Chapter 1
Basic Concepts

This chapter collects well known concepts and results that will play a major role in constructing approximate fixed point theory in the remaining chapters. We note that we will reference the appropriate source papers after Sect. 1.2.8 (before this subsection well known results are presented so that the book is self contained). A brief introduction on fixed point theory is given at the end of this chapter.

1.1 Topological Spaces

1.1.1 The Notion of Topological Spaces

The topology on a set X is usually defined by specifying its open subsets of X.

Definition 1.1 A topology τ on a set X is a family of subsets of X which satisfies the following conditions:

1. The empty set $\emptyset$ and the whole X are both in τ.
2. τ is closed under finite intersections.
3. τ is closed under arbitrary unions.

The pair (X, τ) is called a topological space.

The sets $Y \in \tau$ are called open sets of X and their complements $Z = X \setminus Y$ are closed of X. A subset of X may be neither closed nor open, or both. A set that is both closed and open is called a clopen set.

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Examples 1.1

(i) Let X any set. Then $\tau = \{\emptyset, X\}$ is a topology on X, called the trivial topology on X.
(ii) At the other extreme of the topological spectrum, if X is any nonempty set, then $\tau = P(X)$ the power set of X, is a topology on X, called the discrete topology on X.
(iii) Let $X = \{a, b\}$, and set $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then τ is a topology on X.
(iv) Let (X, d) be a metric space. Let

$$\tau = \Big\{ Y \subseteq X\colon \text{ for all } x \in Y, \text{ there exists } \delta > 0 \text{ such that } B_\delta(x) = \{y \in X\colon d(x, y) < \delta\} \subseteq Y \Big\}.$$

Then τ is a topology, called the metric topology on X induced by d. This is the usual topology one thinks of when dealing with metric spaces, but as we shall see, there can be many more.
(v) Let X be any nonempty set. Then

$$\tau_{cf} = \{\emptyset\} \cup \{Y \subseteq X\colon X \setminus Y \text{ is finite }\}$$

is a topology on X, called the co-finite topology on X.

Definition 1.2 Let (X, τ) be a topological space and $Y \subseteq X$. Then $Y \cap \tau = \{Y \cap U\colon U \in \tau\}$ is called the induced topology on Y.

Definition 1.3 Let (X, τ) be a topological space and $Y \subseteq X$. We define

(i) The interior of a subset $Y \subseteq X$ is the largest open set contained in it. It will be denoted by $\operatorname{int} Y$. Equivalently, $\operatorname{int} Y$ is the union of all open subsets of X contained in Y.
(ii) A point $x \in X$ is a limit point (or accumulation point) of Y if and only if for every open set U containing x, it is true that $U \cap Y$ contains some point distinct from x, i.e., $Y \cap (U \setminus \{x\}) \neq \emptyset$. Note that x need not belong to Y.
(iii) The point $x \in Y$ is an isolated point of Y if there is some open set U such that $U \cap Y = \{x\}$. (In other words, there is some open set containing x but no other points of Y.)
(iv) The closure of a subset Y, written $\overline{Y}$, is the union of Y and its set of limit points,

$$\overline{Y} = Y \cup \{x \in X\colon x \text{ is a limit point of } Y\}.$$

Remark 1.1 It follows from the definition that $x \in \overline{Y}$ if and only if $Y \cap U \neq \emptyset$ for any open set U containing x. Indeed, suppose that $x \in \overline{Y}$ and that U is some open set containing x. Then either $x \in Y$ or x is a limit point of Y (or both), in which case $Y \cap U \neq \emptyset$. On the other hand, suppose that $Y \cap U \neq \emptyset$ for any open set U containing x. Then if x is not an element of Y it is certainly a limit point. Thus $x \in \overline{Y}$.

Proposition 1.1 *Let (X, τ) be a topological space and $Y \subseteq X$. The closure of Y is the smallest closed set containing Y, that is,*

$$\overline{Y} = \bigcap\{Z\colon Z \text{ is closed and } Y \subseteq Z\}.$$

Corollary 1.1 *A subset Y of a topological space is closed if and only if $Y = \overline{Y}$. Moreover, for any subset Y, $\overline{Y} = \overline{\overline{Y}}$.*

Proof If Y is closed, then Y is surely the smallest closed set containing Y. Thus $Y = \overline{Y}$. On the other hand, if $Y = \overline{Y}$ then Y is closed because $\overline{Y}$ is. Now let Y be arbitrary. Then $\overline{Y}$ is closed and so equal to its closure, as above. That is, $\overline{Y} = \overline{\overline{Y}}$. ■

Definition 1.4 Let (X, τ) be a topological space.

1. A subfamily $\mathcal{B}$ of τ is called a base if every open set can be written as a union of sets in $\mathcal{B}$.
2. A subfamily $\mathcal{X}$ is called a subbase if the finite intersections of its sets form a base, i.e. every open set can be written as a union of finite intersections of sets in $\mathcal{X}$.

Examples 1.2

1. The collection $\mathcal{B} = \{(a, b)\colon a, b \in \mathbb{R}, a < b\}$ is a base for the usual topology on $\mathbb{R}$.
2. Let $\mathcal{S}$ be the collection of all semi-infinite intervals of the real line of the forms $(-\infty, a)$, and $(b, +\infty)$, where $a \in \mathbb{R}$. $\mathcal{S}$ is not a base for any topology on $\mathbb{R}$. To show this, suppose it were. Then, for example, $(-\infty, 1)$ and $(0, +\infty)$ would be in the topology generated by $\mathcal{S}$, being unions of a single base element, and so their intersection $(0, 1)$ would be by the axiom 2) of topology. But $(0, 1)$ clearly cannot be written as a union of elements in $\mathcal{S}$.
3. The collection $\mathcal{S}$ is a subbase for the usual topology on $\mathbb{R}$.

Proposition 1.2 *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. $\mathcal{S}$ is a base for a topology τ on X iff the following hold:*

1. *$\mathcal{B}$ covers X, i.e., $\forall x \in X, \exists\, B \in \mathcal{B}$ such that $x \in B$.*
2. *If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $B_3 \in \mathcal{B}$ such that $x \in B_3 \in B_1 \cap B_2$.*

Definition 1.5 Let (X, τ) be a topological space and $\in X$. A subset U of X is called a neighborhood of x if it contains an open set containing the point x. The neighborhood system at x is $\mathcal{N}_x = \{U \subseteq X\colon U$ is a neighborhood of $x\}$.

Theorem 1.1 *Let (X, τ) be a topological space, and $x \in X$. Then:*

(a) *If $U \in \mathcal{N}_x$, then $x \in U$.*
(b) *If $U, V \in \mathcal{N}_x$, then $U \cap V \in \mathcal{N}_x$.*
(c) *If $U \in \mathcal{N}_x$, there exists $V \in \mathcal{N}_x$ such that $U \in \mathcal{N}_y$ for each $y \in V$.*
(d) *If $U \in \mathcal{N}_x$ and $U \subseteq V$, then $V \subseteq \mathcal{N}_x$.*
(e) *$G \subseteq X$ is open if and only if G contains a neighborhood of each of its points.*

Remark 1.2 Conversely, if in a set X a nonempty collection $\mathcal{N}_x$ of subsets of X is assigned to each $x \in X$ so as to satisfy conditions (a) through (d) and if we use (e) to define the notion of an open set, the result is a topology on X in which the neighborhood system at x is precisely $\mathcal{N}_x$.

Definition 1.6 Let (X, τ) be a topological space. A (local) neighborhood base $\mathcal{B}_x$ at a point $x \in X$ (or a fundamental system of neighborhoods of x) is a collection $\mathcal{B}_x \subseteq \mathcal{N}_x$ so that $U \in \mathcal{N}_x$ implies that there exists $B \in \mathcal{B}_x$ so that $B \subseteq U$. We refer to the elements of $\mathcal{B}_x$ as basic neighborhoods of the point x.

Example 1.1 Consider (X, d) be a metric space equipped with the metric topology τ. For each $x \in X$, fix a sequence $(r_n(x))_{n\geq 1}$ of positive real numbers such that $\lim\limits_{n\to\infty} r_n(x) = 0$ and consider $\mathcal{B}_x = \{B_{r_n(x)}(x) : n \geq 1\}$. Then $\mathcal{B}_x$ is a neighborhood base at x for each $x \in X$.

Remark 1.3 Let (X, τ) be a topological space, and for each $x \in X$, suppose that $\mathcal{B}_x$ is a neighborhood base at x. Then $\mathcal{B} = \bigcup\limits_{x\in X} \mathcal{B}_x$ is a base for the topology τ on X.

Definition 1.7 If (X, τ) is a topological space and $x \in X$ and $\mathcal{B}$ is a set of open sets, we say that $\mathcal{B}$ is a local base at x if each element of $\mathcal{B}$ includes x and for every open set U that includes x there is some $V \in \mathcal{B}$ such that $V \subseteq U$.

Remark 1.4 If for each $x \in X$ the set $\mathcal{B}_x$ is a local base at x, then $\bigcup\limits_{x\in X} \mathcal{B}_x$ is a base for the topology of X.

Definition 1.8 Let (X, τ) be a topological space.

1. (X, τ) is said to be T_1 if for every $x, y \in X$ such that $x \neq y$, there are neighborhoods U_x of x and U_y of y with $y \notin U_x$ and $x \notin U_y$.
2. (X, τ) is said to be T_2 (or Hausdorff) if for every $x, y \in X$ such that $x \neq y$, there are neighborhoods U_x of x and U_y of y with $U_x \cap U_y = \emptyset$.

   We say that two subsets Y and Z can be separated by τ if there exist $U, V \in \tau$ with $Y \subseteq U$, $Z \subseteq V$ and $U \cap V = \emptyset$.
3. (X, τ) is said to be regular if whenever $Y \subseteq X$ is closed and $x \notin Y$, Y and $\{x\}$ can be separated.
4. (X, τ) is said to be normal if whenever $Y_1, Y_2 \subseteq X$ are closed and disjoint, then Y_1 and Y_2 can be separated.
5. (X, τ) is said to be T_3 if it is T_1 and regular.
6. (X, τ) is said to be T_4 if it is T_1 and normal.

Definition 1.9 Let (X, τ) be a topological space. An open cover of $Y \subseteq X$ is a collection $\mathcal{G} \subseteq \tau$ such that $Y \subseteq \cup_{G\in\mathcal{G}}$

A subset Y of a topological space (X, τ) is said to be compact if every open cover of X admits a finite subcover.

Proposition 1.3 *Suppose (X, τ) is a topological Hausdorff space.*

1. *Any compact set $Y \subseteq X$ is closed.*
2. *If Y is a compact set, then a subset $Z \subseteq Y$ is compact, if and only if Z is closed (in X).*

Proposition 1.4 *For a subset Y of a topological space (X, τ), the following statements are equivalent.*

1. *Y is compact.*
2. *If $(Z_\alpha)_{\alpha \in I}$ is any family of closed sets such that $Y \cap \bigcap_{\alpha \in I} Z_\alpha = \emptyset$, then $Y \cap \bigcap_{\alpha \in J} Z_\alpha = \emptyset$ for some finite subset $J \subseteq I$.*
3. *If $(Z_\alpha)_{\alpha \in I}$ is any family of closed sets such that $Y \cap \bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$, for every finite subset $J \subseteq I$, then $Y \cap \bigcap_{\alpha \in I} Z_\alpha \neq \emptyset$.*

Proof The statements 2. and 3. are contrapositives. We shall show that 1. and 2. are equivalent. The proof rests on the observation that if $(U_\alpha)_\alpha$ is a collection of sets, then $Y \subseteq \bigcup_\alpha U_\alpha$ if and only if $Y \cap \bigcap_\alpha (X \setminus U_\alpha) = \emptyset$. We first show that 1. implies 2. Suppose that Y is compact and let $(Z_\alpha)_{\alpha \in I}$ be a family of closed sets such that $Y \cap \bigcap_{\alpha \in I} Z_\alpha = \emptyset$. Put $U_\alpha = X \setminus Z_\alpha$. Then each U_α is open, and by the above observation, $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$. But then there is a finite set J such that $Y \subseteq \bigcup_{\alpha \in J} U_\alpha$, and so $Y \cap \bigcap_{\alpha \in J} Z_\alpha = \emptyset$, which proves 2.

Now suppose that 2. holds, and let $(U_\alpha)_\alpha$ be an open cover of Y. Then each $X \setminus U_\alpha$ is closed and $Y \cap \bigcap_{\alpha \in I} (X \setminus U_\alpha) = \emptyset$. By 2., there is a finite set J such that $Y \cap \bigcap_{\alpha \in J} (X \setminus U_\alpha) = \emptyset$. This is equivalent to the statement that $Y \subseteq \bigcup_{\alpha \in J} U_\alpha$. Hence Y is compact. ■

Remark 1.5 A topological space (X, τ) is compact if and only if any family of closed sets $(Z_\alpha)_{\alpha \in I}$ in X having the finite intersection property (i.e., $\bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$ for each finite subset J in I) is such $\bigcap_{\alpha \in I} Z_\alpha \neq \emptyset$.

Proposition 1.5 *A nonempty subset Y of a topological space (X, τ) is compact if and only Y is compact with respect to the induced topology, that is, if and only if (Y, τ_Y) is compact. If (X, τ) is Hausdorff then so (Y, τ_Y).*

Proof Suppose first that Y is compact in (X, τ), and let $(G_\alpha)_{\alpha\in I}$ an open cover of Y in (Y, τ_Y). Then each G_α has the form $G_\alpha = Y \cap U_\alpha$ for some $U_\alpha \in \tau$. It follows that $(U_\alpha)_{\alpha\in I}$ is an open cover of Y in (X, τ). By hypothesis, there is a finite subcover, $U_1, \cdots, U_n$, say. But then $G_1, \cdots, G_n$ is an open cover of Y in (Y, τ_Y), that is, (Y, τ_Y) is compact.

Conversely, suppose that (Y, τ_Y) is compact. Let $(U_\alpha)_{\alpha\in I}$ be an open cover of Y in (X, τ). Set $G_\alpha = Y \cap U_\alpha$. Then $(G_\alpha)_{\alpha\in I}$ is an open cover of (Y, τ_Y). By hypothesis, there is a finite subcover, say, $G_1, \cdots, G_m$. Clearly, $U_1, \cdots, U_m$, is an open cover for Y in (X, τ). That is, Y is compact in (X, τ).

Suppose that (X, τ) is Hausdorff, and let x, y be any two distinct points of Y. Then there is a pair of disjoint open sets U, V in X such that $x \in U$ and $y \in V$. Evidently, $G_1 = Y \cap U$ and $G_2 = Y \cap V$ are open in (Y, τ_Y), are disjoint and $x \in G_1$ and $y \in G_2$. Hence (Y, τ_Y) is Hausdorff, as required. ■

Theorem 1.2 *Let (X, d) be a metric space. Then X, equipped with the metric topology is T_4.*

Theorem 1.3 *Let (X, τ) be a compact, Hausdorff space. Then (X, τ) is T_4.*

Proof Let $Y, Z \subseteq X$ be two closed sets with $Y \cap Z = \emptyset$. We need to find two open sets $U, V \subseteq X$, with $Y \subseteq U$, $Z \subseteq V$, and $U \cap V = \emptyset$. Assume first that Z is a singleton, $Z = \{z\}$.

For every $y \in Y$ we find open sets U_y and V_y, such that $U_y \ni y$, $V_y \ni z$, and $U_y \cap V_y = \emptyset$. Using Proposition 1.3 we know that Y is compact, and since we clearly have $Y \subseteq \bigcup_{y\in Y} U_y$, there exist $y_1, \cdots, y_n \in Y$ such that $\bigcup_{i=1}^{n} U_{y_i} \supseteq Y$. Then we are done by taking $U = \bigcup_{i=1}^{n} U_{y_i}$ and $V = \bigcap_{i=1}^{n} V_{y_i}$.

Having proven the above particular case, we proceed now with the general case. For every $z \in Z$, we use the particular case to find two open sets U_z and V_z with $U_z \supseteq Y$, $V_z \ni z$, and $U_z \cap V_z = \emptyset$. Arguing as above, the set Z is compact, and we have $Z \subseteq \bigcup_{z\in Z} V_z$, so there exists $z_1, \cdots, z_n \in Z$, such that $\bigcap_{i=1}^{n} V_{z_i} \supseteq Z$. Then we are done by taking $U = \bigcap_{i=1}^{n} U_{z_i}$ and $V = \bigcup_{i=1}^{n} V_{z_i}$. ■

Definition 1.10 A topological space (X, τ) is said to be separable if it admits a countable dense subset.

Proposition 1.6 *Let (X, d) be a compact metric space. Then (X, d) is separable.*

Proof For each $n \geq 1$, the collection $\mathcal{G}_n = \{B_{\frac{1}{n}}(x) \colon x \in X\}$ is an open cover of X. Since X is compact, we can find a finite subcover $\{B_{\frac{1}{n}}(x_{(j,n)}) \colon 1 \leq j \leq k_n\}$ of X. It is then clear that if $x \in X$, there exists $1 \leq j \leq k_n$ so that $d(x, x_{(j,n)}) < \frac{1}{n}$. As such, the collection

$$\mathcal{D} = \{x_{(j,n)}\colon 1 \le j \le k_n, 1 \le n\}$$

is a countable, dense set in X, proving that (X, d) is separable. ■

1.1.2 Comparison of Topologies

Any set X may carry several different topologies.

Definition 1.11 Let τ, τ' be two topologies on the same set X. We say that τ is coarser (or weaker) than τ', in symbols $\tau \subseteq \tau'$, if for every subset of X which is open for τ is also open for τ', or equivalently, if for every neighborhood of a point in X with respect to τ is also a neighborhood of that same point in the topology τ'. In this case τ' is said to be finer (or stronger) than τ'.

Two topologies τ and τ' on the same set X coincide when they give the same open sets or the same closed sets or the same neighborhoods of each point, equivalently, when τ is both coarser and finer than τ'.

Two basis of neighborhoods of a set are equivalent when they define the same topology.

Remark 1.6 Given two topologies on the same set, it may very well happen that no-one is finer than the other. If it is possible to establish which one is finer, then we say that the two topologies are comparable.

Example 1.2 The cofinite topology τ_c on $\mathbb{R}$, i.e., $\tau_c = \{U \subseteq \mathbb{R}\colon U = \emptyset \text{ or } \mathbb{R} \setminus U \text{is finite}\}$, and the topology τ_i having $\{(-\infty, a)\colon a \in \mathbb{R}\}$ as a basis are incomparable. In fact, it is easy to see that $\tau_i = \{(-\infty, a)\colon a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ as these are the union of sets in the given basis. In particular, we have that $\mathbb{R} \setminus \{0\}$ is in τ_c but not τ_i. Moreover, we have that $(-\infty, 0)$ is in τ_i but not τ_c. Hence, τ_c and τ_i are incomparable.

Proposition 1.7 *If τ_1, τ_2 are Hausdorff topologies on a set X such that τ_2 is finer than τ_1 and such that (X, τ_2) is compact, then $\tau_1 = \tau_2$.*

Proof Let Y a τ_2-closed set. Since (X, τ_2) is compact then Y is τ_2-compact. Since $\tau_1 \subseteq \tau_2$ it follows that Y is τ_1-compact (any τ_1-open cover of Y is also a τ_2-open cover of Y and has a finite subcover). Since τ_1 is Hausdorff and Y is τ_1-compact then it is also τ_1-closed, which completes the proof (we showed that every τ_2-closed set is a τ_1-closed set). ■

Definition 1.12 Let X be a set and let F be a family of mappings from X into topological spaces:

$$F = \{f_\alpha\colon X \to (Y_\alpha, \tau_\alpha) \;:\; \alpha \in I\}.$$

Let τ be the topology generated by the subbase

$$\{f_\alpha^{-1}(V) \ : V \in \tau_\alpha, \alpha \in I\}.$$

Then τ is the weakest topology on X for which all the f_α are continuous maps (it is the intersection of all topologies having this property). It is called the weak topology induced by F, or the F-topology of X.

Proposition 1.8 *Let F be a family of mappings $X \to (Y_\alpha, \tau_\alpha)$ where X is a set and each (Y_α, τ_α) is a Hausdorff topological space. Suppose F separates points in X i.e., for any $x, y \in X$ with $x \neq y$, there is some $f_\alpha \in F$ such that $f_\alpha(x) \neq f_\alpha(y)$. Then the F-topology on X is Hausdorff.*

Proof Suppose that $x, y \in X$, with $x \neq y$. By hypothesis, there is some $\alpha \in I$ such that $nf_\alpha(x) \neq f_\alpha(y)$. Since (Y_α, τ_α) is Hausdorff, there exist elements $U, V \in \tau_\alpha$ such that $f_\alpha(x) \in U$, $f_\alpha(y) \in V$ and $U \cap V = \emptyset$. But then $f_\alpha^{-1}(U)$ and $f_\alpha^{-1}(V)$ are open with respect to F-topology and $x \in f_\alpha^{-1}(U)$, $y \in f_\alpha^{-1}(V)$ and $f_\alpha^{-1}(U) \cap f_\alpha^{-1}(V) = \emptyset$. ■

Definition 1.13 Let (X, τ) be a topological space. X is called metrizable if it is compatible with some metric d (i.e., τ is generated by the open balls $B_r(x) = \{y \in X, d(x, y) < r\}$).

Proposition 1.9 *Let (X, τ) be a compact topological space. If there is a sequence $\{f_n, n \in \mathbb{N}\}$ of continuous real-valued functions that separates points in X then X is metrizable.*

Proof Since (X, τ) is compact and the f_n are continuous then they are bounded. Thus, we can normalize them such that $\|f_n\|_\infty = \sup_{x \in X} |f_n(x)| \leq 1$. Define:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}.$$

This series converges. In fact, it converges uniformly on $X \times X$ hence the limit is continuous. Because the f_n separate points $d(x, y) = 0$ iff $x = 0$. d is also symmetric and satisfies the triangle inequality.

Thus d is a metric and we denote by τ_d the topology induced by this metric. We need to show that $\tau_d = \tau$. Consider the metric balls:

$$B_r(x) = \{y \in X, d(x, y) < r\}.$$

Since d is τ-continuous on $X \times X$, these balls are τ-open and

$$\tau_d \subseteq \tau.$$

By Proposition 1.7, since τ is compact and τ_d is Hausdorff (like any metric space) then $\tau = \tau_d$. ■

1.1.3 Nets and Convergence in Topology

Nets generalize the notion of sequences so that certain familiar results relating to continuity and compactness of sequences in metric spaces can be proved in arbitrary topological spaces. We now expand our notion of "sequence" $(x_n)_n$ to something for which the index n need not be a natural number, but can instead take values in a (possibly uncountable) partially ordered set.

Definition 1.14 A directed set $(I, \prec)$ consists of a set I with a partial order $\prec$ such that for every pair $\alpha, \beta \in I$, there exists an element $\gamma \in I$ with $\gamma \succ \alpha$ and $\gamma \succ \beta$.

Examples 1.3

1. The natural numbers $\mathbb{N}$ with the relation $\leq$ define a directed set $(I, \prec) = (\mathbb{N}, \leq)$.
2. If (X, τ) is a topological space and $x \in X$, one can define a directed set $(I, \prec)$ where I is the set of all neighborhoods of x in X, and $U \prec V$ for $U, V \in I$ means $V \subseteq U$. This is a directed set because given any pair of neighborhoods $U, V \subseteq X$ of x, the intersection $U \cap V$ is also a neighborhood of x and thus defines an element of I with $U \cap V \subseteq U$ and $U \cap V \subseteq V$. Note that neither of U and V need be contained in the other, so they might not satisfy either $U \prec V$ or $V \prec U$, hence $\prec$ is only a partial order, not a total order. Moreover, for most of the topological spaces we are likely to consider, I is uncountably infinite.
3. Let (X, τ) a topological space and let $x \in X$. Then the set $I_x = \{U \in \tau, x \in U\}$ is a directed set when equipped with the either the subset relation $\subseteq$, or more usefully the superset relation $\supseteq$.
4. If $(I_1, \prec_1)$ and $(I_2, \prec_2)$ are directed sets, then $(I_1 \times I_2, \prec)$ is a directed set where $\prec$ is defined by

$$(a, b) \prec (x, y) \text{ if and only if } a \prec_1 x \text{ and } b \prec_2 y.$$

5. Let I denote the set of all finite partitions of $[0, 1]$, partially ordered by inclusion (i.e., refinement). Let f be a continuous function on $[0, 1]$, then to $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\} \in I$, we associate the quantity $L_P(f) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$. The map $f \mapsto L_P(f)$ is a net (I is a directed set), and from Calculus, $\lim_{P \in I} L_P(f) = \int_0^1 f(x)dx$.

Definition 1.15 Let $\mathcal{P}$ be a property of elements of a directed set $(I, \prec)$. We shall say that:

1. $\mathcal{P}$ holds eventually if there exists $\alpha_0 \in I$ such that $\mathcal{P}$ holds for each $\alpha \succ \alpha_0$,
2. $\mathcal{P}$ holds frequently if for each $\alpha \in I$ there exists $\beta \succ \alpha$ satisfying $\mathcal{P}$.

Thus "eventually" means "for all successors of some element", and "frequently" means "for arbitrary large elements".

Definition 1.16 Given a topological space (X, τ), a net $(x_\alpha)_{\alpha \in I}$ is a function $I \to X : \alpha \longmapsto x_\alpha$, where $(I, \prec)$ is a directed set.

Definition 1.17 We say that a net $(x_\alpha)_{\alpha \in I}$ in X converges to $x \in X$ if for every neighborhood $U \subseteq X$ of x, there exists $\alpha_0 \in I$ such that $x_\alpha \in U$ for every $\alpha \succ \alpha_0$.

Example 1.3 A net $(x_\alpha)_{\alpha \in I}$ with $(I, \prec) = (\mathbb{N}, \leq)$ is simply a sequence, and convergence of this net to x means the same thing as convergence of the sequence.

Definition 1.18 A net $(x_\alpha)_{\alpha \in I}$ has a cluster point (also known as accumulation point) at $x \in X$ if for every neighborhood $U \subseteq X$ of x and for every $\alpha_0 \in I$, there exists $\alpha \succ \alpha_0$ with $x_\alpha \in U$.

Definition 1.19 A net $(y_\beta)_{\beta \in J}$ is a subnet of the net $(x_\alpha)_{\alpha \in I}$ if $y_\beta = x_{\phi(\beta)}$ for some order preserving function $\phi : J \to I$ such that for every $\alpha_0 \in I$, there exists an element $\beta_0 \in J$ for which $\beta \succ \beta_0$ implies $\phi(\beta) \succ \alpha_0$ (cofinal).

Example 1.4 If $(x_n)_n$ is a sequence, any subsequence $(x_{k_n})_n$ becomes a subnet $(y_\beta)_{\beta \in J}$ of the net $(x_n)_{n \in \mathbb{N}}$ by setting $J = \mathbb{N}$ and $\phi : \mathbb{N} \to \mathbb{N} : n \longmapsto k_n$. Note that this remains true if we slightly relax our notion of subsequences so that (k_n) need not be a monotone increasing sequence in $\mathbb{N}$ but satisfies $k_n \to \infty$ as $n \to \infty$. Conversely, any subnet $(y_\beta)_{\beta \in J}$ of a sequence $(x_n)_{n \in \mathbb{N}}$ with $(J, \prec) = (\mathbb{N}, \leq)$ is also a subsequence in this slightly relaxed sense, and can then be reduced to a subsequence in the usual sense by skipping some terms (so that the function $n \longmapsto k_n$ becomes strictly increasing). Note however that a subnet of a sequence need not be a subsequence in general, i.e., it is possible to define a subnet $(y_\beta)_{\beta \in J}$ of a sequence $(x_n)_{n \in \mathbb{N}}$ such that J is uncountable, and one can derive concrete examples of such objects.

Remark 1.7 If $(x_\alpha)_{\alpha \in I}$ is a net converging to x, then every subnet $(x_{\phi(\beta)})_{\beta \in J}$ also converges to x.

Theorem 1.4 *Let Y be a subset of a topological space (X, τ). Then $x \in \overline{Y}$ if and only if there is a net $(x_\alpha)_{\alpha \in I}$ with $x_\alpha \in Y$ such that $x_\alpha \longrightarrow x$.*

Proof We know that a point $x \in X$ belongs to $\overline{Y}$ if and only if every neighborhood of x meets Y. Suppose then that $(x_\alpha)_{\alpha \in I}$ is a net in Y such that $x_\alpha \longrightarrow x$. By definition of convergence, $(x_\alpha)_{\alpha \in I}$ is eventually in every neighborhood of x, so certainly $x \in \overline{Y}$.

Suppose, on the other hand, that $x \in \overline{Y}$. Let $\mathcal{N}_x$ be the collection of all neighborhoods of x ordered by reverse inclusion. Then $\mathcal{N}_x$ is a directed set. We know that for each $V \in \mathcal{N}_x$ the set $V \cap Y$ is nonempty so let x_V be any element of $V \cap Y$. Then $x_V \longrightarrow x$. ■

Lemma 1.1 *Let X be a set, and $(x_\alpha)_{\alpha \in I}$ a net in X. Let $\mathcal{B}$ be a family of subsets of X, satisfying*

1. *x_α is contained frequently in each element of $\mathcal{B}$, and*
2. *the intersection of any two elements of $\mathcal{B}$ contains an element of $\mathcal{B}$.*

Then $(x_\alpha)_{\alpha \in I}$ admits a subnet which is eventually contained in each element of $\mathcal{B}$.

Proof Clearly, the family $\mathcal{B}$ is directed by the inverse inclusion. Consider the set

$$J = \{(\alpha, B) \in I \times \mathcal{B}\colon x_\alpha \in B\}$$

equipped with the coordinate-wise pre-ordering. It is easy to see that J is a directed set. The function $\phi\colon J \to I$, defined by $\phi(\alpha, B) = \alpha$, is nondecreasing and onto, and hence tends to infinity. Consequently, $(x_{\phi(\alpha,B)})_{(\alpha,B)}$ is a subnet of $(x_\alpha)_{\alpha \in I}$. Moreover, given $A \in B$, fix $\alpha_0 \in I$ so that $x_{\alpha_0} \in A$, and observe that if $(\alpha, B) \succ (\alpha_0, A)$ then $x_{\phi(\beta,B)} = x_\beta \in B \subseteq A$. This completes the proof. ■

In metric spaces, a standard theorem states that sequential continuity is equivalent to continuity. In arbitrary topological spaces this no longer true, but we have the following generalization.

Theorem 1.5 *For any two topological spaces X and Y, a map $T\colon X \to Y$ is continuous if and only if for every net $(x_\alpha)_{\alpha \in I}$ in X converging to a point $x \in X$, the net $(T(x_\alpha))_{\alpha \in I}$ in Y converges to $T(x)$.*

Proposition 1.10 *A point x of a topological space (X, τ) is a cluster point of a net $(x_\alpha)_{\alpha \in I}$ in X if and only if there exists a subnet $(x_{\phi(\beta)})_{\beta \in J}$ that converges to x.*

Proof If $(x_{\phi(\beta)})_{\beta \in J}$ is a subnet of $(x_\alpha)_{\alpha \in I}$ converging to x, then for every neighborhood $U \subseteq X$ of x, there exists $\beta_0 \in J$ such that $x_{\phi(\beta)} \in U$ for every $\beta \succ \beta_0$. Then for any $\alpha_0 \in I$, the definition of a subnet implies that we can find $\beta_1 \in J$ with $\phi(\beta) \succ \alpha_0$ for all $\beta \succ \beta_1$, and since J is a directed set, there exists $\beta_2 \in J$ with $\beta_2 \succ \beta_0$ and $\beta_2 \succ \beta_1$. It follows that for $\alpha = \phi(\beta_2), \alpha \succ \alpha_0$ and $x_\alpha = x_{\phi(\beta_2)} \in U$, thus x is a cluster point of $(x_\alpha)_{\alpha \in I}$.

Conversely, if x is a cluster point of $(x_\alpha)_{\alpha \in I}$, we can define a convergent subnet as follows. Define a new directed set

$$J = I \times \{\text{ neighborhoods of } x \text{ in } X\},$$

with the partial order $(\alpha, U) \prec (\beta, V)$ defined to mean both $\alpha \prec \beta$ and $V \subseteq U$. Then for each $(\beta, U) \in J$, the fact that x is a cluster point implies that we can choose $\phi(\beta, U) \in I$ to be any $\alpha \in I$ such that $\alpha \succ \beta$ and $x_\alpha \in U$. This defines a function $\phi\colon J \to I$ such that for any $\alpha_0 \in I$ and any neighborhood $U_0 \subseteq X$ of x, every $(\beta, U) \in J$ with $(\beta, U) \succ (\alpha_0, U_0)$ satisfies $\phi(\beta, U) \succ \beta \succ \alpha_0$, hence $(x_{\phi(\beta,U)})_{\beta \in J}$ is a subnet of $(x_\alpha)_{\alpha \in I}$. Moreover, for any neighborhood $U \subseteq X$ of x, we can choose an arbitrary $\alpha_0 \in I$ and observe that

$$(\beta, V) \succ (\alpha_0, U) \Longrightarrow x_{\phi(\beta,V)} \in V \subseteq U,$$

thus $(x_{\phi(\beta,U)})_{(\beta,U)\in J}$ converges to x. ■

Theorem 1.6 *A topological space (X, τ) is compact if and only if every net in X has a convergent subnet.*

Proof Suppose X is compact but there exists a net $(x_\alpha)_{\alpha\in I}$ in X with no cluster point. The fact that every $x \in X$ is not a cluster point of $(x_\alpha)_{\alpha\in I}$ then means that we can find for each $x \in X$ an open neighborhood $U_x \subseteq X$ of x and an index $\alpha_x \in I$ such that $x_{\alpha_x} \notin U_x$ for all $\alpha \succ \alpha_x$. But $(U_x)_{x\in X}$ is then an open cover of X and therefore has a finite subcover, meaning there is a finite subset $x_1, \cdots, x_N \in X$ such that $X = \bigcup_{n=1}^{N} U_{x_n}$. Since $(I, \prec)$ is a directed set, there also exists an element $\beta \in I$ such that

$$\beta \succ \alpha_{x_n} \quad \text{for each} \quad n = 1, \cdots, N.$$

Then $x_\beta \notin U_{x_n}$ for every $n = 1, \cdots, N$, but since the sets U_{x_n} cover X, this is a contradiction.

Conversely, suppose that every net in X has a cluster point, but that X has a collection O of open sets that cover X such that no finite subcollection in O covers X. Define a directed set where I is the set of all finite subcollections of O, with the ordering relation defined by inclusion, i.e., for $A, B \in I$, $A \prec B$ means $A \subseteq B$. Note that $(I, \prec)$ is a directed set since for any two $A, B \in I$, we have $A \cup B \in I$ with $A \cup B \supset A$ and $A \cup B \supset B$. By assumption, none of the unions $\bigcup_{U\in A}$ for $A \in I$ cover X, so we can choose a point

$$x_A \in X \setminus \bigcup_{U\in A} U \tag{1.1}$$

for each $A \in I$, thus defining a net $(x_A)_{A\in I}$. Then $(x_A)_{A\in I}$ has a cluster point $x \in X$. Since the sets in O cover X, we have $x \in V$ for some $V \in O$, and the collection $\{V\}$ is an element of I, hence there exists $A \succ \{V\}$ such that $x_A \in V$. But this means A is a finite subcollection of O that includes V, thus contradicting (1.1). ■

Theorem 1.7 *Let X be a set and let τ_1 and τ_2 be topologies on X. Then the following are equivalent*

1. $\tau_1 = \tau_2$.
2. *Every $(x_\alpha)_{\alpha\in I}$ in X, converges in τ_1 if and only if it converges in τ_2.*

Proposition 1.11 *A topological space* (X, τ) *is Hausdorff if and only if no net has two distinct limits.*

Proof Suppose (X, τ) is Hausdorff and consider a net $(x_\alpha)_{\alpha \in I}$. Suppose for contradiction that x and y are distinct limits of $(x_\alpha)_{\alpha \in I}$. Take disjoint neighborhoods U of x and V of y. By definition of convergence, there is a α_x such that $x_\alpha \in U$ for all $\alpha \succ \alpha_x$ and a α_y such that $x_\alpha \in V$ for all $\alpha \succ \alpha_y$. In particular we have $x_\alpha \in U \cap V$ for an upper bound α of α_x and α_y in the directed set I, contradicting the disjointness of U and V. Thus $(x_\alpha)_{\alpha \in I}$ cannot have two distinct limits.

Conversely, suppose that (X, τ) is not Hausdorff, so there are two distinct points x and y such that any neighborhood of x intersects any neighborhood of y. So there is a net $(x_{(U,V)})_{\mathcal{N}(x)\times\mathcal{N}(y)}$ such that

$$x_{(U,V)} \in U \cap V$$

for neighborhoods U of x and V of y. Take any neighborhood U_0 of x and any $(U, V) \in \mathcal{N}(x) \times \mathcal{N}(y)$ with $(U, V) \succ (U_0, X)$. By definition we have $U \subseteq U_0$ and thus $x_{(U,V)} \in U \cap V \subseteq U_0$. This proves that $x_{(U,V)} \to x$ and we can similarly show that $x_{(U,V)} \to y$. So the net $(x_{(U,V)})_{\mathcal{N}(x)\times\mathcal{N}(y)}$ has two distinct limits, as required. ■

1.2 Topological Vector Spaces

1.2.1 Linear Topologies

Definition 1.20 Let X be a vector space. A linear topology on X is a topology τ such that the maps

$$X \times X \ni (x, y) \mapsto x + y \in X \tag{1.2}$$

$$\mathbb{K} \times X \ni (\alpha, x) \mapsto \alpha x \in X \tag{1.3}$$

are continuous. For the map (1.2) we use the product topology $\tau \times \tau$. For the map (1.3) we use the product topology $\tau_{\mathbb{K}} \times \tau$, where $\tau_{\mathbb{K}}$ is the standard topology on $\mathbb{K}$.

A topological vector space is a pair (X, τ) consisting of a vector space X and a Hausdorff linear topology τ on X.

Remark 1.8 If (X, τ) is a topological vector space then it is clear from Definition 1.20 that $\sum_{k=1}^{N} \lambda_k^{(n)} x_k^{(n)} \to \sum_{k=1}^{N} \lambda_k x_k$ as $n \to \infty$ with respect to τ if for each $k = 1, \cdots, N$ as $n \to \infty$ we have $\lambda_k^{(n)} \to \lambda_k$ with respect to the euclidean topology on $\mathbb{K}$ and $x_k^{(n)} \to x_k$ with respect to τ.

Examples 1.4

1. Every vector space X over $\mathbb{K}$ endowed with the trivial topology is a topological vector space.
2. The field $\mathbb{K}$, viewed as a vector space over itself, becomes a topological vector space, when equipped with the standard (euclidean) topology $\tau_{\mathbb{K}}$.
3. Every normed vector space endowed with the topology given by the metric induced by the norm is a topological vector space.

Proposition 1.12 *Every vector space X over $\mathbb{K}$ endowed with the discrete topology is not a topological vector space unless $X = \{\theta\}$.*

Proof Assume by a contradiction that it is a topological vector space and take $\theta \neq x \in X$. The sequence $\alpha_n = \frac{1}{n}$ in $\mathbb{K}$ converges to 0 in the euclidean topology. Therefore, since the scalar multiplication is continuous, $\alpha_n x \to \theta$, i.e., for any neighborhood U of θ in X there exists $m \in \mathbb{N}$ such that $\alpha_n x \in U$ for all $n \geq m$. In particular, we can take $U = \{\theta\}$ since it is itself open in the discrete topology. Hence, $\alpha_m x = \theta$, which implies that $x = \theta$ and so a contradiction. ■

Remark 1.9 In terms of net convergence, the continuity requirements for a linear topology on X read:

- Whenever (x_α) and (y_α) are nets in X, such that $x_\alpha \to x$ and $y_\alpha \to y$, it follows that $x_\alpha + y_\alpha \to x + y$.
- Whenever (λ_α) and (x_α) are nets in $\mathbb{K}$ and X, respectively, such that $\lambda_\alpha \to \lambda$ (in $\mathbb{K}$) and $x_\alpha \to x$ (in X), it follows that $\lambda_\alpha x_\alpha \to \lambda x$.

Example 1.5 Let I be an arbitrary nonempty set. The product space $\mathbb{K}^I$ (defined as the space of all functions $I \to \mathbb{K}$) is obviously a vector space (with pointwise addition and scalar multiplication). The product topology turns $\mathbb{K}^I$ into a topological vector space.

Remark 1.10 If X is a vector space, then the following maps are continuous with respect to any linear topology on X :

- The translations $T_y \colon X \to X$, $y \in X$, defined by $T_y(x) = x + y$.
- The dilations $D_\alpha \colon X \to X$, $\alpha \in \mathbb{K}$, defined by $D_\alpha(x) = \alpha x$.

If τ is a linear topology on a vector space X, then τ is translation invariant. That is, a subset $U \subseteq X$ is open if and only if the translation $y + U$ is open for all $y \in X$. Indeed, the continuity of addition implies that for each $y \in X$, the translation $x \mapsto y+x$ is a linear homeomorphism. In particular, every neighborhood of y is of the form $y + U$, where U is a neighborhood of zero. In other words, the neighborhood system at zero determines the neighborhood system at every point of X by translation. Also note that the dilation $x \mapsto \alpha x$ is a linear homeomorphism for any $\alpha \neq 0$. In particular, if U is a neighborhood of zero, then so is αU for all $\alpha \neq 0$.

Example 1.6 If a metric d on a vector space X is translation invariant, i.e., $d(x + z, y + z) = d(x, y)$ for all $x, y \in X$ (i.e., the metric induced by a norm), then the topology induced by the metric is translation invariant and the addition is always continuous. However, the multiplication by scalars does not need to be necessarily continuous (take d to be the discrete metric, then the topology generated by the metric is the discrete topology which is not compatible with the scalar multiplication).

Proposition 1.13 *If Y is a linear subspace of a topological vector space (X, τ), then so its closure $\overline{Y}$. In particular, any maximal proper subspace is either dense or closed.*

Proof We must show that if $x, y \in \overline{Y}$ and $\lambda \in \mathbb{K}$, then $\lambda x + y \in \overline{Y}$. There are nets (x_α) and (y_α) in Y, such that $x_\alpha \longrightarrow x$ and $y_\alpha \longrightarrow y$. By Remark 1.9, we deduce that $tx_\alpha \longrightarrow tx$ and $tx_\alpha + y_\alpha \longrightarrow tx + y$ and we conclude that $tx + y \in \overline{Y}$, as required.

If Y is a maximal proper subspace, the inclusion $Y \subseteq \overline{Y}$ implies either $Y = \overline{Y}$, in which case Y is closed, or $\overline{Y} = X$, in which case Y is dense in X. ■

Notations Given a vector space X, a subset $Y \subseteq X$, and a vector $x \in X$, we denote the translation $T_x(Y)$ simply by $Y + x$ $(x + Y)$, that is,

$$Y + x = x + Y = \{y + x : y \in Y\}.$$

Likewise, for an $\alpha \in \mathbb{K}$ we denote the dilation $D_\alpha(Y)$ simply by αY, that is,

$$\alpha Y = \{\alpha y : y \in Y\}.$$

Given another subset $Z \subseteq X$, we define

$$Y + Z = \{y + z : y \in Y, z \in Z\} = \bigcup_{y \in Y}(y + Z) = \bigcup_{z \in Z}(Y + z).$$

Remark 1.11 In general we only have the inclusion $2Y \subseteq Y + Y$.

Lemma 1.2 *Let τ be a linear topology on the vector space X.*

1. *The algebraic sum of an open set and an arbitrary set is open.*
2. *Nonzero multiples of open sets are open.*
3. *If Y is open, then for any set Z we have $\overline{Z} + Y = Z + Y$.*
4. *The algebraic sum of a compact set and a closed set is closed. (However, the algebraic sum of two closed sets need not be closed.)*
5. *The algebraic sum of two compact sets is compact.*
6. *Scalar multiples of closed sets are closed.*
7. *Scalar multiples of compact sets are compact.*

Proof We shall prove only items 3. and 4.

3. Clearly $Y + Z \subseteq Y + \overline{Z}$. For the reverse inclusion, let $x = z + y$ where $z \in \overline{Z}$ and $y \in Y$. Then there is an open neighborhood U of θ such that $y + U \subseteq Y$. Since

$z \in \overline{Z}$, there exists some $t \in Z \cap (z - U)$. Then $x = z + y = t + z + (y - z) \in t + z + U \subseteq Z + Y$.

4. Let Y be compact and Z be closed, and let a net $(y_\alpha + z_\alpha)_{\alpha \in I}$ satisfy $y_\alpha + z_\alpha \longrightarrow x$. Since Y is compact, we can assume (by passing to a subnet) that $y_\alpha \longrightarrow y \in Y$. The continuity of the algebraic operations yields

$$z_\alpha = (y_\alpha + z_\alpha) - y_\alpha \longrightarrow x - y = z.$$

Since Z is closed, $z \in Z$, so $x = y + z \in Y + Z$, proving that $Y + Z$ is closed.

■

Proposition 1.14 *Let τ be a linear topology on the vector space X.*

1. *For every neighborhood V of θ, there exists a neighborhood W of θ, such that $W + W \subseteq V$.*
2. *For every neighborhood V of θ, and any compact set $C \subseteq \mathbb{K}$, there exists a neighborhood W of θ, such that $\alpha W \subseteq V, \forall\, \alpha \in C$.*

Proof 1. Let $T : X \times X \to X$ denote the addition map (1.2). Since T is continuous at $(\theta, \theta) \in X \times X$, the preimage $T^{-1}(V)$ is a neighborhood of (θ, θ) in the product topology. In particular, there exists neighborhoods W_1, W_2 of θ, such that $W_1 \times W_2 \subseteq T^{-1}(V)$, so if we take $W = W_1 \cap W_2$, then W is still a neighborhood of θ satisfying $W \times W \subseteq T^{-1}(V)$, which is precisely the desired inclusion $W + W \subseteq V$.

2. Let $G : \mathbb{K} \times X \to X$ denote the multiplication map (1.3). Since G is continuous at $(0, \theta) \in \mathbb{K} \times X$, the preimage $G^{-1}(V)$ is a neighborhood of $(0, \theta)$ in the product topology. In particular, there exists a neighborhood I of 0 in $\mathbb{K}$ and a neighborhood W_0 of θ in X such that $I \times W_0 \subseteq G^{-1}(V)$. Let then $\rho > 0$ such that I contains the closed disk $\overline{B_\rho}(0) = \{\alpha \in \mathbb{K} \colon |\alpha| \leq \rho\}$, so that we still have the inclusion $\overline{B_\rho}(0) \times W_0 \subseteq G^{-1}(V)$ i.e.,

$$\alpha \in \mathbb{K}, |\alpha| \leq \rho \Longrightarrow \alpha W_0 \subseteq V. \tag{1.4}$$

Since $C \subseteq \mathbb{K}$ is compact, there is some $R > 0$, such that

$$|\gamma| \leq R, \quad \forall\, \gamma \in C. \tag{1.5}$$

Let us then define $W = (\frac{\rho}{R})W_0$. First of all, since W is a non-zero dilation of W_0, it is a neighborhood of θ. Secondly, if we start with some $\gamma \in C$ and some $w \in W$, written as $w = (\frac{\rho}{R})w_0$ with $w_0 \in W_0$, then

$$\gamma w = (\frac{\rho\alpha}{R})w_0.$$

By (1.5) we know that $\left|\frac{\rho\alpha}{R}\right| \leq \rho$, so by (1.4) we get $\gamma w \in V$. ■

1.2.2 Absorbing and Balancing Sets

Definition 1.21 A subset Y of a vector space X is convex if, whenever Y contains two points x and y, Y also contains the segment or the straight line joining them, i.e.,

$$\forall\, x, y \in Y, \forall\, \alpha, \beta \geq 0 \ \text{ such } \ \text{that}\, \alpha + \beta = 1, \alpha x + \beta y \in Y.$$

Examples 1.5

1. The convex subsets of $\mathbb{R}$ are simply the intervals of $\mathbb{R}$. Examples of convex subsets of $\mathbb{R}^2$ are solid regular polygons. The Platonic solids are convex subsets of $\mathbb{R}^3$. Hyperplanes and half spaces in $\mathbb{R}^n$ are convex.
2. Balls in a normed space are convex.
3. Consider a topological space X and the set $C(X)$ of all real valued functions defined and continuous on X. $C(X)$ with the pointwise addition and scalar multiplication of functions is a vector space. Fixed $g \in C(X)$, the subset $Y := \{f \in C(X) \colon f(x) \geq g(x), \forall x \in X\}$ is convex.
4. Consider the vector space $\mathbb{R}[x]$ of all polynomials in one variable with real coefficients. Fixed $n \in \mathbb{N}$ and $c \in \mathbb{R}$, the subset of all polynomials in $\mathbb{R}[x]$ such that the coefficient of the term of degree n is equal to c is convex.

Proposition 1.15 *Let X be a vector space. The following properties hold.*

(a) *$\emptyset$ and X are convex.*
(b) *Arbitrary intersections of convex sets are convex sets.*
(c) *Unions of convex sets are generally not convex.*
(d) *The sum of two convex sets is convex.*
(e) *A set Y is convex if and only if $\alpha Y + \beta Y = (\alpha + \beta)Y$ for all nonnegative scalars α and β.*
(f) *The image and the preimage of a convex set under a linear map is convex.*

Definition 1.22 Let Y be any subset of a vector space X. We define the convex hull of X, denoted by $\text{conv}(Y)$, to be the set of all finite convex linear combinations of elements of Y, i.e.,

$$\text{conv}(Y) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in Y, \alpha_i \in [0, 1], \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \right\}.$$

Proposition 1.16 *Let Y, Z be arbitrary sets of a vector space X. The following hold.*

(a) $\text{conv}(Y)$ *is convex.*
(b) $Y \subseteq \text{conv}(Y)$.
(c) *A set is convex if and only if it is equal to its own convex hull.*
(d) *If $Y \subseteq Z$ then* $\text{conv}(Y) \subseteq \text{conv}(Z)$.

(*e*) $\operatorname{conv}(\operatorname{conv}(Y)) = \operatorname{conv}(Y)$.
(*f*) $\operatorname{conv}(Y + Z) = \operatorname{conv}(Y) + \operatorname{conv}(Z)$.
(*g*) *The convex hull of Y is the smallest convex set containing Y, i.e.,* $\operatorname{conv}(Y)$ *is the intersection of all convex sets containing Y.*

Definition 1.23 Let X be a vector space.

- A subset $Y \subseteq X$ is said to be absorbing (or radial), if for every $x \in X$, there exists some scalar $\alpha > 0$, such that $\alpha x \in Y$. Roughly speaking, we may say that a subset is absorbing if it can be made by dilation to swallow every point of the whole space.
- A subset $Y \subseteq X$ is said to be balancing (or circled), if for every $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, one has the inclusion $\alpha Y \subseteq Y$. Note that the line segment joining any point x of a balanced set Y to $-x$ lies in Y.
- A subset $Y \subseteq X$ is said to be symmetric, if for every $x \in Y$, one has $(-x) \in Y$, namely $(-Y) = Y$.
- A subset $Y \subseteq X$ is said to be absolutely convex, if it is convex and balanced.
- A subset $Y \subseteq X$ is said to be starshaped about zero if it included the line segment joining each of its points with zero. That is, if for any $x \in Y$ and any $0 \leq \alpha \leq 1$ we have $\alpha x \in Y$.

Remark 1.12 Note that an absorbing set must contain θ, and any set including an absorbing set is itself absorbing. For any absorbing set Y, the set $Y \cap (-Y)$ is nonempty, absorbing, and symmetric. Every circled set is symmetric. Every circled set is star-shaped about θ, as is every convex set containing θ.

Remark 1.13 Given τ a linear topology of a vector space X, all neighborhoods of θ are absorbing. Indeed, if we start with some $x \in X$, the sequence $x_n = \dfrac{1}{n}x$ clearly converges to θ, so every neighborhood of θ will contain (many) terms x_n.

Examples 1.6

1. In a normed space the unit balls centered at the origin are absorbing and balanced.
2. The unit ball B centered at $(\frac{1}{2}, 0) \in \mathbb{R}^2$ is absorbing but not balanced in the real vector space $\mathbb{R}^2$ endowed with the euclidean norm. Indeed, B is a neighborhood of the origin. However, B is not balanced because for example if we take $x = (1, 0) \in B$ and $\alpha = -1$ then $\alpha x \notin B$.
3. The polynomials $\mathbb{R}[\mathbb{X}]$ are a balanced but not absorbing subset of the real space $C([0, 1], \mathbb{R})$ of continuous real valued functions on $[0, 1]$. Indeed, any multiple of a polynomial is still a polynomial but not every continuous function can be written as multiple of a polynomial.
4. The subset $Y = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq |z_2|\}$ of the complex space $\mathbb{C}^2$ with the euclidean topology is balanced but $\operatorname{int}Y$ is not balanced.

Definition 1.24 Given τ a linear topology of a vector space X, a subset $Y \subseteq X$ is said to be a barrel if it has the following properties:

1. Y is absorbing,
2. Y is absolutely convex,
3. Y is closed.

Proposition 1.17 *Let X be a vector space and let τ be a linear topology on X.*

A. *If $\mathcal{B}$ is a neighborhood base at θ, then:*

1. *For every $V \in \mathcal{B}$, there exists $W \in \mathcal{B}$, such that $W + W \subseteq V$.*
2. *For every $V \in \mathcal{B}$ and every compact set $C \subseteq \mathbb{K}$, there exists $W \in \mathcal{B}$, such that $\gamma W \subseteq V, \forall \gamma \in C$.*
3. *For every $x \in X$, the collection $\mathcal{B}_x = \{V + x : V \in \mathcal{B}\}$ is a neighborhood base at x.*
4. *The topology τ is Hausdorff, if and only if $\bigcap_{V \in \mathcal{B}} V = \{\theta\}$.*

B. *There exists a neighborhood base at θ, consisting of open balanced sets.*

Proof

A. Statements 1. and 2. follow immediately from Proposition 1.14. Statement 3. is clear, since translations are homeomorphisms.

4. Denote for simplicity the intersection $\bigcap_{V \in \mathcal{B}} V$ by J, so clearly $\theta \in J$. Assume first τ is Hausdorff. In particular, for each $x \in X \setminus \{\theta\}$, the set $X \setminus \{x\}$ is an open neighborhood of θ, so there exists some $V^x \in \mathcal{B}$ with $V^x \subseteq X \setminus \{x\}$. We then clearly have the inclusion

$$J \subseteq \bigcap_{x \neq \theta} V^x \subseteq \bigcap_{x \neq \theta} (X \setminus \{x\}) = \{\theta\},$$

so $J = \{\theta\}$. Conversely, assume $J = \{\theta\}$, and let us show that τ is Hausdorff. Start with two points $x, y \in X$ with $x \neq y$, so that $x - y \neq \theta$, and let us indicate how to construct two disjoint neighborhoods, one for x and one for y. Using translations, we can assume $y = \theta$. Since $\theta \neq x \notin \bigcap_{V \in \mathcal{B}} V$, there exists some $V \in \mathcal{B}$, such that $x \notin V$. Using 1., there is some $W \in \mathcal{B}$, such that $W + W \subseteq V$, so we still have $x \in W + W$. This clearly forces

$$x + ((-1)V) \cap V = \emptyset. \tag{1.6}$$

Since V is a neighborhood of θ, so is $(-1)V$ (non-zero dilation), then by 3. the left-hand side of (1.5) is a neighborhood of x.

B. Let us take the $\mathcal{D}$ to be the collection of all open balanced sets that contain θ. All we have to prove is the following statement: for every neighborhood V of

θ, there exists $W \in \mathcal{D}$, such that $W \subseteq V$. Using 2. there exists some open set $O \ni \theta$, such that

$$\gamma O \subseteq V, \ \ \forall \gamma \in \mathbb{K}, |\gamma| \leq 1. \tag{1.7}$$

In particular, $\bigcup_{\alpha \in \mathbb{K}, 0<|\alpha|\leq 1} \alpha O$ is an open set contained in V. So $\bigcup_{\alpha \in \mathbb{K}, 0<|\alpha|\leq 1} \alpha O \in V$.

■

Definition 1.25 Assume τ is a linear topology on a vector space X. A subset $Y \subseteq X$ is said to be τ-bounded, if it satisfies the following condition:

for every neighborhood V of θ, there exists $\rho > 0$, such that $Y \subseteq \rho V$.

Example 1.7 Suppose τ is a linear topology on a vector space X. If $x \in X \in$, then $\{x\}$ is bounded. Indeed, let V any neighborhood of θ. Then V is absorbing and so $x \in \rho V$ for all sufficiently large $\rho > 0$, that is, $\{x\}$ is bounded.

Proposition 1.18 *Let X be a vector space X endowed with a linear topology τ. Then*

1. *If $Y \subseteq X$ is τ-bounded, then its closure $\overline{Y}$ is also τ-bounded.*
2. *If $Y, Z \subseteq X$ are τ-bounded, then so is $Y + Z$.*
3. *If $Y \subseteq X$ is τ-bounded and $C \subseteq \mathbb{K}$ is bounded, then so $\bigcup_{\alpha \in C} \alpha Y$.*
4. *All compact subsets in X are τ-bounded.*

Remark 1.14 It follows by induction, that any finite set in a vector space X endowed with a linear topology τ is bounded. Also, taking $Y = \{x\}$ (in the above proposition) we see that any translate of a bounded set is bounded.

Proposition 1.19 *Any convergent sequence in topological vector space is bounded.*

Proof Suppose that $(x_n)_n$ is a sequence in a topological vector space (X, τ) such that $x_n \longrightarrow x$. For each $n \in \mathbb{N}$, set $y_n = x_n - x$, so that $y_n \longrightarrow \theta$. Let V any neighborhood of θ. Let U be any balanced neighborhood of θ such that $U \subseteq V$. Then $U \subseteq \rho U$ for all ρ with $|\rho| \geq 1$. Since $y_n \longrightarrow \theta$, there is $N \in \mathbb{N}$ such that $y_n \in U$ whenever $n > N$. Hence $y_n \in U \subseteq tU \subseteq tV$ whenever $n > N$ and $t \geq 1$. Set $Y = \{y_1, \cdots, y_n\}$ and $Z = \{y_n : n > N\}$. Then Y is a finite set so is bounded and therefore $Y \subseteq tV$ for all sufficiently large t. But then it follows that $Y \cup Z \subseteq tV$ for sufficiently large t, that is, $\{y_n : n \in \mathbb{N}\}$ is τ-bounded and so is $\{x_n : n \in \mathbb{N}\} = x + (Y \cup Z)$. ■

Remark 1.15 A convergent net in a topological vector space need not be bounded. For example, let I be $\mathbb{R}$ equipped with its usual order and let $x_\alpha \in \mathbb{R}$ be given by $x_\alpha = e^{-\alpha}$. Then $(x_\alpha)_{\alpha \in I}$ is an unbounded but convergent net (with limit 0) in the real normed space $\mathbb{R}$.

Proposition 1.20 ("Zero. Bounded" Rule) *Suppose τ is a linear topology in a vector space X. If the net $(\alpha_\lambda)_{\lambda\in\Lambda} \subseteq \mathbb{K}$ converges to 0, and the net $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ is τ-bounded, then $(\alpha_\lambda x_\lambda)_{\lambda\in\Lambda}$ is convergent to θ.*

Proof Start with some neighborhood V of θ. We wish to construct an index $\lambda_V \in \Lambda$ such that

$$\alpha_\lambda x_\lambda \in V, \ \forall\, \lambda \succ \lambda_V. \tag{1.8}$$

Using Proposition 1.17 **B.**, we can assume that V is balanced (otherwise we replace it with a balanced open set $V' \subseteq V$). Using the boundedness condition we find $\rho > 0$, such that

$$x_\lambda \in \rho V, \ \forall\, \lambda \in \Lambda. \tag{1.9}$$

Using the condition $\alpha_\lambda \to 0$, we then choose $\lambda_V \in \Lambda$, so that

$$|\alpha_\lambda| \le \frac{1}{\rho}, \forall\, \lambda \succ \lambda_V.$$

To check (1.8), start with some $\lambda \succ \lambda_V$ and apply (1.9) to write $x_\lambda = \rho v$, for some $v \in V$. Now we have

$$\alpha_\lambda x_\lambda = (\alpha_\lambda \rho) v \in (\alpha_\lambda \rho) V,$$

with $|\alpha_\lambda \rho| \le 1$, so using the fact that V is bounded, it follows that $\alpha_\lambda x_\lambda \in V$. ∎

Definition 1.26 Let (X, τ) be a topological vector space.

1. X is locally bounded if θ has a bounded neighborhood.
2. X is locally compact if θ has a neighborhood whose closure is compact.
3. X is metrizable if it is compatible with some metric d (i.e., τ is generated by the open balls $B_r(x) = \{y \in X, d(x, y) < r\}$).
4. X is normable if it can be endowed with a norm whose induced metric is compatible with τ.
5. X has the Heine-Borel property if every closed and bounded set is compact.

Proposition 1.21 *Let (X, τ) be a topological vector space. For every $x \neq \theta$ the set $Y = \{nx, \ n \in \mathbb{N}\}$ is not bounded.*

Proof By separation, there exists an open neighborhood V of θ that does not contain x, hence $nx \notin nV$, i.e., for every n,

$$Y \not\subseteq nV.$$

∎

Lemma 1.3

1. *Let d be a translation invariant metric on a vector space X, then for all $n \in \mathbb{N}$ and $x \in X$,*

$$d(nx, \theta) \leq nd(x, \theta).$$

2. *If $x_n \to \theta$ in a metrizable topological vector space (X, τ), then there exist positive scalars $\alpha_n \to \infty$ such that $\alpha_n x_n \to \theta$.*

Proof The first part is obvious by successive applications of the triangle inequality,

$$d(nx, \theta) \leq \sum_{k=1}^{n} d(kx, (k-1)x) \leq nd(x, \theta).$$

For the second, we note that since $d(x_n, \theta) \to 0$, there exists a diverging sequence of positive integers n_k, such that

$$d(x_k, \theta) \leq \frac{1}{n_k^2},$$

from which we get that

$$d(n_k x_k, \theta) \leq n_k d(x_k, \theta) \leq \frac{1}{n_k} \to 0.$$

■

Corollary 1.2 *The only bounded subspace of a topological vector space is $\{\theta\}$.*

Proposition 1.22 *Let (X, τ) be a topological vector space and let $Y \subseteq X$. Then, Y is bounded if and only if for every sequence $(x_n)_n \subseteq Y$ and every sequence of scalars $\alpha_n \to 0$, $\alpha_n x_n \to \theta$.*

Proof Suppose that Y is bounded, it suffices to apply Proposition 1.20.

Suppose that for every sequence $(x_n)_n \subseteq Y$ and every sequence of scalars $\alpha_n \to \theta$, $\alpha_n x_n \to \theta$. If Y is not bounded, then there exists an open neighborhood of θ and a sequence $\beta_n \to \infty$, such that no $\beta_n V$ contains Y. Take then a sequence $(x_n)_n \subseteq Y$ such that $x_n \notin \beta_n V$. Thus,

$$\beta_n^{-1} x_n \notin V,$$

which implies that $\beta_n^{-1} x_n \nrightarrow \theta$, which is a contradiction. ■

Theorem 1.8 *Let (X, τ) be a topological vector space. Let $Y, Z \subseteq X$ satisfy:*

$$Y \textit{ is compact}, Z \textit{ is closed and } Y \cap Z = \emptyset.$$

Then there exists an open neighborhood V of θ such that

$$(Y + V) \cap (Z + V) = \emptyset.$$

In other words, there exist disjoint open sets that contain Y and Z.

Proof Let $x \in Y$. Since $X \setminus Z$ is an open neighborhood of x, it follows that there exists a symmetric open neighborhood V_x of θ such that

$$x + V_x + V_x + V_x \subseteq X \setminus Z,$$

i.e.,

$$(x + V_x + V_x + V_x) \cap Z = \emptyset.$$

Since V_x is symmetric,

$$(x + V_x + V_x) \cap (Z + V_x) = \emptyset.$$

For every $x \in Y$ corresponds such a V_x. Since Y is compact, there exists a finite collection $(x_i, V_i)_{1\leq i\leq n}$ such that

$$K \subseteq \bigcup_{i=1}^{n} (x_i + V_i).$$

Define

$$V = \bigcap_{i=1}^{n} V_{x_i}.$$

Then, for every i,

$$(x + V_{x_i} + V_{x_i}) \quad \text{does not} \quad \text{intersect } (Z + V_{x_i}),$$

so

$$(x + V_{x_i} + V) \quad \text{does not} \quad \text{intersect} \quad (Z + V).$$

Taking the union over i:

$$Y + V \subseteq \bigcup_{i=1}^{n} (x_i + V_{x_i} + V) \quad \text{does not} \quad \text{intersect} \quad (Z + V).$$

■

Remark 1.16 A topological vector space is regular.

Proposition 1.23 *Suppose τ is a linear topology in a vector space X.*

1. *For $Y \subseteq X$,*

$$\overline{Y} = \bigcap_{V,\textit{open neighborhood of } \theta} (Y + V).$$

That is, the closure of a set is the intersection of all the open neighborhoods of that set.

2. *For $Y, Z \subseteq X$, $\overline{Y} + \overline{Z} \subseteq \overline{Y + Z}$.*
3. *If $Y \subseteq X$ is a linear subspace, then so is $\overline{Y}$.*
4. *For every $B \subseteq X$: If B is balanced so is $\overline{B}$.*
5. *For every $B \subseteq X$: If B is balanced and $\theta \in \operatorname{int} B$ then $\operatorname{int} B$ is balanced.*
6. *If $Y \subseteq X$ is bounded so is $\overline{Y}$.*

Proof

1. Let $x \in \overline{Y}$. By definition, for every open neighborhood V of θ, $x + V$ intersects Y, of $x \in Y - V$. Thus,

$$x \in \bigcap_{V,\text{open neighborhood of } \theta} (Y - V) = \bigcap_{V,\text{open neighborhood of } \theta} (Y + V).$$

Conversely, suppose that $x \notin \overline{Y}$. Then, there exists an open neighborhood V of θ such that $x + V$ does not intersect Y, i.e., $x \notin Y - V$, hence

$$x \notin \bigcap_{V,\text{open neighborhood of } \theta} (Y + V).$$

2. Let $x \in \overline{Y}$ and $y \in \overline{Z}$. By the continuity of vector addition, for every open neighborhood U of $x + y$ there exists an open neighborhood V of x and an open neighborhood W of y such that

$$V + W \subseteq U.$$

By the definition of $\overline{Y}$ every neighborhood of x intersects Y and by the definition of $\overline{Z}$ every neighborhood of y intersects W: that is, there exist $z \in V \cap Y$ and $t \in W \cap Z$. Then,

$$z \in Y \text{ and } t \in Z \text{ implies } z + t \in Y + Z,$$

and

$$z \in V \text{ and } t \in W \text{ implies } z + t \in V + W \subseteq U.$$

In other words, every neighborhood of $x + y \in \overline{Y} + \overline{Z}$ intersects $Y + Z$, which implies that $x + y \in \overline{Y + Z}$, and therefore

$$\overline{Y} + \overline{Z} \subseteq \overline{Y + Z}.$$

3. Let Y be a linear subspace of X, which means that,

$$Y + Y \subseteq Y \text{ and } \forall\, \alpha \in \mathbb{K}, \alpha Y \subseteq Y.$$

By the previous item,

$$\overline{Y} + \overline{Y} \subseteq \overline{Y + Y} \subseteq \overline{Y}.$$

Since scalar multiplication is a homeomorphism it maps the closure of a set into the closure of its image, namely, for every $\alpha \in \mathbb{K}$,

$$\alpha \overline{Y} \subseteq \overline{Y}.$$

4. Since multiplication by a (non-zero) is a homeomorphism,

$$\alpha \overline{B} = \overline{\alpha B}.$$

If B is balanced, then for $|\alpha| \le 1$,

$$\alpha \overline{B} = \overline{\alpha B} \subseteq \overline{B},$$

hence $\overline{B}$ is balanced.
5. Again, for every $0 < |\alpha| \le 1$,

$$\alpha(\text{int} B) = \text{int}(\alpha B) \subseteq \text{int} B.$$

Since for $\alpha = 0$, $\alpha(\text{int} B) = \{\theta\}$, we must require that $\theta \in \text{int} B$ for the latter to be balanced.
6. Let V be an open neighborhood of θ. Then there exists an open neighborhood W of θ such that $\overline{W} \subseteq V$. Since Y is bounded, $Y \subseteq \alpha W \subseteq \alpha \overline{W} \subseteq \alpha V$ for sufficiently large α. It follows that for large enough α,

$$\overline{Y} \subseteq \alpha \overline{W} \subseteq \alpha V,$$

which proves that $\overline{Y}$ is bounded.

■

Lemma 1.4 *Suppose τ is a linear topology in a vector space X.*

1. *If Y is convex so is $\overline{Y}$.*
2. *If Y is convex so is* $\operatorname{int} Y$.

Proof

1. The convexity of Y implies that for all $\alpha \in [0, 1]$:

$$\alpha Y + (1-\alpha)Y \subseteq Y.$$

Let $\alpha \in [0, 1]$, then

$$\alpha\overline{Y} = \overline{\alpha Y} \text{ and } (1-\alpha)\overline{Y} = \overline{(1-\alpha)Y}.$$

By the second item:

$$\alpha\overline{Y} + (1-\alpha)\overline{Y} = \overline{\alpha Y} + \overline{(1-\alpha)Y} \subseteq \overline{\alpha Y + (1-\alpha)Y} \subseteq \overline{Y},$$

which proves that $\overline{Y}$ is convex.

2. Suppose once again that Y is convex. Let $x, y \in \operatorname{int}Y$. This means that there exist open neighborhoods U, V of θ such that

$$x + U \subseteq Y \text{ and } y + V \subseteq Y.$$

Since Y is convex:

$$\alpha(x+U) + (1-\alpha)(y+V) = (\alpha x + (1-\alpha)y) + \alpha U + (1-\alpha)V \subseteq Y,$$

which proves that $\alpha x + (1-\alpha)y \in \operatorname{int}Y$, namely $\operatorname{int}Y$ is convex.

■

Lemma 1.5 *Suppose τ is a linear topology in a vector space X. If Y is a convex subset of X, then:*

$$0 < \alpha \leq 1 \implies \alpha(\operatorname{int}Y) + (1-\alpha)\overline{Y} \subseteq \operatorname{int}Y. \tag{1.10}$$

In particular, if $\operatorname{int}Y \neq \emptyset$, *then:*

(a) *The interior of Y is dense in $\overline{Y}$, that is,* $\overline{\operatorname{int}Y} = \overline{Y}$.
(b) *The interior of $\overline{Y}$ coincides with the interior of Y, that is,* $\operatorname{int}\overline{Y} = \operatorname{int}Y$.

Proof The case $\alpha = 1$ in (1.10) is immediate. So let $x \in \operatorname{int}Y$, $y \in \overline{Y}$, and let $0 < \alpha < 1$. Choose an open neighborhood U of θ such that $x + U \subseteq Y$. Since $y - \dfrac{\alpha}{1-\alpha}U$ is a neighborhood of y, there is some $z \in Y \cap (y - \dfrac{\alpha}{1-\alpha}U)$, so that $(1-\alpha)(y-z)$ belongs to αU. Since Y is convex, the (nonempty) open set

$V = \alpha(x+U)+(1-\alpha)z = \alpha x+\alpha U+(1-\alpha)z$ lies entirely in Y. Moreover, from

$$\alpha x+(1-\alpha)y = \alpha x+(1-\alpha)(y-z)+\alpha x+(1-\alpha)z \in \alpha x+\alpha U+(1-\alpha)z = V \subseteq Y,$$

we see that $\alpha x+(1-\alpha)y \in \mathrm{int}Y$. This proves (1.10), and letting $\alpha \longrightarrow 0$ proves (a).

For (b), fix $x_0 \in \mathrm{int}Y$ and $x \in \mathrm{int}\overline{Y}$. Pick a neighborhood of θ satisfying $x+W \subseteq \overline{Y}$. Since W is absorbing, there is some $0 < \lambda < 1$ such that $\lambda(x-x_0) \in W$, so $x+\lambda(x-x_0) \in \overline{Y}$. By (1.10), we have $x-\lambda(x-x_0) = \lambda x_0+(1-\lambda)x \in \mathrm{int}Y$. But then, using (1.10) once more, we obtain $x = \frac{1}{2}[x-\lambda(x-x_0)]+\frac{1}{2}[x+\lambda(x-x_0)] \in \mathrm{int}Y$. Therefore, $\mathrm{int}\overline{Y} \subseteq \mathrm{int}Y \subseteq \mathrm{int}\overline{Y}$ so that $\mathrm{int}\overline{Y} = \mathrm{int}Y$. ∎

Definition 1.27 Let τ be a linear topology in a vector space X and $Y \subseteq X$.

1. The closed convex hull of a set Y, denoted $\overline{\mathrm{conv}}(Y)$, is the smallest closed convex set including Y. By Lemma 1.4 1. it is the closure of $\mathrm{conv}(Y)$, that is, $\overline{\mathrm{conv}}(Y) = \overline{\mathrm{conv}(Y)}$.
2. The convex circled hull of Y is the smallest convex and circled set that includes Y. It is the intersection of all convex and circled sets that include Y.
3. The closed convex circled hull of Y is the smallest closed convex circled set including Y. It is the closure of the convex circled hull of Y.

Definition 1.28 Let X be a vector space and let τ be a linear topology on X. Then (X, τ) is said to be locally convex if there is a base of neighborhoods of the origin in X consisting of convex sets.

Proposition 1.24 *A locally convex space (X, τ) always has a base of neighborhoods of the origin consisting of open absorbing absolutely convex subsets.*

Proof Let V be a neighborhood of θ in X. Since (X, τ) is locally convex, there exists W convex neighborhood of θ such that $W \subseteq V$. Moreover, by Remark 1.13, there exists U balanced neighborhood of θ such that $U \subseteq W$. The balancedness of U implies that $U = \bigcup_{\alpha\in\mathbb{K},|\alpha|\le 1} \alpha U$. Thus, using that W is a convex set containing U, we get

$$N := \mathrm{conv}\left(\bigcup_{\alpha\in\mathbb{K},|\alpha|\le 1} \alpha U\right) = \mathrm{conv}(U) \subseteq W \subseteq V$$

and so $\mathrm{int}N \subseteq V$. Hence, the conclusion holds because $\mathrm{int}N$ is clearly open and convex and it is also balanced since $\theta \in \mathrm{int}N$ and N is balanced. ∎

1.2.3 Compactness and Completeness

Definition 1.29 Let (X, τ) be a topological vector space.

1. A net $(x_\alpha)_{\alpha \in I}$ in X is said to be a Cauchy net if for each neighborhood V of θ there exists $\alpha_0 \in I$ such that $x_\alpha - x_\beta \in V$ whenever $\alpha, \beta \succ \alpha_0$.
2. A set $Y \subseteq X$ is complete if each Cauchy net in X converges to a point of Y.
3. A set $Y \subseteq X$ is sequentially complete if each Cauchy sequence in X converges to a point of Y

Example 1.8 Every convergent net is Cauchy.

Proposition 1.25 *A Cauchy sequence (and in particular a converging sequence) in a topological vector space (X, τ) is bounded.*

Proof Let $(x_n)_n$ be a Cauchy sequence. Let W, V be two balanced open neighborhoods of θ satisfying

$$V + V \subseteq W.$$

By the definition of a Cauchy sequence, there exists an N such that for all $m, n \geq N$,

$$x_n - x_m \in V,$$

and in particular

$$\forall\, n > N \quad x_n \in x_N + V.$$

Set $s > 1$ such that $x_N \in sV$ (we know that such an s exists), then for all $n > N$,

$$x_n \in sV + V \subseteq sV + sV \subseteq W.$$

Since for balanced sets $sW \subseteq tW$ for $s < t$, and since every open neighborhood of θ contains an open balanced neighborhood, this proves that the sequence is indeed bounded. ■

Proposition 1.26 *Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of topological vector spaces, and let $X = \prod_{i \in I} X_i$ endowed with the product topology $\tau = \prod_{i \in I} \tau_i$. Then (X, τ) is complete if and only if each factor (X_i, τ_i) is complete.*

Proposition 1.27 *Let (X, τ) be a topological vector space with a countable base of neighborhoods of θ. A set $Y \subseteq X$ is complete if and only if Y is sequentially complete.*

Proof Let $\mathcal{B} = \{V_n : n \in \mathbb{N}\}$ be a countable base of neighborhoods of θ. We can assume that $V_1 \supseteq V_2 \supseteq \cdots$, indeed, otherwise we can substitute $\mathcal{B}$ with the base

$$\{V_1, V_1 \cap V_2, V_1 \cap V_2 \cap V_3, \cdots\}.$$

Let Y be complete, and $(x_n)_n$ a Cauchy sequence in Y. There exists a subnet $(x_{\phi(\alpha)})_{\alpha \in I}$ converging to a point $x \in Y$. Let us construct inductively a sequence

(α_k) in I. Choose α_1 so that $x_{\phi(\alpha)} \in x + V_1$ for each $\alpha \succ \alpha_1$. If we already have $\alpha_1, \cdots, \alpha_k$, choose $\alpha_{k+1} \succ \alpha_k$ so that $\phi(\alpha_{k+1}) \succ \phi(\alpha_k) + 1$ and $x_{\phi(\alpha)} \in x + V_{k+1}$ for each $\alpha \succ \alpha_{k+1}$. It is easy to verify that $(x_{\phi(\alpha_k)})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_n$ that converges to x.

Conversely, Let Y be sequentially complete, and $(x_\alpha)_{\alpha\in I}$ a Cauchy net in Y. Let us construct inductively a sequence $(\alpha_k)_k$ in I. Choose α_1 so that $x_\alpha - x_{\alpha_1} \in V_1$ for each $\alpha \succ \alpha_1$. If we already have $\alpha_1, \cdots, \alpha_k$, choose $\alpha_{k+1} \succ \alpha_k$ so that $x_\alpha - x_{\alpha_{k+1}} \in V_{k+1}$ for each $\alpha \succ \alpha_{k+1}$. Then (x_{α_n}) is a Cauchy sequence since $x_{\alpha_m} - x_{\alpha_n} \in V_n$ whenever $m \geq n$. Consequently, (x_{α_n}) converges to a point $x \in Y$. Now, it is easy to show that $(x_\alpha)_{\alpha\in I}$ converges to x, too. ■

Definition 1.30 A set Y in a topological vector space (X, τ) is totally bounded (or precompact) if for each neighborhood V of θ there is a finite set $F \subseteq X$ such that $Y \subseteq F + V$.

It is easy to see that in normed spaces (or in topological metric spaces) this definition coincides with the usual metric one: for each $\varepsilon > 0$ there is a finite set $F \subseteq X$ such that $\text{dist}(x, F) < \varepsilon$ for each $x \in Y$.

Theorem 1.9 *Let Y be a set in a topological vector space (X, τ). Then Y is totally bounded if and only if each net in Y admits a Cauchy subnet.*

Proof Let $(x_\alpha)_{\alpha\in I}$ be a net in a totally bounded set Y. The family $\mathcal{Z} = \{Z \subseteq Y\}$: $\mathcal{B}$ be a maximal subfamily of $\mathcal{Z}$ that contains Y and is closed under making finite intersections (existence of such $\mathcal{B}$ follows by `Zorn's` lemma). Let us show several properties of $\mathcal{B}$.

(a) if $\mathcal{F}$ is a finite subfamily of $\mathcal{Z}$ such that $\bigcup \mathcal{F} \in \mathcal{B}$, then $\mathcal{F} \cap \mathcal{B} \neq \emptyset$. Let $\mathcal{F} = \{Z_1, \cdots, Z_n\}$. We claim that, for some index k, $Z_k \cap B \in \mathcal{Z}$ for each $B \in \mathcal{B}$. Indeed, if this not the case, for each $i \in \{1, \cdots, n\}$ there exists $B_i \in \mathcal{B}$ such that $Z_i \cap B_i \notin \mathcal{Z}$, but then $\mathcal{B} \ni (\bigcup_{i=1}^{n} Z_i) \cap \bigcap_{i}^{n} B_i \subseteq \bigcup_{i=1}^{n}(Z_i \cap B_i) \notin \mathcal{Z}$, a contradiction. Our claim implies that the family of all finite intersections of elements of $\mathcal{B} \bigcup \{Z_k\}$ is closed under finite intersections and is contained in $\mathcal{Z}$. By maximality of $\mathcal{B}$, we must have $Z_k \in \mathcal{B}$.

(b) For each set $Z \subseteq Y$, the family $\mathcal{B}$ contains either Z or $Y \setminus Z$. If $Z \notin \mathcal{Z}$, then eventually $x_\alpha \in Y \setminus Z$. Since the intersection of $Y \setminus Z$ with any element of $\mathcal{B}$ belongs to $\mathcal{Z}$, the family of finite intersections of $\mathcal{B} \cup \{Y \setminus Z\}$ is contained in $\mathcal{Z}$. Thus $Y \setminus Z \in \mathcal{B}$ by the maximality of $\mathcal{B}$. In the same way we get that $Y \setminus Z \notin \mathcal{Z}$ then $Z \in \mathcal{B}$. Finally, if both Z and $Y \setminus Z$ belong to $\mathcal{Z}$ the one of them belongs to $\mathcal{B}$ by (a) (since $Y \in \mathcal{B}$).

(c) $\mathcal{B}$ contains arbitrarily small elements, in the sense that for each neighborhood V of θ there exists $B \in \mathcal{B}$ such that $B - B \subseteq V$. Given a neighborhood V of θ, there exists a neighborhood W of θ with $W - W \subseteq V$. By total boundedness, there exists a finite set $F = \{y_1, \cdots, y_n\} \subseteq Y$ such that $Y \subseteq F + W$. Denoting $Y_i = (y_i + W) \cap Y (i = 1, \cdots, n)$, we have $Y = \cup_{i=1}^{n} Y_i$. Consider the set

$P = \{i \in \{1, \cdots, n\} \colon Y_i \in \mathcal{Z}\}$ and its complement $\{1, \cdots, n\} \setminus P$. Since $C = \bigcup_{i \in \{1, \cdots, n\} \setminus P} Y_i \notin \mathcal{Z}$, we must have $P \neq \emptyset$. Let $Z = \bigcup_{i \in P} Y_i$. Then $Y \setminus Z \notin \mathcal{Z}$ (since $Y \setminus Z \subseteq C$). By (b), we must have $Z \in \mathcal{B}$. By (a), there exists $k \in P$ with $Y_k \in \mathcal{B}$. Notice that $Y_k - Y_k \subseteq W - W \subseteq V$.

To conclude the proof of this implication, notice that the family $\mathcal{B}$ satisfies the assumptions of Lemma 1.1. Hence there exists a subnet of (x_α) that is eventually contained in each element of $\mathcal{B}$. By (c), this subnet is Cauchy.

Conversely, assume that Y is not totally bounded. There exists a neighborhood V of θ such that $Y \setminus (F + V) \neq \emptyset$ for each finite set $F \subseteq V$. An easy inductive construction gives a sequence $(x_n)_n$ such that $x_{n+1} \notin \{x_1, \cdots, x_n\} + V$ for each n. Since for two indexes $m > n$ we have $x_m - x_n \notin V$, our sequence has no Cauchy subnets. The proof is complete. ■

Theorem 1.10 *A set Y in a topological vector space is compact if and only if Y is totally bounded and complete.*

Proof Let Y be compact. Given an open neighborhood V of θ, the open cover $\{y + V : y \in Y\}$ of Y admits a finite sub cover. This proves that Y is totally bounded. Let $(x_\alpha)_{\alpha \in I}$ be a Cauchy net in Y. By Theorem 1.6 $(x_\alpha)_{\alpha \in I}$ admits a subnet converging to a point of Y. It easily follows that the net $(x_\alpha)_{\alpha \in I}$ converges to the same limit.

Conversely, assume Y is totally bounded and complete. Given a net $(x_\alpha)_{\alpha \in I}$ in Y, it admits a Cauchy subnet by Theorem 1.9. Since Y is complete, this subnet converges to a point of Y. Again, it follows that $(x_\alpha)_{\alpha \in I}$ converges to the same point. By Theorem 1.6, Y is compact. ■

1.2.4 Seminorms and Local Convexity

Definition 1.31 A seminorm on a vector space X is map $p \colon X \to \mathbb{R}$ such that

$$p(x + y) \leq p(x) + p(y),$$

and

$$p(\alpha x) = |\alpha|\, p(x).$$

Definition 1.32 Let $\mathcal{P} := (p_i)_{i \in I}$ be a family of seminorms. It is called separating if to each $x \neq \theta$ corresponds a $p_i \in \mathcal{P}$, such that $p_i(x) \neq 0$. Note that the separation condition is equivalent to

$$p_i(x) = 0, \forall\, i \in I \Rightarrow x = \theta.$$

Examples 1.7

1. Suppose $X = \mathbb{R}^n$ and let Y be a vector subspace of X. Set for any $x \in X$

$$p_Y(x) := \inf_{y \in Y} \|x - y\|$$

   where $\|.\|$ is the Euclidean norm, i.e., $p_Y(x)$ is the distance from the point x to Y in the usual sense. If $\dim(Y) \geq 1$ then p_Y is a seminorm and not a norm (Y is exactly the kernel of p_Y). When $Y = \{\theta\}$, $p_Y(.) = \|.\|$.
2. Let X be a vector space on which is defined a nonnegative sesquilinear Hermitian form $\varphi\colon X \times X \to \mathbb{K}$. Then the function

$$p_\varphi(x) := \varphi(x, x)^{\frac{1}{2}}$$

   is a seminorm. p_φ is a norm if and only if φ is positive definite (i.e., $\varphi(x, x) > 0, \forall\, x \neq \theta$).
3. Let $C(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, we define for any $f \in C(\mathbb{R})$:

$$p_{[a,b]}(f) := \sup_{a \leq t \leq b} |f(t)| \,.$$

   $p_{[a,b]}$ is a seminorm but is never a norm because it might be that $f(t) = 0$ for all $t \in [a, b]$ (and so that $p_{[a,b]}(f) = 0$) but $f \not\equiv 0$. Other seminorms are the following ones:

$$q(f) := |f(0)| \text{ and } q_p(f) := \left(\int_a^b |f(t)|^p\right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty.$$

Proposition 1.28 *Let p be a seminorm on a vector space X.*

1. *p is symmetric.*
2. $p(\theta) = 0$.
3. $|p(x) - p(y)| \leq p(x - y)$.
4. $p(x) \geq 0$.
5. $\ker p$ *is a linear subspace.*

Proof By the properties of the seminorm:

1. $p(x - y) = p(-(y - x)) = |-1|\, p(y - x) = p(y - x)$.
2. $p(\theta) = p(0.x) = 0.p(x) = 0$.
3. This follows from the inequalities

$$p(x) \leq p(y) + p(x - y) \text{ and } p(y) \leq p(x) + p(y - x) = p(x) + p(x - y).$$

4. By the previous item, for every x:

$$0 \leq |p(x) - p(\theta)| \leq p(x).$$

5. If $x, y \in \ker p$:

$$p(\alpha x + \beta y) \leq p(\alpha x) + p(\beta y) = |\alpha|\, p(x) + |\beta|\, p(y) = 0.$$

■

Notation Let X be a vector space and p a seminorm on X. The sets

$$B_1^p = \{x \in X \colon p(x) < 1\} \quad \text{and} \quad \overline{B}_1^p = \{x \in X \colon p(x) \leq 1\},$$

are said to be, respectively, the open and the closed unit semiball of p.

Proposition 1.29 *Let τ be a linear topology on the vector space X. Then the following conditions are equivalent*:

1. *the open unit semiball B_1^p of p is an open set.*
2. *p is continuous at the origin.*
3. *the closed unit semiball $\overline{B}_1^p$ of p is a barrel neighborhood of the origin.*
4. *p is continuous at every point.*

Proof 1. $\Rightarrow$ 2. Suppose that B_1^p is open in the topology τ on X. Then for any $\varepsilon > 0$ we have that $p^{-1}([0, \varepsilon[) = \{x \in X \colon p(x) \leq \varepsilon\} = \varepsilon B_1^p$ is an open neighborhood of the origin in X. This is enough to conclude that $p \colon X \to \mathbb{R}^+$ is continuous at the origin.

2. $\Rightarrow$ 3. Suppose that p is continuous at the origin, then $\overline{B}_1^p = p^{-1}([0, 1])$ is a closed neighborhood of the origin. Since B_1^p is also absorbing and absolutely convex, $\overline{B}_1^p$ is a barrel.

3. $\Rightarrow$ 4. Assume that 3. holds and fix $\theta \neq x \in X$. We have for any $\varepsilon > 0$: $p^{-1}([-\varepsilon + p(x), \varepsilon + p(x)]) = \{y \in X \colon |p(y) - p(x)| \leq \varepsilon\} \supseteq \{y \in X \colon p(y - x) \leq \varepsilon\} = x + \varepsilon \overline{B}_1^p$, which is a closed neighborhood of x since τ is a linear topology on X and by the assumption 3. Hence, p is continuous.

4. $\Rightarrow$ 1. If p is continuous on X then 1. holds because the preimage of an open set under a continuous function is open and $B_1^p = p^{-1}([0, 1[)$. ■

Definition 1.33 Let X be a vector space. For $K \subseteq X$ convex and radial at θ (equivalently, K is absorbing), we define the Minkowski functional of K as

$$p_K(x) = \inf\{t > 0 \colon \frac{x}{t} \in K\}.$$

Intuitively, $p_K(x)$ is the factor by which x must be shrunk in order to reach the boundary of K.

Definition 1.34 (Topology Induced from Seminorms) Let $(p_i)_{i\in I}$ a family of seminorms on a vector space X. Then the ith open strip of radius r centered at $x \in X$ is

$$B_r^i(x) = \{y \in X \colon p_i(x-y) < r\}.$$

Let Λ be the collection of all open strips in X:

$$\Lambda = \{B_r^i(x) \colon i \in I, r > 0, x \in X\}.$$

The topology $\tau(\Lambda)$ generated by Λ is called the topology induced by $(p_i)_{i\in I}$.

The fact that p_i is a seminorm ensures that each open strip $B_r^i(x)$ is convex. Hence all finite intersections of open strips will also be convex.

Theorem 1.11 *Let $(p_i)_{i\in I}$ be a family of seminorms on a vector space X. Then*

$$\mathcal{B} = \left\{ \bigcap_{j=1}^{n} B_r^{i_j}(x) \colon n \in \mathbb{N}, i_j \in I, r > 0, x \in X \right\}$$

forms a base for the topology induced from these seminorms. In fact, if U is open and $x \in U$, then there exists an $r > 0$ and $i_1, \cdots, i_n \in I$ such that

$$\bigcap_{j=1}^{n} B_r^{i_j}(x) \subseteq U.$$

Further, every element of $\mathcal{B}$ is convex.

Proof Suppose $U \subseteq X$ and $x \in U$. In order to show that $\mathcal{B}$ is a base for the topology, we have to show that there exists some set $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By the characterization of the generated topology, U is a union of finite intersections of elements of Λ. Hence we have

$$x \in \bigcap_{j=1}^{n} B_{r_j}^{i_j}(x_j)$$

for some $n > 0, i_j \in I, r_j > 0$, and $x_j \in X$. Then $x \in B_{r_j}^{i_j}(x_j)$, so, by definition $p_{i_j}(x - x_j) < r_j$ for each j. Therefore, if we set

$$r = \min\{r_j - p_{i_j}(x - x_j) \colon j = 1, \cdots, n\},$$

then we have $B_r^{i_j}(x) \subseteq B_{r_j}^{i_j}(x_j)$ for each $j = 1, \cdots, n$. Hence

$$B = \bigcap_{j=1}^{n} B_r^{i_j}(x) \in \mathcal{B},$$

and we have $x \in B \subseteq U$. ■

Proposition 1.30 *Let $(p_i)_{i \in I}$ be a family of seminorms on a vector space X. Then the induced topology on X is Hausdorff if and only if the family $(p_i)_{i \in I}$ is separating.*

Remark 1.17 If any one of the seminorms in our family is a norm, then the corresponding topology is automatically Hausdorff (for example, this is the case for $C_b^\infty(\mathbb{R})$). On the other hand, the topology can be Hausdorff even if no individual seminorms in a norm (consider $L^1_{loc}(\mathbb{R})$).

Examples 1.8

1. Given an open subset Ω of $\mathbb{R}^m$ with the euclidean topology, the space $C(\Omega)$ of real valued continuous functions on Ω with the so-called topology of uniform convergence on compact sets is a locally convex topological vector space. This topology is defined by the family $\mathcal{P}$ of all the seminorms on $C(\Omega)$ given by

$$p_K(f) := \max_{x \in K} |f(x)|, \ \forall\, K \subseteq \Omega \text{ compact}.$$

Moreover, the linear topology $\tau_{\mathcal{P}}$ induced from the family $\mathcal{P}$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_K(f) = 0, \forall\, K$ compact subsets of Ω then in particular $p_{\{x\}}(f) := |f(x)| = 0\ \forall\, x \in \Omega$, which implies $f \equiv 0$ on Ω.

More generally, for any X locally compact we have that $C(X)$ with the topology of uniform convergence on compact subsets of X is a locally convex topological vector space.
2. Let $\mathbb{N}_0$ be the set of all non-negative integers. For any $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_0^m$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. For any $\beta \in \mathbb{N}_0^m$, the symbol D^β denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^{m} \beta_i$, i.e.,

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_m^{\beta_m}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_m}}{\partial x_m^{\beta_m}}.$$

   (a) Let $\Omega \subseteq \mathbb{R}^m$ open in the euclidean topology. For any $k \in \mathbb{N}_0$, let $C^k(\Omega)$ be the set of all real valued $k-$times continuously differentiable functions on Ω, i.e., all the derivatives of f of order $\leq k$ exist (at every point of Ω) and are continuous functions in Ω. Clearly, when $k = 0$ we get the set $C(\Omega)$ for all real valued continuous functions on Ω and when $k = \infty$ we get the so-called set of all infinitely differentiable functions or smooth functions on Ω. For any $k \in \mathbb{N}_0$, $C^k(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$. The topology given by the following family of seminorms on $C^k(\Omega)$:

$$p_{d,K}(f) := \sup_{\substack{\beta\in\mathbb{N}_0^m \\ |\beta|\le d}} \sup_{x\in K} \left|(D^\beta f)(x)\right|, \ \forall\, K \subseteq \Omega \text{ compact } \forall\, d \in \{0, 1, \cdots, k\},$$

makes $C^k(\Omega)$ into a locally convex topological vector space. (Note that when $k = \infty$ we have $m \in \mathbb{N}_0$.)

(b) The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^m$ is defined as the set $\mathcal{S}(\mathbb{R}^m)$ of all real-valued functions which are defined and infinitely differentiable on $\mathbb{R}^m$ and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of x, i.e.,

$$\mathcal{S}(\mathbb{R}^m) = \left\{ f \in C^\infty(\mathbb{R}^m) \colon \sup_{x\in\mathbb{R}^m} \left|x^\alpha D^\beta f(x)\right| < \infty, \ \forall\, \alpha, \beta \in \mathbb{N}_0^m \right\}.$$

If f is a smooth function with compact support in $\mathbb{R}^m$ then $f \in \mathcal{S}(\mathbb{R}^m)$, since any derivative of f is continuous and supported on a compact subset of $\mathbb{R}^m$, so $x^\alpha(D^\beta f(x))$ has a maximum in $\mathbb{R}^m$ by the extreme value theorem.

The Schwartz space $\mathcal{S}(\mathbb{R}^m)$ is a vector space over $\mathbb{R}$ and the topology given by the family $\mathcal{P}$ of seminorms on $\mathcal{S}(\mathbb{R}^m)$:

$$p_{\alpha,\beta} := \sup_{x\in\mathbb{R}^m} \left|x^\alpha D^\beta f(x)\right|, \ \forall\, \alpha, \beta \in \mathbb{N}_0^m$$

makes $\mathcal{S}(\mathbb{R}^m)$ into a locally convex topological vector space. Indeed, the family is clearly separating, because if $p_{\alpha,\beta}(f) = 0, \ \forall\, \alpha, \beta \in \mathbb{N}_0^m$ then in particular $p_{0,0}(f) = \sup_{x\in\mathbb{R}^m} |f(x)| = 0 \ \forall\, x \in \mathbb{R}^m$, which implies $f \equiv 0$ on $\mathbb{R}^m$.

Note that $\mathcal{S}(\mathbb{R}^m)$ is a linear subspace of $C^\infty(\mathbb{R}^m)$, but its topology $\tau_\mathcal{P}$ on $\mathcal{S}(\mathbb{R}^m)$ is finer than the subspace topology induced on it by $C^\infty(\mathbb{R}^m)$.

Theorem 1.12 *Let X be a vector space whose topology is induced from a family of seminorms $(p_i)_{i\in I}$. Then given any net $(x_\alpha)_{\alpha\in J}$ and any $x \in X$, we have*

$$x_\alpha \to x \Leftrightarrow \forall\, i \in I,\ p_i(x - x_\alpha) \to 0.$$

Proof $\Rightarrow$. Suppose that $x_\alpha \to x$, and fix any $i \in I$ and $\varepsilon > 0$. Then $B^i_\varepsilon(x)$ is an open neighborhood of x, so by definition of convergence with respect to a net, there exists an $\alpha_0 \in J$ such that

$$\alpha \succ \alpha_0 \Rightarrow x_\alpha \in B^i_\varepsilon(x).$$

Therefore for all $\alpha \succ \alpha_0$ we have $p_i(x - x_\alpha) < \varepsilon$, so $p_i(x - x_\alpha) \to 0$.

$\Leftarrow$. Suppose that $p_i(x - x_\alpha) \to 0$ for every $i \in I$, and let U be any open neighborhood of x. Then by Theorem 1.11, we can find an $r > 0$ and finitely many $i_1, \cdots, i_n \in I$ such that

$$x \in \bigcap_{j=1}^{n} B^{i_j}_r(x) \subseteq U.$$

Now, given any $j = 1, \cdots, n$ we have $p_{i_j}(x - x_\alpha) \to 0$. Hence, for each j we can find $\alpha_j \in J$ such that

$$\alpha \succ \alpha_j \ \Rightarrow p_{i_j}(x - x_\alpha) < r.$$

Since J is a directed set, there exists some $\alpha_0 \in J$ such that $\alpha_0 \succ \alpha_j$ for $j = 1, \cdots, n$. Thus, for all $\alpha \succ \alpha_0$ we have $p_{i_j}(x - x_\alpha) < r$ for each $j = 1, \cdots, n$, so

$$x_\alpha \in \bigcap_{j=1}^{n} B^{i_j}_r(x) \subseteq U, \quad \alpha \succ \alpha_0.$$

Hence $x_\alpha \to x$. ■

Corollary 1.3 *Let X be a vector space whose topology is induced from a family of seminorms $(p_i)_{i\in I}$, let Y be any topological space, and fix $x \in X$. Then the following two statements are equivalent.*

1. *$T : X \to Y$ is continuous at x.*
2. *For any net $(x_\alpha)_{\alpha\in J}$,*

$$p_i(x - x_\alpha) \to 0 \quad \textit{for each } i \in I \quad \Rightarrow T(x_\alpha) \to T(x) \textit{ in } Y.$$

Proposition 1.31 *Let X be a vector space whose topology is induced from a family of seminorms $(p_i)_{i\in I}$. Then,*

1. *for all $i \in I$, p_i is continuous.*
2. *A set $Y \subseteq X$ is bounded if and only if p_i is bounded on Y for all $i \in I$.*

Proof

1. Let $i \in I$. Because of the reverse triangle inequality, $p_i(x - x_\alpha) \to 0$ implies $p_i(x_\alpha) \to p_i(x)$. Hence each seminorm p_i is continuous with respect to the induced topology.
2. Suppose $Y \subseteq X$ is bounded. Take $i \in I$. Then $B_1^{p_i}$ is a neighborhood of θ. Hence,

$$Y \subseteq \rho B_1^{p_i}$$

for some $\rho > 0$ (by definition of boundedness). Hence, for all $x \in Y$,

$$x \in \{\rho y \in X\colon p(y) < 1\} = \{\rho y \in X\colon p(\rho y) < \rho\} = \{z \in X\colon p(z) < \rho\},$$

i.e., $p(x) < \rho$.

Conversely, if $p_i(Y)$ is bounded for every $i \in I$. Then there are numbers r_i such that

$$\sup_{x \in Y} p_i(x) < r_i.$$

Let U be a neighborhood of θ. Again

$$\bigcap_{j=1}^{n} B_r^{i_j}(\theta) \subseteq U.$$

Choose $m > \frac{M_{i_j}}{r_{i_j}} (1 \le j \le n)$. If $x \in Y$ then $p_{i_j}(\frac{x}{m}) < \frac{M_{i_j}}{m} < r_{i_j} \Rightarrow \frac{x}{m} \in U \Rightarrow x \in mU$.

■

Theorem 1.13 *If X is a vector space whose topology τ is induced from a separating family of seminorms $(p_i)_{i \in I}$, then (X, τ) is a locally convex topological vector space.*

Proof We have already seen that there is a base for the topology τ that consists of convex open sets, so we just have to show that vector addition and scalar multiplication are continuous with respect to this topology.

Suppose that $((\lambda_\alpha, x_\alpha))_{\alpha \in J}$ is any net in $\mathbb{K} \times X$, and that $(\lambda_\alpha, x_\alpha) \to (\lambda, x)$ with respect to the product topology on $\mathbb{K} \times X$. This is equivalent to assuming that $\lambda_\alpha \to \lambda$ in $\mathbb{K}$ and $x_\alpha \to x$ in X. Fix any $i \in I$ and any $\varepsilon > 0$. Suppose that $p_i(x) \neq 0$. Since $p_i(x - x_\alpha) \to 0$, there exist $\alpha_1, \alpha_2 \in J$ such that

$$\alpha \succ \alpha_1 \;\Rightarrow\; |\lambda - \lambda_\alpha| < \min\left\{\frac{\varepsilon}{2p_i(x)}, 1\right\},$$

and

$$\alpha \succ \alpha_2 \quad \Rightarrow \quad p_i(x - x_\alpha) < \frac{\varepsilon}{2(|\lambda| + 1)}.$$

By definition of directed set, there exists $\alpha_0 \succ \alpha_1, \alpha_2$, so both of these inequalities hold for $\alpha \succ \alpha_0$. In particular, $(\lambda_\alpha)_{\alpha \succ \alpha_0}$ is a bounded net, with $|\lambda_\alpha| < |\lambda| + 1$ for all $\alpha \succ \alpha_0$. Hence, for $\alpha \succ \alpha_0$ we have

$$\begin{aligned} p_i(\lambda x - \lambda_\alpha x_\alpha) &\le p_i(\lambda x - \lambda_\alpha x) + p_i(\lambda_\alpha x - \lambda_\alpha x_\alpha) \\ &= |\lambda - \lambda_\alpha|\, p_i(x) + |\lambda_\alpha|\, p_i(x - x_\alpha) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

If $p_i(x) = 0$ then we similarly obtain $p_i(\lambda x - \lambda_\alpha x_\alpha) < \frac{\epsilon}{2}$ for $\alpha \succ \alpha_0$. Thus we have $p_i(\lambda x - \lambda_\alpha x_\alpha) \to 0$. Since this is true for every i, Theorem 1.12 implies that $\lambda_\alpha x_\alpha \to \lambda x$. ■

Theorem 1.14 *The topology of a locally convex topological vector space X is given by the collection of seminorms obtained as Minkowski functionals p_U associated to a local basis at θ consisting of convex balanced open.*

Proof The proof is straightforward. With or without local convexity, every neighborhood of θ contains a balanced neighborhood of θ. Thus, a locally convex topological vector space has a local basis $\mathcal{B}$ at θ of balanced convex open sets.

Every open $U \in \mathcal{B}$ can be recovered from the corresponding seminorm by

$$U = \text{int}U = \{x \in X \colon p_U(x) < 1\}.$$

Oppositely, every seminorm local basis open

$$\{x \in X \colon p_U(x) < r\}$$

is simply rU. Thus, the original topology is at least as fine as the seminorm topology. ■

1.2.5 *Metrizable Topological Vector Spaces*

What does it take for a topological vector space (X, τ) to be metrizable? Suppose there is a metric d compatible with the topology τ. Thus, all open sets are unions of open balls, and in particular, the countable collection of balls $B_{\frac{1}{n}}(\theta)$ forms a local base at the origin.

Theorem 1.15 *A Hausdorff topological vector space is metrizable if and only if zero has a countable neighborhood base. In this case, the topology is generated by a translation invariant metric.*

Proof Let (X, τ) be a topological vector space. If τ is metrizable, then τ has clearly a neighborhood base at θ. For the converse, assume that τ has a countable neighborhood base at θ. Choose a countable base $\{V_n\}$ of circled neighborhoods of θ such that $V_{n+1} + V_{n+1} + V_{n+1} \subseteq V_n$ holds for each n. Now define the function $\rho\colon X \to [0, \infty)$ by

$$\rho(x) = \begin{cases} 1, & \text{if} \quad x \notin V_1, \\ 2^{-k}, & \text{if } x \in V_k \setminus V_{k+1}, \\ 0, & \text{if} \quad x = \theta. \end{cases}$$

Then it is easy to check that for each $x \in X$ we have the following:

1. $\rho(x) \geq 0$ if and only if $x = \theta$.
2. $x \in V_k$ for some k if and only if $\rho(x) \leq 2^{-k}$
3. $\rho(x) = \rho(-x)$ and $\rho(\alpha x) \leq \rho(x)$ for all $|\alpha| \leq 1$.
4. $\lim_{\alpha \to 0} \rho(\alpha x) = 0$.

We also note the following property : $x_n \xrightarrow{\tau} \theta$ if and only $\rho(x_n) \longrightarrow 0$.

Now by means of the function ρ we define the function $\Pi\colon X \to [0, \infty)$ via the formula

$$\Pi(x) = \inf\left\{ \sum_{i=1}^{n} \rho(x_i)\colon x_1, \cdots, x_n \in X. \text{ and } \sum_{i=1}^{n} x_i = x \right\}.$$

The function Π satisfies the following properties.

(a) $\Pi(x) \geq 0$ for each $x \in X$.
(b) $\Pi(x + y) \leq \Pi(x) + \Pi(y)$ for all $x, y \in X$.
(c) $\frac{1}{2}\rho(x) \leq \Pi(x) \leq \rho(x)$ for each $x \in X$ (so $\Pi(x) = 0$ if and only if $x = \theta$).

Property (a) follows immediately from the definition of Π. Property (b) is straightforward. The proof of (c) will be based upon the following property:

$$\text{If } \sum_{i=1}^{n} \rho(x_i) < \frac{1}{2^m}, \textit{ then} \sum_{i=1}^{n} x_i \in V_m. \tag{1.11}$$

To verify (1.11), we use induction on n. For $n = 1$ we have $\rho(x_1) < \frac{1}{2^m}$, and consequently $x_1 \in V_{m+1} \subseteq V_m$ is trivially true. For the induction step, assume that if $\{x_i : i \in I\}$ is any collection of at most n vectors satisfying $\sum_{i \in I} \rho(x_i) < \frac{1}{2^m}$ for some $m \in \mathbb{N}$, then $\sum_{i \in I} x_i \in V_m$. Suppose that $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^m}$ for some $m \in \mathbb{N}$. Clearly, we have $\rho(x_i) < \frac{1}{2^{m+1}}$, so $x_i \in V_{m+1}$ for each $1 \leq n+1$. We now distinguish two cases.

Case 1: $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^{m+1}}$

Clearly $\sum_{i=1}^{n} \rho(x_i) < \frac{1}{2^{m+1}}$, so by the induction hypothesis $\sum_{i=1}^{n} x_i \in V_{m+1}$. Thus

$$\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n} x_i + x_{n+1} \in V_{m+1} + V_{m+1} \subseteq V_m.$$

Case 2: $\sum_{i=1}^{n+1} \rho(x_i) \geq \frac{1}{2^{m+1}}$

Let $1 \leq k \leq n+1$ be the largest k such that $\sum_{i=k}^{n+1} \rho(x_i) \geq \frac{1}{2^{m+1}}$. If $k = n+1$, then $\rho(x_{n+1}) = \frac{1}{2^{m+1}}$, so from $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^m}$ we have $\sum_{i=1}^{n} \rho(x_i) < \frac{1}{2^{m+1}}$. But then, as in Case 1, we get $\sum_{i=1}^{n+1} x_i \in V_m$. Thus, we can assume that $k < n+1$. Assume first that $k > 1$. From the inequalities $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^m}$ and $\sum_{i=k}^{n+1} \rho(x_i) \geq \frac{1}{2^{m+1}}$, we obtain $\sum_{i=1}^{k-1} \rho(x_i) < \frac{1}{2^{m+1}}$. So our induction hypothesis yields $\sum_{i=1}^{k-1} x_i \in V_{m+1}$. Also by the choice of k we have $\sum_{i=k+1}^{n+1} \rho(x_i) < \frac{1}{2^{m+1}}$, and thus by our induction hypothesis also we have $\sum_{i=k+1}^{n+1} x_i \in V_{m+1}$. Therefore, in this case we obtain

$$\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{k-1} x_i + x_k + \sum_{i=k+1}^{n+1} x_i \in V_{m+1} + V_{m+1} + V_{m+1} \subseteq V_m.$$

If $k = 1$, then we have $\sum_{i=2}^{n+1} \rho(x_i) < \frac{1}{2^{m+1}}$, so $\sum_{i=2}^{n+1} x_i \in V_{m+1}$. This implies $\sum_{i=1}^{n+1} x_i = x_1 + \sum_{i=2}^{n+1} x_i \in V_{m+1} + V_{m+1} \subseteq V_m$. This completes the induction and the proof of (1.11).

Next, we verify (c). To this end, let $x \in X$ satisfy $\rho(x) = \frac{1}{2^m}$ for some $m \geq 0$. Also, assume by way of contradiction that the vectors $x_1, \cdots, x_k$ satisfy $\sum_{i=1}^{k} x_i = x$ and $\sum_{i=1}^{k} \rho(x_i) < \frac{1}{2}\rho(x) = \frac{1}{2^{m+1}}$. But then, from (1.11) we get $x = \sum_{i=1}^{k} x_i \in V_{m+1}$, so $\rho(x) \leq \frac{1}{2^{m+1}} < \frac{1}{2^m} = \rho(x)$, which is impossible. This contradiction, establishes the validity of (c).

Finally, for each $x, y \in X$ define $d(x, y) = \Pi(x - y)$ and note that d is a translation invariant metric that generates τ. ■

Definition 1.35 Let (X, τ) be a topological vector space.

1. X is an F-space (completely metrizable topological vector space) if its topology is induced by a complete translationally invariant metric. In other words, a completely metrizable topological vector space is a complete topological vector space having a countable neighborhood base at θ. Every Banach space is an F-space. An F-space is a Banach space if in addition $d(\alpha x, \theta) = |\alpha|\, d(x, \theta)$.
2. X is a Fréchet space if it is a locally convex F-space.

Definition 1.36 A complete topological vector space (Y, Γ) is called a topological completion or simply a completion of another topological vector space (X, τ) if there is a linear homeomorphism $T : X \to Y$ such that $T(X)$ is dense in Y, identifying X with $T(X)$, we can think of X as a subspace of Y.

Theorem 1.16 *Every topological vector space has a unique (up to linear homeomorphism) topological completion.*

It turns out that the existence of a countable local base is also sufficient for metrizability. (It suffices that τ is induced from a separating countable family of seminorms $(p_n)_n$). Indeed, there exists a translation-invariant metric compatible with τ. One can show that the following is a compatible metric:

$$d(x, y) = \max_n \frac{\alpha_n p_n(x - y)}{1 + p_n(x - y)},$$

where $(\alpha_n)_n$ is any sequence of positive numbers that decays to 0 (it is easy to see that the maximum is indeed attained). Clearly, $d(x, x) = 0$. Also, since the p_n's are separating $d(x, y) > 0$ for $x \neq y$. Symmetry, as well as translational invariance are obvious. Finally, the triangle inequality follows from the fact that every p_n is subadditive, and that $a \leq b + c$ implies that

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

It remains to show that this metric is compatible with the topology τ. One can also define the following translation-invariant metric compatible with τ

$$d(x, y) = \sum_{n=1}^{+\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}.$$

Example 1.9 Let $s = \{(x_n)_{n\geq 1} \colon x \in \mathbb{K} \text{ for all } n \geq 1\}$, the space of all scalar sequences. The topology of pointwise convergence is described by the seminorms p_k, $(k \geq 1)$, $p_k((x_n)_{n\geq 1}) = |x_k|$ and the metric is

$$d(x, y) = \sum_{n=1}^{+\infty} 2^{-n} \frac{|x_n - y_n|}{1+|x_n - y_n|}, \quad x = (x_n)_{n\geq 1}, \quad y = (y_n)_{n\geq 1}.$$

The ball $\overline{B}_{\frac{1}{4}}(\theta) = \{x \colon d(x, \theta) \leq \frac{1}{4}\}$ is not convex, since $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$ $\in \overline{B}_{\frac{1}{4}}(\theta)$, but $\frac{3}{4}(1, 0, 0, \cdots) + \frac{1}{4}(0, 1, 0, \cdots) = (\frac{3}{4}, \frac{1}{4}, 0, 0, \cdots) \notin \overline{B}_{\frac{1}{4}}(\theta)$.

Theorem 1.17 *Let (X, τ) be topological vector space that has a countable local base. Then there is a metric d on X such that*:

1. *d is compatible with τ (every τ-open set is a union of d-open balls).*
2. *The open balls $B_r(\theta)$ are balanced.*
3. *d is invariant: $d(x+z, y+z) = d(x, y)$.*
4. *If, in addition, X is locally convex, then d can be chosen such that all open balls are convex.*

Theorem 1.18 *A topological vector space (X, τ) is normable if and only if there exists a convex bounded open neighborhood.*

Proof If (X, τ) is normable then $B_1 = \{x \colon \|x\| < 1\}$ is convex and bounded. Suppose that there exists an open convex and bounded neighborhood V of θ. Set

$$U = \bigcap_{|\alpha|=1} \alpha V.$$

Since U is the intersection of convex sets it is convex. It is balanced because for every $|\beta| \leq 1$,

$$\beta U = \bigcap_{|\alpha|=1} \beta\alpha V = \bigcap_{|\alpha|=1} |\beta|\, \alpha V = |\beta|\, U,$$

and by convexity,

$$|\beta| U = |\beta| U + (1 - |\beta|)\{\theta\} \subseteq U.$$

Since U contains θ, int$U$ is balanced, it is also convex. Then there exists a convex and balanced (and certainly bounded) open neighborhood $W = \text{int}U \subseteq V$. Set

$$\|x\| = p_W(x),$$

where p_W is the Minkowski functional of W. We will show that this indeed a norm. Clearly, $\|x\| = 0$ if and only if $x = \theta$. Since W is balanced then $p_W(\alpha x) = |\alpha|\, p_W(x)$. The triangle inequality follows from the properties of p_W. It remains to show this norm is compatible with the topology τ. This follows from the fact that

$$B_r(\theta) = \{x \colon \|x\| < r\} = \{x \colon p_W(x) < r\} = \{x \colon p_W(\frac{x}{r}) < 1\} \subseteq rW,$$

which means that $B_r(\theta)$ is bounded, hence

$$\left\{B_r(\theta) \colon r > 0\right\}$$

is a local base. ∎

Example 1.10 Let Ω be an open set in $\mathbb{R}^m$. We consider the space $C(\Omega)$ of all continuous functions. Note that the sup-norm does not work here. There exist unbounded continuous functions on open sets.

Every open set Ω in $\mathbb{R}^m$ can be written as

$$\Omega = \bigcup_{n=1}^{\infty} K_n,$$

where $K_n \Subset K_{n+1}$, where the K_n are compact, and $\Subset$ stands for compactly embedded, i.e., K_n is a compact set in the interior of K_{n+1}. We topologize $C(\Omega)$ with the separating family of seminorms,

$$p_n(f) = \max\{|f(x)| : x \in K_n\} = \|f\|_{K_n}.$$

(These are clearly seminorms, and they are separating because for every $f \neq 0$ there exists an n such that $f_{|K_n} \neq 0$).

Since the p_n's are monodically increasing,

$$\bigcap_{d=1}^{D} \bigcap_{k=1}^{n} B^k_{\frac{1}{d}}(\theta) = \bigcap_{d=1}^{D} \bigcap_{k=1}^{n} \{f \colon p_k(f) < \frac{1}{d}\} = B^n_{\frac{1}{D}}(\theta),$$

which means that the $B^n_{\frac{1}{D}}(\theta)$ form a convex local base for $C(\Omega)$. In fact, $B^n_{\frac{1}{D}}(\theta)$ contains a neighborhood obtained by taking n, D to be the greatest of the two, from which follows that

$$B^n_{\frac{1}{n}}(\theta) = \{f : p_n(f) < \frac{1}{n}\}$$

is a convex local base for $C(\Omega)$, and the p_n's are continuous in this topology. We can thus endow this topological space with a compatible metric, for example,

$$d(f, g) = \max_n \frac{2^{-n} p_n(f-g)}{1 + p_n(f-g)}.$$

We will now show that this space is complete. Recall that if a topological vector space has a compatible metric with respect to which is complete, then it is called an **F**-space. If, moreover, the space is locally convex, then it is called a Fréchet space. Thus, $C(\Omega)$ is a Fréchet space. Let $(f_n)_n$ be a Cauchy sequence. This means that for every $\varepsilon > 0$ there exists an N, such that for every $d, n > N$,

$$\max_k \frac{2^{-k} p_k(f_n - f_d)}{1 + p_k(f_n - f_d)} < \varepsilon,$$

and so,

$$(\forall\, k \geq 1) \quad \frac{2^{-k} p_k(f_n - f_d)}{1 + p_k(f_n - f_d)} < \varepsilon,$$

which means that $(f_n)_n$ is a Cauchy sequence in each K_k (endowed with the sup-norm), and hence converges uniformly to a function f. Given ε and let M such that $2^{-M} < \varepsilon$, then

$$\max_{k>M} \frac{2^{-M} p_k(f_n - f)}{1 + p_k(f_n - f)} < \varepsilon,$$

and there exists an N, such that for every $n > N$,

$$\max_{k \leq M} \frac{2^{-M} p_k(f_n - f)}{1 + p_k(f_n - f)} < \varepsilon,$$

which implies that $f_n \longrightarrow f$, hence the space is indeed complete.

The question remains whether $C(\Omega)$ with this topology is normable. For this, the origin must have a convex bounded neighborhood. Recall that a set Y is bounded if and only if $\{p_n(f) : f \in Y\}$ is bounded for every n, i.e., if

$$\{\sup\{|f(x)| : x \in K_n\} : f \in Y\}$$

is a bounded set for every n, or if

$$\forall\, n \geq 1 \quad \sup\{|f(x)| : x \in K_n,\, f \in Y\} < \infty.$$

Because the $B^n_{\frac{1}{n}}(\theta)$ form a base, every neighborhood of θ contains a set

$$B^k_{\frac{1}{k}}(\theta),$$

hence,

$$\sup\{|f(x)| : x \in K_n,\, f \in Y\} \geq \sup\{\|f\|_{K_n} : \|f\|_{K_k} < \frac{1}{k}\}.$$

The right hand side can be made as large as we please for $n > k$, i.e., no set is bounded, and hence the space is not normable.

1.2.6 Finite Dimensional Topological Vector Spaces

Lemma 1.6 *Let (X, τ) be a topological vector space. Any linear map $T : \mathbb{K}^n \to X$ is continuous.*

Proof Denote by $(e_i)_{1\leq i\leq n}$ the standard basis in $\mathbb{K}^n$ and set

$$u_j = T(e_j) \quad j = 1, \cdots, n.$$

By linearity, for any $x = (x_1, \cdots, x_n) = \sum_{j=1}^{n} x_j e_j$

$$T(x) = \sum_{j=1}^{n} x_j u_j.$$

The map $x \mapsto x_j$ (which is linear map $\mathbb{K}^n \to \mathbb{K}$) is continuous and so are addition and scalar multiplication in X. ■

Proposition 1.32 *Let (X, τ) be a topological vector space. Then:*

1. *Every finite dimensional subspace Y of X is a closed subset of X.*
2. *If Y is an n-dimensional subspace of X and $(u_i)_{1\leq i\leq n}$ is a basis for Y, then the map $T : \mathbb{K}^n \to Y$ defined by $T(x_1, \cdots, x_n) = \sum_{j=1}^{n} x_j u_j$ is a topological isomorphism of $\mathbb{K}^n$ equipped with its Euclidean topology, onto X. That is,*

specifically, a net $(x^\alpha)_\alpha = \left(\sum_{j=1}^{n} x_j^\alpha u_j\right)_\alpha$ *converges to an element* $x = \sum_{j=1}^{n} x_j u_j \in Y$ *if and only if each net* $(x_j^\alpha)_\alpha$ *converges to* x_j, $1 \leq j \leq n$.

Proof

1. We prove part 1 by induction on the dimension of the subspace Y. First, if Y has dimension 1, let $y \neq \theta \in Y$ be a basis for Y. If $(\lambda_\alpha y)_\alpha$ is a net in Y that converges to an element $x \in X$, then the net $(\lambda_\alpha)_\alpha$ must be eventually bounded in $\mathbb{K}$, in the sense that there must exist an index α_0 and a constant M such that $|\lambda_\alpha| \leq M$ for all $\alpha \succ \alpha_0$. Indeed, if the net $(\lambda_\alpha)_\alpha$ were not eventually bounded, let $(\lambda_{\alpha_\beta})_\beta$ be a subnet for which $\lim_\beta |\lambda_{\alpha_\beta}| = \infty$. Then

$$\begin{aligned} y &= \lim_\beta \frac{1}{\lambda_{\alpha_\beta}} \lambda_{\alpha_\beta} y \\ &= \lim_\beta \frac{1}{\lambda_{\alpha_\beta}} \lim_\beta \lambda_{\alpha_\beta} y \\ &= 0 \times x \\ &= \theta. \end{aligned}$$

which is a contradiction. So, the net $(\lambda_\alpha)_\alpha$ is bounded. Let $(\lambda_{\alpha_\beta})_\beta$ be a convergent subnet of $(\lambda_\alpha)_\alpha$ with limit λ. Then

$$x = \lim_\alpha \lambda_\alpha y = \lim_\beta \lambda_{\alpha_\beta} = \lambda y.$$

whence $x \in Y$, and Y is closed.

Assume now that any n-1-dimensional subspace is closed, and let Y have dimension $n > 1$. Let $\{y_1, \cdots, y_n\}$ be a basis for Y, and write Y' for the linear span of $y_1, \cdots, y_{n-1}$. Then elements y of Y can be written uniquely in the form $y = y' + \lambda y_n$, for $y' \in Y'$ and $\lambda \in \mathbb{K}$. Suppose that x is an element of the closure of Y, i.e., $x = \lim_\alpha (y'_\alpha + \lambda_\alpha y_n)$. As before, we have that the net $(\lambda_\alpha)_\alpha$ must be bounded. Indeed, if the net $(\lambda_\alpha)_\alpha$ were not bounded, then let $(\lambda_{\alpha_\beta})_\beta$ be a subnet for which $\lim_\beta |\lambda_{\alpha_\beta}| = \infty$. Then

$$\theta = \lim_\beta \frac{1}{\lambda_{\alpha_\beta}} x = \lim_\beta \frac{y'_{\alpha_\beta}}{\lambda_{\alpha_\beta}} + y_n,$$

or

$$y_n = -\lim_\beta \frac{y'_{\alpha_\beta}}{\lambda_{\alpha_\beta}},$$

implying that y_n belongs to the closure of the closed subspace Y', this is impossible, showing that the net $(\lambda_\alpha)_\alpha$ is bounded. Hence, letting $(\lambda_{\alpha_\beta})_\beta$ be a convergent subnet of $(\lambda_\alpha)_\alpha$, say $\lambda = \lim_\beta \lambda_{\alpha_\beta}$, we have

$$x = \lim_\beta (y'_{\alpha_\beta} + \lambda_{\alpha_\beta} y_n),$$

showing that

$$x - \lambda y_n = \lim_\beta y'_{\alpha_\beta},$$

whence, since Y' is closed, there exists a $y' \in Y'$ such that $x - \lambda y_n = y'$. Therefore, $x = y' + \lambda y_n \in Y$, and Y is closed, proving part 1.

2. We prove part 2 for real vector spaces. The map $T : \mathbb{R}^n \to Y$ of part 2 is obviously linear, one to one and onto. Also, it is continuous by previous lemma. Let us show that T^{-1} is continuous. Thus, let $(x^\alpha)_\alpha = \left(\sum_{j=1}^n x_j^\alpha u_j\right)_\alpha$ converge to θ in Y. Suppose, by way of contradiction, that there exists an j for which the net $(x_j^\alpha)_\alpha$ does not converge to 0. Then let $(x_j^{\alpha^\beta})_\beta$ be a subnet for which $\lim_\beta x_j^{\alpha^\beta} = x_j$, where x_j either is $\pm\infty$ or is a nonzero real number. Write $x^\alpha = x_j^\alpha u_j + x'^\alpha$. Then

$$\frac{1}{x_j^{\alpha^\beta}} x^{\alpha^\beta} = u_j + \frac{1}{x_j^{\alpha^\beta}} x'^{\alpha^\beta},$$

whence

$$u_j = -\lim_\beta \frac{1}{x_j^{\alpha^\beta}} x'^{\alpha^\beta},$$

implying that u_j belongs to the (closed) subspace spanned by the vectors

$$u_1, \cdots, u_{j+1}, \cdots, u_n.$$

and this is a contradiction, since the u_j's form a basis of Y. Therefore, each of the nets $(x_j^\alpha)_\alpha$ converges to 0, and T^{-1} is continuous.

■

Corollary 1.4 *There exists a unique topology on $\mathbb{K}^n$ (viewed as a topological vector space), and all n-dimensional topological vector spaces are topologically isomorphic.*

There are no infinite dimensional locally compact topological vector spaces. This is essentially due to F. Riesz.

Theorem 1.19 *A topological space is locally compact if and only if is finite dimensional.*

Proof Let (X, τ) be a topological vector space. If X is finite dimensional, then τ coincides with the Euclidean topology and since the closed balls are compact sets, it follows that (X, τ) is locally compact.

For the converse assume that (X, τ) is locally compact and let V be a compact neighborhood of θ. From $V \subseteq \bigcup_{x \in V}(x + \frac{1}{2}V)$, we see that there exists a finite subset $\{x_1, \cdots, x_k\}$ of V such that

$$V \subseteq \bigcup_{i=1}^{k}(x_i + \frac{1}{2}V) = \{x_1, \cdots, x_k\} + \frac{1}{2}V. \tag{1.12}$$

Let Y be a linear span of $x_1, \cdots, x_k$. From (1.12), we get $V \subseteq Y + \frac{1}{2}V$. This implies $\frac{1}{2}V \subseteq \frac{1}{2}(Y + \frac{1}{2}V) = Y + \frac{1}{2^2}V$, so $V \subseteq Y + (Y + \frac{1}{2^2}V) = Y + \frac{1}{2^2}V$. By induction we see that

$$V \subseteq Y + \frac{1}{2^n}V \tag{1.13}$$

for each n. Next, fix $x \in V$. From (1.13), it follows that for each n there exist $y_n \in Y$ and $v_n \in V$ such that $x = y_n + \frac{1}{2^n}v_n$. Since V is compact, there exists a subnet (v_{n_α}) of the sequence (v_n) such that $v_{n_\alpha} \xrightarrow{\tau} v \in X$ (and clearly $\frac{1}{2^{n_\alpha}} \longrightarrow 0$ in $\mathbb{R}$). So

$$y_{n_\alpha} = x - \frac{1}{2^{n_\alpha}}v_{n_\alpha} \xrightarrow{\tau} x - 0v = x.$$

Since (Proposition 1.32 1.) Y is a closed subspace, $x \in Y$. That is, $V \subseteq Y$. Since V is also an absorbing set, it follows that $X = Y$, so that X is finite dimensional. ■

Theorem 1.20

1. *Let $Y_1, \cdots, Y_n$ be compact convex sets in a vector space (X endowed with a linear topology τ). Then*

$$\operatorname{conv}(Y_1 \cup \cdots \cup Y_n)$$

   is compact.
2. *Let (X, τ) be a locally convex topological vector space. If $Y \subseteq X$ is totally bounded then* $\operatorname{conv}(Y)$ *is totally bounded as well.*
3. *If (X, τ) is a Fréchet space and $K \subseteq X$ is compact then* $\overline{\operatorname{conv}}(K)$ *is compact.*
4. *If $K \subseteq \mathbb{R}^n$ is compact then* $\operatorname{conv}(K)$ *is compact.*

Proof

1. Let $S \subseteq \mathbb{R}^n$ be the simplex

$$S = \{(s_1, \cdots, s_n) \; : \; s_i \geq 0, \; \sum_{i=1}^{n} s_i = 1\}.$$

   Set $Y = Y_1 \times \cdots \times Y_n$ and define the function $\varphi \colon S \times Y \to X$:

$$\varphi(s, y) = \sum_{i=1}^{n} s_i y_i.$$

   Consider the set $K = \varphi(S \times Y)$. It is the continuous image of a compact set and it is therefore compact. Moreover,

$$K \supseteq \operatorname{conv}(Y_1 \cup \cdots \cup Y_n).$$

   It is easy to show that K is convex, and since it includes all the Y_i's it must in fact be equal to $\operatorname{conv}(Y_1 \cup \cdots \cup Y_n)$.
2. Let U be an open neighborhood of θ. Because X is locally convex there exists a convex open neighborhood V of θ such that

$$V + V \subseteq U.$$

   Since Y is totally bounded there exists a finite set F such that

$$Y \subseteq F + V \subseteq \operatorname{conv}(F) + V.$$

   Since the right hand side is convex

$$\operatorname{conv}(Y) \subseteq \operatorname{conv}(F) + V.$$

By the first item $\mathrm{conv}(F)$ is compact, therefore there exists a finite set F' such that

$$\mathrm{conv}(F) = F' + V,$$

i.e.,

$$\mathrm{conv}(Y) \subseteq F' + V + V \subseteq F' + U,$$

which proves that $\mathrm{conv}(Y)$ is totally bounded.

3. In every metric space the closure of a totally bounded set is totally bounded, and if the space is complete it is compact. Since K is compact, then it is totally bounded. By the previous item $\mathrm{conv}(K)$ is totally bounded and hence its closure is compact.
4. $S \subseteq \mathbb{R}^n$ be the convex simplex. One can show that $\mathrm{conv}(K)$ is the image of the continuous map $S \times K$:

$$(s, x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} s_i x_i,$$

whose domain is compact.

■

Corollary 1.5 *Let X be a vector space endowed with a linear topology τ. The convex hull of a finite set (polytope) is compact.*

Example 1.11 (Noncompact Convex Hull) Consider l_2, the space of all square summable sequences. For each n let $u_n = (\underbrace{0, \cdots, 0}_{n-1}, \frac{1}{n}, 0, 0, \cdots)$. Observe that $\|u_n\|_2 = \frac{1}{n}$, so $u_n \xrightarrow{\|.\|_2} \theta$. Consequently,

$$Y = \{u_1, u_2, u_3, \cdots\} \cup \{\theta\}$$

is norm compact subset of l_2. Since $\theta \in Y$, it is easy to see that

$$\mathrm{conv}(Y) = \left\{ \sum_{i=1}^{k} \alpha_i u_i : \alpha_i \geq 0 \text{ for each } i \text{ and } \sum_{i=1}^{k} \alpha_i \leq 1 \right\}.$$

In particular, each vector of $\mathrm{conv}(Y)$ has only finitely many nonzero components. We claim that $\mathrm{conv}(Y)$ is not norm compact. To see this, set

$$x_n = (\frac{1}{2}, \frac{1}{2}.\frac{1}{2^2}, \frac{1}{3}.\frac{1}{2^3}, \cdots, \frac{1}{n}.\frac{1}{2^n}, 0, 0, \cdots) = \sum_{i=1}^{n} \frac{1}{2^i} u_i,$$

so $x_n \in \operatorname{conv}(Y)$. Now $x_n \xrightarrow{\|\cdot\|_2} x = (\frac{1}{2}, \frac{1}{2}.\frac{1}{2^2}, \frac{1}{3}.\frac{1}{2^3}, \cdots, \frac{1}{n}.\frac{1}{2^n}, \frac{1}{n+1}.\frac{1}{2^{n+1}}, \cdots)$ in l_2. But $x \notin \operatorname{conv}(Y)$, so $\operatorname{conv}(Y)$ is not even closed, let alone compact.

Remark 1.18 In the above example, the convex hull of a compact set failed to be closed. The question remains whether the closure of the convex hull is compact. In general, the answer is no. To see this, let X the space of sequences that are eventually zero, equipped with the l_2-norm. Let Y as above, and note that $\overline{\operatorname{conv}}(Y)$ (where the closure is taken in X, not l_2) is not compact either. To see this, observe that the sequence $(x_n)_n$ defined above has no convergent subsequence (in X).

Proposition 1.33 *Let Y and Z are two nonempty convex subsets of a topological vector space (X, τ) such that Y is compact and Z is closed and bounded, then* $\operatorname{conv}(Y \cup Z)$ *is closed.*

Proof Let $x_i = (1 - \alpha_i)y_i + \alpha_i z_i \longrightarrow x$, where $0 \leq \alpha_i \leq 1$, $y_i \in Y$ and $z_i \in Z$ for each i. By passing to a subnet, we can assume that $y_i \longrightarrow y \in Y$ and $\alpha_i \longrightarrow \alpha \in [0, 1]$. If $\alpha > 0$, then $z_i \longrightarrow \dfrac{x - (1-\alpha)y}{\alpha} = z \in Z$, and consequently $x = (1 - \alpha)y + \alpha z \in \operatorname{conv}(Y \cup Z)$.

Now consider the case $\alpha = 0$. The boundedness of Z and Proposition 1.20 imply $\alpha_i z_i \longrightarrow \theta$, so $x_i = (1 - \alpha_i)y_i + \alpha_i z_i \longrightarrow y$. Since the space is Hausdorff, $x = y \in \operatorname{conv}(Y \cup Z)$. ■

1.2.7 The Weak Topology of Topological Vector Spaces and the Weak* Topology of Their Duals

If X is a topological vector space then the weak topology on it is coarser than the origin topology: any set that is open in the original topology is open in the weak topology. From this, it follows that it is easier for a sequence to converge in the weak topology than in the original topology.

We will consider topological vector spaces (X, τ) over the field $\mathbb{K}, \mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For definiteness we assume $\mathbb{K} = \mathbb{C}$.

Remark 1.19 Given a vector space X and a linear functional $\phi \colon X \to \mathbb{K}$, the map $p_\phi = |\phi| : X \ni x \mapsto |\phi(x)| \in [0, \infty[$ defines a seminorm on X.

Definition 1.37 Let (X, τ) be a topological vector space. The topological dual space X' is the set of all continuous linear maps $(X, \tau) \to \mathbb{K}$.

Next, we will discuss the geometric form of the Hahn-Banach theorems. The first geometric version is

Lemma 1.7 *Let (X, τ) be a real topological vector space, and let $V \subseteq X$ be a convex open set which contains θ. If $x_0 \in X \setminus V$, there exists $\psi \in X'$, such that $\psi(x_0) = 1$ and $\psi(x) < 1$, for all $x \in V$.*

It turns out that Lemma 1.7 is a particular case of a more general result:

Theorem 1.21 (Hahn-Banach Separation Theorem-Real Case) *Let (X, τ) be a real topological vector space, let $Z, W \subseteq X$ be nonempty convex sets with Z open, and $Z \cap W = \emptyset$. Then there exists $\psi \in X'$, and a real number α, such that*

$$\psi(z) < \alpha \le \psi(w), \textit{ for all } z \in Z, w \in W.$$

Proof Fix some points $z_0 \in Z$, $w_0 \in W$, and define the set

$$V = Z - W + w_0 - z_0 = \{z - w + w_0 - z_0 \colon z \in Z, w \in W\}.$$

It is straightforward that V is convex and contains θ. The equality

$$V = \bigcup_{w \in W} (Z - w + w_0 - z_0)$$

shows that V is also open. Define the vector $x_0 = w_0 - z_0$. Since $Z \cap W = \emptyset$, it is clear that $x_0 \notin V$. Use Lemma 1.7 to produce $\psi \in X'$ such that

(i) $\psi(x_0) = 1$,
(ii) $\psi(x) < 1$, for all $x \in V$.

By the definition of x_0 and V, we have $\psi(w_0) = \psi(z_0) + 1$, and

$$\psi(z) < \psi(w) + \psi(z_0) - \psi(w_0) + 1, \text{ for all } z \in Z, w \in W,$$

which gives

$$\psi(z) < \psi(w), \text{ for all } z \in Z, w \in W. \tag{1.14}$$

Put

$$\alpha = \inf_{w \in W} \psi(w).$$

The inequality (1.14) gives

$$\psi(z) \le \alpha \le \psi(w), \text{ for all } z \in Z, w \in W. \tag{1.15}$$

The proof will be complete once we prove the following:

$$\psi(z) < \alpha \text{ for all } z \in Z.$$

Suppose the contrary, i.e., there exists some $z_1 \in Z$ with $\psi(z_1) = \alpha$. Using the continuity of the map

$$\mathbb{R} \ni \beta \mapsto z_1 + \beta x_0 \in X,$$

there exists some $\varepsilon > 0$ such that

$$z_1 + \beta x_0 \in Z, \text{ for all } \beta \in [-\varepsilon, \varepsilon].$$

In particular, by (1.15) one has

$$\psi(z_1 + \varepsilon x_0) \leq \alpha,$$

which means that

$$\alpha + \varepsilon \leq \alpha,$$

which is clearly impossible. ■

Theorem 1.22 (Hahn-Banach Separation Theorem-Complex Case) *Let (X, τ) be a complex topological vector space, let $Z, W \subseteq X$ be nonempty convex sets with Z open, and $Z \cap W = \emptyset$. Then there exists $\psi \in X'$, and a real number α, such that*

$$\operatorname{Re}\psi(z) < \alpha \leq \operatorname{Re}\psi(w), \textit{ for all } z \in Z, w \in W.$$

Proof Regard X as a real topological vector space, and apply the real version to produce an $\mathbb{R}$-linear continuous functional $\psi_1 \colon X \to \mathbb{R}$, and a real number α, such that

$$\psi_1(z) < \alpha \leq \psi_1(w), \ x \in X$$

Then the functional $\psi \colon X \to \mathbb{C}$ defined by

$$\psi(x) = \psi_1(x) - i\psi_1(ix), \ x \in X$$

will clearly satisfy the desired properties. ■

Remark 1.20 Geometrically we can say that the hyperplane $\{\operatorname{Re}\psi(x) = \alpha\}$ separates the sets Z, W in broad sense.

There is another version of the Hahn-Banach separation theorem, which holds for locally convex topological vector spaces.

Theorem 1.23 *Let (X, τ) be a locally convex topological vector space. Suppose $C, D \subseteq X$ are convex sets, with C compact, D closed, and $C \cap D = \emptyset$. Then there exists $\psi \in X'$ and two numbers $\alpha, \beta \in \mathbb{R}$, such that*

$$\mathrm{Re}\,\psi(x) \le \alpha < \beta \le \mathrm{Re}\,\psi(y), \ \textit{for all } x \in C, y \in D.$$

Proof Let $W = D - C$. By Lemma 1.2, 4. W is closed. Since $C \cap D = \emptyset$, we have $\theta \notin W$. Since W is closed, its complement $X \setminus W$ will then be a neighborhood of θ. Since X is locally convex, there exists a convex open set Z, with $\theta \in Z \subseteq X \setminus W$. In particular we have $Z \cap W = \emptyset$. Applying the suitable version of the Hahn-Banach separation theorem (real or complex case), we find a linear continuous map $\psi\colon X \to \mathbb{K}$ and a real number γ, such that

$$\mathrm{Re}\,\psi(z) < \gamma \le \mathrm{Re}\,\psi(w), \ \text{for all } z \in Z, w \in W.$$

Notice that $\theta \in Z$, we get $\gamma > 0$. Then the inequality

$$\gamma \le \mathrm{Re}\,\psi(w), \ \text{for all } w \in W,$$

gives

$$\mathrm{Re}\,\psi(y) - \mathrm{Re}\,\psi(x) \ge \gamma > 0, \ \text{for all } x \in C, y \in D.$$

Then if we define

$$\beta = \inf_{y \in D} \mathrm{Re}\,\psi(y) \text{ and } \alpha = \sup_{x \in C} \mathrm{Re}\,\psi(x),$$

we get $\beta \ge \alpha + \gamma$, and we are done. ■

Remark 1.21 Geometrically we can say that the hyperplane $\{\mathrm{Re}\,\psi(x) = \beta\}$ separates the compact sets C and the closed set D in the strict sense.

One important feature of topological duals in the locally convex Hausdorff case is described by the following result.

Proposition 1.34 *If (X, τ) is a locally convex topological vector space, then X' separates the points of X, in the following sense: for any $x, y \in X$, such that $x \ne y$, there exists $\phi \in X'$, such that $\phi(x) \ne \phi(y)$.*

Proof Since X is locally convex and Hausdorff, there exists some open convex set $V \ni y$ such that $x \notin V$. The existence of ϕ then follows from the Hahn-Banach separation theorem. ■

Definition 1.38 Let (X, τ) be a topological vector space. The weak topology on X, which we denote by $\sigma(X, X')$, is the initial topology for X'. That is, $\sigma(X, X')$ is the coarsest topology on X such that each element of X' is continuous $(X, \sigma(X, X')) \to \mathbb{C}$.

Equivalently, the weak topology on X is the seminorm topology given by the seminorms $|\phi|$, $\phi \in X'$.

Remark 1.22

- The topologies τ and $\sigma(X, X')$ are comparable, and τ is at least as fine as $\sigma(X, X')$. That is, $\sigma(X, X') \subseteq \tau$. A vague rule is that the smaller X' is compared to the set of all linear maps $(X, \sigma(X, X')) \to \mathbb{C}$, the smaller $\sigma(X, X')$ will be compared to τ.
- If X' separates X then $(X, \sigma(X, X'))$ is a locally convex topological vector space. It is Hausdorff because $\sigma(X, X')$ is induced by the separating family of seminorms $p_\phi = |\phi|$, $\phi \in X'$. In particular if (X, τ) is a locally convex topological vector space then $(X, \sigma(X, X'))$ is a locally convex topological vector space.

Definition 1.39 Let (X, τ) be a topological vector space and $(x_\alpha)_{\alpha \in I}$ a net in X. We say that

1. The net $(x_\alpha)_{\alpha \in I}$ converges strongly to x and we write

$$x_\alpha \to x \ \textit{ if } (x_\alpha)_{\alpha \in I} \text{ converges to } x \textit{ in the original topology } \tau.$$

2. The net $(x_\alpha)_{\alpha \in I}$ converges weakly to x and we write

$$x_\alpha \rightharpoonup x \ \textit{ if } (x_\alpha)_{\alpha \in I} \text{ converges to } x \text{ in the topology } \sigma(X, X').$$

This condition is equivalent to the condition that $p_\phi(x_\alpha - x) \to 0, \forall \phi \in X'$, which in turn is equivalent to

$$\phi(x_\alpha) \to \phi(x), \quad \forall \phi \in X'.$$

A simple consequence of the fact that $\sigma(X, X') \subseteq \tau$ is that

$$x_\alpha \to x \Longrightarrow x_\alpha \rightharpoonup x,$$

i.e., every strongly convergent net is weakly convergent.

Similarly, we will speak about the strong neighborhood, strongly closed, strongly bounded $\cdots$, and weak neighborhood, weakly closed, weakly bounded $\cdots$

Definition 1.40 We say that $Y \subseteq X$ is weakly bounded if Y is a bounded subset of $(X, \sigma(X, X'))$: for every neighborhood N of θ in $(X, \sigma(X, X'))$ there is some $c \geq 0$ such that $Y \subseteq \{cx : x \in N\} = cN$ (equivalently, $\phi(Y)$ is bounded in $\mathbb{C}$).

Remark 1.23 If (X, τ) is an infinite dimensional locally convex topological vector space, the weak topology $\sigma(X, X')$ has a peculiar property: every weak neighborhood of θ contains a closed infinite dimensional linear subspace. Indeed, if we start with some neighborhood V, then there exist $\phi_1, \cdots, \phi_n \in X'$ and

$\varepsilon_1, \cdots, \varepsilon_n > 0$, such that $\varepsilon_1 B_{P_{\phi_1}}(\theta) \cap \cdots \cap \varepsilon_n B_{P_{\phi_n}}(\theta)$, where for $i = 1, \cdots, n$, $B_{P_{\phi_i}}(\theta) = \{x \in X, |\phi_i(x)| < 1\}$. So V will clearly contain the closed subspace $(\ker \phi_1) \cap \cdots \cap (\ker \phi_n)$. It follows that

$$\dim X \leq n + \dim(\ker \phi_1) \cap \cdots \cap (\ker \phi_n),$$

i.e., $\dim(\ker \phi_1) \cap \cdots \cap (\ker \phi_n) = \infty$. Hence $\sigma(X, X')$ is not locally bounded.

Proposition 1.35 *In any finite-dimensional normed space, the weak topology coincides with the topology generated by any norm.*

Proof Let X be a finite-dimensional vector space, let $(e_1, \cdots, e_d)$ be a basis in X, and let $\phi_1, \cdots, \phi_d$ be its dual basis, defined by $\phi_i(e_j) = \delta_{i,j}$. Then, $\|x\|_\infty = \max_{1 \leq i \leq d} |\phi_i(x)|$ is a norm on X, and since X is finite-dimensional, all linear functionals on X are also continuous.

We know that on finite dimensional vector space two norms are equivalent, so it is enough to compare the weak topology to the topology τ induced by $\| \|_\infty$. It is clear that $\tau \supseteq \sigma(X, X')$. On the other hand,

$$|x|_{\phi_1, \cdots, \phi_d} = \sup_{1 \leq i \leq d} = \|x\|_\infty, x \in X,$$

and hence the open $\|.\|_\infty$-balls around any point and with any radius are open in the weak topology. Hence, $\tau \subseteq \sigma(X, X')$. ■

Theorem 1.24 *Let X be an infinite-dimensional normed space and $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. The closure of the unit sphere in the weak topology is the whole closed unit ball, i.e.,*

$$\overline{S_X}^{\sigma(X, X')} = \{x \in X : \|x\| \leq 1\}.$$

Similarly, one can show that $B_1(\theta) = \{x \in X : \|x\| < 1\}$ has empty interior for $\sigma(X, X')$. In particular it is not open. Despite these facts, there are sets whose weak closure is equivalent to its strong closure.

Remark 1.24 If (X, τ) is a locally convex topological vector space, then for any $Y \subseteq X$, then $\overline{\text{conv}(Y)}^\tau = \overline{\text{conv}(Y)}^{\sigma(X, X')}$.

Theorem 1.25 *If $Y \subseteq X$ is convex and (X, τ) is a locally convex topological vector space, then*

1. *Y is $\sigma(X, X')$-closed (weakly closed) if and only if Y is τ-closed (strongly closed).*
2. *Y is $\sigma(X, X')$-dense if and only if Y is τ-dense.*

Proof

1. Since $\sigma(X, X') \subseteq \tau$, then if Y is $\sigma(X, X')$-closed it is τ-closed. Conversely, if Y is τ-closed and convex, let $x_0 \in X \setminus Y$. Then by the Hahn-Banach separation theorem (for complex vector spaces) there is some $\phi \in X'$ such that

$$\sup_{x \in Y} \mathrm{Re}(\phi(x)) \leq \gamma_1 < \gamma_2 \leq \mathrm{Re}(\phi(x_0)).$$

   Hence the neighborhood of x_0

$$x_0 + V = x_0 + \left\{ x \colon \ |\gamma(x)| \leq \mathrm{Re}(\phi(x_0)) - \gamma_2 \right\}$$

   has empty intersection with Y.
2. Obvious.

■

In particular, in a topological vector space, the closure of convex sets is convex.

If a sequence converges weakly, it need not converge in the original topology, and Mazur's theorem shows that if a sequence in a metrizable locally convex space converges weakly then there is a sequence in the convex hull of the original sequence that converges to the same limit as the weak limit of the original sequence.

Theorem 1.26 *(Mazur)Let X be a metrizable locally convex space. If $x_n \rightharpoonup x$, then there is a sequence $(y_m)_m \subseteq X$ such that each y_m is a convex combination of finitely many x_n and such that $y_m \to x$.*

Proof The convex hull of a subset Y of X is the set of all convex combinations of finitely many elements of Y. The convex hull of a set is convex and contains the set. Let Z be the convex hull of the sequence $(x_n)_n$ and let W the weak closure of Z. Since $x_n \rightharpoonup x$ and $x_n \in Z$, Theorem 1.25 tells us that $W = \overline{Z}$, so $x \in \overline{Z}$. But X is metrizable, so x being in the closure of Z implies that there is a sequence $(y_m)_m \subseteq Z$ such that $y_m \to x$. This sequence $(y_m)_m$ satisfies the claim. ■

Let (X, τ) be a topological vector space. The dual space X' does not come with an a priori topology.

Let $x \in X$, and define $f_x \colon X' \to \mathbb{C}$ by $f_x(\phi) = \phi(x)$. Now f_x is linear. If $\phi_1, \phi_2 \in X'$ are distinct, then $\phi_1 - \phi_2 \neq 0$ so there is some $x \in X$ such that $(\phi_1 - \phi_2)(x) \neq 0$, which tells us that $f_x(\phi_1) \neq f_x(\phi_2)$. Therefore the set $\{f_x : x \in X\}$ is a separating family of seminorms on X', hence generating a topology which makes X' a locally convex topological vector space. We denote this topology by $\sigma(X', X)$ or w^* and it is called the weak* topology on X'. The open sets in the weak* topology are generated by the subbase

$$B_r^x = \{\phi \in X' \colon \ |\phi(x)| < r\}.$$

Lemma 1.8

(a) *The weak topology* $\sigma(X', X)$ *is the weakest topology on* X' *such that each map* f_x *is continuous.*
(b) *A sequence* $(\phi_n)_n$ *converges to* ϕ *in* $\sigma(X', X)$ *if and only if for all* $x \in X$

$$\lim_{n \longrightarrow \infty} \phi_n(x) = \phi(x).$$

(c) *A set* $Y \subseteq X'$ *is bounded* w.r.t. $\sigma(X', X)$ *if and only if for all* $x \in X$

$$\{\phi(x), \phi \in Y\}$$

is bounded in $\mathbb{C}$.

Example 1.12 Recall that $c_0' = l_1$ and $l_1' = l_\infty$. Weak convergence of a sequence $(x_n)_k \subseteq l_1$ to zero (with l_1 viewed as a topological vector space) means that

$$\forall\, y = (y_k)_k \subseteq l_\infty \quad \lim_{k \to \infty} \sum_{k=1}^{+\infty} (x_n)_k y_k = 0.$$

Weak* convergence of a sequence $(x_n)_k \subseteq l_1$ to zero (with l_1 viewed as the dual of the topological vector space c_0) means that

$$\forall\, y = h(y_k)_k \subseteq c_0 \quad \lim_{k \to \infty} \sum_{k=1}^{+\infty} (x_n)_k y_k = 0.$$

Clearly, weak convergence implies weak* convergence (but not the opposite).

A priori, one can look at the second dual Y of the locally convex vector space $(X, \sigma(X', X))$, i.e.,

$$Y = \{\lambda \colon X' \to \mathbb{C}, \text{w.r.t}, \sigma(X', X)\}.$$

By construction, it follows that $X \subseteq Y$,

i.e., X can be embedded into Y. It turns out that $X = Y$, i.e., the dual of $(X, \sigma(X', X))$ can be identified with X.

Theorem 1.27 *If* $\lambda \colon X' \to \mathbb{C}$ *is linear and continuous w.r.t,* $\sigma(X', X)$*, then there exists* $x \in X$ *such that*

$$\lambda(\phi) = \phi(x) \;\; \forall \phi \in X'.$$

Proof By definition of continuity w.r.t, $\sigma(X', X)$, for all $\epsilon > 0$ there are $\delta > 0$ and $x_1, \cdots, x_n$ such that

$$\lambda\{\phi\colon\ |\phi(x_i)| \leq \delta, i = 1, \cdots, n\} \subseteq (-\epsilon, \epsilon).$$

In particular, if ϕ is such that $\phi(x_i) = 0$ for all i, then $\lambda(\phi) = 0$. This show that

$$N_\phi \supseteq \bigcap_{i=1}^{n} N_{x_i}.$$

Consider the linear mapping $T\colon\ X' \to \mathbb{C}^{n+1}$ defined by

$$T(\phi) = (\lambda(\phi), \cdots, \phi(x_1), \cdots, \phi(x_n)).$$

By the assumption, $T(X')$ is a subspace of $\mathbb{C}^{n+1}$ and the point $(1, 0, \cdots, 0)$ is not in $T(X')$. Then there are $\alpha = (\alpha_1, \cdots, \alpha_{n+1}) \in \mathbb{C}^{n+1}$ such

$$\alpha T(X') = \{\alpha_1\lambda(\phi) + \sum_{i=2}^{n+1} \alpha_i\phi(x_{i-1}), \phi \in X'\} = 0 < \Re\alpha_1.$$

It follows that $\alpha_1 \neq 0$ and

$$\lambda\phi = \sum_{i=1}^{n} \frac{\alpha_{i+1}}{\alpha_1}\phi(x_i).$$

■

If X is in particular a normed space, then we know that $(X', \|.\|_{X'})$ is a Banach space. Hence, if τ is the vector topology of X' generated by the norm , $\|.\|_{X'}, \sigma(X', X) \subseteq \tau$.

Definition 1.41 We say that

- The sequence $(\phi_n)_n$ converges strongly to ϕ and we write

$$\phi_n \longrightarrow \phi \ \text{ if } \|\phi_n - \phi\|_{X'} \longrightarrow 0.$$

- The sequence $(\phi_n)_n$ converges weakly to ϕ and we write $\phi_n \rightharpoonup^* \phi$ if $(\phi_n)_n$ converges to ϕ in the topology $\sigma(X', X)$.

The Banach-Alaoglu theorem shows that certain subsets of X' are weak* compact, i.e., they are compact subsets of $\sigma(X', X)$.

Definition 1.42 Let X be a topological vector space and V be a neighborhood of θ. Define the polar of V as

$$K = \left\{ \phi \in X' \colon \ |\phi(x)| \leq 1 \ \ \forall \, x \in V \right\}.$$

Theorem 1.28 (Banach-Alaoglu) *Let X be a topological vector space and V be a neighborhood of θ. Then the polar K of V is compact in the weak* topology $\sigma(X', X)$.*

Proof Since each V local neighborhood absorbing, then there is a $\gamma(x) \in \mathbf{C}$ such that

$$x \in \gamma(x)V.$$

Hence it follows that

$$|\phi(x)| \leq \gamma(x) \ \ x \in X, \ \phi \in K.$$

Consider the topological space

$$P = \prod_{x \in X} \{\alpha \in \mathbf{C} \colon \ |\alpha| \leq \gamma(x)\},$$

with the product topology σ. By Tychonoff's theorem (P, σ) is compact.

By the construction, the elements of P are functions $f \colon X \to \mathbb{C}$ (not necessarily linear) such that

$$|f(x)| \leq \gamma(x).$$

In particular, the set K is the subset of P made of the linear functions.

We first show that K is the subset of P w.r.t the topology σ. This follows from the fact that if f_0 is in the σ closure of $\overline{K}$, then the scalars α, β and point $x, y \in X$ one has that

$$\left\{ |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \varepsilon, |f(x) - f_0(x)| < \varepsilon, |f(y) - f_0(y)| < \varepsilon \right\}$$
$$\bigcap K \neq \emptyset.$$

Take thus ϕ in the intersection, so that

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| = \Big|(f_0(\alpha x + \beta y) - f(\alpha x + \beta y)) + \alpha(f(x) - f_0(x)) + (f(y) - f_0(y))\Big| < (1 + |\alpha)| + |\beta|)\varepsilon.$$

Since ε is arbitrary, f_0 is linear. Moreover, since $|f_0(x)| \leq \gamma(x)$, then for $x \in V$

$$|f_0(x)| \leq 1.$$

It follows that we have two topologies on K :

- the weak* topology $\sigma(X', X)$ inherited by X',
- the product topology σ inherited by P. Since K is closed in (P, σ), then (K, σ) is compact.

To conclude, we need only to show that the two topologies coincide. This follows because the bases of the two topologies are generated by the sets

$$V_{\sigma(X',X)} = \Big\{ |\phi(x_i) - \phi_0(x_i)| < \varepsilon, i = 1 \cdots, n \Big\},$$

$$V_\sigma = \Big\{ |f(x_i) - f_0(x_i)| < \varepsilon, i = 1 \cdots, n \Big\}.$$

There is thus a one to one correspondence among local bases, hence the two topologies coincide. ∎

Theorem 1.29 *Let (X, τ) be a separable topological vector space. Let $K \subseteq X'$ be weakly* compact. Then K is metrizable in the weak* topology.*

Proof Let $\{x_n, n \in \mathbb{N}\}$ be a dense subset of X and $f_{x_n}(\phi) = \phi(x_n)$ for $\phi \in X'$. By the definition of the weak* topology on X', the functionals f_{x_n} are weak* continuous. Also, for every n,

$$f_{x_n}(\phi_1) = f_{x_n}(\phi_2),$$

i.e.,

$$\phi_1(x_n) = \phi_2(x_n),$$

then $\phi_1 = \phi_2$ (continuous functionals that coincide on a dense set).

Thus, $\{f_{x_n}, n \in \mathbb{N}\}$ is a countably family of continuous functionals that separates points in X'. It follows by Proposition 1.9 that K is metrizable. ∎

Remark 1.25

1. The claim is not that X' endowed with the weak* topology is metrizable. For example, this is not true in infinite-dimensional Banach spaces.
2. The topological space $(X', \sigma(X', X))$ is never metrizable, unless X has a countable vector base.

Theorem 1.30 *Let X be a separable topological vector space. If V is a neighborhood of θ and if the sequence $(\phi_n)_n \subseteq X'$ satisfies*

$$|\phi_n(x)| \leq 1, \qquad n \geq 1, x \in V,$$

then there is a subsequence $(\phi_{\alpha(n)})_n$ and some $\phi \in X'$ such that for all $x \in X$,

$$\lim_{n\to\infty} \phi_{\alpha(n)}(x) = \phi(x).$$

Proof The Banach-Alaoglu theorem implies that the polar

$$K = \Big\{\phi \in X' \colon \; |\phi(x)| \leq 1 \;\; \forall\, x \in V\Big\},$$

is weak* compact. K with the subspace topology inherited from $\sigma(X', X)$ is compact, hence by Theorem 1.29 it is metrizable. Since the sequence $(\phi_n)_n$ is contained in K, it has a subsequence $(\phi_{\alpha(n)})_n$ that converges weakly to some $\phi \in K$. For each $x \in X$, the functional $f_x \colon (X', \sigma(X', X)) \to \mathbb{C}$ defined by $f_x(\phi) = \phi(x)$ is continuous, hence for all $x \in X$ we have $f_x(\phi_{\alpha(n)})) \to f_x(\phi)$, which is the claim. ■

Theorem 1.31 *If (X, τ) is locally convex and $Y \subseteq X$, then Y is bounded in (X, τ) if and only if Y is bounded in $(X, \sigma(X, X'))$.*

Dual of Banach Spaces and Reflexive Spaces

A particular case is when X is normed: in this case X' is a Banach space with norm $\|\phi\|_{X'} = \sup_{\|x\|=1} |\phi(x)|$. One can introduce the second dual of X, i.e., denoted by X''. Clearly, there is a canonical immersion J of X into X'', by

$$J \colon X \to X'', \quad J(x)(\phi) = \phi(x), \|J(x)\|_{X''} = \|x\|_X.$$

Since $J \colon X \to X''$ is continuous, it follows that $J(X)$ is a closed subspace of X''. In particular, either $J(X) = X''$ or it is not dense.

Lemma 1.9 (Helly) *Let X be a Banach space, $\phi \in X', i = 1 \cdots, n, n$ linear functionals in X' and $\alpha_i \in \mathbb{C}, i = 1 \cdots, n, n$ scalars. Then the following properties are equivalent*:

1. *for all $\varepsilon > 0$ there is x_ε, $\|x_\varepsilon\| < 1$ such that*

$$|\phi(x_\varepsilon) - \alpha_i| \leq \epsilon \quad i = 1 \cdots, n,$$

2. *for all $\beta_1, \cdots, \beta_n \in \mathbb{C}$*

$$\left|\sum_i^n \beta_i \alpha_i\right| \leq \|\sum_i^n \beta_i \phi_i\|_{X'}.$$

Proof The first implication follows by

$$\left|\sum_i^n \beta_i \alpha_i\right| = \left|\sum_i^n \beta_i (\alpha_i - \phi_i(x_\varepsilon))\right| + \left|\sum_i^n \beta_i \phi_i(x_\varepsilon)\right|$$

$$\leq \varepsilon \sum_i^n |\beta_i| + \|\sum_i^n \beta_i \phi_i\|_{X'},$$

since $\|x_\varepsilon\| \leq 1$. Conversely if 1. does not hold, then this means that the closure of the set

$$(\phi_1, \cdots, \phi_n)\Big\{x : \|x\| \leq 1\Big\} \subseteq \mathbb{C}^n$$

does not contains $(\alpha_1, \cdots, \alpha_n)$. Thus there is $(\beta_1, \cdots, \beta_n) \in \mathbb{C}^n$ such that

$$\max \text{Re}\left\{\sum_i^n \beta_i \phi_i(x), \|x\| \leq 1\right\} < \text{Re}\left\{\sum_i^n \beta_i \alpha_i\right\} \leq \left|\sum_i^n \beta_i \alpha_i\right|.$$

Since $\{x : \|x\| \leq 1\}$ is balanced, it follows that 2. is false. ■

Proposition 1.36 (Goldstine) *If X is a Banach space, then $J(B_X)$ is dense in $B_{X''}$ for the weak* topology.*

Proof If $\xi \in X''$, take a neighborhood of the form

$$V = \Big\{\eta \in X' : |\eta(\phi_i) - \xi(\phi_i)| < \varepsilon, \phi_i \in X', i = 1 \cdots n\Big\}.$$

We need only to find $x \in X$ such that

$$|\phi_i(x) - \xi(\phi_i)| < \varepsilon.$$

Since $\|\xi\|_{X''} \leq 1$, then

$$\left|\sum_i^n \beta_i \xi(\phi_i)\right| \leq \|\sum_i^n \beta_i \phi_i\|_{X'},$$

so that for Lemma 1.9 it follows that there is an $x_\varepsilon \in X$ which belongs to V. ■

Definition 1.43 A Banach space is reflexive if $J(X) = X''$.

It is important to observe that in the previous definition the canonical immersion J is used: even for particular non-reflexive spaces, one can find a continuous linear surjection from X to X''.

Theorem 1.32 (Kakutani) *The Banach space X is reflexive if and only if B_X is compact for the weak topology $\sigma(X, X')$.*

Proof If X is reflexive, then $J: X \to X''$ is continuous, injective and surjective. Hence J^{-1} is linear and continuous w.r.t. the strong topologies of X and X''. Actually both J and J^{-1} are isometries.

It is clear that

$$J\left\{x:\ |\phi(x)| < \varepsilon\right\} = \left\{\eta:\ |\eta\phi| < \varepsilon\right\},$$

so that the topology $J^{-1}(\sigma(X'', X'))$ coincides with the topology $\sigma(X, X')$. Since $B_{X''}$ is weak* compact, so B_X.

Conversely, if B_X is compact, then $J(B_X)$ is closed, and by Proposition 1.36 it coincide with the whole $B_{X''}$. ■

Theorem 1.33 *If X is a Banach space and X' is separable, then X is separable.*

Proof Let $(\phi_n)_n$ be a dense countable set in X'. Let $x_n \in X$, $\|x_n\|_X \leq 1$, be a point where

$$|\phi_n(x_n)| \geq \frac{1}{2}\|\phi_n\|_{X'},$$

and consider the countable set

$$Q = \left\{\sum_{finite} \alpha_i x_i : \alpha_i \textit{ belongs to a countable dense subset of } \mathbb{C}\right\}.$$

Clearly Q is countable and dense in the vector space L generated by $\{x_n\}_n$, so that it remains to prove that L is dense in X.

If L is not dense, then there is a non null continuous functional ϕ such that

$$\phi \neq 0_{X'} \ \ \phi(x_n) = 0 \, \forall \, n.$$

Since $(\phi_n)_n$ is dense, there is n_ϕ such that $\|\phi - \phi_{n_\phi}\|_{X'} < \varepsilon$, so that

$$\|\phi_{n_\phi}\|_{X'} \leq \left|\phi_{n_\phi}(x_{n_\phi})\right| \leq \left|(\phi - \phi_{n_\phi})(x_{n_\phi})\right| + \left|\phi(x_{n_\phi})\right| \leq \varepsilon.$$

Thus $\|\phi_{n_\phi}\|_{X'} \leq 2\varepsilon$, which implies that $\phi = 0_{X'}$. ■

Proposition 1.37 *If $Y \subseteq X$ is a closed subspace of a reflexive space, then Y is reflexive.*

Proof The proof follows by proving that the topology $\sigma(Y, Y')$ coincide with the topology $Y \cap \sigma(X, X')$ and B_Y is closed for $\sigma(X, X')$ (closed for strong topology and convex). ■

Corollary 1.6 *Let X be a normed space. Then, X is separable and reflexive if and only if X' is separable and reflexive.*

Proof Clearly if X is reflexive, the unit ball $B_{X'}$ is compact for the topology $\sigma(X', X'')$ because of the Banach-Alaoglu theorem and the fact $\sigma(X', X'') = \sigma(X', X)$. Moreover if X is reflexive and separable, then X'' is separable, hence by Theorem 1.33 is separable.

Conversely, if X' is reflexive, then X'' is reflexive, so that $M(X)$ is reflexive by Proposition 1.37, hence X is reflexive. Moreover, we know from Theorem 1.33 that X is separable, if X' is separable. ■

Definition 1.44 We say that X Banach space is uniformly convex if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x\|_X, \|y\|_X \leq 1, \|\frac{x+y}{2}\| \geq 1 - \delta \Longrightarrow \|x - y\|_X < \varepsilon.$$

Theorem 1.34 (Milman) *If X is a uniformly convex Banach space, then X is reflexive.*

Proof Let $\xi \in X''$, $\|\xi\|_{X''} = 1$. We want to prove that for all $\varepsilon > 0$ there is $x \in X$, $\|x\|_X \leq 1$ such that

$$\|\xi - J(x)\|_{X''} < \varepsilon.$$

Since $J(X)$ is strongly closed (J is an isometry), then J is surjective.

Let $\phi \in X'$ be such that

$$\|\phi\|_{X'} = 1, \quad \xi\phi > 1 - \delta,$$

where δ is the constant chosen by the uniform convexity estimate corresponding to ε, and consider the neighborhood of ξ of the form

$$V = \left\{\eta \in X'' \colon \left|(\xi - \eta)(\phi) < \frac{\delta}{2}\right|\right\}.$$

By Proposition 1.36, it follows that there is some $x \in B_X$ such that $J(x) \in V$.

Assume that $\xi \notin J(x) + \varepsilon B_{X''}$. Then we obtain a new neighborhood of ξ for the weak* topology which does not contains x. With the same procedure, we can find a new $\overline{x}$ in this new neighborhood. Thus we have

$$|\phi(x) - \xi(\phi)| \le \frac{\delta}{2}, \quad |\phi(\overline{x}) - \xi(\phi)| \le \frac{\delta}{2}.$$

Adding we obtain

$$2\,|\xi(\phi)| \le |\phi(x + \overline{x})| + \delta \le \|x + \overline{x}\| + \delta.$$

Then $\|\frac{x + \overline{x}}{2}\| \ge (1 - \delta)$, so that $\|x + \overline{x}\| < \varepsilon$, which is a contradiction. ■

1.2.8 l_1-Sequences

Definition 1.45 Let $(x_n)_n$ be a bounded sequence in a Banach space X, and $\varepsilon > 0$. We say that $(x_n)_n$ admits ε-l_1-blocks if for every infinite $M \subseteq \mathbb{N}$ there are $a_1, \cdots, \cdots a_r \in \mathbb{K}$ with $\sum |a_\rho| = 1$ and $i_1 < \cdots < i_r$ in M such that $\|\sum a_\rho x_{i_\rho}\| \le \varepsilon$.

Clearly there will be no subsequence of $(x_n)_n$ equivalent to the l_1-basis iff $(x_n)_n$ admits ε-l_1-blocks for arbitrary small $\varepsilon > 0$.

Theorem 1.35 *Let X be a real (for simplicity) Banach space and $(x_n)_n$ a bounded sequence. Suppose that, for some $\varepsilon > 0$, $(x_n)_n$ admits small ε-l_1-blocks. Then there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $(x_{n_k})_k$ is "close to being a weak Cauchy sequence" in the following sense*:

$$\limsup_k \phi(x_{n_k}) - \liminf_k \phi(x_{n_k}) \le 2\varepsilon$$

for every $\phi \in X'$ with $\|\phi\|_{X'} = 1$.

Proof Suppose the theorem were not true. We claim that without loss of generality we may assume that there is a $\delta > 0$ such that

$$\varphi((x_{n_k})_k) := \sup_{\|\phi\|_{X'}=1} \left(\limsup_k \phi(x_{n_k}) - \liminf_k \phi(x_{n_k}) > 2\varepsilon + \delta \right) \tag{1.16}$$

for all subsequences $(x_{n_k})_k$. In fact, if every subsequence contained another subsequence with a φ-value arbitrarily close to 2ε, the diagonal process would even provide one where $\varphi((x_{n_k})_k) \leq 2\varepsilon$ in contrast to our assumption.

Fix a $\tau > 0$ which will be specified later. After passing to a subsequence we may assume that $(x_n)_n$ satisfies the following conditions:

(i) If C and D are finite disjoint subsets of $\mathbb{N}$ there are a $\lambda_0 \in \mathbb{R}$ and an $\phi \in X'$ with $\|\phi\|_{X'} = 1$ such that $\phi(x_n) < \lambda_0$ for $n \in C$ and $\phi(x_n) > \lambda_0 + 2\varepsilon + \delta$ for $n \in D$.

(ii) There are $i_1 < \cdots < i_r$ in $\mathbb{N}$, $a_1, \cdots a_r \in \mathbb{R}$ with

$$\sum |a_\rho| = 1,\ |\sum a_\rho| < \tau,\ \|\sum a_\rho x_{i_\rho}\| \leq \varepsilon$$

For (**i**), define, for $r \in \mathbb{N}$, T_r to be the collection of all $(i_1, \cdots, i_r)$ (with $i_1 < \cdots < i_r$) such that there are a $\lambda_0 \in \mathbb{R}$ and a normalized ϕ such that $\phi(x_{i_\rho}) < \lambda_0$ if ρ is even and $> \lambda_0 + 2\varepsilon + \delta$ otherwise. (1.16) implies that there is an M_0 for which all $(i_1, \cdots, i_r)$ are in T_r for $i_1 < \cdots < i_r$ in M_0. Let us assume that $M_0 = \mathbb{N}$. Let C and D be finite disjoint subsets of $2\mathbb{N} = \{2, 4, \cdots\}$. We may select $i_1 < \cdots < i_r$ in $\mathbb{N}$ such that $C \subseteq \{i_\rho \mid \rho \text{ even }\}$ and $D \subseteq \{i_\rho \mid \rho \text{ odd }\}$. Because of $(i_1, \cdots, i_r) \in T_r$ we have settled (i) provided C and D are in $2\mathbb{N}$, and all what's left to do is to consider $(x_{2n})_n$ instead of $(x_n)_n$.

For (ii), By assumption we find $i_1 < \cdots < i_r$, $a_1, \cdots a_r \in \mathbb{R}$ such that $\sum |a_\rho| = 1$ and $\|\sum a_\rho x_{i_\rho}\| \leq \varepsilon$ with arbitrarily large i_1. Therefore we obtain $i_1^1 < \cdots < i_{r_1}^1 < i_1^2 < \cdots < i_{r_2}^2 < i_1^3 < \cdots < i_{r_3}^3 < \cdots$ and associated a_ρ^i. The numbers $\eta_j := \sum_{\rho=1}^{r_j} a_\rho^j$ all lie in $[-1, 1]$ so that we find $j < k$ with $|\eta_j - \eta_k| \leq 2\tau$. Let $i_1 < \cdots < i_r$ be the family $i_1^j < \cdots < i_{r_j}^j < i_1^k < \cdots < i_{r_k}^k$, and define the $a_1, \cdots a_r$ by $\frac{1}{2}a_1^j, \cdots \frac{1}{2}a_{r_j}^j, -\frac{1}{2}a_1^k, \cdots - \frac{1}{2}a_{r_k}^k$.

We are now ready to derive a contradiction. On the one hand, by (ii), we find $i_1 < \cdots < i_r$, $a_1, \cdots, \cdots a_r \in \mathbb{R}$ such that $\sum |a_\rho| = 1, |\sum a_\rho| \leq \tau$ with $\|\sum a_\rho x_{i_\rho}\| \leq \varepsilon$. On the other hand we may apply (**i**) with $C := \{i_\rho \mid a_\rho < 0\}$ and $D := \{i_\rho \mid a_\rho > 0\}$. We put $\alpha := -\sum_{\rho \in C} a_\rho$, $\beta := \sum_{\rho \in D} a_\rho$, and we note that

$|\alpha - \beta| \leq \tau, \alpha + \beta = 1$ so that $|\beta - \frac{1}{2}| \leq \tau$, hence

$$\varepsilon \geq \| \sum a_\rho x_{i_\rho} \| \leq \sum a_\rho \phi(x_{i_\rho}) \geq -\lambda_0 \alpha + (\lambda_0 + 2\varepsilon + \delta)\beta \geq -|\lambda_0|\tau + \varepsilon + \frac{\delta}{2} - \tau\delta.$$

This expression can be made larger than ε if τ has been chosen sufficiently small (note that the numbers $|\lambda_0|$ are bounded by $\sup_n \|x_n\|$), a contradiction which proves the theorem. ∎

Remark 1.26 Since the unit vector basis $(x_n)_n$ of real l_1 the assumption of the theorem holds with $\varepsilon = 1$ and since for every subsequence $(x_{n_k})_k$ one may find $\|\phi\|_{X'} = 1$ with

$$\limsup_k \phi(x_{n_k}) - \liminf_k \phi(x_{n_k}) = 2$$

there can be no better constant than that given in our theorem.

Theorem 1.36 (Rosenthal's Theorem) *Let X be a Banach space and $(x_n)_n$ a bounded sequence in X. If there exists no subsequence which is a weak Cauchy sequence then one can find a subsequence $(x_{n_k})_k$ which is equivalent with the unit vector basis of l_1 (i.e., $(\lambda_k)_k \mapsto \sum \lambda_k x_{n_k}$, from l_1 to X, is an isomorphism).*

In particular one has: If X does not contain an isomorphic copy of l_1, then every bounded sequence admits a subsequence which is a weak Cauchy sequence.

Proof Rosenthal's theorem is the assertion that $(x_n)_n$ has a weak Cauchy subsequence provided it admits ε-l_1-blocks for all ε. So, it is simple to derive the theorem from Theorem 1.35. If $(x_n)_n$ and thus every subsequence has ε-l_1-blocks for all ε, apply Theorem 1.35 successively with ε running through a sequence tending to zero. The diagonal sequence which is obtained from this construction will be a Cauchy sequence. ∎

Remark 1.27

1. Since weakly convergent sequences are weakly Cauchy it follows immediately that Rosenthal's theorem holds in reflexive spaces.
2. Rosenthal's theorem holds, whenever X is such that X' is separable. Let $(x_n)_n$ be bounded and ϕ be a fixed functional. If we apply the Bolzano-Weierstrass theorem to the scalar sequence $(\phi(x_n))_n$ we get a subsequence $(x_{n_k})_k$ such that $(\phi(x_{n_k}))_k$ converges. Applying the same idea to $(x_{n_k})_k$ with a second functional, say ψ, we get a subsequence of this subsequence such that the application of ψ produces something which is convergent. ϕ, applied to this new subsequence, also gives rise to convergence. Thus we have a subsequence of $(x_n)_n$ where ϕ and ψ converge, and similarly one can achieve this for any prescribed finite number of functionals. Even countably many functionals are manageable, by the diagonal process. Since we are dealing with bounded sequences $(y_n)_n$ (typically

subsequences of the original sequence) the collection of ϕ where $(\phi(y_n))_n$ converges is a norm closed subspace of X'.

There is a generalization of Rosenthal's theorem to Fréchet spaces which, it seems, has been firstly by Díaz [44]. Thus the starting point for proving promised generalizations is to understand what it means for a sequence in a locally convex space be equivalent to the unit basis of l_1.

We denote by l_1^0 the subspace of l_1 formed by elements with only finitely many nonzero coordinates.

Barroso, Kalenda and Lin introduced the following notion of l_1-sequences in topological vector spaces [14].

Definition 1.46 Let (X, τ) be a topological vector space and $(x_n)_n$ a sequence in X. We say that $(x_n)_n$ is an l_1-sequence if the mapping $T_0 : l_1^0 \to X$ defined by

$$T_0((a_i)_{i\geq 1}) = \sum_{i=1}^{\infty} a_i x_i \tag{1.17}$$

is an isomorphism of l_1^0 onto $T_0(l_1^0)$.

The following characterization of l_1-sequences is given in [14].

Proposition 1.38 *Let (X, τ) be a locally convex space and $(x_n)_n$ a bounded sequence in X. The following are equivalent:*

(i) *There is a continuous seminorm p on X such that*

$$p\left(\sum_{i=1}^{n} a_i x_i\right) \geq \sum_{i=1}^{n} |a_i|, \quad n \in \mathbb{N}, a_1, \cdots, a_n \in \mathbb{R}.$$

(ii) *$(x_n)_n$ is an l_1-sequence.*

If X is sequentially complete, then these conditions are equivalent to the following:

(iii) *The mapping $T : l_1 \to X$ defined by $T((a_i)_{i\geq 1}) = \sum_{i=1}^{\infty} a_i x_i$ is a well defined isomorphism of l_1 onto its image in X*

Proof Let $T_0 : l_1^0 \to X$ be defined by (1.17). As $(x_n)_n$ is bounded and X is locally convex, it is easy to check that T_0 is continuous.

Further, if (i) holds, then T_0 is clearly one-to-one and T_0^{-1} is continuous. This proves $(i) \Rightarrow (ii)$.

Conversely, suppose that (ii) holds. Set

$$U = T_0(\{x \in l_1^0 : \|x\|_{l^1} < 1\}).$$

As T_0 is an isomorphism, U is an absolutely convex open subset of $T_0(l_1^0)$. We can find V, an absolutely convex neighborhood of θ in X such that $V \cap T_0(l_1^0) \subset U$. Let p the Minkowski functional of V. Then p is a continuous seminorm witnessing that (i) holds. This proves $(ii) \Rightarrow (i)$.

Now suppose that X is sequentially complete. As T_0 is continuous and linear, it is uniformly continuous and hence it maps Cauchy sequences to Cauchy sequences. In particular the mapping T_0 can be uniquely extended to a continuous linear mapping $T: l_1 \to X$. This is obviously the mapping described in (iii). As l_1^0 is dense in l_1, we get $(ii) \Leftrightarrow (iii)$. ■

The following theorem is a variant of Rosenthal's theorem [14]. Its proof is a slight refinement of the proof of Lemma 3 in [44].

Theorem 1.37 *Let (X, τ) be a metrizable locally convex space. Then each bounded sequence in X contains either a weakly Cauchy subsequence or a subsequence which is an l_1-sequence.*

Proof Let $(\|.\|_n)$ be a sequence of seminorms generating the topology of X. Without loss of generality we may assume that $\|x\|_n \leq \|x\|_{n+1}$ for all n and $x \in X$. Let $U_n = \{x : \|x\|_n < 1\}$ and let $B_n = U_n^0$ be the polar of U_n. Assume that $(x_m)_m$ is a bounded sequence in X such that no its subsequence is an l_1-sequence. For $n = 0, 1, 2, \cdots$ we construct a sequence $(x_m^n)_m$ inductively as follows. Set $x_m^0 = x_m$ for all $m \in \mathbb{N}$. Assume that for a given $n \in \mathbb{N}$ the sequence $(x_m^{n-1})_m$ has been defined. By Rosenthal's theorem one of the following possibilities takes place (elements of X are viewed as functions on B_n):

(i) $(x_m^{n-1})_m$ has a subsequence which is equivalent to the l_1-basis on B_n.
(ii) $(x_m^{n-1})_m$ has a subsequence which point wise converges on B_n.

Let us show that the case (i) cannot occur. Indeed, suppose that (i) holds. Let $(y_m)_m$ be the respective subsequence. The equivalence to the l_1 basis on B_n means that there is some $C > 0$ such that

$$\|\sum_{i=1}^{m} a_i y_i\|_n \geq C \sum_{i=1}^{m} |a_i|$$

for each $m \in \mathbb{N}$ and each choice $a_1, \cdots, a_m \in \mathbb{R}$. By Proposition 1.38 $(y_m)_m$ is an l_1-sequence in X, which is a contradiction.

Thus the possibility (**ii**) takes place. Denote by $(x_m^n)_m$ the respective subsequence. This completes the inductive construction.

Take the diagonal sequence (x_m^m). It is a subsequence of $(x_m)_m$ which pointwise converges on B_n for each $n \in \mathbb{N}$. Moreover, if $\phi \in X'$ is arbitrary, then there is n and $c > 0$ such that $c\phi \in B_n$. In particular, the linear span of the union of all $B_n's$ is the whole dual X'. It follows that the sequence (x_m^m) is weakly Cauchy. The proof is complete. ■

Remark 1.28 Let $X = l_1$ endowed with its weak topology. Let $(e_n)_n$ denote the canonical basic sequence. Then, the sequence $(e_n)_n$ contains neither a weakly Cauchy subsequence nor a subsequence which is an l_1-sequence. Indeed, suppose that $(x_n)_n$ is an l_1-sequence in X. Denote by Y its linear span. By the definition of an l_1-sequence we get that Y is isomorphic to $(l_1^0, \|.\|_1)$, hence it is metrizable. On the other hand, by the definition of X we get that Y is equipped with its weak topology which is not metrizable as Y has infinite dimension.

Further, the sequence $(e_n)_n$ contains no weakly Cauchy subsequence in $(l_1, \|.\|_1)$ and in $(l_1, \sigma(l_1, (l_1)'))$ coincide, we get that $(e_n)_n$ contains no weakly Cauchy subsequence in X. Thus the proof is completed.

The following is given in [14] and is about the coincidence of norm and weak topologies.

Proposition 1.39 *Let* Γ *be an arbitrary set. Then the norm and weak topologies coincide on the positive cone of* $l_1(\Gamma)$.

Proof Denote by C the positive cone of $l_1(\Gamma)$. Since the weak topology is weaker than the norm one, it is enough to prove that the identity of C endowed with the weak topology onto $(C, \|.\|)$ is continuous. Let $x \in C$ and $\varepsilon > 0$ be arbitrary. Fix a nonempty finite set $F \subseteq \Gamma$ such that

$$\sum_{\gamma \in F} x(\gamma) > \|x\| - \frac{\varepsilon}{4}.$$

Set

$$U = \left\{ y \in C\colon\ |y(\gamma) - x(\gamma)| < \frac{\epsilon}{4\,|F|} \text{ for } \gamma \in F \right\},$$

$$V = \left\{ y \in C\colon\ \sum_{\gamma \in \Gamma \setminus F} y(\gamma) - \sum_{\gamma \in \Gamma \setminus F} x(\gamma) < \frac{\epsilon}{4} \right\}.$$

Then both U and V are weak neighborhoods of x in C (recall that the dual of $l_1(\Gamma)$ is represented by $l_\infty(\Gamma)$), hence so $U \cap V$. Moreover, if $y \in U \cap V$, then

$$\begin{aligned} \|y - x\| &= \sum_{\gamma \in F} |y(\gamma) - x(\gamma)| + \sum_{\gamma \in \Gamma \setminus F} |y(\gamma) - x(\gamma)| < \frac{\varepsilon}{4} + \sum_{\gamma \in \Gamma \setminus F} (y(\gamma) + x(\gamma)) \\ &= \frac{\varepsilon}{4} + \sum_{\gamma \in \Gamma \setminus F} (y(\gamma) - x(\gamma)) + 2 \sum_{\gamma \in \Gamma \setminus F} x(\gamma) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2.\frac{\varepsilon}{4}. \end{aligned}$$

This shows that the identity is weak-to-norm continuous at x. The proof is complete. ■

1.2.9 The Fréchet-Urysohn Property

Definition 1.47 Let Y be a subset of a topological (Hausdorff) space X.

(1) Y is countably compact, if every sequence in Y has a cluster-point in Y.
(2) Y is sequentially compact, if every sequence in Y has a convergent subsequence with limit in Y.
(3) Y is relatively countably compact, if every sequence in Y has a cluster-point in X.
(4) Y is relatively sequentially compact, if every sequence in Y has a convergent subsequence with limit in X.

It is easy to see that

(1) Every (relatively) compact set is (relatively) countably compact.
(2) Every (relatively) sequentially compact set is (relatively) countably compact.

Definition 1.48 A topological space (X, τ) is called Fréchet-Urysohn if the closures of subsets of X are described using sequences, i.e., if whenever $Y \subseteq X$ and $x \in X$ such that $x \in \overline{Y}$, there is a sequence $(x_n)_n$ in Y with $x_n \to x$.

Example 1.13 Metrizable spaces and one point compactifications of discrete spaces are Fréchet-Urysohn.

Definition 1.49 A completely regular Hausdorff topological space X is called a g-space, if its relatively countably compact subsets are relatively compact.

Definition 1.50 A Hausdorff topological space X is said to be angelic if for every relatively countably compact set $Y \subseteq X$, the following hold:

(i) Y is relatively compact,
(ii) for each $x \in \overline{Y}$, there exists a sequence $(x_n)_n \subseteq Y$ such that $x_n \longrightarrow x$.

If K is a compact topological space then K is a Fréchet-Urysohn space if and only if it is angelic. It can be said that a Hausdorff topological space X is angelic if and only if X is a g-space for which any compact subspace is a Fréchet-Urysohn space.

The following are some characterizations of Fréchet-Urysohn spaces.

Theorem 1.38 *For a topological vector space (X, τ) the following assertions are equivalent:*

1. *X is Fréchet-Urysohn.*
2. *For every subset Y of X such that $\theta \in \overline{Y}$ there exists a bounded subset Z of Y such that $\theta \in \overline{Z}$.*
3. *For any sequence $(Y_n)_n$ of subsets of X, each with $\theta \in \overline{Y_n}$, there exists a sequence $Z_n \subseteq Y_n$, $n \in \mathbb{N}$, such that $\bigcup_n Z_n$ is bounded and $\theta \in \overline{\bigcup_{n \leq k} Z_k}$ for each $n \in \mathbb{N}$.*

Proof Clearly 1. implies 2. Now assume 2. It is obvious that 3. holds if $\theta \in Y_n$ for infinitely many n. Therefore, we assume that $\theta \in \overline{Y_n} \setminus Y_n$, for each $n \in \mathbb{N}$. Consequently, there exists a null sequence $(x_n)_n$ in $X \setminus \overline{\{\theta\}}$. For each $n \in \mathbb{N}$ there exists a closed neighbourhood U_n of zero such that $\theta \notin U_n + x_n$. Let each $W_n = U_n \cap Y_n$. Clearly θ is in each $\overline{W_n} \setminus W_n$ and not in the set

$$Y = \bigcup_n (W_n + x_n).$$

However, $\theta \in \overline{Y}$: For U, an open neighborhood of θ, there exist $k \in \mathbb{N}$ with $x_k \in U$ and, V, a neighbourhood of θ with $V + x_k \subseteq U$. As there is $y \in V \cap W_k$ we also have $y + x_k \in U \cap Y$. Thus $\theta \in \overline{Y} \setminus Y$. By hypothesis, there is $Z \subseteq Y$ with Z bounded and $\theta \in \overline{Z}$. There exists subsets $Z_n \subseteq W_n = U_n \cap Y_n$ such that

$$Z = \bigcup_n (Z_n + x_n).$$

By construction, θ does not belong to the closed sets

$$\bigcup_{k<n} (U_k + x_k).$$

Therefore θ is not in any $\overline{\bigcup_{k<n} (Z_k + x_k)}$. This and $\theta \in \overline{Z}$ imply that

$$\theta \in \overline{\bigcup_{n \leq k} (Z_k + x_k)},$$

for each $n \in \mathbb{N}$. Let V' and V be any balanced neighborhoods of θ with $V - V \subseteq V'$. Fix $n \in \mathbb{N}$. There exists $m \geq n$, in $\mathbb{N}$, such that $x_k \in V$ for all $k \geq m$. From

$$\theta \in \overline{\bigcup_{m \geq k} (Z_k + x_k)},$$

it follows that there exist $k \geq m$ and $y \in B_k$ with $y + x_k \in V$. From $y \in V - x_k \subseteq V - V \subseteq V'$, we see, for each $n \in \mathbb{N}$, the set V' meets $\bigcup_{n \leq k} Z_k$. As any neighborhood of θ contains V' and V as above, θ is in the closure of each $\bigcup_{n \leq k} Z_k$. Note also that $\bigcup_n Z_n$ is bounded. Indeed, as

$$Z = \bigcup_n (Z_n + x_n)$$

and $W = \{x_m : m \in \mathbb{N}\}$ are bounded and since

$$\bigcup_n Z_n \subseteq \bigcup_n (Z_n + x_n) - \{x_m : m \in \mathbb{N}\} = Z - W,$$

then $\bigcup_n Z_n$ is also bounded too. We have proved that 2. implies 3.

3. implies 1.: Assume that $\theta \in \overline{Y}$, and set $Y_n = nY$, for each $n \in \mathbb{N}$. Since θ is in each $\overline{Y_n}$, there exist $Z_n \subseteq Y_n$, as in 3.. So each $\bigcup_{n \leq k} Z_k$ is nonempty, and, consequently, there exists a strictly increasing sequence $(n_k)_k$ in $\overline{N}$ with Z_{n_k} nonempty. For each k, let $z_k \in Z_{n_k}$. There exists a sequence $(y_k)_k$ in Y such that $z_k = n_k y_k$ for each $k \in \mathbb{N}$. Since $(n_k)_k$ is strictly increasing and $(z_k)_k = (n_k y_k)_k$ is bounded, the sequence $(y_k)_k$ in Y converges to zero in X. The proof is complete. ■

There are many nonmetrizable Fréchet-Urysohn spaces. To provide some examples, we have the following deep result of J. Bourgain, D. H. Fremlin and M. Talagrand [24]:

Theorem 1.39 *Let X be a Polish space (i.e., a separable completely metrizable space). Denote by $B_1(X)$ the space of all real-valued functions on X which are of the first Baire class and equip this space with the topology of pointwise convergence. Suppose that $Y \subseteq B_1(X)$ is relatively countably compact in $B_1(X)$ (i.e., each sequence in Y has a cluster point in $B_1(X)$. Then the closure $\overline{Y}$ of Y in $B_1(X)$ is compact and Fréchet-Urysohn.*

A slightly weaker version is given in [101].

Corollary 1.7 *Let X be a Polish space and Y be a set of real-valued continuous functions on X. Suppose that each sequence in Y has a pointwise convergent subsequence. Then the closure of Y in $\mathbb{R}^p$ is a Fréchet-Urysohn compact space contained in $B_1(X)$.*

Proof Y is obviously contained in $B_1(X)$. Moreover, let $(f_n)_n$ be any sequence in Y. By the assumption there is a subsequence $(f_{n_k})_k$ pointwise converging to some function f. As the functions f_{n_k} are continuous, the limit function f is of the first Baire class. Hence, it is a cluster point of $(f_n)_n$ in $B_1(X)$. So, Y is relatively countably compact in $B_1(X)$. The assertion now follows from Theorem 1.39. ■

We continue by the following example [14].

Proposition 1.40 *Let (X, τ) be a metrizable locally convex space and Y be a bounded subset of X. If Y is τ-separable and contains no l_1-sequence, then the set*

$$\overline{Y - Y}^{\sigma(X,X')} = \overline{\{x - y : x, y \in Y\}}^{\sigma(X,X')}$$

is Fréchet-Urysohn when equipped with the weak topology.

Proof As the closed linear span of Y is separable, we can without loss of generality suppose that X is separable. Let $(\|.\|_n)$, U_n and B_n $(n \in \mathbb{N})$ be as in the proof of Theorem 1.37. Notice that B_n is a metrizable weak*compact subset of X'. Moreover, the linear span of the union of all B_n's is the whole dual X' (see the end of the proof of Theorem 1.37). Let now P be the topological sum of the spaces $(B_n, \sigma(X', X)), n \in \mathbb{N}$. Then P is a Polish space. Denote by $G \colon P \to X'$ the canonical mapping of P onto the union of all B_n's. Then G is continuous from P to $(X', \sigma(X', X))$. Define a mapping $H \colon P \to \mathbb{R}^P$ by the formula $H(x)(p) = G(p)(x)$. Then H is a homeomorphism of $(X, \sigma(X, X'))$ onto $H(X)$ equipped with the pointwise convergence topology. Moreover, the functions from $H(X)$ are continuous on P.

Let $Z = H(Y - Y)$. We claim that each sequence from Z has a pointwise convergent subsequence. To show that it is enough to observe that each sequence in $Y - Y$ has weakly Cauchy subsequence. Indeed, let $(z_n)_n$ be a sequence in $Y - Y$. Then $z_n = x_n - y_n$ for some $x_n, y_n \in Y$. As Y contains no l_1-sequence, by Theorem 1.37, we get a weakly subsequence $(x_{n_k})_k$ of $(x_n)_n$. Applying Theorem 1.37 once more we get a weakly Cauchy subsequence $(y_{n_k})_k$ of $(y_n)_n$. Then $(z_{n_k})_k$ is a weakly Cauchy subsequence of $(z_n)_n$. Thus Z is relatively countably compact in $B_1(P)$, which is the space of all Baire-one functions on P equipped with the topology of pointwise convergence. By Theorem 1.39, the closure of Z in $\mathbb{R}^P$ is a Fréchet-Urysohn compact subset of $B_1(P)$. In particular, the weak closure of $Y - Y$ is Fréchet-Urysohn when equipped with the weak topology. The proof is complete. ■

Note that the result of the above proposition generalizes the following in the context of Banach spaces [101].

Proposition 1.41 *Let X be a Banach space and Y be a bounded subset of X. If X is norm-separable and contains no l_1-sequence, then the set*

$$\overline{Y - Y}^{\sigma(X',X)} = \overline{\{J(x - y) \colon x, y \in Y\}}^{\sigma(X',X)}$$

is Fréchet-Urysohn when equipped with the weak topology, where J denotes the canonical embedding of X into X''. In particular,*

$$\overline{Y - Y}^{\sigma(X,X')} = \overline{\{x - y \colon x, y \in Y\}}^{\sigma(X,X')}$$

is Fréchet-Urysohn when equipped with the weak topology.

We have the following characterization of the Fréchet-Urysohn property in locally convex spaces [14].

Proposition 1.42 *Let (X,τ) be a Hausdorff locally convex space such that there is a metrizable locally convex topology on X compatible with the duality. The following assertion are equivalent.*

(i) *Any bounded subset of X is Fréchet-Urysohn in the weak topology.*
(ii) *Any bounded sequence in X has a weakly Cauchy subsequence.*
If, moreover, τ itself is metrizable, then these assertions are equivalent to the following one:
(iii) *X contains no l_1-sequence.*

Proof Let ρ be a metrizable locally convex topology compatible with the duality . By Theorem 1.37 (X,ρ) contains no l_1-sequence if and only if (X,ρ) satisfies the condition (ii). Further, the validity of (ii) for (X,ρ) is equivalent to its validity for (X,τ). It follows that (ii) holds if and only if (X,ρ) contains no l_1-sequence. In particular, if $\rho=\tau$, we get $(i)\Leftrightarrow(ii)$.

$(ii)\Rightarrow(i)$ Suppose that (ii) holds. Let Y be a bounded subset of (X,τ) and let $x\in X\in$ belong to the weak closure of Y. We need to find a sequence in Y converging to x. We first prove it under the additional assumption that Y is separable. Then Y is bounded and separable in (X,ρ) as well. As (X,ρ) contains no l_1-sequence, by Proposition 1.40 we get that the weak closure of Y-Y is Fréchet-Urysohn in the weak topology. Hence, in particular, there is a sequence in Y weakly converging to x.

To prove the general case it is enough to show that there is a countable set $Z\subseteq Y$ such that x belongs to the weak closure of Z. In other words, it is enough to show that the weak topology on X has countable tightness. To prove that observe that $(X,\sigma(X,X'))$ is canonically homeomorphic to a subspace of $C_p(X',\sigma(X',X))$, which is the space of all continuous functions on the space $(X',\sigma(X',X))$ equipped with the topology of pointwise convergence. Further notice that $(X',\sigma(X',X))$ is σ-compact, this follows by the metrizability of ρ as $X'=\bigcup_{m,n\in\mathbb{N}} mB_n$ using the notation from the proof of Theorem 1.37. Finally, as any finite power of a σ-compact and hence Lindelöf, we can conclude by the Arkhangel'skii-Pytkeev theorem [7].

$(i)\Rightarrow(ii)$ Suppose that (ii) does not hold. Then there is a sequence $(x_n)_n$ in X which is an l_1-sequence in (X,ρ). Let $T_0\colon l_1^0\to X$ be defined as in (1.17). Let S denote the unit sphere in l_1^0. Then θ is in the weak closure of S (as l_1^0 is an infinite dimensional normed space) but it is not the weak limit of any sequence from S (by Schur's theorem [75]). Thus, θ is in the weak closure of $T_0(S)$ without being the weak limit of any sequence from $T_0(S)$. Thus $T_0(S)\cup\{\theta\}$ is a bounded set which is not Fréchet-Urysohn in the weak topology. ∎

The following characterization of Banach spaces not containing l_1 is given in [101].

Theorem 1.40 *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X contains no isomorphic copy of* l_1.
2. *Each bounded separable subset of X is Fréchet-Urysohn in the weak topology.*
3. *For each separable subset* $Y \subseteq X$ *there are relatively weakly closed subsets* $Y_n, n \in \mathbb{N}$ *such that* $Y = \bigcup_{n\in\mathbb{N}} Y_n$ *and each* Y_n *is Fréchet-Urysohn in the weak topology.*

Proof The implication 1. ⇒ 2. follows from Proposition 1.40.

The implication 2. ⇒ 1. follows from the fact that the unit ball of l_1 is not Fréchet-Urysohn (as θ is in the weak closure of the sphere and the sphere is weakly sequentially closed by the Schur theorem [75]).

The implication 2. ⇒ 3. is trivial if we use the fact that a closed ball is weakly closed.

Let us prove 3. ⇒ 2. To show 2. it is enough to prove that the unit ball of any closed separable subspace of X is Fréchet-Urysohn in the weak topology. Let Z be such a subspace. Let $Y_n, n \in \mathbb{N}$ be the cover of Z provided by 3. As each Y_n is weakly closed, it is also norm-closed. By the Baire category theorem some Y_n has a nonempty interior in Y, so it contains a ball. We get that some ball in Y is Fréchet-Urysohn, so the unit ball has this property as well. ■

Remark 1.29 Note that the assertion 3. is a topological property of the space $(X, \sigma(X, X'))$ (as norm separability coincides with weak separability).

As a consequence of Proposition 1.42 we get the following improvement of Theorem 1.40.

Corollary 1.8 *Let X be a Banach space. The following assertions are equivalent.*

1. *X contains no isomorphic copy of* l_1.
2. *The closed unit ball of X is Fréchet-Urysohn in the weak topology.*
3. *There is a sequence* $(Y_n)_{n\geq 1}$ *of weakly closed sets which are Fréchet-Urysohn in the weak topology such that* $X = \bigcup_{n=1}^{\infty} Y_n$.

Proof The equivalence 1. ⇔ 2. follows from Proposition 1.42. The implication 2. ⇒ 3. is trivial. The implication 3. ⇒ 1. follows from Theorem 1.40 (or, alternatively, 3. ⇒ 2.) follows from the Baire category theorem as in Theorem 1.40. ■

Definition 1.51 A Banach space $(X, \|.\|)$ is Asplund if and only if Y' is separable for each separable subspace $Y \subseteq X$.

Remark 1.30 A Banach space X is an *Asplund* space if each convex continuous function $T: X \to \mathbb{R}$ is Fréchet differentiable on a dense G_δ set in X. Also it is known that a Banach space X is *Asplund* if and only if X' has the *RNP* [25].

It is worthwhile to remark that there are separable Banach spaces having no copy of l_1 for which X' is nonseparable [93, 127]. On the other hand, the well-known James's space J is an example of a nonreflexive Banach space without an unconditional basis which does not contain any copy of l_1 and yet has separable dual.

Remark 1.31 Let us remark that the implication $(ii) \Rightarrow (i)$ of Proposition 1.42 does not hold for general locally convex spaces. Indeed, there are Banach spaces X such that the closed unit ball of X' is weak* sequentially compact, but it is not Fréchet-Urysohn in the weak* topology. In particular, the dual closed unit ball is weak* sequentially compact whenever X is $Asplund$ [55], in particular if $X = C(K)$ with K scattered [55]. On the other hand, K is canonically homeomorphic to a subset of the closed unit ball of $C(K)'$ equipped with the weak* topology, so it is enough to observe that there are scattered compact spaces which are not Fréchet-Urysohn. As a concrete example we can take $K = [0, w_1]$, the ordinal interval equipped with the order topology (w_1 is the first uncountable ordinal).

It is worth to compare Theorem 1.40 with a similar characterization of $Asplund$ spaces [101].

Theorem 1.41 *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X is Asplund.*
2. *Each bounded separable subset of X is metrizable in the weak topology.*
3. *For each separable subset $Y \subseteq X$ there are relatively weakly closed subsets $Y_n, n \in \mathbb{N}$, of Y such that $Y = \bigcup_{n\in\mathbb{N}} Y_n$ and each Y_n is metrizable in the weak topology.*

Proof The equivalence of 1. and 2. follows from the well-known fact that the unit ball of Y is metrizable in the weak topology if and only if Y' is separable. The equivalence of 2. and 3. can be proved similarly as corresponding equivalence in the previous theorem. ■

Remark 1.32 There is no analogue of Theorem 1.40 for convex sets. Indeed, let $X = l_1$ and let C be the closed convex hull of the standard basis. Then C contains an l_1-sequence but is Fréchet-Urysohn in the weak topology. In fact, it is even metrizable as it is easy to see that on the positive cone of l_1 the weak and norm topologies coincide.

1.3 Ultrametric Spaces

The origin of ultrametric spaces lies in valuation theory and dates back to Krasner and Monna who developed this theory for ultrametric distances with real values (non-Archimedean analysis). A systematic study of (general) ultrametric spaces was provided [16, 81, 84, 113, 120, 152, 154, 155, 157, 160, 169] and others. This study is concerned with ultrametric whose distance functions take their values in an arbitrary partially ordered set (with a smallest element 0) not just in the real numbers.

Definition 1.52 Let $(\Gamma, \leq)$ be an ordered set with smallest element 0. Let X be a nonempty set. A mapping $d\colon X \times X \longrightarrow \Gamma$ is called an ultrametric distance and (X, d, Γ) an ultrametric space if d has the following properties for all $x, y, z \in X$ and $\gamma \in \Gamma$:

$(d1)$ $d(x, y) = 0$ if and only if $x = y$,
$(d2)$ $d(x, y) = d(y, x)$,
$(d3)$ if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$.

If there is no ambiguity, we simply write X instead of (X, d, Γ).

If Γ is totally ordered , $(d3)$ becomes
$(d3')$ $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$.

Remark 1.33 The ultrametric space (X, d, Γ) is trivial, if there exists $\gamma \in \Gamma$ such that for all $x, y \in X, x \neq y, d(x, y) = \gamma$.

Definition 1.53 Let $(Y, d_{|Y}, \Gamma_Y)$ and (X, d, Γ) be ultrametric spaces such that $Y \subset X$ and $\Gamma_Y \subset \Gamma$. Assume that Γ_Y has the induced order of Γ and the same 0 as Γ and that furthermore, $d_{|Y}(Y \times Y) \subset \Gamma_Y$ and $d_{|Y}(y, y') = d(y, y')$ for all $y, y' \in Y$. Then $(Y, d_{|Y}, \Gamma_Y)$ is said to be a subspace of (X, d, Γ) and X is called an extension of Y. Often we simply write d instead of $d_{|Y}$.

Definition 1.54 Let (X, d, Γ) be an ultrametric space. The space X is said to be solid if for every $\gamma \in \Gamma$ and $x \in X$ there exists $y \in X$ such that $d(x, y) = \gamma$. If X is solid, then $d(X \times X) = \Gamma$.

Definition 1.55 Let (X, d, Γ) be an ultrametric space. Let $\gamma \in \Gamma^\bullet = \Gamma \setminus \{0\}$ and $a \in X$. The set $B_\gamma(a) = \{x \in X \mid d(a, x) \leq \gamma\}$ is called a ball. The element a is said to be a center of $B_\gamma(a)$ and the element γ to be a radius of $B_\gamma(a)$. If $x, y \in X, x \neq y$, then $B(x, y) = B_{d(x,y)}(x)$ is called a principal ball.

Remark 1.34 Let (X, d, Γ) be an ultrametric space. If X is solid, every ball is principal. If Γ is totally ordered, also the converse conclusion holds.

Definition 1.56 Let (X, d, Γ) be an ultrametric space. A nonempty Y of X is said to be convex in X when for all $y_1, y_2 \in Y$ with $y_1 \neq y_2$ the principal ball $B(y_1, y_1) \subseteq Y$.

Remark 1.35 Every principal ball is convex in X and furthermore, if $\bigcap_{i \in I} B(x_i, y_i) \neq \emptyset$ then $\bigcap_{i \in I} B(x_i, y_i)$ is convex in X.

In the following lemma, we list some properties of balls which can easily be verified [161].

Lemma 1.10 *Let (X, d, Γ) be an ultrametric space and let $\gamma, \delta \in \Gamma^\bullet$.*

1. *Let $x, y \in X$.*
   (a) *If $\gamma \leq \delta$ and $B_\gamma(x) \cap B_\gamma(y) \neq \emptyset$, then $B_\gamma(x) \subseteq B_\delta(y)$,*
   (b) *if $B_\delta(y) \subset B_\gamma(x)$, then $\gamma \not\leq \delta$.*
2. *Concerning principal balls, if $x, y, z, u \in X$, $x \neq z$ and $y \neq u$, then*
   (a) *$B(x, z) \subseteq B_\delta(y)$ if and only if $d(x, z) \leq \delta$ and $x \in B_\delta(y)$,*
   (b) *if $B(x, z) \subset B_\delta(y)$, then $d(x, z) < \delta$,*
   (c) *if $B(x, z) = B(y, u)$, then $d(x, z) = d(y, u)$.*
3. *Let X be solid and $x, y \in X$.*
   (a) *$B_\gamma(x) \subseteq B_\delta(y)$ if and only if $\gamma \leq \delta$ and $x \in B_\delta(y)$,*
   (b) *if $B_\gamma(x) \subset B_\delta(y)$, then $\gamma < \delta$.*
   (c) *if $B_\gamma(x) = B_\delta(y)$, then $\gamma = \delta$*
4. *If Γ is totally ordered and $B_\gamma(x) \subset B_\delta(y)$, then $\delta < \gamma$.*

Definition 1.57 Let (X, d, Γ) be an ultrametric space. A set of balls which is totally ordered by inclusion is said to be a chain.

Lemma 1.11 *Let (X, d, Γ) be an ultrametric space. Let $\mathcal{C}$ be a chain of balls of X which does not have a smallest ball. Then there exists a limit ordinal λ and a strictly decreasing family of balls $(B_i)_{i<\lambda}$ such that each $B_i \in \mathcal{C}$ and for every ball $C \in \mathcal{C}$ there exists B_i such that $B_i \supseteq C$ and hence $\bigcap \mathcal{C} = \bigcap_{i<\lambda} B_i$.*

Definition 1.58 Let (X, d, Γ) be an ultrametric space. X is called spherically complete (resp., principally complete) if every chain of balls of X (resp., principal balls of X) has a nonempty intersection.

Remark 1.36 Every spherically complete ultrametric space (X, d, Γ) is principally complete. The converse is true when Γ is totally ordered or the space is solid.

Definition 1.59 An ultrametric space (X, d, Γ) is said to be complete if every chain of balls $\{B_{\gamma_i} \mid i \in I\}$, with $\inf\{\gamma_i \mid i \in I\} = 0$, has a nonempty intersection.

Remark 1.37 A spherically complete ultrametric space (X, d, Γ) is complete. If Γ is totally ordered and if $\Gamma^\bullet$ does not have a smallest element, the ultrametric distance induces on X a uniformity, hence also a topology. In this case, the concept of completeness coincides with that given by the uniformity.

Several examples of different types of ultrametric spaces are discussed in [160]. Some where Γ is totally ordered and others where Γ is not totally ordered.

Examples 1.9

1. Let Δ be a totally ordered Abelian additive group, let ∞ be a symbol such that $\infty \notin \Delta$, and $\delta + \infty = \infty + \delta = \infty$, $\infty + \infty = \infty$, $\delta < \infty$ for all $\delta \in \Delta$. We

denote by 0 the neutral element of Δ, that is $0+\delta = \delta$ for every $\delta \in \Delta$. Let K be a commutative field, let $v\colon K \longrightarrow \Delta \cup \{\infty\}$ be a valuation of K, so we have

($v1$) $v(x) = \infty$ if and only if $x = 0$,
($v2$) $v(xy) = v(x) + v(y)$,
($v3$) $v(x + y) \geq \min\{v(x), v(y)\}$.

Let $\Gamma^\bullet$ be a totally ordered Abelian multiplicative group with neutral element 1, let 0 be a symbol such that $0 \notin \Gamma^\bullet$, $0\gamma = \gamma 0 = 0$, $0.0 = 0$, $0 < \gamma$ for every $\gamma \in \Gamma^\bullet$. Let $\theta\colon \Delta \cup \{\infty\} \longrightarrow \Gamma = \Gamma^\bullet \cup \{0\}$ be an order reversing bijection such that $\theta(\infty) = 0$, $\theta(\delta + \delta') = \theta(\delta).\theta(\delta')$, so $\theta(0) = 1$.

Let $d\colon K \times K \longrightarrow \Gamma$ be defined by $d(x, y) = \theta(v(x - y))$, then (K, d, Γ) is an ultrametric space which is said to be associated to the valued field $(K, v, \Delta \cup \{\infty\})$.

2. Let Γ be a totally ordered set with smallest element 0, let $\Gamma^\bullet = \Gamma\backslash\{0\}$. Let R be a nonempty set with a distinguished element 0. For each $f\colon \Gamma^\bullet \longrightarrow R$, let $\mathrm{supp}(f) = \{\gamma \in \Gamma^\bullet \mid f(\gamma) \neq 0\}$ be the support of f. Let $R[[\Gamma]]$ be the set of all $f\colon \Gamma^\bullet \longrightarrow R$ with support which is empty or anti-well ordered. Let $d\colon R[[\Gamma]] \times R[[\Gamma]] \longrightarrow \Gamma$ be defined by $d(f, f) = 0$ and if $f \neq g$, $d(f, g)$ is the largest element of the set $\{\gamma \in \Gamma^\bullet \mid f(\gamma) \neq g(\gamma)\}$. Then $(R[[\Gamma]], d, \Gamma)$ is an ultrametric space which is solid and spherically complete.
3. Let I be a set with at least two elements, let $(X_i)_{i\in I}$ be a family of sets X_i, each one having at least two elements. Let $X = \prod_{i\in I} X_i$. Let $\mathcal{P}(I)$ be the set of all subsets of I, ordered by inclusion. And let $d\colon X \times X \longrightarrow \mathcal{P}(I)$ be defined by $d(f, g) = \{i \in I \mid f_i \neq g_i\}$, where $f = (f_i)_{i\in I}$ and $g = (g_i)_{i\in I}$. Then $(X, d, \mathcal{P}(I))$ is a solid and spherically complete ultrametric space. If each $X_i = \{0, 1\}$, we obtain the ultrametric space $(\mathcal{P}(I), d, \mathcal{P}(I))$ with $d(A, B) = (A \cup B)\backslash(A \cap B)$ for all $A, B \subseteq I$.
4. Let X be a topological space, let Y be a discrete topological space, let $\mathcal{C}(X, Y)$ denote the set of continuous functions from X to Y and let $\mathcal{C}l(X)$ the set of clopen (i.e., closed and open) subsets of X. The mapping $d\colon \mathcal{C}(X, Y) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}l(X)$ is defined by $d(f, g) = \{x \in X \mid f(x) \neq g(x)\}$. Then $(\mathcal{C}(X, Y), d, \mathcal{C}l(X))$ is a solid ultrametric space, and it is spherically complete if $\mathcal{C}l(X)$ is a complete sub-Boolean-algebra of $\mathcal{P}(X)$.

Definition 1.60 Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Let $(Y, d_{|Y}, \Gamma_Y)$ be a subspace of (X, d, Γ) and assume that $d(Y \times Y) = d(X \times X) = \Gamma$. If for every $x \in X$ and for every $y \in Y$, with $x \neq y$, there exists $y' \in Y$ such that $d(y', x) < d(y, x)$, the extension $Y \prec X$ is called immediate and we write $Y\mathrm{im} \prec X$. The extension $Y \prec X$ is said to be dense (denoted by $Y\mathrm{de} \prec X$), if for every $x \in X$ and for every $0 < \gamma \in \Gamma$ there exists y in Y such that $d(y, x) < \gamma$. Thus if $Y\mathrm{de} \prec X$ then also $Y\mathrm{im} \prec X$.

Remark 1.38 If $\Gamma^\bullet$ does not have a smallest element, Definition 1.60 coincides with that given by the topology of X. We remark that both notions, "immediate" and "dense" can be defined more generally for ultrametric spaces, where Γ is only ordered [155].

The following is given in [161].

Theorem 1.42

1. *Every ultrametric space (X, d, Γ), with Γ totally ordered, has an immediate extension which is spherically complete. (We call such an extension a spherical completion of X.)*
2. *Every ultrametric space (X, d, Γ), with Γ totally ordered, has an extension (X', d, Γ) such that X' is dense in X'. (We call such an extension a completion of X.)*
3. *Let $(Y, d_{|Y}, \Gamma_Y)$ be a subspace of Let (X, d, Γ). Assume that Γ is totally ordered and that $\Gamma^\bullet_{|Y}$ is coinitial in $\Gamma^\bullet$ and that furthermore $d(Y \times Y) = \Gamma_Y, d(X \times X) = \Gamma$. If X is complete, then there exists one and only one completion $\widehat{Y}$ of Y which is a subspace of X.*

Proof The proofs of 1. and 2. are given in [155, 176].

3. Let $\mathcal{S}$ be the set of all ultrametric subspaces $\mathcal{S}$ such that Y is dense in $\mathcal{S}$. Since Y is dense in itself, $\mathcal{S} \neq$. The set $\mathcal{S}$ is ordered by inclusion. Let $\{S_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{S}$. Then $S = \bigcup_{i \in I} S_i$ is a subspace of X and Y is dense in S. Thus $S \in \mathcal{S}$ is an upper bound for all $S_i, i \in I$. By Zorn's lemma, there exists a maximal element in $\mathcal{S}$ which we denote again by S. We show that S is complete. Since $\Gamma^\bullet_{|Y}$ is coinitial in $\Gamma^\bullet$ and $\Gamma^\bullet_{|Y} = \Gamma^\bullet_{|S} = d(S \times S) \setminus \{0\}$ has in $\Gamma^\bullet_{|S}$ the infimum 0 if and only if the infimum of Δ in $\Gamma^\bullet$ is 0, thus we may just write $\inf \Delta = 0$. We assume that S is not complete. Then there exists a chain $\{B^S_{\gamma_i}(a_i) \mid i \in I\}$ of balls in S with

$$\inf\{\gamma_i \mid i \in I\} = 0 \text{ and } \bigcap B^S_{\gamma_i}(a_i) = \emptyset.$$

Since X is complete and for each $i \in I$, $B^S_{\gamma_i}(a_i) = S \cap B^S_{\gamma_i}(a_i)$, where $B_{\gamma_i}(a_i)$ denotes the ball with center a_i and radius γ_i in X, there exists $z \in X$ such that $\{z\} = \bigcap B_{\gamma_i}(a_i)$. Let $S' = S \cup \{z\}$. Then S' is a subspace of X which properly contains S, so also Y. To prove that Y is dense in S', it suffices to show that if $0 < \gamma \in \Gamma$, there exists $y \in Y$ such that $d(y, z) < \gamma$. Since $\inf\{\gamma_i \mid i \in I\} = 0$ there exists γ_i with $0 < \gamma_i < \gamma$. Since Y is dense in S and $a_i \in S$, it follows that there exists $y \in Y$ such that $d(y, a_i) < \gamma_i$. Since, moreover, $z \in B_{\gamma_i}(a_i)$, then $d(z, y) \leq \max\{d(z, a_i), d(y, a_i)\} \leq \gamma_i < \gamma$. Thus Y is dense in S'. So $S' \in \mathcal{S}$, which contradicts the maximality of S in $\mathcal{S}$. We have proved that S is complete, hence a completion of Y in X. It remains to show that Y has at most one completion in X. Assume that $\widehat{Y_1}, \widehat{Y_2}$ are completions of Y in X. Let $\widehat{y_1} \in \widehat{Y_1}$. For each $\gamma \in \Gamma^\bullet$

there exists $y_\gamma \in Y$ such that $d(\widehat{y_1}, y_\gamma) < \gamma$. If $\Gamma^\bullet$ has a smallest element, say γ^* then

$$\widehat{y_1} = y_{\gamma^*} \in Y \subset \widehat{Y_2}.$$

If $\Gamma^\bullet$ does not have a smallest element, then $\inf\{\gamma \mid \gamma \in \Gamma^\bullet\} = 0$, thus there exists $\widehat{y_2} \in \widehat{Y_2}$ with

$$\{\widehat{y_2}\} = \bigcap_{\gamma \in \Gamma^\bullet} B_\gamma(y_\gamma)$$

because $\widehat{Y_2}$ is complete. Hence $\widehat{y_1} = \widehat{y_2} \in \widehat{Y_2}$. This shows that $\widehat{Y_1} \subseteq \widehat{Y_2}$. By the same argumentation, we conclude that $\widehat{Y_2} \subseteq \widehat{Y_1}$, thus $\widehat{Y_1} = \widehat{Y_2}$. ■

Definition 1.61 Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Let λ be a limit ordinal and let $\xi = (x_i)_{i<\lambda}$ be a family of elements of X. We say that ξ is a Cauchy family if for every $\gamma \in \Gamma^\bullet$ there exists $i_0 = i_0(\gamma, \xi) < \lambda$ such that if $i_0 \leq i < \kappa < \lambda$, then $d(x_i, x_k) < \gamma$. The family $\xi = (x_i)_{i<\lambda}$ is said to be pseudo-convergent if there exists $i_0 = i_0(\xi) < \lambda$ such that if $i_0 \leq i < \kappa < \mu < \lambda$, then $d(x_\kappa, x_\mu) < d(x_i, x_\kappa)$.

Remark 1.39 We note that if $\xi = (x_i)_{i<\lambda}$ is pseudo-convergent, the elements x_i, for $i_0(\xi) \leq i < \lambda$ are all distincts and if $i_0(\xi) \leq i < \kappa < \mu < \lambda$, then $d(x_i, x_\kappa) = d(x_\kappa, x_\mu)$, this element is denoted by ξ_i. Hence if $i_0 \leq i < \kappa < \lambda$, then $\xi_i > \xi_\kappa$.

Definition 1.62 Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Let λ be a limit ordinal and let $\xi = (x_i)_{i<\lambda}$ be a Cauchy family of elements of X. The element $y \in X$ is a limit of the family ξ if for every $\gamma \in \Gamma^\bullet$ there exists $i_1 = i_1(\gamma) < \lambda$ such that if $i_1 \leq i < \lambda$, then $d(y, x_i) < \gamma$. The ultrametric space X is complete if and only if every Cauchy family has a limit in X.

Remark 1.40 A Cauchy family $\xi = (x_i)_{i<\lambda}$ has at most one limit. Indeed, if y, z are limits, then $d(y, z) < \gamma$ for all $\gamma \in \Gamma^\bullet$, so $y = z$.

Definition 1.63 Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Let λ be a limit ordinal and let $\xi = (x_i)_{i<\lambda}$ be a pseudo-convergent family of elements of X. The element $y \in X$ is a pseudo-limit of the family $\xi = (x_i)_{i<\lambda}$ if there exists $i_1 = i_1(\xi, y)$, $i_0(\xi) \leq i_1 < \lambda$, such that if $i_1 \leq i < \lambda$ then $d(y, x_i) \leq \xi_i$. If y is a pseudo-limit of ξ, then $z \in X$ is a pseudo-limit of ξ if and only if $d(y, z) < \xi_i$ for all i such that $i_1 \leq i < \lambda$.

The following is a characterization of spherical completeness [151].

Proposition 1.43 *Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Then X is spherically complete if and only if every pseudo-convergent family of X has a pseudo-limit in X.*

1.4 Admissible Functions

Throughout this subsection, we denote by (X, τ) a topological vector space, and by Y a nonempty subset of X.

Below the definition of functions providing the possibility of working with extended real seminorms in topological vector spaces.

Definition 1.64 An admissible function for Y on X is an extended real-valued function $\rho\colon X \longrightarrow [0, \infty]$ such that

1. The mapping $(x, y) \mapsto \rho(x, y)$ is continuous on $Y \times Y$,
2. $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$,
3. $\rho(\lambda x) = |\lambda|\, \rho(x)$, for all $\lambda \in \mathbb{R}$ and $x \in X$,
4. If $x, y \in Y$ and $\rho(x - y) = 0$, then $x = y$.

Remark 1.41 Notice that if ρ is an admissible function for Y on X, then it defines a metric on Y whose induced topology is coarser than τ.

Remark 1.42 It is instructive to compare the notion of continuity in the sense of 1. with the usual one. It is easy to see that if ρ is continuous on X, then $(x, y) \mapsto \rho(x, y)$ is continuous on $Y \times Y$. Furthermore, if 1. – 3. hold then ρ is continuous on Y.

It is not true, in general, that if ρ is continuous on Y, then it satisfies 1. For example, if $X = \mathbb{R}$ and $Y = [0, \infty)$, then the mapping $\rho\colon \mathbb{R} \longrightarrow [0, \infty]$ defined by

$$\rho(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x > 0, \\ \infty, & \text{if } x = 0, \\ 0, & \text{if } x < 0, \end{cases}$$

is continuous on Y. However, the mapping $T\colon Y \times Y \longrightarrow [0, \infty]$ given by $T(x, y) = \rho(x - y)$ is not continuous at the point $(1, 1)$. Indeed, it suffices to see that $(1 - \frac{1}{k}, 1)$ converges to $(1, 1)$ in $Y \times Y$, while that $T(1 - \frac{1}{k}, 1) = 0$ and $T(1, 1) = \infty$.

Barroso [12] proved that the class of admissible functions is sufficiently good to imply that the Schauder-projection operator is continuous.

Proposition 1.44 *Let ρ be an admissible function for Y on X. Then for any $\varepsilon > 0$ and $p \in Y$, the function $g\colon Y \longrightarrow [0, \infty)$ given by*

$$g(x) = \max\{\varepsilon - \rho(x - p), 0\}$$

is continuous on Y.

Proof Firstly, let us recall that the effective domain of ρ is the set

$$\mathtt{D}(\rho) = \{x \in X : \rho(x) < \infty\}.$$

Let x_0 be a point in Y and $\delta > 0$ be arbitrary. By assumption, there exists a neighborhood $U \times V$ of (x_0, p) in $Y \times Y$ such that

$$\rho(x_0 - p) - \delta \leq \rho(x - z) \leq \rho(x_0 - p) + \delta,$$

for all $(x, z) \in U \times V$. If $x_0 - p \notin \mathtt{D}(\rho)$ then $\rho(x_0 - p) = \infty$ and, hence, $\rho(x - p) = \infty$ for all $x \in U$. In consequence, $g(x) = g(x_0) = 0$ for all $x \in U$. In case $x_0 - p \in \mathtt{D}(\rho)$, we can conclude that $x - p \in \mathtt{D}(\rho)$ for all $x \in U$. In this case, it is easy to see that $g(x_0) + \delta \geq g(x)$, for all $x \in U$. On the other hand, if $g(x_0) = 0$, then clearly $g(x) \geq g(x_0) - \delta$ holds for every $x \in U$. Assuming now that $g(x_0) = \varepsilon - \rho(x_0 - p)$, we have $g(x_0) - \delta \leq \varepsilon - \rho(x - p) \leq g(x)$, for all $x \in U$. In any case, we have proven that g is continuous at x_0, and hence continuous in Y. The proof is complete. ■

The following is an example of an admissible function [12].

Proposition 1.45 *Let Y be a compact convex subset of a topological vector space (X, τ) and $\mathcal{F} = \{\rho_n : n \in \mathbb{N}\}$ a countable family of seminorms on X which separate points of $Y - Y$ and such that the topology Γ generated by $\mathcal{F}$ is coarser than τ in Y. Then the function $\rho : X \to [0, \infty]$ defined as*

$$\rho(x) = \sum_{n=1}^{\infty} \rho_n(x), \ \ x \in X$$

is admissible.

Proof Since Y is compact and Γ is coarser than τ, each ρ_n restricted to Y is τ-continuous. Thus we have $\max\{\rho_n(x) : x \in Y < \infty\}$ for all $n \in \mathbb{N}$. By replacing the seminorms ρ_n by suitable positive multiples, if necessary, we may assume that

$$\max\{\rho_n(x) : x \in Y_n\} \leq 2^{-n-1}, \tag{1.18}$$

for all $n \in \mathbb{N}$. Notice that $\rho(x - y) < \infty$ for all $x, y \in Y$. Moreover, one readily checks 2. – 4.. Using now (1.18), we see that the sequence of functions $\rho^n(x - y) = \sum_{i=1}^{n} \rho_i(x - y)$ is Cauchy w.r.t. the topology of uniform convergence on $Y \times Y$. Thus $\rho^n(x - y)$ converges uniformly on $Y \times Y$ to $\rho(x - y)$. Furthermore, to verify that 1– holds, we have only to ensure this for each ρ_n. Let (x_α, y_α) be a net in $Y \times Y$ converging to (x, y). Since τ is finer than Γ on Y, both $\rho_n(x_\alpha - x)$ and $\rho_n(y_\alpha - y)$ converge to 0. We may then apply the triangle inequality to conclude $|\rho_n(x_\alpha, y_\alpha) - \rho_n(x - y)| \to 0$. ■

1.5 Some Fixed Point Theorems

Banach's Contraction Mapping Principle is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. The principle first appeared in explicit form in Banach's thesis [11].

Theorem 1.43 (Banach's Contraction Mapping Principle) *Let (X, d) be a complete metric space and let $T\colon X \longrightarrow X$. If there exists an $0 < k < 1$ such that $d(T(x), T(y)) \le kd(x, y)$ for all $x, y \in X$, then T has a unique fixed point.*

Proof First we consider the case when:

$$\operatorname{diam}(X) := \sup\{d(x, y)\colon x, y \in X\} < \infty.$$

For each $n \in \mathbb{N}$, let $Y_n = T^n(X)$. Then

$$Y_{n+1} = T^{n+1}(X) = T^n(T(X)) \subseteq T^n(X) = Y_n$$

for all $n \in \mathbb{N}$. Therefore, $\{Y_n : n \in \mathbb{N}\}$ is a decreasing sequence of nonempty subsets of X. Next, notice that

$$0 \le \operatorname{diam}(Y_{n+1}) \le k\operatorname{diam}(Y_n) \quad \text{for all } n \in \mathbb{N}$$

and so, by induction,

$$0 \le \operatorname{diam}(Y_{n+1}) \le k^n\operatorname{diam}(Y_n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $\lim_{n\to\infty} \operatorname{diam}(\overline{Y_n}) = \lim_{n\to\infty} \operatorname{diam}(Y_n) = 0$. It then follows from Cantor's intersection property that

$$\bigcap_{n\in\mathbb{N}} \overline{Y_n} = \{x\} \quad \text{for some } x \in X.$$

Moreover, since $x \in \overline{Y_n}$,

$$T(x) \in T(\overline{Y_n}) \subseteq \overline{T(Y_n)} = \overline{Y_{n+1}} \subseteq \overline{Y_n},$$

$T(x) \in \bigcap_{n\in\mathbb{N}} \overline{Y_n} = \{x\}$. That is, $T(x) = x$.

In the case when $\operatorname{diam}(X) = \infty$ some extra work is required. In this case we choose any $x_0 \in X$ and let

$$Z := \overline{\{T^n(x_0)\colon n \in \mathbb{N}\}}.$$

Then $T(Z) \subseteq Z$ and

$$\mathrm{diam}(Z) \leq \frac{d(T(x_0), x_0)}{1-k} < \infty.$$

Hence from the previous argument there exists a point $x \in Z \subseteq X$ such that $T(x) = x$. ■

The Caristi fixed point theorem [35] is known as one of the very interesting and useful generalizations of the Banach's Contraction Mapping Principle for self-mappings on a complete metric space. Neither continuity nor a Lipschitz condition is required.

Theorem 1.44 (Caristi's Fixed Point Theorem) *Let (X, d) be a complete metric space and let $T: X \longrightarrow X$ be a mapping such that*

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x))$$

for all $x \in X$, where $\varphi: X \longrightarrow [0, +\infty)$ is a lower semicontinuous mapping. Then T has at least a fixed point.

The setting of generalized ultrametric spaces offers a highly flexible framework in which to study the fixed point theory is necessary for logic programming semantics [59, 85, 104, 113, 151, 153, 156, 157] and [177].

Definition 1.65 Let (X, d, Γ) be an ultrametric space. A mapping $\varphi: X \longrightarrow X$ is said to be strictly contracting if for all $x, x' \in X$, with $x \neq x'$, $d(\varphi(x), \varphi(x')) < d(x, x')$. An element $z \in X$ with $\varphi(z) = z$ is called a fixed point of φ.

For strictly contracting maps on ultrametric spaces we have the following fixed point theorem [151, 153, 160].

Theorem 1.45 *Assume that (X, d, Γ) is a spherically complete ultrametric space and that $\varphi: X \longrightarrow X$ is strictly contracting. Then φ has exactly one fixed point $z \in X$.*

Proof Assume, $\pi_x = d(x, \varphi(x)) \neq 0$ for every $x \in X$. Let $B_x = B_{\pi_x}$. The set $\mathcal{B} = \{B_x \mid x \in X\}$ is ordered by inclusion. Let $\mathfrak{C}$ be a maximal chain in $\mathcal{B}$. Since X is spherically complete, there exists an element $z \in \bigcap\{B_x \mid B_x \in \mathfrak{C}\}$. Then $B_z \subseteq B_x$ for every $B_x \in \mathfrak{C}$. Indeed, this is obvious, if $z = x$. If $z \neq x$ then $d(\varphi(z), \varphi(x)) \leq d(z, x) \leq \pi_x = d(x, \varphi(x)), \pi_z = d(\varphi(z), z) \leq \pi_x$. Hence $B_z \subseteq B_x$. Since $\mathfrak{C}$ is a maximal chain in $\mathcal{B}$, then B_z is the smallest element of $\mathfrak{C}$. But $\pi_{\varphi(z)} = d(\varphi(z), \varphi(\varphi(z))) < d(z, \varphi(z)) = \pi_z$ and therefore $B_{\varphi(z)} \subsetneq B_z$, contradicting the maximality of $\mathfrak{C}$. Hence there exists an element $x \in X$ with $\varphi(x) = x$. If also $\varphi(y) = y$ for $x \neq y$, then $d(x, y) = d(\varphi(x), \varphi(y)) < d(x, y)$, which is absurd. Thus there exists exactly one fixed point for φ. ■

Remark 1.43 Analysing the proof of Theorem 1.45, we see that to prove the existence of a fixed point for the mapping $\varphi: X \longrightarrow X$, it suffices to assume the following property. For any $x, y \in X, d(\varphi(x), \varphi(y)) \leq d(x, y)$ and for $x \neq \varphi(x), d(\varphi(x), \varphi(\varphi(x))) < d(x, \varphi(x))$.

In the special case when Γ is totally ordered, we obtain the following characterization of principal completeness [153].

Theorem 1.46 *Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. The following conditions are equivalent*:

1. *X is principally complete*
2. *Every strictly contracting mapping $\varphi\colon X \longrightarrow X$ has a fixed point.*

Proof 1. $\Longrightarrow$ 2.: this was proved in Theorem 1.45.

2. $\Longrightarrow$ 1.: We assume that X is not principally complete, so there exists a chain $\mathcal{C}$ of principal balls such that $\bigcap \mathcal{C} = \emptyset$. Hence $\mathcal{C}$ dos not have a smallest ball and therefore the coinitial type λ of $\mathcal{C}$ is a limit ordinal. Then there exists a strictly decreasing family $(B_i)_{i<\lambda}$ of balls $B_i \in \mathcal{C}$ such that $\bigcap_{i<\lambda} B_i = \bigcap \mathcal{C} = \emptyset$. We write $B_i = B_{\gamma_i}(a_i)$ and we define $\varphi\colon X \longrightarrow X$. If $x \in X$ there exists the smallest $\kappa = \kappa(x) < \lambda$ such that $x \notin B_\kappa$, we define $\varphi(x) = a_\kappa$. We show that φ is strictly contracting. Let $x, y \in X, x \neq y$. If $\kappa(x) = \kappa(y)$ then $0 = d(\varphi(x), \varphi(y)) < d(x, y)$. If $\kappa(x) \neq \kappa(y)$, say $\kappa(x) < \kappa(y)$, from $B_{\kappa(x)} \supset B_{\kappa(y)}$ and $x \notin B_{\kappa(x)}, y \in B_{\kappa(x)}$ we get $d(x, y) > \gamma_{\kappa(x)} \geq d(\varphi(x), \varphi(y))$. So φ is strictly contracting. From the definition of φ, it is obvious that φ does not have a fixed point. ■

Brouwer's fixed point theorem, in mathematics, a theorem of algebraic topology that was stated and proved by Brouwer [27, 28]. Inspired by the earlier work of the French mathematician Poincaré, Brouwer investigated the behavior of continuous functions mapping the closed ball of unit radius in n-dimensional Euclidean space into itself.

Theorem 1.47 (Brouwer's Fixed Point Theorem) *Let X be an n-dimensional Euclidean space. Then, any continuous map of $\left\{x \in X\colon \|x\| \leq 1\right\}$ into itself has a fixed point.*

As a consequence, we get

Theorem 1.48 *Any continuous map T of a compact convex K set in n-dimensional Euclidean space X into itself has a fixed point.*

Proof Assume first that $K \subseteq B_X = \left\{x \in X\colon \|x\| \leq 1\right\}$. Define $G\colon B_X \to K$ by taking $G(x)$ to be the unique point $y \in K$ such that $\|x - y\| \leq \|x - z\|$ for all $z \in K$. Such a vector y exists and unique. Note that $G(x) = x = y$ if $x \in K$. Consider $T \circ G\colon B_X \to K$ as a map from B_X into itself. The map $H\colon B_X \to B_X$ defined by $H(x) = T(G(x))$ is continuous because G is continuous. Let $x_n \to x$. We have $\|x_n - G(x_n)\| \leq \|x_n - z\|$ for all $z \in K$. Hence, if y is any limit point of $\{G(x_n)\}$ then $\|x - y\| \leq \|x - z\|$ for all $z \in K$. This proves that $G(x)$ is the only limit of $\{G(x_n)\}$ which lies in the compact set K. Hence $G(x_n) \to G(x)$. By Theorem 1.47

there exists $x \in B_X$ such that $T(G(x)) = x$. Since the range of T is contained in K we get $x \in K$. But then $G(x) = x$ so $T(x) = x$. This proves the theorem when $K \subseteq B_X$. For the general case choose R such that $K \subseteq \Big\{x \in X : \|x\| \le R\Big\}$. Let $K_1 = \{R^{-1}x : x \in K\}$. Then K_1 is a compact convex set and the function $T_1 : K_1 \to K_1$ defined by $T_1(x) = R^{-1}T(Rx)$ is continuous. By the first case there exists $x_1 \in K_1$ such that $R^{-1}T(Rx_1) = x_1$. If $x = Rx_1$ then $T(x) = x$. ■

Remark 1.44 (Kakutani's Example) Theorem 1.47 does not hold in an infinite dimensional Hilbert space:

if $T(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, \cdots)$ then T maps $\Big\{x \in l_2 : \|x\| \le 1\Big\}$ into itself and is continuous. It has no fixed point.

Definition 1.66 A map $T : Y \to X$ where X is a normed space and $Y \subseteq X$ is called compact if $T(Z)$ is relatively compact whenever $Z \subseteq Y$ is bounded.

Brouwer's Theorem was extended to infinite dimensional spaces by Schauder in the following way [174].

Theorem 1.49 (Schauder's Fixed Point Theorem) *Let Y be a closed bounded convex set in a normed space $(X, \|.\|)$ and T a continuous map of Y into itself. If T is compact then it has a fixed point.*

Proof Let $Z \subseteq X$ be compact. Let $\varepsilon > 0$ and $B_\epsilon(x_1), B_\epsilon(x_2), \cdots, B_\epsilon(x_N)$ cover Z where $\{x_1, x_2, \cdots, x_N\} \subseteq Z$. Let $m_i(x) = \max(\varepsilon - \|x - x_i\|, 0)$ and $\varphi(x) = \dfrac{\sum_{i=1}^N m_i(x)x_i}{\sum_{j=1}^N m_j(x)}$ for $x \in Z$. It is obvious that each m_i is continuous and $\sum_{j=1}^{N} m_j(x) > 0$ for all $x \in Z$. Hence φ is continuous. If $x \in Z$ then $m_i(x) \neq 0$ implies $\|x - x_i\| < \varepsilon$ and hence $\left\|\sum_{i=1}^{N} m_i(x)(x_i - x)\right\| < \varepsilon \sum_{i=1}^{N} m_i(x)$ which proves that $\|\varphi(x) - x\| < \varepsilon$ ($m_i(x) \neq 0$ for at least one i). Further $\varphi(Z) \subseteq \text{conv}(Z)$.

Let $W = \overline{T(Y)}$. Then W is a compact subset of Y. For each n let $\varphi_n : W \to \text{conv}(W) \subseteq Y$ be a continuous map such that $\|\varphi_n(x) - x\| < \dfrac{1}{n}$ for all $x \in W$ for all n. This is possible by the reasoning above. Let $T_n = \varphi_n \circ T$ so that T_n is a continuous map : $W \to Y$. So there is a finite set $\{x_1^n, x_2^n, \cdots, x_{N_n}^n\} \subseteq W$ such that $\varphi_n(W) \subseteq W_n := \text{span}(\{x_1^n, x_2^n, \cdots, x_{N_n}^n\})$. Let $Y_n = Y \cap W_n$. Then Y_n is a compact convex set in the finite dimensional space W_n. We claim that T_n maps Y_n into itself. First note that $T(Y_n) \subseteq T(Y) \subseteq W$ so $T_n = \varphi_n \circ T$ is defined on Y_n. Also φ_n takes values in $\text{conv}(\{x_1^n, x_2^n, \cdots, x_{N_n}^n\}) \subseteq W_n$ as well as in Y so it takes values in Y_n. By Theorem 1.48 there exists $y_n \in Y_n$ such that $T_n(y_n) = y_n$. Since $y_n \in Y$ and $T(y_n) \in W$ we have $\|\varphi_n(T(y_n)) - T(y_n)\| < \dfrac{1}{n}$ for all n. In other words $\|y_n - T(y_n)\| < \dfrac{1}{n}$ for all n. Since $(T(y_n))_n \subseteq W$ and W is compact there is a subsequence $(T(y_{n_j}))_{n_j}$ converging to some y. Now

$$\|y_{n_j} - y\| \leq \|T(y_{n_j}) - y\| + \|y_{n_j} - T(y_{n_j})\| < \|T(y_{n_j}) - y\| + \frac{1}{n_j} \to 0.$$

This implies $T(y) = y$. ∎

Lemma 1.12 *Let* $Y_0 = \left\{x = (x_n)_{n\geq 1} \in l_2 \colon |x_n| \leq \frac{1}{n} \text{ for all } n \geq 1\right\}$. *Then any continuous map* $T\colon Y_0 \to Y_0$ *has a fixed point.*

Proof We first prove that the parallelepiped Y_0 is compact in l_2. We have $Y_0 = \bigcap_{n\geq 1} Z_n$, $Z_n = \left\{x = (x_m)_{m\geq 1} \in l_2 \colon |x_n| \leq \frac{1}{n}\right\}$. Since the canonical projection $p_n\colon l_2 \to \mathbb{K}$ is continuous, it follows that $Z_n = p_n^{-1}(\overline{B_{\frac{1}{n}}(0)})$ is closed for all $n \geq 1$, and therefore Y_0 is a closed set. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows that for any $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that $\sum_{k=n_\varepsilon}^{\infty} \frac{1}{k^2} \leq \varepsilon$. Since $|p_n(x)| \leq \frac{1}{n}$ for all $x \in Y_0$ and $n \geq 1$, it follows that $\sum_{k=n_\varepsilon}^{\infty} |p_k(x)|^2 \leq \varepsilon$ for all $x \in Y_0$, i.e., Y_0 is relatively compact in l_2. Hence Y_0 is compact.

Let $Y_n = \{(x_1, x_2, \cdots, x_n, 0, 0, \cdots)\colon x \in Y\}$ and define $T_n\colon Y_n \to Y_n$ by $T_n(x) = (y_1, y_2, \cdots, y_n, 0, 0, \cdots)$ where $y = T(x_1, x_2, \cdots, x_n, 0, 0, \cdots)$. Y_n can be identified with compact convex set in $\mathbb{K}^n$ and T_n is continuous, hence it has a fixed point $x^{(n)}$. Since $(x_n)_{n\geq 1} \subseteq Y_0$ and Y_0 is compact in $(l_2, \|.\|_2)$ there is a subsequence $(x_{n_j})_j$ converging to some $x \in Y$. Let $y^n = T(x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}, 0, 0, \cdots)$ so that $x^{(n)} = T_n(x^{(n)}) = (y_1^{(n)}, y_2^{(n)}, \cdots, y_n^{(n)}, 0, 0, \cdots)$. It is clear that $\lim_{n\to\infty}(x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}, 0, 0, \cdots) = x$ so $\lim_{n\to\infty} y^{(n)} = T(x)$. Hence $x = \lim_{j\to\infty} x^{(n_j)} = \lim_{j\to\infty}(y_1^{(n_j)}, y_2^{(n_j)}, \cdots, y_n^{(n_j)}, 0, 0, \cdots) = \lim_{j\to\infty} y^{(n_j)} = T(x)$. ∎

Lemma 1.13 *If* Z *is a closed convex of* Y_0 *then every continuous map of* Z *into itself has a fixed point.*

Proof For each $x \in Y_0$ there is a unique point $P(x) \in Z$ closet to x and the map $P\colon Y_0 \to Z$ is continuous. If $T\colon Z \to Z$ is continuous then $G\colon Y_0 \to Y_0$ defined by $G = T \circ P$ is continuous. Hence by Lemma 1.12 there exists $x \in Y_0$ such that $T(P(x)) = x$. Since the range of T is contained in Z we see that $x = T(P(x)) \in Z$. But then $P(x) = x$ so $x = T(x)$. ∎

Proposition 1.46 *Let Y be a compact convex set in a locally convex topological vector space (X, τ). If Y has at least two points and $T: Y \to Y$ is continuous then there is a proper subset Y_1 of Y such that $T(Y_1) \subseteq Y_1$ and Y_1 is also compact and convex.*

Proof We reduce the proof to the case when the topology τ of X is replaced by the weak topology. We introduce an ordering for subsets of X' as follows: $Z \leq W$ if for any $\psi \in Z$ and $\varepsilon > 0$ there exists a finite subset $\varphi_1, \varphi_2, \cdots, \varphi_k$ of W and $\delta > 0$ such that $x, y \in Y$ and $|\varphi_i(x) - \varphi_i(y)| < \delta, 1 \leq i \leq k$ imply $|\psi(T(x)) - \psi(T(y))| < \varepsilon$. We observe that if $Z \leq W$ and $\varphi(x) = \varphi(y)$ for all $\varphi \in Z$ then $\psi(T(x)) = \psi(T(y))$. We claim that for any $\psi \in Z$ there exists a countable family $W = \{\varphi_1, \varphi_2, \cdots\}$ such that $\{\psi\} \leq W$. For this let $\varepsilon > 0$. First note that T is weak-weak continuous and Y is compact convex in weak topology. By uniform continuity of $\psi \circ T$ on Y with its weak topology, $|\psi(T(x)) - \psi(T(y))| < \varepsilon$ if $x - y$ belongs to a suitable weak neighbourhood of θ. Hence there exists $\varphi_1, \varphi_2, \cdots, \varphi_k$ and $\delta > 0$ such that $|\varphi_i(x) - \varphi_i(y)| < \delta, 1 \leq i \leq k$ implies $|\psi(T(x)) - \psi(T(y))| < \varepsilon$. Now vary ε over $\{\frac{1}{n}, n \geq 1\}$ to get a countable set $W \subseteq X'$. For any $\varepsilon > 0$ choose n such that $\frac{1}{n} < \varepsilon$. There exists $\varphi_1, \varphi_2, \cdots, \varphi_k$ and $\delta > 0$ such that $|\varphi_i(x) - \varphi_i(y)| < \delta, 1 \leq i \leq k$ implies $|\psi(T(x)) - \psi(T(y))| < \frac{1}{n} < \varepsilon$. It follows that if $|\varphi(x) - \varphi(y)| < \delta$ for all $\varphi \in W$ then $|\psi(T(x)) - \psi(T(y))| < \varepsilon$. Hence $\{\psi\} \leq W$. If we now repeat the argument for each element of W to get another countable set W_1, then repeat the argument for each element of W_1 and so on we end up with countable family W_0 such that with ψ it self, we get a countable subset P of X' which contains ψ with $P \leq P$.

If Y_1 is weakly compact, convex and contained in Y then it is a weakly closed convex set, hence strongly closed. Hence it is a closed convex subset of Y in the strong (i.e., original) topology, hence strongly compact also. Thus, we may and do assume that the topology τ of X is the weak topology. Now suppose $x, y \in Y, x \neq y$. Choose ψ such that $\psi(x) \neq \psi(y)$. Let $P = \{\psi_1 = \psi, \psi_2, \cdots\}$ be a countable subset of X' containing ψ such that $P \leq P$. Now $\psi_n(Y)$ is compact for each $n \geq 1$. Because if $Q = \{\alpha_1\psi, \alpha_2\psi_2, \cdots\}$ with each $\alpha_n > 0$ then $Q \leq Q$, we may suppose $|\psi_n(z)| \leq \frac{1}{n}$ for all $n \geq 1$, for all $z \in Y$. Define $G: Y \to l_2$ by $G(z) = (\psi_n(z))_{n\geq 1}$. G is continuous and its range S is contained in $Y_0 = \left\{x = (x_n)_{n\geq 1} \in l_2 : |x_n| \leq \frac{1}{n} \text{ for all } n \geq 1\right\}$. S has at least two points because $\psi(x) \neq \psi(y)$. Let $T_0: S \to S$ be the map $G \circ T \circ G^{-1}$. In other words, if $s \in S$ we pick $z \in Y$ such that $s = G(z)$ and define $T_0(s) = G(T(z))$. To see that this is well defined note that $s = G(z_1) = G(z_2)$ implies $\psi_n(z_1) = \psi_n(z_2)$ for all n which implies $\psi_n(T(z_1)) = \psi_n(T(z_2))$ for all n (because $P \leq P$) so $G(T(z_1)) = G(T(z_2))$ so T_0 is well defined. The fact that $P \leq P$ also implies that if $\psi_n(z_m) \longrightarrow \psi_n(z)$ as $m \longrightarrow \infty$ for each n then $\psi_n(T(z_m)) \longrightarrow \psi_n(T(z))$ for each n. This means T_0 is continuous. Lemma 1.13

shows that T_0 has a fixed point $s_1 \in S$. Let $Y_1 = G^{-1}(\{s_1\})$. Let $z \in Y_1$ so $G(z) = s_1$. Then $s_1 = T_0(s_1) = G(T(z))$. Hence $T(z) \in Y_1$. Thus $T(Y_1) \subseteq Y_1$. Clearly Y_1 is convex. It is a closed subset of S and hence it is compact. ■

Tychonoff extended Brouwer's result to a compact convex subset of a locally convex topological vector space [190].

Theorem 1.50 (Schauder-Tychonoff's Fixed Point Theorem) *Any continuous map T from a compact convex subset Y of a locally convex topological vector space (X, τ) into Y has a fixed point.*

Proof By Proposition 1.46 there is a minimal nonempty compact convex set Y_0 such that $T(Y_0) \subseteq Y_0$ and Y_0 must be a singleton. ■

The following result [98], called Markov-Kakutani fixed point theorem, is powerful in that it determines a single fixed point for a whole family of mappings, while theorems such as the Schauder-Tychonoff fixed point theorem determine conditions on the space such that the restriction on the mapping is minimal, namely that we only require the mapping T to be continuous.

Theorem 1.51 (Markov-Kakutani's Fixed Point Theorem) *Let Y be a compact convex subset of a locally convex topological vector space (X, τ). Let $T_\alpha : Y \to Y (\alpha \in I)$ be a family of continuous mappings that are affine (which means they satisfy the condition $T_\alpha(\sum_{i=1}^{n} \lambda_i x_i) = \sum_{i=1}^{n} \lambda_i T_\alpha(x_i)$ whenever $n \in \mathbb{N}$, $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{n} \lambda_i = 1$). If $T_\alpha \circ T_\beta = T_\beta \circ T_\alpha$ for all $\alpha, \beta \in I$ then there exists $x \in Y$ such that $T_\alpha(x) = x$ for all $\alpha \in I$.*

Proof For each $\alpha \in I$, let $Z_\alpha = \{x \in Y : T_\alpha(x) = x\}$. From the Schauder-Tychonoff fixed point theorem we know that $Z_\alpha \neq \emptyset$. Since T_α is a continuous affine map, it follows that Z_α is compact and convex. So to restate the conclusion of the theorem we must show that $\bigcap_{\alpha \in I} Z_\alpha \neq \emptyset$. Since Y is compact, we have, by Proposition 1.4 that we need only show that $\bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$ for each nonempty finite subset J of I. To this end, let $J = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a nonempty finite subset of I. We shall proceed by induction.

Let x be any element of Z_{α_1} then

$$T_{\alpha_1}(T_{\alpha_2}(x)) = T_{\alpha_2}(T_{\alpha_1}(x)) = T_{\alpha_2}(x).$$

That is, $T_{\alpha_2}(x)$ is a fixed point of T_{α_1} and so $T_{\alpha_2}(x) \in Z_{\alpha_1}$. Thus, $T_{\alpha_2}(Z_{\alpha_1}) \subseteq Z_{\alpha_1}$. Hence, from the Schauder-Tychonoff fixed point theorem, T_{α_2} has a fixed point in Z_{α_1}. Therefore, $Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset$. Now, suppose that

$$Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j} \text{ where, } 1 \leq j \leq n.$$

Let $Z = Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j}$. Then Z is nonempty, compact and convex. Let x be any element of Z and let $1 \leq i \leq j$ then

$$T_{\alpha_i}(T_{\alpha_{j+1}}(x)) = T_{\alpha_{j+1}}(T_{\alpha_i}(x)) = T_{\alpha_{j+1}}(x).$$

That is, $T_{\alpha_{j+1}}(x)$ is a fixed point of T_{α_i} and so $T_{\alpha_{j+1}}(x) \in Z_{\alpha_i}$. Since $1 \leq i \leq j$ was arbitrary,

$$T_{\alpha_{j+1}}(x) \in Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j} = Z.$$

Hence, from the Schauder-Tychonoff fixed point theorem, $T_{\alpha_{j+1}}$ has a fixed point in Z. Therefore,

$$Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j} \cap Z_{\alpha_{j+1}} \neq \emptyset.$$

By induction, we see that $\bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$. This completes the proof. ■

We shall need some facts about the Kuratowski measure of noncompactnessμ introduced by Kuratowski [122]. This measure of noncompactness is used by Darbo [40], Furi and Vignoli [61], Nussbaum [136], Petryshyn [150], and others.

The concept of Kuratowski's measure of noncompactness is defined below.

Definition 1.67 Let (X, d) a metric space. If Y is a bounded subset of X (i.e., $\operatorname{diam}(Y) = \sup\{d(x, y) : x, y \in Y\} < \infty$), the Kuratowski measure of noncompactness of Y is defined by

$$\mu(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{i=1}^{n} Y_i \text{ for some } Y_i \text{ with } \operatorname{diam}(Y_i) \leq \delta, 1 \leq i \leq n < \infty\right\}.$$

We give the following properties of μ. For the proofs see [136].

Proposition 1.47 *Let (X, d) be a metric space. If Y is a bounded subset of X, then* $\mu(Y) = \mu(\overline{Y})$.

Proposition 1.48 *Let (X, d) be a complete metric space. Then*

1. *for every bounded subset Y of X, $\mu(Y) = 0$ if and only if $\overline{Y}$ is compact.*
2. *If $(Y_n)_{n \geq 1}$ is a decreasing sequence of closed, bounded nonempty subsets of X and if $\lim_{n \longrightarrow \infty} \mu(Y_n) = 0$, then $Y = \bigcap_{n \geq 1} Y_n$ is compact and nonempty.*

If $(X, \|.\|)$ is a normed space, the norm $\|.\|$ gives a metric on X and one can take the Kuratowski measure of noncompactness μ on X with respect to this metric.

Proposition 1.49 *Let* $(X, \|.\|)$ *be a normed space,* Y, Z *two bounded subsets of* X, $x_0 \in X$ *and* $\lambda \in \mathbb{K}$. *Then*

1. $\mu(\lambda Y) = |\lambda|\mu(Y)$.
2. $\mu(\mathrm{conv}(Y)) = \mu(Y)$.
3. $\mu(Y + Z) \leq \mu(Y) + \mu(Z)$.
4. $\mu(Y \cup \{x_0\}) = \mu(Y)$.

Closely associated with the measure of noncompactness is the concept of k-set contraction.

Definition 1.68 If Y_1 is a subset of a metric space (X_1, d_1), and (X_2, d_2) is a second metric space and $T: Y_1 \to X_2$ is a continuous map, we shall say that T is a k-set-contraction if $\mu_2(T(Z)) \leq \mu_1(Z)$, for all bounded sets $Z \subseteq Y_1$, where μ_i denotes the Kuratowski measure of noncompactness on (X_i, d_i).

Theorem 1.52 (Darbo's Fixed Point Theorem) *Let* Y *be a closed bounded convex set in a Banach space* $(X, \|.\|)$ *and let* $T: Y \to Y$ *be a* k*-set-contraction with* $k < 1$. *Then* T *has a fixed point in* Y *[40].*

There is a more useful generalization of Darbo's fixed point theorem.

Theorem 1.53 *Let* Y *be a closed bounded convex set in a Banach space* $(X, \|.\|)$ *and* $T: Y \to Y$ *a continuous map. Define* $Y_1 = \overline{\mathrm{conv}}(T(Y))$ *and* $Y_n = \overline{\mathrm{conv}}(T(Y_{n-1}))$ *for* $n > 1$ *and assume that if* $\lim\limits_{n \longrightarrow \infty} \mu(Y_n) = 0$ *where* μ *denotes the Kuratowski measure of noncompactness on* X. *Then* T *has a fixed point in* Y.

If T in Theorem 1.53 is a k-set contraction with $k < 1$, then if $\lim\limits_{n \longrightarrow \infty} \mu(Y_n) = 0$, but the conditions of Theorem 1.53 may be satisfied in cases of interest for which T is not a k-set contraction with $k < 1$.

The following result is an extension of Darbo's fixed point theorem [61, 136, 172].

Theorem 1.54 (Sadovskii's Fixed Point Theorem) *Let* Y *be a closed bounded convex set in a Banach space* $(X, \|.\|)$ *and let* $T: Y \to Y$ *be a continuous* μ*-condensing map (i.e.,* $\mu(T(Z)) < \mu(Z)$*, for all bounded sets* $Z \subseteq Y$ *for which* $\mu(Z) > 0$*). Then* T *has a fixed point in* Y.

1.6 Nonexpansive Mappings

Definition 1.69 A mapping T is nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all x, y in its domain.

Definition 1.70 Let X be a Banach space and Y be a nonempty bounded closed convex subset of X. We say that Y has the fixed point property for nonexpansive mapping if for every nonexpansive mapping $T: Y \longrightarrow Y$, Y contains a fixed point x^* (i.e., $T(x^*) = x^*$), X has the fixed point property (***FPP*** for short) if any nonempty bounded closed convex subset of X has the fixed point property for nonexpansive mapping, X has the weak fixed point property (***WFPP*** for short) if any weakly compact convex subset of X has the fixed point property for nonexpansive mapping.

Remark 1.45 For a reflexive Banach space, ***FPP*** and ***WFPP*** are obviously the same.

Definition 1.71 Let Y be a nonempty set. A nonempty subset Y_0 of Y is called invariant under T or T-invariant for a mapping $T: Y \longrightarrow Y$ if $T(Y_0) \subset Y_0$. Let $\mathcal{Y}$ be a class of subsets of Y. We say that an element $Y_0 \in \mathcal{Y}$ is $\mathcal{Y}$-minimal for T if there exists no proper T-invariant subset of Y_0 in the class $\mathcal{Y}$.

We are interested mainly in the case that Y is a subset of a Banach space X and $\mathcal{Y}$ is the class of weakly compact subsets of X or the class of closed convex subsets of X.

Remark 1.46 If Y is a closed convex subset of a Banach space X and $T: Y \longrightarrow Y$, then a decreasing sequence of nonempty, closed, convex, T-invariant sets may be obtained by setting

$$Y_0 = Y \text{ and } Y_{n+1} = \overline{\text{conv}}(T(Y_n)) \;\; \forall n \geq 1.$$

We set

$$\widehat{Y} = \bigcap_{n=1}^{\infty} Y_n.$$

The set $\widehat{Y}$ is closed, convex and T-invariant. But it may be empty. Of course this situation cannot occur if Y is weakly compact.

Proposition 1.50 *If X is a Banach space, $Y \subseteq X$ is a nonempty, weakly compact, convex set and $T: Y \longrightarrow Y$, then there exists a nonempty, closed, convex set $\widehat{Y} \subseteq Y$ which is minimal invariant for T.*

Proof Let Γ be the family of all nonempty, closed, convex subsets of Y which are T-invariant. We order Γ by reverse inclusion, namely if $Y_1, Y_2 \in \Gamma$, then

$$Y_1 \leq Y_2 \Longleftrightarrow Y_2 \subset Y_1.$$

By the finite intersection property for the weak topology, every chain in Γ has an upper bound (namely the intersection of the elements in the chain). So by the Zorn lemma, Γ has a maximal element $\widehat{Y} \in \Gamma$. Evidently $\widehat{Y}$ is T-invariant. ∎

Remark 1.47 Note that if $\widehat{Y} \subseteq Y$ is a nonempty, closed, convex and minimal T-invariant set, then

$$\widehat{Y} = \overline{\text{conv}}(T(\widehat{Y})).$$

If $\widehat{Y} \in \Gamma$ in Proposition 1.50 is a singleton, i.e., $\widehat{Y} = \{y\}$, then

$$T(y) = y,$$

i.e., it is a fixed point of T.

The famous question whether a Banach space has the fixed point property had remained open for a long time. It has been answered in the negative by Sadovski [172] and Alspach [4] who constructed the following examples, respectively.

Examples 1.10

1. Let $X = c_0$ and $Y = \{x \in c_0, \|x\|_\infty \leq 1\}$. Define $T : Y \longrightarrow Y$ by

$$T(x) = (1, x_1, x_2, x_3, \ldots), \text{ for all } x = (x_1, x_2, x_3, \ldots) \in Y.$$

2. Let $X = L^1(0, 1)$ and

$$Y = \left\{ x \in X,\ 0 \leq x(t) \leq 1 \text{ and } \int_0^1 x(t)dt = \frac{1}{2} \right\}.$$

Define $T : Y \longrightarrow Y$ by

$$T(x)(t) = \begin{cases} \min\{1, 2x(2t)\}, & \text{if } 0 \leq t \leq \dfrac{1}{2}, \\ \max\{0, 2x(2t-1) - 1\}, & \text{if } \dfrac{1}{2} < t \leq 1. \end{cases}$$

Then Y is bounded, closed, and convex, and T is an isometry $\|T(x) - T(y)\|_1 = \|x - y\|_1$, for all $x, y \in Y$ and is fixed point free.

Namely, c_0 and $L^1(0, 1)$ do not have the fixed point property. The above two examples suggest that to obtain positive results for the existence of fixed points for nonexpansive mappings, it is necessary to impose some restrictions either on T or on the Banach space X.

The following well-known result is due to Kirk [107].

Theorem 1.55 *Let X be a reflexive Banach space and Y a closed bounded convex subset of X. Let Y have normal structure. If $T : Y \to Y$ is nonexpansive, then T has a fixed point.*

Remark 1.48 Theorem 1.55 remains true if X is any Banach space and Y is a convex weakly compact subset having normal structure.

An immediate consequence of Theorem 1.55 is the following well-known result, which was proved independently by Browder [29], Göhde [69] and Kirk [107].

Theorem 1.56 *Let X be a uniformly convex Banach space and Y a nonempty closed bounded convex subset of X. If $T : Y \to Y$ is nonexpansive, then T has a fixed point.*

Remark 1.49 For nonexpansive maps, no characterization of ***FPP*** or ***WFPP*** seems to be known [21].