo determinato" is much smaller than the fraction of women who were "ricercatrici a tempo indeterminato" ten years ago (29% vs. 46%); the second is that the portion of women in RTD-B positions, who are tenured and in 3 years become associate professor permanent jobs, is smaller than the portion of women in RTD-A positions, who are truly temporary (winners are appointed for 3 years which can be extended for 2 more years, then the contract is over).

So, formally, in Italy, there is no rule which prevents women from entering academy, but an invisible door (a glass one, indeed) keeps them off and this happens at the very beginning of the career. Unexpectedly, this happens also in fields like mathematics which till some years ago were more open to female participation.

As already noticed in the previous paragraph, the reduction of the rate of female mathematicians at the lowest level of the academic career creates a deceptive effect on the trend of the GCI: paradoxically, since the portion of females at the first step of the ladder diminishes, the Glass Ceiling phenomenon is less evident.

Moreover, notice that the fact that the percentage of women in RTI positions is increasing also points in the direction of an analogous of the Glass Ceiling phenomenon at the level of associate professor: since 2010 none entered this particular post of employment anymore, the only variations are due to retirements (which statistically affect men and women in comparable proportion) and promotions to associate professor level (which are more frequent for men than for women).

4 Good Practices and Affirmative Actions for the General Public

In the last years, many strategies of very dissimilar nature have been suggested in order to eliminate, or at least reduce, gender gap in academia in general and in STEM disciplines in particular. They include the creation of a process that estimates gender equality in universities by measuring several parameters (just to give an example, these evaluation systems can range from the adoption of a Gender Equality Plan to the implementation of a systematized policy on the model of English Athena Swan Awards); the introduction of some kind of quotas in the recruitment process, or at least a sort of reward in FFO (Fondo di Finanziamento Ordinario, the amount of money the Ministry annually gives to state universities) for public universities which decrease gender inequality; the computation of 18 months of career break for each maternity leave; the development of a mentoring scheme for female M.Sc. and Ph.D. students and Post-docs.

In this paragraph, I am going to speak of a different kind of actions that can be undertaken in order to increase the participation of women to academic staff by means of good practices whose goal is the popularization to a wider public of the perception of the existence of women doing research in mathematics.

In the last decades of the twentieth century, in Italy a large majority of high school professors in mathematics and physics were women, but most of their students had a very low awareness of the existence of female professional mathematicians. Asking young people for a list of women in mathematics would probably come out with a couple of lines (usually featuring Ipatia and Maria Gaetana Agnesi), and only a few ones who were most interested in mathematics could be able to add one among Sophie Germain, Sofja Kovalevskaja and Emmy Noether.

The increased attention to gender inequality which developed during the last years, brought to a different perception of the presence of women in the history of mathematics (and science in general); thanks to an impressing commitment of intergovernmental organizations, learning societies, activists for women's rights, scholars and experts in women's studies, authors, screenwriters, directors and producers, names like Ada Lovelace, Katherine Johnson, Maryam Myrzakhani, are now a common heritage for most learned people.

The tools which can be used to give to the general public, and to girls and female teenagers in particular, an opportunity to become more familiar with the idea of women working in mathematical research are of a very different nature: events organized for special days, films, plays, articles on newspapers and magazines, science girl camps and many other such initiatives, all help to spread the familiarity with women in mathematics.

The introduction of days dedicated to women in several fields of science followed different paths: in 2009 with a pledge on a British civil action site, Pledgebank, the blogger, journalist, and social software consultant Suw Charman-Anderson founded Ada Lovelace Day (which is held on the second Tuesday of October) in order to celebrate the achievements of women in STEM (science, technology, economy, and mathematics); the 2020 edition saw more than 60 events taking place worldwide.

In 2015, the United Nations General Assembly declared February 11th the International Day of Women and Girls in Science; the strength and commitment of UNESCO and UN-Women, which organize the day in collaboration with many institutions and civil society partners, quickly made this date an important pivot for the promotion of women in science.

Nevertheless, the most specific initiative concerning women in mathematics is almost a new-born, since it was established at the World Meeting for Women in Mathematics-(WM)^2-on July 31th, 2018. On that occasion, the Women's Committee of the Iranian Mathematical Society proposed that May 12th, the birthday of Maryam Mirzakhani, the first woman to receive a Fields Medal, would be used for celebrating women in mathematics. In its first edition, in 2019, more than 100 events took place in 36 countries, (see [13] for a report on the organization, an account on the happenings in each continent and planning for future years).

In 2020, the website of the initiative, funded by the International Mathematical Union Committee for Women in Mathematics, European Women in Mathematics and the Association for Women in Mathematics, was turned perennial and adapted to annual events, so that it could support the celebrations taking place each year. Even if the COVID-19 pandemic made the organization of in-presence conferences, exhibitions, and film projections very complicated or even impossible, more than 150 events were planned worldwide (in over 100 countries) and the participation of a large audience was possible thanks to the fact that more than one-third of the events were online ones (see the report available at [14]).

One of the key points of the success of May 12th, 2020, is "Secrets of the Surface: the Mathematical Vision of Maryam Mirzakhani", a documentary film by George Csicsery about the life and work of the Iranian Fields Medalist.

In 2020 Zala films, the production society, decided to support the May 12th initiative: between April 1st and May 19th, both individual and institutions were allowed to access the film freely just by filling a form on the May 12th website: they received more than 20,000 requests. Zala films also offers a very stimulating discussion guide for educators in order to support them in presenting the film to students involved in a women and gender study curriculum; in my opinion it could also be used fruitfully for senior students of Italian high schools (in particular the ones attending liceo scientifico) (Fig. 4).

(WM)^2 was also the occasion for the premiere of the first edition of "Journey of Women in Mathematics," a 20 minutes film created by the IMU Committee for Women in Mathematics, filmed and edited by Micro-Documentaries, funded by a grant of the Simons Foundation. In the first part, three women mathematicians (Neela Nataraj from India, Aminatou Pecha from Cameroon, and Carolina Araujo from Brasil) are featured at their home institutions, while the second part, shot at (WM)^2, shows the atmosphere of the event and contains six interviews of women from Latin America; the film is freely available at the IMU website ([15]).

Of course, there are many other films that showed a wider audience the work of women in mathematics: just to make an example, Hidden Figures, the biography of three Afro-American female mathematicians (Katherine Johnson, Dorothy Vaughan, and Mary Jackson), who worked at NASA during the Space Race, grossed $236 million worldwide and received three nominations at the 89th Academy Awards; in its first screening on Italian TV (Rai1) in 2019 it reached over 4.3 million single spectators.

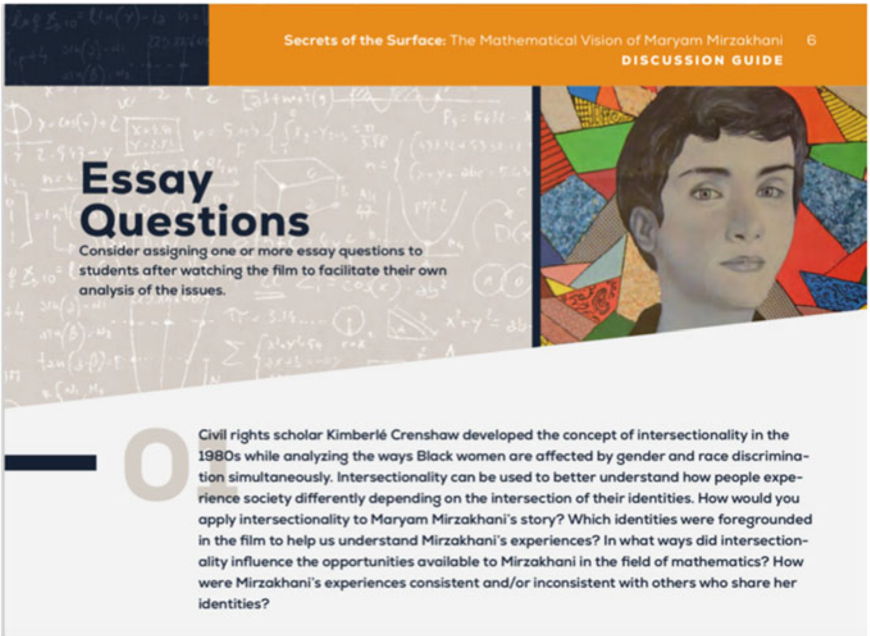

Fig. 4 Essay Questions page from Secrets of the Surface Discussion Guide-Courtesy of Zala films

A different strategy to ease the approach of girls and young women to STEM in general and maths in particular is the creation of science girl camps in which groups of children or teenagers get in touch with scientific subjects suitable for their age (the groups can consist of girls only or of some boys joined with a majority of girls) (Fig. 5).

In June 2020, the Italian Ministry for Equal Opportunities opened a call for the organization of summer camps addressed to groups of pupils aged 3–18 with the purpose of overcoming some of the by-products of the pandemic: 3 million euros were made available for schools, universities, municipalities, non-profit associations with a strong background in education with the aim to run at least 2 weeks of activities focused on STEM disciplines. Events with a longer tradition like "Pinkcamp" at Università dell'Aquila (which was established in 2018) or new-comers such as "STEM in Ancona!" (a pun with the double sense of "Stem" which means "Let's remain!" in the local dialect) offered several scores of secondary schools students the possibility to improve their knowledge of STEM subjects (mathematics, chemistry, physics, and computer science) and to realize that "women" and "science" can be an impressive and sound couple.

Fig. 5 Flyer of STEM in Ancona activity

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Women in Charge of Mathematics

Elisabetta Strickland

In 1937 a collection of biographical essays was published by Eric Temple Bell, a Scottish born mathematician and science fiction writer. It covered the lives of about 40 mathematicians, from ancient times to the beginning of the twentieth century. The book inspired many boys to become mathematicians, but we believe it did not inspire many girls, as the only woman mentioned was Sofia Kovalevskaya, the brilliant Russian mathematician and the first woman to obtain a doctorate in mathematics.

Things did not change a lot after about almost 70 years, as Ioan James, a British mathematician working in topology, published in 2003 a collection of biographies about "Remarkable mathematicians: from Euler to von Neumann," where, in addition to Kovalevskaya, were mentioned Sophie Germain (see Fig. 1), the outstanding French mathematician, and Emmy Noether, known as "the mother of modern algebra".

A strong change took place in 2014, when Maryam Mirzakhani, a mathematician born in Iran, full professor at Stanford University, was awarded the Fields Medal, the most coveted prize in mathematics, for her research [1].

This award is as important for mathematics as the Nobel Prize is for other sciences and Mirzakhani was the first woman to win the Medal in its 80-years history. Born in Tehran on May 12th, 1977, she was the first girl to compete for Iran in the International Mathematical Olympiad and she won gold medals in Hong Kong in 1994 and in Toronto in 1995. This was a remarkable achievement. Mirzakhani specialized in the geometry and dynamics of complex curved surfaces. She died in 2017 from breast cancer at the age of just 40.

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Fig. 1 The stamp issued by the French post in honor of the mathematician Sophie Germain in 2016. https://i.ebayimg.com/images/g/4j4AAOSwbb5eOaCb/s-l1600.jpg

Mirzakhani, when she received the award in Seoul (see Fig. 2), said that she hoped that her work would inspire more women in mathematics and for sure her example has been a strong one.

After that exploit, Karen Uhlenbeck (see Fig. 3), the American mathematician known for her pioneering work in geometry, analysis, and mathematical physics, in 2019 was the first woman in the 16-year history of the Abel Prize, named in commemoration of the outstanding Norwegian mathematician Niels Henrick Abel, to receive it.

Uhlenbeck in 1990 presented a plenary lecture at the International Congress of Mathematicians, the ICM, the largest and most important gathering of mathematicians in the world; she was the second woman to give a plenary lecture, the first being Emmy Noether in 1932. This indicates how difficult it has been for women to reach the pinnacle in a male-dominated field.

At the World Meeting for Women in Mathematics in Rio de Janeiro in 2018, Mirzakhani's birth date, May 12th, was chosen for the celebration of women in mathematics. The aim was to inspire to follow careers in math and to encourage an open and inclusive environment for all. Many events took place in the last 2 years throughout the world as part of the celebrations.

All these events not only support those who participate in them directly but also help influence the mathematics culture more generally, so that young women entering the field today encounter an environment that is more nurturing than that of the past.

Currently, there is an international dialogue around the lack of representation of women at the highest levels: across academia, government and industry. These

Fig. 2 Maryam Mirzakhani, first and only woman Fields Medalist, at the ICM 2014 in Seoul, together with the other Fields Medal winners: Arthur Avila, Manjul Bhargava and Martin Hairer and Ingrid Daubechies, President of IMU (2010–2014). https://kongres-magazine.eu/wp-content/uploads/2014/10/SEOUL-ICM-2014_0813-348.jpg

Fig. 3 Karen Ulhenbeck, American mathematician, first woman Abel Prize Winner in 2019. https://www.europeanwomeninmaths.org/marini-uhlenbeck/

institutions need to continue to review their organizational cultures and adjust their internal promotional practices, otherwise increases to the numbers of women who move through the career pipeline will still fail to affect the representation of women at the top.

This is the reason why all associations of women in math, national and international, have to work in continuing to foster and maintain the pool of female mathematicians: this is a key piece in bringing about these long-term goals.

Moreover, in this era of big data and fast-paced technological changes, both of which require mathematical expertise, we cannot afford to leave so much of the population behind. The deficit of women in STEM (Science, Technology, Engineering, Mathematics) and particularly women in math, is not just a women's issue [2].

We know very well that these fields have remained predominantly male with historically low participation among women since their origins and we also know that scholars and policymakers have been exploring the various reasons for the continued existence of the gender disparity in STEM fields and studies suggest that many factors contribute to the attitude toward the achievement of young people in mathematics and in general in science, including encouragement from parents, interactions with teachers, curriculum content, high school achievement in mathematics and resources available at home.

Research findings are mixed concerning when boys' and girls' attitudes about mathematics diverge. Few differences are found in girls' and boys' attitude toward mathematics in the years of early secondary school. Student's aspirations to pursue careers in mathematics influence both the courses they choose to take in this area and the level of effort they put forth in these courses.

Apparently, girls begin to lose self-confidence in middle school because they believe that men possess more intelligence in technological fields, while boys are more likely to gain skills because they are culturally and socially encouraged to work in scientific areas, but research shows that girls can develop these same skills if they have the same form of training.

At the post-secondary level, women are less likely than men to earn a degree in mathematics.

This of course is a problem, as it has been estimated that doubling women's high skills would largely benefit the economy.

The differences in salary among graduates are related to the differences in occupations entered by women and men; women are less likely than men to be employed in scientific occupations and so there remains a wage gap between men and women in comparable positions [3].

UNESCO, among other agencies, has been outspoken about the underrepresentation of women in mathematics globally, even if it is not possible to use the same indicators to determine the situation in every country. The significant statistic might be the percentage of women teaching at the university level, or the proportion of women at research institutes and academies of sciences, or the percentage of women who publish, or the proportion of women who go abroad for post-graduate study and conferences, or the percentage of women awarded grants by national and international funding agencies.

This is just to say that indices have different meanings in different countries and the prestige of various positions and honors can vary considerably, but in any case, the main fact is that the underrepresentation of women in careers in mathematics is worldwide.

Luckily changes in society and the ubiquity of computers in everyday life are pushing women toward a deeper understanding of scientific matters at large, inducing a breaking down of gender distinctions in information technology. Both genders have acquired skills, competencies and confidence in using a variety of technological, mobile and application tools for personal, educational and professional use, but the gap still remains when it comes to enrollment of girls in mathematics and computer science classes: as a matter of fact for higher educational programs in information and communications technology, women make up only 3% of graduates globally.

Stereotypes about what someone in these fields should look and act like may cause established members of such areas to overlook individuals who are highly competent. The stereotypical mathematician is usually thought to be male and perceived incongruity between gender and a particular role or occupation can result in negative evaluation.

In addition, negative stereotypes about women's quantitative abilities may lead people to devalue their work or discourage these women from continuing in mathematics.

There have been several controversial statements about innate ability and success in mathematics. A notable example is given by Lawrence Summers, former president of Harvard University, who, in a 2005 speech, suggested that "the underrepresentation of women in science and engineering could be due to a different availability of aptitude at the high-end positions". Summers after this statement had to step down as president [4].

Fortunately, although women entering traditionally male professions face negative stereotypes suggesting that they are not "real" women, these stereotypes do not seem to deter women to the same degree that similar stereotypes may deter men from pursuing nontraditional professions. There are historical evidence that women flock to male-identified occupations once opportunities are available.

On the other hand, examples of occupations changing from predominantly female to predominantly male are very rare in human history.

Women in mathematics in addition are more likely than established men to help career women who display sufficient qualifications.

A very good example is offered by the reaction of the members of the European Women in Mathematics association (EWM) (see Fig. 4) during the Covid-19 pandemic. The virus and the full and partial lockdowns that swept across Europe and the world caused an impact on research and training in academia which was disastrous: conferences were canceled and collaborations stood still.

Time slated for research splintered among the competing demands of homeschooling, eldercare, and quarantines. Networking and mentorship stalled, common

Fig. 4 General Assembly in 2018 at Graz, Austria, of the European Women in Mathematics (EWM). Courtesy of European Women in Mathematics. Permission granted by the Organizer of the 2018 General Assembly, Karin Baur

but often unaddressed mental health issues mushroomed, at a time when getting help was harder than ever.

But the crisis was not experienced equally, untenured faculty lost more. Women lost more and caregivers lost more. The more vulnerable the population, the greater the disadvantage. Therefore a Working Group on the Corona Crisis was formed (https://www.womeninmathematics.org) in order to choose how to respond and to support current employees in temporary positions and future job applicants in mathematics in light of the crisis. The main concern was that we could lose talented women mathematicians during and following the crisis, that women could choose to leave their profession or reduce their hours, that women in temporary positions could choose security and settle for lesser positions, that young women may opt not to pursue careers in science.

As we know, the COVID-19 pandemic has exacerbated existing gender inequities in mathematics and of course in other sciences. And gender-blind measures do not correct gender inequity. It appeared to the members of the Working Group that to those who say we should relax and trust the system, it should be answered that the system has not produced a gender-balance representation to date and it would be naïve to expect an automatic correction in the face of enormous burdens.

Therefore the women involved in this project advocated proactive measures in order to encourage universities, government and funding agencies to invest in extending the contracts of researchers in temporary positions to offset the loss of productivity during the crisis, to encourage universities and funding agencies to award release from teaching or teaching reductions to untenured mathematicians who lost significant research time to digital teaching and caregivers responsibilities, of course giving particular consideration to women.

In addition, evaluators of hiring, tenure, prize, grant, and other committees should be reminded that the crisis has impacted individual differently and that more flexibility in deadlines and meeting times should be advocated, especially for women with dependent children.

These proactive measures have been listed in a letter that was sent to all authorities in institutions and academia all over Europe, after being signed by a very long list of members of EWM and others (https://www.europeanwomeninmaths.org/signatories-ewm-open-letter/).

This meant that women in established positions could do something important for other women in untenured positions, for the very simple reason that Europe needs more women in the sciences and the only solution is to shape smart policy to recruit and retain a diverse group of talented young scientists.

This example gives also a clear idea of how important is that women can reach leading roles in the world of mathematics, in order to promote gender-balanced policies.

Actually, one of the proposed methods for alleviating stereotype threat is through introducing role models. One study found that women who took a math test that was administered by a female experimenter had a better performance when compared to women whose test was administered by a male experimenter. Additionally, these researchers found that it was not the physical presence of the female experimenter but rather learning about her apparent competence in math that buffered participants against stereotype threat.

Female mathematicians who read about a successful woman, even though these successes were not directly related to performance in math, perform better on a math test. Both female and male role models can be effective in recruiting women to STEM fields, but female role models are more effective at promoting the retention of women in these fields. And of course female teachers can also act as role models for young girls, as reports have shown that the presence of female teachers positively influences girls' perceptions of STEM and increases their interest in STEM careers [5].

So at this point, we would like to give a look to what has been achieved by women in top positions in mathematics who therefore became role models, often overcoming institutionalized infrastructures, behaviors and beliefs, so that women could continue advancing.

We are going to focus on this aspect because the role-model intervention has a positive and significant effect on mathematics enjoyment, importance attached to math, expectations of success in math and women's aspirations in this field and help to reduce gender stereotypes.

Removing the barriers that prevent women from accessing the sector of mathematics and in general science, research and technology sectors will be the key to changing the current academic orientation, which is essential for fighting new forms of gender inequalities [6].

A good way of overcoming stereotype barriers is through the intervention of female role models, who can increase the sense of belonging to mathematics and reinforce the idea that hard work is the way to succeed [7].

Indeed, not only do role models help broaden the perspectives of who can work in mathematics, they also expand perceptions of researchers of their own potential. Therefore women are more motivated (in terms of expectation of success, enjoyment, and importance) to engage in subjects like math, after interacting with female role models. Being exposed to the professional and personal experiences of actual female role models with a successful professional trajectory in mathematics is the optimal way to encourage women to pursue emerging high-growth roles, requiring math skills.

Last but not least, an increase in women's presence within professions in math and in general in the STEM area is particularly important so as to enable women to seize the new opportunities offered by digital transformation. If women continue to be underrepresented in STEM fields, they may fall further behind in the labor market: the World Economic Forum (WEF) suggests that there is an urgent need to increase the supply and visibility of women with technical skills to close the gender gap in the professions of the future.

In this regard, it has been estimated [8] that, globally, between 40 million and 160 million women may need to undergo a transition between occupations by 2030, often into higher-skilled roles. To make these transitions, women will need new skills. In particular, they will need to overcome their low participation in STEM fields compared to men, as an important barrier that, if not broken, will make it harder for women to make transitions.

We are going to consider two main structures in the world of mathematics, the International Mathematical Union and the European Mathematical Society, where women in charge, meaning that they were elected Presidents, appeared only recently.

The International Mathematical Union is an international non-governmental and non-profit scientific organization. IMU's objectives are to promote international cooperation in mathematics, to support and assist the International Congress of Mathematicians (ICM) and other scientific meetings or conferences, to encourage and support other international activities considered likely to contribute to the development of mathematical science in any of its aspects, pure, applied, or educational. The IMU was officially founded in September 1920 in Strasbourg.

Shortly before the ICM, the General Assembly takes place, which is a gathering of a kind of parliament of mathematics. Usually, when the Program Committee is established, the Adhering Organizations of the IMU and mathematical societies

Fig. 5 Ingrid Daubechies, Belgian physicist and mathematician, first woman President of the International Mathematical Union (IMU). https://www.europeanwomeninmaths.org/wp-content/uploads/2018/08/087916_daubechies007-high-rez-1170x750.jpg

worldwide are invited to nominate plenary and sectional speakers and nominations should be made to the Chair of the Program Committee within the month of November of the year before the one in which the ICM takes place. The next one is going to be held in Saint Petersburg, Russia, between 6 and 14 July 2022. Moreover, the IMU grants a number of prestigious prizes and awards every 4 years at the ICM. The IMU members worldwide are 88. All this gives a clear idea of the importance of this organization and its enormous prestige.

There is nothing in mathematics comparable with the honor of being invited as speaker at the ICM and of course the Fields Medals which are awarded each time are the most coveted prizes.

Among the Presidents of IMU, which were 18 since its foundation, only one has been a woman, Ingrid Daubechies, a mathematician and physicist, who served from 2010 to 2014 (see Fig. 5). Just recently women appeared among the Vice-Presidents: Christiane Rousseau, Alicia Dickenstein, and Nalini Joshi, in chronological order since 2010. So if we are speaking of role models, Ingrid Daubechies is a very good one: taking care of the most important duties in mathematics is quite challenging.

At this point, we would like to say something about her, in order to understand how she reached this prominent position. She was born in 1954 at Houthalen, in Belgium. She is best known for her work with wavelets in image compression.

Her study of the mathematical methods that enhance image-compression technology gave her an international reputation, which made her member of the National

Academy of Engineering in the US, the National Academy of Sciences and the American Academy of Arts and Sciences. She is also a 1992 MacArthur Fellow.

The name Daubechies is widely associated with the orthogonal Daubechies wavelet and the biorthogonal CDF (Cohen-Daubechies-Feauveau) wavelet. A wavelet from this family of wavelets is now used in the JPEG 2000 standard.

Her research involves the use of automatic methods from both mathematics, technology and biology to extract information from samples like bones and teeth. She also developed sophisticated image processing techniques used to find out the authenticity and age of some world's famous works of art including paintings by Vincent van Gogh and Rembrandt.

What took Ingrid Daubechies to the Chair as President of IMU? It is quite a fascinating story and it is worthwhile to say something about it.

She is the daughter of Marcel Daubechies, a civil mining engineer, and Simonne Duran, a criminologist. She remembers that when she was a little girl and could not sleep, she did not count numbers, as you would expect by a child, but started to multiply numbers by two from memory, so she already familiarized herself with the properties of exponential growth.

After finishing the Lyceum in Hasselt, she entered the Vrije Universiteit Brussels at 17 and there completed her undergraduate studies in physics in 1975. She obtained her Ph.D. in theoretical physics in 1980 at Free University Brussels, after a collaboration with Alex Grossmann in quantum mechanics. She continued her career until 1985 in Brussels, first as assistant professor, then as associate professor.

After she went as a guest-researcher at the Courant Institute of Mathematical Sciences in New York and there she made her best-known discovery: based on quadrature mirror filter-technology, she constructed compactly supported continuous wavelets that would require only a finite amount of processing, in this way enabling wavelet theory to enter the realm of digital signal processing (see Fig. 6).

In July 1987, Daubechies joined the Murray Hill AT&T Bell Laboratories' New Jersey facility, where in 1988 she published the result of her research on orthonormal bases of compactly supported wavelets [9]. From 1991 to 1994, she taught as a professor at Rutgers University in the Mathematics Department. In 1994 she moved to Princeton University, where in 2004 she became the first female full professor of mathematics at Princeton.

After moving to Duke University in 2011 at the Department of Mathematics and Electrical and Computer Engineering, she founded together with Heekyoung Hahn the Duke Summer Workshop in Mathematics (SWIM) for female rising high school seniors. Moreover, she has been on the board of directors of Enhancing Diversity in Graduate Education (EDGE), a program that helps women entering in graduate studies in the mathematical sciences.

At this point, it is clear why she became the first woman to be President of the International Mathematical Union (2011–2014).

It was under her direction that for the first time a Fields Medal was awarded to a woman, Maryam Mirzakhani, in 2014 at the ICM in Seoul, South Korea. Of course, the procedure to award the Medal officially does not take under consideration the gender of the winners, but it is a fact that she brought good luck to this enormous

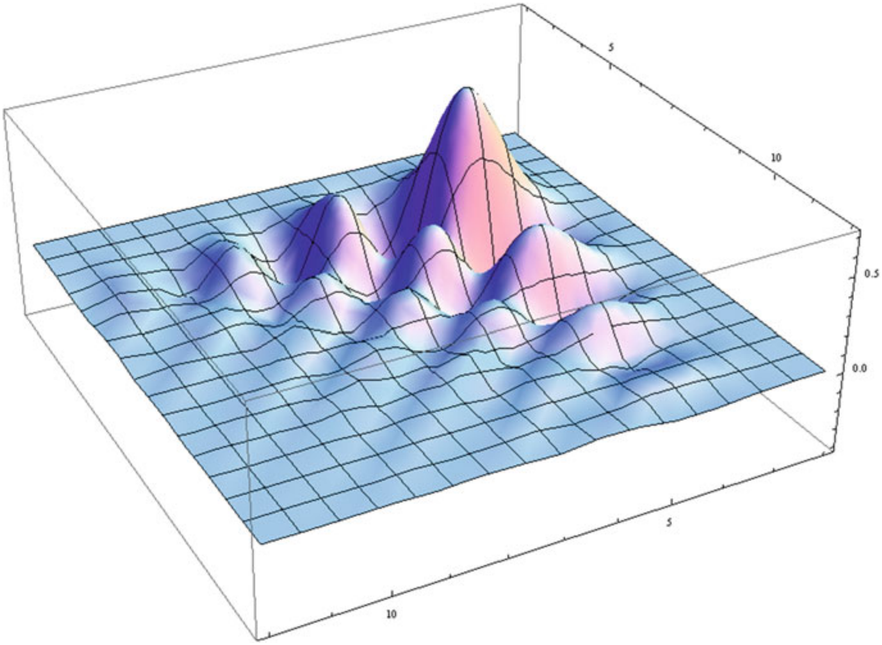

Fig. 6 Daubechies wavelets, mathematical methods used in image-compression technology. https://en.wikipedia.org/wiki/Daubechies_wavelet

achievement, which was extremely important to promote mathematics among young female researchers.

Among the long list of prizes that Daubechies received during her career, we would like to mention for its gender significance the 2019 L'Oréal-UNESCO International Award For Women in Science: since 1998, the award annually recognizes five outstanding women in chemistry, physics, materials science, mathematics and computer science worldwide.

Daubechies was chosen for North America, along with Najat Aoun Saliba (Africa and Arab States), Maki Kawai (Asia Pacific), Karen Hallberg (Latin America) and Claire Voisin (Europe).

One could think that so much work could have prevented her from having a family, but not in her case: she has been married since 1985 to mathematician Robert Calderbank, and they have two children, Michael and Carolyn. So she represents a successful example of work-life balance.

Another achievement of a similar kind was the one of Marta Sanz-Solé, who became the first and up to now only woman president of the European Mathematical Society (EMS), from 2011 to 2014 (see Fig. 7).

The EMS is a European organization dedicated to the development of mathematics in Europe. Its members are different mathematical societies in Europe, academic institutions, and individual mathematicians. The current president is

Fig. 7 Marta Sanz-Solé, Catalan mathematician, first and only woman President of the European Mathematical Society (2001–2020). https://www.ara.cat/2016/07/07/videos/avancaments/Marta-Sanz-Sole-mentalitat-que-Trobem_1609069082_30008435_1132x636.jpg

Volker Mehrmann, professor at the Institute for Mathematics at the Technical University of Berlin. Before him eight Presidents took care of the EMS, the first one was Friedrich Hirzebruch in 1990.

The precursor to the EMS, the European Mathematical Council, was founded in 1978 at the International Congress of Mathematicians in Helsinki. The informal federation of mathematical societies was chaired by Sir Michael Atiyah. The EMS as we know it now was founded in 1990 in Madralin, near Warsaw, Poland, and the first European Congress of Mathematics (ECM) was held at the Sorbonne and Panthéon-Sorbonne universities in Paris in 1992.

The Society seeks to serve all kinds of mathematicians in university, research institute, and other forms of higher education. Its aims are to promote mathematical research, both pure and applied, assist and advise on problems of mathematical education, concern itself with the broader relations of mathematics to society, foster interaction between mathematicians of different countries, establish a sense of identity amongst European mathematicians, represent the mathematical community in supra-national institutions. The EMS is itself an Affiliate of the International Mathematical Union.

The governing body of the EMS is its Council, which comprises delegates representing all of the societies which are themselves members of the EMS, along with delegates representing the institutional and individual EMS members. The Council meets every 2 years and appoints the President and Executive Committee, who are responsible for the running of the society.

Besides the Executive Committee, the EMS has standing committees on: Applied Mathematics, Developing Countries, Mathematical Education, ERCOM (Directors of European Research Centers in the Mathematical Sciences), Ethics, European Solidarity, Meetings, Publications and Electronic Dissemination, Raising Public Awareness of Mathematics, Women in Mathematics. The EMS is headquartered at the University of Helsinki.

One of the important issues of the Society is the organization of the European Congress of Mathematics (ECM), which is held every 4 years, at which ten EMS Prizes are awarded to "recognize excellent contributions in Mathematics by young researchers not older than 35 years".

In addition, since 2000, the Felix Klein Prize has been awarded to "a young scientist or a small group of young scientists (normally under the age of 38) for using sophisticated methods to give an outstanding solution which meets with the complete satisfaction of industry, to a concrete and difficult industrial problem," and since 2012 the Otto Neugenbauer Prize has been awarded to researchers "for highly original and influential work in the field of history of mathematics."

We have pointed out these prizes because in many cases (Maxim Kontsevich 1992, Richard Borcherds 1992, Timothy Gowers 1996, Grigori Perelman (declined) 1996, Wendelin Werner 2000, Elon Lindenstrauss 2004, Andrei Okounkov 2004, Stanislav Smirnov 2004, Artur Avila 2008, Cédric Villani 2008, Alessio Figalli 2012, Peter Scholze 2016) the winners later have been awarded the Fields Medal at the ICM, so the EMS prizes represent a springboard for reaching the coveted prize.

We described the EMS in order to make clear how important it was for a woman to become its President and the significance that this represented for all the community of women in mathematics.

Marta Sanz-Solé, born in January 1952 in Sabadell, Barcelona, is a Catalan mathematician specialized in probability theory. She obtained her Ph.D. in 1978 from the University of Barcelona under the supervision of David Nualart.

Currently she is professor at the University of Barcelona and head of a research group on stochastic processes. She was Dean of the Faculty of Mathematics at UB from 1993 to 1996 and Vice-President of the Division of Experimental Sciences and Mathematics from 2000 to 2003.

In May 2015 she was appointed chair of the scientific Committee of the Graduate School of Mathematics and from May 2018 until October 2019, she held the position of Director.

Her research interests are in stochastic analysis, in particular stochastic differential and partial differential equations.

Sanz-Solé served in the Executive Committee of the European Mathematical Society in 1997–2004. She was elected President in 2010 and, as we already pointed out, held the post from January 2011 to December 2014. She is a member of several international committees overseeing the mathematical sciences, such as the Board of Directors of the Institut Henri Poincaré in Paris and the Scientific Committee of CIRM (Centre des Rencontres Mathématiques, Luminy, France) and in June 2015 she was appointed member of the Abel Committee for the Abel Prize 2016, 2017.

For her scientific contributions and relevant international positions and service, she was awarded the Real Sociedad Matematica Espanola Medal in 2017. In 2019

Fig. 8 Ingrid Daubechies and Martha Sanz-Solé pictured at the ICM in Hyderabad, India, in 2010. https://owpdb.mfo.de/detail?photo_id=13137

she became numerary member of the Royal Academy of Sciences and Arts of Barcelona.

During her term as President of EMS, two women were awarded the EMS prizes in Krakow, 2012, Sophie Morel and Corinna Ulcigrai. Even if the choice is up to the panel which has the responsibility of choosing the winners, having a woman as President probably inspires to be for a change more gender oriented (see Fig. 8).

It is quite obvious from our excursus through the lives of these two women that in order to reach a really relevant position that allows to be in charge of mathematics, a broad experience in research and responsibilities in mathematics is absolutely necessary, but of course all this comes together with an attitude toward taking care in a positive and effective way of the goals at stake.

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Part XII
Comics and Mathematics

Without Title

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2012–2021: A Comics&Science Experience

Roberto Natalini and Andrea Plazzi

It possibly began on November 1, 2012. Time: Noon. Place: the luxurious and much sought-after by speakers Palazzo Ducale's main hall in Lucca (Italy). The panel's title reads "What we talk about when we talk about comics and science."

The host event was the Lucca Comics&Games festival, on its way to become the largest convention of its kind in the Western World: roughly half a million people gathering in five days to celebrate their love for comics, *manga*, sci-fi, TV series, movies, role-playing games, boardgames, videogames, *any kind* of games. In short: a massive crowd—the like of which is rarely to be seen in public places for less than national strikes, crucial political rallies or important sport events—looking for any conceivable form of entertainment, and laying their claim to the redemption of the so-called "nerd culture" as—above all—a *passionate attitude*.

And this is the key to why the year before, we had started musing over a crucial consideration: Where if not among the hundreds of thousands attending Lucca Comics and Games were we going to find people willing to devote themselves to something—anything—with *that kind* of attention, concentration, enthusiasm for discovery, imagination and improvisation bordering obsession?

So, we actually did not do much more than giving a—not very inventive but clear and effective—name to what was already there.

Let's start from the very beginning, when humans used to express themselves by gestures and inarticulate ramblings. Some time after that, they began to keep score of the animals they killed, the passing days and seasons, carving notches on wooden blocks or ordering pebbles and stones (Greek: *khalix*; Latin: *calculus*) in lines. That

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was probably when modeling—the assumption that “signs” can tell us meaningful things about the actual objects they stand for—and Visual Arts began traveling down the same path.

Very soon we realized how things had not changed *that* much: when we asked cartoonist Giuseppe Palumbo to tell a story about Archimedes, the very first image he turned out was the Greek genius focused on tracing geometric figures on his native Siracusa’s shores. Which coincidentally just fits with Henry Poincaré’s famous statement of 2000 years later, about how “la géométrie est l’art de bien raisonner sur des figures mal faites” (“Geometry is the art of correct reasoning from incorrectly drawn figures”).

In simpler words: Science and the signs we need to “tell science” have always been there. An old and ever-present concept.

Andrea has a mathematical background and has been stricken on his way to comics, sometimes a comics character himself. Roberto is a professional applied mathematician who, in order to properly do his job, discovered how sorely he needed to *communicate* to a larger audience what his work *is*.

Back in 2012, we started the *Comics&Science* (C&S) section of the Lucca Comics and Games cultural program, with a small number of simple, targeted events (panels, book presentations): the results were encouraging, and we persuaded ourselves we had to be up to the “comics” part.

Leo Ortolani—“Leo,” for all his fans—was and still is one of Italy’s most popular cartoonists, with a streak of trademark irreverent humor, since the early 1990s. Moreover, he is a trained geologist who never forgot his training, and lovingly targeting science with his scorching jokes is something coming easy to him, with wildly funny outcomes.

For *C&S* he fine-tuned one of his many brilliant storytelling devices, giving birth to “MISTERIUS, the show with no idea of what it’s babbling about, just like you,” a laughable avatar of all those pseudo-scientific TV formats trying to captivate audiences with Holy Grail stories, chemtrails and how the Pyramids were built by the aliens, all in one.

Leo knows well how *not* to pull punches (see Fig. 1): His MISTERIUS comic book for *C&S* is a phantasmagorical explosion of unlikely characters more or less taken from real life, ineptly grappling with science. On the “more” side, we find the french mathematician—and celebrated Fields medalist—Cédric Villani, who in November 2013 felt the thrill of walking down the never-so-crowded Lucca medieval streets and alleys being recognized as an Ortolani character meeting his creator.

C&S’ cornerstone is an easy one to state, and somewhat more difficult to implement: talking about science at a state-of-the-art level with comics by the best cartoonists around, plus editorials and pieces delivered in a layman language and, at the same time, always—always—as scientifically accurate as possible. Stories reaching out to their audience for what they are, nothing more and not the slightest bit less: engaging, entertaining comics aspiring to be artistically relevant.

Fig. 1 Leo Ortolani's take on Mathematics (from "MISTERIUS—Speciale Scienza!"; *Comics&Science*, October 2013)

C&S is not interested in indulging in detailed, meticulous descriptions of scientific facts, history or ideas: The kind of more literal, specifically educational approach TV shows and so many popular science books are best suited for.

This is how what we aim at, and what we think of what we do, is described in official press releases:

> Comics&Science's goal is to promote the link between Science and Entertainment, strongly believing that both are crucial formative factors for all citizens' development. The "Comics" tag in its name clearly refers to our medium of choice, fully embraced by production and publishing choices only slightly revising the typical, classical comic book format, well-known and deeply loved worldwide by generations.

But what did we actually do? What do we feel proud of?

Feedbacks, to start with: both from "hardcore," "pure" comics fans and from new readers we look for in different, more strictly scientific venues.

As in research centers like Geneva-based CERN (no introduction needed), which affably welcomed quite a peculiar *Comic&Science* delegation: a minivan fully packed of cartoonists which sparkled several projects.

Like *Oramai* by cartoonist Tuono Pettinato (2014; the same year he was awarded the "Premio Gran Guinigi," Italy's main recognition in the comics field), a story about time's paradoxes and ultimate nature, stemmed from discussions around today's theoretical physics more abstract and philosophical aspects.

We also had the pleasure to see a reader "crossing the line" becoming a valued addition to our roster: Francesco Artibani made the history of Disney comics in the last 25 years, and his name is well recognized wherever Disney comics are printed. He is also a *C&S* fan and helped in taking aboard Silver (Guido Silvestri's *nom de plume*; see Fig. 2), one of the living "gods of Italian comics." So we had the additional pleasure to discover how Silver is a passionate fan, devouring popular science book after popular science book. His main concern was—and still his—

Fig. 2 Silver's trademark character "Lupo Alberto," as a testimonial for what drives science and knowledge: passion and curiosity (from "Materia oscura;" *Comics&Science* 001/2016, April 2016)

how in recent years the "social" dimension of communication is giving "hoax" a new meaning, making "fake news" and "information disorder" key words when crucial topics like vaccines and GMOs are concerned.

Having Italian CNR—Consiglio Nazionale delle Ricerche ("National Research Council of Italy") as a publisher might come in handy, especially if you are Director one of its historical institutes (IAC—Istituto per le Applicazioni del Calcolo, "Institute for Applied Computing"), which is what Roberto happens to be. CNR promotes and carries out research projects in 27 main research areas, with almost 100 institutes operating as part of each one of them. *Nature* magazine recognized the Council as one of the top 10 innovation centers in the world, so it is no surprise when something happening inside its revered halls gets some attention.

It is what happened with the Pisa-based CNR-IIT—Istituto di Informatica e Telematica ("Information technology and data communication Institute"), a name which in Italy spells "In-ter-net:" CNR-IIT is the Italian arm of ICANN, operating the DNS—Domain Name System when the ".it" domain names are concerned, and it was CNR-IIT to bring the Internet to Italy back in 1986–1987, making the first connection possible.

So, in 2016–2017 it was CNR-IIT's choice to have special, exclusive *C&S* productions as part of the many events celebrating the 30th anniversary of the Italian Internet, starting with an "Internet Issue" featuring—again—the comic genius of Leo Ortolani, teaming up with Federico Bertolucci, an Italian superstar in his own field, with five nominations to the Eisner Awards (the "Nobel Prize of cartoonists") under his belt and, in 2019, the Italian "Romics d'oro," awarded each year in Rome.

It was the beginning of a long continuing streak of *C&S*-inspired productions: two more comic books, a card game, and a series of educational comics and illustrations in digital form followed, while a videogame is entering its beta testing phase at the time of writing.

A very typical feature of scientific research, as opposite to the prevailing competition in other sectors, is its openness to collaboration and partnerships. So, joining forces with "sister" institutions has always been the rule for CNR and *C&S* since the start. Institutions like Universities, research centers or the Pisa-based "Museo degli Strumenti per il Calcolo," a very peculiar museum built around the mission of bringing computer science and its "key players"—computers—at a general public level, beyond the skin-deep, if not shallow "pseudo-knowledge" coming from being passive users of modern technology.

It all started from the question, "How is it possible for machines to do calculations?" A very simple but far from trivial one, like its many possible answers. Some of them are even surrounded by modern myths, if not legends, which is what comics writer Alfredo Castelli thrives on. Castelli is one of the venerable fathers of contemporary Italian comics and jumped aboard the *C&S* boat with *Il segreto di Babbage* ("Babbage's secret"), joined by the rising star of young artist Gabriele Peddes.

Then it came the aforementioned *Archimede Infinito 2.0* by Giuseppe Palumbo (see Fig. 3), recounting one of the most incredible, far-out, 100%-true stories ever to be told, a milestone event in science, history, and archeology. An epic ride across centuries, from Third Century B.C. to 1998 when, during one of the harshest auctions ever, a still-today unknown US millionaire won over the greek Government for the possession of the invaluable Archimedes' "Codex C," and to an accelerator in Harvard, where high-energy X-ray fluorescence techniques were developed in order to recover the secrets still encoded in the parchment. While we cannot be but partial, we found this issue a very well done—if not spectacular—embodiment of the "*C&S* principles."

It is 2018 when *Educazione subatomica* ("Subatomic Education") hits the stands. Zerocalcare (Michele Rech) is possibly the only cartoonist attaining an actual "stardom status" in Italy, a sought-after public figure, looked at for opinions and viewpoints about controversial political and social issues. Following his own curiosity over the mysteries of quantum physics, he spent a day at the premises of the ELETTRA and FERMI accelerators, in the Trieste area (Northeastern Italy), which are among the most powerful tools at our disposal when it comes to investigate matter at the micro- and nano-scale. Totally captured by the environment and by "the Sspirit of Research" (see Fig. 4), Zerocalcare delivered a compelling and deeply personal account of what reasearch is for hard-working active researchers, and how they should be—and often are not—considered and perceived by the general public taking advantage of their discoveries. A heartfelt and inspired report mixing the highest *C&S* standard and the trademark Zerocalcare style together, with humor, brilliant jokes and darker musings about how blind humans can be.

We said "no literal or educational approach to Science."

Licia Troisi is a writer of fantasy novels selling million of copies worldwide. She has also got a PhD in astrophysics and that is where her brilliant "fantasy metaphor" for a star's life cycle comes from (and if you are wondering what a "fantasy metaphor" actually is you will have to read it): *La fanciulla e il drago* ("The Dragon and the Maiden") does not call for any specific scientific knowledge, only the will

Fig. 3 A *Comics&Science* issue devoted to Giuseppe Palumbo re-telling the story of Achimedes' "Codex C" (from "Archimede Infinito 2.0;" *Comics&Science* 002/2017, October 2017)

of plunging into the spectacular images summoned by Licia Troisi and visually rendered by the extremely talented Carmine Di Giandomenico (a comics superstar on his own, working for global powerhorses like Marvel and DC comics), artist Alessandro Micelli and the out-of-scale coloring by Leo Colapietro (see Fig. 5).

In 2019 the Periodic Table of Elements turned 150 and *C&S* celebrated the anniversary joining forces (remember? Openness and collaboration) with CNR-

Fig. 4 Zerocalcare's moving rendition of the Spirit of Research (from "Educazione subatomica;" *Comics&Science* 002/2018, October 2018)

Fig. 5 A spectacular two-pager from Licia Troisi's story: layouts by Carmine Di Giandomenico, art by Alessandro Micelli, colors by Leo Colapietro (from "La fanciulla e il drago;" *Comics&Science* 001/2019, April 2019)

ICCOM—Istituto di Chimica dei Composti Organo-metallici ("Chemical Institute for Organ-metallic Compounds") and the Società Italiana di Chimica ("Italian Society for Chemistry"). The outcome was a story by writer Giovanni Eccher and artist Sergio Ponchione, where very peculiar young people are supposed to learn how to use their very peculiar abilities by enrolling in a very peculiar school. Which might be ringing some bell to comics fans (see Fig. 6).

One should not look for any thin blue line connecting these *C&S* dots: each and every one of these works of art springs from talented and creative cartoonists, who could not differ more one from another, getting in touch and mingling with researchers who, in turn, are actively working in any possible venue of contemporary Science.

What makes *C&S* an actual, coherent line of books, is its approach to any given topic: 20 to 24 pages of pure and simple comics, followed by roughly the same amount of editorials, articles and pieces covering that topic. A very simple, down-to-earth answer to the "popularization paradox:" The more scientific content is simplified for the sake of comprehension, the more it becomes something else, and we fail to communicate.

It must not be necessarily so.

With this in mind, we recently produced very different projects.

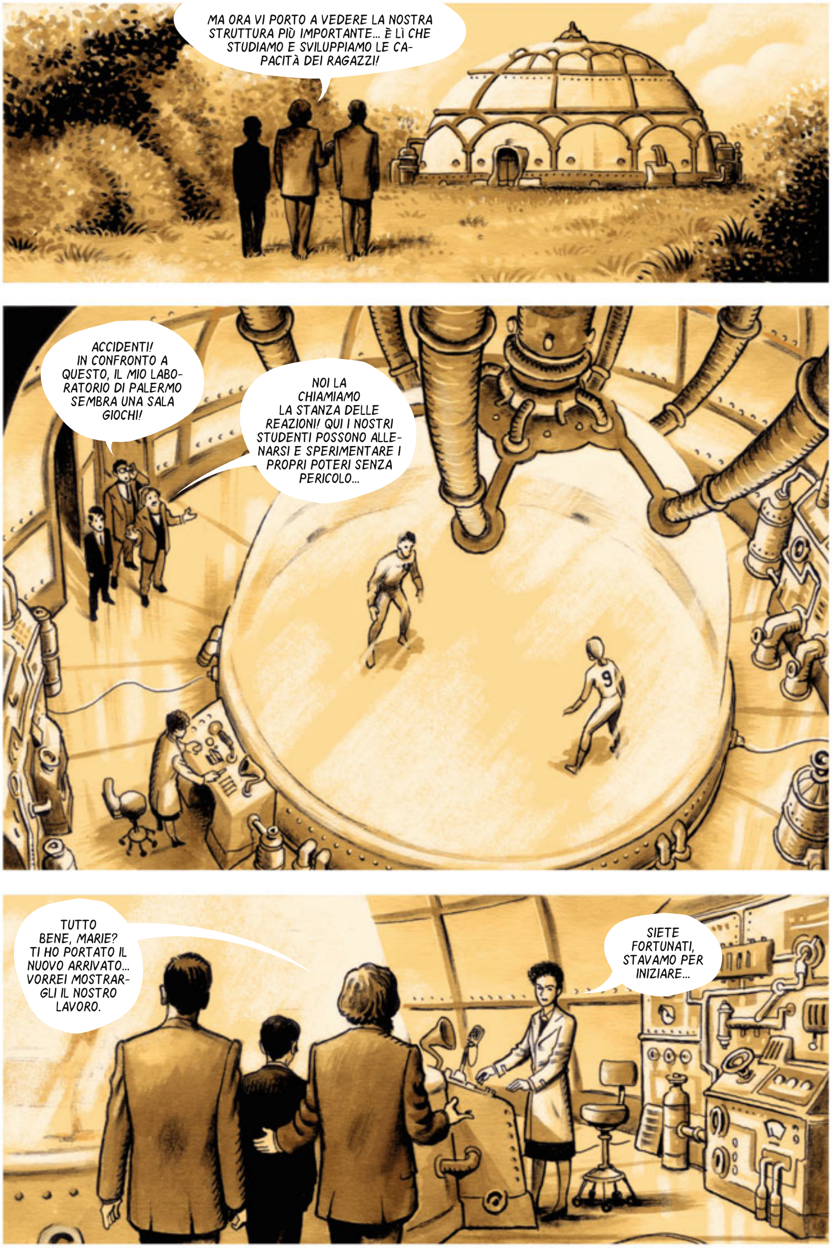

Fig. 6 The "Reaction chamber" devised by Giovanni Eccher and Sergio Ponchione for "gifted elements" (from "L'Accademia del Professor M per elementi dotati;" *Comics&Science* 002/2019, October 2019)

To start with, we tried our hand at Artificial Intelligence, a fascinating and crucial issue, already permeating every aspect of modern Science. Blossomed as a distinct discipline from Alan Turing's researches during the Fifties, AI has progressed by bumps and jumps, with years of stagnation often following (and followed by) important breakthroughs. All the while movies and Science Fiction have been giving voice to great hopes and not lesser fears, with fascinating tales not always so far from reality.

Today's AI is part of our everyday lives: we find it in cell phones, computers, biomedical imaging's analysis and in natural language recognition. Something almost unthinkable until not so many years ago. What happened? How did we get where we stand now? And most of all: what looms at the horizon? Again, AI's path is paved with fears and hopes: complex and crucial themes for science and society at large. Strictly cooperating with AIxIA—Associazione Italiana per l'Intelligenza Artificiale ("Italian Association for Artificial Intelligence"), we targeted these topics with the help of Diego Cajelli and Andrea Scoppetta. Cajelli is and experienced comics writer, routinely handling important Italian comics properties like most Bonelli characters (Bonelli is by far Italy's leading comics publisher) and the iconic "Diabolik" series. He also teaches "Crossmedial Storytelling" at the Sacro Cuore University in Milan. Scoppetta is a cartoonist, illustrator and animator who contributed to world-renowed productions by Disney/Pixar and Dreamworks. *N3well's visit* is their *C&S* take on AI and the very classic, evergreen theme of the "thinking machine" (see Figs. 7 and 8): N3well is a robot and much, much more, as readers will discover following *him* looking for *his* origins. Something normal for human beings and downright surprising for an artificial mind. With a heartfelt tip of the hat to Isaac Asimov's centenary.

From today to a remote past as a way to "Imagine Math": Leonardo "Pisano" ("from Pisa"), better known under his *Filius Bonaccii*, or Fibonacci, family name, was allegedly born in 1170. In 2020, 850 years after, *C&S* joined the town of Pisa and honorable institutions like the local University, "Scuola Normale Superiore" and—again—"Museo degli strumenti per il calcolo" in a series of events celebrating the anniversary. Fibonacci's *Liber Abbaci* (1208) brought the Indo-Arabic positional notation for numbers to Europe, a durable legacy which still today—every day—tells how much Mathematics, technology and Science as a whole owe him. First of all, his book was intended to be of help to merchants and businessmen, illustrating practical problems and how to solve them using the "new numbers" and the "new ways" of handling them (which today we call algorithms). So, illustrator and cartoonist Claudia Flandoli concocted *Il libro di Leonardo* ("Leonardo's Book;" see Fig. 9), a brilliant rendition of Leonardo's early years as a young man returning to his native town, telling his friend Sara—and all of Western World and us with her—how everybody's lives are going to be changed forever by what he learned from "the Arabic scholars."

One specific *C&S* aspect might turn out to be even more relevant than its comic books' success. We call it "fertility." Since its inception, *C&S* inspired—we like to say "catalyzed"—many other different, often fully spontaneous and independent projects in its own vein. As an example, we recall here only four of them.

Fig. 7 Diego Cajelli (story) and Andrea Scoppetta (art) and their difficult child (from "N3well;" *Comics&Science* 001/2020, April 2020)

Fig. 8 How N3well became all too human (ibid.)

Fig. 9 How young Fibonacci became acquainted with rabbits, as told by Claudia Flandoli (from "Il libro di Leonardo;" *Comics&Science* 002/2020, October 2020)

– "Archimedia:" A two-page short story in comics form for *Archimede*, a historical Italian journal devoted to Mathematics and mainly aimed to teachers.
– Two series of book mass-marketed through the national network of newsstands: *I manga delle Scienze* ("Science Manga") and *I grandi della Scienza a fumetti* ("Great Scientist in comics").
– The series of Disney tales "Topolino Comic & Science" (almost a namesake), which Roberto Natalini, Alberto Saracco, world-renowned theoretical physicist Carlo Rovelli, Fields medalist Alessio Figalli and other researchers (all die-hard Disney fans) personally contributed to, both as advisors and writers, suggesting themes and topics.

These positive feedbacks have been very encouraging, taking us—almost by the hand, in a way—to a single story we were eager to tell, after years of it lingering in our minds and conversations: the life and works of Italian mathematician and physicist Vito Volterra.

One of the founding fathers of Functional Analysis, Volterra pioneered more than one crucial field of research, from Integral and Integro-Differential Equation to Bio-Mathematics (population dynamics, predator-prey models), bringing his new, visionary approach to both fundamental and applied scientific research, which—way back in 1923—led him to found what today is CNR—Consiglio Nazionale delle Ricerche, a model *ante litteram* for many European research institutions and agencies to come.

He was also passionately politically engaged, "Senatore del Regno per meriti scientifici" (Senator of the then-Kingdom of Italy for scientific distinction) from the age of 40, strenuously opposing the rising fascist regime, which succeeded in marginalizing and then expelling him from his academic positions, upon his refusal of taking a "solemn fidelity oath" (1931).

In a joint venture with Italian major publisher Feltrinelli Editore, *C&S* editorial board edited and produced a graphic novel, a biographical comic in book form telling the story and the political hardships of this illustrious the Twentieth Century Italian mathematician.

Cartoonists Alessandro Bilotta (writer) and Dario Grillotti (artist), both acclaimed professionals in their own field, joined forces giving birth to a compelling tale of knowledge, Science and civil passion as ways to improve our lives, making them better and worth living.

It is quite obvious how the main reason for *C&S* to work out so well is that something was "in the air," in some sense, while kind of an astonishment for the unreasonable effectiveness of the very basic idea of using comics in order to boost interest for science, still stands.

What we see, from this viewpoint, is that comics, like mathematics, are not simply a language—a "structured" way to tell or explain things in a very specific way—but a way to look at the world, telling its stories with a concise, terse approach.

Archimedes used to carve his diagrams in sand and—we like to think—would subscribe to that (Fig. 10).

Fig. 10 (from "Archimede Infinito" a short story by Giuseppe Palumbo; *Archimede 1/2016*, March 2016)

Is Math Useful?

Alberto Saracco

Introduction

"Is math useful?" might sound as a trick question. And it is. Of course math is useful, we live in a data-filled world and every aspect of life is totally entwined with math applications, both trivial and subtle applications, of both basic and advanced math. But we need to ask once again that question, in order to truly understand what is math useful for and what being useful means. Moreover, is it knowledge of math useful for a class of specialists, or for political leaders or for all people at large? Being more on a concrete level, why does math need to have a central role in education? Each section will be titled by a question. And each section will not give an answer, but—at least I hope—provide some food for thought to the reader, in order to try to come up with his or her own answers.

I feel that these kind of questions are at home in a book devoted to the interplays between mathematics and culture: what is the space we should give to math in culture and what is math's role in becoming a complete citizen?

1 Is Math Useful?

> The mass of mathematical truth is obvious and imposing; its practical applications, the bridges and steam-engines and dynamos, obtrude themselves on the dullest imagination. The public does not need to be convinced that there is something in mathematics. All this is in its way very comforting to mathematicians, but it is hardly possible for a genuine mathematician to be content with it.
>
> A mathematician's Apology [9] §2—G. H. Hardy

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Fig. 1 *Math describes reality*, according to young Roby Vic, a Disney version of the ESA astronaut Roberto Vittori in the story *3 gradini per le stelle—Paperino Paperotto, Roby Vic e i conti... alla rovescia* (*3 steps to the stars—Donald Duckling, Roby Vic and the countdowns*) [20] ©Disney

> [...] the most 'useful' subjects are quite commonly just those which it is most useless for most of us to learn. It is useful to have an adequate supply of physiologists and engineers; but physiology and engineering are not useful studies for ordinary men[...]
>
> A mathematician's Apology [9] §20—G. H. Hardy

The English mathematician Hardy dealt very well with the subject of this paper, and I will often cite his famous *Apology*,[1] written over 70 years ago. Since we are dwarves sitting on the shoulders of giants, I hope I will be able to see a little further and give some new ideas on the subject.

More precisely I would like to deal with the problem posed by the above two quotes: no one is usually fool enough to deny the usefulness of mathematics to our society, but the usefulness to a society is not at all the same as the usefulness to an individual. *Is math useful to me?* will be our second to last section.

Before getting there, tough, we have a long way. We first have to understand what is the usefulness of math and how math is (and can be) used.

2 How Is Math Used in War Time?

> Ten, twenty, thirty, forty, fifty or more
> The bloody Red Baron was running up the score
> Eighty men died trying to end that spree
> Of the bloody Red Baron of Germany
>
> Snoopy vs the Red Baron (1966)—The Royal Guardsmen[2]

[1] For an interpretation of Hardy's *Apology* we refer the reader to [2].

[2] Usually the books of the series *Imagine Math* are the proceedings of the meeting on mathematics and culture held in Venice. This year, due to the pandemic of Sars-Cov2, the meeting has not taken

Fig. 2 Snoopy vs the Red Baron (Peanuts comic book, photos courtesy of Milena Crovini and Andrea Cittadini Bellini)

Math has always been considered a strong ally in war time. Archimedes used math (and physics) to construct parabolic glasses in order to set on fire Roman ships. Math has been used to compute the line of firing of cannons: modern ballistics was born due to an English mathematician, Robins, who in 1742 wrote New Principles of Gunnery, a treaty which was used till World War II. Mathematicians have always been considered precious for war and enrolled for their logical and computing abilities. The English mathematician Littlewood, close friend, and fellow mathematician of the already cited Hardy served in the Royal Garrison Artillery during World War I.

Up to World War I, though, the math used in war time was quite elementary: basic geometry and physics.

During World War II math played a fundamental role and many different areas of mathematics turned out to be useful for winning the war. Everyone knows the story of how Alan Turing cracked Enigma, the Nazi cryptography tool, and heavily contributed to lead the Allies to a victory. Both cryptography and decryptography are based on deep math.

Moreover there is some statistics in figuring out the number of tanks produced by the Nazi: the problem of estimating the maximum of a discrete uniform distribution from sampling without replacement. In simple terms, suppose there exists an unknown number of items which are sequentially numbered from 1 to N. A random sample of these items is taken and their sequence numbers observed; the problem is to estimate N from these observed numbers. This problem is called the German tank problem, since it was of uttermost importance to the Allies: they wanted to estimate the number of German tanks just by knowing the serial numbers of the few tanks captured.

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place. Anyhow you may think of me beginning my lecture playing with planes while the song *Snoopy vs the Red Baron* is being played. I suggest you to listen to this song while beginning to read this chapter, to put yourself in the right mood.

Fig. 3 Area of damage of damaged airplanes returned to base during WWII (image from Wikipedia). Where do the airplanes need to be reinforced?

But probably the nicest use of math in WWII is that Wald did to evaluate where planes would need additional armor against enemy's shootings. Data of damaged airplanes was collected by US military, leading to the picture of Fig. 3. The US military concluded that the area in need of ticker armor where the ones with the most shoots. Wald concluded the opposite: the areas in need for additional armor were precisely the one with least red dots, since the sample was made up just of planes who survived the enemy's fire. The planes which were shot elsewhere did not made their way back home: they simply crushed down, as if they were shot by the bloody Red Baron. So the parts to reinforce were exactly those which when shot did not allow the plane to come back and its damage to enter the stats.

Math is no doubt useful in war time, both for computational purposes (and in this I put also physics and computer science) and for the mathematical-logical thinking.

What Is Math Useful For?

I have no doubt that the reader, when reading the title of this chapter, immediately thought that math *is useful* and did not think about math in war time, which—depending on how you put it—may be described as useful or bloody dangerous. So, why did I choose such a subject to begin my chapter? I will let Hardy answer:

> I once said that 'a science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life'
>
> A mathematician's Apology [9] 21—G. H. Hardy

This extremely pessimistic phrase was spoken by Hardy in 1915, when times were dark and there was little space for hope and for the future of humankind. Nevertheless, way too often the *usefulness* of something has

(continued)

indeed had the effect to accentuate inequalities or favor wars, as Hardy stated. Hence, I feel that we should start discussing usefulness of math by starting from its darkest sides, not hiding them under the carpet, but being well conscious of their existence.

It is also worth noticing that, usually, when a war time example of usefulness of math is made, it is usually a situation in which the good Allies used math to win against the Nazi.

It is as if, when talking about the applications of math to the real world, we try to hide the darker sides of math's applications, and—if ever we talk about math and war—cite only occasion in which math has been used to make the Allies won WWII vs the Nazi, i.e., show war-time-math as the hero in a classic war movie.

Of course, reality is much more various than a movie, and in war math helped killing people as much as saving them.

We must deal with this whenever we try to answer the question whether math is useful or not: math is a tool, and like most tools it can be used in a wise or a wicked way.

3 How Can Pure Math be Useful?

There are uncountable[3] applications of math in every day's life. While most of the applications known to the wide public rely on basic math, or on math born explicitly for applications, I would like to give some example of pure math which later on turned into applied math. As the online comics Abstruse Goose puts it, *all math is eventually applied math* (see Fig. 4).

If you feel a geek online comics is not a good enough reference,[4] I will go with Galileo:

> [Nature's book] is written in mathematical language, and its characters are triangles, circles and other geometrical figures.[5]
>
> *Il Saggiatore*—Galileo Galilei

Our limits in applying mathematics in describing the world are just those of our knowledge (deep and true knowledge) of it.[6] We may think of our knowledge as a

[3] Obviously this is an hyperbole, since everything in the real world is not only countable, but finite!

[4] And you would be wrong. Comics are totally part of culture and they also had a good place in this series of books, see, e.g., [1, 14].

[5] *[Il libro della natura] è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche*, in the original.

[6] Is it "the world" or "math"? I will leave the answer to the reader.

Fig. 4 All math is applied math...eventually—Personal reinterpretation of the web-comics Abstruse Goose *Impure Math* drawn by Sofia Saracco [3]

box of tools. As soon as we have a tool in it, we may find some uses for the tool. When we do not have the tool (or we even ignore its existence), we cannot find uses for the tool. And mathematical tools, being completely general and abstract, have a wide range of possible applications. What is really needed is to create the mathematical tools (doing mathematical research, both pure and applied) and handle them to people who may need them.

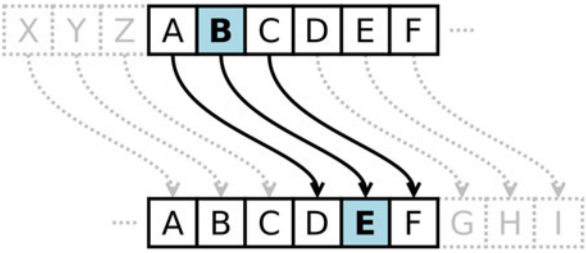
Thus, in this section I go with some example of pure math applications to the real world.

3.1 Number Theory and Cryptography

The Abstruse Goose comics (Fig. 4) gets it right: no matter how pure and far from applications a part of mathematics may be, once it is in our tool box it is only a matter of time since it will find some applications.[7]

[7] Again an hyperbole: some theorems are just too weird to find an application...or are they just too weird for now? Maybe just because we do not understand that result well enough?

Fig. 5 Caesar's cryptographic method: just replace a letter with the one three places after in alphabetical order (image from Wikipedia)



3.1.1 Number Theory

So it is just appropriate to begin with an example about number theory, the field of research of Hardy, who was absolutely proud of doing research in a pure area of math, with no applications whatsoever.

Hardy in his *A mathematician's apology* [9] makes quite a point of personal pride in number theory being a completely pure (and useless) branch of mathematics. And he was quite right! Number theory deals with the distribution and the properties of prime numbers and so it is a subject of great charm, ancient (the proof that there are infinite prime numbers and Eratosthenes' sieve date back to Ancient Greece) and full of elementary problems (e.g., Goldbach's conjecture), which are easy to state and extremely difficult to solve. These characteristics lead to a heap of amateurs trying to solve very difficult conjectures in this field across centuries.[8] Few succeeded, most did not, and a lot of apparently simple conjectures are still unsolved.

So Number Theory always had a great appeal, both to professional mathematicians, amateurs and the wide audience. But no one ever questioned its being a totally pure and abstract area of mathematics, whose interest belonged all to the world of pure ideas and not to our material world.

At least that was so until computer age begun and some old Number Theory theorems by Fermat become useful for cryptography.

3.1.2 Cryptography

Also Cryptography is an ancient subject. Sending secret messages has always been of crucial importance in war time (as we already stated in Sect. 2) and the first use of Cryptography dates back to Julius Caesar, who sent messages replacing each letter with the one 3 places after that in alphabetical order (see Fig. 5).

To read the original message, one should just reverse the arrows and replace each letter with the one 3 places before in alphabetical order.

This method of cryptography has several problems, of course.

[8] And this often lead to frustration professional number theorist who continue to receive "proofs" from amateurs, see, e.g., the nice *Dialogue on Prime Numbers* written by Zaccagnini [21].

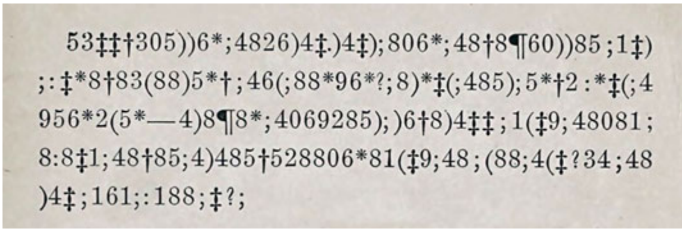

53‡‡†305))6*;4826)4‡.)4‡);806*;48†8¶60))85;1‡)
;:‡*8†83(88)5*†;46(;88*96*?;8)*‡(;485);5*†2:*‡(;4
956*2(5*—4)8¶8*;4069285);)6†8)4‡‡;1(‡9;48081;
8:8‡1;48†85;4)485†528806*81(‡9;48;(88;4(‡?34;48
)4‡;161;:188;‡?;

Fig. 6 The cryptogram in Poe's novel *The Gold-Bug* [13]

First of all, if one knows how to encrypt a message, also knows how to decrypt the message. Secondly, the possible shifts are just one less than the number of characters in your alphabet (not counting the 0-shift which does not encrypt): not too many to check fast by hand. Moreover, even if not a simple shift is used to encrypt but rather any permutation, if the message is long enough, a simple statistical analysis of most frequently occurring characters may yield to an easy decryption (as it is done in E. A. Poe's *The Gold-Bug*, see Fig. 6). Finally, both the receiver and the sender must know the crypting and decrypting keys in order to have a crypted communication between them. And how do they exchange the keys?

With the computer era, decrypting messages has become easier and easier. The faster the computers, the better encryption methods had to be.

The breaking of the Nazi encrypting machine *Enigma* by a huge group lead by Alan Turing was a key turning point of WWII.

3.1.3 Number Theory and Cryptography

Number theory was used to solve one of the biggest problem in cryptography. Namely, number theory allows for a method in which the encryption key is public but the decryption key is private, thus allowing anyone to send messages to the receiver (e.g., your password to a website or the PIN of your credit card to your bank) without having to worry about a third party decrypting the message.

This has been a major breakthrough in cryptography and its applications are amusing.

From the theoretical point of view, the RSA method is really simple. One needs to find two distinct prime numbers, p and q and computes $n = pq$ and $m = (p - 1)(q - 1)$. Then a number a such that $(a, m) = 1$ is chosen and the number b such that $ab \equiv 1 \pmod{m}$ is computed.

The pair (n, a) is the public key and is known to everyone. The number b is the private key and it is secret. The message is translated into a number $x < n$ and the sender sends the encrypted message $y < n$, where $y \equiv x^a \pmod{n}$. The receiver computes

$$y^b \equiv (x^a)^b = x^{ab} \equiv x\ (mod\ n)$$

thanks to Euler Theorem, thus getting to know the original message.

The operations of taking a power up to a congruence class is not much time consuming and can be easily done by a computer. Finding out from n its prime factors p, q is completely a different task, in term of computation time.

Indeed, nowadays all the factorization algorithms are too slow. To be more precise, all known algorithms to factor a number n in primes have a computational complexity (and hence time needed) growing exponentially with n, hence, for n big enough, the computation time to factor n is way too large. Obviously there is no guarantee the currently known algorithms are the best possible algorithms.[9] Only 15 years ago a new algorithm for finding out whether a number n is prime or not, whose complexity grows polynomially with n, was discovered. A new factorization algorithm may be found, and this could change drastically the size of n needed for a secure encrypting, or maybe even make RSA obsolete and useless.

Of course, even just considering the algorithms known nowadays, the greater the computational power of computers, the bigger the two primes p and q need to be. Nowadays the RSA key is 128 bits long (or 256 bits for TopSecret tasks).

3.2 Radon-Nikodym Antitransformation and Computed Axial Tomography

Looking inside a body may be a difficult task. Our body is not transparent and cutting a person in order to see what is on the inside may not always be a good idea. Radiography, using X-rays, helped in seeing bones, since the rest of the body is transparent to X-rays, but they are not a good means to inspect soft parts of our bodies.

Medicine was in search for a tool we were apparently lacking: a way to see inside our bodies without tearing them apart. The tool was only apparently lacking. Indeed math has invented ways to transform local information into global ones and vice versa: transformation and antitransformation. There are several of them, and they answer to different needs, but actually what a transformation does is taking as an input a function or a series of numbers, and giving back another function or a series of numbers; the antitrasformation goes the opposite direction and is an inverse to the transformation. Usually these tools work computing integrals.

Sending rays through your body and see how much they were absorbed was not a new idea (indeed it was used with X-rays and radiography), but it is just in the early Seventies that a physicist (Allan Cormack) and an engineer (Godfrey Housefield) had the idea of using Radon-Nikodim transformation and its inverse in order to compute from the information of rays absorbed in the various direction a 3D model of the inside of a body. This application of math eventually led to the Nobel Prize

[9] Should the 1 million dollar conjecture P=NP be proved to be true, there is a polynomial time algorithm to factor numbers in primes.

for Medicine (in 1979) for the two and gave a huge tool of diagnosis to hospitals all around the world.

When Radon-Nikodim transformation was developed in 1917, it was a completely pure and "useless" tool of high mathematics. Of course, computers were far from being invented, at that time, and practical applications of the Radon anti-transformation were unforeseeable. But, as we said, all mathematics is eventually applied mathematics. And you never know when something you know may actually turn useful. In any case, it is better to know more than less.

4 Why Politicians Should Know Math?

All of the above is a bunch of examples showing what is math useful to us as a species, as a community. But of course, we may ask ourselves if math knowledge should not be simply limited to mathematicians, engineers, and other people who may use it in their work for the benefit of the community at large. After all, we do not need people to know exactly how a bridge is constructed, how a TV works, or how to repair a broken engine. For that, we use people who know how to do it. Why should math be different?

I will address this question in the following sections. First, let us consider why politicians should know math.

Our modern world is a world filled with data and numbers. Decisions must be made based on those numbers and those data. But interpreting data is far from obvious, as the "survivor's bias" example should have already clarified. An inability to correctly interpret data may turn into a disaster. Indeed statistics is quite difficult.

4.1 Education System in the US, Covid-19 Death Toll and Simpson's Paradox

For many years, Wisconsin's students performed consistently better than Texas' students in standardized tests (see Fig. 7).

One could conclude that Wisconsin's education system is way better than Texas', and a politician willing to improve Texas' education system may be tempted to copy the one of Wisconsin. But is that a good idea?

Knowing the mean performance of a huge number of students for a long time may sound as pretty solid evidence towards this claim. But statistics is full of surprises.

Namely, if we divide the data of the students of the two States among different ethnic group (and we know that ethnic group correlates with wealth which correlates with education level), a surprise pops out: white Texas students outperform white

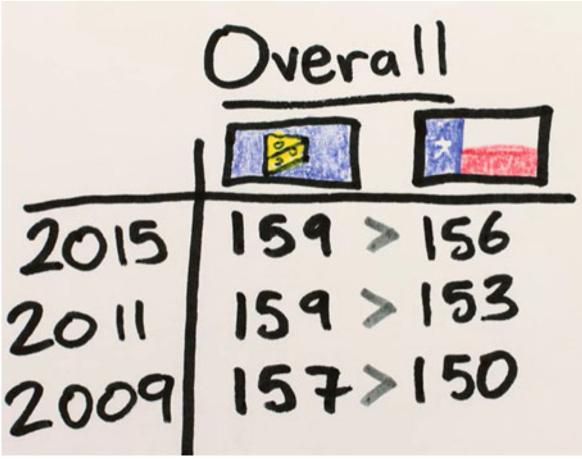


Fig. 7 Data of Texas and Wisconsin overall results in standardized tests (source MinutePhysics [11])

Wisconsin students, black Texas students outperform black Wisconsin students and Hispanic Texas students outperform Hispanic Wisconsin students (see Fig. 8).

So it actually looks that, when seen broken by race, data suggest that Texas' education system is better than Winsconsin's. How can data tell two different things? First of all, one of the problem is that the mean of some data is not the same as the mean of the means: it depends on how many data are there in every subgroup in which data have been divided. Wisconsin's population is much whiter than Texas': thus the overall mean of Wisconsin is much more tilted towards the white mean (which is the ethnic group performing better in the test) than it is the mean of Texas.

A similar situation happened when comparing the death toll in Italy and China at the beginning of the Covid-19 pandemic. Indeed, the overall fatality rate of the disease in Italy was bigger than the overall fatality rate of the virus in China, but—when people infected with Sars-CoV2 were split in age groups—in every single age group the fatality rate was greater in China than in Italy (see Fig. 9).

In this case the problem is that in Italy there were much more old people sick with Covid-19 than there were in China. And Covid-19 has a higher fatality rate in older patients. So, this explains the apparent discrepancy of the data.

This phenomenon, where there is a positive correlation overall, while—when data is divided in groups—there is a negative correlation, is called Simpson's paradox. Simpson's paradox is one of the things a politician should be aware of, before taking action according to data.

But actually the problem is deeper than that. Indeed, is Wisconsin's school system better or worse than Texas'? Is this second way to see data the correct one? If we say that ethnicity correlates with wealth (or with parents' education level) and this last thing correlates with results in tests, why are we using ethnic group and not wealth (or parents' education level) directly in order to interpret our data? A third look at data may be needed.

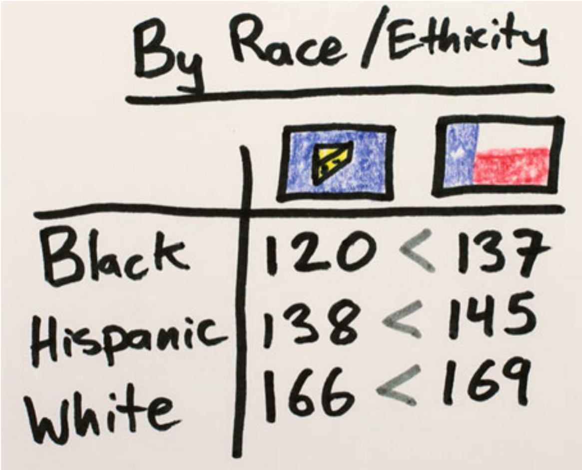


Fig. 8 Data of Texas and Wisconsin results in standardized tests, divided by race (source minutephysics [11])

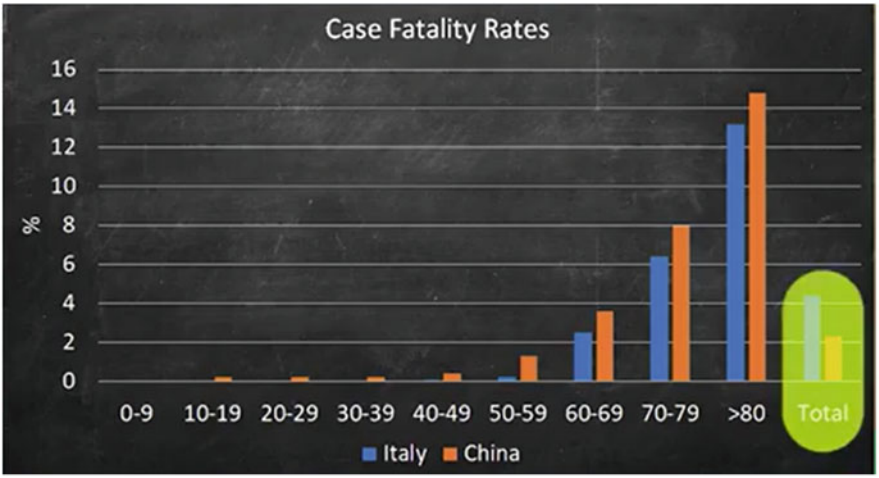


Fig. 9 Data of Italy and China fatality rates of Covid-19 cases (image from [6])

The point is: if you believe data are objective and need not to be interpreted and analyzed with a close look, you are likely to be fooled by data. If you know how statistics (and math) works, you are more likely not to get fooled and to take a second (or even a third) look at data before taking action (and possibly going in the wrong direction).

This is why politicians should have a good base in math and know how to analyze data. Before making decisions, the least you must have is correct data and info, and possibly understand correctly what they mean.

5 How Do Politicians Use Math?

You might say that politicians do not really have to know and understand math, in order not to fall into such errors, but just to have good advisors who do know math. And indeed they have. Plenty of them. And here we get to the problem.

First, as we have seen, data and numbers are far from being *objective*: data must be interpreted and investigated in order to understand what they say, but they are also easily bent to furnish support to almost any political view. So, sadly, way too often the scientific advisors of politicians try to cherry pick data or to present data in such a way to give a scientific-looking aspect to the political ideas they want to communicate. This when data are not right-away invented. But cheating too much is not even needed: the same data, presented with different words, from a different point of view, can lead to very different conclusions. And we must bear in mind that politicians often have a very skilled advisory group whose only purpose is to find the best way to present data.

Another new interesting tool of math (or computer science) often used by politicians is given by big data: *sentiment analysis* and *trending topics* are fundamental in political communication. In our modern world we have an incredible amount of data about almost everyone: use of credit cards, posts, or comments on social networks, our GPS position in real time, the shopping habits (both online and in physical stores), internet usage... The math of big data can extract patterns out of all this huge amount of data. And this is how your phone can suggest you the fastest path to go back home or where to buy the book you really want to read or the item you really needed. This can be useful, but of course all this information can be (and is!) used to make enormous profits.

Politicians are informed real time about the hottest arguments of discussion (*trending topics*) and on what most people think about the argument (*sentiment analysis*), and so they are ready to band-wagon on the hottest topic with the coolest opinion. It almost does not matter whether the opinions expressed are coherent with one another or not: what is really important is to say something on the trending topic of the day, with their opinion being shared and viewed by the highest possible number of people. In time of election, people will recognize your name, and you will have bigger chances of being voted, hence more votes. This is the core of marketing, applied to politics. Not so great if you think politics should be about solving the community's problems, but that is how it is. And there is a lot of math in that.

5.1 Paradoxes of Elections

So, we have decided that politicians' biggest task is getting elected (even if they are really interested in doing their work for the benefit of the community: in order to do that, they need to be elected). Alas, the outcome of elections is far from being determined from what voters think, and the electoral system is crucial for the result. This is exactly the reason why politicians spend so much time discussing the electoral system. This subsection is mainly based on my paper [18], on mathematical paradoxes of elections. I refer the reader to that paper for greater details.

Unluckily, no electoral system is perfect. In 1951, the economist Kenneth Arrow [5] considered a very general definition of electoral system as a function (which he calls *social choice function*) from the individual preferences among the alternatives of the electors to a single preference of the social group, where a *preference* is a total ordering of the alternatives. Arrow introduces three desirable properties of the social choice function:

A1 (sovereignity of electors) The function is surjective, i.e., if the electors agree on the desired outcome, they can vote (choose their individual preferences) in order to have that outcome;
A2 (positive correlation) If in a certain situation the social choice function says x is better than y, in any other situation in which the only change in any elector's preference is that their ranking of x gets higher, then x is still better than y;
A3 (Invariance under irrelevant alternatives) The relative position of x and y according to the social choice function depends only on the relative positions of x and y for each elector and not on the opinion on a third alternative z.

Arrow then proves that if there are at least three alternatives, the only social choice functions satisfying the above three axioms are dictatorships: the social choice function is simply a projection on one on the factors, or differently said the "will" of the people is the "will" of a single individual, the dictator (see Fig. 10).

Fig. 10 The only election satisfying Arrow's axioms is a dictatorship:
US: "If you vote YES, my proposals will be accepted. If you vote NO…"
GG: "they will be rejected?"
US: "No, they will be accepted, and your vote will be kept in an appropriate dossier"
©Disney [12]

This theorem actually means that, if more than two alternatives are allowed, our electoral system must not satisfy one of the above properties if we do not want to have a dictator. Electoral systems usually do not satisfy the axiom **A3** and the outcome of election can be modified by the presence or absence of otherwise totally irrelevant political forces.

Politicians (or their advisors) know well this fact, and this explains all the fight on whether some little meaningless party should be allowed or not to participate in an election.

The Theorem of Arrow is based on Condorcet paradoxes, i.e., situations where you have 3 alternatives A-B-C and A wins vs B, B wins vs C, and C wins vs A, in a rock-paper-scissor way.

The Theorem of Arrow may suggest that a system with only two alternatives to choose from is the best one. The most useful electoral system to force politicians to gather in only two major party, thus having a system with only two alternatives and a way out of Arrow's paradox, is a one-district one-seat system, where the party who gets the most votes in the district takes the seat, and all other votes for the other parties are meaningless.

Unluckily the system one-district one-seat has one big weakness: the outcome of the election strongly depends on the shape of the districts and a party which has the power to decide the shape of the districts may win the election in a 1 vs 1 race with as little as slightly more than 25% of the votes. This is due to the fact that the party can lose 0–100% in slightly less than half the districts and win by barely one vote over 50% in the remaining (slightly more than half) districts, thus winning the election. Of course it is impossible to have complete information about votes, but big data analysis gives parties quite a good level of knowledge about voting intentions, thus allowing an easy win even if the electorate strongly favors the other party.

This art of carefully shaping the districts in order to win is called *gerrymandering* in honor to the salamander-shaped district designed by Governor Elbridge Gerry to win an election (see Fig. 11), but is still well used nowadays in the US, by both parties and districts of really weird shapes are not at all uncommon. Mathematical research on the subject of gerrymandering is very active, to limit gerrymandering, both by finding the objective subdivision on districts or by giving measures to find out whether there has been some gerrymandering going on in order to cheat or if the subdivision is fair. The two main approaches to the problem are an analytical approach using isoperimetric-like techniques and a discrete geometry approach, using weighted graphs. I proposed an approach of this second kind [17].

All this is why it is usually forbidden to change the rules of an election (or the shape of districts) too near to the upcoming election. But of course, regulations do not completely stop politicians to use math for their own benefit.

Fig. 11 The satirical panel, with the salamander-shaped district, published on the Boston Centinel to mock Governor Gerry in 1812. Image of public domain, from Wikipedia

6 Is Math Useful to Me?

Let us now address the main point of this paper: how is math useful to *me*? Why should *I* learn math? Can't just a few people be knowledgeable of math for the whole society's benefit?

I once heard that when a kid learns to read by itself, it does no longer have to depend on others to read and can find out what is written around without having to trust others to read correctly to them. Math is a powerful tool to read the world, and knowing math gives you the power to independently analyze the complexity of the world, without blindly trusting others to do that for you.

Be aware! I am not saying you should not trust others or the scientific community, not at all! What I am saying is that, being able to do the math by yourself or—even better—to mathematize a problem and to take a look at it through the magnifying glass of math is always a good idea, not only to find out a correct answer yourself, but mainly to find out who is trying to trick you and who you should trust.

So a first answer is that the more you know, the less you are likely to be fooled by people who want to gain something by fooling you, by politicians or lobbies who want to push their own ideas, or simply by arguments which may look plausible until inspected closely. Knowing math, or even better being able to reason with a math-oriented mind, is fundamental for every single individual.

Of course the problem is that it is not enough being able to reason correctly and foresee what is coming, if the majority of the population does not share this ability and is easily fooled into non rational behaviors. This is always true, but even more in a period where acting fast and correctly is the key to avoiding a disaster.

In October 2020, at a EU meeting, the German Chancellor Angela Merkel said "*Once we got to this point, closures are the only possible choice: we should have acted earlier, but people would have hardly understood. They need to see hospitals'* *beds full...*".[10] Angela Merkel has a Ph.D. in physical chemistry and knew pretty well what was going on. She even gave, some month before, a very nice explanation of the meaning of the index R_t in an epidemics. But that was not enough. She was knowledgeable, she was powerful, but still she could not act without her citizens being fully aware of what was going on. She was in the sad situation where she could foresee hospitals' beds getting full and death tolls raising to very high levels without being able to act tempestively.

The consequences of a low mathematical literacy of the vast majority of the population are terrible: more deaths, more pressure on the health system, more economical consequences... In order to avoid this in the future (the damages of the present situation cannot be undone, alas), a wide-spread math literacy is needed. Math allows you to see that an exponential growth of an epidemics means the hospitals' beds will be full and to act timely, in order not to have them full.

Without people clearly understanding that, the action needed to solve the problem is also the action that will make people shout "*Nothing happened! There was nothing to be worried about! We should have not done this*".

So, not only math gives you instruments to understand, analyze the world and not getting fooled, but a wide-spread knowledge of mathematics will turn into a huge benefit for the whole society.

7 Is Math Usefulness Relevant to Learners?

I hope we have cleared that math is useful to everyone and to the population at large. Given that, the fact that something is useful to someone, it does not straightforwardly imply that someone will be interested in learning it, much more so when you are dealing with children or kids or young adults. Showing the utility of something may make want some students to learn something, but most of them will be simply bored as hell.

I will just quote Paul Lockhart, who makes his Salviati go right to the point[11]

> It may be true that you have to be able to read in order to fill out forms at the DMV, but that is not why we teach children to read. We teach them to read for the higher purpose of allowing them access to beautiful and meaningful ideas. Not only would it be cruel to teach reading in such a way— to force third graders to fill out purchase orders and tax forms— it would not work! We **learn** things **because they interest us now, not because they might be useful later**.
>
> A Mathematician's Lament [10]—P. Lockhart

[10] Translated in English by the author.

[11] The emphasis is mine.

We learn because we are interested, because we are amused by something, not because we have to or because it will do us some good or it will make us a better person!

Luckily math is filled with interesting ideas and theories! We must never forget this, and when trying to appeal a young learner *usefulness* should not be our guide through mathematics. Math was not born *because* it was useful, but because it is amusing. Math is filled with interesting problems, which can be given to kids and adult of different ages and knowledge, in order to hook them into mathematics.

For the sake of completeness, I will just present an example, but many more can be found in the very indicated reading of Lockhart's pamphlet [10].

Second Degree Polynomials Usually, when studying second degree polynomials, students are presented with a huge nomenclature (pure polynomial, spurious polynomial…) and a vast casuistry to solve particular polynomial equations, then they are given a lot of exercises to practice the method that was given. After that, they are given the general formula

$$ax^2 + bx + c = 0 \quad \Leftrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(possibly also with a second variant in case b happens to be even) and then a new round of dumb exercises, each one equal to the previous. Totally boring.

I mean, I know that during the history of math all these different kinds of equations were solved (and given funny names), but math is not about zoology or funny names: math is about the struggle to find a path that lead to the solution of problems. Not necessarily the smartest and shortest path. Not immediately, anyhow.

A possible different teaching sequence would be to start from problems: give the students some problems, which once mathematized turn out into solving a second degree polynomial. Some of the problems should be easily solvable (i.e., lead to an equation of the form $x^2 = d$ or $x^2 + bx = 0$), some should lead to a complete equation with no vanishing terms. The students will find by themselves how to solve the simpler ones, and maybe will even give a try to the more difficult ones. Guided by the teacher, working in groups, they may rediscover themselves the formula (maybe by completing the square) and teach it to other groups. A discovery made by themselves, while trying to solve a problem and effectively experiencing the hardness of the problem and the sense of joy that comes with the solution, will leave the students something more than just a formula to blindly apply. Most of all, it will leave them with the sense of doing math.

And after the group work, a good recap by the teacher would be nice, so to put all the ideas (which came from the students) in order. Doing like that, probably a lot more of them will remember the formula, but most important will know how to find it again if needed.

So…Is Math Usefulness Relevant to Learners?
In my opinion it is very little relevant for their decision to willingly learn maths, and both teachers and popularizers of mathematics should not focus too much on applications and usefulness of mathematics, since applications and usefulness are often not immediate, but rather on the joy and the challenges of doing mathematics.

8 Is Math Popularization Useful?: A Math Popularizer's Apology

To end this chapter, I would like to give an apology to the activity of math popularization. Hardy had quite harsh words for this activity, and I feel most of my colleagues agree with him: any amount of time spent in popularizing math is time stolen from doing actual math, and probably, if you do that, it is just because you are not good enough to do actual math.

> If then I find myself writing, not mathematics, but 'about' mathematics, it is a confession of weakness, for which I may rightly be scorned or pitied by younger and more vigorous mathematicians. I write about mathematics because, like any other mathematician who has passed sixty, I have no longer the freshness of mind, the energy, or the patience to carry on effectively with my proper job.
>
> A Mathematician's Apology [9] §1—G. H. Hardy

I am sorry for my colleagues, but if this vision of popularization or communication of mathematics was ok in the Forties of the last century, it is no longer so nowadays. For a better and longer essay on this subject, I refer the reader to the article by Silvia Benvenuti and Roberto Natalini [7].

There are several top mathematicians deeply involved in communication of math (just think at the Fields medalists Cedric Villani and Alessio Figalli, to name two). Moreover, in the 2011 *European's Charter of Researchers* [8] it is clearly written that scientists should be directly involved in communicating their own researches to the wide public in order to favor the creation of a scientific mind.

> Researchers should ensure that their research activities are made known to society at large in such a way that they can be understood by non-specialists, thereby improving the public's understanding of science. Direct engagement with the public will help researchers to better understand public interest in priorities for science and technology and also the public's concerns.
>
> European's Charter of Researchers [8]

The reason for that is precisely what we tried to outline and suggest in this chapter: a scientific-leaned mind is needed for the well-being of the society at large,

Fig. 12 A comic by Saturday Morning Breakfast Cereal [19] getting right to the point of how math is perceived

and—given how modern democracies work—it is a need of the whole society and not just for a few enlightened who are part of the governing class.

The idea many researchers have of themselves and of research is that they are needed by society (which is true) and they have no urge to explain to society why they are needed (and this is false). This idea that what matters for research is getting it done and not being presented to society at large is deeply fixed into researchers' minds, but it is false, in the sense that the society must be aware of the fact that investing (money, time, and effort) in research, both applied and pure, is what we need to do. And this is even more true when we talk of an inherently abstract subject as mathematics, whose practical implications are neither immediate nor evident.

I perfectly know the feeling of frustration when you are telling someone you are a mathematician, or you teach math and the response you get is that shown in the comic by SMBC (see Fig. 12).

You usually get on the defensive and have trouble to communicate the beauty of mathematics, or even—if you are tired—do not get at all into the subject. Or sometimes, people just say that they understand math is useful, but *not for them / they do not get it / they are not a math person* (chose one or more).

Communicating to the society is tough. And society at large are not people who willingly go to an event of science (or math) popularization, or not just them: society at large, like it or not, is mainly composed by people who have a problematic

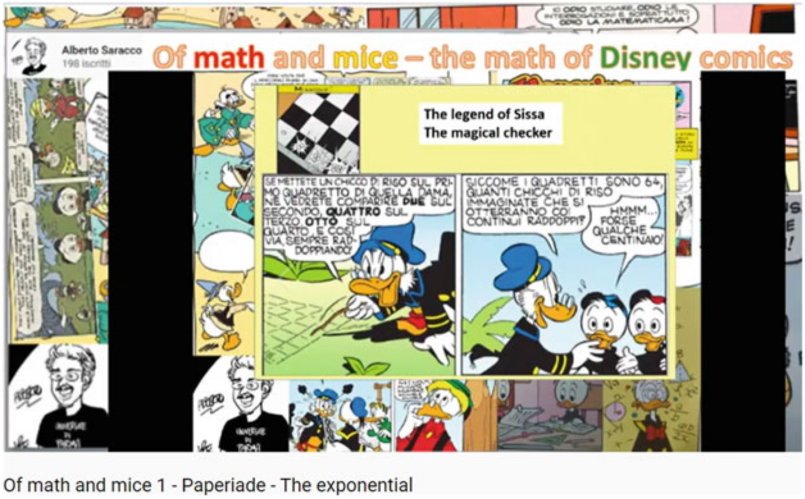


Fig. 13 The first video of the YouTube project *Of math and mice—the mathematics of Disney comics* [15], devoted to popularizing math using Disney comics, translation in English of the corresponding Italian project *Un matematico prestato alla Disney* [16], both available on my YouTube channel *Alberto Saracco*. The comics in the picture is ©Disney

relationship with mathematics, and they will not come to an event where you talk math to them.

> Part of the problem is that nobody has the faintest idea what it is that mathematicians do.
>
> A Mathematician's Lament [10]—P. Lockhart

Doing mathematics is an activity quite similar to that of an artist or a writer: there is a lot of technique involved, but also a lot of artistic out-of-the-box thinking and imagination. People are scared by technique (the only thing about math they know) and are not willing to know more about math.

It is up to you as a mathematician to get people to know what mathematicians do. They will not come to you. You have to go to them, by using their passions to talk about math. It is a while, since I started doing that with comics, using Disney comics to talk about math (see [14], but also my YouTube playlists on the subject [15, 16], see Fig. 13).

I noticed that when I put Disney comics or characters in the title of one of my talks, the audience is thrice as big as it usually is. And often many of the people in the audience were not interested at all about math at the beginning of the talk, but exited the talk with better feelings about the subject.

We need these kind of things in order to get math near to people who would not approach math. And, as I said, it is of fundamental importance. If we want society to grow and fully use math, it is up to us. As Francesca Arici, member of the *Raising Public Awareness* European Commettee, said

> Don't be afraid to use metaphors and don't be afraid to lie a little bit: the objective is to communicate math and not to make people see we can do math and prove theorems in a rigorous way.
>
> F. Arici [4]

Figures and acknowledgements

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Figure 4 is an elaboration based on the web-comics Abstruse Goose and Fig. 12 is from the web-comics Saturday Morning Breakfast Cereal. I thank both the authors for their fun science-filled comics.

Figure 6 is taken form [13].

Figures 7 and 8 are taken form the video [11] by MinutePhysics.

Figure 9 is taken from the video [6].

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