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Topology and Approximate Fixed Points

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Topology and Approximate Fixed Points

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Afif Ben Amar:

To the memory of my father, Fathi Ben Amar, who expired on December 23, 2020.

Donal O'Regan:

This book is dedicated to my wife Alice (Ali) and to the memory of my parents Cornelius (Con) and Eileen (Eily) O'Regan.

Preface

A point x is a fixed point of a mapping T if $T(x) = x$. Fixed point theorems assert that (under various conditions) a fixed point exists. For instance, we will be concerned with a (continuous) mapping T of a subset Y of a metric space X (with a metric d) and with points which are almost fixed, in the sense that

$$d(T(x), x) < \varepsilon.$$

We call such a point "ε-fixed." Where convexity is mentioned, we assume that Y is a subset of a topological vector space. There are other concepts of "almost fixed." In general, almost fixed points have usually appeared in a secondary role, in discussions of fixed points. This amounts almost to a reversal of reality. In most proofs of fixed point theorems, the constructive part of the argument yields almost fixed points, and a non-constructive compactness argument then gives the existence of a fixed point. Thus, almost fixed points unlike fixed points can be found numerically. For most applications involving computation, it is important to know just what can be calculated (applications to economics). There are also cases where the existence of a fixed point is non-trivial or uncertain, whereas almost fixed are easily found, so perhaps this means that almost fixed are the natural objects to use.

At the same time, an active branch of current research is devoted to the existence of approximate fixed points for single-valued maps. Basically, given a bounded, closed convex set Y of a topological vector space X and a map $T: Y \to Y$, one wants to find a sequence $(x_n)_n \subseteq Y$ such that

$$x_n - T(x_n) \to \theta.$$

A sequence with this property will be called an approximate fixed point sequence.

The main motivation for this topic is purely mathematical and comes from several instances of the failure of the fixed point property in convex sets that are no longer assumed to be compact. This gives rise to the natural question of whether a given space without the fixed point property might still have the approximate fixed point property. Approximate fixed point results have a lot of applications in many interesting problems. They arise naturally in the study of some problems in economics and game theory, and one can apply them to asymptotic fixed point theory and to study the existence of limiting-weak solutions for differential equations in reflexive Banach spaces.

The book has five main chapters.

In Chap. 1, we present basic notions in topologic space theory and introduce the classical topological vector spaces, locally convex spaces, and ultrametric spaces. Special attention is devoted to weak topology and weak* topology and their properties related to compactness, to l_1-sequences, in particular, Rosenthal's theorem, and the Fréchet-Urysohn property. This includes the most recent work in great detail. Also, we give a brief survey on classical fixed point theorems.

Chapter 2 introduces the reader to the almost fixed point theory in metric spaces (normed spaces). Results on the existence of ε-fixed points and approximate fixed point sequences for different classes of mapping are presented, in particular where there are no fixed points. Special effort is devoted to approximate fixed points of nonexpansive mappings in unbounded sets.

In Chap. 3, we indicate how the fixed point of a strictly contracting self-mapping of a spherically ultrametric space can either be reached or approximated. One of the merits is that we deal with ultrametric spaces having sets of distances that are not necessarily totally ordered, but the results then apply to general kinds of algorithms.

Chapter 4 deals with synthetic approaches to problems of fixed points. We are concerned with the theory of regular-global-inf functions which satisfy conditions weaker than continuity and with an original synthetic approach based on convergence with continuity (by sequence). Some results and applications to fixed point theorems for different classes of mappings and in different classes of topological spaces are discussed.

Chapter 5 is devoted to almost fixed and approximate fixed point theories in topological vector spaces. First, we introduce the notion of the (convexly) almost fixed point property, and by using the KKM principle for the closed and open valued cases, we present existence results for almost fixed points of different classes of lower semicontinuous and upper semicontinuous multifunctions on convex subsets of topological vector spaces and having totally bounded ranges. These results are applied to obtain the most well-known fixed point theorems in analytical fixed point theory. Second, we discuss the approximate fixed point property for a (closed) convex (bounded) subset of topological vector spaces. We present some recent existence results of approximate fixed point nets (approximate fixed point sequences) (weak approximate fixed point sequences) for different classes of mappings (continuous, Lipschitz, sequentially continuous, affine, demicontinuous, strongly continuous, the range is totally bounded) of a (closed) convex (bounded) subset of topological vector spaces. These results are related to the nature of the convex set and to the properties of the ambient space. Applications to asymptotic fixed point theory and the existence of limiting-weak solutions for differential equations in reflexive Banach spaces are given.

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Contents

List of Abbreviations

AFPP	approximate fixed point property
approx. L.f.p.p.	approximate Lipschitz fixed point property
BAPP	bounded approximate fixed point property
c.t.b.	convexly totally bounded
FPP	fixed point property
L.A.E.	Lipschitz absolute extensor
L.A.R.	Lipschitz absolute retract
L.f.p.p.	Lipschitz fixed point property
l.s.c.	lower semicontinuous
r.g.i.	regular-global-inf
s.c.t.b.	strongly convexly totally bounded
SCS	strong convex structure
TCS	Takahashi convex metric space
u.s.c.	upper semicontinuous
WFPP	weak fixed point property

List of Symbols

(X, τ):	a topological space
$B_r(x)$:	the open ball centered at x with radius r
$(I, \prec)$:	a directed set
$(x_\alpha)_{\alpha \in I}$:	a net
$\text{conv}(Y)$:	the convex hull of a set Y
$\overline{\text{conv}}(Y)$:	the closed convex hull of a set Y
$\|.\|$:	a norm
$(X, \|.\|)$:	a normed space with its norm
B_X:	the closed unit ball of X
p_K:	the Minkowski functional of a set K
$\mathcal{S}(\mathbb{R}^m)$:	the Schwartz space
X':	the topological dual of X
$\sigma(X, X')$:	the weak topology on X
"$\rightharpoonup$":	weak convergence
S_X:	the unit sphere of X
$\sigma(X', X)$:	the weak* topology on X'
"$\rightharpoonup^*$":	weak* convergence
X'':	the second topological dual of X
l_1^0:	the subspace of l_1 by elements with only finitely many nonzero coordinates
$B_1(X)$:	the space of real-valued functions on X which are of the first Baire class
$F_\varepsilon(T)$:	the approximate fixed point set of a mapping T
$\text{Fix}(T)$:	the fixed point set of a mapping T
L_c:	the level set
$\text{Gr}(T)$:	the graph of a multifunction T
$\text{KKM}(X, Y)$:	a class of multifunctions
$\mathcal{U}_c^k(Y, Z)$:	an admissible class of multifunctions
$\mathcal{B}$:	a better admissible class of multifunctions
$\sigma(X, Z)$:	the weak topology on X induced by Z

Chapter 1
Basic Concepts

This chapter collects well known concepts and results that will play a major role in constructing approximate fixed point theory in the remaining chapters. We note that we will reference the appropriate source papers after Sect. 1.2.8 (before this subsection well known results are presented so that the book is self contained). A brief introduction on fixed point theory is given at the end of this chapter.

1.1 Topological Spaces

1.1.1 The Notion of Topological Spaces

The topology on a set X is usually defined by specifying its open subsets of X.

Definition 1.1 A topology τ on a set X is a family of subsets of X which satisfies the following conditions:

1. The empty set $\emptyset$ and the whole X are both in τ.
2. τ is closed under finite intersections.
3. τ is closed under arbitrary unions.

The pair (X, τ) is called a topological space.

The sets $Y \in \tau$ are called open sets of X and their complements $Z = X \setminus Y$ are closed of X. A subset of X may be neither closed nor open, or both. A set that is both closed and open is called a clopen set.

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Examples 1.1

(i) Let X any set. Then $\tau = \{\emptyset, X\}$ is a topology on X, called the trivial topology on X.
(ii) At the other extreme of the topological spectrum, if X is any nonempty set, then $\tau = P(X)$ the power set of X, is a topology on X, called the discrete topology on X.
(iii) Let $X = \{a, b\}$, and set $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then τ is a topology on X.
(iv) Let (X, d) be a metric space. Let

$$\tau = \Big\{Y \subseteq X\colon \text{ for all } x \in Y, \text{ there exists } \delta > 0 \text{ such that } B_\delta(x) = \{y \in X\colon d(x, y) < \delta\} \subseteq Y\Big\}.$$

Then τ is a topology, called the metric topology on X induced by d. This is the usual topology one thinks of when dealing with metric spaces, but as we shall see, there can be many more.
(v) Let X be any nonempty set. Then

$$\tau_{cf} = \{\emptyset\} \cup \{Y \subseteq X\colon X \setminus Y \text{ is finite }\}$$

is a topology on X, called the co-finite topology on X.

Definition 1.2 Let (X, τ) be a topological space and $Y \subseteq X$. Then $Y \cap \tau = \{Y \cap U\colon U \in \tau\}$ is called the induced topology on Y.

Definition 1.3 Let (X, τ) be a topological space and $Y \subseteq X$. We define

(i) The interior of a subset $Y \subseteq X$ is the largest open set contained in it. It will be denoted by $\operatorname{int} Y$. Equivalently, $\operatorname{int} Y$ is the union of all open subsets of X contained in Y.
(ii) A point $x \in X$ is a limit point (or accumulation point) of Y if and only if for every open set U containing x, it is true that $U \cap Y$ contains some point distinct from x, i.e., $Y \cap (U \setminus \{x\}) \neq \emptyset$. Note that x need not belong to Y.
(iii) The point $x \in Y$ is an isolated point of Y if there is some open set U such that $U \cap Y = \{x\}$. (In other words, there is some open set containing x but no other points of Y.)
(iv) The closure of a subset Y, written $\overline{Y}$, is the union of Y and its set of limit points,

$$\overline{Y} = Y \cup \{x \in X\colon x \text{ is a limit point of } Y\}.$$

Remark 1.1 It follows from the definition that $x \in \overline{Y}$ if and only if $Y \cap U \neq \emptyset$ for any open set U containing x. Indeed, suppose that $x \in \overline{Y}$ and that U is some open set containing x. Then either $x \in Y$ or x is a limit point of Y (or both), in which case $Y \cap U \neq \emptyset$. On the other hand, suppose that $Y \cap U \neq \emptyset$ for any open set U containing x. Then if x is not an element of Y it is certainly a limit point. Thus $x \in \overline{Y}$.

Proposition 1.1 *Let (X, τ) be a topological space and $Y \subseteq X$. The closure of Y is the smallest closed set containing Y, that is,*

$$\overline{Y} = \bigcap \{Z\colon Z \text{ is closed and } Y \subseteq Z\}.$$

Corollary 1.1 *A subset Y of a topological space is closed if and only if $Y = \overline{Y}$. Moreover, for any subset Y, $\overline{Y} = \overline{\overline{Y}}$.*

Proof If Y is closed, then Y is surely the smallest closed set containing Y. Thus $Y = \overline{Y}$. On the other hand, if $Y = \overline{Y}$ then Y is closed because $\overline{Y}$ is. Now let Y be arbitrary. Then $\overline{Y}$ is closed and so equal to its closure, as above. That is, $\overline{Y} = \overline{\overline{Y}}$. ■

Definition 1.4 Let (X, τ) be a topological space.

1. A subfamily $\mathcal{B}$ of τ is called a base if every open set can be written as a union of sets in $\mathcal{B}$.
2. A subfamily $\mathcal{X}$ is called a subbase if the finite intersections of its sets form a base, i.e. every open set can be written as a union of finite intersections of sets in $\mathcal{X}$.

Examples 1.2

1. The collection $\mathcal{B} = \{(a, b)\colon a, b \in \mathbb{R}, a < b\}$ is a base for the usual topology on $\mathbb{R}$.
2. Let $\mathcal{S}$ be the collection of all semi-infinite intervals of the real line of the forms $(-\infty, a)$, and $(b, +\infty)$, where $a \in \mathbb{R}$. $\mathcal{S}$ is not a base for any topology on $\mathbb{R}$. To show this, suppose it were. Then, for example, $(-\infty, 1)$ and $(0, +\infty)$ would be in the topology generated by $\mathcal{S}$, being unions of a single base element, and so their intersection $(0, 1)$ would be by the axiom 2) of topology. But $(0, 1)$ clearly cannot be written as a union of elements in $\mathcal{S}$.
3. The collection $\mathcal{S}$ is a subbase for the usual topology on $\mathbb{R}$.

Proposition 1.2 *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. $\mathcal{S}$ is a base for a topology τ on X iff the following hold:*

1. *$\mathcal{B}$ covers X, i.e., $\forall x \in X, \exists\, B \in \mathcal{B}$ such that $x \in B$.*
2. *If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $B_3 \in \mathcal{B}$ such that $x \in B_3 \in B_1 \cap B_2$.*

Definition 1.5 Let (X, τ) be a topological space and $\in X$. A subset U of X is called a neighborhood of x if it contains an open set containing the point x. The neighborhood system at x is $\mathcal{N}_x = \{U \subseteq X\colon U \text{ is a neighborhood of } x\}$.

Theorem 1.1 *Let (X, τ) be a topological space, and $x \in X$. Then:*

(*a*) *If $U \in \mathcal{N}_x$, then $x \in U$.*
(*b*) *If $U, V \in \mathcal{N}_x$, then $U \cap V \in \mathcal{N}_x$.*
(*c*) *If $U \in \mathcal{N}_x$, there exists $V \in \mathcal{N}_x$ such that $U \in \mathcal{N}_y$ for each $y \in V$.*
(*d*) *If $U \in \mathcal{N}_x$ and $U \subseteq V$, then $V \subseteq \mathcal{N}_x$.*
(*e*) *$G \subseteq X$ is open if and only if G contains a neighborhood of each of its points.*

Remark 1.2 Conversely, if in a set X a nonempty collection $\mathcal{N}_x$ of subsets of X is assigned to each $x \in X$ so as to satisfy conditions (a) through (d) and if we use (e) to define the notion of an open set, the result is a topology on X in which the neighborhood system at x is precisely $\mathcal{N}_x$.

Definition 1.6 Let (X, τ) be a topological space. A (local) neighborhood base $\mathcal{B}_x$ at a point $x \in X$ (or a fundamental system of neighborhoods of x) is a collection $\mathcal{B}_x \subseteq \mathcal{N}_x$ so that $U \in \mathcal{N}_x$ implies that there exists $B \in \mathcal{B}_x$ so that $B \subseteq U$. We refer to the elements of $\mathcal{B}_x$ as basic neighborhoods of the point x.

Example 1.1 Consider (X, d) be a metric space equipped with the metric topology τ. For each $x \in X$, fix a sequence $(r_n(x))_{n\geq 1}$ of positive real numbers such that $\lim_{n\to\infty} r_n(x) = 0$ and consider $\mathcal{B}_x = \{B_{r_n(x)}(x) \colon n \geq 1\}$. Then $\mathcal{B}_x$ is a neighborhood base at x for each $x \in X$.

Remark 1.3 Let (X, τ) be a topological space, and for each $x \in X$, suppose that $\mathcal{B}_x$ is a neighborhood base at x. Then $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}_x$ is a base for the topology τ on X.

Definition 1.7 If (X, τ) is a topological space and $x \in X$ and $\mathcal{B}$ is a set of open sets, we say that $\mathcal{B}$ is a local base at x if each element of $\mathcal{B}$ includes x and for every open set U that includes x there is some $V \in \mathcal{B}$ such that $V \subseteq U$.

Remark 1.4 If for each $x \in X$ the set $\mathcal{B}_x$ is a local base at x, then $\bigcup_{x\in X} \mathcal{B}_x$ is a base for the topology of X.

Definition 1.8 Let (X, τ) be a topological space.

1. (X, τ) is said to be T_1 if for every $x, y \in X$ such that $x \neq y$, there are neighborhoods U_x of x and U_y of y with $y \notin U_x$ and $x \notin U_y$.
2. (X, τ) is said to be T_2 (or Hausdorff) if for every $x, y \in X$ such that $x \neq y$, there are neighborhoods U_x of x and U_y of y with $U_x \cap U_y = \emptyset$.
   We say that two subsets Y and Z can be separated by τ if there exist $U, V \in \tau$ with $Y \subseteq U$, $Z \subseteq V$ and $U \cap V = \emptyset$.
3. (X, τ) is said to be regular if whenever $Y \subseteq X$ is closed and $x \notin Y$, Y and $\{x\}$ can be separated.
4. (X, τ) is said to be normal if whenever $Y_1, Y_2 \subseteq X$ are closed and disjoint, then Y_1 and Y_2 can be separated.
5. (X, τ) is said to be T_3 if it is T_1 and regular.
6. (X, τ) is said to be T_4 if it is T_1 and normal.

Definition 1.9 Let (X, τ) be a topological space. An open cover of $Y \subseteq X$ is a collection $\mathcal{G} \subseteq \tau$ such that $Y \subseteq \cup_{G\in\mathcal{G}}$

A subset Y of a topological space (X, τ) is said to be compact if every open cover of X admits a finite subcover.

Proposition 1.3 *Suppose (X, τ) is a topological Hausdorff space.*

1. *Any compact set $Y \subseteq X$ is closed.*
2. *If Y is a compact set, then a subset $Z \subseteq Y$ is compact, if and only if Z is closed (in X).*

Proposition 1.4 *For a subset Y of a topological space (X, τ), the following statements are equivalent.*

1. *Y is compact.*
2. *If $(Z_\alpha)_{\alpha \in I}$ is any family of closed sets such that $Y \cap \bigcap_{\alpha \in I} Z_\alpha = \emptyset$, then $Y \cap \bigcap_{\alpha \in J} Z_\alpha = \emptyset$ for some finite subset $J \subseteq I$.*
3. *If $(Z_\alpha)_{\alpha \in I}$ is any family of closed sets such that $Y \cap \bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$, for every finite subset $J \subseteq I$, then $Y \cap \bigcap_{\alpha \in I} Z_\alpha \neq \emptyset$.*

Proof The statements 2. and 3. are contrapositives. We shall show that 1. and 2. are equivalent. The proof rests on the observation that if $(U_\alpha)_\alpha$ is a collection of sets, then $Y \subseteq \bigcup_\alpha U_\alpha$ if and only if $Y \cap \bigcap_\alpha (X \setminus U_\alpha) = \emptyset$. We first show that 1. implies 2. Suppose that Y is compact and let $(Z_\alpha)_{\alpha \in I}$ be a family of closed sets such that $Y \cap \bigcap_{\alpha \in I} Z_\alpha = \emptyset$. Put $U_\alpha = X \setminus Z_\alpha$. Then each U_α is open, and by the above observation, $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$. But then there is a finite set J such that $Y \subseteq \bigcup_{\alpha \in J} U_\alpha$, and so $Y \cap \bigcap_{\alpha \in J} Z_\alpha = \emptyset$, which proves 2.

Now suppose that 2. holds, and let $(U_\alpha)_\alpha$ be an open cover of Y. Then each $X \setminus U_\alpha$ is closed and $Y \cap \bigcap_{\alpha \in I} (X \setminus U_\alpha) = \emptyset$. By 2., there is a finite set J such that $Y \cap \bigcap_{\alpha \in J} (X \setminus U_\alpha) = \emptyset$. This is equivalent to the statement that $Y \subseteq \bigcup_{\alpha \in J} U_\alpha$. Hence Y is compact. ■

Remark 1.5 A topological space (X, τ) is compact if and only if any family of closed sets $(Z_\alpha)_{\alpha \in I}$ in X having the finite intersection property (i.e., $\bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$ for each finite subset J in I) is such $\bigcap_{\alpha \in I} Z_\alpha \neq \emptyset$.

Proposition 1.5 *A nonempty subset Y of a topological space (X, τ) is compact if and only Y is compact with respect to the induced topology, that is, if and only if (Y, τ_Y) is compact. If (X, τ) is Hausdorff then so (Y, τ_Y).*

Proof Suppose first that Y is compact in (X, τ), and let $(G_\alpha)_{\alpha \in I}$ an open cover of Y in (Y, τ_Y). Then each G_α has the form $G_\alpha = Y \cap U_\alpha$ for some $U_\alpha \in \tau$. It follows that $(U_\alpha)_{\alpha \in I}$ is an open cover of Y in (X, τ). By hypothesis, there is a finite subcover, $U_1, \cdots, U_n$, say. But then $G_1, \cdots, G_n$ is an open cover of Y in (Y, τ_Y), that is, (Y, τ_Y) is compact.

Conversely, suppose that (Y, τ_Y) is compact. Let $(U_\alpha)_{\alpha \in I}$ be an open cover of Y in (X, τ). Set $G_\alpha = Y \cap U_\alpha$. Then $(G_\alpha)_{\alpha \in I}$ is an open cover of (Y, τ_Y). By hypothesis, there is a finite subcover, say, $G_1, \cdots, G_m$. Clearly, $U_1, \cdots, U_m$, is an open cover for Y in (X, τ). That is, Y is compact in (X, τ).

Suppose that (X, τ) is Hausdorff, and let x, y be any two distinct points of Y. Then there is a pair of disjoint open sets U, V in X such that $x \in U$ and $y \in V$. Evidently, $G_1 = Y \cap U$ and $G_2 = Y \cap V$ are open in (Y, τ_Y), are disjoint and $x \in G_1$ and $y \in G_2$. Hence (Y, τ_Y) is Hausdorff, as required. ■

Theorem 1.2 *Let (X, d) be a metric space. Then X, equipped with the metric topology is T_4.*

Theorem 1.3 *Let (X, τ) be a compact, Hausdorff space. Then (X, τ) is T_4.*

Proof Let $Y, Z \subseteq X$ be two closed sets with $Y \cap Z = \emptyset$. We need to find two open sets $U, V \subseteq X$, with $Y \subseteq U$, $Z \subseteq V$, and $U \cap V = \emptyset$. Assume first that Z is a singleton, $Z = \{z\}$.

For every $y \in Y$ we find open sets U_y and V_y, such that $U_y \ni y$, $V_y \ni z$, and $U_y \cap V_y = \emptyset$. Using Proposition 1.3 we know that Y is compact, and since we clearly have $Y \subseteq \bigcup_{y \in Y} U_y$, there exist $y_1, \cdots, y_n \in Y$ such that $\bigcup_{i=1}^{n} U_{y_i} \supseteq Y$. Then we are done by taking $U = \bigcup_{i=1}^{n} U_{y_i}$ and $V = \bigcap_{i=1}^{n} V_{y_i}$.

Having proven the above particular case, we proceed now with the general case. For every $z \in Z$, we use the particular case to find two open sets U_z and V_z with $U_z \supseteq Y$, $V_z \ni z$, and $U_z \cap V_z = \emptyset$. Arguing as above, the set Z is compact, and we have $Z \subseteq \bigcup_{z \in Z} V_z$, so there exists $z_1, \cdots, z_n \in Z$, such that $\bigcap_{i=1}^{n} V_{z_i} \supseteq Z$. Then we are done by taking $U = \bigcap_{i=1}^{n} U_{z_i}$ and $V = \bigcup_{i=1}^{n} V_{z_i}$. ■

Definition 1.10 A topological space (X, τ) is said to be separable if it admits a countable dense subset.

Proposition 1.6 *Let (X, d) be a compact metric space. Then (X, d) is separable.*

Proof For each $n \geq 1$, the collection $\mathcal{G}_n = \{B_{\frac{1}{n}}(x) \colon x \in X\}$ is an open cover of X. Since X is compact, we can find a finite subcover $\{B_{\frac{1}{n}}(x_{(j,n)}) \colon 1 \leq j \leq k_n\}$ of X. It is then clear that if $x \in X$, there exists $1 \leq j \leq k_n$ so that $d(x, x_{(j,n)}) < \frac{1}{n}$. As such, the collection

$$\mathcal{D} = \{x_{(j,n)} : 1 \le j \le k_n, 1 \le n\}$$

is a countable, dense set in X, proving that (X, d) is separable. ■

1.1.2 Comparison of Topologies

Any set X may carry several different topologies.

Definition 1.11 Let τ, τ' be two topologies on the same set X. We say that τ is coarser (or weaker) than τ', in symbols $\tau \subseteq \tau'$, if for every subset of X which is open for τ is also open for τ', or equivalently, if for every neighborhood of a point in X with respect to τ is also a neighborhood of that same point in the topology τ'. In this case τ' is said to be finer (or stronger) than τ'.

Two topologies τ and τ' on the same set X coincide when they give the same open sets or the same closed sets or the same neighborhoods of each point, equivalently, when τ is both coarser and finer than τ'.

Two basis of neighborhoods of a set are equivalent when they define the same topology.

Remark 1.6 Given two topologies on the same set, it may very well happen that no-one is finer than the other. If it is possible to establish which one is finer, then we say that the two topologies are comparable.

Example 1.2 The cofinite topology τ_c on $\mathbb{R}$, i.e., $\tau_c = \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } \mathbb{R} \setminus U \text{is finite}\}$, and the topology τ_i having $\{(-\infty, a) : a \in \mathbb{R}\}$ as a basis are incomparable. In fact, it is easy to see that $\tau_i = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ as these are the union of sets in the given basis. In particular, we have that $\mathbb{R} \setminus \{0\}$ is in τ_c but not τ_i. Moreover, we have that $(-\infty, 0)$ is in τ_i but not τ_c. Hence, τ_c and τ_i are incomparable.

Proposition 1.7 *If τ_1, τ_2 are Hausdorff topologies on a set X such that τ_2 is finer than τ_1 and such that (X, τ_2) is compact, then $\tau_1 = \tau_2$.*

Proof Let Y a τ_2-closed set. Since (X, τ_2) is compact then Y is τ_2-compact. Since $\tau_1 \subseteq \tau_2$ it follows that Y is τ_1-compact (any τ_1-open cover of Y is also a τ_2-open cover of Y and has a finite subcover). Since τ_1 is Hausdorff and Y is τ_1-compact then it is also τ_1-closed, which completes the proof (we showed that every τ_2-closed set is a τ_1-closed set). ■

Definition 1.12 Let X be a set and let F be a family of mappings from X into topological spaces:

$$F = \{f_\alpha : X \to (Y_\alpha, \tau_\alpha) \ : \ \alpha \in I\}.$$

Let τ be the topology generated by the subbase

$$\{f_\alpha^{-1}(V) \ : V \in \tau_\alpha, \alpha \in I\}.$$

Then τ is the weakest topology on X for which all the f_α are continuous maps (it is the intersection of all topologies having this property). It is called the weak topology induced by F, or the F-topology of X.

Proposition 1.8 *Let F be a family of mappings $X \to (Y_\alpha, \tau_\alpha)$ where X is a set and each (Y_α, τ_α) is a Hausdorff topological space. Suppose F separates points in X i.e., for any $x, y \in X$ with $x \neq y$, there is some $f_\alpha \in F$ such that $f_\alpha(x) \neq f_\alpha(y)$. Then the F-topology on X is Hausdorff.*

Proof Suppose that $x, y \in X$, with $x \neq y$. By hypothesis, there is some $\alpha \in I$ such that $nf_\alpha(x) \neq f_\alpha(y)$. Since (Y_α, τ_α) is Hausdorff, there exist elements $U, V \in \tau_\alpha$ such that $f_\alpha(x) \in U$, $f_\alpha(y) \in V$ and $U \cap V = \emptyset$. But then $f_\alpha^{-1}(U)$ and $f_\alpha^{-1}(V)$ are open with respect to F-topology and $x \in f_\alpha^{-1}(U)$, $y \in f_\alpha^{-1}(V)$ and $f_\alpha^{-1}(U) \cap f_\alpha^{-1}(V) = \emptyset$. ■

Definition 1.13 Let (X, τ) be a topological space. X is called metrizable if it is compatible with some metric d (i.e., τ is generated by the open balls $B_r(x) = \{y \in X, d(x, y) < r\}$).

Proposition 1.9 *Let (X, τ) be a compact topological space. If there is a sequence $\{f_n, n \in \mathbb{N}\}$ of continuous real-valued functions that separates points in X then X is metrizable.*

Proof Since (X, τ) is compact and the f_n are continuous then they are bounded. Thus, we can normalize them such that $\|f_n\|_\infty = \sup_{x \in X} |f_n(x)| \leq 1$. Define:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}.$$

This series converges. In fact, it converges uniformly on $X \times X$ hence the limit is continuous. Because the f_n separate points $d(x, y) = 0$ iff $x = 0$. d is also symmetric and satisfies the triangle inequality.

Thus d is a metric and we denote by τ_d the topology induced by this metric. We need to show that $\tau_d = \tau$. Consider the metric balls:

$$B_r(x) = \{y \in X, d(x, y) < r\}.$$

Since d is τ-continuous on $X \times X$, these balls are τ-open and

$$\tau_d \subseteq \tau.$$

By Proposition 1.7, since τ is compact and τ_d is Hausdorff (like any metric space) then $\tau = \tau_d$. ■

1.1.3 Nets and Convergence in Topology

Nets generalize the notion of sequences so that certain familiar results relating to continuity and compactness of sequences in metric spaces can be proved in arbitrary topological spaces. We now expand our notion of "sequence" $(x_n)_n$ to something for which the index n need not be a natural number, but can instead take values in a (possibly uncountable) partially ordered set.

Definition 1.14 A directed set $(I, \prec)$ consists of a set I with a partial order $\prec$ such that for every pair $\alpha, \beta \in I$, there exists an element $\gamma \in I$ with $\gamma \succ \alpha$ and $\gamma \succ \beta$.

Examples 1.3

1. The natural numbers $\mathbb{N}$ with the relation $\leq$ define a directed set $(I, \prec) = (\mathbb{N}, \leq)$.
2. If (X, τ) is a topological space and $x \in X$, one can define a directed set $(I, \prec)$ where I is the set of all neighborhoods of x in X, and $U \prec V$ for $U, V \in I$ means $V \subseteq U$. This is a directed set because given any pair of neighborhoods $U, V \subseteq X$ of x, the intersection $U \cap V$ is also a neighborhood of x and thus defines an element of I with $U \cap V \subseteq U$ and $U \cap V \subseteq V$. Note that neither of U and V need be contained in the other, so they might not satisfy either $U \prec V$ or $V \prec U$, hence $\prec$ is only a partial order, not a total order. Moreover, for most of the topological spaces we are likely to consider, I is uncountably infinite.
3. Let (X, τ) a topological space and let $x \in X$. Then the set $I_x = \{U \in \tau, x \in U\}$ is a directed set when equipped with the either the subset relation $\subseteq$, or more usefully the superset relation $\supseteq$.
4. If $(I_1, \prec_1)$ and $(I_2, \prec_2)$ are directed sets, then $(I_1 \times I_2, \prec)$ is a directed set where $\prec$ is defined by

$$(a, b) \prec (x, y) \text{ if and only if } a \prec_1 x \text{ and } b \prec_2 y.$$

5. Let I denote the set of all finite partitions of $[0, 1]$, partially ordered by inclusion (i.e., refinement). Let f be a continuous function on $[0, 1]$, then to $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\} \in I$, we associate the quantity $L_P(f) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$. The map $f \mapsto L_P(f)$ is a net (I is a directed set), and from Calculus, $\lim_{P \in I} L_P(f) = \int_0^1 f(x)dx$.

Definition 1.15 Let $\mathcal{P}$ be a property of elements of a directed set $(I, \prec)$. We shall say that:

1. $\mathcal{P}$ holds eventually if there exists $\alpha_0 \in I$ such that $\mathcal{P}$ holds for each $\alpha \succ \alpha_0$,
2. $\mathcal{P}$ holds frequently if for each $\alpha \in I$ there exists $\beta \succ \alpha$ satisfying $\mathcal{P}$.

Thus "eventually" means "for all successors of some element", and "frequently" means "for arbitrary large elements".

Definition 1.16 Given a topological space (X, τ), a net $(x_\alpha)_{\alpha \in I}$ is a function $I \to X : \alpha \longmapsto x_\alpha$, where $(I, \prec)$ is a directed set.

Definition 1.17 We say that a net $(x_\alpha)_{\alpha \in I}$ in X converges to $x \in X$ if for every neighborhood $U \subseteq X$ of x, there exists $\alpha_0 \in I$ such that $x_\alpha \in U$ for every $\alpha \succ \alpha_0$.

Example 1.3 A net $(x_\alpha)_{\alpha \in I}$ with $(I, \prec) = (\mathbb{N}, \leq)$ is simply a sequence, and convergence of this net to x means the same thing as convergence of the sequence.

Definition 1.18 A net $(x_\alpha)_{\alpha \in I}$ has a cluster point (also known as accumulation point) at $x \in X$ if for every neighborhood $U \subseteq X$ of x and for every $\alpha_0 \in I$, there exists $\alpha \succ \alpha_0$ with $x_\alpha \in U$.

Definition 1.19 A net $(y_\beta)_{\beta \in J}$ is a subnet of the net $(x_\alpha)_{\alpha \in I}$ if $y_\beta = x_{\phi(\beta)}$ for some order preserving function $\phi : J \to I$ such that for every $\alpha_0 \in I$, there exists an element $\beta_0 \in J$ for which $\beta \succ \beta_0$ implies $\phi(\beta) \succ \alpha_0$ (cofinal).

Example 1.4 If $(x_n)_n$ is a sequence, any subsequence $(x_{k_n})_n$ becomes a subnet $(y_\beta)_{\beta \in J}$ of the net $(x_n)_{n \in \mathbb{N}}$ by setting $J = \mathbb{N}$ and $\phi : \mathbb{N} \to \mathbb{N} : n \longmapsto k_n$. Note that this remains true if we slightly relax our notion of subsequences so that (k_n) need not be a monotone increasing sequence in $\mathbb{N}$ but satisfies $k_n \to \infty$ as $n \to \infty$. Conversely, any subnet $(y_\beta)_{\beta \in J}$ of a sequence $(x_n)_{n \in \mathbb{N}}$ with $(J, \prec) = (\mathbb{N}, \leq)$ is also a subsequence in this slightly relaxed sense, and can then be reduced to a subsequence in the usual sense by skipping some terms (so that the function $n \longmapsto k_n$ becomes strictly increasing). Note however that a subnet of a sequence need not be a subsequence in general, i.e., it is possible to define a subnet $(y_\beta)_{\beta \in J}$ of a sequence $(x_n)_{n \in \mathbb{N}}$ such that J is uncountable, and one can derive concrete examples of such objects.

Remark 1.7 If $(x_\alpha)_{\alpha \in I}$ is a net converging to x, then every subnet $(x_{\phi(\beta)})_{\beta \in J}$ also converges to x.

Theorem 1.4 *Let Y be a subset of a topological space (X, τ). Then $x \in \overline{Y}$ if and only if there is a net $(x_\alpha)_{\alpha \in I}$ with $x_\alpha \in Y$ such that $x_\alpha \longrightarrow x$.*

Proof We know that a point $x \in X$ belongs to $\overline{Y}$ if and only if every neighborhood of x meets Y. Suppose then that $(x_\alpha)_{\alpha \in I}$ is a net in Y such that $x_\alpha \longrightarrow x$. By definition of convergence, $(x_\alpha)_{\alpha \in I}$ is eventually in every neighborhood of x, so certainly $x \in \overline{Y}$.

Suppose, on the other hand, that $x \in \overline{Y}$. Let $\mathcal{N}_x$ be the collection of all neighborhoods of x ordered by reverse inclusion. Then $\mathcal{N}_x$ is a directed set. We know that for each $V \in \mathcal{N}_x$ the set $V \cap Y$ is nonempty so let x_V be any element of $V \cap Y$. Then $x_V \longrightarrow x$. ■

Lemma 1.1 *Let X be a set, and $(x_\alpha)_{\alpha\in I}$ a net in X. Let $\mathcal{B}$ be a family of subsets of X, satisfying*

1. *x_α is contained frequently in each element of $\mathcal{B}$, and*
2. *the intersection of any two elements of $\mathcal{B}$ contains an element of $\mathcal{B}$.*

Then $(x_\alpha)_{\alpha\in I}$ admits a subnet which is eventually contained in each element of $\mathcal{B}$.

Proof Clearly, the family $\mathcal{B}$ is directed by the inverse inclusion. Consider the set

$$J = \{(\alpha, B) \in I \times \mathcal{B} \colon x_\alpha \in B\}$$

equipped with the coordinate-wise pre-ordering. It is easy to see that J is a directed set. The function $\phi\colon J \to I$, defined by $\phi(\alpha, B) = \alpha$, is nondecreasing and onto, and hence tends to infinity. Consequently, $(x_{\phi(\alpha,B)})_{(\alpha,B)}$ is a subnet of $(x_\alpha)_{\alpha\in I}$. Moreover, given $A \in B$, fix $\alpha_0 \in I$ so that $x_{\alpha_0} \in A$, and observe that if $(\alpha, B) \succ (\alpha_0, A)$ then $x_{\phi(\beta,B)} = x_\beta \in B \subseteq A$. This completes the proof. ■

In metric spaces, a standard theorem states that sequential continuity is equivalent to continuity. In arbitrary topological spaces this no longer true, but we have the following generalization.

Theorem 1.5 *For any two topological spaces X and Y, a map $T\colon X \to Y$ is continuous if and only if for every net $(x_\alpha)_{\alpha\in I}$ in X converging to a point $x \in X$, the net $(T(x_\alpha))_{\alpha\in I}$ in Y converges to $T(x)$.*

Proposition 1.10 *A point x of a topological space (X, τ) is a cluster point of a net $(x_\alpha)_{\alpha\in I}$ in X if and only if there exists a subnet $(x_{\phi(\beta)})_{\beta\in J}$ that converges to x.*

Proof If $(x_{\phi(\beta)})_{\beta\in J}$ is a subnet of $(x_\alpha)_{\alpha\in I}$ converging to x, then for every neighborhood $U \subseteq X$ of x, there exists $\beta_0 \in J$ such that $x_{\phi(\beta)} \in U$ for every $\beta \succ \beta_0$. Then for any $\alpha_0 \in I$, the definition of a subnet implies that we can find $\beta_1 \in J$ with $\phi(\beta) \succ \alpha_0$ for all $\beta \succ \beta_1$, and since J is a directed set, there exists $\beta_2 \in J$ with $\beta_2 \succ \beta_0$ and $\beta_2 \succ \beta_1$. It follows that for $\alpha = \phi(\beta_2), \alpha \succ \alpha_0$ and $x_\alpha = x_{\phi(\beta_2)} \in U$, thus x is a cluster point of $(x_\alpha)_{\alpha\in I}$.

Conversely, if x is a cluster point of $(x_\alpha)_{\alpha\in I}$, we can define a convergent subnet as follows. Define a new directed set

$$J = I \times \{\text{ neighborhoods of } x \text{ in } X\},$$

with the partial order $(\alpha, U) \prec (\beta, V)$ defined to mean both $\alpha \prec \beta$ and $V \subseteq U$. Then for each $(\beta, U) \in J$, the fact that x is a cluster point implies that we can choose $\phi(\beta, U) \in I$ to be any $\alpha \in I$ such that $\alpha \succ \beta$ and $x_\alpha \in U$. This defines a function $\phi\colon J \to I$ such that for any $\alpha_0 \in I$ and any neighborhood $U_0 \subseteq X$ of x, every $(\beta, U) \in J$ with $(\beta, U) \succ (\alpha_0, U_0)$ satisfies $\phi(\beta, U) \succ \beta \succ \alpha_0$, hence $(x_{\phi(\beta,U)})_{\beta\in J}$ is a subnet of $(x_\alpha)_{\alpha\in I}$. Moreover, for any neighborhood $U \subseteq X$ of x, we can choose an arbitrary $\alpha_0 \in I$ and observe that

$$(\beta, V) \succ (\alpha_0, U) \Longrightarrow x_{\phi(\beta,V)} \in V \subseteq U,$$

thus $(x_{\phi(\beta,U)})_{(\beta,U)\in J}$ converges to x. ■

Theorem 1.6 *A topological space (X, τ) is compact if and only if every net in X has a convergent subnet.*

Proof Suppose X is compact but there exists a net $(x_\alpha)_{\alpha\in I}$ in X with no cluster point. The fact that every $x \in X$ is not a cluster point of $(x_\alpha)_{\alpha\in I}$ then means that we can find for each $x \in X$ an open neighborhood $U_x \subseteq X$ of x and an index $\alpha_x \in I$ such that $x_{\alpha_x} \notin U_x$ for all $\alpha \succ \alpha_x$. But $(U_x)_{x\in X}$ is then an open cover of X and therefore has a finite subcover, meaning there is a finite subset $x_1, \cdots, x_N \in X$ such that $X = \bigcup_{n=1}^{N} U_{x_n}$. Since $(I, \prec)$ is a directed set, there also exists an element $\beta \in I$ such that

$$\beta \succ \alpha_{x_n} \text{ for each } n = 1, \cdots, N.$$

Then $x_\beta \notin U_{x_n}$ for every $n = 1, \cdots, N$, but since the sets U_{x_n} cover X, this is a contradiction.

Conversely, suppose that every net in X has a cluster point, but that X has a collection O of open sets that cover X such that no finite subcollection in O covers X. Define a directed set where I is the set of all finite subcollections of O, with the ordering relation defined by inclusion, i.e., for $A, B \in I$, $A \prec B$ means $A \subseteq B$. Note that $(I, \prec)$ is a directed set since for any two $A, B \in I$, we have $A \cup B \in I$ with $A \cup B \supset A$ and $A \cup B \supset B$. By assumption, none of the unions $\bigcup_{U\in A}$ for $A \in I$ cover X, so we can choose a point

$$x_A \in X \setminus \bigcup_{U\in A} U \tag{1.1}$$

for each $A \in I$, thus defining a net $(x_A)_{A\in I}$. Then $(x_A)_{A\in I}$ has a cluster point $x \in X$. Since the sets in O cover X, we have $x \in V$ for some $V \in O$, and the collection $\{V\}$ is an element of I, hence there exists $A \succ \{V\}$ such that $x_A \in V$. But this means A is a finite subcollection of O that includes V, thus contradicting (1.1). ■

Theorem 1.7 *Let X be a set and let τ_1 and τ_2 be topologies on X. Then the following are equivalent*

1. $\tau_1 = \tau_2$.
2. *Every $(x_\alpha)_{\alpha\in I}$ in X, converges in τ_1 if and only if it converges in τ_2.*

Proposition 1.11 *A topological space (X, τ) is Hausdorff if and only if no net has two distinct limits.*

Proof Suppose (X, τ) is Hausdorff and consider a net $(x_\alpha)_{\alpha\in I}$. Suppose for contradiction that x and y are distinct limits of $(x_\alpha)_{\alpha\in I}$. Take disjoint neighborhoods U of x and V of y. By definition of convergence, there is a α_x such that $x_\alpha \in U$ for all $\alpha \succ \alpha_x$ and a α_y such that $x_\alpha \in V$ for all $\alpha \succ \alpha_y$. In particular we have $x_\alpha \in U \cap V$ for an upper bound α of α_x and α_y in the directed set I, contradicting the disjointness of U and V. Thus $(x_\alpha)_{\alpha\in I}$ cannot have two distinct limits.

Conversely, suppose that (X, τ) is not Hausdorff, so there are two distinct points x and y such that any neighborhood of x intersects any neighborhood of y. So there is a net $(x_{(U,V)})_{\mathcal{N}(x)\times\mathcal{N}(y)}$ such that

$$x_{(U,V)} \in U \cap V$$

for neighborhoods U of x and V of y. Take any neighborhood U_0 of x and any $(U, V) \in \mathcal{N}(x) \times \mathcal{N}(y)$ with $(U, V) \succ (U_0, X)$. By definition we have $U \subseteq U_0$ and thus $x_{(U,V)} \in U \cap V \subseteq U_0$. This proves that $x_{(U,V)} \to x$ and we can similarly show that $x_{(U,V)} \to y$. So the net $(x_{(U,V)})_{\mathcal{N}(x)\times\mathcal{N}(y)}$ has two distinct limits, as required. ■

1.2 Topological Vector Spaces

1.2.1 Linear Topologies

Definition 1.20 Let X be a vector space. A linear topology on X is a topology τ such that the maps

$$X \times X \ni (x, y) \mapsto x + y \in X \tag{1.2}$$

$$\mathbb{K} \times X \ni (\alpha, x) \mapsto \alpha x \in X \tag{1.3}$$

are continuous. For the map (1.2) we use the product topology $\tau \times \tau$. For the map (1.3) we use the product topology $\tau_{\mathbb{K}} \times \tau$, where $\tau_{\mathbb{K}}$ is the standard topology on $\mathbb{K}$.

A topological vector space is a pair (X, τ) consisting of a vector space X and a Hausdorff linear topology τ on X.

Remark 1.8 If (X, τ) is a topological vector space then it is clear from Definition 1.20 that $\sum_{k=1}^{N} \lambda_k^{(n)} x_k^{(n)} \to \sum_{k=1}^{N} \lambda_k x_k$ as $n \to \infty$ with respect to τ if for each $k = 1, \cdots, N$ as $n \to \infty$ we have $\lambda_k^{(n)} \to \lambda_k$ with respect to the euclidean topology on $\mathbb{K}$ and $x_k^{(n)} \to x_k$ with respect to τ.

Examples 1.4

1. Every vector space X over $\mathbb{K}$ endowed with the trivial topology is a topological vector space.
2. The field $\mathbb{K}$, viewed as a vector space over itself, becomes a topological vector space, when equipped with the standard (euclidean) topology $\tau_{\mathbb{K}}$.
3. Every normed vector space endowed with the topology given by the metric induced by the norm is a topological vector space.

Proposition 1.12 *Every vector space X over $\mathbb{K}$ endowed with the discrete topology is not a topological vector space unless $X = \{\theta\}$.*

Proof Assume by a contradiction that it is a topological vector space and take $\theta \neq x \in X$. The sequence $\alpha_n = \frac{1}{n}$ in $\mathbb{K}$ converges to 0 in the euclidean topology. Therefore, since the scalar multiplication is continuous, $\alpha_n x \to \theta$, i.e., for any neighborhood U of θ in X there exists $m \in \mathbb{N}$ such that $\alpha_n x \in U$ for all $n \geq m$. In particular, we can take $U = \{\theta\}$ since it is itself open in the discrete topology. Hence, $\alpha_m x = \theta$, which implies that $x = \theta$ and so a contradiction. ■

Remark 1.9 In terms of net convergence, the continuity requirements for a linear topology on X read:

- Whenever (x_α) and (y_α) are nets in X, such that $x_\alpha \to x$ and $y_\alpha \to y$, it follows that $x_\alpha + y_\alpha \to x + y$.
- Whenever (λ_α) and (x_α) are nets in $\mathbb{K}$ and X, respectively, such that $\lambda_\alpha \to \lambda$ (in $\mathbb{K}$) and $x_\alpha \to x$ (in X), it follows that $\lambda_\alpha x_\alpha \to \lambda x$.

Example 1.5 Let I be an arbitrary nonempty set. The product space $\mathbb{K}^I$ (defined as the space of all functions $I \to \mathbb{K}$) is obviously a vector space (with pointwise addition and scalar multiplication). The product topology turns $\mathbb{K}^I$ into a topological vector space.

Remark 1.10 If X is a vector space, then the following maps are continuous with respect to any linear topology on X :

- The translations $T_y : X \to X$, $y \in X$, defined by $T_y(x) = x + y$.
- The dilations $D_\alpha : X \to X$, $\alpha \in \mathbb{K}$, defined by $D_\alpha(x) = \alpha x$.

If τ is a linear topology on a vector space X, then τ is translation invariant. That is, a subset $U \subseteq X$ is open if and only if the translation $y + U$ is open for all $y \in X$. Indeed, the continuity of addition implies that for each $y \in X$, the translation $x \mapsto y+x$ is a linear homeomorphism. In particular, every neighborhood of y is of the form $y + U$, where U is a neighborhood of zero. In other words, the neighborhood system at zero determines the neighborhood system at every point of X by translation. Also note that the dilation $x \mapsto \alpha x$ is a linear homeomorphism for any $\alpha \neq 0$. In particular, if U is a neighborhood of zero, then so is αU for all $\alpha \neq 0$.

Example 1.6 If a metric d on a vector space X is translation invariant, i.e., $d(x + z, y + z) = d(x, y)$ for all $x, y \in X$ (i.e., the metric induced by a norm), then the topology induced by the metric is translation invariant and the addition is always continuous. However, the multiplication by scalars does not need to be necessarily continuous (take d to be the discrete metric, then the topology generated by the metric is the discrete topology which is not compatible with the scalar multiplication).

Proposition 1.13 *If Y is a linear subspace of a topological vector space (X, τ), then so its closure $\overline{Y}$. In particular, any maximal proper subspace is either dense or closed.*

Proof We must show that if $x, y \in \overline{Y}$ and $\lambda \in \mathbb{K}$, then $\lambda x + y \in \overline{Y}$. There are nets (x_α) and (y_α) in Y, such that $x_\alpha \longrightarrow x$ and $y_\alpha \longrightarrow y$. By Remark 1.9, we deduce that $tx_\alpha \longrightarrow tx$ and $tx_\alpha + y_\alpha \longrightarrow tx + y$ and we conclude that $tx + y \in \overline{Y}$, as required.

If Y is a maximal proper subspace, the inclusion $Y \subseteq \overline{Y}$ implies either $Y = \overline{Y}$, in which case Y is closed, or $\overline{Y} = X$, in which case Y is dense in X. ■

Notations Given a vector space X, a subset $Y \subseteq X$, and a vector $x \in X$, we denote the translation $T_x(Y)$ simply by $Y + x$ $(x + Y)$, that is,

$$Y + x = x + Y = \{y + x : y \in Y\}.$$

Likewise, for an $\alpha \in \mathbb{K}$ we denote the dilation $D_\alpha(Y)$ simply by αY, that is,

$$\alpha Y = \{\alpha y : y \in Y\}.$$

Given another subset $Z \subseteq X$, we define

$$Y + Z = \{y + z : y \in Y, z \in Z\} = \bigcup_{y \in Y}(y + Z) = \bigcup_{z \in Z}(Y + z).$$

Remark 1.11 In general we only have the inclusion $2Y \subseteq Y + Y$.

Lemma 1.2 *Let τ be a linear topology on the vector space X.*

1. *The algebraic sum of an open set and an arbitrary set is open.*
2. *Nonzero multiples of open sets are open.*
3. *If Y is open, then for any set Z we have $\overline{Z} + Y = Z + Y$.*
4. *The algebraic sum of a compact set and a closed set is closed. (However, the algebraic sum of two closed sets need not be closed.)*
5. *The algebraic sum of two compact sets is compact.*
6. *Scalar multiples of closed sets are closed.*
7. *Scalar multiples of compact sets are compact.*

Proof We shall prove only items 3. and 4.

3. Clearly $Y + Z \subseteq Y + \overline{Z}$. For the reverse inclusion, let $x = z + y$ where $z \in \overline{Z}$ and $y \in Y$. Then there is an open neighborhood U of θ such that $y + U \subseteq Y$. Since

$z \in \overline{Z}$, there exists some $t \in Z \cap (z - U)$. Then $x = z + y = t + z + (y - z) \in t + z + U \subseteq Z + Y$.

4. Let Y be compact and Z be closed, and let a net $(y_\alpha + z_\alpha)_{\alpha \in I}$ satisfy $y_\alpha + z_\alpha \longrightarrow x$. Since Y is compact, we can assume (by passing to a subnet) that $y_\alpha \longrightarrow y \in Y$. The continuity of the algebraic operations yields

$$z_\alpha = (y_\alpha + z_\alpha) - y_\alpha \longrightarrow x - y = z.$$

Since Z is closed, $z \in Z$, so $x = y + z \in Y + Z$, proving that $Y + Z$ is closed.

■

Proposition 1.14 *Let τ be a linear topology on the vector space X.*

1. *For every neighborhood V of θ, there exists a neighborhood W of θ, such that $W + W \subseteq V$.*
2. *For every neighborhood V of θ, and any compact set $C \subseteq \mathbb{K}$, there exists a neighborhood W of θ, such that $\alpha W \subseteq V, \forall\, \alpha \in C$.*

Proof 1. Let $T : X \times X \to X$ denote the addition map (1.2). Since T is continuous at $(\theta, \theta) \in X \times X$, the preimage $T^{-1}(V)$ is a neighborhood of (θ, θ) in the product topology. In particular, there exists neighborhoods W_1, W_2 of θ, such that $W_1 \times W_2 \subseteq T^{-1}(V)$, so if we take $W = W_1 \cap W_2$, then W is still a neighborhood of θ satisfying $W \times W \subseteq T^{-1}(V)$, which is precisely the desired inclusion $W + W \subseteq V$.

2. Let $G : \mathbb{K} \times X \to X$ denote the multiplication map (1.3). Since G is continuous at $(0, \theta) \in \mathbb{K} \times X$, the preimage $G^{-1}(V)$ is a neighborhood of $(0, \theta)$ in the product topology. In particular, there exists a neighborhood I of 0 in $\mathbb{K}$ and a neighborhood W_0 of θ in X such that $I \times W_0 \subseteq G^{-1}(V)$. Let then $\rho > 0$ such that I contains the closed disk $\overline{B_\rho}(0) = \{\alpha \in \mathbb{K}\colon |\alpha| \leq \rho\}$, so that we still have the inclusion $\overline{B_\rho}(0) \times W_0 \subseteq G^{-1}(V)$ i.e.,

$$\alpha \in \mathbb{K}, |\alpha| \leq \rho \Longrightarrow \alpha W_0 \subseteq V. \tag{1.4}$$

Since $C \subseteq \mathbb{K}$ is compact, there is some $R > 0$, such that

$$|\gamma| \leq R, \quad \forall\, \gamma \in C. \tag{1.5}$$

Let us then define $W = (\frac{\rho}{R})W_0$. First of all, since W is a non-zero dilation of W_0, it is a neighborhood of θ. Secondly, if we start with some $\gamma \in C$ and some $w \in W$, written as $w = (\frac{\rho}{R})w_0$ with $w_0 \in W_0$, then

$$\gamma w = (\frac{\rho\alpha}{R})w_0.$$

By (1.5) we know that $\left|\frac{\rho\alpha}{R}\right| \leq \rho$, so by (1.4) we get $\gamma w \in V$. ■

1.2.2 Absorbing and Balancing Sets

Definition 1.21 A subset Y of a vector space X is convex if, whenever Y contains two points x and y, Y also contains the segment or the straight line joining them, i.e.,

$$\forall\, x, y \in Y, \forall\, \alpha, \beta \geq 0 \text{ such that } \alpha + \beta = 1, \alpha x + \beta y \in Y.$$

Examples 1.5

1. The convex subsets of $\mathbb{R}$ are simply the intervals of $\mathbb{R}$. Examples of convex subsets of $\mathbb{R}^2$ are solid regular polygons. The Platonic solids are convex subsets of $\mathbb{R}^3$. Hyperplanes and half spaces in $\mathbb{R}^n$ are convex.
2. Balls in a normed space are convex.
3. Consider a topological space X and the set $C(X)$ of all real valued functions defined and continuous on X. $C(X)$ with the pointwise addition and scalar multiplication of functions is a vector space. Fixed $g \in C(X)$, the subset $Y := \{f \in C(X) \colon f(x) \geq g(x), \forall x \in X\}$ is convex.
4. Consider the vector space $\mathbb{R}[x]$ of all polynomials in one variable with real coefficients. Fixed $n \in \mathbb{N}$ and $c \in \mathbb{R}$, the subset of all polynomials in $\mathbb{R}[x]$ such that the coefficient of the term of degree n is equal to c is convex.

Proposition 1.15 *Let X be a vector space. The following properties hold.*

(a) *$\emptyset$ and X are convex.*
(b) *Arbitrary intersections of convex sets are convex sets.*
(c) *Unions of convex sets are generally not convex.*
(d) *The sum of two convex sets is convex.*
(e) *A set Y is convex if and only if $\alpha Y + \beta Y = (\alpha + \beta)Y$ for all nonnegative scalars α and β.*
(f) *The image and the preimage of a convex set under a linear map is convex.*

Definition 1.22 Let Y be any subset of a vector space X. We define the convex hull of X, denoted by conv(Y), to be the set of all finite convex linear combinations of elements of Y, i.e.,

$$\operatorname{conv}(Y) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in Y, \alpha_i \in [0, 1], \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \right\}.$$

Proposition 1.16 *Let Y, Z be arbitrary sets of a vector space X. The following hold.*

(a) conv(Y) *is convex.*
(b) $Y \subseteq \operatorname{conv}(Y)$.
(c) *A set is convex if and only if it is equal to its own convex hull.*
(d) *If $Y \subseteq Z$ then* $\operatorname{conv}(Y) \subseteq \operatorname{conv}(Z)$.

(*e*) $\text{conv}(\text{conv}(Y)) = \text{conv}(Y)$.
(*f*) $\text{conv}(Y + Z) = \text{conv}(Y) + \text{conv}(Z)$.
(*g*) *The convex hull of Y is the smallest convex set containing Y, i.e.,* $\text{conv}(Y)$ *is the intersection of all convex sets containing Y.*

Definition 1.23 Let X be a vector space.

- A subset $Y \subseteq X$ is said to be absorbing (or radial), if for every $x \in X$, there exists some scalar $\alpha > 0$, such that $\alpha x \in Y$. Roughly speaking, we may say that a subset is absorbing if it can be made by dilation to swallow every point of the whole space.
- A subset $Y \subseteq X$ is said to be balancing (or circled), if for every $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, one has the inclusion $\alpha Y \subseteq Y$. Note that the line segment joining any point x of a balanced set Y to $-x$ lies in Y.
- A subset $Y \subseteq X$ is said to be symmetric, if for every $x \in Y$, one has $(-x) \in Y$, namely $(-Y) = Y$.
- A subset $Y \subseteq X$ is said to be absolutely convex, if it is convex and balanced.
- A subset $Y \subseteq X$ is said to be starshaped about zero if it included the line segment joining each of its points with zero. That is, if for any $x \in Y$ and any $0 \leq \alpha \leq 1$ we have $\alpha x \in Y$.

Remark 1.12 Note that an absorbing set must contain θ, and any set including an absorbing set is itself absorbing. For any absorbing set Y, the set $Y \cap (-Y)$ is nonempty, absorbing, and symmetric. Every circled set is symmetric. Every circled set is star-shaped about θ, as is every convex set containing θ.

Remark 1.13 Given τ a linear topology of a vector space X, all neighborhoods of θ are absorbing. Indeed, if we start with some $x \in X$, the sequence $x_n = \dfrac{1}{n}x$ clearly converges to θ, so every neighborhood of θ will contain (many) terms x_n.

Examples 1.6

1. In a normed space the unit balls centered at the origin are absorbing and balanced.
2. The unit ball B centered at $(\frac{1}{2}, 0) \in \mathbb{R}^2$ is absorbing but not balanced in the real vector space $\mathbb{R}^2$ endowed with the euclidean norm. Indeed, B is a neighborhood of the origin. However, B is not balanced because for example if we take $x = (1, 0) \in B$ and $\alpha = -1$ then $\alpha x \notin B$.
3. The polynomials $\mathbb{R}[\mathbb{X}]$ are a balanced but not absorbing subset of the real space $C([0, 1], \mathbb{R})$ of continuous real valued functions on $[0, 1]$. Indeed, any multiple of a polynomial is still a polynomial but not every continuous function can be written as multiple of a polynomial.
4. The subset $Y = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq |z_2|\}$ of the complex space $\mathbb{C}^2$ with the euclidean topology is balanced but $\text{int}Y$ is not balanced.

Definition 1.24 Given τ a linear topology of a vector space X, a subset $Y \subseteq X$ is said to be a barrel if it has the following properties:

1. Y is absorbing,
2. Y is absolutely convex,
3. Y is closed.

Proposition 1.17 *Let X be a vector space and let τ be a linear topology on X.*

A. *If $\mathcal{B}$ is a neighborhood base at θ, then:*

1. *For every $V \in \mathcal{B}$, there exists $W \in \mathcal{B}$, such that $W + W \subseteq V$.*
2. *For every $V \in \mathcal{B}$ and every compact set $C \subseteq \mathbb{K}$, there exists $W \in \mathcal{B}$, such that $\gamma W \subseteq V, \forall \gamma \in C$.*
3. *For every $x \in X$, the collection $\mathcal{B}_x = \{V + x : V \in \mathcal{B}\}$ is a neighborhood base at x.*
4. *The topology τ is Hausdorff, if and only if $\bigcap_{V \in \mathcal{B}} V = \{\theta\}$.*

B. *There exists a neighborhood base at θ, consisting of open balanced sets.*

Proof

A. Statements 1. and 2. follow immediately from Proposition 1.14. Statement 3. is clear, since translations are homeomorphisms.

4. Denote for simplicity the intersection $\bigcap_{V \in \mathcal{B}} V$ by J, so clearly $\theta \in J$. Assume first τ is Hausdorff. In particular, for each $x \in X \setminus \{\theta\}$, the set $X \setminus \{x\}$ is an open neighborhood of θ, so there exists some $V^x \in \mathcal{B}$ with $V^x \subseteq X \setminus \{x\}$. We then clearly have the inclusion

$$J \subseteq \bigcap_{x \neq \theta} V^x \subseteq \bigcap_{x \neq \theta} (X \setminus \{x\}) = \{\theta\},$$

so $J = \{\theta\}$. Conversely, assume $J = \{\theta\}$, and let us show that τ is Hausdorff. Start with two points $x, y \in X$ with $x \neq y$, so that $x - y \neq \theta$, and let us indicate how to construct two disjoint neighborhoods, one for x and one for y. Using translations, we can assume $y = \theta$. Since $\theta \neq x \notin \bigcap_{V \in \mathcal{B}} V$, there exists some $V \in \mathcal{B}$, such that $x \notin V$. Using 1., there is some $W \in \mathcal{B}$, such that $W + W \subseteq V$, so we still have $x \in W + W$. This clearly forces

$$x + ((-1)V) \cap V = \emptyset. \tag{1.6}$$

Since V is a neighborhood of θ, so is $(-1)V$ (non-zero dilation), then by 3. the left-hand side of (1.5) is a neighborhood of x.

B. Let us take the $\mathcal{D}$ to be the collection of all open balanced sets that contain θ. All we have to prove is the following statement: for every neighborhood V of

θ, there exists $W \in \mathcal{D}$, such that $W \subseteq V$. Using 2. there exists some open set $O \ni \theta$, such that

$$\gamma O \subseteq V, \ \ \forall \gamma \in \mathbb{K}, |\gamma| \leq 1. \tag{1.7}$$

In particular, $\bigcup_{\alpha \in \mathbb{K}, 0<|\alpha|\leq 1} \alpha O$ is an open set contained in V. So $\bigcup_{\alpha \in \mathbb{K}, 0<|\alpha|\leq 1} \alpha O \in V$.

■

Definition 1.25 Assume τ is a linear topology on a vector space X. A subset $Y \subseteq X$ is said to be τ-bounded, if it satisfies the following condition:

for every neighborhood V of θ, there exists $\rho > 0$, such that $Y \subseteq \rho V$.

Example 1.7 Suppose τ is a linear topology on a vector space X. If $x \in X \in$, then $\{x\}$ is bounded. Indeed, let V any neighborhood of θ. Then V is absorbing and so $x \in \rho V$ for all sufficiently large $\rho > 0$, that is, $\{x\}$ is bounded.

Proposition 1.18 *Let X be a vector space X endowed with a linear topology τ. Then*

1. *If $Y \subseteq X$ is τ-bounded, then its closure $\overline{Y}$ is also τ-bounded.*
2. *If $Y, Z \subseteq X$ are τ-bounded, then so is $Y + Z$.*
3. *If $Y \subseteq X$ is τ-bounded and $C \subseteq \mathbb{K}$ is bounded, then so $\bigcup_{\alpha \in C} \alpha Y$.*
4. *All compact subsets in X are τ-bounded.*

Remark 1.14 It follows by induction, that any finite set in a vector space X endowed with a linear topology τ is bounded. Also, taking $Y = \{x\}$ (in the above proposition) we see that any translate of a bounded set is bounded.

Proposition 1.19 *Any convergent sequence in topological vector space is bounded.*

Proof Suppose that $(x_n)_n$ is a sequence in a topological vector space (X, τ) such that $x_n \longrightarrow x$. For each $n \in \mathbb{N}$, set $y_n = x_n - x$, so that $y_n \longrightarrow \theta$. Let V any neighborhood of θ. Let U be any balanced neighborhood of θ such that $U \subseteq V$. Then $U \subseteq \rho U$ for all ρ with $|\rho| \geq 1$. Since $y_n \longrightarrow \theta$, there is $N \in \mathbb{N}$ such that $y_n \in U$ whenever $n > N$. Hence $y_n \in U \subseteq tU \subseteq tV$ whenever $n > N$ and $t \geq 1$. Set $Y = \{y_1, \cdots, y_n\}$ and $Z = \{y_n : n > N\}$. Then Y is a finite set so is bounded and therefore $Y \subseteq tV$ for all sufficiently large t. But then it follows that $Y \cup Z \subseteq tV$ for sufficiently large t, that is, $\{y_n : n \in \mathbb{N}\}$ is τ-bounded and so is $\{x_n : n \in \mathbb{N}\} = x + (Y \cup Z)$. ■

Remark 1.15 A convergent net in a topological vector space need not be bounded. For example, let I be $\mathbb{R}$ equipped with its usual order and let $x_\alpha \in \mathbb{R}$ be given by $x_\alpha = e^{-\alpha}$. Then $(x_\alpha)_{\alpha \in I}$ is an unbounded but convergent net (with limit 0) in the real normed space $\mathbb{R}$.

Proposition 1.20 ("Zero. Bounded" Rule) *Suppose τ is a linear topology in a vector space X. If the net $(\alpha_\lambda)_{\lambda\in\Lambda} \subseteq \mathbb{K}$ converges to 0, and the net $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ is τ-bounded, then $(\alpha_\lambda x_\lambda)_{\lambda\in\Lambda}$ is convergent to θ.*

Proof Start with some neighborhood V of θ. We wish to construct an index $\lambda_V \in \Lambda$ such that

$$\alpha_\lambda x_\lambda \in V, \ \forall\, \lambda \succ \lambda_V. \tag{1.8}$$

Using Proposition 1.17 **B.**, we can assume that V is balanced (otherwise we replace it with a balanced open set $V' \subseteq V$). Using the boundedness condition we find $\rho > 0$, such that

$$x_\lambda \in \rho V, \ \forall\, \lambda \in \Lambda. \tag{1.9}$$

Using the condition $\alpha_\lambda \to 0$, we then choose $\lambda_V \in \Lambda$, so that

$$|\alpha_\lambda| \le \frac{1}{\rho}, \forall\, \lambda \succ \lambda_V.$$

To check (1.8), start with some $\lambda \succ \lambda_V$ and apply (1.9) to write $x_\lambda = \rho v$, for some $v \in V$. Now we have

$$\alpha_\lambda x_\lambda = (\alpha_\lambda \rho) v \in (\alpha_\lambda \rho) V,$$

with $|\alpha_\lambda \rho| \le 1$, so using the fact that V is bounded, it follows that $\alpha_\lambda x_\lambda \in V$. ■

Definition 1.26 Let (X, τ) be a topological vector space.

1. X is locally bounded if θ has a bounded neighborhood.
2. X is locally compact if θ has a neighborhood whose closure is compact.
3. X is metrizable if it is compatible with some metric d (i.e., τ is generated by the open balls $B_r(x) = \{y \in X, d(x, y) < r\}$).
4. X is normable if it can be endowed with a norm whose induced metric is compatible with τ.
5. X has the Heine-Borel property if every closed and bounded set is compact.

Proposition 1.21 *Let (X, τ) be a topological vector space. For every $x \neq \theta$ the set $Y = \{nx,\ n \in \mathbb{N}\}$ is not bounded.*

Proof By separation, there exists an open neighborhood V of θ that does not contain x, hence $nx \notin nV$, i.e., for every n,

$$Y \not\subseteq nV.$$

■

Lemma 1.3

1. *Let d be a translation invariant metric on a vector space X, then for all $n \in \mathbb{N}$ and $x \in X$,*

$$d(nx, \theta) \leq nd(x, \theta).$$

2. *If $x_n \to \theta$ in a metrizable topological vector space (X, τ), then there exist positive scalars $\alpha_n \to \infty$ such that $\alpha_n x_n \to \theta$.*

Proof The first part is obvious by successive applications of the triangle inequality,

$$d(nx, \theta) \leq \sum_{k=1}^{n} d(kx, (k-1)x) \leq nd(x, \theta).$$

For the second, we note that since $d(x_n, \theta) \to 0$, there exists a diverging sequence of positive integers n_k, such that

$$d(x_k, \theta) \leq \frac{1}{n_k^2},$$

from which we get that

$$d(n_k x_k, \theta) \leq n_k d(x_k, \theta) \leq \frac{1}{n_k} \to 0.$$

■

Corollary 1.2 *The only bounded subspace of a topological vector space is $\{\theta\}$.*

Proposition 1.22 *Let (X, τ) be a topological vector space and let $Y \subseteq X$. Then, Y is bounded if and only if for every sequence $(x_n)_n \subseteq Y$ and every sequence of scalars $\alpha_n \to 0$, $\alpha_n x_n \to \theta$.*

Proof Suppose that Y is bounded, it suffices to apply Proposition 1.20.

Suppose that for every sequence $(x_n)_n \subseteq Y$ and every sequence of scalars $\alpha_n \to \theta$, $\alpha_n x_n \to \theta$. If Y is not bounded, then there exists an open neighborhood of θ and a sequence $\beta_n \to \infty$, such that no $\beta_n V$ contains Y. Take then a sequence $(x_n)_n \subseteq Y$ such that $x_n \notin \beta_n V$. Thus,

$$\beta_n^{-1} x_n \notin V,$$

which implies that $\beta_n^{-1} x_n \nrightarrow \theta$, which is a contradiction. ■

Theorem 1.8 *Let (X, τ) be a topological vector space. Let $Y, Z \subseteq X$ satisfy:*

$$Y \text{ is compact}, Z \text{ is closed and } Y \cap Z = \emptyset.$$

Then there exists an open neighborhood V of θ such that

$$(Y + V) \cap (Z + V) = \emptyset.$$

In other words, there exist disjoint open sets that contain Y and Z.

Proof Let $x \in Y$. Since $X \setminus Z$ is an open neighborhood of x, it follows that there exists a symmetric open neighborhood V_x of θ such that

$$x + V_x + V_x + V_x \subseteq X \setminus Z,$$

i.e.,

$$(x + V_x + V_x + V_x) \cap Z = \emptyset.$$

Since V_x is symmetric,

$$(x + V_x + V_x) \cap (Z + V_x) = \emptyset.$$

For every $x \in Y$ corresponds such a V_x. Since Y is compact, there exists a finite collection $(x_i, V_i)_{1 \le i \le n}$ such that

$$K \subseteq \bigcup_{i=1}^{n} (x_i + V_i).$$

Define

$$V = \bigcap_{i=1}^{n} V_{x_i}.$$

Then, for every i,

$$(x + V_{x_i} + V_{x_i}) \quad \text{does not intersect} \; (Z + V_{x_i}),$$

so

$$(x + V_{x_i} + V) \quad \text{does not intersect} \quad (Z + V).$$

Taking the union over i:

$$Y + V \subseteq \bigcup_{i=1}^{n} (x_i + V_{x_i} + V) \quad \text{does not intersect} \quad (Z + V).$$

■

Remark 1.16 A topological vector space is regular.

Proposition 1.23 *Suppose τ is a linear topology in a vector space X.*

1. *For $Y \subseteq X$,*

$$\overline{Y} = \bigcap_{V,\text{open neighborhood of } \theta} (Y + V).$$

That is, the closure of a set is the intersection of all the open neighborhoods of that set.

2. *For $Y, Z \subseteq X$, $\overline{Y} + \overline{Z} \subseteq \overline{Y + Z}$.*
3. *If $Y \subseteq X$ is a linear subspace, then so is $\overline{Y}$.*
4. *For every $B \subseteq X$: If B is balanced so is $\overline{B}$.*
5. *For every $B \subseteq X$: If B is balanced and $\theta \in \operatorname{int} B$ then $\operatorname{int} B$ is balanced.*
6. *If $Y \subseteq X$ is bounded so is $\overline{Y}$.*

Proof

1. Let $x \in \overline{Y}$. By definition, for every open neighborhood V of θ, $x + V$ intersects Y, of $x \in Y - V$. Thus,

$$x \in \bigcap_{V,\text{open neighborhood of } \theta} (Y - V) = \bigcap_{V,\text{open neighborhood of } \theta} (Y + V).$$

Conversely, suppose that $x \notin \overline{Y}$. Then, there exists an open neighborhood V of θ such that $x + V$ does not intersect Y, i.e., $x \notin Y - V$, hence

$$x \notin \bigcap_{V,\text{open neighborhood of } \theta} (Y + V).$$

2. Let $x \in \overline{Y}$ and $y \in \overline{Z}$. By the continuity of vector addition, for every open neighborhood U of $x + y$ there exists an open neighborhood V of x and an open neighborhood W of y such that

$$V + W \subseteq U.$$

By the definition of $\overline{Y}$ every neighborhood of x intersects Y and by the definition of $\overline{Z}$ every neighborhood of y intersects W: that is, there exist $z \in V \cap Y$ and $t \in W \cap Z$. Then,

$$z \in Y \text{ and } t \in Z \text{ implies } z + t \in Y + Z,$$

and

$$z \in V \text{ and } t \in W \text{ implies } z + t \in V + W \subseteq U.$$

In other words, every neighborhood of $x + y \in \overline{Y} + \overline{Z}$ intersects $Y + Z$, which implies that $x + y \in \overline{Y + Z}$, and therefore

$$\overline{Y} + \overline{Z} \subseteq \overline{Y + Z}.$$

3. Let Y be a linear subspace of X, which means that,

$$Y + Y \subseteq Y \quad \text{and} \quad \forall\, \alpha \in \mathbb{K}, \alpha Y \subseteq Y.$$

By the previous item,

$$\overline{Y} + \overline{Y} \subseteq \overline{Y + Y} \subseteq \overline{Y}.$$

Since scalar multiplication is a homeomorphism it maps the closure of a set into the closure of its image, namely, for every $\alpha \in \mathbb{K}$,

$$\alpha \overline{Y} \subseteq \overline{Y}.$$

4. Since multiplication by a (non-zero) is a homeomorphism,

$$\alpha \overline{B} = \overline{\alpha B}.$$

If B is balanced, then for $|\alpha| \leq 1$,

$$\alpha \overline{B} = \overline{\alpha B} \subseteq \overline{B},$$

hence $\overline{B}$ is balanced.

5. Again, for every $0 < |\alpha| \leq 1$,

$$\alpha(\text{int}B) = \text{int}(\alpha B) \subseteq \text{int}B.$$

Since for $\alpha = 0$, $\alpha(\text{int}B) = \{\theta\}$, we must require that $\theta \in \text{int}B$ for the latter to be balanced.

6. Let V be an open neighborhood of θ. Then there exists an open neighborhood W of θ such that $\overline{W} \subseteq V$. Since Y is bounded, $Y \subseteq \alpha W \subseteq \alpha \overline{W} \subseteq \alpha V$ for sufficiently large α. It follows that for large enough α,

$$\overline{Y} \subseteq \alpha \overline{W} \subseteq \alpha V,$$

which proves that $\overline{Y}$ is bounded.

■

Lemma 1.4 *Suppose τ is a linear topology in a vector space X.*

1. *If Y is convex so is $\overline{Y}$.*
2. *If Y is convex so is* $\operatorname{int} Y$.

Proof

1. The convexity of Y implies that for all $\alpha \in [0, 1]$:
$$\alpha Y + (1-\alpha)Y \subseteq Y.$$
Let $\alpha \in [0, 1]$, then
$$\alpha\overline{Y} = \overline{\alpha Y} \text{ and } (1-\alpha)\overline{Y} = \overline{(1-\alpha)Y}.$$
By the second item:
$$\alpha\overline{Y} + (1-\alpha)\overline{Y} = \overline{\alpha Y} + \overline{(1-\alpha)Y} \subseteq \overline{\alpha Y + (1-\alpha)Y} \subseteq \overline{Y},$$
which proves that $\overline{Y}$ is convex.
2. Suppose once again that Y is convex. Let $x, y \in \operatorname{int}Y$. This means that there exist open neighborhoods U, V of θ such that
$$x + U \subseteq Y \text{ and } y + V \subseteq Y.$$
Since Y is convex:
$$\alpha(x + U) + (1-\alpha)(y + V) = (\alpha x + (1-\alpha)y) + \alpha U + (1-\alpha)V \subseteq Y,$$
which proves that $\alpha x + (1-\alpha)y \in \operatorname{int}Y$, namely $\operatorname{int}Y$ is convex.

■

Lemma 1.5 *Suppose τ is a linear topology in a vector space X. If Y is a convex subset of X, then*:

$$0 < \alpha \leq 1 \implies \alpha(\operatorname{int}Y) + (1-\alpha)\overline{Y} \subseteq \operatorname{int}Y. \tag{1.10}$$

In particular, if $\operatorname{int}Y \neq \emptyset$, *then*:

(a) *The interior of Y is dense in $\overline{Y}$, that is,* $\overline{\operatorname{int}Y} = \overline{Y}$.
(b) *The interior of $\overline{Y}$ coincides with the interior of Y, that is,* $\operatorname{int}\overline{Y} = \operatorname{int}Y$.

Proof The case $\alpha = 1$ in (1.10) is immediate. So let $x \in \operatorname{int}Y$, $y \in \overline{Y}$, and let $0 < \alpha < 1$. Choose an open neighborhood U of θ such that $x + U \subseteq Y$. Since $y - \dfrac{\alpha}{1-\alpha}U$ is a neighborhood of y, there is some $z \in Y \cap (y - \dfrac{\alpha}{1-\alpha}U)$, so that $(1-\alpha)(y - z)$ belongs to αU. Since Y is convex, the (nonempty) open set

$V = \alpha(x+U)+(1-\alpha)z = \alpha x+\alpha U+(1-\alpha)z$ lies entirely in Y. Moreover, from

$$\alpha x+(1-\alpha)y = \alpha x+(1-\alpha)(y-z)+\alpha x+(1-\alpha)z \in \alpha x+\alpha U+(1-\alpha)z = V \subseteq Y,$$

we see that $\alpha x+(1-\alpha)y \in \operatorname{int}Y$. This proves (1.10), and letting $\alpha \longrightarrow 0$ proves (a).

For (b), fix $x_0 \in \operatorname{int}Y$ and $x \in \operatorname{int}\overline{Y}$. Pick a neighborhood of θ satisfying $x+W \subseteq \overline{Y}$. Since W is absorbing, there is some $0 < \lambda < 1$ such that $\lambda(x-x_0) \in W$, so $x + \lambda(x-x_0) \in \overline{Y}$. By (1.10), we have $x-\lambda(x-x_0) = \lambda x_0+(1-\lambda)x \in \operatorname{int}Y$. But then, using (1.10) once more, we obtain $x = \frac{1}{2}[x-\lambda(x-x_0)] + \frac{1}{2}[x+\lambda(x-x_0)] \in \operatorname{int}Y$. Therefore, $\operatorname{int}\overline{Y} \subseteq \operatorname{int}Y \subseteq \operatorname{int}\overline{Y}$ so that $\operatorname{int}\overline{Y} = \operatorname{int}Y$. ■

Definition 1.27 Let τ be a linear topology in a vector space X and $Y \subseteq X$.

1. The closed convex hull of a set Y, denoted $\overline{\operatorname{conv}}(Y)$, is the smallest closed convex set including Y. By Lemma 1.4 1. it is the closure of $\operatorname{conv}(Y)$, that is, $\overline{\operatorname{conv}}(Y) = \overline{\operatorname{conv}(Y)}$.
2. The convex circled hull of Y is the smallest convex and circled set that includes Y. It is the intersection of all convex and circled sets that include Y.
3. The closed convex circled hull of Y is the smallest closed convex circled set including Y. It is the closure of the convex circled hull of Y.

Definition 1.28 Let X be a vector space and let τ be a linear topology on X. Then (X, τ) is said to be locally convex if there is a base of neighborhoods of the origin in X consisting of convex sets.

Proposition 1.24 *A locally convex space (X, τ) always has a base of neighborhoods of the origin consisting of open absorbing absolutely convex subsets.*

Proof Let V be a neighborhood of θ in X. Since (X, τ) is locally convex, there exists W convex neighborhood of θ such that $W \subseteq V$. Moreover, by Remark 1.13, there exists U balanced neighborhood of θ such that $U \subseteq W$. The balancedness of U implies that $U = \bigcup_{\alpha\in\mathbb{K},|\alpha|\le 1} \alpha U$. Thus, using that W is a convex set containing U, we get

$$N := \operatorname{conv}\left(\bigcup_{\alpha\in\mathbb{K},|\alpha|\le 1} \alpha U\right) = \operatorname{conv}(U) \subseteq W \subseteq V$$

and so $\operatorname{int}N \subseteq V$. Hence, the conclusion holds because $\operatorname{int}N$ is clearly open and convex and it is also balanced since $\theta \in \operatorname{int}N$ and N is balanced. ■

1.2.3 Compactness and Completeness

Definition 1.29 Let (X, τ) be a topological vector space.

1. A net $(x_\alpha)_{\alpha\in I}$ in X is said to be a Cauchy net if for each neighborhood V of θ there exists $\alpha_0 \in I$ such that $x_\alpha - x_\beta \in V$ whenever $\alpha, \beta \succ \alpha_0$.
2. A set $Y \subseteq X$ is complete if each Cauchy net in X converges to a point of Y.
3. A set $Y \subseteq X$ is sequentially complete if each Cauchy sequence in X converges to a point of Y

Example 1.8 Every convergent net is Cauchy.

Proposition 1.25 *A Cauchy sequence (and in particular a converging sequence) in a topological vector space (X, τ) is bounded.*

Proof Let $(x_n)_n$ be a Cauchy sequence. Let W, V be two balanced open neighborhoods of θ satisfying

$$V + V \subseteq W.$$

By the definition of a Cauchy sequence, there exists an N such that for all $m, n \geq N$,

$$x_n - x_m \in V,$$

and in particular

$$\forall\, n > N \quad x_n \in x_N + V.$$

Set $s > 1$ such that $x_N \in sV$ (we know that such an s exists), then for all $n > N$,

$$x_n \in sV + V \subseteq sV + sV \subseteq W.$$

Since for balanced sets $sW \subseteq tW$ for $s < t$, and since every open neighborhood of θ contains an open balanced neighborhood, this proves that the sequence is indeed bounded. ■

Proposition 1.26 *Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of topological vector spaces, and let $X = \prod_{i\in I} X_i$ endowed with the product topology $\tau = \prod_{i\in I} \tau_i$. Then (X, τ) is complete if and only if each factor (X_i, τ_i) is complete.*

Proposition 1.27 *Let (X, τ) be a topological vector space with a countable base of neighborhoods of θ. A set $Y \subseteq X$ is complete if and only if Y is sequentially complete.*

Proof Let $\mathcal{B} = \{V_n : n \in \mathbb{N}\}$ be a countable base of neighborhoods of θ. We can assume that $V_1 \supseteq V_2 \supseteq \cdots$, indeed, otherwise we can substitute $\mathcal{B}$ with the base

$$\{V_1, V_1 \cap V_2, V_1 \cap V_2 \cap V_3, \cdots\}.$$

Let Y be complete, and $(x_n)_n$ a Cauchy sequence in Y. There exists a subnet $(x_{\phi(\alpha)})_{\alpha\in I}$ converging to a point $x \in Y$. Let us construct inductively a sequence

(α_k) in I. Choose α_1 so that $x_{\phi(\alpha)} \in x + V_1$ for each $\alpha \succ \alpha_1$. If we already have $\alpha_1, \cdots, \alpha_k$, choose $\alpha_{k+1} \succ \alpha_k$ so that $\phi(\alpha_{k+1}) \succ \phi(\alpha_k) + 1$ and $x_{\phi(\alpha)} \in x + V_{k+1}$ for each $\alpha \succ \alpha_{k+1}$. It is easy to verify that $(x_{\phi(\alpha_k)})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_n$ that converges to x.

Conversely, Let Y be sequentially complete, and $(x_\alpha)_{\alpha\in I}$ a Cauchy net in Y. Let us construct inductively a sequence $(\alpha_k)_k$ in I. Choose α_1 so that $x_\alpha - x_{\alpha_1} \in V_1$ for each $\alpha \succ \alpha_1$. If we already have $\alpha_1, \cdots, \alpha_k$, choose $\alpha_{k+1} \succ \alpha_k$ so that $x_\alpha - x_{\alpha_{k+1}} \in V_{k+1}$ for each $\alpha \succ \alpha_{k+1}$. Then (x_{α_n}) is a Cauchy sequence since $x_{\alpha_m} - x_{\alpha_n} \in V_n$ whenever $m \geq n$. Consequently, (x_{α_n}) converges to a point $x \in Y$. Now, it is easy to show that $(x_\alpha)_{\alpha\in I}$ converges to x, too. ■

Definition 1.30 A set Y in a topological vector space (X, τ) is totally bounded (or precompact) if for each neighborhood V of θ there is a finite set $F \subseteq X$ such that $Y \subseteq F + V$.

It is easy to see that in normed spaces (or in topological metric spaces) this definition coincides with the usual metric one: for each $\varepsilon > 0$ there is a finite set $F \subseteq X$ such that $\operatorname{dist}(x, F) < \varepsilon$ for each $x \in Y$.

Theorem 1.9 *Let Y be a set in a topological vector space (X, τ). Then Y is totally bounded if and only if each net in Y admits a Cauchy subnet.*

Proof Let $(x_\alpha)_{\alpha\in I}$ be a net in a totally bounded set Y. The family $\mathcal{Z} = \{Z \subseteq Y\}$: $\mathcal{B}$ be a maximal subfamily of $\mathcal{Z}$ that contains Y and is closed under making finite intersections (existence of such $\mathcal{B}$ follows by `Zorn's` lemma). Let us show several properties of $\mathcal{B}$.

(a) if $\mathcal{F}$ is a finite subfamily of $\mathcal{Z}$ such that $\bigcup \mathcal{F} \in \mathcal{B}$, then $\mathcal{F} \cap \mathcal{B} \neq \emptyset$. Let $\mathcal{F} = \{Z_1, \cdots, Z_n\}$. We claim that, for some index k, $Z_k \cap B \in \mathcal{Z}$ for each $B \in \mathcal{B}$. Indeed, if this not the case, for each $i \in \{1, \cdots, n\}$ there exists $B_i \in \mathcal{B}$ such that $Z_i \cap B_i \notin \mathcal{Z}$, but then $\mathcal{B} \ni (\bigcup_{i=1}^{n} Z_i) \cap \bigcap_{i}^{n} B_i \subseteq \bigcup_{i=1}^{n}(Z_i \cap B_i) \notin \mathcal{Z}$, a contradiction. Our claim implies that the family of all finite intersections of elements of $\mathcal{B} \bigcup \{Z_k\}$ is closed under finite intersections and is contained in $\mathcal{Z}$. By maximality of $\mathcal{B}$, we must have $Z_k \in \mathcal{B}$.

(b) For each set $Z \subseteq Y$, the family $\mathcal{B}$ contains either Z or $Y \setminus Z$. If $Z \notin \mathcal{Z}$, then eventually $x_\alpha \in Y \setminus Z$. Since the intersection of $Y \setminus Z$ with any element of $\mathcal{B}$ belongs to $\mathcal{Z}$, the family of finite intersections of $\mathcal{B} \cup \{Y \setminus Z\}$ is contained in $\mathcal{Z}$. Thus $Y \setminus Z \in \mathcal{B}$ by the maximality of $\mathcal{B}$. In the same way we get that $Y \setminus Z \notin \mathcal{Z}$ then $Z \in \mathcal{B}$. Finally, if both Z and $Y \setminus Z$ belong to $\mathcal{Z}$ the one of them belongs to $\mathcal{B}$ by (a) (since $Y \in \mathcal{B}$).

(c) $\mathcal{B}$ contains arbitrarily small elements, in the sense that for each neighborhood V of θ there exists $B \in \mathcal{B}$ such that $B - B \subseteq V$. Given a neighborhood V of θ, there exists a neighborhood W of θ with $W - W \subseteq V$. By total boundedness, there exists a finite set $F = \{y_1, \cdots, y_n\} \subseteq Y$ such that $Y \subseteq F + W$. Denoting $Y_i = (y_i + W) \cap Y (i = 1, \cdots, n)$, we have $Y = \cup_{i=1}^{n} Y_i$. Consider the set

$P = \{i \in \{1, \cdots, n\} \colon Y_i \in \mathcal{Z}\}$ and its complement $\{1, \cdots, n\} \setminus P$. Since $C = \bigcup_{i \in \{1, \cdots, n\} \setminus P} Y_i \notin \mathcal{Z}$, we must have $P \neq \emptyset$. Let $Z = \bigcup_{i \in P} Y_i$. Then $Y \setminus Z \notin \mathcal{Z}$ (since $Y \setminus Z \subseteq C$). By (b), we must have $Z \in \mathcal{B}$. By (a), there exists $k \in P$ with $Y_k \in \mathcal{B}$. Notice that $Y_k - Y_k \subseteq W - W \subseteq V$.

To conclude the proof of this implication, notice that the family $\mathcal{B}$ satisfies the assumptions of Lemma 1.1. Hence there exists a subnet of (x_α) that is eventually contained in each element of $\mathcal{B}$. By (c), this subnet is Cauchy.

Conversely, assume that Y is not totally bounded. There exists a neighborhood V of θ such that $Y \setminus (F + V) \neq \emptyset$ for each finite set $F \subseteq V$. An easy inductive construction gives a sequence $(x_n)_n$ such that $x_{n+1} \notin \{x_1, \cdots, x_n\} + V$ for each n. Since for two indexes $m > n$ we have $x_m - x_n \notin V$, our sequence has no Cauchy subnets. The proof is complete. ■

Theorem 1.10 *A set Y in a topological vector space is compact if and only if Y is totally bounded and complete.*

Proof Let Y be compact. Given an open neighborhood V of θ, the open cover $\{y + V \colon y \in Y\}$ of Y admits a finite sub cover. This proves that Y is totally bounded. Let $(x_\alpha)_{\alpha \in I}$ be a Cauchy net in Y. By Theorem 1.6 $(x_\alpha)_{\alpha \in I}$ admits a subnet converging to a point of Y. It easily follows that the net $(x_\alpha)_{\alpha \in I}$ converges to the same limit.

Conversely, assume Y is totally bounded and complete. Given a net $(x_\alpha)_{\alpha \in I}$ in Y, it admits a Cauchy subnet by Theorem 1.9. Since Y is complete, this subnet converges to a point of Y. Again, it follows that $(x_\alpha)_{\alpha \in I}$ converges to the same point. By Theorem 1.6, Y is compact. ■

1.2.4 Seminorms and Local Convexity

Definition 1.31 A seminorm on a vector space X is map $p \colon X \to \mathbb{R}$ such that

$$p(x + y) \leq p(x) + p(y),$$

and

$$p(\alpha x) = |\alpha| \, p(x).$$

Definition 1.32 Let $\mathcal{P} := (p_i)_{i \in I}$ be a family of seminorms. It is called separating if to each $x \neq \theta$ corresponds a $p_i \in \mathcal{P}$, such that $p_i(x) \neq 0$. Note that the separation condition is equivalent to

$$p_i(x) = 0, \forall\, i \in I \Rightarrow x = \theta.$$

Examples 1.7

1. Suppose $X = \mathbb{R}^n$ and let Y be a vector subspace of X. Set for any $x \in X$

$$p_Y(x) := \inf_{y \in Y} \|x - y\|$$

where $\|.\|$ is the Euclidean norm, i.e., $p_Y(x)$ is the distance from the point x to Y in the usual sense. If $\dim(Y) \geq 1$ then p_Y is a seminorm and not a norm (Y is exactly the kernel of p_Y). When $Y = \{\theta\}$, $p_Y(.) = \|.\|$.
2. Let X be a vector space on which is defined a nonnegative sesquilinear Hermitian form $\varphi\colon X \times X \to \mathbb{K}$. Then the function

$$p_\varphi(x) := \varphi(x, x)^{\frac{1}{2}}$$

is a seminorm. p_φ is a norm if and only if φ is positive definite (i.e., $\varphi(x, x) > 0, \forall\, x \neq \theta$).
3. Let $C(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, we define for any $f \in C(\mathbb{R})$:

$$p_{[a,b]}(f) := \sup_{a \leq t \leq b} |f(t)|\,.$$

$p_{[a,b]}$ is a seminorm but is never a norm because it might be that $f(t) = 0$ for all $t \in [a, b]$ (and so that $p_{[a,b]}(f) = 0$) but $f \not\equiv 0$. Other seminorms are the following ones:

$$q(f) := |f(0)| \text{ and } q_p(f) := \left(\int_a^b |f(t)|^p\right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty.$$

Proposition 1.28 *Let p be a seminorm on a vector space X.*

1. p is symmetric.
2. $p(\theta) = 0$.
3. $|p(x) - p(y)| \leq p(x - y)$.
4. $p(x) \geq 0$.
5. ker *p is a linear subspace.*

Proof By the properties of the seminorm:

1. $p(x - y) = p(-(y - x)) = |-1|\, p(y - x) = p(y - x)$.
2. $p(\theta) = p(0.x) = 0.p(x) = 0$.
3. This follows from the inequalities

$$p(x) \leq p(y) + p(x - y) \text{ and } p(y) \leq p(x) + p(y - x) = p(x) + p(x - y).$$

4. By the previous item, for every x:

$$0 \le |p(x) - p(\theta)| \le p(x).$$

5. If $x, y \in \ker p$:

$$p(\alpha x + \beta y) \le p(\alpha x) + p(\beta y) = |\alpha|\, p(x) + |\beta|\, p(y) = 0.$$

■

Notation Let X be a vector space and p a seminorm on X. The sets

$$B_1^p = \{x \in X \colon p(x) < 1\} \quad \text{and} \quad \overline{B}_1^p = \{x \in X \colon p(x) \le 1\},$$

are said to be, respectively, the open and the closed unit semiball of p.

Proposition 1.29 *Let τ be a linear topology on the vector space X. Then the following conditions are equivalent*:

1. *the open unit semiball B_1^p of p is an open set.*
2. *p is continuous at the origin.*
3. *the closed unit semiball $\overline{B}_1^p$ of p is a barrel neighborhood of the origin.*
4. *p is continuous at every point.*

Proof 1. $\Rightarrow$ 2. Suppose that B_1^p is open in the topology τ on X. Then for any $\varepsilon > 0$ we have that $p^{-1}([0, \varepsilon[) = \{x \in X \colon p(x) \le \varepsilon\} = \varepsilon B_1^p$ is an open neighborhood of the origin in X. This is enough to conclude that $p \colon X \to \mathbb{R}^+$ is continuous at the origin.

2. $\Rightarrow$ 3. Suppose that p is continuous at the origin, then $\overline{B}_1^p = p^{-1}([0, 1])$ is a closed neighborhood of the origin. Since B_1^p is also absorbing and absolutely convex, $\overline{B}_1^p$ is a barrel.

3. $\Rightarrow$ 4. Assume that 3. holds and fix $\theta \neq x \in X$. We have for any $\varepsilon > 0$: $p^{-1}([-\varepsilon + p(x), \varepsilon + p(x)]) = \{y \in X \colon |p(y) - p(x)| \le \varepsilon\} \supseteq \{y \in X \colon p(y - x) \le \varepsilon\} = x + \varepsilon \overline{B}_1^p$, which is a closed neighborhood of x since τ is a linear topology on X and by the assumption 3. Hence, p is continuous.

4. $\Rightarrow$ 1. If p is continuous on X then 1. holds because the preimage of an open set under a continuous function is open and $B_1^p = p^{-1}([0, 1[)$. ■

Definition 1.33 Let X be a vector space. For $K \subseteq X$ convex and radial at θ (equivalently, K is absorbing), we define the Minkowski functional of K as

$$p_K(x) = \inf\{t > 0 \colon \frac{x}{t} \in K\}.$$

Intuitively, $p_K(x)$ is the factor by which x must be shrunk in order to reach the boundary of K.

Definition 1.34 (Topology Induced from Seminorms) Let $(p_i)_{i\in I}$ a family of seminorms on a vector space X. Then the ith open strip of radius r centered at $x \in X$ is

$$B_r^i(x) = \{y \in X \colon p_i(x-y) < r\}.$$

Let Λ be the collection of all open strips in X:

$$\Lambda = \{B_r^i(x) \colon i \in I, r > 0, x \in X\}.$$

The topology $\tau(\Lambda)$ generated by Λ is called the topology induced by $(p_i)_{i\in I}$.

The fact that p_i is a seminorm ensures that each open strip $B_r^i(x)$ is convex. Hence all finite intersections of open strips will also be convex.

Theorem 1.11 *Let $(p_i)_{i\in I}$ be a family of seminorms on a vector space X. Then*

$$\mathcal{B} = \left\{ \bigcap_{j=1}^{n} B_r^{i_j}(x) \colon n \in \mathbb{N}, i_j \in I, r > 0, x \in X \right\}$$

forms a base for the topology induced from these seminorms. In fact, if U is open and $x \in U$, then there exists an $r > 0$ and $i_1, \cdots, i_n \in I$ such that

$$\bigcap_{j=1}^{n} B_r^{i_j}(x) \subseteq U.$$

Further, every element of $\mathcal{B}$ is convex.

Proof Suppose $U \subseteq X$ and $x \in U$. In order to show that $\mathcal{B}$ is a base for the topology, we have to show that there exists some set $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By the characterization of the generated topology, U is a union of finite intersections of elements of Λ. Hence we have

$$x \in \bigcap_{j=1}^{n} B_{r_j}^{i_j}(x_j)$$

for some $n > 0, i_j \in I, r_j > 0$, and $x_j \in X$. Then $x \in B_{r_j}^{i_j}(x_j)$, so, by definition $p_{i_j}(x - x_j) < r_j$ for each j. Therefore, if we set

$$r = \min\{r_j - p_{i_j}(x - x_j) \colon j = 1, \cdots, n\},$$

then we have $B_r^{i_j}(x) \subseteq B_{r_j}^{i_j}(x_j)$ for each $j = 1, \cdots, n$. Hence

$$B = \bigcap_{j=1}^{n} B_r^{i_j}(x) \in \mathcal{B},$$

and we have $x \in B \subseteq U$. ■

Proposition 1.30 *Let $(p_i)_{i\in I}$ be a family of seminorms on a vector space X. Then the induced topology on X is Hausdorff if and only if the family $(p_i)_{i\in I}$ is separating.*

Remark 1.17 If any one of the seminorms in our family is a norm, then the corresponding topology is automatically Hausdorff (for example, this is the case for $C_b^\infty(\mathbb{R})$). On the other hand, the topology can be Hausdorff even if no individual seminorms in a norm (consider $L^1_{loc}(\mathbb{R})$).

Examples 1.8

1. Given an open subset Ω of $\mathbb{R}^m$ with the euclidean topology, the space $C(\Omega)$ of real valued continuous functions on Ω with the so-called topology of uniform convergence on compact sets is a locally convex topological vector space. This topology is defined by the family $\mathcal{P}$ of all the seminorms on $C(\Omega)$ given by

$$p_K(f) := \max_{x\in K} |f(x)|, \ \forall\, K \subseteq \Omega \text{ compact}.$$

Moreover, the linear topology $\tau_{\mathcal{P}}$ induced from the family $\mathcal{P}$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_K(f) = 0, \forall\, K$ compact subsets of Ω then in particular $p_{\{x\}}(f) := |f(x)| = 0\ \forall\, x \in \Omega$, which implies $f \equiv 0$ on Ω.

More generally, for any X locally compact we have that $C(X)$ with the topology of uniform convergence on compact subsets of X is a locally convex topological vector space.

2. Let $\mathbb{N}_0$ be the set of all non-negative integers. For any $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_0^m$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. For any $\beta \in \mathbb{N}_0^m$, the symbol D^β denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^{m} \beta_i$, i.e.,

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_m^{\beta_m}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_m}}{\partial x_m^{\beta_m}}.$$

   (a) Let $\Omega \subseteq \mathbb{R}^m$ open in the euclidean topology. For any $k \in \mathbb{N}_0$, let $C^k(\Omega)$ be the set of all real valued $k-$times continuously differentiable functions on Ω, i.e., all the derivatives of f of order $\leq k$ exist (at every point of Ω) and are continuous functions in Ω. Clearly, when $k = 0$ we get the set $C(\Omega)$ for all real valued continuous functions on Ω and when $k = \infty$ we get the so-called set of all infinitely differentiable functions or smooth functions on Ω. For any $k \in \mathbb{N}_0$, $C^k(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$. The topology given by the following family of seminorms on $C^k(\Omega)$:

$$p_{d,K}(f) := \sup_{\substack{\beta \in \mathbb{N}_0^m \\ |\beta| \le d}} \sup_{x \in K} \left|(D^\beta f)(x)\right|, \ \forall\, K \subseteq \Omega \text{ compact } \forall\, d \in \{0, 1, \cdots, k\},$$

makes $C^k(\Omega)$ into a locally convex topological vector space. (Note that when $k = \infty$ we have $m \in \mathbb{N}_0$.)

(b) The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^m$ is defined as the set $\mathcal{S}(\mathbb{R}^m)$ of all real-valued functions which are defined and infinitely differentiable on $\mathbb{R}^m$ and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of x, i.e.,

$$\mathcal{S}(\mathbb{R}^m) = \left\{ f \in C^\infty(\mathbb{R}^m) \colon \sup_{x \in \mathbb{R}^m} \left|x^\alpha D^\beta f(x)\right| < \infty, \ \forall\, \alpha, \beta \in \mathbb{N}_0^m \right\}.$$

If f is a smooth function with compact support in $\mathbb{R}^m$ then $f \in \mathcal{S}(\mathbb{R}^m)$, since any derivative of f is continuous and supported on a compact subset of $\mathbb{R}^m$, so $x^\alpha(D^\beta f(x))$ has a maximum in $\mathbb{R}^m$ by the extreme value theorem.

The Schwartz space $\mathcal{S}(\mathbb{R}^m)$ is a vector space over $\mathbb{R}$ and the topology given by the family $\mathcal{P}$ of seminorms on $\mathcal{S}(\mathbb{R}^m)$:

$$p_{\alpha,\beta} := \sup_{x \in \mathbb{R}^m} \left|x^\alpha D^\beta f(x)\right|, \ \forall\, \alpha, \beta \in \mathbb{N}_0^m$$

makes $\mathcal{S}(\mathbb{R}^m)$ into a locally convex topological vector space. Indeed, the family is clearly separating, because if $p_{\alpha,\beta}(f) = 0$, $\forall\, \alpha, \beta \in \mathbb{N}_0^m$ then in particular $p_{0,0}(f) = \sup_{x \in \mathbb{R}^m} |f(x)| = 0 \ \forall\, x \in \mathbb{R}^m$, which implies $f \equiv 0$ on $\mathbb{R}^m$.

Note that $\mathcal{S}(\mathbb{R}^m)$ is a linear subspace of $C^\infty(\mathbb{R}^m)$, but its topology $\tau_{\mathcal{P}}$ on $\mathcal{S}(\mathbb{R}^m)$ is finer than the subspace topology induced on it by $C^\infty(\mathbb{R}^m)$.

Theorem 1.12 *Let X be a vector space whose topology is induced from a family of seminorms $(p_i)_{i \in I}$. Then given any net $(x_\alpha)_{\alpha \in J}$ and any $x \in X$, we have*

$$x_\alpha \to x \Leftrightarrow \forall\, i \in I,\ p_i(x - x_\alpha) \to 0.$$

Proof $\Rightarrow$. Suppose that $x_\alpha \to x$, and fix any $i \in I$ and $\varepsilon > 0$. Then $B^i_\varepsilon(x)$ is an open neighborhood of x, so by definition of convergence with respect to a net, there exists an $\alpha_0 \in J$ such that

$$\alpha \succ \alpha_0 \Rightarrow x_\alpha \in B^i_\varepsilon(x).$$

Therefore for all $\alpha \succ \alpha_0$ we have $p_i(x - x_\alpha) < \varepsilon$, so $p_i(x - x_\alpha) \to 0$.

$\Leftarrow$. Suppose that $p_i(x - x_\alpha) \to 0$ for every $i \in I$, and let U be any open neighborhood of x. Then by Theorem 1.11, we can find an $r > 0$ and finitely many $i_1, \cdots, i_n \in I$ such that

$$x \in \bigcap_{j=1}^{n} B^{i_j}_r(x) \subseteq U.$$

Now, given any $j = 1, \cdots, n$ we have $p_{i_j}(x - x_\alpha) \to 0$. Hence, for each j we can find $\alpha_j \in J$ such that

$$\alpha \succ \alpha_j \ \Rightarrow p_{i_j}(x - x_\alpha) < r.$$

Since J is a directed set, there exists some $\alpha_0 \in J$ such that $\alpha_0 \succ \alpha_j$ for $j = 1, \cdots, n$. Thus, for all $\alpha \succ \alpha_0$ we have $p_{i_j}(x - x_\alpha) < r$ for each $j = 1, \cdots, n$, so

$$x_\alpha \in \bigcap_{j=1}^{n} B^{i_j}_r(x) \subseteq U, \quad \alpha \succ \alpha_0.$$

Hence $x_\alpha \to x$. ■

Corollary 1.3 *Let X be a vector space whose topology is induced from a family of seminorms $(p_i)_{i\in I}$, let Y be any topological space, and fix $x \in X$. Then the following two statements are equivalent.*

1. *$T : X \to Y$ is continuous at x.*
2. *For any net $(x_\alpha)_{\alpha\in J}$,*

$$p_i(x - x_\alpha) \to 0 \quad \text{for each } i \in I \quad \Rightarrow T(x_\alpha) \to T(x) \text{ in } Y.$$

Proposition 1.31 *Let X be a vector space whose topology is induced from a family of seminorms $(p_i)_{i\in I}$. Then,*

1. *for all $i \in I$, p_i is continuous.*
2. *A set $Y \subseteq X$ is bounded if and only if p_i is bounded on Y for all $i \in I$.*

Proof

1. Let $i \in I$. Because of the reverse triangle inequality, $p_i(x - x_\alpha) \to 0$ implies $p_i(x_\alpha) \to p_i(x)$. Hence each seminorm p_i is continuous with respect to the induced topology.
2. Suppose $Y \subseteq X$ is bounded. Take $i \in I$. Then $B_1^{p_i}$ is a neighborhood of θ. Hence,

$$Y \subseteq \rho B_1^{p_i}$$

for some $\rho > 0$ (by definition of boundedness). Hence, for all $x \in Y$,

$$x \in \{\rho y \in X : p(y) < 1\} = \{\rho y \in X : p(\rho y) < \rho\} = \{z \in X : p(z) < \rho\},$$

i.e., $p(x) < \rho$.

Conversely, if $p_i(Y)$ is bounded for every $i \in I$. Then there are numbers r_i such that

$$\sup_{x \in Y} p_i(x) < r_i.$$

Let U be a neighborhood of θ. Again

$$\bigcap_{j=1}^{n} B_r^{i_j}(\theta) \subseteq U.$$

Choose $m > \dfrac{M_{i_j}}{r_{i_j}} (1 \leq j \leq n)$. If $x \in Y$ then $p_{i_j}(\dfrac{x}{m}) < \dfrac{M_{i_j}}{m} < r_{i_j} \Rightarrow \dfrac{x}{m} \in U \Rightarrow x \in mU$.

■

Theorem 1.13 *If X is a vector space whose topology τ is induced from a separating family of seminorms $(p_i)_{i \in I}$, then (X, τ) is a locally convex topological vector space.*

Proof We have already seen that there is a base for the topology τ that consists of convex open sets, so we just have to show that vector addition and scalar multiplication are continuous with respect to this topology.

Suppose that $((\lambda_\alpha, x_\alpha))_{\alpha \in J}$ is any net in $\mathbb{K} \times X$, and that $(\lambda_\alpha, x_\alpha) \to (\lambda, x)$ with respect to the product topology on $\mathbb{K} \times X$. This is equivalent to assuming that $\lambda_\alpha \to \lambda$ in $\mathbb{K}$ and $x_\alpha \to x$ in X. Fix any $i \in I$ and any $\varepsilon > 0$. Suppose that $p_i(x) \neq 0$. Since $p_i(x - x_\alpha) \to 0$, there exist $\alpha_1, \alpha_2 \in J$ such that

$$\alpha \succ \alpha_1 \;\; \Rightarrow \;\; |\lambda - \lambda_\alpha| < \min\left\{\frac{\varepsilon}{2p_i(x)}, 1\right\},$$

and

$$\alpha \succ \alpha_2 \quad \Rightarrow \quad p_i(x - x_\alpha) < \frac{\varepsilon}{2(|\lambda| + 1)}.$$

By definition of directed set, there exists $\alpha_0 \succ \alpha_1, \alpha_2$, so both of these inequalities hold for $\alpha \succ \alpha_0$. In particular, $(\lambda_\alpha)_{\alpha \succ \alpha_0}$ is a bounded net, with $|\lambda_\alpha| < |\lambda| + 1$ for all $\alpha \succ \alpha_0$. Hence, for $\alpha \succ \alpha_0$ we have

$$\begin{aligned} p_i(\lambda x - \lambda_\alpha x_\alpha) &\leq p_i(\lambda x - \lambda_\alpha x) + p_i(\lambda_\alpha x - \lambda_\alpha x_\alpha) \\ &= |\lambda - \lambda_\alpha|\, p_i(x) + |\lambda_\alpha|\, p_i(x - x_\alpha) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

If $p_i(x) = 0$ then we similarly obtain $p_i(\lambda x - \lambda_\alpha x_\alpha) < \dfrac{\epsilon}{2}$ for $\alpha \succ \alpha_0$. Thus we have $p_i(\lambda x - \lambda_\alpha x_\alpha) \to 0$. Since this is true for every i, Theorem 1.12 implies that $\lambda_\alpha x_\alpha \to \lambda x$. ■

Theorem 1.14 *The topology of a locally convex topological vector space X is given by the collection of seminorms obtained as Minkowski functionals p_U associated to a local basis at θ consisting of convex balanced open.*

Proof The proof is straightforward. With or without local convexity, every neighborhood of θ contains a balanced neighborhood of θ. Thus, a locally convex topological vector space has a local basis $\mathcal{B}$ at θ of balanced convex open sets.

Every open $U \in \mathcal{B}$ can be recovered from the corresponding seminorm by

$$U = \text{int}U = \{x \in X \colon\, p_U(x) < 1\}.$$

Oppositely, every seminorm local basis open

$$\{x \in X \colon\, p_U(x) < r\}$$

is simply rU. Thus, the original topology is at least as fine as the seminorm topology. ■

1.2.5 *Metrizable Topological Vector Spaces*

What does it take for a topological vector space (X, τ) to be metrizable? Suppose there is a metric d compatible with the topology τ. Thus, all open sets are unions of open balls, and in particular, the countable collection of balls $B_{\frac{1}{n}}(\theta)$ forms a local base at the origin.

Theorem 1.15 *A Hausdorff topological vector space is metrizable if and only if zero has a countable neighborhood base. In this case, the topology is generated by a translation invariant metric.*

Proof Let (X, τ) be a topological vector space. If τ is metrizable, then τ has clearly a neighborhood base at θ. For the converse, assume that τ has a countable neighborhood base at θ. Choose a countable base $\{V_n\}$ of circled neighborhoods of θ such that $V_{n+1} + V_{n+1} + V_{n+1} \subseteq V_n$ holds for each n. Now define the function $\rho\colon X \to [0, \infty)$ by

$$\rho(x) = \begin{cases} 1, & \text{if } \quad x \notin V_1, \\ 2^{-k}, & \text{if } x \in V_k \setminus V_{k+1}, \\ 0, & \text{if } \quad x = \theta. \end{cases}$$

Then it is easy to check that for each $x \in X$ we have the following:

1. $\rho(x) \geq 0$ if and only if $x = \theta$.
2. $x \in V_k$ for some k if and only if $\rho(x) \leq 2^{-k}$
3. $\rho(x) = \rho(-x)$ and $\rho(\alpha x) \leq \rho(x)$ for all $|\alpha| \leq 1$.
4. $\lim\limits_{\alpha \to 0} \rho(\alpha x) = 0$.

We also note the following property : $x_n \xrightarrow{\tau} \theta$ if and only $\rho(x_n) \longrightarrow 0$.

Now by means of the function ρ we define the function $\Pi\colon X \to [0, \infty)$ via the formula

$$\Pi(x) = \inf\left\{ \sum_{i=1}^{n} \rho(x_i)\colon x_1, \cdots, x_n \in X. \text{ and } \sum_{i=1}^{n} x_i = x \right\}.$$

The function Π satisfies the following properties.

(a) $\Pi(x) \geq 0$ for each $x \in X$.
(b) $\Pi(x + y) \leq \Pi(x) + \Pi(y)$ for all $x, y \in X$.
(c) $\frac{1}{2}\rho(x) \leq \Pi(x) \leq \rho(x)$ for each $x \in X$ (so $\Pi(x) = 0$ if and only if $x = \theta$).

Property (a) follows immediately from the definition of Π. Property (b) is straightforward. The proof of (c) will be based upon the following property:

$$\text{If } \sum_{i=1}^{n} \rho(x_i) < \frac{1}{2^m}, \textit{ then } \sum_{i=1}^{n} x_i \in V_m. \tag{1.11}$$

To verify (1.11), we use induction on n. For $n = 1$ we have $\rho(x_1) < \frac{1}{2^m}$, and consequently $x_1 \in V_{m+1} \subseteq V_m$ is trivially true. For the induction step, assume that if $\{x_i : i \in I\}$ is any collection of at most n vectors satisfying $\sum_{i \in I} \rho(x_i) < \frac{1}{2^m}$ for some $m \in \mathbb{N}$, then $\sum_{i \in I} x_i \in V_m$. Suppose that $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^m}$ for some $m \in \mathbb{N}$. Clearly, we have $\rho(x_i) < \frac{1}{2^{m+1}}$, so $x_i \in V_{m+1}$ for each $1 \leq n+1$. We now distinguish two cases.

Case 1: $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^{m+1}}$

Clearly $\sum_{i=1}^{n} \rho(x_i) < \frac{1}{2^{m+1}}$, so by the induction hypothesis $\sum_{i=1}^{n} x_i \in V_{m+1}$. Thus

$$\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n} x_i + x_{n+1} \in V_{m+1} + V_{m+1} \subseteq V_m.$$

Case 2: $\sum_{i=1}^{n+1} \rho(x_i) \geq \frac{1}{2^{m+1}}$

Let $1 \leq k \leq n+1$ be the largest k such that $\sum_{i=k}^{n+1} \rho(x_i) \geq \frac{1}{2^{m+1}}$. If $k = n+1$, then $\rho(x_{n+1}) = \frac{1}{2^{m+1}}$, so from $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^m}$ we have $\sum_{i=1}^{n} \rho(x_i) < \frac{1}{2^{m+1}}$. But then, as in Case 1, we get $\sum_{i=1}^{n+1} x_i \in V_m$. Thus, we can assume that $k < n+1$. Assume first that $k > 1$. From the inequalities $\sum_{i=1}^{n+1} \rho(x_i) < \frac{1}{2^m}$ and $\sum_{i=k}^{n+1} \rho(x_i) \geq \frac{1}{2^{m+1}}$, we obtain $\sum_{i=1}^{k-1} \rho(x_i) < \frac{1}{2^{m+1}}$. So our induction hypothesis yields $\sum_{i=1}^{k-1} x_i \in V_{m+1}$. Also by the choice of k we have $\sum_{i=k+1}^{n+1} \rho(x_i) < \frac{1}{2^{m+1}}$, and thus by our induction hypothesis also we have $\sum_{i=k+1}^{n+1} x_i \in V_{m+1}$. Therefore, in this case we obtain

$$\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{k-1} x_i + x_k + \sum_{i=k+1}^{n+1} x_i \in V_{m+1} + V_{m+1} + V_{m+1} \subseteq V_m.$$

If $k = 1$, then we have $\sum_{i=2}^{n+1} \rho(x_i) < \frac{1}{2^{m+1}}$, so $\sum_{i=2}^{n+1} x_i \in V_{m+1}$. This implies $\sum_{i=1}^{n+1} x_i = x_1 + \sum_{i=2}^{n+1} x_i \in V_{m+1} + V_{m+1} \subseteq V_m$. This completes the induction and the proof of (1.11).

Next, we verify (c). To this end, let $x \in X$ satisfy $\rho(x) = \frac{1}{2^m}$ for some $m \geq 0$. Also, assume by way of contradiction that the vectors $x_1, \cdots, x_k$ satisfy $\sum_{i=1}^{k} x_i = x$ and $\sum_{i=1}^{k} \rho(x_i) < \frac{1}{2}\rho(x) = \frac{1}{2^{m+1}}$. But then, from (1.11) we get $x = \sum_{i=1}^{k} x_i \in V_{m+1}$, so $\rho(x) \leq \frac{1}{2^{m+1}} < \frac{1}{2^m} = \rho(x)$, which is impossible. This contradiction, establishes the validity of (c).

Finally, for each $x, y \in X$ define $d(x, y) = \Pi(x - y)$ and note that d is a translation invariant metric that generates τ. ■

Definition 1.35 Let (X, τ) be a topological vector space.

1. X is an F-space (completely metrizable topological vector space) if its topology is induced by a complete translationally invariant metric. In other words, a completely metrizable topological vector space is a complete topological vector space having a countable neighborhood base at θ. Every Banach space is an F-space. An F-space is a Banach space if in addition $d(\alpha x, \theta) = |\alpha|\, d(x, \theta)$.
2. X is a Fréchet space if it is a locally convex F-space.

Definition 1.36 A complete topological vector space (Y, Γ) is called a topological completion or simply a completion of another topological vector space (X, τ) if there is a linear homeomorphism $T : X \to Y$ such that $T(X)$ is dense in Y, identifying X with $T(X)$, we can think of X as a subspace of Y.

Theorem 1.16 *Every topological vector space has a unique (up to linear homeomorphism) topological completion.*

It turns out that the existence of a countable local base is also sufficient for metrizability. (It suffices that τ is induced from a separating countable family of seminorms $(p_n)_n$). Indeed, there exists a translation-invariant metric compatible with τ. One can show that the following is a compatible metric:

$$d(x, y) = \max_n \frac{\alpha_n p_n(x - y)}{1 + p_n(x - y)},$$

where $(\alpha_n)_n$ is any sequence of positive numbers that decays to 0 (it is easy to see that the maximum is indeed attained). Clearly, $d(x, x) = 0$. Also, since the p_n's are separating $d(x, y) > 0$ for $x \neq y$. Symmetry, as well as translational invariance are obvious. Finally, the triangle inequality follows from the fact that every p_n is subadditive, and that $a \leq b + c$ implies that

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

It remains to show that this metric is compatible with the topology τ. One can also define the following translation-invariant metric compatible with τ

$$d(x, y) = \sum_{n=1}^{+\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}.$$

Example 1.9 Let $s = \{(x_n)_{n\geq 1} : x \in \mathbb{K} \text{ for all } n \geq 1\}$, the space of all scalar sequences. The topology of pointwise convergence is described by the seminorms p_k, $(k \geq 1)$, $p_k((x_n)_{n\geq 1}) = |x_k|$ and the metric is

$$d(x, y) = \sum_{n=1}^{+\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad x = (x_n)_{n\geq 1}, \quad y = (y_n)_{n\geq 1}.$$

The ball $\overline{B}_{\frac{1}{4}}(\theta) = \{x : d(x, \theta) \leq \frac{1}{4}\}$ is not convex, since $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$ $\in \overline{B}_{\frac{1}{4}}(\theta)$, but $\frac{3}{4}(1, 0, 0, \cdots) + \frac{1}{4}(0, 1, 0, \cdots) = (\frac{3}{4}, \frac{1}{4}, 0, 0, \cdots) \notin \overline{B}_{\frac{1}{4}}(\theta)$.

Theorem 1.17 *Let (X, τ) be topological vector space that has a countable local base. Then there is a metric d on X such that*:

1. *d is compatible with τ (every τ-open set is a union of d-open balls).*
2. *The open balls $B_r(\theta)$ are balanced.*
3. *d is invariant: $d(x + z, y + z) = d(x, y)$.*
4. *If, in addition, X is locally convex, then d can be chosen such that all open balls are convex.*

Theorem 1.18 *A topological vector space (X, τ) is normable if and only if there exists a convex bounded open neighborhood.*

Proof If (X, τ) is normable then $B_1 = \{x : \|x\| < 1\}$ is convex and bounded. Suppose that there exists an open convex and bounded neighborhood V of θ. Set

$$U = \bigcap_{|\alpha|=1} \alpha V.$$

Since U is the intersection of convex sets it is convex. It is balanced because for every $|\beta| \leq 1$,

$$\beta U = \bigcap_{|\alpha|=1} \beta\alpha V = \bigcap_{|\alpha|=1} |\beta|\alpha V = |\beta| U,$$

and by convexity,

$$|\beta|\, U = |\beta|\, U + (1 - |\beta|)\{\theta\} \subseteq U.$$

Since U contains θ, int$U$ is balanced, it is also convex. Then there exists a convex and balanced (and certainly bounded) open neighborhood $W = \mathrm{int}U \subseteq V$. Set

$$\|x\| = p_W(x),$$

where p_W is the Minkowski functional of W. We will show that this indeed a norm. Clearly, $\|x\| = 0$ if and only if $x = \theta$. Since W is balanced then $p_W(\alpha x) = |\alpha|\, p_W(x)$. The triangle inequality follows from the properties of p_W. It remains to show this norm is compatible with the topology τ. This follows from the fact that

$$B_r(\theta) = \{x : \|x\| < r\} = \{x : p_W(x) < r\} = \{x : p_W(\frac{x}{r}) < 1\} \subseteq rW,$$

which means that $B_r(\theta)$ is bounded, hence

$$\{B_r(\theta) : r > 0\}$$

is a local base. ■

Example 1.10 Let Ω be an open set in $\mathbb{R}^m$. We consider the space $C(\Omega)$ of all continuous functions. Note that the sup-norm does not work here. There exist unbounded continuous functions on open sets.

Every open set Ω in $\mathbb{R}^m$ can be written as

$$\Omega = \bigcup_{n=1}^{\infty} K_n,$$

where $K_n \Subset K_{n+1}$, where the K_n are compact, and $\Subset$ stands for compactly embedded, i.e., K_n is a compact set in the interior of K_{n+1}. We topologize $C(\Omega)$ with the separating family of seminorms,

$$p_n(f) = \max\{|f(x)| : x \in K_n\} = \|f\|_{K_n}.$$

(These are clearly seminorms, and they are separating because for every $f \neq 0$ there exists an n such that $f_{|K_n} \neq 0$).

Since the p_n's are monodically increasing,

$$\bigcap_{d=1}^{D} \bigcap_{k=1}^{n} B^k_{\frac{1}{d}}(\theta) = \bigcap_{d=1}^{D} \bigcap_{k=1}^{n} \{f : p_k(f) < \frac{1}{d}\} = B^n_{\frac{1}{D}}(\theta),$$

which means that the $B^n_{\frac{1}{D}}(\theta)$ form a convex local base for $C(\Omega)$. In fact, $B^n_{\frac{1}{D}}(\theta)$ contains a neighborhood obtained by taking n, D to be the greatest of the two, from which follows that

$$B^n_{\frac{1}{n}}(\theta) = \{f : p_n(f) < \frac{1}{n}\}$$

is a convex local base for $C(\Omega)$, and the p_n's are continuous in this topology. We can thus endow this topological space with a compatible metric, for example,

$$d(f, g) = \max_n \frac{2^{-n} p_n(f - g)}{1 + p_n(f - g)}.$$

We will now show that this space is complete. Recall that if a topological vector space has a compatible metric with respect to which is complete, then it is called an **F**-space. If, moreover, the space is locally convex, then it is called a Fréchet space. Thus, $C(\Omega)$ is a Fréchet space. Let $(f_n)_n$ be a Cauchy sequence. This means that for every $\varepsilon > 0$ there exists an N, such that for every $d, n > N$,

$$\max_k \frac{2^{-k} p_k(f_n - f_d)}{1 + p_k(f_n - f_d)} < \varepsilon,$$

and so,

$$(\forall\, k \geq 1) \quad \frac{2^{-k} p_k(f_n - f_d)}{1 + p_k(f_n - f_d)} < \varepsilon,$$

which means that $(f_n)_n$ is a Cauchy sequence in each K_k (endowed with the sup-norm), and hence converges uniformly to a function f. Given ε and let M such that $2^{-M} < \varepsilon$, then

$$\max_{k>M} \frac{2^{-M} p_k(f_n - f)}{1 + p_k(f_n - f)} < \varepsilon,$$

and there exists an N, such that for every $n > N$,

$$\max_{k\leq M} \frac{2^{-M} p_k(f_n - f)}{1 + p_k(f_n - f)} < \varepsilon,$$

which implies that $f_n \longrightarrow f$, hence the space is indeed complete.

The question remains whether $C(\Omega)$ with this topology is normable. For this, the origin must have a convex bounded neighborhood. Recall that a set Y is bounded if and only if $\{p_n(f) : f \in Y\}$ is bounded for every n, i.e., if

$$\{\sup\{|f(x)| : x \in K_n\} : f \in Y\}$$

is a bounded set for every n, or if

$$\forall\, n \geq 1 \quad \sup\{|f(x)| : x \in K_n,\, f \in Y\} < \infty.$$

Because the $B^n_{\frac{1}{n}}(\theta)$ form a base, every neighborhood of θ contains a set

$$B^k_{\frac{1}{k}}(\theta),$$

hence,

$$\sup\{|f(x)| : x \in K_n,\, f \in Y\} \geq \sup\{\|f\|_{K_n} : \|f\|_{K_k} < \frac{1}{k}\}.$$

The right hand side can be made as large as we please for $n > k$, i.e., no set is bounded, and hence the space is not normable.

1.2.6 Finite Dimensional Topological Vector Spaces

Lemma 1.6 *Let (X, τ) be a topological vector space. Any linear map $T : \mathbb{K}^n \to X$ is continuous.*

Proof Denote by $(e_i)_{1 \leq i \leq n}$ the standard basis in $\mathbb{K}^n$ and set

$$u_j = T(e_j) \quad j = 1, \cdots, n.$$

By linearity, for any $x = (x_1, \cdots, x_n) = \sum_{j=1}^{n} x_j e_j$

$$T(x) = \sum_{j=1}^{n} x_j u_j.$$

The map $x \mapsto x_j$ (which is linear map $\mathbb{K}^n \to \mathbb{K}$) is continuous and so are addition and scalar multiplication in X. ■

Proposition 1.32 *Let (X, τ) be a topological vector space. Then*:

1. *Every finite dimensional subspace Y of X is a closed subset of X.*
2. *If Y is an n-dimensional subspace of X and $(u_i)_{1 \leq i \leq n}$ is a basis for Y, then the map $T : \mathbb{K}^n \to Y$ defined by $T(x_1, \cdots, x_n) = \sum_{j=1}^{n} x_j u_j$ is a topological isomorphism of $\mathbb{K}^n$ equipped with its Euclidean topology, onto X. That is,*

specifically, a net $(x^\alpha)_\alpha = \left(\sum_{j=1}^n x_j^\alpha u_j\right)_\alpha$ *converges to an element* $x = \sum_{j=1}^n x_j u_j \in Y$ *if and only if each net* $(x_j^\alpha)_\alpha$ *converges to* x_j, $1 \le j \le n$.

Proof

1. We prove part 1 by induction on the dimension of the subspace Y. First, if Y has dimension 1, let $y \neq \theta \in Y$ be a basis for Y. If $(\lambda_\alpha y)_\alpha$ is a net in Y that converges to an element $x \in X$, then the net $(\lambda_\alpha)_\alpha$ must be eventually bounded in $\mathbb{K}$, in the sense that there must exist an index α_0 and a constant M such that $|\lambda_\alpha| \le M$ for all $\alpha \succ \alpha_0$. Indeed, if the net $(\lambda_\alpha)_\alpha$ were not eventually bounded, let $(\lambda_{\alpha_\beta})_\beta$ be a subnet for which $\lim_\beta \left|\lambda_{\alpha_\beta}\right| = \infty$. Then

$$\begin{aligned} y &= \lim_\beta \frac{1}{\lambda_{\alpha_\beta}} \lambda_{\alpha_\beta} y \\ &= \lim_\beta \frac{1}{\lambda_{\alpha_\beta}} \lim_\beta \lambda_{\alpha_\beta} y \\ &= 0 \times x \\ &= \theta. \end{aligned}$$

which is a contradiction. So, the net $(\lambda_\alpha)_\alpha$ is bounded. Let $(\lambda_{\alpha_\beta})_\beta$ be a convergent subnet of $(\lambda_\alpha)_\alpha$ with limit λ. Then

$$x = \lim_\alpha \lambda_\alpha y = \lim_\beta \lambda_{\alpha_\beta} = \lambda y.$$

whence $x \in Y$, and Y is closed.

Assume now that any n-1-dimensional subspace is closed, and let Y have dimension $n > 1$. Let $\{y_1, \cdots, y_n\}$ be a basis for Y, and write Y' for the linear span of $y_1, \cdots, y_{n-1}$. Then elements y of Y can be written uniquely in the form $y = y' + \lambda y_n$, for $y' \in Y'$ and $\lambda \in \mathbb{K}$. Suppose that x is an element of the closure of Y, i.e., $x = \lim_\alpha (y'_\alpha + \lambda_\alpha y_n)$. As before, we have that the net $(\lambda_\alpha)_\alpha$ must be bounded. Indeed, if the net $(\lambda_\alpha)_\alpha$ were not bounded, then let $(\lambda_{\alpha_\beta})_\beta$ be a subnet for which $\lim_\beta \left|\lambda_{\alpha_\beta}\right| = \infty$. Then

$$\theta = \lim_\beta \frac{1}{\lambda_{\alpha_\beta}} x = \lim_\beta \frac{y'_{\alpha_\beta}}{\lambda_{\alpha_\beta}} + y_n,$$

or

$$y_n = -\lim_\beta \frac{y'_{\alpha_\beta}}{\lambda_{\alpha_\beta}},$$

implying that y_n belongs to the closure of the closed subspace Y', this is impossible, showing that the net $(\lambda_\alpha)_\alpha$ is bounded. Hence, letting $(\lambda_{\alpha_\beta})_\beta$ be a convergent subnet of $(\lambda_\alpha)_\alpha$, say $\lambda = \lim_\beta \lambda_{\alpha_\beta}$, we have

$$x = \lim_\beta (y'_{\alpha_\beta} + \lambda_{\alpha_\beta} y_n),$$

showing that

$$x - \lambda y_n = \lim_\beta y'_{\alpha_\beta},$$

whence, since Y' is closed, there exists a $y' \in Y'$ such that $x - \lambda y_n = y'$. Therefore, $x = y' + \lambda y_n \in Y$, and Y is closed, proving part 1.

2. We prove part 2 for real vector spaces. The map $T: \mathbb{R}^n \to Y$ of part 2 is obviously linear, one to one and onto. Also, it is continuous by previous lemma. Let us show that T^{-1} is continuous. Thus, let $(x^\alpha)_\alpha = \left(\sum_{j=1}^n x_j^\alpha u_j\right)_\alpha$ converge to θ in Y. Suppose, by way of contradiction, that there exists an j for which the net $(x_j^\alpha)_\alpha$ does not converge to 0. Then let $(x_j^{\alpha^\beta})_\beta$ be a subnet for which $\lim_\beta x_j^{\alpha^\beta} = x_j$, where x_j either is $\pm\infty$ or is a nonzero real number. Write $x^\alpha = x_j^\alpha u_j + x'^\alpha$. Then

$$\frac{1}{x_j^{\alpha^\beta}} x^{\alpha^\beta} = u_j + \frac{1}{x_j^{\alpha^\beta}} x'^{\alpha^\beta},$$

whence

$$u_j = -\lim_\beta \frac{1}{x_j^{\alpha^\beta}} x'^{\alpha^\beta},$$

implying that u_j belongs to the (closed) subspace spanned by the vectors

$$u_1, \cdots, u_{j+1}, \cdots, u_n.$$

and this is a contradiction, since the u_j's form a basis of Y. Therefore, each of the nets $(x_j^\alpha)_\alpha$ converges to 0, and T^{-1} is continuous.

■

Corollary 1.4 *There exists a unique topology on $\mathbb{K}^n$ (viewed as a topological vector space), and all n-dimensional topological vector spaces are topologically isomorphic.*

There are no infinite dimensional locally compact topological vector spaces. This is essentially due to F. Riesz.

Theorem 1.19 *A topological space is locally compact if and only if is finite dimensional.*

Proof Let (X, τ) be a topological vector space. If X is finite dimensional, then τ coincides with the Euclidean topology and since the closed balls are compact sets, it follows that (X, τ) is locally compact.

For the converse assume that (X, τ) is locally compact and let V be a compact neighborhood of θ. From $V \subseteq \bigcup_{x \in V}(x + \frac{1}{2}V)$, we see that there exists a finite subset $\{x_1, \cdots, x_k\}$ of V such that

$$V \subseteq \bigcup_{i=1}^{k}(x_i + \frac{1}{2}V) = \{x_1, \cdots, x_k\} + \frac{1}{2}V. \tag{1.12}$$

Let Y be a linear span of $x_1, \cdots, x_k$. From (1.12), we get $V \subseteq Y + \frac{1}{2}V$. This implies $\frac{1}{2}V \subseteq \frac{1}{2}(Y + \frac{1}{2}V) = Y + \frac{1}{2^2}V$, so $V \subseteq Y + (Y + \frac{1}{2^2}V) = Y + \frac{1}{2^2}V$. By induction we see that

$$V \subseteq Y + \frac{1}{2^n}V \tag{1.13}$$

for each n. Next, fix $x \in V$. From (1.13), it follows that for each n there exist $y_n \in Y$ and $v_n \in V$ such that $x = y_n + \frac{1}{2^n}v_n$. Since V is compact, there exists a subnet (v_{n_α}) of the sequence (v_n) such that $v_{n_\alpha} \xrightarrow{\tau} v \in X$ (and clearly $\frac{1}{2^{n_\alpha}} \longrightarrow 0$ in $\mathbb{R}$). So

$$y_{n_\alpha} = x - \frac{1}{2^{n_\alpha}}v_{n_\alpha} \xrightarrow{\tau} x - 0v = x.$$

Since (Proposition 1.32 1.) Y is a closed subspace, $x \in Y$. That is, $V \subseteq Y$. Since V is also an absorbing set, it follows that $X = Y$, so that X is finite dimensional. ■

Theorem 1.20

1. *Let $Y_1, \cdots, Y_n$ be compact convex sets in a vector space (X endowed with a linear topology τ). Then*

$$\text{conv}(Y_1 \cup \cdots \cup Y_n)$$

is compact.
2. *Let (X, τ) be a locally convex topological vector space. If $Y \subseteq X$ is totally bounded then* $\text{conv}(Y)$ *is totally bounded as well.*
3. *If (X, τ) is a Fréchet space and $K \subseteq X$ is compact then* $\overline{\text{conv}}(K)$ *is compact.*
4. *If $K \subseteq \mathbb{R}^n$ is compact then* $\text{conv}(K)$ *is compact.*

Proof

1. Let $S \subseteq \mathbb{R}^n$ be the simplex

$$S = \{(s_1, \cdots, s_n) \ : \ s_i \geq 0, \ \sum_{i=1}^{n} s_i = 1\}.$$

Set $Y = Y_1 \times \cdots \times Y_n$ and define the function $\varphi\colon S \times Y \to X$:

$$\varphi(s, y) = \sum_{i=1}^{n} s_i y_i.$$

Consider the set $K = \varphi(S \times Y)$. It is the continuous image of a compact set and it is therefore compact. Moreover,

$$K \supseteq \text{conv}(Y_1 \cup \cdots \cup Y_n).$$

It is easy to show that K is convex, and since it includes all the Y_i's it must in fact be equal to $\text{conv}(Y_1 \cup \cdots \cup Y_n)$.
2. Let U be an open neighborhood of θ. Because X is locally convex there exists a convex open neighborhood V of θ such that

$$V + V \subseteq U.$$

Since Y is totally bounded there exists a finite set F such that

$$Y \subseteq F + V \subseteq \text{conv}(F) + V.$$

Since the right hand side is convex

$$\text{conv}(Y) \subseteq \text{conv}(F) + V.$$

By the first item $\text{conv}(F)$ is compact, therefore there exists a finite set F' such that

$$\text{conv}(F) = F' + V,$$

i.e.,

$$\text{conv}(Y) \subseteq F' + V + V \subseteq F' + U,$$

which proves that $\text{conv}(Y)$ is totally bounded.

3. In every metric space the closure of a totally bounded set is totally bounded, and if the space is complete it is compact. Since K is compact, then it is totally bounded. By the previous item $\text{conv}(K)$ is totally bounded and hence its closure is compact.
4. $S \subseteq \mathbb{R}^n$ be the convex simplex. One can show that $\text{conv}(K)$ is the image of the continuous map $S \times K$:

$$(s, x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} s_i x_i,$$

whose domain is compact.

■

Corollary 1.5 *Let X be a vector space endowed with a linear topology τ. The convex hull of a finite set (polytope) is compact.*

Example 1.11 (Noncompact Convex Hull) Consider l_2, the space of all square summable sequences. For each n let $u_n = (\underbrace{0, \cdots, 0}_{n-1}, \frac{1}{n}, 0, 0, \cdots)$. Observe that $\|u_n\|_2 = \frac{1}{n}$, so $u_n \xrightarrow{\|.\|_2} \theta$. Consequently,

$$Y = \{u_1, u_2, u_3, \cdots\} \cup \{\theta\}$$

is norm compact subset of l_2. Since $\theta \in Y$, it is easy to see that

$$\text{conv}(Y) = \left\{ \sum_{i=1}^{k} \alpha_i u_i : \alpha_i \geq 0 \text{ for each } i \text{ and } \sum_{i=1}^{k} \alpha_i \leq 1 \right\}.$$

In particular, each vector of $\text{conv}(Y)$ has only finitely many nonzero components. We claim that $\text{conv}(Y)$ is not norm compact. To see this, set

$$x_n = (\frac{1}{2}, \frac{1}{2}.\frac{1}{2^2}, \frac{1}{3}.\frac{1}{2^3}, \cdots, \frac{1}{n}.\frac{1}{2^n}, 0, 0, \cdots) = \sum_{i=1}^{n} \frac{1}{2^i} u_i,$$

so $x_n \in \mathrm{conv}(Y)$. Now $x_n \overset{\|\cdot\|_2}{\longrightarrow} x = (\frac{1}{2}, \frac{1}{2}.\frac{1}{2^2}, \frac{1}{3}.\frac{1}{2^3}, \cdots, \frac{1}{n}.\frac{1}{2^n}, \frac{1}{n+1}.\frac{1}{2^{n+1}}, \cdots)$ in l_2. But $x \notin \mathrm{conv}(Y)$, so $\mathrm{conv}(Y)$ is not even closed, let alone compact.

Remark 1.18 In the above example, the convex hull of a compact set failed to be closed. The question remains whether the closure of the convex hull is compact. In general, the answer is no. To see this, let X the space of sequences that are eventually zero, equipped with the l_2-norm. Let Y as above, and note that $\overline{\mathrm{conv}}(Y)$ (where the closure is taken in X, not l_2) is not compact either. To see this, observe that the sequence $(x_n)_n$ defined above has no convergent subsequence (in X).

Proposition 1.33 *Let Y and Z are two nonempty convex subsets of a topological vector space (X, τ) such that Y is compact and Z is closed and bounded, then* $\mathrm{conv}(Y \cup Z)$ *is closed.*

Proof Let $x_i = (1 - \alpha_i)y_i + \alpha_i z_i \longrightarrow x$, where $0 \le \alpha_i \le 1$, $y_i \in Y$ and $z_i \in Z$ for each i. By passing to a subnet, we can assume that $y_i \longrightarrow y \in Y$ and $\alpha_i \longrightarrow \alpha \in [0, 1]$. If $\alpha > 0$, then $z_i \longrightarrow \dfrac{x - (1-\alpha)y}{\alpha} = z \in Z$, and consequently $x = (1-\alpha)y + \alpha z \in \mathrm{conv}(Y \cup Z)$.

Now consider the case $\alpha = 0$. The boundedness of Z and Proposition 1.20 imply $\alpha_i z_i \longrightarrow \theta$, so $x_i = (1 - \alpha_i)y_i + \alpha_i z_i \longrightarrow y$. Since the space is Hausdorff, $x = y \in \mathrm{conv}(Y \cup Z)$. ■

1.2.7 The Weak Topology of Topological Vector Spaces and the Weak Topology of Their Duals*

If X is a topological vector space then the weak topology on it is coarser than the origin topology: any set that is open in the original topology is open in the weak topology. From this, it follows that it is easier for a sequence to converge in the weak topology than in the original topology.

We will consider topological vector spaces (X, τ) over the field $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For definiteness we assume $\mathbb{K} = \mathbb{C}$.

Remark 1.19 Given a vector space X and a linear functional $\phi \colon X \to \mathbb{K}$, the map $p_\phi = |\phi| : X \ni x \mapsto |\phi(x)| \in [0, \infty[$ defines a seminorm on X.

Definition 1.37 Let (X, τ) be a topological vector space. The topological dual space X' is the set of all continuous linear maps $(X, \tau) \to \mathbb{K}$.

Next, we will discuss the geometric form of the Hahn-Banach theorems. The first geometric version is

Lemma 1.7 *Let (X, τ) be a real topological vector space, and let $V \subseteq X$ be a convex open set which contains θ. If $x_0 \in X \setminus V$, there exists $\psi \in X'$, such that $\psi(x_0) = 1$ and $\psi(x) < 1$, for all $x \in V$.*

It turns out that Lemma 1.7 is a particular case of a more general result:

Theorem 1.21 (Hahn-Banach Separation Theorem-Real Case) *Let (X, τ) be a real topological vector space, let $Z, W \subseteq X$ be nonempty convex sets with Z open, and $Z \cap W = \emptyset$. Then there exists $\psi \in X'$, and a real number α, such that*

$$\psi(z) < \alpha \leq \psi(w),\ \textit{for all } z \in Z, w \in W.$$

Proof Fix some points $z_0 \in Z$, $w_0 \in W$, and define the set

$$V = Z - W + w_0 - z_0 = \{z - w + w_0 - z_0 \colon z \in Z, w \in W\}.$$

It is straightforward that V is convex and contains θ. The equality

$$V = \bigcup_{w \in W} (Z - w + w_0 - z_0)$$

shows that V is also open. Define the vector $x_0 = w_0 - z_0$. Since $Z \cap W = \emptyset$, it is clear that $x_0 \notin V$. Use Lemma 1.7 to produce $\psi \in X'$ such that

(i) $\psi(x_0) = 1$,
(ii) $\psi(x) < 1$, for all $x \in V$.

By the definition of x_0 and V, we have $\psi(w_0) = \psi(z_0) + 1$, and

$$\psi(z) < \psi(w) + \psi(z_0) - \psi(w_0) + 1,\ \text{for all } z \in Z, w \in W,$$

which gives

$$\psi(z) < \psi(w),\ \text{for all } z \in Z, w \in W. \tag{1.14}$$

Put

$$\alpha = \inf_{w \in W} \psi(w).$$

The inequality (1.14) gives

$$\psi(z) \leq \alpha \leq \psi(w),\ \text{for all } z \in Z, w \in W. \tag{1.15}$$

The proof will be complete once we prove the following:

$$\psi(z) < \alpha \text{ for all } z \in Z.$$

Suppose the contrary, i.e., there exists some $z_1 \in Z$ with $\psi(z_1) = \alpha$. Using the continuity of the map

$$\mathbb{R} \ni \beta \mapsto z_1 + \beta x_0 \in X,$$

there exists some $\varepsilon > 0$ such that

$$z_1 + \beta x_0 \in Z, \text{ for all } \beta \in [-\varepsilon, \varepsilon].$$

In particular, by (1.15) one has

$$\psi(z_1 + \varepsilon x_0) \leq \alpha,$$

which means that

$$\alpha + \varepsilon \leq \alpha,$$

which is clearly impossible. ■

Theorem 1.22 (Hahn-Banach Separation Theorem-Complex Case) *Let (X, τ) be a complex topological vector space, let $Z, W \subseteq X$ be nonempty convex sets with Z open, and $Z \cap W = \emptyset$. Then there exists $\psi \in X'$, and a real number α, such that*

$$\operatorname{Re} \psi(z) < \alpha \leq \operatorname{Re} \psi(w), \ \textit{for all } z \in Z, w \in W.$$

Proof Regard X as a real topological vector space, and apply the real version to produce an $\mathbb{R}$-linear continuous functional $\psi_1 : X \to \mathbb{R}$, and a real number α, such that

$$\psi_1(z) < \alpha \leq \psi_1(w), \ x \in X$$

Then the functional $\psi : X \to \mathbb{C}$ defined by

$$\psi(x) = \psi_1(x) - i\psi_1(ix), \ x \in X$$

will clearly satisfy the desired properties. ■

Remark 1.20 Geometrically we can say that the hyperplane $\{\operatorname{Re} \psi(x) = \alpha\}$ separates the sets Z, W in broad sense.

There is another version of the Hahn-Banach separation theorem, which holds for locally convex topological vector spaces.

Theorem 1.23 *Let (X, τ) be a locally convex topological vector space. Suppose $C, D \subseteq X$ are convex sets, with C compact, D closed, and $C \cap D = \emptyset$. Then there exists $\psi \in X'$ and two numbers $\alpha, \beta \in \mathbb{R}$, such that*

$$\operatorname{Re}\psi(x) \leq \alpha < \beta \leq \operatorname{Re}\psi(y), \ \textit{for all } x \in C, y \in D.$$

Proof Let $W = D - C$. By Lemma 1.2, 4. W is closed. Since $C \cap D = \emptyset$, we have $\theta \notin W$. Since W is closed, its complement $X \setminus W$ will then be a neighborhood of θ. Since X is locally convex, there exists a convex open set Z, with $\theta \in Z \subseteq X \setminus W$. In particular we have $Z \cap W = \emptyset$. Applying the suitable version of the Hahn-Banach separation theorem (real or complex case), we find a linear continuous map $\psi : X \to \mathbb{K}$ and a real number γ, such that

$$\operatorname{Re}\psi(z) < \gamma \leq \operatorname{Re}\psi(w), \ \text{for all } z \in Z, w \in W.$$

Notice that $\theta \in Z$, we get $\gamma > 0$. Then the inequality

$$\gamma \leq \operatorname{Re}\psi(w), \ \text{for all } w \in W,$$

gives

$$\operatorname{Re}\psi(y) - \operatorname{Re}\psi(x) \geq \gamma > 0, \ \text{for all } x \in C, y \in D.$$

Then if we define

$$\beta = \inf_{y\in D} \operatorname{Re}\psi(y) \text{ and } \alpha = \sup_{x \in C} \operatorname{Re}\psi(x),$$

we get $\beta \geq \alpha + \gamma$, and we are done. ■

Remark 1.21 Geometrically we can say that the hyperplane $\{\operatorname{Re}\psi(x) = \beta\}$ separates the compact sets C and the closed set D in the strict sense.

One important feature of topological duals in the locally convex Hausdorff case is described by the following result.

Proposition 1.34 *If (X, τ) is a locally convex topological vector space, then X' separates the points of X, in the following sense: for any $x, y \in X$, such that $x \neq y$, there exists $\phi \in X'$, such that $\phi(x) \neq \phi(y)$.*

Proof Since X is locally convex and Hausdorff, there exists some open convex set $V \ni y$ such that $x \notin V$. The existence of ϕ then follows from the Hahn-Banach separation theorem. ■

Definition 1.38 Let (X, τ) be a topological vector space. The weak topology on X, which we denote by $\sigma(X, X')$, is the initial topology for X'. That is, $\sigma(X, X')$ is the coarsest topology on X such that each element of X' is continuous $(X, \sigma(X, X')) \to \mathbb{C}$.

Equivalently, the weak topology on X is the seminorm topology given by the seminorms $|\phi|$, $\phi \in X'$.

Remark 1.22

- The topologies τ and $\sigma(X, X')$ are comparable, and τ is at least as fine as $\sigma(X, X')$. That is, $\sigma(X, X') \subseteq \tau$. A vague rule is that the smaller X' is compared to the set of all linear maps $(X, \sigma(X, X')) \to \mathbb{C}$, the smaller $\sigma(X, X')$ will be compared to τ.
- If X' separates X then $(X, \sigma(X, X'))$ is a locally convex topological vector space. It is Hausdorff because $\sigma(X, X')$ is induced by the separating family of seminorms $p_\phi = |\phi|, \phi \in X'$. In particular if (X, τ) is a locally convex topological vector space then $(X, \sigma(X, X'))$ is a locally convex topological vector space.

Definition 1.39 Let (X, τ) be a topological vector space and $(x_\alpha)_{\alpha \in I}$ a net in X. We say that

1. The net $(x_\alpha)_{\alpha \in I}$ converges strongly to x and we write

$$x_\alpha \to x \;\; \textit{if}\; (x_\alpha)_{\alpha\in I} \text{ converges to } x \textit{ in the original topology } \tau.$$

2. The net $(x_\alpha)_{\alpha \in I}$ converges weakly to x and we write

$$x_\alpha \rightharpoonup x \;\; \textit{if}\; (x_\alpha)_{\alpha\in I} \text{ converges to } x \text{ in the topology } \sigma(X, X').$$

This condition is equivalent to the condition that $p_\phi(x_\alpha - x) \to 0, \forall \phi \in X'$, which in turn is equivalent to

$$\phi(x_\alpha) \to \phi(x), \quad \forall \phi \in X'.$$

A simple consequence of the fact that $\sigma(X, X') \subseteq \tau$ is that

$$x_\alpha \to x \Longrightarrow x_\alpha \rightharpoonup x,$$

i.e., every strongly convergent net is weakly convergent.

Similarly, we will speak about the strong neighborhood, strongly closed, strongly bounded $\cdots$, and weak neighborhood, weakly closed, weakly bounded $\cdots$

Definition 1.40 We say that $Y \subseteq X$ is weakly bounded if Y is a bounded subset of $(X, \sigma(X, X'))$: for every neighborhood N of θ in $(X, \sigma(X, X'))$ there is some $c \geq 0$ such that $Y \subseteq \{cx : x \in N\} = cN$ (equivalently, $\phi(Y)$ is bounded in $\mathbb{C}$).

Remark 1.23 If (X, τ) is an infinite dimensional locally convex topological vector space, the weak topology $\sigma(X, X')$ has a peculiar property: every weak neighborhood of θ contains a closed infinite dimensional linear subspace. Indeed, if we start with some neighborhood V, then there exist $\phi_1, \cdots, \phi_n \in X'$ and

$\varepsilon_1, \cdots, \varepsilon_n > 0$, such that $\varepsilon_1 B_{P_{\phi_1}}(\theta) \cap \cdots \cap \varepsilon_n B_{P_{\phi_n}}(\theta)$, where for $i = 1, \cdots, n$, $B_{P_{\phi_i}}(\theta) = \{x \in X, |\phi_i(x)| < 1\}$. So V will clearly contain the closed subspace $(\ker \phi_1) \cap \cdots \cap (\ker \phi_n)$. It follows that

$$\dim X \leq n + \dim(\ker \phi_1) \cap \cdots \cap (\ker \phi_n),$$

i.e., $\dim(\ker \phi_1) \cap \cdots \cap (\ker \phi_n) = \infty$. Hence $\sigma(X, X')$ is not locally bounded.

Proposition 1.35 *In any finite-dimensional normed space, the weak topology coincides with the topology generated by any norm.*

Proof Let X be a finite-dimensional vector space, let $(e_1, \cdots, e_d)$ be a basis in X, and let $\phi_1, \cdots, \phi_d$ be its dual basis, defined by $\phi_i(e_j) = \delta_{i,j}$. Then, $\|x\|_\infty = \max\limits_{1\leq i\leq d} |\phi_i(x)|$ is a norm on X, and since X is finite-dimensional, all linear functionals on X are also continuous.

We know that on finite dimensional vector space two norms are equivalent, so it is enough to compare the weak topology to the topology τ induced by $\|\|_\infty$. It is clear that $\tau \supseteq \sigma(X, X')$. On the other hand,

$$|x|_{\phi_1,\cdots,\phi_d} = \sup_{1\leq i\leq d} = \|x\|_\infty, x \in X,$$

and hence the open $\|.\|_\infty$-balls around any point and with any radius are open in the weak topology. Hence, $\tau \subseteq \sigma(X, X')$. ∎

Theorem 1.24 *Let X be an infinite-dimensional normed space and $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. The closure of the unit sphere in the weak topology is the whole closed unit ball, i.e.,*

$$\overline{S_X}^{\sigma(X,X')} = \{x \in X : \|x\| \leq 1\}.$$

Similarly, one can show that $B_1(\theta) = \{x \in X : \|x\| < 1\}$ has empty interior for $\sigma(X, X')$. In particular it is not open. Despite these facts, there are sets whose weak closure is equivalent to its strong closure.

Remark 1.24 If (X, τ) is a locally convex topological vector space, then for any $Y \subseteq X$, then $\overline{\text{conv}(Y)}^\tau = \overline{\text{conv}(Y)}^{\sigma(X,X')}$.

Theorem 1.25 *If $Y \subseteq X$ is convex and (X, τ) is a locally convex topological vector space, then*

1. *Y is $\sigma(X, X')$-closed (weakly closed) if and only if Y is τ-closed (strongly closed).*
2. *Y is $\sigma(X, X')$-dense if and only if Y is τ-dense.*

Proof

1. Since $\sigma(X, X') \subseteq \tau$, then if Y is $\sigma(X, X')$-closed it is τ-closed. Conversely, if Y is τ-closed and convex, let $x_0 \in X \setminus Y$. Then by the Hahn-Banach separation theorem (for complex vector spaces) there is some $\phi \in X'$ such that

$$\sup_{x \in Y} \mathrm{Re}(\phi(x)) \leq \gamma_1 < \gamma_2 \leq \mathrm{Re}(\phi(x_0)).$$

Hence the neighborhood of x_0

$$x_0 + V = x_0 + \left\{ x \colon |\gamma(x)| \leq \mathrm{Re}(\phi(x_0)) - \gamma_2 \right\}$$

has empty intersection with Y.
2. Obvious.

■

In particular, in a topological vector space, the closure of convex sets is convex.

If a sequence converges weakly, it need not converge in the original topology, and Mazur's theorem shows that if a sequence in a metrizable locally convex space converges weakly then there is a sequence in the convex hull of the original sequence that converges to the same limit as the weak limit of the original sequence.

Theorem 1.26 *(Mazur)Let X be a metrizable locally convex space. If $x_n \rightharpoonup x$, then there is a sequence $(y_m)_m \subseteq X$ such that each y_m is a convex combination of finitely many x_n and such that $y_m \to x$.*

Proof The convex hull of a subset Y of X is the set of all convex combinations of finitely many elements of Y. The convex hull of a set is convex and contains the set. Let Z be the convex hull of the sequence $(x_n)_n$ and let W the weak closure of Z. Since $x_n \rightharpoonup x$ and $x_n \in Z$, Theorem 1.25 tells us that $W = \overline{Z}$, so $x \in \overline{Z}$. But X is metrizable, so x being in the closure of Z implies that there is a sequence $(y_m)_m \subseteq Z$ such that $y_m \to x$. This sequence $(y_m)_m$ satisfies the claim. ■

Let (X, τ) be a topological vector space. The dual space X' does not come with an a priori topology.

Let $x \in X$, and define $f_x \colon X' \to \mathbb{C}$ by $f_x(\phi) = \phi(x)$. Now f_x is linear. If $\phi_1, \phi_2 \in X'$ are distinct, then $\phi_1 - \phi_2 \neq 0$ so there is some $x \in X$ such that $(\phi_1 - \phi_2)(x) \neq 0$, which tells us that $f_x(\phi_1) \neq f_x(\phi_2)$. Therefore the set $\{f_x : x \in X\}$ is a separating family of seminorms on X', hence generating a topology which makes X' a locally convex topological vector space. We denote this topology by $\sigma(X', X)$ or w^* and it is called the weak* topology on X'. The open sets in the weak* topology are generated by the subbase

$$B_r^x = \{\phi \in X' \colon |\phi(x)| < r\}.$$

Lemma 1.8

(a) *The weak topology $\sigma(X', X)$ is the weakest topology on X' such that each map f_x is continuous.*
(b) *A sequence $(\phi_n)_n$ converges to ϕ in $\sigma(X', X)$ if and only if for all $x \in X$*

$$\lim_{n \longrightarrow \infty} \phi_n(x) = \phi(x).$$

(c) *A set $Y \subseteq X'$ is bounded* w.r.t. $\sigma(X', X)$ *if and only if for all $x \in X$*

$$\{\phi(x), \phi \in Y\}$$

is bounded in $\mathbb{C}$.

Example 1.12 Recall that $c_0' = l_1$ and $l_1' = l_\infty$. Weak convergence of a sequence $(x_n)_k \subseteq l_1$ to zero (with l_1 viewed as a topological vector space) means that

$$\forall\, y = (y_k)_k \subseteq l_\infty \quad \lim_{k \to \infty} \sum_{k=1}^{+\infty} (x_n)_k y_k = 0.$$

Weak* convergence of a sequence $(x_n)_k \subseteq l_1$ to zero (with l_1 viewed as the dual of the topological vector space c_0) means that

$$\forall\, y = h(y_k)_k \subseteq c_0 \quad \lim_{k \to \infty} \sum_{k=1}^{+\infty} (x_n)_k y_k = 0.$$

Clearly, weak convergence implies weak* convergence (but not the opposite).

A priori, one can look at the second dual Y of the locally convex vector space $(X, \sigma(X', X))$, i.e.,

$$Y = \{\lambda \colon X' \to \mathbb{C}, \text{w.r.t}, \sigma(X', X)\}.$$

By construction, it follows that $X \subseteq Y$,

i.e., X can be embedded into Y. It turns out that $X = Y$, i.e., the dual of $(X, \sigma(X', X))$ can be identified with X.

Theorem 1.27 *If $\lambda \colon X' \to \mathbb{C}$ is linear and continuous w.r.t, $\sigma(X', X)$, then there exists $x \in X$ such that*

$$\lambda(\phi) = \phi(x) \ \ \forall \phi \in X'.$$

Proof By definition of continuity w.r.t, $\sigma(X', X)$, for all $\epsilon > 0$ there are $\delta > 0$ and $x_1, \cdots, x_n$ such that

$$\lambda\{\phi\colon\ |\phi(x_i)| \leq \delta, i = 1, \cdots, n\} \subseteq (-\epsilon, \epsilon).$$

In particular, if ϕ is such that $\phi(x_i) = 0$ for all i, then $\lambda(\phi) = 0$. This show that

$$N_\phi \supseteq \bigcap_{i=1}^{n} N_{x_i}.$$

Consider the linear mapping $T\colon\ X' \to \mathbb{C}^{n+1}$ defined by

$$T(\phi) = (\lambda(\phi), \cdots, \phi(x_1), \cdots, \phi(x_n)).$$

By the assumption, $T(X')$ is a subspace of $\mathbb{C}^{n+1}$ and the point $(1, 0, \cdots, 0)$ is not in $T(X')$. Then there are $\alpha = (\alpha_1, \cdots, \alpha_{n+1}) \in \mathbb{C}^{n+1}$ such

$$\alpha T(X') = \{\alpha_1 \lambda(\phi) + \sum_{i=2}^{n+1} \alpha_i \phi(x_{i-1}), \phi \in X'\} = 0 < \Re\alpha_1.$$

It follows that $\alpha_1 \neq 0$ and

$$\lambda\phi = \sum_{i=1}^{n} \frac{\alpha_{i+1}}{\alpha_1}\phi(x_i).$$

■

If X is in particular a normed space, then we know that $(X', \|.\|_{X'})$ is a Banach space. Hence, if τ is the vector topology of X' generated by the norm , $\|.\|_{X'}, \sigma(X', X) \subseteq \tau$.

Definition 1.41 We say that

- The sequence $(\phi_n)_n$ converges strongly to ϕ and we write

$$\phi_n \longrightarrow \phi \ \text{ if } \|\phi_n - \phi\|_{X'} \longrightarrow 0.$$

- The sequence $(\phi_n)_n$ converges weakly to ϕ and we write $\phi_n \rightharpoonup^* \phi$ if $(\phi_n)_n$ converges to ϕ in the topology $\sigma(X', X)$.

The Banach-Alaoglu theorem shows that certain subsets of X' are weak* compact, i.e., they are compact subsets of $\sigma(X', X)$.

Definition 1.42 Let X be a topological vector space and V be a neighborhood of θ. Define the polar of V as

$$K = \Big\{\phi \in X' \colon \ |\phi(x)| \le 1 \ \ \forall\, x \in V\Big\}.$$

Theorem 1.28 (Banach-Alaoglu) *Let X be a topological vector space and V be a neighborhood of θ. Then the polar K of V is compact in the weak* topology $\sigma(X', X)$.*

Proof Since each V local neighborhood absorbing, then there is a $\gamma(x) \in \mathbf{C}$ such that

$$x \in \gamma(x)V.$$

Hence it follows that

$$|\phi(x)| \le \gamma(x) \ \ x \in X, \ \phi \in K.$$

Consider the topological space

$$P = \prod_{x \in X} \{\alpha \in \mathbf{C} \colon \ |\alpha| \le \gamma(x)\},$$

with the product topology σ. By Tychonoff's theorem (P, σ) is compact.

By the construction, the elements of P are functions $f \colon X \to \mathbb{C}$ (not necessarily linear) such that

$$|f(x)| \le \gamma(x).$$

In particular, the set K is the subset of P made of the linear functions.

We first show that K is the subset of P w.r.t the topology σ. This follows from the fact that if f_0 is in the σ closure of $\overline{K}$, then the scalars α, β and point $x, y \in X$ one has that

$$\Big\{ |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \varepsilon, |f(x) - f_0(x)| < \varepsilon, |f(y) - f_0(y)| < \varepsilon \Big\}$$
$$\bigcap K \ne \emptyset.$$

Take thus ϕ in the intersection, so that

$$|f_0(\alpha x+\beta y)-\alpha f_0(x)-\beta f_0(y)| = \Big|(f_0(\alpha x+\beta y)-f(\alpha x+\beta y)) + \alpha(f(x)-f_0(x))+(f(y)-f_0(y))\Big| < (1+|\alpha)|+|\beta|)\varepsilon.$$

Since ε is arbitrary, f_0 is linear. Moreover, since $|f_0(x)| \le \gamma(x)$, then for $x \in V$

$$|f_0(x)| \le 1.$$

It follows that we have two topologies on K :

- the weak* topology $\sigma(X', X)$ inherited by X',
- the product topology σ inherited by P. Since K is closed in (P,σ), then (K,σ) is compact.

To conclude, we need only to show that the two topologies coincide. This follows because the bases of the two topologies are generated by the sets

$$V_{\sigma(X',X)} = \Big\{ |\phi(x_i)-\phi_0(x_i)| < \varepsilon, i = 1\cdots, n\Big\},$$

$$V_\sigma = \Big\{ |f(x_i)-f_0(x_i)| < \varepsilon, i = 1\cdots, n\Big\}.$$

There is thus a one to one correspondence among local bases, hence the two topologies coincide. ■

Theorem 1.29 *Let (X,τ) be a separable topological vector space. Let $K \subseteq X'$ be weakly* compact. Then K is metrizable in the weak* topology.*

Proof Let $\{x_n, n \in \mathbb{N}\}$ be a dense subset of X and $f_{x_n}(\phi) = \phi(x_n)$ for $\phi \in X'$. By the definition of the weak* topology on X', the functionals f_{x_n} are weak* continuous. Also, for every n,

$$f_{x_n}(\phi_1) = f_{x_n}(\phi_2),$$

i.e.,

$$\phi_1(x_n) = \phi_2(x_n),$$

then $\phi_1 = \phi_2$ (continuous functionals that coincide on a dense set).

Thus, $\{f_{x_n}, n \in \mathbb{N}\}$ is a countably family of continuous functionals that separates points in X'. It follows by Proposition 1.9 that K is metrizable. ■

Remark 1.25

1. The claim is not that X' endowed with the weak* topology is metrizable. For example, this is not true in infinite-dimensional Banach spaces.
2. The topological space $(X', \sigma(X', X))$ is never metrizable, unless X has a countable vector base.

Theorem 1.30 *Let X be a separable topological vector space. If V is a neighborhood of θ and if the sequence $(\phi_n)_n \subseteq X'$ satisfies*

$$|\phi_n(x)| \leq 1, \qquad n \geq 1, x \in V,$$

then there is a subsequence $(\phi_{\alpha(n)})_n$ and some $\phi \in X'$ such that for all $x \in X$,

$$\lim_{n\to\infty} \phi_{\alpha(n)}(x) = \phi(x).$$

Proof The Banach-Alaoglu theorem implies that the polar

$$K = \Big\{\phi \in X' \colon\ |\phi(x)| \leq 1 \ \ \forall\, x \in V\Big\},$$

is weak* compact. K with the subspace topology inherited from $\sigma(X', X)$ is compact, hence by Theorem 1.29 it is metrizable. Since the sequence $(\phi_n)_n$ is contained in K, it has a subsequence $(\phi_{\alpha(n)})_n$ that converges weakly to some $\phi \in K$. For each $x \in X$, the functional $f_x \colon (X', \sigma(X', X)) \to \mathbb{C}$ defined by $f_x(\phi) = \phi(x)$ is continuous, hence for all $x \in X$ we have $f_x(\phi_{\alpha(n)})) \to f_x(\phi)$, which is the claim. ■

Theorem 1.31 *If (X, τ) is locally convex and $Y \subseteq X$, then Y is bounded in (X, τ) if and only if Y is bounded in $(X, \sigma(X, X'))$.*

Dual of Banach Spaces and Reflexive Spaces

A particular case is when X is normed: in this case X' is a Banach space with norm $\|\phi\|_{X'} = \sup_{\|x\|=1} |\phi(x)|$. One can introduce the second dual of X, i.e., denoted by X''. Clearly, there is a canonical immersion J of X into X'', by

$$J \colon X \to X'', \quad J(x)(\phi) = \phi(x), \|J(x)\|_{X''} = \|x\|_X.$$

Since $J \colon X \to X''$ is continuous, it follows that $J(X)$ is a closed subspace of X''. In particular, either $J(X) = X''$ or it is not dense.

Lemma 1.9 (Helly) *Let X be a Banach space, $\phi \in X'$, $i = 1 \cdots, n$, n linear functionals in X' and $\alpha_i \in \mathbb{C}$, $i = 1 \cdots, n$, n scalars. Then the following properties are equivalent:*

1. *for all $\varepsilon > 0$ there is x_ε, $\|x_\varepsilon\| < 1$ such that*

$$|\phi(x_\varepsilon) - \alpha_i| \le \epsilon \quad i = 1 \cdots, n,$$

2. *for all $\beta_1, \cdots, \beta_n \in \mathbb{C}$*

$$\left|\sum_i^n \beta_i \alpha_i\right| \le \|\sum_i^n \beta_i \phi_i\|_{X'}.$$

Proof The first implication follows by

$$\left|\sum_i^n \beta_i \alpha_i\right| = \left|\sum_i^n \beta_i (\alpha_i - \phi_i(x_\varepsilon))\right| + \left|\sum_i^n \beta_i \phi_i(x_\varepsilon)\right|$$

$$\le \varepsilon \sum_i^n |\beta_i| + \|\sum_i^n \beta_i \phi_i\|_{X'},$$

since $\|x_\varepsilon\| \le 1$. Conversely if 1. does not hold, then this means that the closure of the set

$$(\phi_1, \cdots, \phi_n)\left\{x : \|x\| \le 1\right\} \subseteq \mathbb{C}^n$$

does not contains $(\alpha_1, \cdots, \alpha_n)$. Thus there is $(\beta_1, \cdots, \beta_n) \in \mathbb{C}^n$ such that

$$\max \mathrm{Re}\left\{\sum_i^n \beta_i \phi_i(x), \|x\| \le 1\right\} < \mathrm{Re}\left\{\sum_i^n \beta_i \alpha_i\right\} \le \left|\sum_i^n \beta_i \alpha_i\right|.$$

Since $\{x : \|x\| \le 1\}$ is balanced, it follows that 2. is false. ■

Proposition 1.36 (Goldstine) *If X is a Banach space, then $J(B_X)$ is dense in $B_{X''}$ for the weak* topology.*

Proof If $\xi \in X''$, take a neighborhood of the form

$$V = \left\{\eta \in X' : |\eta(\phi_i) - \xi(\phi_i)| < \varepsilon, \phi_i \in X', i = 1 \cdots n\right\}.$$

We need only to find $x \in X$ such that

$$|\phi_i(x) - \xi(\phi_i)| < \varepsilon.$$

Since $\|\xi\|_{X''} \leq 1$, then

$$\left|\sum_i^n \beta_i \xi(\phi_i)\right| \leq \|\sum_i^n \beta_i \phi_i\|_{X'},$$

so that for Lemma 1.9 it follows that there is an $x_\varepsilon \in X$ which belongs to V. ∎

Definition 1.43 A Banach space is reflexive if $J(X) = X''$.

It is important to observe that in the previous definition the canonical immersion J is used: even for particular non-reflexive spaces, one can find a continuous linear surjection from X to X''.

Theorem 1.32 (Kakutani) *The Banach space X is reflexive if and only if B_X is compact for the weak topology $\sigma(X, X')$.*

Proof If X is reflexive, then $J : X \to X''$ is continuous, injective and surjective. Hence J^{-1} is linear and continuous w.r.t. the strong topologies of X and X''. Actually both J and J^{-1} are isometries.

It is clear that

$$J\left\{x : |\phi(x)| < \varepsilon\right\} = \left\{\eta : |\eta\phi| < \varepsilon\right\},$$

so that the topology $J^{-1}(\sigma(X'', X'))$ coincides with the topology $\sigma(X, X')$. Since $B_{X''}$ is weak* compact, so B_X.

Conversely, if B_X is compact, then $J(B_X)$ is closed, and by Proposition 1.36 it coincide with the whole $B_{X''}$. ∎

Theorem 1.33 *If X is a Banach space and X' is separable, then X is separable.*

Proof Let $(\phi_n)_n$ be a dense countable set in X'. Let $x_n \in X$, $\|x_n\|_X \leq 1$, be a point where

$$|\phi_n(x_n)| \geq \frac{1}{2}\|\phi_n\|_{X'},$$

and consider the countable set

$$Q = \left\{\sum_{finite} \alpha_i x_i : \alpha_i \textit{ belongs to a countable dense subset of } \mathbb{C}\right\}.$$

Clearly Q is countable and dense in the vector space L generated by $\{x_n\}_n$, so that it remains to prove that L is dense in X.

If L is not dense, then there is a non null continuous functional ϕ such that

$$\phi \neq 0_{X'} \;\; \phi(x_n) = 0 \,\forall\, n.$$

Since $(\phi_n)_n$ is dense, there is n_ϕ such that $\|\phi - \phi_{n_\phi}\|_{X'} < \varepsilon$, so that

$$\|\phi_{n_\phi}\|_{X'} \leq \left|\phi_{n_\phi}(x_{n_\phi})\right| \leq \left|(\phi - \phi_{n_\phi})(x_{n_\phi})\right| + \left|\phi(x_{n_\phi})\right| \leq \varepsilon.$$

Thus $\|\phi_{n_\phi}\|_{X'} \leq 2\varepsilon$, which implies that $\phi = 0_{X'}$. ■

Proposition 1.37 *If $Y \subseteq X$ is a closed subspace of a reflexive space, then Y is reflexive.*

Proof The proof follows by proving that the topology $\sigma(Y, Y')$ coincide with the topology $Y \cap \sigma(X, X')$ and B_Y is closed for $\sigma(X, X')$ (closed for strong topology and convex). ■

Corollary 1.6 *Let X be a normed space. Then, X is separable and reflexive if and only if X' is separable and reflexive.*

Proof Clearly if X is reflexive, the unit ball $B_{X'}$ is compact for the topology $\sigma(X', X'')$ because of the Banach-Alaoglu theorem and the fact $\sigma(X', X'') = \sigma(X', X)$. Moreover if X is reflexive and separable, then X'' is separable, hence by Theorem 1.33 is separable.

Conversely, if X' is reflexive, then X'' is reflexive, so that $M(X)$ is reflexive by Proposition 1.37, hence X is reflexive. Moreover, we know from Theorem 1.33 that X is separable, if X' is separable. ■

Definition 1.44 We say that X Banach space is uniformly convex if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x\|_X, \|y\|_X \leq 1, \|\frac{x+y}{2}\| \geq 1 - \delta \Longrightarrow \|x - y\|_X < \varepsilon.$$

Theorem 1.34 (Milman) *If X is a uniformly convex Banach space, then X is reflexive.*

Proof Let $\xi \in X''$, $\|\xi\|_{X''} = 1$. We want to prove that for all $\varepsilon > 0$ there is $x \in X$, $\|x\|_X \leq 1$ such that

$$\|\xi - J(x)\|_{X''} < \varepsilon.$$

Since $J(X)$ is strongly closed (J is an isometry), then J is surjective.

Let $\phi \in X'$ be such that

$$\|\phi\|_{X'} = 1, \quad \xi\phi > 1 - \delta,$$

where δ is the constant chosen by the uniform convexity estimate corresponding to ε, and consider the neighborhood of ξ of the form

$$V = \left\{\eta \in X'' : \left|(\xi - \eta)(\phi) < \frac{\delta}{2}\right|\right\}.$$

By Proposition 1.36, it follows that there is some $x \in B_X$ such that $J(x) \in V$.

Assume that $\xi \notin J(x) + \varepsilon B_{X''}$. Then we obtain a new neighborhood of ξ for the weak* topology which does not contains x. With the same procedure, we can find a new $\overline{x}$ in this new neighborhood. Thus we have

$$|\phi(x) - \xi(\phi)| \leq \frac{\delta}{2}, \quad |\phi(\overline{x}) - \xi(\phi)| \leq \frac{\delta}{2}.$$

Adding we obtain

$$2\,|\xi(\phi)| \leq |\phi(x + \overline{x})| + \delta \leq \|x + \overline{x}\| + \delta.$$

Then $\|\frac{x + \overline{x}}{2}\| \geq (1 - \delta)$, so that $\|x + \overline{x}\| < \varepsilon$, which is a contradiction. ■

1.2.8 l_1-Sequences

Definition 1.45 Let $(x_n)_n$ be a bounded sequence in a Banach space X, and $\varepsilon > 0$. We say that $(x_n)_n$ admits ε-l_1-blocks if for every infinite $M \subseteq \mathbb{N}$ there are $a_1, \cdots, \cdots a_r \in \mathbb{K}$ with $\sum |a_\rho| = 1$ and $i_1 < \cdots < i_r$ in M such that $\|\sum a_\rho x_{i_\rho}\| \leq \varepsilon$.

Clearly there will be no subsequence of $(x_n)_n$ equivalent to the l_1-basis iff $(x_n)_n$ admits ε-l_1-blocks for arbitrary small $\varepsilon > 0$.

Theorem 1.35 *Let X be a real (for simplicity) Banach space and $(x_n)_n$ a bounded sequence. Suppose that, for some $\varepsilon > 0$, $(x_n)_n$ admits small ε-l_1-blocks. Then there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $(x_{n_k})_k$ is "close to being a weak Cauchy sequence" in the following sense*:

$$\limsup_k \phi(x_{n_k}) - \liminf_k \phi(x_{n_k}) \leq 2\varepsilon$$

for every $\phi \in X'$ with $\|\phi\|_{X'} = 1$.

Proof Suppose the theorem were not true. We claim that without loss of generality we may assume that there is a $\delta > 0$ such that

$$\varphi((x_{n_k})_k) := \sup_{\|\phi\|_{X'}=1} \left(\limsup_k \phi(x_{n_k}) - \liminf_k \phi(x_{n_k}) > 2\varepsilon + \delta \right) \tag{1.16}$$

for all subsequences $(x_{n_k})_k$. In fact, if every subsequence contained another subsequence with a φ-value arbitrarily close to 2ε, the diagonal process would even provide one where $\varphi((x_{n_k})_k) \leq 2\varepsilon$ in contrast to our assumption.

Fix a $\tau > 0$ which will be specified later. After passing to a subsequence we may assume that $(x_n)_n$ satisfies the following conditions:

(i) If C and D are finite disjoint subsets of $\mathbb{N}$ there are a $\lambda_0 \in \mathbb{R}$ and an $\phi \in X'$ with $\|\phi\|_{X'} = 1$ such that $\phi(x_n) < \lambda_0$ for $n \in C$ and $\phi(x_n) > \lambda_0 + 2\varepsilon + \delta$ for $n \in D$.

(ii) There are $i_1 < \cdots < i_r$ in $\mathbb{N}$, $a_1, \cdots a_r \in \mathbb{R}$ with

$$\sum |a_\rho| = 1,\ |\sum a_\rho| < \tau,\ \|\sum a_\rho x_{i_\rho}\| \leq \varepsilon$$

For (**i**), define, for $r \in \mathbb{N}$, T_r to be the collection of all $(i_1, \cdots, i_r)$ (with $i_1 < \cdots < i_r$) such that there are a $\lambda_0 \in \mathbb{R}$ and a normalized ϕ such that $\phi(x_{i_\rho}) < \lambda_0$ if ρ is even and $> \lambda_0 + 2\varepsilon + \delta$ otherwise. (1.16) implies that there is an M_0 for which all $(i_1, \cdots, i_r)$ are in T_r for $i_1 < \cdots < i_r$ in M_0. Let us assume that $M_0 = \mathbb{N}$. Let C and D be finite disjoint subsets of $2\mathbb{N} = \{2, 4, \cdots\}$. We may select $i_1 < \cdots < i_r$ in $\mathbb{N}$ such that $C \subseteq \{i_\rho \mid \rho \text{ even }\}$ and $D \subseteq \{i_\rho \mid \rho \text{ odd }\}$. Because of $(i_1, \cdots, i_r) \in T_r$ we have settled (i) provided C and D are in $2\mathbb{N}$, and all what's left to do is to consider $(x_{2n})_n$ instead of $(x_n)_n$.

For (ii), By assumption we find $i_1 < \cdots < i_r$, $a_1, \cdots a_r \in \mathbb{R}$ such that $\sum |a_\rho| = 1$ and $\|\sum a_\rho x_{i_\rho}\| \leq \varepsilon$ with arbitrarily large i_1. Therefore we obtain $i_1^1 < \cdots < i_{r_1}^1 < i_1^2 < \cdots < i_{r_2}^2 < i_1^3 < \cdots < i_{r_3}^3 < \cdots$ and associated a_ρ^i. The numbers $\eta_j := \sum_{\rho=1}^{r_j} a_\rho^j$ all lie in $[-1, 1]$ so that we find $j < k$ with $|\eta_j - \eta_k| \leq 2\tau$. Let $i_1 < \cdots < i_r$ be the family $i_1^j < \cdots < i_{r_j}^j < i_1^k < \cdots < i_{r_k}^k$, and define the $a_1, \cdots a_r$ by $\frac{1}{2}a_1^j, \cdots \frac{1}{2}a_{r_j}^j, -\frac{1}{2}a_1^k, \cdots -\frac{1}{2}a_{r_k}^k$.

We are now ready to derive a contradiction. On the one hand, by (ii), we find $i_1 < \cdots < i_r$, $a_1, \cdots, \cdots a_r \in \mathbb{R}$ such that $\sum |a_\rho| = 1, |\sum a_\rho| \leq \tau$ with $\|\sum a_\rho x_{i_\rho}\| \leq \varepsilon$. On the other hand we may apply (**i**) with $C := \{i_\rho \mid a_\rho < 0\}$ and $D := \{i_\rho \mid a_\rho > 0\}$. We put $\alpha := -\sum_{\rho \in C} a_\rho$, $\beta := \sum_{\rho \in D} a_\rho$, and we note that

$|\alpha - \beta| \le \tau, \alpha + \beta = 1$ so that $|\beta - \frac{1}{2}| \le \tau$, hence

$$\varepsilon \ge \| \sum a_\rho x_{i_\rho} \| \le \sum a_\rho \phi(x_{i_\rho}) \ge -\lambda_0 \alpha + (\lambda_0 + 2\varepsilon + \delta)\beta \ge -|\lambda_0|\tau + \varepsilon + \frac{\delta}{2} - \tau\delta.$$

This expression can be made larger than ε if τ has been chosen sufficiently small (note that the numbers $|\lambda_0|$ are bounded by $\sup_n \|x_n\|$), a contradiction which proves the theorem. ■

Remark 1.26 Since the unit vector basis $(x_n)_n$ of real l_1 the assumption of the theorem holds with $\varepsilon = 1$ and since for every subsequence $(x_{n_k})_k$ one may find $\|\phi\|_{X'} = 1$ with

$$\limsup_k \phi(x_{n_k}) - \liminf_k \phi(x_{n_k}) = 2$$

there can be no better constant than that given in our theorem.

Theorem 1.36 (Rosenthal's Theorem) *Let X be a Banach space and $(x_n)_n$ a bounded sequence in X. If there exists no subsequence which is a weak Cauchy sequence then one can find a subsequence $(x_{n_k})_k$ which is equivalent with the unit vector basis of l_1 (i.e., $(\lambda_k)_k \mapsto \sum \lambda_k x_{n_k}$, from l_1 to X, is an isomorphism).*

In particular one has: If X does not contain an isomorphic copy of l_1, then every bounded sequence admits a subsequence which is a weak Cauchy sequence.

Proof Rosenthal's theorem is the assertion that $(x_n)_n$ has a weak Cauchy subsequence provided it admits ε-l_1-blocks for all ε. So, it is simple to derive the theorem from Theorem 1.35. If $(x_n)_n$ and thus every subsequence has ε-l_1-blocks for all ε, apply Theorem 1.35 successively with ε running through a sequence tending to zero. The diagonal sequence which is obtained from this construction will be a Cauchy sequence. ■

Remark 1.27

1. Since weakly convergent sequences are weakly Cauchy it follows immediately that Rosenthal's theorem holds in reflexive spaces.
2. Rosenthal's theorem holds, whenever X is such that X' is separable. Let $(x_n)_n$ be bounded and ϕ be a fixed functional. If we apply the Bolzano-Weierstrass theorem to the scalar sequence $(\phi(x_n))_n$ we get a subsequence $(x_{n_k})_k$ such that $(\phi(x_{n_k}))_k$ converges. Applying the same idea to $(x_{n_k})_k$ with a second functional, say ψ, we get a subsequence of this subsequence such that the application of ψ produces something which is convergent. ϕ, applied to this new subsequence, also gives rise to convergence. Thus we have a subsequence of $(x_n)_n$ where ϕ and ψ converge, and similarly one can achieve this for any prescribed finite number of functionals. Even countably many functionals are manageable, by the diagonal process. Since we are dealing with bounded sequences $(y_n)_n$ (typically

subsequences of the original sequence) the collection of ϕ where $(\phi(y_n))_n$ converges is a norm closed subspace of X'.

There is a generalization of Rosenthal's theorem to Fréchet spaces which, it seems, has been firstly by Díaz [44]. Thus the starting point for proving promised generalizations is to understand what it means for a sequence in a locally convex space be equivalent to the unit basis of l_1.

We denote by l_1^0 the subspace of l_1 formed by elements with only finitely many nonzero coordinates.

Barroso, Kalenda and Lin introduced the following notion of l_1-sequences in topological vector spaces [14].

Definition 1.46 Let (X, τ) be a topological vector space and $(x_n)_n$ a sequence in X. We say that $(x_n)_n$ is an l_1-sequence if the mapping $T_0 \colon l_1^0 \to X$ defined by

$$T_0((a_i)_{i\geq 1}) = \sum_{i=1}^{\infty} a_i x_i \tag{1.17}$$

is an isomorphism of l_1^0 onto $T_0(l_1^0)$.

The following characterization of l_1-sequences is given in [14].

Proposition 1.38 *Let (X, τ) be a locally convex space and $(x_n)_n$ a bounded sequence in X. The following are equivalent:*

(i) *There is a continuous seminorm p on X such that*

$$p\left(\sum_{i=1}^{n} a_i x_i\right) \geq \sum_{i=1}^{n} |a_i|, \quad n \in \mathbb{N}, a_1, \cdots, a_n \in \mathbb{R}.$$

(ii) *$(x_n)_n$ is an l_1-sequence.*

If X is sequentially complete, then these conditions are equivalent to the following:

(iii) *The mapping $T : l_1 \to X$ defined by $T((a_i)_{i\geq 1}) = \sum_{i=1}^{\infty} a_i x_i$ is a well defined isomorphism of l_1 onto its image in X*

Proof Let $T_0 \colon l_1^0 \to X$ be defined by (1.17). As $(x_n)_n$ is bounded and X is locally convex, it is easy to check that T_0 is continuous.

Further, if (i) holds, then T_0 is clearly one-to-one and T_0^{-1} is continuous. This proves $(i) \Rightarrow (ii)$.

Conversely, suppose that (ii) holds. Set

$$U = T_0(\{x \in l_1^0 \colon \|x\|_{l^1} < 1\}).$$

As T_0 is an isomorphism, U is an absolutely convex open subset of $T_0(l_1^0)$. We can find V, an absolutely convex neighborhood of θ in X such that $V \cap T_0(l_1^0) \subset U$. Let p the Minkowski functional of V. Then p is a continuous seminorm witnessing that (i) holds. This proves $(ii) \Rightarrow (i)$.

Now suppose that X is sequentially complete. As T_0 is continuous and linear, it is uniformly continuous and hence it maps Cauchy sequences to Cauchy sequences. In particular the mapping T_0 can be uniquely extended to a continuous linear mapping $T: l_1 \to X$. This is obviously the mapping described in (iii). As l_1^0 is dense in l_1, we get $(ii) \Leftrightarrow (iii)$. ■

The following theorem is a variant of Rosenthal's theorem [14]. Its proof is a slight refinement of the proof of Lemma 3 in [44].

Theorem 1.37 *Let (X, τ) be a metrizable locally convex space. Then each bounded sequence in X contains either a weakly Cauchy subsequence or a subsequence which is an l_1-sequence.*

Proof Let $(\|.\|_n)$ be a sequence of seminorms generating the topology of X. Without loss of generality we may assume that $\|x\|_n \leq \|x\|_{n+1}$ for all n and $x \in X$. Let $U_n = \{x: \|x\|_n < 1\}$ and let $B_n = U_n^0$ be the polar of U_n. Assume that $(x_m)_m$ is a bounded sequence in X such that no its subsequence is an l_1-sequence. For $n = 0, 1, 2, \cdots$ we construct a sequence $(x_m^n)_m$ inductively as follows. Set $x_m^0 = x_m$ for all $m \in \mathbb{N}$. Assume that for a given $n \in \mathbb{N}$ the sequence $(x_m^{n-1})_m$ has been defined. By Rosenthal's theorem one of the following possibilities takes place (elements of X are viewed as functions on B_n):

(i) $(x_m^{n-1})_m$ has a subsequence which is equivalent to the l_1-basis on B_n.
(ii) $(x_m^{n-1})_m$ has a subsequence which point wise converges on B_n.

Let us show that the case (i) cannot occur. Indeed, suppose that (i) holds. Let $(y_m)_m$ be the respective subsequence. The equivalence to the l_1 basis on B_n means that there is some $C > 0$ such that

$$\|\sum_{i=1}^{m} a_i y_i\|_n \geq C \sum_{i=1}^{m} |a_i|$$

for each $m \in \mathbb{N}$ and each choice $a_1, \cdots, a_m \in \mathbb{R}$. By Proposition 1.38 $(y_m)_m$ is an l_1-sequence in X, which is a contradiction.

Thus the possibility (**ii**) takes place. Denote by $(x_m^n)_m$ the respective subsequence. This completes the inductive construction.

Take the diagonal sequence (x_m^m). It is a subsequence of $(x_m)_m$ which pointwise converges on B_n for each $n \in \mathbb{N}$. Moreover, if $\phi \in X'$ is arbitrary, then there is n and $c > 0$ such that $c\phi \in B_n$. In particular, the linear span of the union of all $B_n's$ is the whole dual X'. It follows that the sequence (x_m^m) is weakly Cauchy. The proof is complete. ■

Remark 1.28 Let $X = l_1$ endowed with its weak topology. Let $(e_n)_n$ denote the canonical basic sequence. Then, the sequence $(e_n)_n$ contains neither a weakly Cauchy subsequence nor a subsequence which is an l_1-sequence. Indeed, suppose that $(x_n)_n$ is an l_1-sequence in X. Denote by Y its linear span. By the definition of an l_1-sequence we get that Y is isomorphic to $(l_1^0, \|.\|_1)$, hence it is metrizable. On the other hand, by the definition of X we get that Y is equipped with its weak topology which is not metrizable as Y has infinite dimension.

Further, the sequence $(e_n)_n$ contains no weakly Cauchy subsequence in $(l_1, \|.\|_1)$ and in $(l_1, \sigma(l_1, (l_1)'))$ coincide, we get that $(e_n)_n$ contains no weakly Cauchy subsequence in X. Thus the proof is completed.

The following is given in [14] and is about the coincidence of norm and weak topologies.

Proposition 1.39 *Let Γ be an arbitrary set. Then the norm and weak topologies coincide on the positive cone of $l_1(\Gamma)$.*

Proof Denote by C the positive cone of $l_1(\Gamma)$. Since the weak topology is weaker than the norm one, it is enough to prove that the identity of C endowed with the weak topology onto $(C, \|.\|)$ is continuous. Let $x \in C$ and $\varepsilon > 0$ be arbitrary. Fix a nonempty finite set $F \subseteq \Gamma$ such that

$$\sum_{\gamma \in F} x(\gamma) > \|x\| - \frac{\varepsilon}{4}.$$

Set

$$U = \left\{ y \in C \colon\ |y(\gamma) - x(\gamma)| < \frac{\epsilon}{4\,|F|} \text{ for } \gamma \in F \right\},$$

$$V = \left\{ y \in C \colon\ \sum_{\gamma \in \Gamma \setminus F} y(\gamma) - \sum_{\gamma \in \Gamma \setminus F} x(\gamma) < \frac{\epsilon}{4} \right\}.$$

Then both U and V are weak neighborhoods of x in C (recall that the dual of $l_1(\Gamma)$ is represented by $l_\infty(\Gamma)$), hence so $U \cap V$. Moreover, if $y \in U \cap V$, then

$$\begin{aligned} \|y - x\| &= \sum_{\gamma \in F} |y(\gamma) - x(\gamma)| + \sum_{\gamma \in \Gamma \setminus F} |y(\gamma) - x(\gamma)| < \frac{\varepsilon}{4} + \sum_{\gamma \in \Gamma \setminus F} (y(\gamma) + x(\gamma)) \\ &= \frac{\varepsilon}{4} + \sum_{\gamma \in \Gamma \setminus F} (y(\gamma) - x(\gamma)) + 2 \sum_{\gamma \in \Gamma \setminus F} x(\gamma) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2.\frac{\varepsilon}{4}. \end{aligned}$$

This shows that the identity is weak-to-norm continuous at x. The proof is complete. ■

1.2.9 The Fréchet-Urysohn Property

Definition 1.47 Let Y be a subset of a topological (Hausdorff) space X.

(1) Y is countably compact, if every sequence in Y has a cluster-point in Y.
(2) Y is sequentially compact, if every sequence in Y has a convergent subsequence with limit in Y.
(3) Y is relatively countably compact, if every sequence in Y has a cluster-point in X.
(4) Y is relatively sequentially compact, if every sequence in Y has a convergent subsequence with limit in X.

It is easy to see that

(1) Every (relatively) compact set is (relatively) countably compact.
(2) Every (relatively) sequentially compact set is (relatively) countably compact.

Definition 1.48 A topological space (X, τ) is called Fréchet-Urysohn if the closures of subsets of X are described using sequences, i.e., if whenever $Y \subseteq X$ and $x \in X$ such that $x \in \overline{Y}$, there is a sequence $(x_n)_n$ in Y with $x_n \to x$.

Example 1.13 Metrizable spaces and one point compactifications of discrete spaces are Fréchet-Urysohn.

Definition 1.49 A completely regular Hausdorff topological space X is called a g-space, if its relatively countably compact subsets are relatively compact.

Definition 1.50 A Hausdorff topological space X is said to be angelic if for every relatively countably compact set $Y \subseteq X$, the following hold:

(i) Y is relatively compact,
(ii) for each $x \in \overline{Y}$, there exists a sequence $(x_n)_n \subseteq Y$ such that $x_n \longrightarrow x$.

If K is a compact topological space then K is a Fréchet-Urysohn space if and only if it is angelic. It can be said that a Hausdorff topological space X is angelic if and only if X is a g-space for which any compact subspace is a Fréchet-Urysohn space.

The following are some characterizations of Fréchet-Urysohn spaces.

Theorem 1.38 *For a topological vector space (X, τ) the following assertions are equivalent:*

1. *X is Fréchet-Urysohn.*
2. *For every subset Y of X such that $\theta \in \overline{Y}$ there exists a bounded subset Z of Y such that $\theta \in \overline{Z}$.*
3. *For any sequence $(Y_n)_n$ of subsets of X, each with $\theta \in \overline{Y_n}$, there exists a sequence $Z_n \subseteq Y_n$, $n \in \mathbb{N}$, such that $\bigcup_n Z_n$ is bounded and $\theta \in \overline{\bigcup_{n\leq k} Z_k}$ for each $n \in \mathbb{N}$.*

Proof Clearly 1. implies 2. Now assume 2. It is obvious that 3. holds if $\theta \in Y_n$ for infinitely many n. Therefore, we assume that $\theta \in \overline{Y_n} \setminus Y_n$, for each $n \in \mathbb{N}$. Consequently, there exists a null sequence $(x_n)_n$ in $X \setminus \overline{\{\theta\}}$. For each $n \in \mathbb{N}$ there exists a closed neighbourhood U_n of zero such that $\theta \notin U_n + x_n$. Let each $W_n = U_n \cap Y_n$. Clearly θ is in each $\overline{W_n} \setminus W_n$ and not in the set

$$Y = \bigcup_n (W_n + x_n).$$

However, $\theta \in \overline{Y}$: For U, an open neighborhood of θ, there exist $k \in \mathbb{N}$ with $x_k \in U$ and, V, a neighbourhood of θ with $V + x_k \subseteq U$. As there is $y \in V \cap W_k$ we also have $y + x_k \in U \cap Y$. Thus $\theta \in \overline{Y} \setminus Y$. By hypothesis, there is $Z \subseteq Y$ with Z bounded and $\theta \in \overline{Z}$. There exists subsets $Z_n \subseteq W_n = U_n \cap Y_n$ such that

$$Z = \bigcup_n (Z_n + x_n).$$

By construction, θ does not belong to the closed sets

$$\bigcup_{k<n} (U_k + x_k).$$

Therefore θ is not in any $\overline{\bigcup_{k<n} (Z_k + x_k)}$. This and $\theta \in \overline{Z}$ imply that

$$\theta \in \overline{\bigcup_{n \leq k} (Z_k + x_k)},$$

for each $n \in \mathbb{N}$. Let V' and V be any balanced neighborhoods of θ with $V - V \subseteq V'$. Fix $n \in \mathbb{N}$. There exists $m \geq n$, in $\mathbb{N}$, such that $x_k \in V$ for all $k \geq m$. From

$$\theta \in \overline{\bigcup_{m \geq k} (Z_k + x_k)},$$

it follows that there exist $k \geq m$ and $y \in B_k$ with $y + x_k \in V$. From $y \in V - x_k \subseteq V - V \subseteq V'$, we see, for each $n \in \mathbb{N}$, the set V' meets $\bigcup_{n \leq k} Z_k$. As any neighborhood of θ contains V' and V as above, θ is in the closure of each $\bigcup_{n \leq k} Z_k$. Note also that $\bigcup_n Z_n$ is bounded. Indeed, as

$$Z = \bigcup_n (Z_n + x_n)$$

and $W = \{x_m : m \in \mathbb{N}\}$ are bounded and since

$$\bigcup_n Z_n \subseteq \bigcup_n (Z_n + x_n) - \{x_m : m \in \mathbb{N}\} = Z - W,$$

then $\bigcup_n Z_n$ is also bounded too. We have proved that 2. implies 3.

3. implies 1.: Assume that $\theta \in \overline{Y}$, and set $Y_n = nY$, for each $n \in \mathbb{N}$. Since θ is in each $\overline{Y_n}$, there exist $Z_n \subseteq Y_n$, as in 3.. So each $\bigcup_{n \leq k} Z_k$ is nonempty, and, consequently, there exists a strictly increasing sequence $(n_k)_k$ in $\overline{N}$ with Z_{n_k} nonempty. For each k, let $z_k \in Z_{n_k}$. There exists a sequence $(y_k)_k$ in Y such that $z_k = n_k y_k$ for each $k \in \mathbb{N}$. Since $(n_k)_k$ is strictly increasing and $(z_k)_k = (n_k y_k)_k$ is bounded, the sequence $(y_k)_k$ in Y converges to zero in X. The proof is complete. ■

There are many nonmetrizable Fréchet-Urysohn spaces. To provide some examples, we have the following deep result of J. Bourgain, D. H. Fremlin and M. Talagrand [24]:

Theorem 1.39 *Let X be a Polish space (i.e., a separable completely metrizable space). Denote by $B_1(X)$ the space of all real-valued functions on X which are of the first Baire class and equip this space with the topology of pointwise convergence. Suppose that $Y \subseteq B_1(X)$ is relatively countably compact in $B_1(X)$ (i.e., each sequence in Y has a cluster point in $B_1(X)$. Then the closure $\overline{Y}$ of Y in $B_1(X)$ is compact and Fréchet-Urysohn.*

A slightly weaker version is given in [101].

Corollary 1.7 *Let X be a Polish space and Y be a set of real-valued continuous functions on X. Suppose that each sequence in Y has a pointwise convergent subsequence. Then the closure of Y in $\mathbb{R}^p$ is a Fréchet-Urysohn compact space contained in $B_1(X)$.*

Proof Y is obviously contained in $B_1(X)$. Moreover, let $(f_n)_n$ be any sequence in Y. By the assumption there is a subsequence $(f_{n_k})_k$ pointwise converging to some function f. As the functions f_{n_k} are continuous, the limit function f is of the first Baire class. Hence, it is a cluster point of $(f_n)_n$ in $B_1(X)$. So, Y is relatively countably compact in $B_1(X)$. The assertion now follows from Theorem 1.39. ■

We continue by the following example [14].

Proposition 1.40 *Let (X, τ) be a metrizable locally convex space and Y be a bounded subset of X. If Y is τ-separable and contains no l_1-sequence, then the set*

$$\overline{Y - Y}^{\sigma(X,X')} = \overline{\{x - y : x, y \in Y\}}^{\sigma(X,X')}$$

is Fréchet-Urysohn when equipped with the weak topology.

Proof As the closed linear span of Y is separable, we can without loss of generality suppose that X is separable. Let $(\|.\|_n)$, U_n and B_n $(n \in \mathbb{N})$ be as in the proof of Theorem 1.37. Notice that B_n is a metrizable weak*compact subset of X'. Moreover, the linear span of the union of all $B_n's$ is the whole dual X' (see the end of the proof of Theorem 1.37). Let now P be the topological sum of the spaces $(B_n, \sigma(X', X)), n \in \mathbb{N}$. Then P is a Polish space. Denote by $G\colon P \to X'$ the canonical mapping of P onto the union of all $B_n's$. Then G is continuous from P to $(X', \sigma(X', X))$. Define a mapping $H\colon P \to \mathbb{R}^P$ by the formula $H(x)(p) = G(p)(x)$. Then H is a homeomorphism of $(X, \sigma(X, X'))$ onto $H(X)$ equipped with the pointwise convergence topology. Moreover, the functions from $H(X)$ are continuous on P.

Let $Z = H(Y - Y)$. We claim that each sequence from Z has a pointwise convergent subsequence. To show that it is enough to observe that each sequence in $Y - Y$ has weakly Cauchy subsequence. Indeed, let $(z_n)_n$ be a sequence in $Y - Y$. Then $z_n = x_n - y_n$ for some $x_n, y_n \in Y$. As Y contains no l_1-sequence, by Theorem 1.37, we get a weakly subsequence $(x_{n_k})_k$ of $(x_n)_n$. Applying Theorem 1.37 once more we get a weakly Cauchy subsequence $(y_{n_k})_k$ of $(y_n)_n$. Then $(z_{n_k})_k$ is a weakly Cauchy subsequence of $(z_n)_n$. Thus Z is relatively countably compact in $B_1(P)$, which is the space of all Baire-one functions on P equipped with the topology of pointwise convergence. By Theorem 1.39, the closure of Z in $\mathbb{R}^P$ is a Fréchet-Urysohn compact subset of $B_1(P)$. In particular, the weak closure of $Y - Y$ is Fréchet-Urysohn when equipped with the weak topology. The proof is complete. ■

Note that the result of the above proposition generalizes the following in the context of Banach spaces [101].

Proposition 1.41 *Let X be a Banach space and Y be a bounded subset of X. If X is norm-separable and contains no l_1-sequence, then the set*

$$\overline{Y - Y}^{\sigma(X',X)} = \overline{\{J(x - y)\colon x, y \in Y\}}^{\sigma(X',X)}$$

is Fréchet-Urysohn when equipped with the weak topology, where J denotes the canonical embedding of X into X''. In particular,*

$$\overline{Y - Y}^{\sigma(X,X')} = \overline{\{x - y\colon x, y \in Y\}}^{\sigma(X,X')}$$

is Fréchet-Urysohn when equipped with the weak topology.

We have the following characterization of the Fréchet-Urysohn property in locally convex spaces [14].

Proposition 1.42 *Let (X, τ) be a Hausdorff locally convex space such that there is a metrizable locally convex topology on X compatible with the duality. The following assertion are equivalent.*

(i) *Any bounded subset of X is Fréchet-Urysohn in the weak topology.*
(ii) *Any bounded sequence in X has a weakly Cauchy subsequence.*
If, moreover, τ itself is metrizable, then these assertions are equivalent to the following one:
(iii) *X contains no l_1-sequence.*

Proof Let ρ be a metrizable locally convex topology compatible with the duality . By Theorem 1.37 (X, ρ) contains no l_1-sequence if and only if (X, ρ) satisfies the condition (ii). Further, the validity of (ii) for (X, ρ) is equivalent to its validity for (X, τ). It follows that (ii) holds if and only if (X, ρ) contains no l_1-sequence. In particular, if $\rho = \tau$, we get $(i) \Leftrightarrow (ii)$.

$(ii) \Rightarrow (i)$ Suppose that (ii) holds. Let Y be a bounded subset of (X, τ) and let $x \in X \in$ belong to the weak closure of Y. We need to find a sequence in Y converging to x. We first prove it under the additional assumption that Y is separable. Then Y is bounded and separable in (X, ρ) as well. As (X, ρ) contains no l_1-sequence, by Proposition 1.40 we get that the weak closure of Y-Y is Fréchet-Urysohn in the weak topology. Hence, in particular, there is a sequence in Y weakly converging to x.

To prove the general case it is enough to show that there is a countable set $Z \subseteq Y$ such that x belongs to the weak closure of Z. In other words, it is enough to show that the weak topology on X has countable tightness. To prove that observe that $(X, \sigma(X, X'))$ is canonically homeomorphic to a subspace of $C_p(X', \sigma(X', X))$, which is the space of all continuous functions on the space $(X', \sigma(X', X))$ equipped with the topology of pointwise convergence. Further notice that $(X', \sigma(X', X))$ is σ-compact, this follows by the metrizability of ρ as $X' = \bigcup_{m,n \in \mathbb{N}} mB_n$ using the notation from the proof of Theorem 1.37. Finally, as any finite power of a σ-compact and hence Lindelöf, we can conclude by the Arkhangel'skii-Pytkeev theorem [7].

$(i) \Rightarrow (ii)$ Suppose that (ii) does not hold. Then there is a sequence $(x_n)_n$ in X which is an l_1-sequence in (X, ρ). Let $T_0 : l_1^0 \to X$ be defined as in (1.17). Let S denote the unit sphere in l_1^0. Then θ is in the weak closure of S (as l_1^0 is an infinite dimensional normed space) but it is not the weak limit of any sequence from S (by Schur's theorem [75]). Thus, θ is in the weak closure of $T_0(S)$ without being the weak limit of any sequence from $T_0(S)$. Thus $T_0(S) \cup \{\theta\}$ is a bounded set which is not Fréchet-Urysohn in the weak topology. ■

The following characterization of Banach spaces not containing l_1 is given in [101].

Theorem 1.40 *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X contains no isomorphic copy of* l_1.
2. *Each bounded separable subset of X is Fréchet-Urysohn in the weak topology.*
3. *For each separable subset* $Y \subseteq X$ *there are relatively weakly closed subsets* $Y_n, n \in \mathbb{N}$ *such that* $Y = \bigcup_{n\in\mathbb{N}} Y_n$ *and each* Y_n *is Fréchet-Urysohn in the weak topology.*

Proof The implication 1. $\Rightarrow$ 2. follows from Proposition 1.40.

The implication 2. $\Rightarrow$ 1. follows from the fact that the unit ball of l_1 is not Fréchet-Urysohn (as θ is in the weak closure of the sphere and the sphere is weakly sequentially closed by the Schur theorem [75]).

The implication 2. $\Rightarrow$ 3. is trivial if we use the fact that a closed ball is weakly closed.

Let us prove 3. $\Rightarrow$ 2. To show 2. it is enough to prove that the unit ball of any closed separable subspace of X is Fréchet-Urysohn in the weak topology. Let Z be such a subspace. Let $Y_n, n \in \mathbb{N}$ be the cover of Z provided by 3. As each Y_n is weakly closed, it is also norm-closed. By the Baire category theorem some Y_n has a nonempty interior in Y, so it contains a ball. We get that some ball in Y is Fréchet-Urysohn, so the unit ball has this property as well. ■

Remark 1.29 Note that the assertion 3. is a topological property of the space $(X, \sigma(X, X'))$ (as norm separability coincides with weak separability).

As a consequence of Proposition 1.42 we get the following improvement of Theorem 1.40.

Corollary 1.8 *Let X be a Banach space. The following assertions are equivalent.*

1. *X contains no isomorphic copy of* l_1.
2. *The closed unit ball of X is Fréchet-Urysohn in the weak topology.*
3. *There is a sequence* $(Y_n)_{n\geq 1}$ *of weakly closed sets which are Fréchet-Urysohn in the weak topology such that* $X = \bigcup_{n=1}^{\infty} Y_n$.

Proof The equivalence 1. $\Leftrightarrow$ 2. follows from Proposition 1.42. The implication 2. $\Rightarrow$ 3. is trivial. The implication 3. $\Rightarrow$ 1. follows from Theorem 1.40 (or, alternatively, 3. $\Rightarrow$ 2.) follows from the Baire category theorem as in Theorem 1.40. ■

Definition 1.51 A Banach space $(X, \|.\|)$ is Asplund if and only if Y' is separable for each separable subspace $Y \subseteq X$.

Remark 1.30 A Banach space X is an *Asplund* space if each convex continuous function $T: X \to \mathbb{R}$ is Fréchet differentiable on a dense G_δ set in X. Also it is known that a Banach space X is *Asplund* if and only if X' has the *RNP* [25].

It is worthwhile to remark that there are separable Banach spaces having no copy of l_1 for which X' is nonseparable [93, 127]. On the other hand, the well-known James's space J is an example of a nonreflexive Banach space without an unconditional basis which does not contain any copy of l_1 and yet has separable dual.

Remark 1.31 Let us remark that the implication $(ii) \Rightarrow (i)$ of Proposition 1.42 does not hold for general locally convex spaces. Indeed, there are Banach spaces X such that the closed unit ball of X' is weak* sequentially compact, but it is not Fréchet-Urysohn in the weak* topology. In particular, the dual closed unit ball is weak* sequentially compact whenever X is *Asplund* [55], in particular if $X = C(K)$ with K scattered [55]. On the other hand, K is canonically homeomorphic to a subset of the closed unit ball of $C(K)'$ equipped with the weak* topology, so it is enough to observe that there are scattered compact spaces which are not Fréchet-Urysohn. As a concrete example we can take $K = [0, w_1]$, the ordinal interval equipped with the order topology (w_1 is the first uncountable ordinal).

It is worth to compare Theorem 1.40 with a similar characterization of *Asplund* spaces [101].

Theorem 1.41 *Let X be a Banach space. Then the following assertions are equivalent.*

1. *X is Asplund.*
2. *Each bounded separable subset of X is metrizable in the weak topology.*
3. *For each separable subset $Y \subseteq X$ there are relatively weakly closed subsets $Y_n, n \in \mathbb{N}$, of Y such that $Y = \bigcup_{n\in\mathbb{N}} Y_n$ and each Y_n is metrizable in the weak topology.*

Proof The equivalence of 1. and 2. follows from the well-known fact that the unit ball of Y is metrizable in the weak topology if and only if Y' is separable. The equivalence of 2. and 3. can be proved similarly as corresponding equivalence in the previous theorem. ■

Remark 1.32 There is no analogue of Theorem 1.40 for convex sets. Indeed, let $X = l_1$ and let C be the closed convex hull of the standard basis. Then C contains an l_1-sequence but is Fréchet-Urysohn in the weak topology. In fact, it is even metrizable as it is easy to see that on the positive cone of l_1 the weak and norm topologies coincide.

1.3 Ultrametric Spaces

The origin of ultrametric spaces lies in valuation theory and dates back to Krasner and Monna who developed this theory for ultrametric distances with real values (non-Archimedean analysis). A systematic study of (general) ultrametric spaces was provided [16, 81, 84, 113, 120, 152, 154, 155, 157, 160, 169] and others. This study is concerned with ultrametric whose distance functions take their values in an arbitrary partially ordered set (with a smallest element 0) not just in the real numbers.

Definition 1.52 Let $(\Gamma, \leq)$ be an ordered set with smallest element 0. Let X be a nonempty set. A mapping $d\colon X \times X \longrightarrow \Gamma$ is called an ultrametric distance and (X, d, Γ) an ultrametric space if d has the following properties for all $x, y, z \in X$ and $\gamma \in \Gamma$:

$(d1)$ $d(x, y) = 0$ if and only if $x = y$,
$(d2)$ $d(x, y) = d(y, x)$,
$(d3)$ if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$.

If there is no ambiguity, we simply write X instead of (X, d, Γ).

If Γ is totally ordered , $(d3)$ becomes

$(d3')$ $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$.

Remark 1.33 The ultrametric space (X, d, Γ) is trivial, if there exists $\gamma \in \Gamma$ such that for all $x, y \in X, x \neq y, d(x, y) = \gamma$.

Definition 1.53 Let $(Y, d_{|Y}, \Gamma_Y)$ and (X, d, Γ) be ultrametric spaces such that $Y \subset X$ and $\Gamma_Y \subset \Gamma$. Assume that Γ_Y has the induced order of Γ and the same 0 as Γ and that furthermore, $d_{|Y}(Y \times Y) \subset \Gamma_Y$ and $d_{|Y}(y, y') = d(y, y')$ for all $y, y' \in Y$. Then $(Y, d_{|Y}, \Gamma_Y)$ is said to be a subspace of (X, d, Γ) and X is called an extension of Y. Often we simply write d instead of $d_{|Y}$.

Definition 1.54 Let (X, d, Γ) be an ultrametric space. The space X is said to be solid if for every $\gamma \in \Gamma$ and $x \in X$ there exists $y \in X$ such that $d(x, y) = \gamma$. If X is solid, then $d(X \times X) = \Gamma$.

Definition 1.55 Let (X, d, Γ) be an ultrametric space. Let $\gamma \in \Gamma^\bullet = \Gamma \setminus \{0\}$ and $a \in X$. The set $B_\gamma(a) = \{x \in X \mid d(a, x) \leq \gamma\}$ is called a ball. The element a is said to be a center of $B_\gamma(a)$ and the element γ to be a radius of $B_\gamma(a)$. If $x, y \in X, x \neq y$, then $B(x, y) = B_{d(x,y)}(x)$ is called a principal ball.

Remark 1.34 Let (X, d, Γ) be an ultrametric space. If X is solid, every ball is principal. If Γ is totally ordered, also the converse conclusion holds.

Definition 1.56 Let (X, d, Γ) be an ultrametric space. A nonempty Y of X is said to be convex in X when for all $y_1, y_2 \in Y$ with $y_1 \neq y_2$ the principal ball $B(y_1, y_1) \subseteq Y$.

Remark 1.35 Every principal ball is convex in X and furthermore, if $\bigcap_{i \in I} B(x_i, y_i) \neq \emptyset$ then $\bigcap_{i \in I} B(x_i, y_i)$ is convex in X.

In the following lemma, we list some properties of balls which can easily be verified [161].

Lemma 1.10 *Let (X, d, Γ) be an ultrametric space and let $\gamma, \delta \in \Gamma^{\bullet}$.*

1. *Let $x, y \in X$.*
   (a) *If $\gamma \leq \delta$ and $B_{\gamma}(x) \cap B_{\gamma}(y) \neq \emptyset$, then $B_{\gamma}(x) \subseteq B_{\delta}(y)$,*
   (b) *if $B_{\delta}(y) \subset B_{\gamma}(x)$, then $\gamma \nleq \delta$.*
2. *Concerning principal balls, if $x, y, z, u \in X$, $x \neq z$ and $y \neq u$, then*
   (a) *$B(x, z) \subseteq B_{\delta}(y)$ if and only if $d(x, z) \leq \delta$ and $x \in B_{\delta}(y)$,*
   (b) *if $B(x, z) \subset B_{\delta}(y)$, then $d(x, z) < \delta$,*
   (c) *if $B(x, z) = B(y, u)$, then $d(x, z) = d(y, u)$.*
3. *Let X be solid and $x, y \in X$.*
   (a) *$B_{\gamma}(x) \subseteq B_{\delta}(y)$ if and only if $\gamma \leq \delta$ and $x \in B_{\delta}(y)$,*
   (b) *if $B_{\gamma}(x) \subset B_{\delta}(y)$, then $\gamma < \delta$.*
   (c) *if $B_{\gamma}(x) = B_{\delta}(y)$, then $\gamma = \delta$*
4. *If Γ is totally ordered and $B_{\gamma}(x) \subset B_{\delta}(y)$, then $\delta < \gamma$.*

Definition 1.57 Let (X, d, Γ) be an ultrametric space. A set of balls which is totally ordered by inclusion is said to be a chain.

Lemma 1.11 *Let (X, d, Γ) be an ultrametric space. Let $\mathcal{C}$ be a chain of balls of X which does not have a smallest ball. Then there exists a limit ordinal λ and a strictly decreasing family of balls $(B_i)_{i<\lambda}$ such that each $B_i \in \mathcal{C}$ and for every ball $C \in \mathcal{C}$ there exists B_i such that $B_i \supseteq C$ and hence $\bigcap \mathcal{C} = \bigcap_{i<\lambda} B_i$.*

Definition 1.58 Let (X, d, Γ) be an ultrametric space. X is called spherically complete (resp., principally complete) if every chain of balls of X (resp., principal balls of X) has a nonempty intersection.

Remark 1.36 Every spherically complete ultrametric space (X, d, Γ) is principally complete. The converse is true when Γ is totally ordered or the space is solid.

Definition 1.59 An ultrametric space (X, d, Γ) is said to be complete if every chain of balls $\{B_{\gamma_i} \mid i \in I\}$, with $\inf\{\gamma_i \mid i \in I\} = 0$, has a nonempty intersection.

Remark 1.37 A spherically complete ultrametric space (X, d, Γ) is complete. If Γ is totally ordered and if $\Gamma^{\bullet}$ does not have a smallest element, the ultrametric distance induces on X a uniformity, hence also a topology. In this case, the concept of completeness coincides with that given by the uniformity.

Several examples of different types of ultrametric spaces are discussed in [160]. Some where Γ is totally ordered and others where Γ is not totally ordered.

Examples 1.9

1. Let Δ be a totally ordered Abelian additive group, let ∞ be a symbol such that $\infty \notin \Delta$, and $\delta + \infty = \infty + \delta = \infty$, $\infty + \infty = \infty$, $\delta < \infty$ for all $\delta \in \Delta$. We

denote by 0 the neutral element of Δ, that is $0+\delta=\delta$ for every $\delta\in\Delta$. Let K be a commutative field, let $v\colon K\longrightarrow\Delta\cup\{\infty\}$ be a valuation of K, so we have

($v1$) $v(x)=\infty$ if and only if $x=0$,
($v2$) $v(xy)=v(x)+v(y)$,
($v3$) $v(x+y)\geq\min\{v(x),v(y)\}$.

Let $\Gamma^\bullet$ be a totally ordered Abelian multiplicative group with neutral element 1, let 0 be a symbol such that $0\notin\Gamma^\bullet$, $0\gamma=\gamma 0=0$, $0.0=0$, $0<\gamma$ for every $\gamma\in\Gamma^\bullet$. Let $\theta\colon\Delta\cup\{\infty\}\longrightarrow\Gamma=\Gamma^\bullet\cup\{0\}$ be an order reversing bijection such that $\theta(\infty)=0$, $\theta(\delta+\delta')=\theta(\delta).\theta(\delta')$, so $\theta(0)=1$.

Let $d\colon K\times K\longrightarrow\Gamma$ be defined by $d(x,y)=\theta(v(x-y))$, then (K,d,Γ) is an ultrametric space which is said to be associated to the valued field $(K,v,\Delta\cup\{\infty\})$.

2. Let Γ be a totally ordered set with smallest element 0, let $\Gamma^\bullet=\Gamma\backslash\{0\}$. Let R be a nonempty set with a distinguished element 0. For each $f\colon\Gamma^\bullet\longrightarrow R$, let $\mathrm{supp}(f)=\{\gamma\in\Gamma^\bullet\mid f(\gamma)\neq 0\}$ be the support of f. Let $R[[\Gamma]]$ be the set of all $f\colon\Gamma^\bullet\longrightarrow R$ with support which is empty or anti-well ordered. Let $d\colon R[[\Gamma]]\times R[[\Gamma]]\longrightarrow\Gamma$ be defined by $d(f,f)=0$ and if $f\neq g$, $d(f,g)$ is the largest element of the set $\{\gamma\in\Gamma^\bullet\mid f(\gamma)\neq g(\gamma)\}$. Then $(R[[\Gamma]],d,\Gamma)$ is an ultrametric space which is solid and spherically complete.
3. Let I be a set with at least two elements, let $(X_i)_{i\in I}$ be a family of sets X_i, each one having at least two elements. Let $X=\prod_{i\in I}X_i$. Let $\mathcal{P}(I)$ be the set of all subsets of I, ordered by inclusion. And let $d\colon X\times X\longrightarrow\mathcal{P}(I)$ be defined by $d(f,g)=\{i\in I\mid f_i\neq g_i\}$, where $f=(f_i)_{i\in I}$ and $g=(g_i)_{i\in I}$. Then $(X,d,\mathcal{P}(I))$ is a solid and spherically complete ultrametric space. If each $X_i=\{0,1\}$, we obtain the ultrametric space $(\mathcal{P}(I),d,\mathcal{P}(I))$ with $d(A,B)=(A\cup B)\backslash(A\cap B)$ for all $A,B\subseteq I$.
4. Let X be a topological space, let Y be a discrete topological space, let $\mathcal{C}(X,Y)$ denote the set of continuous functions from X to Y and let $\mathcal{C}l(X)$ the set of clopen (i.e., closed and open) subsets of X. The mapping $d\colon\mathcal{C}(X,Y)\times\mathcal{C}(X,Y)\longrightarrow\mathcal{C}l(X)$ is defined by $d(f,g)=\{x\in X\mid f(x)\neq g(x)\}$. Then $(\mathcal{C}(X,Y),d,\mathcal{C}l(X))$ is a solid ultrametric space, and it is spherically complete if $\mathcal{C}l(X)$ is a complete sub-Boolean-algebra of $\mathcal{P}(X)$.

Definition 1.60 Let (X,d,Γ) be an ultrametric space and assume that Γ is totally ordered. Let $(Y,d_{|Y},\Gamma_Y)$ be a subspace of (X,d,Γ) and assume that $d(Y\times Y)=d(X\times X)=\Gamma$. If for every $x\in X$ and for every $y\in Y$, with $x\neq y$, there exists $y'\in Y$ such that $d(y',x)<d(y,x)$, the extension $Y\prec X$ is called immediate and we write $Y\mathrm{im}\prec X$. The extension $Y\prec X$ is said to be dense (denoted by $Y\mathrm{de}\prec X$), if for every $x\in X$ and for every $0<\gamma\in\Gamma$ there exists y in Y such that $d(y,x)<\gamma$. Thus if $Y\mathrm{de}\prec X$ then also $Y\mathrm{im}\prec X$.

Remark 1.38 If $\Gamma^\bullet$ does not have a smallest element, Definition 1.60 coincides with that given by the topology of X. We remark that both notions, "immediate" and "dense" can be defined more generally for ultrametric spaces, where Γ is only ordered [155].

The following is given in [161].

Theorem 1.42

1. *Every ultrametric space (X, d, Γ), with Γ totally ordered, has an immediate extension which is spherically complete. (We call such an extension a spherical completion of X.)*
2. *Every ultrametric space (X, d, Γ), with Γ totally ordered, has an extension (X', d, Γ) such that X' is dense in X'. (We call such an extension a completion of X.)*
3. *Let $(Y, d_{|Y}, \Gamma_Y)$ be a subspace of Let (X, d, Γ). Assume that Γ is totally ordered and that $\Gamma^\bullet_{|Y}$ is coinitial in $\Gamma^\bullet$ and that furthermore $d(Y \times Y) = \Gamma_Y, d(X \times X) = \Gamma$. If X is complete, then there exists one and only one completion $\widehat{Y}$ of Y which is a subspace of X.*

Proof The proofs of 1. and 2. are given in [155, 176].

3. Let $\mathcal{S}$ be the set of all ultrametric subspaces $\mathcal{S}$ such that Y is dense in $\mathcal{S}$. Since Y is dense in itself, $\mathcal{S} \neq$. The set $\mathcal{S}$ is ordered by inclusion. Let $\{S_i \mid i \in I\}$ be a totally ordered subset of $\mathcal{S}$. Then $S = \bigcup_{i \in I} S_i$ is a subspace of X and Y is dense in S. Thus $S \in \mathcal{S}$ is an upper bound for all $S_i, i \in I$. By Zorn's lemma, there exists a maximal element in $\mathcal{S}$ which we denote again by S. We show that S is complete. Since $\Gamma^\bullet_{|Y}$ is coinitial in $\Gamma^\bullet$ and $\Gamma^\bullet_{|Y} = \Gamma^\bullet_{|S} = d(S \times S) \setminus \{0\}$ has in $\Gamma^\bullet_{|S}$ the infimum 0 if and only if the infimum of Δ in $\Gamma^\bullet$ is 0, thus we may just write $\inf \Delta = 0$. We assume that S is not complete. Then there exists a chain $\{B^S_{\gamma_i}(a_i) \mid i \in I\}$ of balls in S with

$$\inf\{\gamma_i \mid i \in I\} = 0 \text{ and } \bigcap B^S_{\gamma_i}(a_i) = \emptyset.$$

Since X is complete and for each $i \in I$, $B^S_{\gamma_i}(a_i) = S \cap B^S_{\gamma_i}(a_i)$, where $B_{\gamma_i}(a_i)$ denotes the ball with center a_i and radius γ_i in X, there exists $z \in X$ such that $\{z\} = \bigcap B_{\gamma_i}(a_i)$. Let $S' = S \cup \{z\}$. Then S' is a subspace of X which properly contains S, so also Y. To prove that Y is dense in S', it suffices to show that if $0 < \gamma \in \Gamma$, there exists $y \in Y$ such that $d(y, z) < \gamma$. Since $\inf\{\gamma_i \mid i \in I\} = 0$ there exists γ_i with $0 < \gamma_i < \gamma$. Since Y is dense in S and $a_i \in S$, it follows that there exists $y \in Y$ such that $d(y, a_i) < \gamma_i$. Since, moreover, $z \in B_{\gamma_i}(a_i)$, then $d(z, y) \leq \max\{d(z, a_i), d(y, a_i)\} \leq \gamma_i < \gamma$. Thus Y is dense in S'. So $S' \in \mathcal{S}$, which contradicts the maximality of S in $\mathcal{S}$. We have proved that S is complete, hence a completion of Y in X. It remains to show that Y has at most one completion in X. Assume that $\widehat{Y}_1, \widehat{Y}_2$ are completions of Y in X. Let $\widehat{y}_1 \in \widehat{Y}_1$. For each $\gamma \in \Gamma^\bullet$

there exists $y_\gamma \in Y$ such that $d(\widehat{y_1}, y_\gamma) < \gamma$. If $\Gamma^\bullet$ has a smallest element, say γ^* then

$$\widehat{y_1} = y_{\gamma^*} \in Y \subset \widehat{Y_2}.$$

If $\Gamma^\bullet$ does not have a smallest element, then $\inf\{\gamma \mid \gamma \in \Gamma^\bullet\} = 0$, thus there exists $\widehat{y_2} \in \widehat{Y_2}$ with

$$\{\widehat{y_2}\} = \bigcap_{\gamma \in \Gamma^\bullet} B_\gamma(y_\gamma)$$

because $\widehat{Y_2}$ is complete. Hence $\widehat{y_1} = \widehat{y_2} \in \widehat{Y_2}$. This shows that $\widehat{Y_1} \subseteq \widehat{Y_2}$. By the same argumentation, we conclude that $\widehat{Y_2} \subseteq \widehat{Y_1}$, thus $\widehat{Y_1} = \widehat{Y_2}$. ■

Definition 1.61 Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Let λ be a limit ordinal and let $\xi = (x_i)_{i<\lambda}$ be a family of elements of X. We say that ξ is a Cauchy family if for every $\gamma \in \Gamma^\bullet$ there exists $i_0 = i_0(\gamma, \xi) < \lambda$ such that if $i_0 \leq i < \kappa < \lambda$, then $d(x_i, x_k) < \gamma$. The family $\xi = (x_i)_{i<\lambda}$ is said to be pseudo-convergent if there exists $i_0 = i_0(\xi) < \lambda$ such that if $i_0 \leq i < \kappa < \mu < \lambda$, then $d(x_\kappa, x_\mu) < d(x_i, x_\kappa)$.

Remark 1.39 We note that if $\xi = (x_i)_{i<\lambda}$ is pseudo-convergent, the elements x_i, for $i_0(\xi) \leq i < \lambda$ are all distincts and if $i_0(\xi) \leq i < \kappa < \mu < \lambda$, then $d(x_i, x_\kappa) = d(x_\kappa, x_\mu)$, this element is denoted by ξ_i. Hence if $i_0 \leq i < \kappa < \lambda$, then $\xi_i > \xi_\kappa$.

Definition 1.62 Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Let λ be a limit ordinal and let $\xi = (x_i)_{i<\lambda}$ be a Cauchy family of elements of X. The element $y \in X$ is a limit of the family ξ if for every $\gamma \in \Gamma^\bullet$ there exists $i_1 = i_1(\gamma) < \lambda$ such that if $i_1 \leq i < \lambda$, then $d(y, x_i) < \gamma$. The ultrametric space X is complete if and only if every Cauchy family has a limit in X.

Remark 1.40 A Cauchy family $\xi = (x_i)_{i<\lambda}$ has at most one limit. Indeed, if y, z are limits, then $d(y, z) < \gamma$ for all $\gamma \in \Gamma^\bullet$, so $y = z$.

Definition 1.63 Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Let λ be a limit ordinal and let $\xi = (x_i)_{i<\lambda}$ be a pseudo-convergent family of elements of X. The element $y \in X$ is a pseudo-limit of the family $\xi = (x_i)_{i<\lambda}$ if there exists $i_1 = i_1(\xi, y)$, $i_0(\xi) \leq i_1 < \lambda$, such that if $i_1 \leq i < \lambda$ then $d(y, x_i) \leq \xi_i$. If y is a pseudo-limit of ξ, then $z \in X$ is a pseudo-limit of ξ if and only if $d(y, z) < \xi_i$ for all i such that $i_1 \leq i < \lambda$.

The following is a characterization of spherical completeness [151].

Proposition 1.43 *Let (X, d, Γ) be an ultrametric space and assume that Γ is totally ordered. Then X is spherically complete if and only if every pseudo-convergent family of X has a pseudo-limit in X.*

1.4 Admissible Functions

Throughout this subsection, we denote by (X, τ) a topological vector space, and by Y a nonempty subset of X.

Below the definition of functions providing the possibility of working with extended real seminorms in topological vector spaces.

Definition 1.64 An admissible function for Y on X is an extended real-valued function $\rho\colon X \longrightarrow [0, \infty]$ such that

1. The mapping $(x, y) \mapsto \rho(x, y)$ is continuous on $Y \times Y$,
2. $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$,
3. $\rho(\lambda x) = |\lambda|\, \rho(x)$, for all $\lambda \in \mathbb{R}$ and $x \in X$,
4. If $x, y \in Y$ and $\rho(x - y) = 0$, then $x = y$.

Remark 1.41 Notice that if ρ is an admissible function for Y on X, then it defines a metric on Y whose induced topology is coarser than τ.

Remark 1.42 It is instructive to compare the notion of continuity in the sense of 1. with the usual one. It is easy to see that if ρ is continuous on X, then $(x, y) \mapsto \rho(x, y)$ is continuous on $Y \times Y$. Furthermore, if 1. − 3. hold then ρ is continuous on Y.

It is not true, in general, that if ρ is continuous on Y, then it satisfies 1. For example, if $X = \mathbb{R}$ and $Y = [0, \infty)$, then the mapping $\rho\colon \mathbb{R} \longrightarrow [0, \infty]$ defined by

$$\rho(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x > 0, \\ \infty, & \text{if } x = 0, \\ 0, & \text{if } x < 0, \end{cases}$$

is continuous on Y. However, the mapping $T\colon Y \times Y \longrightarrow [0, \infty]$ given by $T(x, y) = \rho(x - y)$ is not continuous at the point $(1, 1)$. Indeed, it suffices to see that $(1 - \frac{1}{k}, 1)$ converges to $(1, 1)$ in $Y \times Y$, while that $T(1 - \frac{1}{k}, 1) = 0$ and $T(1, 1) = \infty$.

Barroso [12] proved that the class of admissible functions is sufficiently good to imply that the Schauder-projection operator is continuous.

Proposition 1.44 *Let ρ be an admissible function for Y on X. Then for any $\varepsilon > 0$ and $p \in Y$, the function $g\colon Y \longrightarrow [0, \infty)$ given by*

$$g(x) = \max\{\varepsilon - \rho(x - p), 0\}$$

is continuous on Y.

Proof Firstly, let us recall that the effective domain of ρ is the set

$$\mathtt{D}(\rho) = \{x \in X : \rho(x) < \infty\}.$$

Let x_0 be a point in Y and $\delta > 0$ be arbitrary. By assumption, there exists a neighborhood $U \times V$ of (x_0, p) in $Y \times Y$ such that

$$\rho(x_0 - p) - \delta \leq \rho(x - z) \leq \rho(x_0 - p) + \delta,$$

for all $(x, z) \in U \times V$. If $x_0 - p \notin \mathtt{D}(\rho)$ then $\rho(x_0 - p) = \infty$ and, hence, $\rho(x - p) = \infty$ for all $x \in U$. In consequence, $g(x) = g(x_0) = 0$ for all $x \in U$. In case $x_0 - p \in \mathtt{D}(\rho)$, we can conclude that $x - p \in \mathtt{D}(\rho)$ for all $x \in U$. In this case, it is easy to see that $g(x_0) + \delta \geq g(x)$, for all $x \in U$. On the other hand, if $g(x_0) = 0$, then clearly $g(x) \geq g(x_0) - \delta$ holds for every $x \in U$. Assuming now that $g(x_0) = \varepsilon - \rho(x_0 - p)$, we have $g(x_0) - \delta \leq \varepsilon - \rho(x - p) \leq g(x)$, for all $x \in U$. In any case, we have proven that g is continuous at x_0, and hence continuous in Y. The proof is complete. ■

The following is an example of an admissible function [12].

Proposition 1.45 *Let Y be a compact convex subset of a topological vector space (X, τ) and $\mathcal{F} = \{\rho_n : n \in \mathbb{N}\}$ a countable family of seminorms on X which separate points of $Y - Y$ and such that the topology Γ generated by $\mathcal{F}$ is coarser than τ in Y. Then the function $\rho : X \to [0, \infty]$ defined as*

$$\rho(x) = \sum_{n=1}^{\infty} \rho_n(x), \ \ x \in X$$

is admissible.

Proof Since Y is compact and Γ is coarser than τ, each ρ_n restricted to Y is τ-continuous. Thus we have $\max\{\rho_n(x) : x \in Y < \infty\}$ for all $n \in \mathbb{N}$. By replacing the seminorms ρ_n by suitable positive multiples, if necessary, we may assume that

$$\max\{\rho_n(x) : x \in Y_n\} \leq 2^{-n-1}, \tag{1.18}$$

for all $n \in \mathbb{N}$. Notice that $\rho(x - y) < \infty$ for all $x, y \in Y$. Moreover, one readily checks 2. −4.. Using now (1.18), we see that the sequence of functions $\rho^n(x - y) = \sum_{i=1}^{n} \rho_i(x - y)$ is Cauchy w.r.t. the topology of uniform convergence on $Y \times Y$. Thus $\rho^n(x - y)$ converges uniformly on $Y \times Y$ to $\rho(x - y)$. Furthermore, to verify that 1− holds, we have only to ensure this for each ρ_n. Let (x_α, y_α) be a net in $Y \times Y$ converging to (x, y). Since τ is finer than Γ on Y, both $\rho_n(x_\alpha - x)$ and $\rho_n(y_\alpha - y)$ converge to 0. We may then apply the triangle inequality to conclude $|\rho_n(x_\alpha, y_\alpha) - \rho_n(x - y)| \to 0$. ■

1.5 Some Fixed Point Theorems

Banach's Contraction Mapping Principle is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. The principle first appeared in explicit form in Banach's thesis [11].

Theorem 1.43 (Banach's Contraction Mapping Principle) *Let (X, d) be a complete metric space and let $T\colon X \longrightarrow X$. If there exists an $0 < k < 1$ such that $d(T(x), T(y)) \leq kd(x, y)$ for all $x, y \in X$, then T has a unique fixed point.*

Proof First we consider the case when:

$$\operatorname{diam}(X) := \sup\{d(x, y)\colon x, y \in X\} < \infty.$$

For each $n \in \mathbb{N}$, let $Y_n = T^n(X)$. Then

$$Y_{n+1} = T^{n+1}(X) = T^n(T(X)) \subseteq T^n(X) = Y_n$$

for all $n \in \mathbb{N}$. Therefore, $\{Y_n : n \in \mathbb{N}\}$ is a decreasing sequence of nonempty subsets of X. Next, notice that

$$0 \leq \operatorname{diam}(Y_{n+1}) \leq k\operatorname{diam}(Y_n) \quad \text{for all } n \in \mathbb{N}$$

and so, by induction,

$$0 \leq \operatorname{diam}(Y_{n+1}) \leq k^n\operatorname{diam}(Y_n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $\lim_{n\to\infty} \operatorname{diam}(\overline{Y_n}) = \lim_{n\to\infty} \operatorname{diam}(Y_n) = 0$. It then follows from Cantor's intersection property that

$$\bigcap_{n\in\mathbb{N}} \overline{Y_n} = \{x\} \quad \text{for some } x \in X.$$

Moreover, since $x \in \overline{Y_n}$,

$$T(x) \in T(\overline{Y_n}) \subseteq \overline{T(Y_n)} = \overline{Y_{n+1}} \subseteq \overline{Y_n},$$

$T(x) \in \bigcap_{n\in\mathbb{N}} \overline{Y_n} = \{x\}$. That is, $T(x) = x$.

In the case when $\operatorname{diam}(X) = \infty$ some extra work is required. In this case we choose any $x_0 \in X$ and let

$$Z := \overline{\{T^n(x_0)\colon n \in \mathbb{N}\}}.$$

Then $T(Z) \subseteq Z$ and

$$\operatorname{diam}(Z) \leq \frac{d(T(x_0), x_0)}{1-k} < \infty.$$

Hence from the previous argument there exists a point $x \in Z \subseteq X$ such that $T(x) = x$. ∎

The Caristi fixed point theorem [35] is known as one of the very interesting and useful generalizations of the Banach's Contraction Mapping Principle for self-mappings on a complete metric space. Neither continuity nor a Lipschitz condition is required.

Theorem 1.44 (Caristi's Fixed Point Theorem) *Let (X, d) be a complete metric space and let $T: X \longrightarrow X$ be a mapping such that*

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x))$$

for all $x \in X$, where $\varphi: X \longrightarrow [0, +\infty)$ is a lower semicontinuous mapping. Then T has at least a fixed point.

The setting of generalized ultrametric spaces offers a highly flexible framework in which to study the fixed point theory is necessary for logic programming semantics [59, 85, 104, 113, 151, 153, 156, 157] and [177].

Definition 1.65 Let (X, d, Γ) be an ultrametric space. A mapping $\varphi: X \longrightarrow X$ is said to be strictly contracting if for all $x, x' \in X$, with $x \neq x', d(\varphi(x), \varphi(x')) < d(x, x')$. An element $z \in X$ with $\varphi(z) = z$ is called a fixed point of φ.

For strictly contracting maps on ultrametric spaces we have the following fixed point theorem [151, 153, 160].

Theorem 1.45 *Assume that (X, d, Γ) is a spherically complete ultrametric space and that $\varphi: X \longrightarrow X$ is strictly contracting. Then φ has exactly one fixed point $z \in X$.*

Proof Assume, $\pi_x = d(x, \varphi(x)) \neq 0$ for every $x \in X$. Let $B_x = B_{\pi_x}$. The set $\mathcal{B} = \{B_x \mid x \in X\}$ is ordered by inclusion. Let $\mathfrak{C}$ be a maximal chain in $\mathcal{B}$. Since X is spherically complete, there exists an element $z \in \bigcap\{B_x \mid B_x \in \mathfrak{C}\}$. Then $B_z \subseteq B_x$ for every $B_x \in \mathfrak{C}$. Indeed, this is obvious, if $z = x$. If $z \neq x$ then $d(\varphi(z), \varphi(x)) \leq d(z, x) \leq \pi_x = d(x, \varphi(x)), \pi_z = d(\varphi(z), z) \leq \pi_x$. Hence $B_z \subseteq B_x$. Since $\mathfrak{C}$ is a maximal chain in $\mathcal{B}$, then B_z is the smallest element of $\mathfrak{C}$. But $\pi_{\varphi(z)} = d(\varphi(z), \varphi(\varphi(z))) < d(z, \varphi(z)) = \pi_z$ and therefore $B_{\varphi(z)} \subsetneq B_z$, contradicting the maximality of $\mathfrak{C}$. Hence there exists an element $x \in X$ with $\varphi(x) = x$. If also $\varphi(y) = y$ for $x \neq y$, then $d(x, y) = d(\varphi(x), \varphi(y)) < d(x, y)$, which is absurd. Thus there exists exactly one fixed point for φ. ∎

Remark 1.43 Analysing the proof of Theorem 1.45, we see that to prove the existence of a fixed point for the mapping $\varphi: X \longrightarrow X$, it suffices to assume the following property. For any $x, y \in X, d(\varphi(x), \varphi(y)) \leq d(x, y)$ and for $x \neq \varphi(x), d(\varphi(x), \varphi(\varphi(x))) < d(x, \varphi(x))$.

In the special case when Γ is totally ordered, we obtain the following characterization of principal completeness [153].

Theorem 1.46 *Let* (X, d, Γ) *be an ultrametric space and assume that* Γ *is totally ordered. The following conditions are equivalent*:

1. *X is principally complete*
2. *Every strictly contracting mapping* $\varphi\colon X \longrightarrow X$ *has a fixed point.*

Proof 1. $\Longrightarrow$ 2.: this was proved in Theorem 1.45.

2. $\Longrightarrow$ 1.: We assume that X is not principally complete, so there exists a chain $\mathcal{C}$ of principal balls such that $\bigcap \mathcal{C} = \emptyset$. Hence $\mathcal{C}$ dos not have a smallest ball and therefore the coinitial type λ of $\mathcal{C}$ is a limit ordinal. Then there exists a strictly decreasing family $(B_i)_{i<\lambda}$ of balls $B_i \in \mathcal{C}$ such that $\bigcap_{i<\lambda} B_i = \bigcap \mathcal{C} = \emptyset$. We write $B_i = B_{\gamma_i}(a_i)$ and we define $\varphi\colon X \longrightarrow X$. If $x \in X$ there exists the smallest $\kappa = \kappa(x) < \lambda$ such that $x \notin B_\kappa$, we define $\varphi(x) = a_\kappa$. We show that φ is strictly contracting. Let $x, y \in X, x \neq y$. If $\kappa(x) = \kappa(y)$ then $0 = d(\varphi(x), \varphi(y)) < d(x, y)$. If $\kappa(x) \neq \kappa(y)$, say $\kappa(x) < \kappa(y)$, from $B_{\kappa(x)} \supset B_{\kappa(y)}$ and $x \notin B_{\kappa(x)}, y \in B_{\kappa(x)}$ we get $d(x, y) > \gamma_{\kappa(x)} \geq d(\varphi(x), \varphi(y))$. So φ is strictly contracting. From the definition of φ, it is obvious that φ does not have a fixed point. ■

Brouwer's fixed point theorem, in mathematics, a theorem of algebraic topology that was stated and proved by Brouwer [27, 28]. Inspired by the earlier work of the French mathematician Poincaré, Brouwer investigated the behavior of continuous functions mapping the closed ball of unit radius in n-dimensional Euclidean space into itself.

Theorem 1.47 (Brouwer's Fixed Point Theorem) *Let X be an n-dimensional Euclidean space. Then, any continuous map of* $\left\{x \in X\colon \|x\| \leq 1\right\}$ *into itself has a fixed point.*

As a consequence, we get

Theorem 1.48 *Any continuous map T of a compact convex K set in n-dimensional Euclidean space X into itself has a fixed point.*

Proof Assume first that $K \subseteq B_X = \left\{x \in X\colon \|x\| \leq 1\right\}$. Define $G\colon B_X \to K$ by taking $G(x)$ to be the unique point $y \in K$ such that $\|x-y\| \leq \|x-z\|$ for all $z \in K$. Such a vector y exists and unique. Note that $G(x) = x = y$ if $x \in K$. Consider $T \circ G\colon B_X \to K$ as a map from B_X into itself. The map $H\colon B_X \to B_X$ defined by $H(x) = T(G(x))$ is continuous because G is continuous. Let $x_n \to x$. We have $\|x_n - G(x_n)\| \leq \|x_n - z\|$ for all $z \in K$. Hence, if y is any limit point of $\{G(x_n)\}$ then $\|x - y\| \leq \|x - z\|$ for all $z \in K$. This proves that $G(x)$ is the only limit of $\{G(x_n)\}$ which lies in the compact set K. Hence $G(x_n) \to G(x)$. By Theorem 1.47

there exists $x \in B_X$ such that $T(G(x)) = x$. Since the range of T is contained in K we get $x \in K$. But then $G(x) = x$ so $T(x) = x$. This proves the theorem when $K \subseteq B_X$. For the general case choose R such that $K \subseteq \left\{x \in X \colon \|x\| \leq R\right\}$. Let $K_1 = \{R^{-1}x \colon x \in K\}$. Then K_1 is a compact convex set and the function $T_1 \colon K_1 \to K_1$ defined by $T_1(x) = R^{-1}T(Rx)$ is continuous. By the first case there exists $x_1 \in K_1$ such that $R^{-1}T(Rx_1) = x_1$. If $x = Rx_1$ then $T(x) = x$. ■

Remark 1.44 (Kakutani's Example) Theorem 1.47 does not hold in an infinite dimensional Hilbert space:

if $T(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, \cdots)$ then T maps $\left\{x \in l_2 \colon \|x\| \leq 1\right\}$ into itself and is continuous. It has no fixed point.

Definition 1.66 A map $T \colon Y \to X$ where X is a normed space and $Y \subseteq X$ is called compact if $T(Z)$ is relatively compact whenever $Z \subseteq Y$ is bounded.

Brouwer's Theorem was extended to infinite dimensional spaces by Schauder in the following way [174].

Theorem 1.49 (Schauder's Fixed Point Theorem) *Let Y be a closed bounded convex set in a normed space $(X, \|.\|)$ and T a continuous map of Y into itself. If T is compact then it has a fixed point.*

Proof Let $Z \subseteq X$ be compact. Let $\varepsilon > 0$ and $B_\epsilon(x_1), B_\epsilon(x_2), \cdots, B_\epsilon(x_N)$ cover Z where $\{x_1, x_2, \cdots, x_N\} \subseteq Z$. Let $m_i(x) = \max(\varepsilon - \|x - x_i\|, 0)$ and $\varphi(x) = \dfrac{\sum_{i=1}^N m_i(x)x_i}{\sum_{j=1}^N m_j(x)}$ for $x \in Z$. It is obvious that each m_i is continuous and $\sum_{j=1}^N m_j(x) > 0$ for all $x \in Z$. Hence φ is continuous. If $x \in Z$ then $m_i(x) \neq 0$ implies $\|x - x_i\| < \varepsilon$ and hence $\left\|\sum_{i=1}^N m_i(x)(x_i - x)\right\| < \varepsilon \sum_{i=1}^N m_i(x)$ which proves that $\|\varphi(x) - x\| < \varepsilon$ ($m_i(x) \neq 0$ for at least one i). Further $\varphi(Z) \subseteq \text{conv}(Z)$.

Let $W = \overline{T(Y)}$. Then W is a compact subset of Y. For each n let $\varphi_n \colon W \to \text{conv}(W) \subseteq Y$ be a continuous map such that $\|\varphi_n(x) - x\| < \dfrac{1}{n}$ for all $x \in W$ for all n. This is possible by the reasoning above. Let $T_n = \varphi_n \circ T$ so that T_n is a continuous map : $W \to Y$. So there is a finite set $\{x_1^n, x_2^n, \cdots, x_{N_n}^n\} \subseteq W$ such that $\varphi_n(W) \subseteq W_n := \text{span}(\{x_1^n, x_2^n, \cdots, x_{N_n}^n\})$. Let $Y_n = Y \cap W_n$. Then Y_n is a compact convex set in the finite dimensional space W_n. We claim that T_n maps Y_n into itself. First note that $T(Y_n) \subseteq T(Y) \subseteq W$ so $T_n = \varphi_n \circ T$ is defined on Y_n. Also φ_n takes values in $\text{conv}(\{x_1^n, x_2^n, \cdots, x_{N_n}^n\}) \subseteq W_n$ as well as in Y so it takes values in Y_n. By Theorem 1.48 there exists $y_n \in Y_n$ such that $T_n(y_n) = y_n$. Since $y_n \in Y$ and $T(y_n) \in W$ we have $\|\varphi_n(T(y_n)) - T(y_n)\| < \dfrac{1}{n}$ for all n. In other words $\|y_n - T(y_n)\| < \dfrac{1}{n}$ for all n. Since $(T(y_n))_n \subseteq W$ and W is compact there is a subsequence $(T(y_{n_j}))_{n_j}$ converging to some y. Now

$$\|y_{n_j} - y\| \leq \|T(y_{n_j}) - y\| + \|y_{n_j} - T(y_{n_j})\| < \|T(y_{n_j}) - y\| + \frac{1}{n_j} \to 0.$$

This implies $T(y) = y$. ■

Lemma 1.12 *Let* $Y_0 = \left\{x = (x_n)_{n\geq 1} \in l_2 \colon |x_n| \leq \frac{1}{n} \text{ for all } n \geq 1\right\}$. *Then any continuous map* $T\colon Y_0 \to Y_0$ *has a fixed point.*

Proof We first prove that the parallelepiped Y_0 is compact in l_2. We have $Y_0 = \bigcap_{n\geq 1} Z_n$, $Z_n = \left\{x = (x_m)_{m\geq 1} \in l_2 \colon |x_n| \leq \frac{1}{n}\right\}$. Since the canonical projection $p_n \colon l_2 \to \mathbb{K}$ is continuous, it follows that $Z_n = p_n^{-1}(\overline{B_{\frac{1}{n}}(0)})$ is closed for all $n \geq 1$, and therefore Y_0 is a closed set. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows that for any $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that $\sum_{k=n_\varepsilon}^{\infty} \frac{1}{k^2} \leq \varepsilon$. Since $|p_n(x)| \leq \frac{1}{n}$ for all $x \in Y_0$ and $n \geq 1$, it follows that $\sum_{k=n_\varepsilon}^{\infty} |p_k(x)|^2 \leq \varepsilon$ for all $x \in Y_0$, i.e., Y_0 is relatively compact in l_2. Hence Y_0 is compact.

Let $Y_n = \{(x_1, x_2, \cdots, x_n, 0, 0, \cdots) \colon x \in Y\}$ and define $T_n \colon Y_n \to Y_n$ by $T_n(x) = (y_1, y_2, \cdots, y_n, 0, 0, \cdots)$ where $y = T(x_1, x_2, \cdots, x_n, 0, 0, \cdots)$. Y_n can be identified with compact convex set in $\mathbb{K}^n$ and T_n is continuous, hence it has a fixed point $x^{(n)}$. Since $(x_n)_{n\geq 1} \subseteq Y_0$ and Y_0 is compact in $(l_2, \|.\|_2)$ there is a subsequence $(x_{n_j})_j$ converging to some $x \in Y$. Let $y^n = T(x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}, 0, 0, \cdots)$ so that $x^{(n)} = T_n(x^{(n)}) = (y_1^{(n)}, y_2^{(n)}, \cdots, y_n^{(n)}, 0, 0, \cdots)$. It is clear that $\lim_{n\to\infty} (x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}, 0, 0, \cdots) = x$ so $\lim_{n\to\infty} y^{(n)} = T(x)$. Hence $x = \lim_{j\to\infty} x^{(n_j)} = \lim_{j\to\infty} (y_1^{(n_j)}, y_2^{(n_j)}, \cdots, y_n^{(n_j)}, 0, 0, \cdots) = \lim_{j\to\infty} y^{(n_j)} = T(x)$. ■

Lemma 1.13 *If* Z *is a closed convex of* Y_0 *then every continuous map of* Z *into itself has a fixed point.*

Proof For each $x \in Y_0$ there is a unique point $P(x) \in Z$ closet to x and the map $P\colon Y_0 \to Z$ is continuous. If $T\colon Z \to Z$ is continuous then $G\colon Y_0 \to Y_0$ defined by $G = T \circ P$ is continuous. Hence by Lemma 1.12 there exists $x \in Y_0$ such that $T(P(x)) = x$. Since the range of T is contained in Z we see that $x = T(P(x)) \in Z$. But then $P(x) = x$ so $x = T(x)$. ■

Proposition 1.46 *Let Y be a compact convex set in a locally convex topological vector space (X, τ). If Y has at least two points and $T: Y \to Y$ is continuous then there is a proper subset Y_1 of Y such that $T(Y_1) \subseteq Y_1$ and Y_1 is also compact and convex.*

Proof We reduce the proof to the case when the topology τ of X is replaced by the weak topology. We introduce an ordering for subsets of X' as follows: $Z \leq W$ if for any $\psi \in Z$ and $\varepsilon > 0$ there exists a finite subset $\varphi_1, \varphi_2, \cdots, \varphi_k$ of W and $\delta > 0$ such that $x, y \in Y$ and $|\varphi_i(x) - \varphi_i(y)| < \delta, 1 \leq i \leq k$ imply $|\psi(T(x)) - \psi(T(y))| < \varepsilon$. We observe that if $Z \leq W$ and $\varphi(x) = \varphi(y)$ for all $\varphi \in Z$ then $\psi(T(x)) = \psi(T(y))$. We claim that for any $\psi \in Z$ there exists a countable family $W = \{\varphi_1, \varphi_2, \cdots\}$ such that $\{\psi\} \leq W$. For this let $\varepsilon > 0$. First note that T is weak-weak continuous and Y is compact convex in weak topology. By uniform continuity of $\psi \circ T$ on Y with its weak topology, $|\psi(T(x)) - \psi(T(y))| < \varepsilon$ if $x - y$ belongs to a suitable weak neighbourhood of θ. Hence there exists $\varphi_1, \varphi_2, \cdots, \varphi_k$ and $\delta > 0$ such that $|\varphi_i(x) - \varphi_i(y)| < \delta, 1 \leq i \leq k$ implies $|\psi(T(x)) - \psi(T(y))| < \varepsilon$. Now vary ε over $\{\frac{1}{n}, n \geq 1\}$ to get a countable set $W \subseteq X'$. For any $\varepsilon > 0$ choose n such that $\frac{1}{n} < \varepsilon$. There exists $\varphi_1, \varphi_2, \cdots, \varphi_k$ and $\delta > 0$ such that $|\varphi_i(x) - \varphi_i(y)| < \delta, 1 \leq i \leq k$ implies $|\psi(T(x)) - \psi(T(y))| < \frac{1}{n} < \varepsilon$. It follows that if $|\varphi(x) - \varphi(y)| < \delta$ for all $\varphi \in W$ then $|\psi(T(x)) - \psi(T(y))| < \varepsilon$. Hence $\{\psi\} \leq W$. If we now repeat the argument for each element of W to get another countable set W_1, then repeat the argument for each element of W_1 and so on we end up with countable family W_0 such that with ψ it self, we get a countable subset P of X' which contains ψ with $P \leq P$.

If Y_1 is weakly compact, convex and contained in Y then it is a weakly closed convex set, hence strongly closed. Hence it is a closed convex subset of Y in the strong (i.e., original) topology, hence strongly compact also. Thus, we may and do assume that the topology τ of X is the weak topology. Now suppose $x, y \in Y, x \neq y$. Choose ψ such that $\psi(x) \neq \psi(y)$. Let $P = \{\psi_1 = \psi, \psi_2, \cdots\}$ be a countable subset of X' containing ψ such that $P \leq P$. Now $\psi_n(Y)$ is compact for each $n \geq 1$. Because if $Q = \{\alpha_1\psi, \alpha_2\psi_2, \cdots\}$ with each $\alpha_n > 0$ then $Q \leq Q$, we may suppose $|\psi_n(z)| \leq \frac{1}{n}$ for all $n \geq 1$, for all $z \in Y$. Define $G: Y \to l_2$ by $G(z) = (\psi_n(z))_{n\geq 1}$. G is continuous and its range S is contained in $Y_0 = \left\{x = (x_n)_{n\geq 1} \in l_2 : |x_n| \leq \frac{1}{n} \text{ for all } n \geq 1\right\}$. S has at least two points because $\psi(x) \neq \psi(y)$. Let $T_0: S \to S$ be the map $G \circ T \circ G^{-1}$. In other words, if $s \in S$ we pick $z \in Y$ such that $s = G(z)$ and define $T_0(s) = G(T(z))$. To see that this is well defined note that $s = G(z_1) = G(z_2)$ implies $\psi_n(z_1) = \psi_n(z_2)$ for all n which implies $\psi_n(T(z_1)) = \psi_n(T(z_2))$ for all n (because $P \leq P$) so $G(T(z_1)) = G(T(z_2))$ so T_0 is well defined. The fact that $P \leq P$ also implies that if $\psi_n(z_m) \longrightarrow \psi_n(z)$ as $m \longrightarrow \infty$ for each n then $\psi_n(T(z_m)) \longrightarrow \psi_n(T(z))$ for each n. This means T_0 is continuous. Lemma 1.13

shows that T_0 has a fixed point $s_1 \in S$. Let $Y_1 = G^{-1}(\{s_1\})$. Let $z \in Y_1$ so $G(z) = s_1$. Then $s_1 = T_0(s_1) = G(T(z))$. Hence $T(z) \in Y_1$. Thus $T(Y_1) \subseteq Y_1$. Clearly Y_1 is convex. It is a closed subset of S and hence it is compact. ∎

Tychonoff extended Brouwer's result to a compact convex subset of a locally convex topological vector space [190].

Theorem 1.50 (Schauder-Tychonoff's Fixed Point Theorem) *Any continuous map T from a compact convex subset Y of a locally convex topological vector space (X, τ) into Y has a fixed point.*

Proof By Proposition 1.46 there is a minimal nonempty compact convex set Y_0 such that $T(Y_0) \subseteq Y_0$ and Y_0 must be a singleton. ∎

The following result [98], called Markov-Kakutani fixed point theorem, is powerful in that it determines a single fixed point for a whole family of mappings, while theorems such as the Schauder-Tychonoff fixed point theorem determine conditions on the space such that the restriction on the mapping is minimal, namely that we only require the mapping T to be continuous.

Theorem 1.51 (Markov-Kakutani's Fixed Point Theorem) *Let Y be a compact convex subset of a locally convex topological vector space (X, τ). Let $T_\alpha : Y \to Y (\alpha \in I)$ be a family of continuous mappings that are affine (which means they satisfy the condition $T_\alpha(\sum_{i=1}^{n} \lambda_i x_i) = \sum_{i=1}^{n} \lambda_i T_\alpha(x_i)$ whenever $n \in \mathbb{N}$, $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{n} \lambda_i = 1$). If $T_\alpha \circ T_\beta = T_\beta \circ T_\alpha$ for all $\alpha, \beta \in I$ then there exists $x \in Y$ such that $T_\alpha(x) = x$ for all $\alpha \in I$.*

Proof For each $\alpha \in I$, let $Z_\alpha = \{x \in Y : T_\alpha(x) = x\}$. From the Schauder-Tychonoff fixed point theorem we know that $Z_\alpha \neq \emptyset$. Since T_α is a continuous affine map, it follows that Z_α is compact and convex. So to restate the conclusion of the theorem we must show that $\bigcap_{\alpha \in I} Z_\alpha \neq \emptyset$. Since Y is compact, we have, by Proposition 1.4 that we need only show that $\bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$ for each nonempty finite subset J of I. To this end, let $J = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a nonempty finite subset of I. We shall proceed by induction.

Let x be any element of Z_{α_1} then

$$T_{\alpha_1}(T_{\alpha_2}(x)) = T_{\alpha_2}(T_{\alpha_1}(x)) = T_{\alpha_2}(x).$$

That is, $T_{\alpha_2}(x)$ is a fixed point of T_{α_1} and so $T_{\alpha_2}(x) \in Z_{\alpha_1}$. Thus, $T_{\alpha_2}(Z_{\alpha_1}) \subseteq Z_{\alpha_1}$. Hence, from the Schauder-Tychonoff fixed point theorem, T_{α_2} has a fixed point in Z_{α_1}. Therefore, $Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset$. Now, suppose that

$$Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j} \text{ where, } 1 \leq j \leq n.$$

Let $Z = Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j}$. Then Z is nonempty, compact and convex. Let x be any element of Z and let $1 \leq i \leq j$ then

$$T_{\alpha_i}(T_{\alpha_{j+1}}(x)) = T_{\alpha_{j+1}}(T_{\alpha_i}(x)) = T_{\alpha_{j+1}}(x).$$

That is, $T_{\alpha_{j+1}}(x)$ is a fixed point of T_{α_i} and so $T_{\alpha_{j+1}}(x) \in Z_{\alpha_i}$. Since $1 \leq i \leq j$ was arbitrary,

$$T_{\alpha_{j+1}}(x) \in Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j} = Z.$$

Hence, from the Schauder-Tychonoff fixed point theorem, $T_{\alpha_{j+1}}$ has a fixed point in Z. Therefore,

$$Z_{\alpha_1} \cap Z_{\alpha_2} \neq \emptyset \cap \cdots \cap Z_{\alpha_j} \cap Z_{\alpha_{j+1}} \neq \emptyset.$$

By induction, we see that $\bigcap_{\alpha \in J} Z_\alpha \neq \emptyset$. This completes the proof. ■

We shall need some facts about the Kuratowski measure of noncompactnessμ introduced by Kuratowski [122]. This measure of noncompactness is used by Darbo [40], Furi and Vignoli [61], Nussbaum [136], Petryshyn [150], and others.

The concept of Kuratowski's measure of noncompactness is defined below.

Definition 1.67 Let (X, d) a metric space. If Y is a bounded subset of X (i.e., $\text{diam}(Y) = \sup\{d(x, y) \colon x, y \in Y\} < \infty$), the Kuratowski measure of noncompactness of Y is defined by

$$\mu(Y) = \inf\left\{\delta > 0 \colon Y = \bigcup_{i=1}^{n} Y_i \text{ for some } Y_i \text{ with } \text{diam}(Y_i) \leq \delta, 1 \leq i \leq n < \infty\right\}.$$

We give the following properties of μ. For the proofs see [136].

Proposition 1.47 *Let (X, d) be a metric space. If Y is a bounded subset of X, then $\mu(Y) = \mu(\overline{Y})$.*

Proposition 1.48 *Let (X, d) be a complete metric space. Then*

1. *for every bounded subset Y of X, $\mu(Y) = 0$ if and only if $\overline{Y}$ is compact.*
2. *If $(Y_n)_{n \geq 1}$ is a decreasing sequence of closed, bounded nonempty subsets of X and if $\lim_{n \longrightarrow \infty} \mu(Y_n) = 0$, then $Y = \bigcap_{n \geq 1} Y_n$ is compact and nonempty.*

If $(X, \|.\|)$ is a normed space, the norm $\|.\|$ gives a metric on X and one can take the Kuratowski measure of noncompactness μ on X with respect to this metric.

Proposition 1.49 *Let $(X, \|.\|)$ be a normed space, Y, Z two bounded subsets of X, $x_0 \in X$ and $\lambda \in \mathbb{K}$. Then*

1. $\mu(\lambda Y) = |\lambda|\mu(Y)$.
2. $\mu(\text{conv}(Y)) = \mu(Y)$.
3. $\mu(Y + Z) \leq \mu(Y) + \mu(Z)$.
4. $\mu(Y \cup \{x_0\}) = \mu(Y)$.

Closely associated with the measure of noncompactness is the concept of k-set contraction.

Definition 1.68 If Y_1 is a subset of a metric space (X_1, d_1), and (X_2, d_2) is a second metric space and $T: Y_1 \to X_2$ is a continuous map, we shall say that T is a k-set-contraction if $\mu_2(T(Z)) \leq \mu_1(Z)$, for all bounded sets $Z \subseteq Y_1$, where μ_i denotes the Kuratowski measure of noncompactness on (X_i, d_i).

Theorem 1.52 (Darbo's Fixed Point Theorem) *Let Y be a closed bounded convex set in a Banach space $(X, \|.\|)$ and let $T: Y \to Y$ be a k-set-contraction with $k < 1$. Then T has a fixed point in Y [40].*

There is a more useful generalization of Darbo's fixed point theorem.

Theorem 1.53 *Let Y be a closed bounded convex set in a Banach space $(X, \|.\|)$ and $T: Y \to Y$ a continuous map. Define $Y_1 = \overline{\text{conv}}(T(Y))$ and $Y_n = \overline{\text{conv}}(T(Y_{n-1}))$ for $n > 1$ and assume that if $\lim_{n\longrightarrow\infty} \mu(Y_n) = 0$ where μ denotes the Kuratowski measure of noncompactness on X. Then T has a fixed point in Y.*

If T in Theorem 1.53 is a k-set contraction with $k < 1$, then if $\lim_{n\longrightarrow\infty} \mu(Y_n) = 0$, but the conditions of Theorem 1.53 may be satisfied in cases of interest for which T is not a k-set contraction with $k < 1$.

The following result is an extension of Darbo's fixed point theorem [61, 136, 172].

Theorem 1.54 (Sadovskii's Fixed Point Theorem) *Let Y be a closed bounded convex set in a Banach space $(X, \|.\|)$ and let $T: Y \to Y$ be a continuous μ-condensing map (i.e., $\mu(T(Z)) < \mu(Z)$, for all bounded sets $Z \subseteq Y$ for which $\mu(Z) > 0$). Then T has a fixed point in Y.*

1.6 Nonexpansive Mappings

Definition 1.69 A mapping T is nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all x, y in its domain.

Definition 1.70 Let X be a Banach space and Y be a nonempty bounded closed convex subset of X. We say that Y has the fixed point property for nonexpansive mapping if for every nonexpansive mapping $T: Y \longrightarrow Y$, Y contains a fixed point x^* (i.e., $T(x^*) = x^*$), X has the fixed point property (***FPP*** for short) if any nonempty bounded closed convex subset of X has the fixed point property for nonexpansive mapping, X has the weak fixed point property (***WFPP*** for short) if any weakly compact convex subset of X has the fixed point property for nonexpansive mapping.

Remark 1.45 For a reflexive Banach space, ***FPP*** and ***WFPP*** are obviously the same.

Definition 1.71 Let Y be a nonempty set. A nonempty subset Y_0 of Y is called invariant under T or T-invariant for a mapping $T: Y \longrightarrow Y$ if $T(Y_0) \subset Y_0$. Let $\mathcal{Y}$ be a class of subsets of Y. We say that an element $Y_0 \in \mathcal{Y}$ is $\mathcal{Y}$-minimal for T if there exists no proper T-invariant subset of Y_0 in the class $\mathcal{Y}$.

We are interested mainly in the case that Y is a subset of a Banach space X and $\mathcal{Y}$ is the class of weakly compact subsets of X or the class of closed convex subsets of X.

Remark 1.46 If Y is a closed convex subset of a Banach space X and $T: Y \longrightarrow Y$, then a decreasing sequence of nonempty, closed, convex, T-invariant sets may be obtained by setting

$$Y_0 = Y \ \text{ and } \ Y_{n+1} = \overline{\operatorname{conv}}(T(Y_n)) \ \ \forall n \geq 1.$$

We set

$$\widehat{Y} = \bigcap_{n=1}^{\infty} Y_n.$$

The set $\widehat{Y}$ is closed, convex and T-invariant. But it may be empty. Of course this situation cannot occur if Y is weakly compact.

Proposition 1.50 *If X is a Banach space, $Y \subseteq X$ is a nonempty, weakly compact, convex set and $T: Y \longrightarrow Y$, then there exists a nonempty, closed, convex set $\widehat{Y} \subseteq Y$ which is minimal invariant for T.*

Proof Let Γ be the family of all nonempty, closed, convex subsets of Y which are T-invariant. We order Γ by reverse inclusion, namely if $Y_1, Y_2 \in \Gamma$, then

$$Y_1 \leq Y_2 \Longleftrightarrow Y_2 \subset Y_1.$$

By the finite intersection property for the weak topology, every chain in Γ has an upper bound (namely the intersection of the elements in the chain). So by the Zorn lemma, Γ has a maximal element $\widehat{Y} \in \Gamma$. Evidently $\widehat{Y}$ is T-invariant. ∎

Remark 1.47 Note that if $\widehat{Y} \subseteq Y$ is a nonempty, closed, convex and minimal T-invariant set, then

$$\widehat{Y} = \overline{\operatorname{conv}}(T(\widehat{Y})).$$

If $\widehat{Y} \in \Gamma$ in Proposition 1.50 is a singleton, i.e., $\widehat{Y} = \{y\}$, then

$$T(y) = y,$$

i.e., it is a fixed point of T.

The famous question whether a Banach space has the fixed point property had remained open for a long time. It has been answered in the negative by Sadovski [172] and Alspach [4] who constructed the following examples, respectively.

Examples 1.10

1. Let $X = c_0$ and $Y = \{x \in c_0, \|x\|_\infty \leq 1\}$. Define $T: Y \longrightarrow Y$ by

$$T(x) = (1, x_1, x_2, x_3, \ldots), \text{ for all } x = (x_1, x_2, x_3, \ldots) \in Y.$$

2. Let $X = L^1(0, 1)$ and

$$Y = \left\{x \in X,\ 0 \leq x(t) \leq 1 \text{ and } \int_0^1 x(t)dt = \frac{1}{2}\right\}.$$

Define $T: Y \longrightarrow Y$ by

$$T(x)(t) = \begin{cases} \min\{1, 2x(2t)\}, \text{ if } 0 \leq t \leq \dfrac{1}{2}, \\ \max\{0, 2x(2t-1)-1\}, \text{ if } \dfrac{1}{2} < t \leq 1. \end{cases}$$

Then Y is bounded, closed, and convex, and T is an isometry $\|T(x) - T(y)\|_1 = \|x - y\|_1$, for all $x, y \in Y$ and is fixed point free.

Namely, c_0 and $L^1(0, 1)$ do not have the fixed point property. The above two examples suggest that to obtain positive results for the existence of fixed points for nonexpansive mappings, it is necessary to impose some restrictions either on T or on the Banach space X.

The following well-known result is due to Kirk [107].

Theorem 1.55 *Let X be a reflexive Banach space and Y a closed bounded convex subset of X. Let Y have normal structure. If $T: Y \to Y$ is nonexpansive, then T has a fixed point.*

Remark 1.48 Theorem 1.55 remains true if X is any Banach space and Y is a convex weakly compact subset having normal structure.

An immediate consequence of Theorem 1.55 is the following well-known result, which was proved independently by Browder [29], Göhde [69] and Kirk [107].

Theorem 1.56 *Let X be a uniformly convex Banach space and Y a nonempty closed bounded convex subset of X. If $T : Y \to Y$ is nonexpansive, then T has a fixed point.*

Remark 1.49 For nonexpansive maps, no characterization of ***FPP*** or ***WFPP*** seems to be known [21].

Chapter 2
Almost Fixed Points

This chapter presents various almost fixed points results from the literature. In proofs of many fixed point theorems, almost fixed points have usually appeared in an auxiliary role. In certain cases, almost fixed points, unlike fixed points, can be obtained numerically, and in some other cases, the existence of a fixed point is non-trivial or uncertain, whereas almost fixed points are easily found. Therefore, almost fixed points seem to be natural objects in many applications.

We will concerned with a (continuous) mapping T of a metric space (X, d) and with points which are almost fixed, in the sense that

$$d(T(x), x) < \varepsilon.$$

We call such a point "ε-fixed".

2.1 Relation Between ε-Fixed and Fixed Points

Theorem 2.1 *Let (X, d) be a metric space, Ω be a subset of X and let $T : \Omega \longrightarrow X$ be a continuous map. If x is fixed for T then any point $y \in \Omega$ sufficiently close to x is ε-fixed [183].*

Proof $d(T(y), y) \leq d(T(y), T(x)) + d(T(x), x) + d(x, y)$, which is less than ε for y sufficiently close to x. ■

Here is a general argument showing that ε-fixed points can be found constructively where fixed points exist in compact sets [183].

Theorem 2.2 *Let (X, d) be a metric space and Ω be a compact subset of X and let $T : \Omega \to X$ be a continuous map. If T has ε-fixed points for all $\varepsilon > 0$ then T has a fixed point.*

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Proof We have points x_n such that $d(T(x_n), x_n) \leq \dfrac{1}{n}$. By compactness we can assume that $x_n \longrightarrow y$ in Ω. Then by continuity, $d(T(y), y) = 0$. ■

Note that this argument is not constructive, even if we can find $\dfrac{1}{n}$-fixed point by some effective method (in a finite number of steps) it does not give us an effective way to find (or even to approximate) the fixed point.

We can't drop the word compact in Theorem 2.2.

Example 2.1 ([135]) Let $T(x_1, x_2, \ldots) = (1 - \|x\|, x_1, x_2, \ldots)$ in the closed unit ball of l^2. Then T has no fixed point, but $x_n = k^{\frac{-1}{2}}$ for $1 \leq k \leq n$ and $x_n = \theta$ for $n > k$ then $\|T(x) - x\| = 2^{\frac{1}{2}}k^{\frac{-1}{2}}$ so that T has ε-fixed points for all $\varepsilon > 0$.

Must an ε-fixed point of T be a fixed point of some mapping close to T?

Theorem 2.3 *Let $(X, \|.\|)$, Ω a convex subset of X, and x be an ε-fixed point of a mapping T of Ω into Ω. Then there exists a mapping S of Ω into Ω, uniformly within ε of T, for which x is a fixed point [183].*

Proof Let $\|T(x) - x\| = \delta < \varepsilon$. Define $Sy = x$ if $\|x - T(y)\| \leq \delta$ and $S(y) = T(y) + \delta \dfrac{x - T(y)}{\|x - T(y)\|}$ otherwise. ■

The convexity condition cannot be omitted from Theorem 2.3:

Example 2.2 ([183]) Let Ω be the unit circle in $\mathbb{C}^1$ with a small arc near 1 removed: $\Omega = \{e^{i\theta} : \dfrac{\varepsilon}{2} \leq \theta \leq 2\pi - \dfrac{\varepsilon}{2}\}$ (for small ε). Let T be complex conjugation, $T(z) = \overline{z}$. Then $p = e^{i\frac{\varepsilon}{2}}$ is ε-fixed but no mapping with ε of T (uniformly) can have a fixed point near p.

2.2 Finding ε-Fixed Points Constructively Where Fixed Points Are "Known" to Exist

In dealing with problems of analysis we should always distinguish between the calculation of an approximate solution (one which nearly satisfies the requirements) and the approximate calculation of an exact solution. Most algorithms for "calculating fixed points" actually yield ε-fixed points which may or may not be close to exact fixed points [102]. But sometimes there are reasons why the ε-fixed point found must be close to a fixed point [180, 189]

The following example in [183] demonstrates that an ε-fixed point need not be near a fixed point.

Example 2.3 Consider the following mapping of the closed interval $[0, 1]$, $T\colon [0, 1] \to [0, 1]$ defined by $T(x) = \max(0, \min(x + \alpha, 1))$, where $|\alpha| < \varepsilon$. Then if $\alpha \neq 0$, all points in $[0, 1]$ are ε-fixed but most of them are a long way from the fixed point.

This difficulty in approximating exact fixed points applies to all the deep fixed point theorems which are commonly used in applications: those on continuous mappings of compact sets (Brouwer, Schauder), Kakutani's theorem on many-valued mappings, and so on. (The exception is Banach's Contraction Mapping Principle, which does give approximations to an exact fixed point).

To see that ε-fixed points can be found in these cases, it is often enough to follow a standard proof of the existence of a fixed point up to the place, one line before the end, just before compactness is mentioned, where the proof asserts that an ε-fixed point exists.

Theorem 2.4 *Let (X, d) be a metric space and Ω be a compact subset of X and let $T: \Omega \longrightarrow X$ be a continuous map. If T has a fixed point ξ, then we can find an ε-fixed point (for any $\varepsilon > 0$) constructively.*

Proof Choose $\delta > 0$ so that $\delta < \frac{\varepsilon}{2}$ and so that $d(T(x), T(y)) < \frac{\varepsilon}{2}$ for $d(x, y) < \delta$. Choose a finite δ-net $\{x_1, x_2, \ldots, x_k\}$ in Ω. Calculate $d(x_i, T(x_i))$ for $1 \leq i \leq k$ and one of these values must be less than ε for if $d(x_i, \xi) < \delta$ then

$$d(x_i, T(x_i)) \leq d(x_i, \xi) + d(\xi, T(\xi)) + d(T(\xi), T(x_i)) \leq \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2}.$$

■

A practical case where fixed points can be found constructively is Banach's Contraction Mapping Principle (or a close relative). If $d(T(x), T(y)) \leq cd(x, y)$ (for all x, y) where $0 < c < 1$, then if ξ is the fixed point and x is any starting point then we have $d(T^n(x), \xi) \leq c^n \frac{d(x, T(x))}{1 - c}$ so that we can actually assert that $T^n(x)$ is close to ξ for n sufficiently large.

There is one interesting case where the ε-fixed points are easily found, but getting exact fixed points (even non-constructively) is tricky.

Definition 2.1 Let Y be a subset of a Banach space and $T: Y \to Y$ a nonexpansive mapping. An approximate fixed point set for T is a set of the type

$$F_\varepsilon(T) = \{x : \|(T(x) - x)\| \leq \varepsilon\} \qquad \text{for some } \varepsilon \geq 0.$$

The set Y is said to have the approximate fixed point property for nonexpansive mapping (***AFPP*** for short) if $F_\varepsilon(T) \neq \emptyset$ for each $\varepsilon \geq 0$ and for each nonexpansive mapping $T: Y \to Y$ that is, $\inf\{\|(T(x) - x)\| : x \in Y\} = 0$.

Proposition 2.1 *Let Y be a bounded and convex subset of a Banach space and $T: Y \to Y$ a nonexpansive mapping. Then Y has the approximate fixed point property.*

Proof Consider the mapping $T_\lambda := \lambda I + (1-\lambda)T$ for $\lambda \in (0,1)$, where I denotes the identity map of Y. Then T_λ is a contraction mapping for each λ and by Theorem 1.43 it has a fixed point $x_\lambda \in Y$. Thus

$$\|x_\lambda - T(x_\lambda)\| = (1-\lambda)\|x_\lambda - T(x_\lambda)\| \to 0 \text{ as } \lambda \to 1.$$

∎

The sequence of successive approximations for nonexpansive mappings, unlike contraction mappings, may fail to converge. For example, if

$$T : \mathbb{R} \to \mathbb{R} \text{ given by } T(x) = 1 - x.$$

Then for $x_0 = 1$ we have $T^{2n}(x_0) = 1$ and $T^{2n+1}(x_0) = 0$ for $n \geq 1$. Also, rotation about the origin in the plane is another example where $(T^n(x_0))_n (x_0 \neq \theta)$ does not converge.

More generally, if Y is a convex set in a normed space X and $T : Y \to Y$ is a nonexpansive mapping, then for $\lambda \in (0,1)$,

$$T_\lambda = \lambda I + (1-\lambda)T$$

is a nonexpansive map and has the same fixed points as T.

For fixed $x_0 \in Y$, $(T_\lambda^n(x_0))_n$ is defined $T_\lambda^{n+1}(x_0) = \lambda x_n + (1-\lambda)T(x_n)$, where $x_n = T_\lambda^n(x_0)$.

An early result, concerning the convergence of the sequence of successive approximations, is due to Krasnoselskii [118].

Theorem 2.5 *Let X be a uniformly convex Banach space and Y a closed convex bounded subset of X. If $T : Y \to Y$ is nonexpansive and $T(Y)$ is relatively compact, then for any $x_0 \in Y$, the sequence $(T_{\frac{1}{2}}^n(x_0))_n$ of iterates of x_0 under $T_{\frac{1}{2}} := \frac{1}{2}(I+T)$ converges to a fixed point of T.*

Schaefer [173] observed that the same result holds for any T_λ with $\lambda \in (0,1)$, and Edelstein [52] proved that strict convexity of X suffices.

The important and natural question is whether strict convexity can be removed. This question was resolved in the affirmative in the following theorem [90].

Theorem 2.6 *Let Y be a nonempty subset of a Banach space X and let $T : Y \to X$ be a nonexpansive mapping. For $x_0 \in Y$, define the sequence $(x_n)_n$ by*

$$x_{n+1} := (1-c_n)x_n + c_n T(x_n), \tag{2.1}$$

where the real sequence $(c_n)_n$ satisfies the following conditions:

(a) $\sum_{n=0}^{\infty} c_n = \infty,$

(*b*) $0 \leq c_n \leq 1$ *for all positive integers n,*
(*c*) $x_n \in Y$ *for all positive integers n.*

If $(x_n)_n$ *is bounded, then* $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

The iteration method of Theorem 2.6 is referred to as the Mann iteration method in light of [129]. One consequence of this theorem is that if Y is closed and T is completely continuous, then T has a fixed point and the sequence $(x_n)_n$ defined by (2.1) converges strongly to a fixed point of T [90].

Any sequence satisfying the conclusion of Theorem 2.6, i.e., $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$, is called an approximate fixed point sequence for T.

The importance of approximate fixed point sequences is that once a sequence has been constructed and proved to be an appropriate fixed point sequence for a nonexpansive map T, the convergence of that sequence to a fixed point of T is then achieved under some mild compactness-type assumptions either on T or on its domain.

The concept of asymptotic regularity is due to Browder and Petryshyn in [32].

Definition 2.2 A mapping $T: X \to X$ of a metric space (X, d) into itself is said to be asymptotically regular at $x \in X$ if $d(T^{n+1}(x), T^n(x)) \to 0$ as $n \to \infty$, it is said to be asymptotically regular on X if it is so at each $x \in X$.

Results on the asymptotic regularity of T_λ were first obtained by Browder and Petryshyn in [32]. They showed that if X is uniformly convex and $T: Y \to Y$ is a nonexpansive selfmapping on a closed, bounded, convex subset Y, then T_λ is asymptotically regular.

The asymptotic regularity is relevant to the existence of fixed points is seen from the following simple observation [32].

Proposition 2.2 *If* $T: X \to X$ *is continuous on a metric space* (X, d) *and asymptotically regular at* $x \in X$ *, then any cluster point of* $\{T^n(x)\}$ *is a fixed point of* T*.*

Proof Let $(T^{n_k}(x))_k$ be a subsequence of $(T^n(x))_n$ converging to $y \in X$. By continuity $T^{n_k+1}(x) \to T(y)$ and by asymptotic regularity $T^{n_k+1}(x) \to y$, so that $T(y) = y$. ■

It follows that for continuous T asymptotic regularity of T_λ at any $x \in Y$ implies that $T_\lambda(y) = y$ for any cluster point y of $\{T_\lambda^n(x)\}$.

Asymptotic regularity is not only useful in proving that fixed points exist but also in showing that in certain cases, the sequence of iterates at a point converges to the fixed point as in the following result [54].

Proposition 2.3 *Let* T *be a linear mapping of a normed space* X *into itself and suppose that* T *is power bounded, i.e., for some* $c \geq 0$,

$$\|T^n\| \leq c \quad (n = 1, 2, \cdots),$$

and asymptotically regular. If $\overline{\text{conv}}\{T^n(x)\}$ contains a fixed point z of T for some $x \in X$, then $(T^n(x))_n$ converges strongly to z.

Proof Let $\varepsilon > 0$ be given and suppose that y is a point in $\text{conv}\{T^n(x)\}$ with

$$\|z - y\| < \frac{\varepsilon}{2(c+1)}.$$

Setting $y = \sum_{k=1}^{m} \lambda_k T^k(x)$ we obtain

$$T^n(x-z) = T^n(x-y) + T^n(y-z) = T^n\left(x - \sum_{k=1}^{m} \lambda_k T^k(x)\right) + T^n(y-z)$$

$$= \sum_{k=1}^{m} \lambda_k (T^n(x) - T^{n+k}(x)) + T^n(y-z).$$

Hence

$$\|T^n(x-z)\| \leq \left\| \sum_{k=1}^{m} \lambda_k (T^n(x) - T^{n+k}(x)) \right\| + \frac{\varepsilon c}{2(c+1)}.$$

Now by asymptotic regularity, a positive integer N exists with the property that $n \geq N$ implies

$$\|T^n(x) - T^{n+k}(x)\| < \frac{\varepsilon}{2} \ (k = 1, 2, \cdots, m).$$

It follows that

$$\|T^n(x-z)\| < \sum_{k=1}^{m} \lambda_k \left(\frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq N$ so that $T^n(x-z) = T^n(x) - z \to 0$ as $n \to \infty$, proving the proposition. ■

Definition 2.3 A mapping T of a subset Y of a normed linear space into itself is said to be uniformly asymptotically regular if for any $\delta > 0$ there exists an N such that for all $x \in Y$ and for all $n \geq N$, $\|T^{n+1}(x) - T^n(x)\| < \delta$.

The following lemma was given in [54].

Lemma 2.1 *Let T be a nonexpansive selfmapping of a convex subset Y of a normed space X. For $\lambda \in (0, 1)$ define a nonexpansive mapping $T_\lambda = \lambda I + (1-\lambda)T$ of Y into itself. Let Z be a subset of Y such that for some a, $\|T_\lambda(x) - x\| < a$ for all*

x in Z such that for some $\delta > 0$ and any positive integer n there exists an x in Z (depending on n) with

$$\|T_\lambda^{n+1}(x) - T_\lambda^n(x)\| > \delta. \tag{2.2}$$

Then $\{T_\lambda^n(x) \colon x \in Z, n \in \mathbb{N}\}$ is unbounded.

Proof Assume by way of contradiction, that $\operatorname{diam}\{T_\lambda^n(x) \colon x \in Z, n \in \mathbb{N}\} \le \rho$. Let M and N be positive integers

$$M\delta > \rho + 1 \text{ and } N > \max\left\{M, \frac{Ma}{(1-\lambda)\lambda^M}\right\}.$$

Suppose that x satisfies (2.2) with $n = N$. Then since T_λ is nonexpansive, (2.2) must hold for all positive integers $i \le N$. We simplify the notation by writing $x_i = T_\lambda^i(x)$ and $y_i = T(T_\lambda^i(x))$. These points satisfy the following conditions:

$$\|x_{i+1} - x_i\| \text{ is a monotone non-increasing sequence with} \tag{2.3}$$

$$a \ge \|x_1 - x_0\| \ge \cdots \ge \|x_{N+1} - x_N\| > \delta,$$

$$\|y_{i+1} - y_i\| \le \|x_{i+1} - x_i\| \text{ for all } i = 0, 1, \cdots, N (\text{ by the nonexpansiveness of } T), \tag{2.4}$$

and

$$x_{i+1} = \lambda x_i + (1-\lambda) y_i \text{ so that } y_i = \frac{1}{1-\lambda} x_{i+1} - \frac{\lambda}{1-\lambda} x_i. \tag{2.5}$$

Note that (2.4) and (2.5) imply

$$\left\| \frac{1}{1-\lambda}(x_{i+1} - x_i) - \frac{\lambda}{1-\lambda}(x_i - x_{i-1}) \right\| = \|y_i - y_{i-1}\| \le \|x_i - x_{i-1}\| \tag{2.6}$$

for all $i = 1, 2, \cdots, N$. Also for any integer

$$L \ge \frac{a}{(1-\lambda)\lambda^M},$$

$[\delta, a]$ can be covered by L subintervals each of length $(1-\lambda)\lambda^M$. Hence by (2.3) and the fact that

$$N > \max\left\{M, M\frac{a}{(1-\lambda)\lambda^M}\right\},$$

we can find a subinterval $I = [b, b + (1-\lambda)\lambda^M]$ of $[\delta, a]$ such that I contains at least M of the numbers $\|x_{i+1} - x_i\|$, i.e., for some K,

$$\|x_{K+i+1} - x_{K+i}\| \in I \text{ for } i = 1, 2, \cdots, M. \tag{2.7}$$

By a result of Banach and Mazur, we can embed the linear span of

$$\{x_i, y_i : 0 \leq i \leq N+1\}$$

by a linear isometry into $C[0, 1]$. Viewing the x_i as continuous real functions on $[0, 1]$, it is clear that a $\xi \in [0, 1]$ exists such that $|x_{K+M+1}(\xi) - x_{K+M}(\xi)| \geq b$. We will assume that

$$x_{K+M+1}(\xi) - x_{K+M}(\xi) \geq b$$

as a similar argument holds in the other case. Then (2.6) and (2.7) imply

$$\frac{b}{1-\lambda} - \frac{\lambda}{1-\lambda}(x_{K+M}(\xi) - x_{K+M-1}(\xi)) \leq b + (1-\lambda)\lambda^M,$$

so that

$$x_{K+M}(\xi) - x_{K+M-1}(\xi) \geq b - (1-\lambda)^2\lambda^M \left(\frac{1}{\lambda}\right).$$

Similarly

$$x_{K+M-1}(\xi) - x_{K+M-2}(\xi) \geq b - (1-\lambda)^2\lambda^M \left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right) = b - (1-\lambda)\lambda^M \frac{(1-\lambda^2)}{\lambda^2},$$

and in general,

$$x_{K+M+1-i}(\xi) - x_{K+M-i}(\xi) \geq b - (1-\lambda)\lambda^M \frac{(1-\lambda^i)}{\lambda^i}, \quad \text{for } i = 0, 1, \cdots, M-1.$$

Thus

$$\begin{aligned} x_{K+M+1}(\xi) &\geq x_{K+M}(\xi) + b \\ &\vdots \\ &\geq x_{K+M+1-i}(\xi) + ib - (1-\lambda)\lambda^M \left(\frac{1-\lambda}{\lambda} + \cdots + \frac{1-\lambda^{i-1}}{\lambda^{i-1}}\right) \\ &\vdots \end{aligned}$$

$$\geq x_{K+1}(\xi) + Mb - (1-\lambda)\lambda^M \left(\frac{1-\lambda}{\lambda} + \cdots + \frac{1-\lambda^{M-1}}{\lambda^{M-1}} \right)$$
$$\geq x_{K+1}(\xi) + Mb - 1,$$

since

$$(1-\lambda)\lambda^M \left(\frac{1-\lambda}{\lambda} + \cdots + \frac{1-\lambda^{M-1}}{\lambda^{M-1}} \right) \leq (1-\lambda)\lambda(\lambda^{M-2} + \cdots + 1) \leq 1.$$

But $b \geq \delta$ implies that $Mb \geq M\delta > \rho + 1$, and so $|x_{K+M+1}(\xi) - x_{K+1}(\xi)| > \rho$ contradicting the assumption that $\operatorname{diam}\{T_\lambda^n(x) \colon x \in Z, n \in \mathbb{N}\} \leq \rho$. ■

As consequence, the following result is proved in [54].

Theorem 2.7 *Let T be a nonexpansive selfmapping of a convex subset Y of a normed space X. For $\lambda \in (0,1)$ define a nonexpansive mapping $T_\lambda = \lambda I + (1-\lambda)T$ of Y into itself. Then if the set $\{T_\lambda^n(x) \colon n \in \mathbb{N}\}$ is bounded for some $x \in Y$, T_λ is asymptotically regular at x. Moreover, if Y is a bounded subset of X, then T_λ is uniformly asymptotically regular on Y.*

Proof Both statements follow immediately from Lemma 2.1, the first by setting $\{x\} = Z$ in the lemma and the second by setting $Y = Z$. ■

Remark 2.1 It should be noted that by Theorem 2.7 the open question of whether every nonexpansive mapping of a weakly compact convex subset of a normed space into itself has a fixed point is equivalent to the question whether every uniformly asymptotically regular such map has one.

Using a simple variant of the proof of Lemma 2.1, Edelstein and 0'Brien [54] were able to derive a much stronger version of the Krasnoselskii result.

Theorem 2.8 *Let T be a nonexpansive selfmapping of a convex subset Y (not necessarily bounded) of a normed space X. For $\lambda \in (0,1)$ define a nonexpansive mapping $T_\lambda = \lambda I + (1-\lambda)T$ of Y into itself. Suppose that for some x in Y, $\{T_\lambda^n(x)\}$ has a cluster point $y \in Y$. Then $T_\lambda(y) = y = T(y)$ and $T_\lambda^n(x) \to y$. In particular, if the range of T is contained in a compact subset of Y, then $(T_\lambda^n(x))_n$ converges strongly to a fixed point of T for any $x \in Y$.*

Proof It was shown in [51] that y is also a cluster point of $\{T_\lambda^n(x)\}$ and that $\|T_\lambda^{n+1}(y) - T_\lambda^n(y)\| = \|T_\lambda(y) - y\|$ for all n. As In Lemma 2.1 letting $x_n = T_\lambda^n(y)$ and $y_n = T(T_\lambda^n(y))$ we obtain a set of points, which by embedding can again be assumed to be a subset of $C[0,1]$, which satisfy the following:

1. $\|x_{i+1} - x_i\| = \|y_{i+1} - y_i\| = \rho \geq 0$, for all $i = 0, 1, \cdots$,
2. $x_{i+1} = \lambda x_i + (1-\lambda)y_i$, and
3. $\|y_i - y_{i-1}\| = \left\| \frac{1}{1-\lambda}(x_{i+1} - x_i) - \frac{\lambda}{1-\lambda}(x_i - x_{i-1}) \right\| = \rho.$

Now for any fixed N, 1. implies that for some $\xi \in [0, 1]$, $|(x_{N+1} - x_N)(\xi)| = \rho$. By 1. and 3. we must have $(x_{n+1} - x_n)(\xi) = (x_{N+1} - x_N)(\xi)$ for all $n \leq N$. Without loss of generality assume $(x_{N+1} - x_N)(\xi) = \rho$. Then for all $n \leq N$

$$(x_{N+1})(\xi) = x_n(\xi) + (N + 1 - n)\rho.$$

But if $\rho > 0$, we obtain a contradiction to the existence of a cluster point for $\{x_n\}$. Hence $\rho = 0$ and $T_\lambda(y) = y = T(y)$. That $T_\lambda^n(x) \to y$ now follows easily from the non expansiveness of T_λ.

If the range of T is contained in a compact set, then $\{T_\lambda^n(x)\}$ is bounded and so by Theorem 2.7, T_λ is asymptotically regular at x. Since

$$T_\lambda^{n+1}(x) - T_\lambda^n(x) = (1 - \lambda)[T(T_\lambda^n(x)) - T_\lambda^n(x)],$$

by asymptotic regularity any cluster point of $\{T(T_\lambda^n(x))\}$ will be a cluster point of $\{T_\lambda^n(x)\}$ and the theorem follows. ■

In [54], a number of results were extended, previously known only for uniformly convex spaces, or in some cases, strictly convex spaces, to arbitrary normed linear spaces.

Theorem 2.9 *Let X be a normed space, Y a closed bounded convex subset of X and T a nonexpansive mapping of Y to Y. Suppose that either*

1. *T is demicompact at θ, or*
2. *$I - T$ maps closed bounded subsets of X into closed subsets of X, or*
3. *T is set-condensing or ball-condensing.*

Then for every $x \in Y$, $(T_\lambda^n(x))_n$ converges strongly to a fixed point of T.

Proof

1. Let $x_n = T_\lambda^n(x)$. Then

$$x_n - T(x_n) = \frac{1}{1 - \lambda}(x_n - T_\lambda(x_n)) = \frac{1}{1 - \lambda}(T_\lambda^n(x) - T_\lambda^{n+1}(x)).$$

   Hence by Theorem 2.7 and the demicompactness of T at θ, $\{T_\lambda^n(x)\}$ has a cluster point in Y. The result follows by Theorem 2.8.
2. For any x consider the set $Z = \overline{\{T_\lambda^n(x)\}}$ (the strong closure). By Theorem 2.7 $\theta \in (I - T_\lambda)(Z)$ since $(I - T_\lambda)(Z)$ is closed. Hence there is a subsequence $T_\lambda^{n_k}(x) \to y \in Y$ where y is a point such that $(I - T_\lambda)(y) = \theta$. Thus $T_\lambda^n(x) \to y$.
3. The condition that T be set-condensing or ball-condensing implies that $\{T_\lambda^n(x)\}$ has a cluster point and the result follows from Theorem 2.8.

■

The next result concerns affine, nonexpansive mappings. In [46] Dotson showed that if X is a uniformly convex Banach space and $T: X \to X$ is linear and nonexpansive, then $(T_\lambda^n(x))_n$ converges strongly to a fixed point of T for any $x \in X$. Combining Theorem 2.7 and Proposition 2.3 yields the same conclusion for any normed linear space X.

Theorem 2.10 *If $T: Y \to Y$ is an affine, nonexpansive mapping of a weakly compact convex subset Y of a normed space X into itself, then for each x in Y, $(T_\lambda^n(x))_n$ converges strongly to a fixed point of T [54].*

Proof Since T is affine and nonexpansive, it has a fixed point in Y which by translation we may assume as θ. Then T extends to a linear map of $W = \mathrm{Sp}(Y)$ into W, so that T_λ can be considered as a linear mapping of W into W, which is asymptotically regular at x by Theorem 2.7. Also, since Y is weakly compact, $\{T_\lambda^n(x)\}$ has a weak cluster point z, in $\overline{\mathrm{conv}}\{T_\lambda^n(x)\}$. The result will follow from Proposition 2.3 by showing that z is a fixed point of T_λ.

Fix $\varepsilon > 0$ and by asymptotic regularity choose N such that

$$\|T_\lambda^{n+1}(x) - T_\lambda^n(x)\| < \frac{\varepsilon}{2}$$

for all $n \geq N$. Since $z \in \overline{\mathrm{conv}}\{T_\lambda^n(x) : n \geq N\}$, there exists

$$y = \sum_{i=0}^{m} \lambda_i T_\lambda^{N+1}(x) \in \overline{\mathrm{conv}}\{T_\lambda^n(x) : n \geq N\} \text{ with } \|z - y\| < \frac{\varepsilon}{4}.$$

By the affineness of T_λ,

$$\|T_\lambda(y) - y\| \leq \sum_{i=0}^{m} \lambda_i \|T_\lambda^{N+i+1}(x) - T_\lambda^{N+i}(x)\| < \frac{\varepsilon}{2}$$

so that

$$\|T_\lambda(z) - z\| \leq \|T_\lambda(z) - T_\lambda(y)\| + \|T_\lambda(y) - y\| + \|y - z\| < \varepsilon.$$

It follows that $T_\lambda(z) = z$. ■

When Y is only assumed to be weakly compact, it is known that in general $\{T_\lambda^n(x)\}$ will not have any stronger cluster points [66]. However, Theorem 2.7 allows to conclude weak convergence for spaces which satisfy the following condition introduced by Opial [141]:

Definition 2.4 A normed space X satisfies the Opial's condition if whenever $x_n \rightharpoonup \theta$ and $x \neq \theta$ we have $\liminf \|x_n\| < \liminf \|x - x_n\|$.

For a normed space X, by Hahn-Banach's theorem, for a given x in X, there exists at least one $\varphi \in X'$ such that $\|\varphi\| = \|x\|$ and $\varphi(x) = \|x\|^2$. For each x in X define

$$J(x) = \left\{\varphi \in X', \|\varphi\| = \|x\| \text{ and } \varphi(x) = \|x\|^2\right\}.$$

The mapping $J: X \to 2^{X'}$ is called the normalized duality mapping of X.

Every Hilbert space and $l^p(1 < p < \infty)$ space satisfy Opial's condition. This condition has been used in the study of the existence of fixed points for nonexpansive maps. For example, Gossez and Lami Dozo [72] have shown that for any normed space X, the weakly sequentially continuous duality map implies that X satisfies Opial's condition which in turn implies that X has normal structure, but that none of the converse implications hold.

We have the following definition.

Definition 2.5 Let X be a normed space and $Y \subseteq X$. A mapping $T: Y \to X$ is said to be demiclosed if for any sequence $(x_n)_n$ in Y with $x_n \rightharpoonup x_0$ in Y and $T(x_n) \to y$ in Y, then $T(x_0) = y$.

It follows [32] that if T is asymptotically regular and $I - T$ is demiclosed, then any weak cluster point of $\{T^n(x)\}$ is a fixed point. It is also known [141] that if a space satisfies Opial's condition , then $I - T$ is demiclosed for any nonexpansive map T from a closed and bounded convex set into itself. In [54] the following was proved.

Theorem 2.11 *Let X be a normed space which satisfies Opial's condition and let T be a nonexpansive mapping of a weakly compact convex subset Y of X into itself. Then for any $x \in Y$, $(T_\lambda^n(x))_n$ converges weakly to a fixed point of T.*

Proof By the above quoted results any weak cluster point of $\{T_\lambda^n(x)\}$ is a fixed point. If there exist two distinct weak cluster points of $\{T_\lambda^n(x)\}$, say y_1 and y_2 and two subsequences $(T_\lambda^{n_k}(x))_k$ converging weakly to y_1 and $(T_\lambda^{n_l}(x))_l$ converging weakly to y_2, then since $\|T_\lambda^n(x) - y_i\|$ is non-increasing, Opial's condition implies that

$$\lim \|T_\lambda^n(x) - y_1\| = \lim \|T_\lambda^{n_k}(x) - y_1\| < \lim \|T_\lambda^{n_k}(x) - y_2\| = \lim \|T_\lambda^n(x) - y_2\|,$$

and similarly,

$$\lim \|T_\lambda^n(x) - y_2\| = \lim \|T_\lambda^{n_l}(x) - y_2\| < \lim \|T_\lambda^{n_l}(x) - y_1\| = \lim \|T_\lambda^n(x) - y_1\|.$$

The contradiction shows that exactly one weak cluster point exists and by weak compactness $T_\lambda^n(x) \rightharpoonup y$. ∎

In [65], a new class of mappings was introduced.

Definition 2.6 Let Y be a nonempty closed convex subset of Banach space X. For a continuous strictly increasing function $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = 0$ we say $T: Y \to Y$ is an α-most convex mapping if for all $x, y \in Y$ and all $\lambda \in [0, 1]$ we have

$$J_T(\lambda x + (1-\lambda)y) \le \alpha(\max\{J_T(x), J_T(y)\}),$$

where J_T is defined by

$$J_T(x) := \|T(x) - x\|, \quad \text{for all } x \in Y.$$

In the case when $\alpha(t) = rt$, for some $r > 0$, we say T is r-almost convex, and simply refer to T as almost convex where $r = 1$. That is, T is almost convex whenever

$$J_T(\lambda x + (1-\lambda)y) \le \max\{J_T(x), J_T(y)\},$$

for all $x, y \in Y$ and all $\lambda \in [0, 1]$.

Affine maps are clearly almost convex, indeed they satisfy the seemingly stronger inequality,

$$J_T(\lambda x + (1-\lambda)y) \le \lambda J_T(x) + (1-\lambda)J_T(y).$$

On the other hand, in [103], was proved that any α-most convex map is of "convex type", that is, if $J_T(x_n) \to 0$ and $J_T(y_n) \to 0$ then $J_T\left(\frac{1}{2}(x_n + y_n)\right) \to 0$, so the midpoint of two approximate fixed point sequences is itself an approximate fixed point for T.

Remark 2.2 α-most (or, quasi) convex functions have been considered in optimization theory [36, 39], where α is referred to as a "forcing function" and is often also required to be convex.

Beyond the affine mappings already mentioned, instances of α-most convex maps include the following [65].

Examples 2.1

1. $T: [0, 1] \to [0, 1]: x \mapsto x(1-x)$ is not affine, but $J_T(x) = |x - T(x)| = x^2$ is a convex function, and so T is almost convex.
2. $T: B_{c_0} \to B_{c_0}$ defined by

$$T((x_n)_n) := (x_1 - sgn(x_1)\|(x_n)_n\|_\infty, x_2, x_3, \cdots)$$

is almost convex, as $J_T(x) = \|x\|_\infty$ is a convex function.

3. Let $(\varphi_n : \mathbb{R} \to \mathbb{R})$ be a family of functions which are equicontinuous at 0 and satisfy

$$\varphi_n(0) \to 0, \quad \varphi_n(x) \leq x, \text{ and } \varphi_n'' \leq 0,$$

then $T : (x_n)_n \mapsto (\varphi_n(x_n))_n$ is an almost convex mapping from c_0 into c_0.

4. A self mapping T of a metric space (Y, d) is a contraction in the sense of Bianchini [168] whenever there exists a number h, $0 < h < 1$, such that, for each $x, y \in X$,

$$d(T(x), T(y)) \leq h \max d(T(x), x), d(T(y), y).$$

If Y is a convex subset of a Banach space X, then this type of mapping is α-almost convex .

Indeed,

$$\begin{aligned} J_T(\lambda x + (1-\lambda)y) &\leq \lambda J_T(x) + (1-\lambda)J_T(y) \\ &+ \lambda h \max\{J_T(x), J_T(\lambda x + (1-\lambda)y)\} \\ &+ (1-\lambda)h \max\{J_T(x), J_T(\lambda x + (1-\lambda)y)\} \\ &\leq 2(\lambda J_T(x) + (1-\lambda)J_T(y)) + hJ_T(\lambda x + (1-\lambda)y)). \end{aligned}$$

Therefore

$$J_T(\lambda x + (1-\lambda)y) \leq \frac{2}{1-h} \max\{J_T(x), J_T(y)\}.$$

5. Let Y be a convex nonempty subset of a Banach space X. Every k-Lipchitzian mapping $T : Y \to Y$ which satisfies

$$\|x - y\| \leq \gamma(\max\{J_T(x), J_T(y)\})$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous strictly increasing function with $\gamma(0) = 0$ for all $x, y \in Y$ is α-almost convex.

Indeed

$$\begin{aligned} J_T(\lambda x + (1-\lambda)y) &\leq \lambda J_T(x) + (1-\lambda)J_T(y) \\ &+ \lambda\|T(x) - T(\lambda x + (1-\lambda)y)\| \\ &+ (1-\lambda)\|T(y) - T(\lambda x + (1-\lambda)y)\| \\ &\leq \beta(\max\{J_T(x), J_T(y)\}), \end{aligned}$$

where $\beta(t) = t + \dfrac{k}{2}\gamma(t)$.

6. Every strict contraction $T: Y \to X$ where Y is a convex nonempty subset of a Banach space X satisfies the above condition, and therefore it is an α-almost convex mapping.

   Indeed, we can take $\gamma(t) = \dfrac{2}{1-k}t$ and hence $\beta(t) = \dfrac{1}{1-k}t$ where $0 < k < 1$ is the contraction constant of T.
7. Similar, though more tedious, calculations to those of the last three examples establish that if $T: Y \to X$ is a generalized nonexpansive map, that is,

$$\begin{aligned}\|T(x) - T(y)\| &\le a\|x - y\| + b(\|x - T(x)\| + \|y - T(y)\|)\\ &+ c(\|x - T(y)\| + \|y - T(x)\|)\end{aligned}$$

   where a, b and c are positive constants with $a + 2b + 2c \le 1$, and if either this last inequality is strict, or $b \neq 0$, then T is r-almost convex. Indeed,

$$\begin{aligned}J_T(\lambda x + (1-\lambda)y) &\le \frac{(1+b+c)(1-c)}{(1-b-c)(1-a-2c)} \max\{J_T(x), J_T(y)\}\\ &\le \frac{3}{2b} \max\{J_T(x), J_T(y)\}.\end{aligned}$$

8. A mapping T of a closed convex subset of a Banach space X is said to be of type Γ [33] if there exists a continuous strictly increasing convex function $\gamma: \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$ for which

$$\gamma(\|\lambda T(x) + (1-\lambda)T(y) - T(\lambda x + (1-\lambda)y)\|) \le |\|x - y\| - \|T(x) - T(y)\||.$$

   Such maps are α-almost convex, where $\alpha(t) = t + \gamma^{-1}(2t)$.

   To see this, note that γ^{-1} is strictly increasing and that

$$\begin{aligned}J_T(\lambda x + (1-\lambda)y) &= \|\lambda x + (1-\lambda)y - T(\lambda x + (1-\lambda)y)\|\\ &\le \|\lambda x + (1-\lambda)y - \lambda Tx + (1-\lambda)Ty)\|\\ &+ \gamma^{-1}(|\|x - y\| - \|T(x) - T(y)\||)\\ &\le \lambda J_T(x) + (1-\lambda)J_T(y)\\ &+ \gamma^{-1}(\|x - T(x)\| + \|y - T(y)\|)\\ &\le \alpha(\max\{J_T(x), J_T(y)\}).\end{aligned}$$

   As a consequence of this last example and [33] we have:
9. All nonexpansive self maps of closed bounded convex subsets in a uniformly convex space are α-almost convex.

Remark 2.3 It is worth noting that the class of maps which are α-almost convex on a given domain Y is stable under equivalent renormings. Indeed, if $m\|x\| \leq \|x\|' \leq M\|x\|$ and T is α-almost convex with respect to the norm $\|.\|$ then it is α'-almost convex with respect to $\|.\|'$, where $\alpha'(t) = M\alpha(\frac{t}{m})$.

Many other examples of α-almost convex mappings are a consequence of the following result [65].

Proposition 2.4 *If Y is a closed bounded convex set of a Banach space X, and $T: Y \to Y$, then at least one of the following applies.*

1. *T is r-almost convex, for some $r > 0$, or*
2. $\inf\{J_T(x): x \in Y\} = 0$. *That is, T admits approximate fixed points in Y.*

Proof If T is not r-almost convex for any $r > 0$, then for each $n \in \mathbb{N}$ taking $r = n$, we see that there must exist points x_n and y_n in Y and $\lambda_n \in [0, 1]$ such that

$$\infty > \text{diam}Y \geq J_T(\lambda_n x_n + (1-\lambda_n)y_n) \geq n \max\{J_T(x_n), J_T(y_n)\},$$

so $J_T(x_n)$ and $J_T(y_n)$ tend to 0 as $n \to \infty$. ■

Using the previous proposition with non-zero minimal displacement given in [67], we see that there r-almost convex self maps of weak compact convex sets (including B_{l_2}) with $\inf J_T(x) > 0$. In particular such maps are fixed point free, and can not be weakly continuous. Indeed, examples 2., 3. above show that unlike affine maps, almost convex maps need not be weakly continuous. To see this note that in c_0 the standard basis vectors $e_n \rightharpoonup \theta$, but

$$\begin{aligned} T(e_n) &= (-1, 0, \cdots, 1, 0, \cdots), \quad \text{where the 1 occurs in the n'th position} \\ &\rightharpoonup (-1, 0, 0, \cdots) \neq T(\theta) = \theta. \end{aligned}$$

Nonetheless, we have the following [65].

Proposition 2.5 *Let X be a Banach space and let Y be a nonempty closed convex subset of X. If $T: Y \to X$ is norm continuous and almost convex then $J_T(x) := \|T(x) - x\|$ is weak lower semicontinuous.*

Proof Suppose that $(x_n)_n$ is a sequence in Y such that $x_n \rightharpoonup x$. Given $\varepsilon > 0$, choose a subsequence $(x_{n_k})_k$ such that $J_T(x_{n_k}) < \liminf J_T(x_n) + \frac{\varepsilon}{2}$, for all k, and let $\delta > 0$ be such that $|J_T(y) - J_T(x)| < \frac{\varepsilon}{2}$ whenever $\|y - x\| < \delta$ (possible, as T and hence J_T is norm continuous at x). Since $x_{n_k} \rightharpoonup x$, by Mazur's theorem, there exists $x_{n_{k_1}}, x_{n_{k_2}}, \cdots, x_{n_{k_m}}$ and $\lambda_1, \lambda_2, \cdots, \lambda_m \in (0, 1]$ with $\sum \lambda_i = 1$ such that

$\|x - \sum \lambda_i x_{n_{k_i}}\| < \delta$. But, then

$$\begin{aligned}
J_T(x) &< J_T\left(\sum_{i=1}^{m} \lambda_i x_{n_{k_i}}\right) + \frac{\varepsilon}{2} \\
&= J_T\left(\lambda_i x_{n_{k_1}} + (1-\lambda_1)\sum_{i=2}^{m} \frac{\lambda_i}{(1-\lambda_1)} x_{n_{k_i}}\right) + \frac{\varepsilon}{2} \\
&\le \max\left\{J_T(x_{n_{k_1}}), J_T\left(\sum_{i=2}^{m} \frac{\lambda_i}{(1-\lambda_1)} x_{n_{k_i}}\right)\right\} + \frac{\varepsilon}{2} \\
&\le \cdots \\
&\le \max\{J_T(x_{n_{k_1}}), \cdots, J_T(x_{n_{k_m}})\} + \frac{\varepsilon}{2} \\
&< \liminf_n J_T(x_n) + \varepsilon,
\end{aligned}$$

and so we conclude that J_T is weak lower semicontinuous. ■

Corollary 2.1 *For X, Y and T as above, if in addition Y is weak compact, then*

$$M(T) := \{x \in Y : J_T(x) = \inf_{y \in Y} J_T(y)\}$$

is a nonempty weak compact convex subset of Y. Indeed the same is true of any of the sub-level sets for J_T.

In particular, such a T has a fixed point if and only if

$$\inf_{x \in Y} J_T(x) = 0,$$

that is, if and only if T admits an approximate fixed point sequence in Y. And, in this case $\mathrm{Fix}(T) := M(T)$ *is a nonempty weak compact convex set.*

We do not know if $J_T(x)$ is weak lower semicontinuous for arbitrary α-almost convex maps, however, we have the following demiclosedness result [65].

Proposition 2.6 *Let X be a Banach space and let Y be a nonempty closed convex subset of X. If $T : Y \to X$ is norm continuous and α-almost convex then $I - T$ is demiclosed at θ.*

Proof Suppose $x_n \rightharpoonup x_0$ and $|J_T(x_n)| = \|(I - T)(x_n)\| \to 0$. We may assume without loss of generality that

$$J_T(x_n) > 0$$

for all positive integers n.

Fix $\varepsilon > 0$. Since T is continuous, there exists $\delta > 0$ such that

$$J_T(x_0) < J_T(y) + \frac{\varepsilon}{2},$$

whenever $y \in Y$ and $\|y - x_0\| < \delta$.

On the other hand, since α is continuous at 0 and $\alpha(0) = 0$, there exists a positive integer n_1 such that

$$0 < \alpha(J_T(x_{n_1})) < \frac{\varepsilon}{2}.$$

As $J_T(x_n) \to 0$ and $\alpha(J_T(x_n)) \to 0$, there exists $n_2 > n_1$ such that

$$0 < J_T(x_{n_2}) < \min\{J_T(x_{n_1}), \alpha(J_T(x_{n_1}))\}$$

and

$$0 < \alpha(J_T(x_{n_2})) < \min\{J_T(x_{n_1}), \alpha(J_T(x_{n_1}))\}.$$

Thus, by induction we can get a subsequence $(x_{n_k})_k$ of $(x_n)_n$ satisfying

$$0 < J_T(x_{n_{k+1}}) < \min\{J_T(x_{n_k}), \alpha(J_T(x_{n_k}))\}$$

and

$$0 < \alpha(J_T(x_{n_{k+1}})) < \min\{J_T(x_{n_k}), \alpha(J_T(x_{n_k}))\},$$

for all positive integer k.

We assert that if $m \geq 2$ and $\sum_{k=1}^{m} \lambda_k x_{n_k}$ is a convex combination of $x_{n_1}, x_{n_2}, \cdots, x_{n_m}$ then

$$J_T\left(\sum_{k=1}^{m} \lambda_k x_{n_k}\right) \leq \alpha(J_T(x_{n_1})).$$

Indeed, for $m = 2$ we have

$$J_T(\lambda_1 x_{n_1} + \lambda_2 x_{n_2}) \leq \alpha(\max\{J_T(x_{n_1}),\, J_T(x_{n_2})) = \alpha(J_T(x_{n_1})).$$

If we suppose that the assertion is true for $k = m - 1$, then

$$J_T\left(\sum_{k=1}^{m}\lambda_k x_{n_k}\right) = J_T\left(\lambda_1 x_{n_1} + (1-\lambda_1)\sum_{k=2}^{m}\frac{\lambda_k}{(1-\lambda_1)}x_{n_k}\right) \tag{2.8}$$

$$\leq \alpha\left(\max\left\{J_T(x_{n_1}),\, J_T\left(\sum_{k=2}^{m}\frac{\lambda_k}{(1-\lambda_1)}x_{n_k}\right)\right\}\right) \tag{2.9}$$

$$\leq \alpha(\max\{J_T(x_{n_1}), \alpha(J_T(x_{n_2}))\}) = \alpha(J_T(x_{n_1})). \tag{2.10}$$

To complete the proof we need only observe that by Mazur's theorem, there exists a convex combination $\sum_{k=1}^{m}\lambda_k x_{n_k}$ such that

$$\|\sum_{k=1}^{m}\lambda_k x_{n_k} - x_0\| < \delta,$$

and then

$$J_T(x_0) < J_T\left(\sum_{k=1}^{m}\lambda_k x_{n_k}\right) + \frac{\varepsilon}{2} \leq \alpha(J_T(x_{n_1})) + \frac{\varepsilon}{2} < \varepsilon,$$

which concludes the proof. ■

As an immediate consequence we have the following fixed point result for α-almost convex maps [65].

Proposition 2.7 *Let X be a Banach space, let Y be a nonempty weak compact convex subset of X, and let $T: Y \to X$ be norm continuous and α-almost convex. Then T has a fixed point in Y if and only if* $\inf\{J_T(x) \colon x \in Y\} = 0$.

Proof ($\Longrightarrow$) is obvious.

($\Longleftarrow$) Since $\inf\{J_T(x) \colon x \in Y\} = 0$, we can find an approximate fixed point sequence $(x_n)_n$ in Y which without loss of generality we can assume is weakly convergent to $x_0 \in Y$. The above proposition now applies to yield the result. ■

As an immediate consequence of Proposition 2.6 we have the following [65].

Proposition 2.8 *Let Y be a nonempty weak compact convex subset of the Banach space X, and let $T: Y \to X$ be norm continuous, α-almost convex, and asymptotically regular at $x_0 \in Y$, that is $J_T(T^n(x_0)) \longrightarrow 0$ (for example, if $T = \frac{1}{2}(I + V)$, where V is α-almost convex and nonexpansive). Then the iterates $T^n(x_0)$ weakly converge to a fixed point of T if either*

(i) *T is a contraction, that is, $\|T(x) - T(y)\| < \|x - y\|$ whenever $x \neq y$, or*
(ii) *X satisfies the Opial's condition.*

Proof

(i) Suppose this were not the case, then we can find subsequences $T^{n_k}(x_0) \rightharpoonup y_0$ and $T^{m_k}(x_0) \rightharpoonup z_0 \neq y_0$. By the demiclosedness both y_0 and z_0 are fixed points of T, a contradiction, since contractions can have at most one fixed point.

(ii) This follows from standard arguments similar to those used in the nonexpansive case [64].

■

The following characterization of reflexivity follows from the theorem of Mil'man and Mil'man [132] and the above considerations [65].

Proposition 2.9 *The Banach space X is reflexive if and only if whenever Y is a nonempty closed bounded convex subset of X and $T: Y \to Y$ is norm continuous, α-almost convex with* $\inf\{J_T(x): x \in Y\} = 0$ *it follows that T has a fixed point.*

2.3 Approximate Fixed Points of Nonexpansive Mappings in Unbounded Sets

It is less obvious that some unbounded convex sets have the approximate fixed point property for nonexpansive mappings.

Definition 2.7 A set Y of a Banach space X is called linearly bounded if it has a bounded intersection with all lines in X (Y does not contain any half-line).

In [165], Reich characterized closed convex subsets of reflexive Banach spaces which possess the approximate fixed point property for nonexpansive mappings.

Theorem 2.12 *A closed convex subset of a reflexive Banach space has the* **AFPP** *if it is linearly bounded.*

Proof Let Y be a closed convex subset of a (real) reflexive Banach space X, and let X' be the dual of X. To show necessity, assume that $\{y + tz: 0 \leq t < \infty\} \subseteq Y$ for some $z \neq \theta$. If x is in Y, then $(1 - \frac{1}{t})x + \frac{(y + tz)}{t}$ belongs to Y for all $t \geq 1$. Therefore we can define a mapping $S: Y \to Y$ by $S(x) = x + z$. This mapping is nonexpansive and $\|x - S(x)\| = \|z\|$ for all $x \in Y$.

Conversely, let $T: Y \to Y$ be any nonexpansive mapping, and denote $\inf\{\|(T(x) - x)\|:$
$x \in Y\}$ by d. It is known [116, 164] that for each $x \in Y$ there is a functional $j \in X'$ with $\|j\| = d$ such that $\left(\frac{x - T^n(x)}{n}, j\right) \geq d^2$ for all $n \geq 1$. It is also known [163] that $\lim_{n\to\infty} \frac{\|T^n(x)\|}{n} = d$. Let a subsequence $\left(\frac{T^n(x)}{n}\right)_{n\geq 1}$ converge weakly to w. Clearly $\|w\| \leq d$. On the other hand, $\|w\|d = \|w\|\|j\| \geq (-w, j) \geq d^2$, so that

$\|w\| = d$. Now let y be any point in Y. Since $(1 - \frac{1}{n})y + \frac{T^n(x)}{n}$ belongs to Y for each $n \geq 1$, we see that $y + w$ also belongs to Y. Consequently, we may conclude that the points $y + mw$ belong to Y for all $m \geq 1$. If Y is linearly bounded, then this fact implies that $w = \theta$, so that $d = 0$ too. This completes the proof. ■

Remark 2.4 Theorem 2.12 cannot be extended to all Banach spaces. To see this, let $X = l^1$, $Y = \{x = (x_1, x_2, \cdots) \in l_1 \colon \|x_n\| \leq 1 \text{ for all } n\}$, and define $T \colon Y \to Y$ by $T(x_1, x_2, \cdots) = (1, x_1, x_2, \cdots)$. Then Y is linearly bounded and T is an isometry, but $\inf\{\|(T(x) - x)\| \colon x \in Y\} = 1$.

Remark 2.5 If X is finite-dimensional and Y is linearly bounded, then Y is, in fact, bounded. Hence in this case either Y is bounded and has the fixed property, or it is unbounded and does not even have the ***AFPP***.

In [179] Shafrir presented a more general geometric characterization of the **AFPP** that is valid in an arbitrary Banach space. This result is true even for a more general class of metric spaces with a convexity structure, namely hyperbolic spaces introduced by Kirk [108].

Let (X, ρ) be a complete metric space. We say that a mapping $c \colon \mathbb{R} \to X$ is a metric embedding of $\mathbb{R}$ into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of $\mathbb{R}$ under a metric embedding is called a metric line. The image of a real interval $[a, b]$ under such a mapping is called a metric segment.

Assume that there is a family M of metric lines in X such that for every $x, y \in X, x \neq y$, there is a unique metric line in M that passes through x and y. The closed metric segment connecting x and y will be denoted by $[x, y]$. For every $0 \leq t \leq 1$ we shall denote by $(1 - t)x \oplus ty$ the unique point $z \in [x, y]$ satisfying $\rho(x, z) = t\rho(x, y)$ and $\rho(z, y) = (1 - t)\rho(x, y)$.

Definition 2.8 We shall say that X, or more precisely (X, ρ, M), is a hyperbolic space if

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho(y, z)$$

for all x, y and z in X.

An equivalent requirement is that

$$\rho\left((1 - t)x \oplus tz, (1 - t)y \oplus tw\right) \leq (1 - t)\rho(x, y) + t\rho(z, w)$$

for all x, y, z and w in X and all $0 \leq t \leq 1$.

Hyperbolic spaces were studied in [166]. The following are some examples of these spaces.

Examples 2.2

1. All Banach spaces are also hyperbolic spaces. So is the Hilbert ball B equipped with the hyperbolic metric [68] and B^n, the Cartesian product of n Hilbert balls, equipped with the hyperbolic metric [121].
2. The open unit ball of $L(H)$, the space of all bounded linear self-mappings of a complex Hilbert space H, with the hyperbolic metric.
3. Hadamard manifolds and simply connected Riemannian manifolds of nonpositive curvature are finite dimensional examples of hyperbolic spaces [10].
4. New examples of hyperbolic spaces can be constructed from old ones by a product procedure which is described in [166].

Definition 2.9 Let (X, ρ, M) be a hyperbolic space. A subset $Y \subseteq X$ is said to be convex if $[x, y] \subseteq Y$ whenever $x, y \in Y$. A mapping $T : Y \to Y$ is said to be nonexpansive if $\rho(T(x), T(y)) \leq \rho(x, y)$ for all $x, y \in Y$. We shall say that has the ***AFPP*** if $\inf\{\rho(x, T(x)),\ x \in Y\} = 0$ for every nonexpansive mapping $T : Y \to Y$.

In [179], Shafrir introduced the concepts of the directional curve, directional sequence, and directionally bounded convex subsets of hyperbolic spaces.

Definition 2.10 Let (X, ρ, M) be a hyperbolic space. A curve $\gamma : [0, \infty) \to X$ is said to be directional (with constant b) if there is $b \geq 0$ such that

$$t - s - b \leq \rho(\gamma(s), \gamma(t)) \leq t - s$$

for all $t \geq s \geq 0$. A sequence $(x_n)_{n\geq 1} \subseteq X$ is said to be directional if:

(1) $\rho(x_1, x_n) \to \infty$ as $n \to \infty$,
(2) there is $b \geq 0$ such that

$$\rho(x_{n_1}, x_{n_l}) \geq \sum_{i=1}^{l-1} \rho(x_{n_i}, x_{n_{i+1}}) - b$$

for all $x_{n_1} < x_{n_2} < \cdots < x_{n_l}$.

Definition 2.11 A convex subset Y of a hyperbolic space (X, ρ, M) is called directionally bounded if it contains no directional curves.

Lemma 2.2 *A convex subset of a hyperbolic space (X, ρ, M) is directionally bounded if and only if it contains no directional sequences (does not contain any approximate metric half-line) [179].*

Proof Suppose Y contains a directional curve $\gamma(t)$ with a constant b. Choose any positive sequence $(t_n)_{n\geq 1}$ such $t_n \uparrow \infty$ and define

$$x_i = \gamma(t),\ \ i \geq 1.$$

For $x_{n_1} < x_{n_2} < \cdots < x_{n_l}$ we have:

$$\begin{aligned}\rho(x_{n_l}, x_{n_1}) &= \rho(\gamma(t_{n_l}), \gamma(t_{n_1})) \geq t_{n_l} - t_{n_1} - b \\ &= \sum_{i=1}^{l-1}(t_{n_{i+1}} - t_{n_i}) - b \geq \sum_{i=1}^{l-1} \rho(x_{n_i}, x_{n_{i+1}}) - b.\end{aligned}$$

Conversely, if Y contains a directional sequence $(t_n)_{n\geq 1}$ with constant b we define

$$t_1 = 0, \quad t_n = \sum_{i=1}^{n-1} \rho(x_i, x_{i+1})$$

for $n \geq 2$, and $\gamma(t_n) = x_n$ for $n \geq 1$. We extend γ to all of $\mathbb{R}^+$ by $\gamma(t) = (1 - a_t)x_n \oplus a_t x_{n+1}$ where $t_n \leq t < t_{n+1}$ and $a_t = \dfrac{t - t_n}{\rho(x_n, x_{n+1})}$. If $t_{n+1} > t \geq t_n \geq t_{m+1} > s \geq t_m$, then

$$\begin{aligned}\rho(\gamma(t), \gamma(s)) &\geq \rho(\gamma(t_{n+1}), \gamma(s)) - \rho(\gamma(t_{n+1}), \gamma(t)) \\ &\geq \rho(\gamma(t_{n+1}), \gamma(t_m)) - \rho(\gamma(t_s), \gamma(t_m)) - \rho(\gamma(t_{n+1}), \gamma(t)) \\ &= \rho(x_m, x_{n+1}) - (s - t_m) - (t_{n+1} - t) \\ &\geq \sum_{i=m}^{n} \rho(x_i, x_{i+1}) - b - (s - t_m) - (t_{n+1} - t) \\ &= t - s - b\end{aligned}$$

■

Let Y be a closed convex subset of a hyperbolic space (X, ρ, M) and $T : Y \to Y$ a nonexpansive mapping. For any $x \in Y$ and $t > 0$ consider the mapping $S : Y \to Y$ defined by

$$S(y) = \frac{1}{t+1} x \oplus \frac{t}{t+1} T(y).$$

T is a strict contraction, hence by Banach's Contraction Mapping Principle, it has a unique fixed point in Y which we shall denote by $J_t(x)$. The mappings $\{J_t\}_{t>0}$ thus defined are easily seen to be nonexpansive and are called the resolvents of T, just as in Banach spaces.

The resolvent identity

$$J_t(x) = J_t\left(\frac{s}{t} x \oplus \left(1 - \frac{s}{t}\right) J_t(x)\right)$$

for any $x \in Y$ and $0 < s \leq t$ can be easily verified.

For u and v in X and $s > 0$ we shall denote by $(1+s)u \ominus sv$ the unique point w on the metric line connecting u and v that satisfies $\rho(w,u) = s\rho(u,v)$ and $\rho(w,v) = (1+s)\rho(u,v)$.

The following is given in [179].

Theorem 2.13 *A convex subset Y of a hyperbolic space (X, ρ, M) has the* **AFPP** *if it is directionally bounded.*

The following lemmas simplify the proof of the above theorem.

Lemma 2.3 ([166]) $\forall x \in Y$,

$$\lim_{t\to\infty} \frac{\rho(x, J_t(x))}{t} = \inf_{y\in Y} \rho(y, T(y)).$$

Proof By the resolvent identity we have for $t \geq s > 0$,

$$\rho(x, J_s(x)) \geq \rho(x, J_t(x)) - \rho(J_t(x), J_s(x)) \geq \frac{s}{t}\rho(x, J_t(x)),$$

hence $\left\{\frac{\rho(x, J_t(x))}{t} | t > 0\right\}$ is nonincreasing and

$$\lim_{t\to\infty} \frac{\rho(x, J_t(x))}{t} = L$$

exists. Since $\frac{\rho(x, J_t(x))}{t} = \rho(J_t(x), TJ_t(x))$, it is clear that $L \geq d = \inf\{\rho(y, T(y)) \mid y \in Y\}$.

In order to prove the reverse inequality we fix $y \in Y$ and $s > 0$. For $t \geq s$ we have

$$\frac{s}{t}x \oplus \left(1 - \frac{s}{t}\right) J_t(x) = (1+s)J_t(x) \ominus sTJ_t(x)$$

hence

$$\begin{aligned}
\rho(y, J_t(x)) &\leq \rho\left(1+s)y \ominus sT(y), \frac{s}{t}x \oplus \left(1-\frac{s}{t}\right) J_t(x)\right) \\
&\leq \frac{s}{t}\rho((1+s)y \ominus sT(y), x) + \left(1-\frac{s}{t}\right)\rho((1+s)y \ominus sT(y), J_t(x)) \\
&\leq \frac{s}{t}\rho((1+s)y \ominus sT(y), y) + \frac{s}{t}\rho(y, x) \\
&\quad + \left(1-\frac{s}{t}\right)\rho((1+s)y \ominus sT(y), y) + \left(1-\frac{s}{t}\right)\rho(y, J_t(x)) \\
&= s\rho(y, T(y)) + \frac{s}{t}\rho(y, x) + \left(1-\frac{s}{t}\right)\rho(y, J_t(x)).
\end{aligned}$$

So we conclude that $\frac{\rho(y, J_t(x))}{t} \leq \rho(y, T(y)) + \frac{\rho(y, x)}{t}$. Letting $t \to \infty$ we get $L \leq \rho(y, T(y))$. Since y was arbitrary, $L \leq d$ and the result follows.

For $x \in Y$ and a positive sequence $(t_i)_{i\geq 1}$ we construct a sequence $(y_i)_{i\geq 1} \subseteq Y$ as follows:

$$y_1 = J_t(x), \ \ y_{i+1} = \frac{s_i}{s_i + 1} y_i \oplus \frac{1}{s_{i+1} t_{i+1}} J_{t_{i+1}}(x), \quad i \geq 1, \tag{2.11}$$

where

$$s_j = \sum_{k=1}^{j} \frac{1}{t_k}, \quad j \geq 1.$$

■

Note that in normed spaces

$$y_j = \frac{\left(\sum_{i=1}^{j} \frac{J_{t_i}(x)}{t_i}\right)}{s_j}.$$

Lemma 2.4 *Let $(y_i)_{i\geq 1}$ be defined by (2.11). Then, for $m \geq 1$ and $t \geq \max\{t_i \mid 1 \leq i \leq m\}$,*

$$\rho(y_m, J_t(x)) \leq \left(1 - \frac{m}{s_m t}\right) \rho(x, J_t(x)).$$

Proof We use induction on m. The case $m = 1$ is clear from the resolvent identity. Suppose the result is true for m. Then

$$\begin{aligned}
\rho(y_{m+1}, J_t(x)) &\leq \frac{s_m}{s_{m+1}} \rho(y_m, J_t(x)) + \frac{1}{s_{m+1} t_{m+1}} \rho(J_{t_{m+1}}(x), J_t(x)) \\
&\leq \frac{s_m}{s_{m+1}} \rho(x, J_t(x)) - \frac{m}{s_{m+1} t} \rho(x, J_t(x)) + \frac{1 - \frac{t_{m+1}}{t}}{s_{m+1} t_{m+1}} \rho(x, J_t(x)) \\
&= \rho(x, J_t(x)) - \frac{m+1}{s_{m+1} t} \rho(x, J_t(x)).
\end{aligned}$$

■

Lemma 2.5 *Let $(y_i)_{i\geq 1}$ be defined by (2.11). Then for every $m \geq 1$,*

$$\rho(y_m, x) \geq \frac{md}{s_m}, \quad \textit{where } d = \inf_{y \in Y} \rho(y, T(y)).$$

Proof Fix any $t \geq \max\{t_i \mid 1 \leq i \leq m\}$. Then by Lemma 2.4,

$$\rho(y_m, x) \geq \rho(x, J_t(x)) - \rho(y_m, J_t(x)) \geq \frac{m}{s_m} \frac{\rho(x, J_t(x))}{t} \geq \frac{md}{s_m}.$$

■

Proof (Theorem 2.13) First we prove the necessity part. Suppose Y contains a directional curve $\gamma(t)$ with a constant b. We define $T: Y \to Y$ by $T(x) = \gamma(A_x + 1 + b)$ where $A_x \equiv \rho(\gamma(0), x)$. It is easy to see that T is nonexpansive. In addition, for each $x \in Y$ we have

$$\begin{aligned} \rho(T(x), x) &= \rho(\gamma(A_x + 1 + b), x) \\ &\geq \rho(\gamma(A_x + 1 + b), \gamma(0)) - A_x \\ &\geq A_x + 1 + b - b - A_x = 1 \end{aligned}$$

hence Y does not have the **AFPP**.

Now we prove the sufficiency part. If (a closed) Y does not have the **AFPP**, then there is a nonexpansive mapping $T: Y \to Y$ such that $\inf_{y \in Y} \rho(y, T(y)) = d > 0$. We shall show that Y is not directionally bounded. Fix any $x \in Y$. We shall construct a sequence $(y_i)_{i \geq 1}$ defined by (2.11) with an appropriate choice of $(t_i)_{i \geq 1}$. We choose t_1 such that

$$\frac{\rho(x, J_{t_1}(x))}{t_1} \leq d + \frac{1}{2},$$

so $y_1 = J_{t_1}(x)$. Having chosen $t_1, t_2, \cdots, t_m$, and therefore $y_1, y_2, \cdots, y_m$, we next choose t_{m+1} such that

(i) $t_{m+1} \geq 2t_m$
(ii) $\dfrac{\rho(J_{t_{m+1}}(x), y_m)}{t_{m+1}} \leq d + \dfrac{1}{2^{m+1}}$,

and define y_{m+1} as in (2.11). The existence of t_{m+1} is guaranteed by Lemma 2.3. We claim that $(y_m)_{m \geq 1}$ thus defined is a directional sequence.

It is enough to show that

$$\left(\sum_{i=1}^{n-1} \rho(y_i, y_{i+1}) - \rho(y_1, y_n) \right)_{n \geq 2}$$

is bounded.

By our construction

$$\rho(y_i, y_{i+1}) = \frac{\rho(J_{t_{i+1}}(x), y_i)}{s_{i+1}t_{i+1}} \le \frac{(d + \frac{1}{2^{i+1}})}{s_{i+1}}$$

for every i, so by Lemma 2.5 we have, for $n \ge 2$,

$$\begin{aligned}
\sum_{i=1}^{n-1} \rho(y_i, y_{i+1}) - \rho(y_1, y_n) &\le \sum_{i=1}^{n-1} \frac{(d + \frac{1}{2^{i+1}})}{s_{i+1}} - \frac{dn}{s_n} + t_1(d + \frac{1}{2}) \\
&\le \sum_{i=1}^{n-1} \left[\frac{(d + \frac{1}{2^i})}{s_i} - \frac{d}{s_n} \right] \le t_1 + d \sum_{i=1}^{n-1} \left(\frac{1}{s_i} - \frac{1}{s_n} \right) \\
&\le t_1 + t_1^2 d \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{t_j} \\
&\le t_1 + t_1^2 d \sum_{i=1}^{n-1} \frac{2}{t_{i+1}} \le t_1(2d + 1).
\end{aligned}$$

Hence $(y_i)_{i \ge 1}$ is indeed directional, and by Lemma 2.2 this completes the proof. ■

In an infinite-dimensional real Banach space, there is a useful criterion that enables us to check whether a convex subset is directionally bounded and hence has the **AFPP** by Theorem 2.13. We denote by S_X and $S_{X'}$ the unit spheres of X and X' respectively.

The following is given in [179].

Lemma 2.6 *Let $\gamma(t)$ be a directional curve with constant b in a Banach space X. Then there is a functional $\varphi \in S_{X'}$ such that*

$$t - s - b \le \varphi(\gamma(t) - \gamma(s)) \le t - s \quad \textit{for all } 0 \le s \le t.$$

Proof For $r > b$ consider $\varphi_r \in J\left(\frac{(\gamma(r) - \gamma(0))}{\|\gamma(r) - \gamma(0)\|}\right)$ (J is the normalized duality mapping of X) and let φ a weak* limit of a subset of $\{\varphi_r, r > b\}$ as r tends to infinity. For $t \ge s \ge 0$ take $r > \max(t, b)$. Then

$$\begin{aligned}
\varphi_r(\gamma(t) - \gamma(s)) &= \varphi_r(\gamma(r) - \gamma(0)) - \varphi_r(\gamma(r) - \gamma(t)) - \varphi_r(\gamma(s) - \gamma(0)) \\
&\ge \|\gamma(r) - \gamma(0)\| - \|\gamma(r) - \gamma(t)\| - \|\gamma(s) - \gamma(0)\| \\
&\ge r - b - (r - t) - s = t - s - b.
\end{aligned}$$

We conclude that $\varphi(\gamma(t) - \gamma(s)) \ge t - s - b$ for any $t \ge s \ge 0$ and that $\varphi \in S_{X'}$. The result follows. ■

As a consequence of the above lemma and Theorem 2.13, Shafrir proved the following [179].

Theorem 2.14 *A convex subset Y of an infinite-dimensional real Banach space X has the* **AFPP** *if and only if for every sequence $(x_n)_{n\geq 1} \subseteq Y$ such that $\|x_n\| \to \infty$ as $n \to \infty$ and every $\varphi \in S_{X'}$,*

$$\limsup_{n\to\infty} \varphi\left(\frac{x_n}{\|x_n\|}\right) < 1.$$

Proof If Y does not have the **AFPP**, then by Theorem 2.13 it contains a directional curve $\gamma(t)$. Taking $x_n = \gamma(n)$ and the functional $\varphi \in S_{X'}$ given by Lemma 2.6 we certainly have $\lim_{n\to\infty} \varphi\left(\frac{x_n}{\|x_n\|}\right) = 1$.

The proof of necessity is similar to the proof of sufficiency of Theorem 2.13. Suppose we have an unbounded sequence $(x_n)_{n\geq 1} \subseteq Y$ and a bounded functional $\varphi \in S_{X'}$ such that $\lim_{n\to\infty} \varphi\left(\frac{x_n}{\|x_n\|}\right) = 1$. We may assume that $\varphi\left(\frac{x_n}{\|x_n\|}\right) \geq 1 - \frac{1}{2^n}$ for all n. We now define inductively sequences $(n_i)_{i\geq 1}$ and $(y_i)_{i\geq 1} \subseteq Y$ such that $(y_i)_{i\geq 1}$ is a directional sequence. We set $n_1 = 1$, $y_1 = x_1$ and for $i \geq 1$ we choose n_{i+1} such that

$$\|x_{n_{i+1}}\| \geq 2\|x_{n_i}\| \text{ and } \frac{\|x_{n_{i+1}} - y_i\|}{\|x_{n_{i+1}}\|} \leq 1 + \frac{1}{2^{i+1}}$$

and set

$$y_{i+1} \equiv \frac{\left(\sum_{j=1}^{i+1} \frac{x_{n_j}}{\|x_{n_j}\|}\right)}{\sum_{j=1}^{i+1} \frac{1}{\|x_{n_j}\|}}.$$

A computation similar to the one given in the proof of Theorem 2.13 shows that $(y_i)_{i\geq 1}$ is a directional sequence, hence Y does not have the **AFPP**. ■

Remark 2.6 In Theorem 2.14 we may replace "$\varphi \in S_{X'}$" by "φ which is an extreme point of $S_{X'}$". Indeed, if Y contains a directional curve $\gamma(t)$ then

$$a = \inf_{\psi\in S_{X'}} \sup_{t>0} \big(t - \psi(\gamma(t) - \gamma(0))\big)$$

is finite by Lemma 2.6. Note that "sup" can be replaced by "lim" since the function $H(t) = t - \psi(\gamma(t) - \gamma(0))$ is nondecreasing for $t > 0$. The infimum a is attained since if the sequence

$$a_n = \sup_{t>0} \big(t - \psi_n(\gamma(t) - \gamma(0))\big)$$

converges to a, then for any $t > 0, t - \psi(\gamma(t) - \gamma(0)) \leq a$ where ψ is a weak* accumulation point of the net $\{\psi_n\}$. The set

$$Z = \{\varphi \in S_{X'}, \sup_{t>0} \big(t - \varphi(\gamma(t) - \gamma(0))\big) = a\}$$

is a nonempty weak*-compact and convex subset of $S_{X'}$. Hence by the Krein-Milman theorem [119] it contains an extreme point. Since Z is clearly an extremal subset of $S_{X'}$, i.e., $\varphi, \psi \in S_{X'}$ and $\dfrac{\varphi + \psi}{2} \in Z \Rightarrow \varphi, \psi \in Z$, this extreme point is also an extreme point of $S_{X'}$.

The following example [179] illustrates Theorem 2.14 and Remark 2.6.

Example 2.4 Consider the subset Y of c_0 given by

$$Y = \big\{(x_i)_{i\geq 1} \in c_0, |x_i| \leq a_i \text{ for every } i\big\},$$

where $(a_n)_{n\geq 1}$ is a positive unbounded sequence. We claim that Y has the ***AFPP***. Indeed $c_0' = l_1$, the extreme points of l_1 are $(\pm e_n)_{n\geq 1}$ where $(e_n)_{n\geq 1}$ denote the coordinate functionals, all of them are bounded on Y, so for any such functional φ and any sequence $(y_n)_{n\geq 1} \subseteq Y$ for which $\|y_n\| \to \infty$, $\dfrac{\varphi(y_n)}{\|y_n\|} \to 0$. The result follows from Remark 2.6.

The next result shows that we can replace directionally bounded by linearly bounded in Theorem 2.13 if and only if the Banach space X is reflexive [179].

Proposition 2.10 *In a Banach space X, every closed and convex subset that is linearly bounded is directionally bounded if and only if X is reflexive.*

Proof If X is reflexive and $Y \subseteq X$ is a closed and convex subset which is not directionally bounded, then there is a directional curve $\gamma(t)$ contained in Y. Let $t_n \uparrow \infty$ be a sequence such that

$$\frac{\gamma(t_n)}{\|\gamma(t_n)\|} \rightharpoonup v.$$

By Lemma 2.6 there is $\varphi \in S_{X'}$ such $\dfrac{\varphi(\gamma(t_n))}{\|\gamma(t_n)\|} \to 1$, hence $\varphi(v) = 1$ and $v \in S_X$. Fix any $y \in Y$. We claim that the half line $\{y + sv \mid s > 0\}$ is contained in Y. Indeed, for any $s > 0$,

$$\frac{(\|\gamma(t_n)\| - s)y}{\|\gamma(t_n)\|} + s\frac{\gamma(t_n)}{\|\gamma(t_n)\|} \rightharpoonup y + sv \in Y.$$

Conversely, suppose X is not reflexive. Then by James' theorem [92] there is a functional $\varphi \in S_{X'}$ which does not attain its maximum on S_X. We choose a sequence

$(y_i)_{i\geq 1}$ from S_X such that

$$\sum_{i=1}^{\infty}(1-\varphi(y_i)) \leq 1.$$

We define

$$x_n = \sum_{i=1}^{n} y_i \quad \text{for } n \geq 1,$$

and set $Y = \overline{\text{conv}}\{x_n, n \geq 1\}$. Y is not directionally bounded since it contains the directional sequence $(x_n)_{n\geq 1}$:

$$\sum_{i=1}^{n-1} |x_i - x_{i+1}| - |x_1 - x_n| = \sum_{i=2}^{n} |y_i| - \left|\sum_{i=2}^{n} y_i\right| \leq n - 1 - \sum_{i=2}^{n} \varphi(y_i) \leq 1.$$

We claim that Y is linearly bounded. For

$$z = \sum_{i=1}^{n} c_i x_i, \quad \text{where } c_i \geq 0, \quad \text{for all } i \text{ and } \sum_{i=1}^{n} c_i = 1,$$

we have

$$\sum_{i=1}^{n} i c_i \geq \varphi(z) \geq \sum_{i=1}^{n} c_i(i-1) = \sum_{i=1}^{n} i c_i - 1$$

so

$$\frac{\varphi(z)}{|z|} \geq \frac{\left(\sum_{i=1}^{n} i c_i - 1\right)}{\sum_{i=1}^{n} c_i |x_i|} \geq 1 - \frac{1}{\sum_{i=1}^{n} i c_i}.$$

It follows that if Y contains a half line $\{y + sv \mid s > 0\}$ then

$$\lim_{s\to\infty} \frac{\varphi(y+sv)}{|y+sv\|} = 1,$$

so $\varphi(v) = 1$, contrary to our assumption. ■

To deal with the construction of unbounded convex subsets which have the **AFPP**, the following definition appears in [179].

Definition 2.12 Let X be a Banach space. A sequence $(x_n)_{n\geq 1} \subseteq S_X$ is called a (P)-sequence if for every functional $\varphi \in S_{X'}$, there is a functional $\psi \in S_{X'}$ such that

$$\limsup_{n\to\infty} \varphi(x_n) < \liminf_{n\to\infty} \psi(x_n).$$

Clearly, a subsequence of a (P)-sequence is again a (P)-sequence. The following is a straightforward consequence of the Hahn-Banach theorem.

Lemma 2.7 *Suppose Y is a closed subspace of a Banach space X and $(x_n)_{n\geq 1} \subseteq S_Y$ is a (P)-sequence in Y. Then $(x_n)_{n\geq 1}$ is a* (P)*-sequence in X as well.*

Proof Let $\varphi \in S_{X'}$ be given. Denote by a the norm of φ when restricted to Y. Clearly, $a \leq 1$. If $a = 0$, we choose any $\psi \in S_{X'}$ with $\liminf_{n\to\infty} \psi(x_n) > 0$, the existence of such a functional ψ follows directly from the definition of a (P)-sequence. If $0 < a < 1$, we choose $\psi \in S_{Y'}$ so that $\limsup_{n\to\infty} \frac{\varphi(x_n)}{a} < \liminf_{n\to\infty} \psi(x_n)$. In both cases we use the Hahn-Banach theorem to extend g to a norm-one functional on X. ■

Quite often, we will make use of the following description of (P)-sequences. Note that a sequence $(x_n)_{n\geq 1}$ satisfying (1) is called a Pryce sequence [175].

Lemma 2.8 *Let X be a Banach space and let $(x_n)_{n\geq 1} \subseteq S_X$ satisfy*

$$\sup_{\varphi\in S_{X'}} \limsup_{n\to\infty} \varphi(x_n) = \sup_{\varphi\in S_{X'}} \liminf_{n\to\infty} \varphi(x_n). \tag{1}$$

If the supremum on the left-hand side is not attained, then $(x_n)_{n\geq 1}$ is a (P)*-sequence. Conversely, if $(x_n)_{n\geq 1}$ is a* (P)*-sequence, then* (1) *is satisfied and neither of the suprema is attained [130].*

Proof Suppose (1) is satisfied is satisfied and the supremum on the left-hand side is not attained. Let $\varphi \in S_{X'}$. Then

$$\limsup_{n\to\infty} \varphi(x_n) < \sup_{\phi\in S_{X'}} \limsup_{n\to\infty} \phi(x_n) = \sup_{\phi\in S_{X'}} \liminf_{n\to\infty} \phi(x_n).$$

Therefore there exists a functional $\psi \in S_{X'}$ such that $\limsup_{n\to\infty} \varphi(x_n) < \liminf_{n\to\infty} \psi(x_n)$.

For the converse, observe that, trivially, for any bounded sequence $(x_n)_{n\geq 1}$ and $\varphi \in S_{X'}$,

$$\limsup_{n\to\infty} \varphi(x_n) \geq \liminf_{n\to\infty} \varphi(x_n). \tag{2}$$

Hence $\sup_{\varphi\in S_{X'}} \limsup_{n\to\infty} \varphi(x_n) \geq \sup_{\varphi\in S_{X'}} \liminf_{n\to\infty} \varphi(x_n)$. The definition of a (P)-sequence provides the other inequality needed for (1). By (1) and (2), if $\sup_{\phi\in S_{X'}} \liminf_{n\to\infty} \phi(x_n)$

is attained at some $\varphi \in S_{X'}$, then $\sup_{\phi \in S_{X'}} \limsup_{n\to\infty} \phi(x_n)$ is attained at this φ as well. But for this particular φ, this contradicts the existence of the functional ψ from the definition of a (P)-sequence. ∎

Definition 2.13 Let X be a Banach space. We call a bounded sequence $(x_n)_{n\geq 1}$ in X **norm attaining** if $\sup_{\varphi \in S_{X'}} \liminf_{n\to\infty} \varphi(x_n)$ is attained on $S_{X'}$.

The following lemma shows that (P)-sequences and sequences which do not attain their norm are closely related [130].

Lemma 2.9 *Let X be a Banach space and let $(x_n)_{n\geq 1} \subseteq S_X$. If $(x_n)_{n\geq 1}$ is a* (P)-*sequence, then no subsequence thereof is norm attaining. Conversely, if $(x_n)_{n\geq 1}$ contains no norm attaining subsequences, then it contains a* (P)-*sequence.*

Consequently, a Banach space contains no (P)-*sequence if and only if every bounded sequence in X contains a norm attaining subsequence.*

Proof Every subsequence of a (P)-sequence is also a (P)-sequence, so it is not norm attaining by Lemma 2.8.

Suppose the sequence $(x_n)_{n\geq 1}$ contains no norm attaining subsequences. By [130], it contains a subsequence $(x_{n_k})_{k\geq 1}$ such that

$$\sup_{\varphi \in S_{X'}} \limsup_{k\to\infty} \varphi(x_{n_k}) = \sup_{\varphi \in S_{X'}} \liminf_{k\to\infty} \varphi(x_{n_k}).$$

By Lemma 2.8, $(x_{n_k})_{k\geq 1}$ is a (P)-sequence.

The last statement of the lemma for norm-one sequences is just a reformulation of the previous two. Hence, to finish it is enough to observe that if there is a bounded sequence $(x_n)_{n\geq 1}$ with no norm attaining subsequences, then there is a normalized sequence which has this property as well. Clearly, $(x_n)_{n\geq 1}$ contains a subsequence $(x_{n_k})_{k\geq 1}$ with $\lim_{k\to\infty} \|x_{n_k}\| = a > 0$. Then $\limsup_{k\to\infty} \varphi\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) = \limsup_{k\to\infty} \frac{\varphi(x_{n_k})}{a}$ for all $\varphi \in X'$. Hence, if the supremum $\sup_{\varphi \in S_{X'}} \limsup_{k\to\infty} \varphi(x_{n_k})$ is not attained, then $\sup_{\varphi \in S_{X'}} \limsup_{k\to\infty} \varphi\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right)$ is not attained either. ∎

The following provides the existence of an unbounded convex subset Y which is directionally bounded [179].

Lemma 2.10 *If a Banach space X contains a* (P)-*sequence then X contains an unbounded convex subset Y which is directionally bounded.*

Proof Let $(x_n)_{n\geq 1} \subseteq S_X$ be a (P)-sequence, define $y_n = nx_n$ and consider $Y = \operatorname{conv}\{y_n, n \geq 1\}$. We claim that Y is directionally bounded. If not, there is $\varphi \in S_{X'}$ and $(z_n)_{n\geq 1}$ in Y such that $|z_n| \to \infty$ and $\lim_{n\to\infty} \frac{\varphi(z_n)}{|z_n|} = 1$. But since $(x_n)_{n\geq 1}$ is a (P)-sequence, there is $\psi \in S_{X'}$ such that $\psi(x_n) > (1 + 3\varepsilon)\varphi(x_n)$ for $n \geq n_0$

and some $\varepsilon > 0$. This implies that $\psi(z_n) > (1+2\varepsilon)\varphi(z_n)$ for $n \geq n_1$. We get for $n \geq n_2$,

$$1+\varepsilon < \frac{\psi(z_n)}{|z_n|} \leq 1,$$

a contradiction. ■

In [179], Shafrir asked whether in every Banach space there is an unbounded closed convex subset which has the **AFPP**? He gave the following partial answer [179].

Proposition 2.11 *If a Banach space X does not contain an isomorphic copy of l_1, then there is a closed convex unbounded subset Y of X which has the* **AFPP**.

Proof It is clear that we may assume that X is separable. First, we consider the case when X is reflexive. By Proposition 2.10, it is sufficient to find an unbounded closed and convex subset that is linearly bounded. Let $(x_n)_{n\geq 1}$ be a dense sequence in S_X and for any $n \geq 1$ choose $\varphi_n \in J(x_n)$. Note that for any $y \in X$, $\|y\| = \sup\{\varphi_n(y) \mid n \geq 1\}$. Next we define a sequence $(y_n)_{n\geq 1}$ as follows:

$$y_n \in \bigcap_{i=1}^{n} \ker \varphi_i \quad \text{and} \quad \|y_n\| = n.$$

We set $Y = \overline{\operatorname{conv}}\{y_n \mid n \geq 1\}$ and claim that Y is linearly bounded. Indeed, if for some v such that $\|v\| = 1$, $\{y + sv \mid s > 0\} \subseteq Y$, then there exists some n_0 such that

$$\varphi_{n_0}(v) > 0,$$

hence

$$\varphi_{n_0}(y + nv) \to \infty \quad \text{as } n \to \infty.$$

But by our construction,

$$\sup_{x\in Y} \varphi_{n_0}(x) \leq n_0 - 1,$$

a contradiction.

Now, consider the case when X is not reflexive. Since X does not contain l_1 isomorphically, the Odel-Rosenthal theorem [45] states that S_X is weak*-sequentially dense in $S_{X''}$. Let $\psi \in S_{X''}$ be such that the maximum of ψ on $S_{X'}$ is not attained. Choose $(x_n)_{n\geq 1} \subseteq S_X$ such that $x_n \rightharpoonup^* \psi$ as $n \to \infty$ in X''. Clearly $(x_n)_{n\geq 1}$ is a (P)-sequence and the result follows by Lemma 2.10. ■

It is interesting to note that although the above proposition does not cover the case $X = l_1$, its conclusion is still valid in this case, too. Shafrir constructed such a subset in l_1 [179].

Example 2.5 To show that there is an unbounded convex subset of l_1 which has the **AFPP**, by Lemma 2.10 it is enough to find *(P)*-sequence in l_1. Let $(x_n)_{n\geq 1}$ denote the standard base of l_1 and let $\alpha = (\alpha_n)_{n\geq 1} \subseteq l_1$ be such that

$$\alpha_i > 0 \qquad \text{for all } i \text{ and } \sum_{i=1}^{\infty} \alpha_n = 1.$$

Consider the sequence $(x_n)_{n\geq 1}$ where $x_n = \alpha - e_n$. For $m \geq 1$ let $\varphi_m = (a_n^{(m)})_{n\geq 1} \in S_{l_\infty}$ be defined by $a_n^{(m)} = 1$ for $n \leq m$ and $a_n^{(m)} = -1$ for $n > m$. We have

$$\varphi_m(x_n) = 1 + \sum_{i=1}^{m} \alpha_i - \sum_{i=m+1}^{\infty} \alpha_n$$

for $n > m$. Hence

$$\sup_{m\geq 1} \lim_{n\to\infty} \varphi_m(x_n) = 2 = \lim_{n\to\infty} \|x_n\|.$$

We claim that for every $\varphi \in S_{l_\infty}$, $\limsup_{n\to\infty} \varphi(x_n) < 2$. For $\varphi = (a_n)_{n\geq 1} \in S_{l_\infty}$ we have

$$\varphi(x_n) = \varphi(\alpha) - a_n \qquad \text{for all } n.$$

If $a_n > 0$ for all n then $\varphi(x_n) \leq 1$ for all n. Otherwise, if $a_l \leq 0$ for some l then $\varphi(\alpha) \leq 1 - \alpha_i$ and so for all n, $\varphi(x_n) \leq 2 - \alpha_l$. In any case $\limsup_{n\to\infty} \varphi(x_n) < 2$, hence $\left(\frac{x_n}{\|x_n\|}\right)_{n\geq 1}$ is a *(P)*-sequence.

In [130], Matoušková and Reich answered Shafrir's question in the affirmative. It has been done by providing the following characterizations of reflexive Banach spaces.

Theorem 2.15 *For a Banach space X the following are equivalent*:

- (i) *X is reflexive,*
- (ii) *every bounded sequence $(x_n)_{n\geq 1}$ contains a norm attaining subsequence, that is, a subsequence $(x_{n_k})_{k\geq 1}$ for which $\sup_{\varphi\in S_{X'}} \limsup_{k\to\infty} \varphi(x_{n_k})$ is attained.*
- (iii) *X does not contain any* (P)*-sequence.*

Proof $(i) \Rightarrow (ii)$. Let $(x_n)_{n\geq 1}$ be a bounded sequence in a reflexive Banach space X. There is a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ such that $x_{n_k} \rightharpoonup x \in X$. Choose $\varphi \in S_{X'}$ so that $\varphi(x) = \|x\|$. Then for any $\psi \in S_{X'}$, we have $\limsup_{k\to\infty} \psi(x_{n_k}) = \psi(x) \leq \|x\|$.

$(ii) \Rightarrow (iii)$. This is proved in Lemma 2.9. $(iii) \Rightarrow (i)$. Suppose X does not contain any (P)-sequences and suppose for a contradiction that X is not reflexive. Let Y be a separable nonreflexive subspace of X. By Lemma 2.7, Y also does not contain any (P)-sequences. This means that Y contains an isomorphic copy of l_1: if it did not, then according to the proof of Proposition 2.11, Y would contain a (P)-sequence. Let $(x_n)_{n\geq 1}$ be an isomorphic l_1-basis in Y. $(x_n)_{n\geq 1}$ contains a subsequence which converges pointwise on Y'. As this subsequence is again an l_1-basis, we may assume that $(x_n)_{n\geq 1}$ already has this property. Let $T: l_1 \hookrightarrow Y$ be the embedding for which $T(e_n) = x_n$, here $(e_n)_{n\geq 1}$ is the coordinate functionals. Then the dual mapping $T^*: Y' \to l_\infty$ is surjective and we can choose $\varphi \in Y'$ so that $T^*(\varphi) = (-1, 1, -1, 1, \cdots)$. Then $\varphi(x_n) = \varphi(T(e_n)) = T^*(\varphi(e_n))$. Hence $\varphi(x_n) = (-1)^n$, which is a contradiction. Consequently, X is reflexive. ■

Corollary 2.2 *Let X be an infinite-dimensional Banach space. Then X contains an unbounded closed convex set with the* **AFPP**.

Proof If X is reflexive, then X contains such a set according to Theorems 2.12 and 2.14. If X is not reflexive, then it contains, by Theorem 2.15, a (P)-sequence $(x_n)_{n\geq 1}$. By Theorem 2.13 and Lemma 2.10, $Y = \overline{\text{conv}}\{nx_n \,|\, n \geq 1\}$ has the **AFPP**. ■

In [167], Reich and Zaslavski showed the existence of an open and everywhere dense set in the space of all nonexpansive self-mappings of any closed and convex (not necessarily bounded) set in a hyperbolic space (endowed with the natural metric of uniform convergence on bounded subsets) such that all its elements have the **AFPP**.

We need some preliminary results.

Let (X, ρ, M) be a hyperbolic space and let Y be a nonempty, closed and ρ-convex subset of X.

For each $x \in Y$ and each $r > 0$, set

$$B(x, r) = \{y \in Y, : \rho(x, y) \leq r\}.$$

Denote by $\mathcal{A}$ the set of all nonexpansive self-mappings of Y. Fix $\omega \in Y$.

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{U}(n) = \left\{(T, S) \in \mathcal{A} \times \mathcal{A} \colon \rho(T(x), S(x)) \leq \frac{1}{n} \text{ for all } x \in B(\omega, n)\right\},$$

where n is a natural number. It is not difficult to see that the uniform space $\mathcal{A}$ is metrizable and complete.

Definition 2.14 We say that a mapping $T \in \mathcal{A}$ has the bounded approximate fixed point property (or the ***BAFP*** property, for short) if there is a nonempty bounded set $Y_0 \subseteq Y$ such that for each $\varepsilon > 0$, T has an ε-fixed point in Y_0, that is, a point $x_\varepsilon \in Y_0$ which satisfies $\rho(x_\varepsilon, T(x_\varepsilon)) \leq \varepsilon$.

We have the following [167].

Proposition 2.12 *Assume that $T \in \mathcal{A}$ and that $Y_0 \subseteq Y$ is a nonempty, closed, ρ-convex and bounded subset of Y such that*

$$T(Y_0) \subseteq Y_0. \tag{2.12}$$

Then T has the **BAFP** .

Proof Let $\varepsilon > 0$ be given. Set

$$d_0 = \sup\{\rho(y, z) \colon y, z \in Y_0\}. \tag{2.13}$$

Choose $\gamma \in (0, 1)$ such that

$$\gamma(d_0 + 1) < \varepsilon \tag{2.14}$$

and fix

$$\widetilde{x} \in Y_0. \tag{2.15}$$

For each $x \in Y$, set

$$\widetilde{T}(x) = (1 - \gamma)T(x) \oplus \gamma\widetilde{x}. \tag{2.16}$$

By (2.12), (2.15) and (2.16),

$$\widetilde{T}(Y_0) \subseteq Y_0. \tag{2.17}$$

Since $T \in \mathcal{A}$, by (2.16), for all $x, y \in Y_0$,

$$\begin{aligned} \rho(\widetilde{T}(x), \widetilde{T}(y)) &= \rho((1 - \gamma)T(x) \oplus \gamma\widetilde{x}, (1 - \gamma)T(y) \oplus \gamma\widetilde{x}) \\ &\leq (1 - \gamma)\rho(T(x), T(y)) \leq (1 - \gamma)\rho(x, y). \end{aligned} \tag{2.18}$$

By (2.17), (2.18) and Banach's Contraction Mapping Principle, there is a point x_ε such that

$$x_\varepsilon \in Y_0 \text{ and } \widetilde{T}(x_\varepsilon) = x_\varepsilon. \tag{2.19}$$

By (2.19), (2.16), (2.13) and (2.14),

$$\rho(x_\varepsilon, T(x_\varepsilon)) = \rho(\widetilde{T}(x_\varepsilon), T(x_\varepsilon)) = \rho((1-\gamma)T(x_\varepsilon) \oplus \gamma\widetilde{x}, T(x_\varepsilon))$$
$$\leq \gamma\rho(\widetilde{x}, T(x_\varepsilon)) \leq \gamma d_0 < \varepsilon.$$

Proposition 2.12 is proved. ■

Proposition 2.12 immediately implies the following result.

Corollary 2.3 *Assume that Y is bounded. Then any $T \in \mathcal{A}$ has the* **BAFP** *property.*

Corollary 2.3 does not, of course, hold if the set Y is unbounded. For example, if Y is a Banach space and T is a translation mapping, then T does not possess the **BAFP** property.

As a consequence of Proposition 2.12, the following result is proved [167].

Theorem 2.16 *There exists an open and everywhere dense set $\mathcal{B} \subseteq \mathcal{A}$ such that each $T \in \mathcal{B}$ has the* **BAFP** *property.*

Proof In view of Proposition 2.12, in order to prove this theorem it is sufficient to show that there exists an open and everywhere dense set $\mathcal{B}_0 \subseteq \mathcal{A}$ such that for each $T \in \mathcal{B}_0$, there is a nonempty, closed, ρ-convex and bounded set $Y_T \subseteq Y$ such that

$$T(Y_T) \subseteq Y_T.$$

It is not difficult to see that in order to prove this assertion, it suffices to show that given a mapping $T \in \mathcal{A}$ and a natural number n, there exists $\widetilde{T} \in \mathcal{A}$ and a natural number k such that the following two properties hold:

(i) $(T, \widetilde{T}) \in \mathcal{U}(n)$,
(ii) there is a nonempty, closed, ρ-convex and bounded set $Z \subseteq Y$ such that

$$S(Z) \subseteq Z \text{ for each } S \in \mathcal{A} \text{ satisfying } (S, \widetilde{T}) \in \mathcal{U}(k).$$

Choose a number $\gamma \in (0, 1)$ such that

$$\gamma(n + \rho(T(\omega), \omega)) < \frac{1}{2n}. \tag{2.20}$$

Set

$$\widetilde{T}(x) = (1-\gamma)T(x) \oplus \gamma\omega, \quad x \in Y. \tag{2.21}$$

Since $T \in \mathcal{A}$, by (2.21), for all $x, y \in Y$

$$\rho(\widetilde{T}(x), \widetilde{T}(y)) \leq (1-\gamma)\rho(T(x), T(y)) \leq (1-\gamma)\rho(x, y). \tag{2.22}$$

By (2.22) and Banach's Contraction Mapping Principle, there is a point $\widetilde{x}$ such that

$$\widetilde{T}(\widetilde{x}) = \widetilde{x}. \tag{2.23}$$

Since $T \in \mathcal{A}$, by (2.20) and (2.21), we have for all $x \in B(\omega, n)$,

$$\begin{aligned}\rho(T(x), \widetilde{T}(x)) &= \rho(T(x), (1-\gamma)T(x) \oplus \gamma\omega) \\ &\le \gamma\rho(T(x), \omega) \le \gamma\,[\rho(T(x), T(\omega)) + \rho(T(\omega), \omega)] \\ &\le \gamma\rho(x, \omega) + \gamma\rho(T(\omega), \omega) \le \gamma(n + \rho(T(\omega), \omega)) < \frac{1}{2n}.\end{aligned}$$

Thus

$$(T, \widetilde{T}) \in \mathcal{U}(n). \tag{2.24}$$

Next, choose a natural number k such that

$$k > \rho(\widetilde{x}, \omega) + 1 \quad \texttt{and} \quad \frac{1}{k} < \gamma. \tag{2.25}$$

By (2.25), for each $x \in B(\widetilde{x}, 1)$ we have

$$\rho(x, \omega) \le \rho(x, \widetilde{x}) + \rho(\widetilde{x}, \omega) \le 1 + \rho(\widetilde{x}, \omega) < k.$$

Hence

$$B(\widetilde{x}, 1) \subseteq B(\omega, k). \tag{2.26}$$

Let

$$x \in B(\widetilde{x}, 1) \tag{2.27}$$

and let $S \in \mathcal{U}(k)$ satisfy

$$(S, \widetilde{T}) \in \mathcal{U}(k). \tag{2.28}$$

By (2.28), (2.27), (2.23), (2.22) and (2.25)

$$\begin{aligned}\rho(S(x), \widetilde{x}) &\le \rho(S(x), \widetilde{T}(x)) + \rho(\widetilde{T}(x), \widetilde{x}) \le \frac{1}{k} + \rho(\widetilde{T}(x), \widetilde{T}(\widetilde{x})) \\ &\le \frac{1}{k} + (1-\gamma)\rho(x, \widetilde{x}) \le \frac{1}{k} + (1-\gamma) < 1.\end{aligned}$$

Thus

$$S(B(\widetilde{x}, 1)) \subseteq B(\widetilde{x}, 1)$$

for all $S \in \mathcal{A}$ satisfying (2.28). When combined with (2.24) and (2.26), this inclusion completes the proof of Theorem 2.16. ■

2.4 Finding ε-Fixed Points Where There Are No Fixed Points

In the last section we found ε-fixed points of a nonexpansive self-mapping of any closed convex set (closed ball) of a Banach space even when there are no fixed points, and the same argument obviously works for an open ball or in an incomplete normed space.

If we remove a completeness or compactness assumption from the statement of a fixed point theorem, we don't normally expect that we will still get fixed points. Indeed, it is usually easy to produce examples of fixed-point-free mappings by moving points "towards some missing limit point" or "towards infinity". But in many of these cases (if the set is bounded) we can easily show that ε-fixed points exist.

We have the following example.

Example 2.6 Let

$$Y = \{x \in C[0, 1] \colon 0 \le x \le 1, x(0) = 0, x(1) = 1\}.$$

Y is a closed, convex and bounded subset of the space $C[0, 1]$ of all real continuous functions on [0, 1] and $T \colon Y \to Y$ defined by

$$Tx(t) = tx(t)$$

satisfies $\|T(x) - T(y)\| < \|x - y\|$ for $x \neq y$ in Y. However, T has no fixed points. It does have, though, ε-fixed points. If x_n is the nth power $x_n(t) = t^n$, then $x_n - T(x_n) \to \theta$.

In [183], Smart presented a theorem given by Fort [60] for continuous mappings of an open disc in $\mathbb{R}^2$, but the proof extends directly to $\mathbb{R}^n$, and with a slight change in the statement, to any normed space:

Theorem 2.17 *If either*

1. *Ω is an open ball in $\mathbb{R}^n$ and T maps Ω into Ω, or*
2. *Ω is an open ball in a normed space and T maps Ω into a precompact subset of $\overline{\Omega}$,*

then T has an ε-fixed point for each $\varepsilon > 0$.

Proof

1. We can assume that Ω is a unit ball centered at θ. Then $S = (1 - \varepsilon)T$ maps $(1 - \varepsilon)\overline{\Omega}$ into itself so has a fixed point x by Brouwer's theorem. Since S is uniformly within ε of T, x is an ε-fixed point of T.
2. Similar, using the completion of Ω and Schauder's theorem.

■

Remark 2.7 Fort's theorem can be extended in various ways. For example, in 1), Ω can be any convex precompact set, while in 2., Ω can be any convex set if T maps Ω into a precompact subset of $\overline{\Omega}$. The proofs use Schauder's projection in a standard way. Or in 2) we could assume merely that T maps each smaller concentric ball into a compact set and the proof is unaltered. Or we can replace T with a suitable multifunction U and obtain a point x such that $d(x, Ux) < \varepsilon$.

2.5 Families of Mappings

It is natural to ask when a family of mappings has a common ε-fixed point, that is, a point which is ε-fixed for all the mappings in the family. Consider first various powers of a mapping T. When Fort's theorem gives us an x such that $T(x) \approx x$ (where $\approx$ means "is approximately equal to") it is tempting to argue that then also $T(x) \approx T^2(x) \approx \ldots \approx T^n(x)$. This suggests: in the cases covered by Fort's theorem, can we say that, for every n and every $\varepsilon > 0$, there exists x such that $\|T^r(x) - x\| \leq \varepsilon$ for $1 \leq r \leq n$?

In some special cases, there is an affirmative answer [183].

Theorem 2.18 *Let $\varepsilon > 0$ and n be given. Then there is a point which is ε-fixed point for all T^r $(1 \leq r \leq n)$ if either*

1. *T is a uniformly continuous mapping of an open ball in $\mathbb{R}^k$*
   or
2. *T is a uniformly continuous mapping of an open ball Ω in a normed space, into a precompact subset of $\overline{\Omega}$,*
3. *T is a continuous mapping of $(0, 1)$ into $(0, 1)$.*

Proof We can prove 1. and 2. to the completion of Ω and 3. by an easy elementary argument. ■

Unfortunately, in spite of this evidence, the answer to the last question is "no". We give an example to show that the condition "uniformly continuous" cannot be omitted from Theorem 2.18 1., 2.). Consider the open disc $\Omega = \{x : \|x\| < 1\}$ in $\mathbb{R}^2$. We define a homeomorphism of Ω into Ω by $h : (r, \theta) \longrightarrow (r, \theta + (1 - r)^{-1})$ (in polar coordinates consider the image S under h of the radius $J = \{(r, \theta) : \theta = 0, 0 \leq r < 1\}$. Thus $S = hJ$ is a spiral approaching the unit circle. There is a continuous map $U : S \longrightarrow S$ in which each point is moved out along the spiral until its argument has increased by π. Clearly U has no ε-fixed points for small ε (actually for $\varepsilon < (\pi+1)^{-1}$). Also there is a retraction r of Ω onto J, thus $hrh^{-1} = R$ retracts Ω onto S. We can see that:

Example 2.7 ([183]) UR is a continuous mapping of the open disc Ω into Ω for sufficiently small ε and no point is ε-fixed point for both UR and $(UR)^2$. In fact $(UR)^2(x) = URUR(x) = UUR(x)$. Thus if x is ε-fixed point for UR and for $(UR)^2$ we have $\|UR(x) - UUR(x)\| < 2\varepsilon$ which is impossible if $2\varepsilon < (\pi+1)^{-1}$.

Remark 2.8 The theorem of Kakutani on common fixed points for an equicontinuous group of affine mapping of a closed ball has an analogue giving common ε-fixed points for an open ball, and we simply extend the mappings to the closed ball and use Kakutani's theorem. Similar remarks apply to the theorems on common fixed points of families of nonexpansive mappings given by Kirk. To obtain an ε-fixed analogue of the Markov-Kakutani theorem (Theorem 1.51) we apparently have to assume that the family of mappings is equicontinuous: otherwise, we obtain only a common fixed point for any finite subset of the family.

The examples of Huneke[87] and Boyce [26] give us commuting mappings of [0, 1] with no common fixed point, by compactness we can see that they have no common ε-fixed points for ε sufficiently small.

The following result is proved in [111].

Theorem 2.19 *Suppose Y is a nonempty bounded convex subset of a Banach space, and suppose T and G are two commuting nonexpansive mappings of $Y \to Y$ at least one of which is α-almost convex. Then $F_\varepsilon(T) \cap G_\varepsilon(T) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof Suppose T is α-almost convex. Let $\varepsilon > 0$, and let $G_\lambda = (1-\lambda)I + \lambda G$. By Theorem 2.25 it is possible to choose $N \in \mathbb{N}$ so large that

$$\|G_\lambda^n(x) - G \circ G_\lambda^n(x)\| \le \varepsilon$$

for all $x \in Y$ and all $n \ge N$. For any $y \in Y$,

$$\begin{aligned} J_T(G(y)) &= \|G(y) - T \circ G(y)\| \\ &= \|G(y) - G \circ T(y)\| \\ &\le \|y - T(y)\| \\ &= J_T(y). \end{aligned}$$

Therefore for any $x \in Y$,

$$\begin{aligned} J_T(G_\lambda^n x) &= J_T((1-\lambda)G_\lambda^{n-1}(x) + \lambda G \circ G_\lambda^{n-1}(x)) \\ &\le \alpha(\max\{J_T(G_\lambda^{n-1}x), J_T(G \circ G_\lambda^{n-1}(x))\}) \\ &= \alpha(J_T(G_\lambda^{n-1}(x))) \end{aligned}$$

$$\leq \cdots$$
$$\leq \alpha^n(J_T(x)).$$

Since α^n is continuous at 0 it is possible to choose $\delta > 0$ so that $J_T(x) \leq \delta \Rightarrow \alpha^n(J_T(x)) \leq \varepsilon$. Further we may assume $\delta \leq \varepsilon$. Therefore if $x \in F_\delta(T)$ and $n \geq N$, then $G_\lambda^n(x) \in F_\varepsilon(T) \cap G_\varepsilon(T)$. Since $F_\delta(T) \neq \emptyset$, for each $\delta > 0$ the proof is complete. ■

2.6 Mappings Without ε-Fixed Points

Definition 2.15 Let X be a topological space and Y a subset of X. Then a continuous mapping $R\colon X \to Y$ is a retractionif the restriction of R to Y is the identity map on Y, that is, $R(y) = y$ for all $y \in Y$. Y is called a retract of X if such a retraction exists.

Any nonempty space retracts to a point in the obvious way (the constant map yields a retraction). If X is Hausdorff, then any retract of X must be closed.

If a mapping of a compact set has no fixed point then for ε sufficiently small it has no ε-fixed points. In [106], Kinoshita gave an example of a mapping of a compact contractible subset Y of $\mathbb{R}^3$ (the identity map 1_Y of Y is homotopic to a constant map) with no ε-fixed points for small ε. On the other hand, if a set is not compact a fixed-point-free mapping may well have ε-fixed points for all $\varepsilon > 0$. Kakutani [99] and Nirenberg [135] gave examples of fixed-point-free mappings of the unit ball of Hilbert spaces but both these examples have ε-fixed points for all $\varepsilon > 0$. These examples are based on the existence of a retraction of the closed unit ball onto the unit sphere [48]. This retraction is used in [183].

Theorem 2.20 *There is a mapping of the closed unit ball of any infinite-dimensional normed space with ε-fixed points for $\varepsilon < 1 - \delta$, for any given $\delta, 0 < \delta < 1$.*

Proof We retract the ball of radius δ onto its surface and follow this by retracting the annulus $A = \{x\colon \delta \leq \|x\| \leq 1\}$ radially onto the unit sphere which is radial in A. The combined effect is a retraction R of the unit ball onto the unit sphere which is radial in A. Then the map $T\colon x \longrightarrow -R(x)$ has the required property since if $\|x\| \leq \delta$, $\|T(x) - x\| \geq \|T(x)\| - \|x\| \geq 1 - \delta$, while if $\|x\| \geq \delta$, $R(x) = \dfrac{x}{\|x\|}$ so that $\|T(x) - x\| = \|-R(x) - x\| = (\|x\|^{-1} + 1)\|x\| > 1$. ■

The following simple example illustrates the previous theorem [183].

Example 2.8 In the unit ball of c_0 the mapping T is free of ε-fixed points for $\varepsilon \leq \dfrac{1}{4}$, where $T(x_1, x_2, \ldots) = (1 - \|x\|, \sqrt{|x_1|}, \sqrt{|x_2|}, \ldots)$. For if $\|x\| =$

$\sup |x_i| \geq \frac{1}{4}$, some $|x_i|$ is at least $\frac{1}{4}$ so that for some i, $|x_i| \geq \frac{1}{4}$ and $|x_{i+1}| < \frac{1}{4}$. Thus $\left|\sqrt{|x_i|} - x_{i+1}\right| > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. On the other hand if $\|x\| < \frac{1}{4}$, then $|(1 - \|x\|) - x_1| \geq 1 - \|x\| - |x_1| \geq 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$. Thus $\|T(x) - x\| > \frac{1}{4}$ in either case.

There are unbounded sets where each continuous mapping has a fixed point [182]. However, in all convex unbounded sets there are mappings without fixed points [115].

Definition 2.16 Let Ω be a nonempty subset of a normed space $(X, \|.\|)$. We say that Ω has ε-fixed property if each continuous mapping of Ω into Ω has ε-fixed points for all $\varepsilon > 0$.

The following result is given in [183].

Theorem 2.21 *Let Ω be a nonempty subset of a normed space $(X, \|.\|)$. If K is a retract of Ω under a uniformly continuous retraction mapping r and if Ω has the ε-fixed property then so does K.*

Proof Suppose that $\|r(x) - r(y)\| < \varepsilon$ for $\|x - y\| < \delta$. If T is any mapping of K into K then Tr maps Ω into K, thus $Tr = rTr$. Since Tr maps Ω into Ω, there exists u with $\|Tr(u) - u\| < \delta$. Thus $\|Tr(u) - r(u)\| = \|rTr(u) - r(u)\| < \varepsilon$ so that $r(u)$ is ε-fixed point for T. ■

Remark 2.9 The sets Ω and S used in Example 2.7 show that Theorem 2.21 needs the word "uniformly".

2.7 Sets of ε-Fixed Points

In a Euclidean space (or any strictly convex normed space) the set of fixed points of a nonexpansive mapping is convex. This property does not extend to the set of ε-fixed points:

Example 2.9 ([183]) A mapping of the plane such that $\|T(p) - T(q)\| \leq \frac{\|p - q\|}{\sqrt{2}}$ for all points p and q, the points $(2, 2)$ and $(2, -2)$ are $\frac{3}{2}$-fixed points but $(2, 0)$ is not a $\frac{3}{2}$-fixed point.

We take $T(x, y) = \frac{1}{2}(y, y)$ and find that

$$\|T(2, 2) - (2, 2)\| = \sqrt{2} = \|T(2, -2)\|$$

but

$$\|T(2,0)-(2,0)\| = \sqrt{2} = 2.$$

We consider continuity properties of sets of ε-fixed points, taken as functions of the mapping. We write

$$F_\varepsilon(T) = \{x : d(T(x), x) \le \varepsilon\} \qquad \text{for } \varepsilon \ge 0,$$

and

$$G_\varepsilon(T) = \{x : d(T(x), x) \le \varepsilon\} \qquad \text{for } \varepsilon > 0,$$

for set of ε-fixed points of T.

The following two results are proved in [183].

Theorem 2.22 *Let Ω be a nonempty subset of a metric space (X, d). If T and T_n are mappings of Ω and $T_n \longrightarrow T$ uniformly then*

$$F_\varepsilon(T) \supset \limsup F_\varepsilon(T_n) \supset \liminf G_\varepsilon(T_n) \supset G_\varepsilon(T).$$

Proof If we have a sequence of points $x_n \longrightarrow x$ with $d(T_n(x_n), x_n) \le \varepsilon$ then

$$d(T(x), x) \le d(T(x), T(x_n)) + d(T(x_n), T_n(x_n)) + d(T_n(x_n), x_n) + d(x_n, x),$$

so that (letting $n \longrightarrow \infty$) we have $d(T(x), x) \le \varepsilon$. If on the other hand $d(T(x), x) < \varepsilon$ then for n sufficiently large

$$d(T_n(x), x) \le d(T_n(x), T(x)) + d(T(x), x) < \varepsilon.$$

■

Theorem 2.23 *Let Ω be a nonempty subset of a normed space $(X, \|.\|)$. We consider $C(\Omega)$ the set of continuous mappings on Ω to Ω, with the metric $d(S, T) = \inf\{1, \sup_x \|S(x) - T(x)\|\}$, which gives uniform convergence. Then in $C(\Omega) \times \Omega \times [0, \infty[$, we have*

1. *the set of triples (S, x, ε) satisfying $\|S(x) - x\| \le \varepsilon$ is closed,*
2. *the set of triples (S, x, ε) satisfying $\|S(x) - x\| < \varepsilon$ is open.*

Proof Consider the set where $f(S, x, \varepsilon) = \|S(x) - x\| - \varepsilon$ is non-positive, or is negative. ■

Theorem 2.23 suggests that $F_\varepsilon(T)$ and $G_\varepsilon(T)$ are "nearly" continuous functions of T. The following construction of an object which is continuous, by using all the ε-fixed point is given in [183].

Definition 2.17 If Ω is bounded we consider the almost fixed pyramid for T:

$$P(T) = \{(x, \varepsilon)\colon \|T(x) - x\| \le \varepsilon \le m\},$$

where $m = \mathrm{diam}(\Omega)$.

Clearly $P(T)$ is the union of all sets $F_\varepsilon(T) \times \{\varepsilon\}$ in $\Omega \times [0, m]$.

Theorem 2.24 *If S and T are mappings of a bounded set Ω then $H(P(T), P(S)) \le \|T - S\|_\infty$ where H is the Hausdorff metric and $\|.\|_\infty$ is the uniform norm [183].*

Proof If $\|T - S\|_\infty = \theta$ then each point (x, ε) of $P(T)$ is within θ of the point $(x, \varepsilon + \theta)$ of $P(S)$. Similarly each point of $P(S)$ is within θ of a point of $P(T)$. ■

Remark 2.10 It might in fact be preferable to study the structure of the possibly smaller set obtained by taking the closure of the set

$$F_\varepsilon^0(T) = \{x \in \Omega\colon \|x - T(x)\| < \varepsilon\}.$$

As the following example in [34] illustrates, the set $\overline{F_\varepsilon^0(T)}$ can be much nicer than $F_\varepsilon(T)$.

Example 2.10 Let C be the rectangle $[0, 2] \times [-1, 1]$ in the Euclidean space $\mathbb{R}^2$, and define

$$T(x, y) = (x - \min(x, 1), 0).$$

It is easy to see that T is nonexpansive and that the set $F_1(T)$ consists of the closed unit disk intersected with the right half-plane along with the segment

$$\{(x, 0)\colon 1 \le x \le 2\}.$$

However $\overline{F_1^0(T)}$ consists of just the closed unit disk intersected with the right half-plane.

Little is known about the structure of the sets $F_\varepsilon(T)$ in general. The following result of Edelstein and O'Brien [54] shows that there is always a nonexpansive mapping of Y into $F_\varepsilon(T)$, although there is nothing to assure that this mapping is a retraction, or that such a retraction exists.

Theorem 2.25 *Suppose Y is a nonempty bounded convex subset of a Banach space and suppose $T\colon Y \to Y$ is nonexpansive. Then $T_\lambda := \lambda I + (1 - \lambda)T$ is uniformly asymptotically regular for each $\lambda \in (0, 1)$. That is, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|T_\lambda^n(x) - T_\lambda^{n+1}(x)\| \le \varepsilon$ for all $n \ge N$ and all $x \in Y$. In particular, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \ge N$, $T_\lambda^n\colon Y \to F_\varepsilon(T)$.*

Definition 2.18 A path in a metric space (X, d) is a continuous image of the unit interval $I = [0, 1] \subseteq \mathbb{R}$. If $S = f(I)$ is a path then its length is defined as

$$l(S) = \sup_{(x_i)} \sum_{i=0}^{N-1} d(f(x_i), f(x_{i+1}))$$

where $0 = x_0 < x_1 < \cdots < x_N = 1$ is any partition of $[0, 1]$. If $l(S) < \infty$ then the path is said to be rectifiable .

Definition 2.19 A metric space (X, d) is said to be a length space if the distance between each two points x, y of X is the infimum of the lengths of rectifiable paths joining them. In this case, d is said to be a length metric (otherwise known an inner metric or intrinsic metric).

Definition 2.20 A length space X is called a geodesic if there is a path S joining each two points $x, y \in X$ for which $l(S) = d(x, y)$. Such a path is often called a metric segment (or geodesic segment) with endpoints x and y.

There is a simple criterion which assures the existence of metric segments.

Definition 2.21 A metric space (X, d) is said to be metrically convex if given any two points $p, q \in M$ there exists a point $z \in X$, $p \neq z \neq q$, such that

$$d(p, z) + d(z, q) = d(p, q).$$

We have the following fact, first noticed by Bruck [34].

Theorem 2.26 *Suppose Y is a nonempty bounded convex subset of a Banach space and suppose $T : Y \to Y$ is nonexpansive. Then for each $\varepsilon > 0$, $F_\varepsilon(T)$ is nonempty and rectifiably pathwise connected.*

Proof For $\lambda \in (0, 1)$, let $T_\lambda = \lambda I + (1 - \lambda)T$. If $x \in F_{(1-\lambda)\varepsilon}(T_\lambda)$ then $x \in F_\varepsilon(T)$. Also if y is on the segment joining x and $T_\lambda(x)$ then

$$\begin{aligned}\|y - T_\lambda(y)\| &\leq \|y - T_\lambda(x)\| + \|T_\lambda(x) - T_\lambda(y)\| \\ &\leq \|y - T_\lambda(x)\| + \|x - y\| \\ &= \|x - T_\lambda(x)\|.\end{aligned}$$

Thus if $x \in F_\varepsilon(T)$ then every point on the segment joining x and $f(x)$ lies in $F_\varepsilon(T)$. To see that Theorem 2.25 implies $F_\varepsilon(T)$ is pathwise connected, let $u, v \in F_\varepsilon(T)$ and choose N so large that $T_\lambda^N(Y) \subseteq F_\varepsilon(T)$. Then the image under T_λ^N of the segment joining u and v maps into a path joining $T_\lambda^N(u)$ and $T_\lambda^N(v)$. Moreover the segments joining $T_\lambda^i(u)$ and $T_\lambda^{i+1}(u)$, $i = 0, \cdots, N - 1$ all lie in $F_\varepsilon(T)$. Similarly the segments joining $T_\lambda^i(v)$ and $T_\lambda^{i+1}(v)$, $i = 0, \cdots, N - 1$. By piecing these together one obtains a path S in $F_\varepsilon(T)$ joining u and v. Moreover, $l(s) \leq 2\varepsilon N + \|u - v\|$. ■

Chapter 3
Approximate Fixed Points in Ultrametric Spaces

A strictly contracting mapping of a spherically complete ultrametric space has a unique fixed point. In this chapter, we indicate how to reach or approximate this fixed point. In general, the fixed point can be approached by a pseudo-convergent family.

3.1 The Process of Approximation

First, we deal with ultrametric spaces having sets of distances that are not necessarily totally ordered.

Let (X, d, Γ) be a principally complete ultrametric space. We shall assume that $\Gamma^{\bullet}$ does not have a smallest element. To exclude the trivial case, we also assume that X has at least two elements.

Let $\varphi : X \longrightarrow X$ be a strictly contracting mapping, so by Theorem 1.45, φ has a unique fixed point, which we denote by z.

Definition 3.1 If λ is an ordinal number, let $l(\lambda)$ denote the set of ordinal numbers $\mu < \lambda$. As it is known, λ may be identified with $l(\lambda)$ and the cardinal of λ is $\text{card}\lambda = \text{card } l(\lambda)$. Let κ be a limit ordinal with $\text{card } \kappa > \text{card } \Gamma$. For every ordinal λ such that $\lambda < \kappa$, let $\mathcal{P}$ be the set of all families $\alpha = (a_i)_{i<\lambda} \in X^{l(\lambda)}$ which satisfy the following conditions:

1. if $i + 1 < \lambda$, then $a_{i+1} = \varphi(a_i) \neq a_i$,
2. $(d(a_i), a_{i+1})_{i+1<\lambda}$ is strictly decreasing,
3. if μ is a limit ordinal, $\mu < \lambda$, then $d(a_\mu, a_i) < d(a_i, a_{i+1})$ for all $i < \mu$.

If $\lambda = 1$, $\mathcal{P}_1$ is naturally identified with X, so $\mathcal{P}_1 \neq \emptyset$. Let $\mathcal{P}$ be the union of the sets $\mathcal{P}_\lambda$ for $\lambda < \kappa$. If $y \in X$, let $\mathcal{P}_y$ be the set of families in $\mathcal{P}$ with $a_0 = y$. We say that $\alpha = (a_i)_{i<\lambda}$ reaches z if there exists $i_0 < \lambda$ such that $a_{i_0} = z$.

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Remark 3.1

(i) If $\alpha = (a_i)_{i<\lambda}$ reaches z, then $\varphi(a_{i_0}) = z = a_{i_0}$, hence by 1., $i_0 + 1 = \lambda$. Thus λ is not a limit ordinal.

(ii) Let λ be a limit ordinal, let $\alpha = (a_i)_{i<\lambda} \in \mathcal{P}$ and for every $i < \lambda$ let $B_i = B_i(\alpha) = B_{d(a_i,\varphi(a_i))}(a_i)$, so by 1., B_i is a principal ball. By Lemma 1.10, we have $B_{i+1} \subseteq B_i$ and, in fact, $B_{i+1} \subset B_i$, because $d(a_i, a_{i+1}) > d(a_{i+1}, a_{i+2})$, so $a_i \notin B_{i+1}$. For every limit ordinal $\mu \leq \lambda$, let $I_\mu(\alpha) = \bigcap_{i<\lambda} B_i(\alpha)$. Since X is principally complete, it follows that $I_\mu(\alpha) \neq \emptyset$.

Definition 3.2 We say that $\alpha = (a_i)_{i<\lambda} \in \mathcal{P}$ is an asymptotic approximation to z (or more simply, an approximation to z) if λ is a limit ordinal and $I_\lambda(\alpha) = \bigcap_{i<\lambda} B_i(\alpha) = \{z\}$.

Remark 3.2 We note that an approximation to z does not reach z, because λ is a limit ordinal.

The next result is due to Priess-Crampe and Ribenboim [161] and will be called the Approximation Theorem. In important definite situations, the Approximation Theorem provides an algorithm that directly can be implemented for the calculation of asymptotic solutions, so for example in the case of some ordinary differential equations or their systems, in Hardy fields.

Theorem 3.1 *Let $Y \subseteq X$, $Y \neq \emptyset$. Assume that z cannot be reached by any $\alpha \in \mathcal{P}$ such that $a_0 \in Y \setminus \{z\}$. Then for every $y \in Y \setminus \{z\}$ there exists an asymptotic approximation $\alpha = (a_i)_{i<\lambda}$ to z such that $a_0 = y$.*

Proof The proof requires some preliminary considerations about the set $\mathcal{P}$. Let $\alpha = (a_i)_{i<\lambda}$ and $\alpha' = (a'_i)_{i<\lambda'}$ be families in $\mathcal{P}$. We define $\alpha \leq \alpha'$ when $\lambda \leq \lambda'$ and $a'_i = a_i$ for all $i < \lambda$. It is immediate to verify that $\leq$ is an order relation. Moreover, for every λ, the order restricted to $\mathcal{P}_\lambda$ is trivial. Let $y \in Y$, $y \neq z$. We claim that the ordered set $\mathcal{P}_y$ is inductive. Indeed, let C be a nonempty set, for every $c \in C$ let

$$\alpha^c = (a_i^c)_{i<\lambda_c} \in \mathcal{P}_y,$$

assume that if $c \neq c'$, then $\alpha^c \neq \alpha^{c'}$ and that the set $A = \{\alpha^c / c \in C\}$ is a totally ordered subset of $\mathcal{P}_y$. It follows that if $\alpha^c \neq \alpha^{c'}$, then $\lambda^c \neq \lambda^{c'}$. We recall that since $\alpha^c \in \mathcal{P}_y$, it follows that $\lambda^c < \kappa$. We consider two cases.

1. $L = \{\lambda_c/c \in C\}$ has a largest element λ^{c_1}. Then $\alpha^c \neq \alpha^{c_1}$ for every $c \in C$, otherwise there exists $c_2 \in C$ such that $\alpha^{c_1} < \alpha^{c_2}$, hence $\lambda_{c_1} < \lambda_{c_2}$, which is a contradiction. In this case, α^{c_1} is an upper bound for A.
2. L does not have a largest element. Since $\lambda_c < \kappa$ for every $c \in C$, there exists the smallest element μ such that $\lambda_c < \mu$ for every $c \in C$. So $\mu \leq \kappa$. If $\mu = \nu + 1$, then by the minimality of μ, there exists $c_1 \in C$ such that $\nu \leq \lambda_{c_1}$ and therefore $\nu = \lambda_{c_1}$, because $\mu > \lambda_{c_1}$. In this case, λ_{c_1} is the largest element in L,

which has been excluded. We have shown that μ is a limit ordinal. Now if i is an ordinal such that $i < \mu$, by the minimality of μ there exists $\lambda_c \in L$ such that $i \leq \lambda_c < \mu$. Since L does not have a largest element, there exists $c^* \in C$ such that $i \leq \lambda_c < \lambda_{c^*} < \mu$. We define $\widetilde{a_i} = a_i^{c^*}$. It is immediate to verify that $\widetilde{a_i}$ is well-defined, independently of the choice of $c^* \in C$ such that $\lambda_c < \lambda_{c^*} < \mu$. By 2., the family $(d(a_i^{c^*}, a_{i+1}^{c^*}))_{i<\mu}$ of elements of Γ is strictly decreasing, hence all these elements are pairwise distinct. So card$\mu \leq \Gamma$. Since card$\Gamma <$ cardκ thus $\mu < \kappa$, which implies that $\widetilde{\alpha} = (\widetilde{a_i})_{i<\mu}$ belongs to $\mathcal{P}$. Furthermore, $\alpha^c < \widetilde{\alpha}$ for every $c \in C$. Hence $\widetilde{\alpha}$ is an upper bound for A. This concludes the proof that $\mathcal{P}_y$ is inductive. By Zorn's lemma, there exists a maximal $\alpha \in \mathcal{P}_y$. That is, for every $y \in Y \setminus \{z\}$ there exists a maximal $\alpha \in \mathcal{P}$ such that $a_0 = y$.

We assume that z is not reached by any family in $\mathcal{P}_y$ for every $y \in Y \setminus \{z\}$. By 2., for every $y \in Y \setminus \{z\}$ there exists a maximal $\alpha = (a_i)_{i<\lambda} \in \mathcal{P}$ such that $a_0 = y$. First we observe that λ is a limit ordinal. We assume the contrary, let $\lambda = i_0 + 1$. Since z is not reached by α then $a_{i_0} \neq z$, so $a_{i_0} \neq \varphi(a_{i_0})$, hence

$$d(\varphi(a_{i_0}), \varphi^2(a_{i_0})) < d(a_{i_0}, \varphi(a_{i_0})).$$

Let $\alpha' = (a'_i)_{i<\lambda+1}$, where $a'_i = a_i$ for all $i < \lambda$ and $a'_\lambda = \varphi(a_{i_0})$. So $\alpha' \in \mathcal{P}, \alpha < \alpha'$. This is impossible, because α is maximal in $\mathcal{P}$. Thus as stated, λ is a limit ordinal. Since X is principally complete and each $B_i(\alpha)$ is a principal ball of X, then $I_\lambda(\alpha) = \bigcap_{i<\lambda} B_i(\alpha) \neq \emptyset$. We show that $I_\lambda(\alpha) = \{z\}$. Let $t \in I_\lambda(\alpha)$. We note that $t \neq a_i$ for all $i < \lambda$. Indeed, if there exists $i_0 < \lambda$ such that $t = a_{i_0}$ then $t \notin B_{i_0+1}$ which is a contradiction. Now we show that $\varphi(t) \in I_\lambda(\alpha)$. We have

$$d(\varphi(t), a_{i+1}) = d(\varphi(t), \varphi(a_i)) < d(t, a_i) \leq d(a_i, a_{i+1})$$

for all $i < \lambda$. It follows that $d(t, \varphi(t)) \leq d(a_i, a_{i+1})$ for every $i < \lambda$. Hence $d(t, \varphi(t)) < d(a_i, a_{i+1})$ for every $i < \lambda$. Let $\alpha' = (a'_i)_{i<\lambda+1}$ defined by $a'_i = a_i$ for all $i < \lambda$ and $a'_\lambda = t$. So $\alpha' \in \mathcal{P}$, because $d(t, a_i) \leq d(a_i, a_{i+1})$ for every $i < \lambda$. We have $\alpha < \alpha'$, which is contrary to the maximality of α. This shows that $t = \varphi(t)$, so $t = z$ and we deduce that $I_\lambda(\alpha) = \{z\}$. Hence α is an asymptotic approximation to z. ■

Remark 3.3 Under the assumptions of the Approximation Theorem, if $y \in Y \setminus \{z\}$ there exists the smallest limit ordinal λ for which there exists an approximation $\alpha = (a_i)_{i<\lambda}$ to z such that $a_0 = y$. So the set

$$\mathcal{M} = \{\alpha = (a_i)_{i<\lambda} \mid \alpha \text{ is an approximation to } z \text{ and } a_0 = y\}$$

is not empty. For $\alpha \in \mathcal{M}$ and each limit ordinal $\mu < \lambda$ the set $I_\mu(\alpha)$ properly contains z. The set $I_\mu(\alpha)$ may be considered a measure of the accuracy of the approximation α, when restricted to $\alpha_{|\mu} = (a_i)_{i<\mu}$

As a consequence, we have

Corollary 3.1 *If $y \in X$, $y \neq z$, then either there exists $\alpha \in \mathcal{P}_y$ which reaches z, or if this is not the case, there exists an approximation $\alpha \in \mathcal{P}_y$ to z [161].*

Proof The corollary is a special case of the Approximation Theorem, taking $Y = \{y\}$, where $y \neq z$. ■

Remark 3.4 The proof of the Approximation Theorem suggests the method to reach or to approximate the fixed point. Let $y \in Y$. If $y = z$ there is nothing to do. If $y \neq z$ let $a_0 = y$ and $a_1 = \varphi(a_0) \neq a_0$. If $a_1 = z$, then z has been reached by the family consisting only of a_0, a_1. If $a_1 \neq z$ let $a_2 = \varphi(a_1) \neq a_1$. The procedure may be iterated. It may happen that there exists $n_0 > 2$ such that $a_{n_0} = z$, so z has been reached. Or, for every $n < \omega$, $a_n \neq z$. Let $\alpha = (a_n)_{n<\omega}$. If the set $I_\omega(\alpha)$ consists of only one element, this element is the fixed point z. If $I_\omega(\alpha)$ has more than one element, we may choose any one of the elements of $I_\omega(\alpha)$ and call it a_ω. Then $a_{\omega+1} = \varphi(a_\omega)$ if $a_\omega \neq \varphi(a_\omega)$, $a_{\omega+2} = \varphi(a_{\omega+1})$ if $a_{\omega+} \neq \varphi(a_{\omega+1})$, etc. It may happen that there exists $n \geq 0$ such that $a_{\omega+n} = z$, or one needs to consider $I_{2\omega}(\alpha')$, where $\alpha' = (a'_i)_{i<2\omega}$, with $a'_i = a_i$ for $i < \omega$ and a'_i, defined as indicated for $\omega \leq i \leq 2\omega$. Even though there exists a family $\alpha \in \mathcal{P}$ which reaches or approximates z, in general it is not possible to predict what will happen, in particular, when the algorithm will stop.

3.2 The Case When Γ Is Totally Ordered

Henceforth we shall assume that Γ is totally ordered and that $\Gamma^\bullet$ does not have a smallest element.

We shall use the following notations:

- $\mathcal{A}$ = set of all approximations α to z,
- $\mathcal{PC}$ = set of all pseudo-convergent families in X.

The following is given in [161].

Proposition 3.1

1. $\mathcal{A} \subseteq \mathcal{PC}$.
2. Let $\mathcal{PLA}$ be the set of all pseudo-limits of all $\alpha \in \mathcal{A}$. Then $\mathcal{PLA} = \{z\}$.
3. If $\alpha \in \mathcal{A}$, if $\alpha' \in \mathcal{P}$ and $\alpha < \alpha'$, then α' reaches z.

Proof

1. We show that α is a pseudo-convergent family in X. Since α is an approximation to z then λ is a limit ordinal. We shall prove that if $i < \mu < \nu < \lambda$, then $d(a_i, a_\mu) > d(a_\mu, a_\nu)$. For this purpose, we prove that

$$d(a_i, a_{i+1}) = d(a_i, a_\mu).$$

The proof is by induction on μ. It is trivial if $\mu = i + 1$.

Let $i + 1 < \mu$. We consider two cases:

(a) $\mu = \kappa + 1$: By induction, $d(a_i, a_{i+1}) = d(a_i, a_\kappa)$, since $a_i \neq a_\kappa$, it follows that $d(a_i, a_\kappa) > d(a_{i+1}, a_\mu)$ and therefore $d(a_i, a_{i+1}) = d(a_i, a_\mu)$.

(b) μ is a limit ordinal: By construction of α, we have

$$d(a_i, a_\mu) \leq d(a_i, a_{i+1})$$

for all $i < \lambda$. If $d(a_i, a_\mu) < d(a_i, a_{i+1})$ for some $i < \lambda$, then

$$d(a_i, a_{i+1}) = d(a_i, a_\mu).$$

Since $a_\mu \neq a_i$, it follows

$$d(a_{\mu+1}, a_{i+1}) < d(a_\mu, a_i) < d(a_\mu, a_{i+1}),$$

hence $d(a_i, a_{i+1}) = d(a_\mu, a_{\mu+1})$. This is absurd, so $d(a_i, a_{i+1}) = d(a_\mu, a_{i+1})$ for all $i < \mu$. This concludes the proof by induction. In a similar way $d(a_\mu, a_\nu) = d(a_\mu, a_{\mu+1})$ for $\mu < \nu < \lambda$. It follows that if $i < \mu < \nu < \lambda$, then

$$d(a_i, a_{i+1}) = d(a_\mu, a_i) > d(a_\mu, a_\nu) = d(a_\mu, a_{\mu+1}).$$

So we have proved that α is a pseudo-convergent family in X.

2. Assume that $\alpha \in \mathcal{A}$, then $I_\lambda(\alpha) = \{z\}$, so

$$d(z, a_i) \leq d(a_i, a_{i+1}) = d(a_\mu, a_i)$$

for all $i < \mu < \lambda$. Thus z is a pseudo-limit of the pseudo-convergent family α. Let $t \in X, t \neq z$, then $t \in I_\lambda(\alpha)$. So there exists $i_0 < \lambda$ such that $d(t, a_{i_0}) \not\leq d(a_{i_0+1}, a_{i_0})$, that is, $t \notin B_{i_0}(\alpha)$. Hence for every i such that $i_0 < i < \lambda$, we also have $t \notin B_i(\alpha)$, that is, $d(t, a_i) \not\leq d(a_i, a_{i+1}) = d(a_\mu, a_i)$ for $i < \mu < \lambda$. So t is not a pseudo-limit of α.
3. Let $\alpha' \in \mathcal{P}$ be such that $\alpha < \alpha'$. Since $\alpha' \in \mathcal{P}$ we have for every $i < \lambda, d(a'_\lambda, a'_i) \leq d(a'_i, a'_{i+1})$ or equivalently, $d(a'_\lambda, a_i) \leq d(a_i, a_{i+1})$ because $a'_i = a_i, a'_{i+1} = a_{i+1}$. Hence $a'_\lambda \in I_\lambda(\alpha) = \{z\}$. So α' reaches z. Let $\alpha = (a_i)_{i<\lambda} \in \mathcal{P}$ and let $\Sigma_\alpha = \{d(a_i, \varphi(a_i)) \mid i < \lambda\}$. We note that $0 \in \Sigma_\alpha$ if and only if α reaches z and, in this case, λ is not a limit ordinal. Let $\Lambda_\varphi = \{d(x, \varphi(x)) \mid x \in X, x \neq z\}$. Then $\Sigma_\alpha \setminus \{0\} \subseteq \Lambda_\varphi \subseteq \Gamma^\bullet$. Let (Y, d, Γ) be subspace of (X, d, Γ). If φ is such that $\varphi(Y) \subseteq Y$, let $\Lambda_\varphi^Y = \{d(y, \varphi(y)) \mid y \in Y, y \neq z\}$. Since X is principally complete (and Γ is totally ordered), X is spherically complete. If, moreover, $d(Y \times Y) \setminus \{0\}$ is coinitial in $d(X \times X) \setminus \{0\}$, then by Theorem 1.42, Y has one and exactly one completion $\widehat{Y}$ in X.

■

Proposition 3.2 *Let α be an approximation to z. Then we have the following.*

1. *Σ_α is coinitial in Λ_φ.*
2. *Assume that (Y, d, Γ) is a subspace of (X, d, Γ) and that $\varphi(Y) \subseteq Y$. Assume, moreover, that $d(Y \times Y) \setminus \{0\}$ is coinitial in $d(X \times X) \setminus \{0\}$. If $z \in \widehat{Y} \setminus Y$, then Λ_φ^Y is coinitial in Λ_φ.*
3. *If Σ_α is coinitial in $\Gamma^\bullet$, then α is a Cauchy family and $z = \lim \alpha$.*
4. *If X is solid, then $\Lambda_\varphi = \Gamma^\bullet$, furthermore, α is a Cauchy family and $z = \lim \alpha$.*

Proof

1. Assume that Σ_α is coinitial in Λ_φ. So there exists $x \in X, x \neq z$ such that $d(x, \varphi(x)) < d(a_i, a_{i+1})$ for all $i < \lambda$. Let $\alpha' = (a'_i)_{i<\lambda+1}$ be defined by $a'_i = a_i$ for all $i < \lambda$ and $a'_\lambda = x$. Then $\alpha' \in \mathcal{P}$ and $\alpha < \alpha'$. By Proposition 3.1(3), α' reaches z, while α does not reach z. So $x = a'_\lambda = z$, and this is absurd.
2. Since $z \in \widehat{Y} \setminus Y$, there exists a limit ordinal ρ and a Cauchy family $(y_\nu)_{\nu<\rho}$, with $y_\nu \in Y$, such that $z = \lim\limits_{\nu<\rho} y_\nu$. Let $d(x, \varphi(x)) \in \Lambda_\varphi$. Since $z = \lim\limits_{\nu<\rho} y_\nu$, there exists $\nu < \rho$ such that $d(y_\nu, z) \leq d(x, \varphi(x))$. Thus
$$d(\varphi(y_\nu), \varphi(z)) = d(\varphi(y_\nu), z) < d(y_\nu, z) \leq d(x, \varphi(x)),$$
which implies that
$$d(\varphi(y_\nu), y_\nu) = d(y_\nu, z) \leq d(x, \varphi(x)).$$
Hence Λ_φ^Y is coinitial in Λ_φ.
3. Let $\gamma \in \Gamma^\bullet$, by assumption there exists $i_0 < \lambda$ such that
$$d(a_{i_0}, \varphi(a_{i_0})) = d(a_{i_0}, a_{i_0+1}) \leq \gamma.$$
By Proposition 3.1, α is pseudo-convergent. Hence
$$d(a_i, a_\mu) < d(a_{i_0}, a_{i_0+1}) \leq \gamma$$
for all i, μ such that $i_0 < i < \mu < \lambda$. By assumption, $z \in I_\lambda(\alpha)$, so
$$d(z, a_i) \leq d(a_i, \varphi(a_i)) < \gamma$$
for every i such that $i_0 < i < \lambda$. This shows that α is a Cauchy family and $z = \lim \alpha$.
4. Let $0 < \gamma \in \Gamma$. Since X is solid, there exists $x \in X$ such that $d(x, z) = \gamma$. So $x \neq z$, hence
$$d(z, \varphi(x)) = d(\varphi(z), \varphi(x)) < d(z, x),$$

which implies that

$$d(x, \varphi(x)) = d(z, x) = \gamma.$$

Thus $\Lambda_\varphi = \Gamma^\bullet$. By 1., Σ_α is coinitial in $\Lambda_\varphi = \Gamma^\bullet$. Hence by 3., α is a Cauchy family in X and $z = \lim \alpha$.

■

If the ultrametric space (Y, d, Γ) is not spherically complete and $\varphi \colon Y \longrightarrow Y$ is strictly contracting, the following result guarantees an appropriate extension of φ [152].

Theorem 3.2 *Let (X, d, Γ) be an ultrametric space and that (X, d, Γ) is a spherically complete. Let Y be a subspace of X. If $\psi \colon Y \longrightarrow Y$ is strictly contracting, there exists $\varphi \colon X \longrightarrow X$ such that φ is strictly contracting and extends ψ. If, moreover, $d(Y \times Y) \setminus \{0\}$ is coinitial in $d(X \times X) \setminus \{0\}$, the restriction $\varphi_{|\widehat{Y}}$ of φ to the completion $\widehat{Y}$ of Y is uniquely determined.*

Proof Let $b \in X \setminus Y$. For every $x \in X$ let $\pi_x = d(b, x) \neq 0$. For each $x \in Y$ consider in X the ball $B'_{\pi_x}((\psi(x))$. If $\pi_y \leq \pi_x$ then $B'_{\pi_y}((\psi(y))) \subseteq B'_{\pi_x}((\psi(x)))$ because

$$d(\psi(y), \psi(x)) \leq d(y, x) \leq \max\{\pi_y, \pi_x\} = \pi_x.$$

Since X is spherically complete, the chain of balls $(B'_{\pi_x}((\psi(x)))_{x\in X}$ has a nonempty intersection, let $b' \in \bigcap_{x\in Y} B'_{\pi_x}((\psi(x))$ and define $\varphi(b') = b'$. We show that $\varphi \colon Y \cup \{b\} \longrightarrow X$ satisfies $d(b', \psi(x)) < d(b, x) = \pi_x$ for all $x \in Y$. Since the extension X of Y is immediate, there exists $y \in Y$ such that $\pi_y < \pi_x$. Hence $d(\psi(x), \psi(y)) < d(x, y) = d(b, x)$. Since $b' \in B'_{\pi_y}((\psi(y))$, then $d(b', \psi(y)) \leq \pi_y < d(b, x)$. Thus

$$d(b', \psi(x)) \leq \max\{d(b', \psi(y)), d(\psi(y), \psi(x))\} < d(b, x).$$

The proof of the extension of $\varphi \colon X \longrightarrow X$ may be concluded by applying Zorn's lemma.

Let $\widehat{Y}$ be the completion of Y in X and assume that φ and φ' are extensions of ψ to $\widehat{Y}$ which are strictly contracting. Assume there exists $z' \in \widehat{Y} \setminus Y$ such that $\varphi(z') \neq \varphi'(z')$. Since φ and φ' are strictly contracting, $d(\varphi(z'), \psi(x)) < d(z', x)$ and $d(\varphi'(z'), \psi(x)) < d(z', x)$ for all $x \in X$. Hence

$$d(\varphi(z'), \varphi'(z')) \leq \max\{d(\varphi(z'), \psi(x)), d(\varphi'(z'), \psi(x))\} < d(z', x)$$

for all $x \in Y$ which is impossible, since the set $\{d(z', x) \mid x \in Y\}$ is coinitial in $\Gamma \setminus \{0\}$. So $\varphi = \varphi'$. ■

In the next theorem [161], we shall study the following situation: (Y, d, Γ) is an ultrametric space, the mapping $\psi\colon Y \longrightarrow Y$ is strictly contracting, and the spherically complete ultrametric space (X, d, Γ) is an extension of Y, furthermore, we assume that $d(Y \times Y) \setminus \{0\}$ is coinitial in $d(X \times X) \setminus \{0\}$. (For example, X could be the spherical completion of Y, see Theorem 1.42.) By Theorem 1.42, Y has exactly one completion $\widehat{Y}$ in X. In general, different extensions of ψ to strictly contracting mappings of X will lead to different fixed points of these mappings. But if $z \in \widehat{Y}$ then, since all these extensions coincide on $\widehat{Y}$, z is the fixed point of all these mappings.

Theorem 3.3 *Let Y, X and the mappings ψ, φ be as described above. Assume that $\alpha = (a_{i'})_{i'<\lambda'}$, with $a_{i'} \in Y$, is (with respect to φ) an approximation to $z \in X \setminus Y$ and that, furthermore, Σ_α is coinitial in $\Gamma^\bullet$. Then $z \in \widehat{Y}$ and conversely, if $z \in \widehat{Y} \setminus Y$, then there exists an approximation $\beta = (b_i)_{i<\lambda}$ to z such that $b_0 = a_0$ and $b_i \in Y$ for all $i < \lambda$.*

Proof By Proposition 3.2, α is a Cauchy family and $z = \lim \alpha \in \widehat{Y}$. We now refer to the proof of Theorem 3.1. Let κ, $\mathcal{P}$ and the order relation on $\mathcal{P}$ be as described there. Let Υ be the set of all $\beta = (b_i)_{i<\lambda}$ of $\mathcal{P}$ such that $b_0 = a_0$, $b_i \in Y$ for every $i < \lambda$ and $z \in \bigcap_{i<\lambda} B_i$, where $B_i = B_{d(b_i, \psi(b_i))}(b_i)$. (We note that $b_i \neq z$ for every $i < \lambda$ because $z \in \widehat{Y} \setminus Y$.)

First we show that Υ, which the restriction of the order of $\mathcal{P}$, is inductive. Let C be a nonempty set, for every $c \in C$ let $\beta^c = \beta = (b_i^c)_{i<\lambda_c} \in \Upsilon$. Assume that $\beta^c \neq \beta^{c'}$, if $c \neq c'$, and that $\mathcal{B} = \{\beta^c \mid c \in C\}$ is totally ordered. If $\mathbf{L} = \{\lambda_c \mid c \in C\}$ has a largest λ_{c_1}, it follows, as shown in the proof of Theorem 3.1, that β^{c_1} is an upper bound for $\mathcal{B}$.

Thus there remains the case that $\mathbf{L}$ does not have a largest element. We conclude, as in part (b) of the proof of Theorem 3.1, that there exists the smallest ordinal μ such that $\lambda_c < \mu$ for every $c \in C$, that $\mu \le k$ and that μ is a limit ordinal. Now we define in a similar way, as explained there, a family $\widetilde{\beta} = (\widetilde{b}_i)_{i<\mu}$ which belongs to $\mathcal{P}$ and which furthermore has the following properties: $\widetilde{b}_0 = a_0$, $\widetilde{b}_i \in Y$ for every $i < \mu$ and $z \in \bigcap_{i<\mu} \widetilde{B}_i$, with

$$\widetilde{B}_i = B_{d(\widetilde{b}_i, \psi(\widetilde{b}_i))}(\widetilde{b}_i).$$

Thus $\widetilde{\beta} \in \Upsilon$ is an upper bound for $\mathcal{B}$. Hence Υ is inductive.

Moreover, $\Upsilon \neq \emptyset$, because $(a_{i'})_{i'<\omega_0} \in \Upsilon$. Thus by Zorn's lemma, Υ has a maximal element $\beta = (b_i)_{i<\lambda}$. Then λ is a limit ordinal. Indeed, if not, let $\lambda = i_0+1$. Since $b_{i_0} \in Y$, also $\psi(b_{i_0}) \in Y$, so

$$b_{i_0} \neq z, \psi(b_{i_0}) \neq z, \psi(b_{i_0}) \neq b_{i_0}.$$

Therefore

$$d(z, \psi^2(b_{i_0})) < d(z, \psi(b_{i_0})) < d(z, b_{i_0}),$$

hence

$$d(\psi(b_{i_0}), \psi^2(b_{i_0})) = d(z, \psi(b_{i_0})) < d(z, b_{i_0}) = d(b_{i_0}, \psi(b_{i_0})).$$

Thus if $b_i^* = b_i$ for $i < \lambda$ and $b_\lambda^* = \psi(b_{i_0})$, then $\beta < \beta^* = (b_i^*)_{i<\lambda+1}$, furthermore $z \in B_{d(b_\lambda^*, \psi(b_\lambda^*))}(b_\lambda^*)$, so $\beta^* \in \Upsilon$ contrary to the maximality of $\beta \in \Upsilon$. Hence λ is a limit ordinal. Since $z \in \bigcap_{i<\lambda} B_i(\beta)$, we have $z \in I_\lambda(\beta)$. Assume that there exists $t \in X$ such that $t \neq z$ and $t \in I_\lambda(\beta)$. Then $0 < d(t, z)$. Since $z \in \widehat{Y} \setminus Y$, there exists a Cauchy family $(y_\nu)_{\nu<\rho}$ in Y, ρ a limit ordinal, such that $z = \lim_{\nu<\rho} y_\nu$. Thus there exists $\nu_0 < \rho$ such that $d(z, y_{\nu_0}) \leq d(t, z)$. Then

$$d(\psi(y_{\nu_0}), z) = d(\varphi(y_{\nu_0}), z) < d(y_{\nu_0}, z) \leq d(t, z).$$

So

$$d(\psi(y_{\nu_0}), y_{\nu_0}) = d(y_{\nu_0}, z) \leq d(t, z).$$

It follows that if $b_i' = b_i$ for $i < \lambda$ and $b_\lambda' = y_{\nu_0}$ then $\beta' = (b_i')_{i<\lambda+1} > \beta$ and, moreover, $\beta' \in \Upsilon$, because $z \in B_\lambda = B_{d(y_{\nu_0}, \psi(y_{\nu_0}))}(y_{\nu_0})$. This contradicts the maximality of β in Υ. Hence $I_\lambda(\beta) = \{z\}$. ■

The given results in this chapter are used in [158, 159] to provide solutions or approximations to solutions of twisted polynomial equations and polynomial differential equations.

Chapter 4
Synthetic Approaches to Problems of Fixed Points

In this chapter, we introduce synthetic approaches to fixed point problems involving regular-global-inf functions. Such functions satisfy a condition weaker than continuity. Additionally, under appropriate assumptions, it assures that approximate fixed point sequences always approach the fixed point set.

4.1 Regular-Global-Inf Functions

It is well-known that continuity is an ideal property [38], while in some applications the mapping under consideration may not be continuous, yet at the same time it may be "not very discontinuous". In [5] Angrisani introduced regular-global-inf functions. Such functions satisfy a condition weaker than continuity, yet in many circumstances, it is precisely the condition needed to assure either the uniqueness or compactness of the set of solutions in fixed point and optimization problems.

We begin by a definition given in [6].

Definition 4.1 Let X be a topological space and $T: X \longrightarrow \mathbb{R}$. The function T is said to be a regular-global-inf (r.g.i.) at $x \in X$ if $T(x) > \inf_X(T)$ implies that there exist $\varepsilon > 0$ such that $\varepsilon < T(x) - \inf_X(T)$ and a neighborhood N_x such that $T(y) > T(x) - \varepsilon$ for each $y \in N_x$. If this condition holds for each $x \in X$, then T is said to be an r.g.i. on X.

An equivalent condition to be r.g.i. on a metric space for $\inf_X(T) \neq -\infty$ is proved by Kirk and Saliga [112].

A. Ben Amar, D. O'Regan, *Topology and Approximate Fixed Points*, Developments in Mathematics 71, https://doi.org/10.1007/978-3-030-92204-7_4

Proposition 4.1 *Let X be a metric space and $T\colon X \longrightarrow \mathbb{R}$. Then T is an* r.g.i. *on X if and only if, for any sequence $(x_n)_n \subset X$, the conditions*

$$\lim_{n\longrightarrow\infty} T(x_n) = \inf_X(T), \quad \lim_{n\longrightarrow\infty} x_n = x \tag{1}$$

imply $T(x) = \inf_X(T)$.

Proof Assume T is an r.g.i. on X and let $(x_n)_n \subset X$ satisfy $\lim_{n\longrightarrow\infty} T(x_n) = \inf_X(T)$ and $\lim_{n\longrightarrow\infty} x_n = x \in X$. Suppose $T(x) > \inf_X(T)$. Then there exist $\varepsilon > 0$, $\varepsilon < T(x) - \inf_X(T)$, and a neighborhood N_x such that for all $y \in N_x$

$$T(y) > T(x) - \varepsilon > \inf_X(T).$$

This implies that

$$\liminf T(x_n) \geq T(x) - \varepsilon > \inf_X(T),$$

a contradiction.

Now suppose the condition of the proposition holds and assume there exists $x \in X$ such that T is not an r.g.i. at x. Then $T(x) > \inf_X(T)$. Let $\varepsilon_n > 0$ satisfy $\varepsilon_n < T(x) - \inf_X(T)$ with $\lim_{n\longrightarrow\infty} \varepsilon_n = T(x) - \inf_X(T)$. Then for each n there exists $y_n \in X$ with $d(x, y_n) < \dfrac{1}{n} \in X$ such that $T(y_n) \leq T(x) - \varepsilon_n$. This implies $\lim_{n\longrightarrow\infty} T(y_n) = \inf_X(T)$ and $\lim_{n\longrightarrow\infty}(y_n) = x$. Therefore $T(x) = \inf_X(T)$ a contradiction. ■

Remark 4.1 If (X, d) is a metric space with $T\colon X \longrightarrow \mathbb{R}$, and if $c \in \mathbb{R}$, set

$$L_c := \{x \in X\colon T(x) \leq c\}.$$

L_c is called level set. It follows that T is an r.g.i. at $x \in X$ if and only if $T(x) > \inf_X(T)$ implies $\operatorname{dist}(x, L_c) > 0$ for some $c > \inf_X(T)$.

As before we use the symbol μ to denote the usual Kuratowski measure of noncompactness. Conventionally, the form $c \longrightarrow (\inf_X(T))^+$ when $\inf_X(T) = -\infty$ has the same significance as the form $c \longrightarrow \inf_X(T)$.

The following is a well-known result of Kuratowski [122].

Proposition 4.2 *Let (X, d) be a complete metric space and let $(C_n)_n$ be a decreasing sequence of nonempty closed subsets of X with the property*

$$\lim_{n\longrightarrow\infty} \mu(C_n) = 0.$$

Then $\bigcap_n C_n = C$ *is nonempty and compact, and moreover* $\lim_{n\longrightarrow\infty} H(C_n, C) = 0$ *in the classical Hausdorff metric* H.

The following two theorems are due to Angrisani [5].

Theorem 4.1 *Let* $T : X \longrightarrow \mathbb{R}$ *be an* r.g.i. *function defined on a complete metric space* X. *If* $\lim_{c\longrightarrow(\inf_X(T))^+} \mu(L_c) = 0$, *then the set of the global minimum points of* T *is nonempty and compact.*

Proof We construct a monotone decreasing sequence $(c_n)_n(c_n < c_{n-1})$, so that $\lim_{n\longrightarrow+\infty} c_n = \inf_X(T)$ and $\mu(L_{c_n}) \le \frac{1}{n}$. We construct a monotone decreasing sequence $(a_n)_n$ so that $a_n > \inf_X(T)$ and L_{a_n} is a nonempty set, so that $\mu(L_{a_n}) \le \frac{1}{n}$. We can also consider some monotone decreasing sequence $(m_n)_n$ so that $\lim_{n\longrightarrow+\infty} m_n = \inf_X(T)$. So we put $c_n = \min\{a_n, m_n\}$, for each $n \in \mathbb{N}$.

We take a sequence (in X) $(c_n)_n$, so that, for each $n \in \mathbb{N}$, we have $b_n \in L_{c_n}$, i.e., we have $T(b_n) \le c_n$. It is proposed to show that a sequence of sets $(M_n)_n$ can be constructed so that, for each $n \in \mathbb{N}$, we have $M_{n-1} \supseteq M_n$, $\text{diam} M_n \le \frac{1}{n}$ and

$$(*) \qquad (b_n)_n \text{ frequently belongs to each } M_i.$$

So, from $(*)$, it follows that we can construct a subsequence $(b_{k_n})_n$ of $(b_n)_n$ so that, for each $n \in \mathbb{N}$, we have $b_{k_n} \in M_n$. By $M_{n-1} \supseteq M_n$ and $\text{diam} M_n \le \frac{1}{n}$, such a subsequence will be a Cauchy sequence, which therefore converges at a point we shall show to be a global minimum for T. Now we show that the sequence of sets $(M_n)_n$ can be constructed.

We show that, if we have $i \in \mathbb{N}$ and a set M_i so that $(b_n)_n$ frequently belongs to M_i, we can construct a set M_{i+1} so that $M_i \supseteq M_{i+1}$, $\text{diam}(M_{i+1}) \le \frac{1}{i}$ and $(b_n)_n$ frequently belongs to M_{i+1}. For the fact that $(b_n)_n$ frequently belongs to the sets of level and their diameter tends to nought (for the first step induction), we can prove by induction that we can construct the sequence $(M_i)_i$ verifying the said properties. Let $i \in \mathbb{N}$ and let M_i be a set so that $(b_n)_n$ frequently belongs to M_i. Let $M_{i+1} = L'_{c_i} \cap M'_{i+1}$, where $L'_{c_i} = L_{c_i} \cap M_i$ and M'_{i+1} is constructed as follow:

For the fact that $\mu(L_{c_i}) \le \frac{1}{i}$, we have $\mu(L'_{c_i}) \le \frac{1}{i}$. So there is a cover of L'_{c_i} with sets of diameter less than or equal to $\frac{1}{i}$. Let M'_{i+1} be a set of such cover to which frequently belong the terms of the sequence $(b_n)_n$. We recall that $(b_n)_n$ frequently belongs to L'_{c_i} for the fact that $(b_n)_n$ definitely belongs to L_{c_i} and frequently belongs to M_i and $L'_{c_i} = L_{c_i} \cap M_i$.

For construction $M_i \supseteq M_{i+1}$.

For the fact that M'_{i+1} has diameter less than or equal to $\frac{1}{i}$, also M_{i+1} has diameter less than or equal to $\frac{1}{i}$. So M_{i+1} has the required properties and we can construct the sequence of sets $(M_n)_n$ with the required properties. So, as we have seen, we can chose a subsequence $(b_{k_n})_n = (y_n)_n$ so that $y_n \in M_n$ and $(y_n)_n$ is a Cauchy sequence, i.e., (because of the completeness of the space) $(y_n)_n$ has a limit. Let y^* be its limit. For every $c > \inf_X(T)$, $(y_n)_n$ definitely belongs to L_c, by $\lim_{n \longrightarrow +\infty} T(b_n) = \inf_X(T)$ (being $T(b_n) \leq c_n$ and $\lim_{n \longrightarrow +\infty} c_n = \inf_X(T)$). So y^* belongs to the topological closure of L_c, i.e., $d(y^*, L_c) = 0$, for each $c > \inf_X(T)$. Thus owing to the fact that T is r.g.i. in X and therefore in y^*, $F(y^*) = \inf_X(T)$. This implies that the set $L_{\min T}$ of the global minimum points of T is nonempty. Clearly we have $L_c \supseteq L_{\min T}$ for each $c > \inf_X(T)$. Thus, $\mu(L_{\min T}) = 0$, i.e., $L_{\min T}$ is relatively compact.

Now we shall show that $L_{\min T}$ is closed and therefore is compact. If $x^* \in \mathrm{Ch}(L_{\min T})$, i.e., $d(x^*, L_{\min T}) = 0$, then we have $d(x^*, L_c) = 0$ for each $c \geq \inf_X(T)$. So, by T is r.g.i. in X, $T(x^*) = \inf_X(T)$. This concludes the proof. ■

Remark 4.2 The last theorem assures that if T is a mapping of compact metric space into itself with $\inf_X(T)) = 0$, and if $F(x) = d(x, T(x))$, $x \in X$ is an r.g.i. on X, then the fixed point set of T is nonempty and compact even when T is discontinuous.

Theorem 4.2 *Let $T : X \longrightarrow \mathbb{R}$ be a function defined on a complete metric space X such as we have $\lim_{c \longrightarrow (\inf_X(T))^+} \mathrm{diam}(L_c) = 0$. Thus T has one (and only one) global minimum point, if, and only if, it is* r.g.i.

Proof Suppose that T is r.g.i.

Because the fact that, for every set A, $\mu(A) \leq \mathrm{diam}(A)$, the hypotheses of Theorem 4.1 are verified and thus the set $L_{\min T}$ of the global minimum points of T is nonempty. But $L_{\min T}$ must have a vanishing diameter. So there is one and only one global minimum point of T.

Conversely let us suppose that there is a global minimum point x^* of T.

First note that it is the only global minimum point of T. So if x is a point of X different from x^*, we have $T(x) > T(x^*) = \inf_X(T)$. For every $c \geq \inf_X(T)$, we have $x^* \in L_c$. Let S_c be the closed sphere of center x^* and radius $\mathrm{diam}(L_c)$. For every $c \geq \inf_X(T)$, we have $S_c \supseteq L_c$. So $d(x, x^*) \geq d(x, L_c) \geq d(x, S_c) \geq d(x, x^*) - \mathrm{diam}(L_c)$. So, for $c \longrightarrow \inf_X(T)$, $d(x, L_c) \longrightarrow d(x, x^*) > 0$. This implies that T is r.g.i. ■

The following is given in [6].

Theorem 4.3 *Let* $T : X \longrightarrow \mathbb{R}$ *be an* r.g.i. *function defined on a complete metric space* X *so that we have* $\lim_{c \longrightarrow (\inf_X(T))^+} \mathrm{diam}(L_c) = 0$. *Then* T *has one and only one global minimum point.*

Proof Let $\lim_{n \longrightarrow +\infty} c_n = \inf_X(T)$ and $x_n \in L_{c_n}$, as $\lim_{n \longrightarrow \infty} \mathrm{diam}(L_{c_n}) = 0$, we have that $(x_n)_n$ is a Cauchy sequence. Thus, for the completeness of X, $(x_n)_n$ has a limit. Let us call $\underline{x}$ the limit of $(x_n)_n$. Thus $T(\underline{x}) = \inf_X(T)$. If, by reduction ad absurdum, this were not so, as T is r.g.i, it should be $d(\underline{x}, L_{c_n}) > 0$ for at least one n, in contrast to the fact that $x_n \longrightarrow \underline{x}$ and $x_m \in L_{c_n}$ for $m \geq n$. Furthermore $\lim_{c \longrightarrow (\inf_X(T))^+} \mathrm{diam}(L_c) = 0$ implies that $\mathrm{diam}(L_{\inf_X(T)}) = 0$ and therefore $\underline{x}$ is the only point of global minimum. ■

Remark 4.3 Let X be a complete metric space with a distance $d(.,.)$ and T a selfmapping in X. As observed in [5], Theorem 4.3 can be applied in the following way.

If we define $F(x) = d(x, T(x))$, T has one and only one fixed point in case the following three facts occur:

1. $\inf_X(F) = 0$,
2. $\lim_{c \longrightarrow 0^+} \mathrm{diam}(L_c) = 0$,
3. F is r.g.i. in X.

In the study of contractive-type mappings, the goal usually is to show that the Picard iterates of the mapping under consideration converges to a fixed point. Motivated by this fact and by Theorem 4.2, Angrisani and Calvelli [6] gave the following result on diameters of level sets.

Theorem 4.4 *If* X *is a metric space and* T *a selfmapping in* X *with:*

$$(**) \qquad \exists \alpha < 1 \, \forall x, y \in X$$

$$d(T(x), T(y)) \leq \alpha \max\{d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))\}$$

then as $F(x) = d(x, T(x))$ *and* $L_c = \{x, F(x) \leq c\}$, *we have* $\inf_X(F) = 0$, $\lim_{c \longrightarrow 0^+} \mathrm{diam}(L_c) = 0$ *and* F *is* r.g.i. *in* X.

Proof We first verify that $\inf_X(F) = 0$. Applying $(**)$ we have

$$\begin{aligned} &d(T^i(x), T^j(x)) \leq \\ &\alpha \max\{d(T^{i-1}(x), T^{j-1}(x)), d(T^{i-1}(x), T^i(x)), d(T^{j-1}(x), T^j(x)), \\ &d(T^{i-1}(x), T^j(x)), d(T^{j-1}(x), T^i(x))\}. \end{aligned}$$

Therefore, if $0 < i \leq j-1$, $d(T^i(x), T^j(x)) \leq \alpha \max_{i-1\leq m,n\leq j} d(T^m(x), T^n(x))$. Therefore, by induction on i, we have

$$(***) \qquad d(T^i(x), T^j(x)) \leq \alpha^i \max_{0\leq m,n\leq j} d(T^m(x), T^n(x)).$$

The maximum at the second member of the $(***)$ will be reached by certain numbers p, q so that $0 \leq p \leq q \leq j$ gives $d(T^p(x), T^q(x))$. If $p > 0$, we have

$$d(T^p(x), T^q(x)) \leq d(T^i(x), T^j(x)) \leq \alpha \max_{i-1\leq m,n\leq j} d(T^m(x), T^n(x)) d(T^i(x), T^j(x))$$

$$\leq \alpha \max_{0\leq m,n\leq j} d(T^m(x), T^n(x)) = \alpha d(T^p(x), T^q(x))$$

and therefore $d(T^i(x), T^j(x)) \leq \alpha^i d(T^p(x), T^q(x)) \leq d(T^p(x), T^q(x)) = 0$. If on the other hand $p = 0$, we have

$$d(x, T^q(x)) \leq d(x, T(x)) + d(T(x), T^q(x)) \leq d(x, T(x)) + \alpha \max_{0\leq m,n\leq j} d(T^m(x), T^n(x))$$

$$\leq d(x, T(x)) + \alpha \max_{0\leq m,n\leq j} d(T^m(x), T^n(x)) = d(x, T(x)) + \alpha d(x, T^q(x)).$$

Therefore

$$\max_{0\leq m,n\leq j} d(T^m(x), T^n(x)) = d(x, T^q(x)) \leq (\frac{1}{1-\alpha}) d(x, T(x)).$$

Therefore, if $p = 0$ or if $p > 0$, we have $d(T^i(x), T^j(x)) \leq (\frac{\alpha^i}{1-\alpha}) d(x, T(x))$ and in particular $d(T^n(x), T^{n+1}(x)) \leq (\frac{\alpha^n}{1-\alpha}) d(x, T(x))$. Consequently $\inf_X(F) = 0$.

The rest of the proof proceeds in the following way. Let $x, y \in L_c$, thus $d(x, y) \leq d(x, T(x)) + d(T(x), T(y)) + d(y, T(y)) \leq 2c + d(T(x), T(y))$

$$\begin{aligned} d(T(x), T(y)) &\leq \alpha. \max\{d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))\} \\ &\leq \alpha. \max\{d(x, y), d(x, T(x)), d(y, T(y)), d(x, y) + d(y, T(y)), d(y, x) \\ &\quad + d(x, T(x)) \\ &\leq\leq \alpha. \max\{d(x, y), c, c, d(x, y) + c, d(x, y) + c\} \leq \alpha(x, y) + c, \end{aligned}$$

therefore $d(x, y) \leq \alpha(x, y) + 3c$, and therefore $d(x, y) \leq \dfrac{3c}{1-\alpha} \longrightarrow_{c\longrightarrow 0^+} 0$, and consequently $\lim_{c\longrightarrow 0^+} \mathrm{diam}(L_c) = 0$.

Let us suppose that there is $\underline{x} \in X$ so that F is not r.g.i. in $\underline{x}$. This fact implies that there exists a sequence $(x_n)_n$ tending to $\underline{x}$ so that $x_n \in L_{\frac{1}{n}}$ (and therefore $\lim_{n \longrightarrow \infty} F(x_n) = 0$). This gives:

$$\begin{aligned} &d(\underline{x}, T(\underline{x})) - d(\underline{x}, x_n) - d(x_n, T(x_n)) \leq d(T(\underline{x}), T(x_n)) \\ &\leq \alpha. \max\{d(\underline{x}, x_n), d(\underline{x}, T(\underline{x})), d(x_n, T(x_n)), d(\underline{x}, T(x_n)), d(x_n, T(\underline{x}))\} \\ &\leq \alpha. \max\{d(\underline{x}, x_n), d(\underline{x}, T(\underline{x})), d(x_n, T(x_n)), d(\underline{x}, x_n) + d(x_n, T(x_n)), d(x_n, \underline{x}) \\ &\quad + d(\underline{x}, T(\underline{x}))\}. \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we obtain $d(\underline{x}, T(\underline{x})) \leq \alpha d(\underline{x}, T(\underline{x}))$, i.e., $F(\underline{x}) = d(\underline{x}, T(\underline{x})) = 0$, in contrast to the hypothesis that F is not r.g.i. in X, which is absurd. ■

Without continuity and compactness conditions, we obtain [6].

Corollary 4.1 *Let X be a complete metric space and T a selfmapping with*

$$d(T(x), T(y)) \leq \alpha. \max\{d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))\}. \tag{4.1}$$

Thus T has one and only one fixed point.

Proof Given that $F(x) = d(x, T(x))$, for Theorem 4.4, we have $\inf_X(F) = 0$, $\lim_{c \longrightarrow 0^+} \text{diam}(L_c) = 0$ and F is r.g.i. in X. Therefore, for Theorem 4.3, F has a single global minimum point $\underline{x}$ (on which F takes the value of 0). Therefore $d(x, T(x)) = 0$ if and only if $x = \underline{x}$, i.e., $\underline{x}$ is the only fixed point of T. ■

Remark 4.4 Many properties of "contractivity" in [168] imply (4.1).

Definition 4.2 (Orbits) Let X be a set and $T: X \longrightarrow X$. For $x, y \in X$, the orbit of T at x is

$$O(x) = \{x, T(x), T^2(x), \cdots\}$$

and

$$O(x, y) = O(x) \cup O(y).$$

Walter [192] proved a far-reaching extension of Banach's Contraction Mapping Principle. We use this fact to show that Theorem 4.2 extends to a much wider class of mappings under the additional assumption that the orbits of T are bounded. We state Walter's result below.

Theorem 4.5 *Let X be a metric space and $\phi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous nondecreasing function and satisfies $\phi(s) < s$ for $s > 0$. Assume $T: X \longrightarrow X$*

has bounded orbits and satisfies the following condition. For each $x, y \in X$,

$$d(T(x), T(y)) \leq \phi(\mathrm{diam}(O(x, y))).$$

Then T *has a unique fixed point* $z \in X$ *and* $\lim_{k \longrightarrow \infty} T^k(x) = z$ *for each* $x \in X$.

Using this fact we obtain the following where we use some ideas of Hegedüs in [82].

Theorem 4.6 *Let* X *be a complete metric space and suppose* $T: X \longrightarrow X$ *has bounded orbits and satisfies*: *there exists* $\alpha < 1$ *such that for each* $x, y \in X$,

$$d(T(x), T(y)) \leq \alpha \mathrm{diam}(O(x, y)). \quad (*)$$

Suppose $(x_n)_n \subset X$ *satisfies* $\lim_{n \longrightarrow \infty} d(x_n, T(x_n)) = 0$. *Then* T *has a unique fixed point* $z \in X$ *and* $\lim_{c \longrightarrow 0^+} \mathrm{diam}(L_c) = 0$. *Moreover,* $\lim_{n \longrightarrow \infty} d(x_n, T(x_n)) = 0$ *if and only if* $\lim_{n \longrightarrow \infty} x_n = z$.

Proof The existence of a unique fixed point z with $\lim_{n \longrightarrow \infty} T^n(x) = z$ for each $x \in X$ follows from Theorem 4.5. Let $\varepsilon > 0$ and suppose $d(u, z) \leq \varepsilon$. Then since $T(z) = z$,

$$d(u, T(u)) \leq d(u, z) + d(T(u), T(z)) \leq \varepsilon + \alpha \mathrm{diam}(O(u) \cup \{z\}).$$

Similarly, if $d(u, T(u)) \leq \varepsilon$, then

$$d(u, z) \leq d(u, T(u)) + d(T(u), T(z)) \leq \varepsilon + \alpha \mathrm{diam}(O(u) \cup \{z\}).$$

We complete the argument by showing that $\mathrm{diam}(O(u) \cup \{z\})$ depends on ε and tends to 0 as $\varepsilon \longrightarrow 0^+$. There are two cases.

1. $\mathrm{diam}(O(u) \cup \{z\}) = \sup_p d(T^p(u), z)$. In this case let $\varepsilon' > 0$ be arbitrary and choose p so that $\sup_p d(T^p(u), z) \leq d(T^p(u), z) + \varepsilon'$. Then if $p = 0$, we have

$$\mathrm{diam}(O(u) \cup \{z\}) \leq d(u, z) + \varepsilon' \leq \varepsilon + \varepsilon'$$

in which case $\mathrm{diam}(O(u) \cup \{z\}) \leq \varepsilon$ and we are finished. On the other hand, if $p \geq 1$,

$$\begin{aligned}\mathrm{diam}(O(u) \cup \{z\}) &\leq d(T^p(u), T(z)) + \varepsilon' \leq \alpha \mathrm{diam}(O(T^{p-1}(u)) \cup \{z\}) + \varepsilon' \\ &\leq \alpha \mathrm{diam}(O(u) \cup \{z\}) + \varepsilon'.\end{aligned}$$

This implies

$$\operatorname{diam}(O(u) \cup \{z\}) = 0.$$

2. $\operatorname{diam}(O(u) \cup \{z\}) = \sup_p d(T^p(u), u)$. Since $\lim_{p \longrightarrow \infty} d(T^p(u), u) = d(z, u) \leq \varepsilon$, we may assume there exists $q \geq 1$ such that $\sup_p d(T^p(u), u) = d(T^q(u), u)$, in which case we have

$$\begin{aligned}\operatorname{diam}(O(u) \cup \{z\}) &\leq d(u, z) + d(T^q(u), T(z)) \leq \alpha \operatorname{diam}(O(T^{q-1}(u)) \cup \{z\}) + \varepsilon \\ &\leq \alpha \operatorname{diam}(O(u) \cup \{z\}) + \varepsilon.\end{aligned}$$

In this case we have

$$\operatorname{diam}(O(u) \cup \{z\}) \leq \frac{\varepsilon}{1-\alpha}.$$

Therefore,

$$d(u, z) \leq \varepsilon \Longrightarrow d(u, T(u)) \leq \varepsilon + \frac{\alpha\varepsilon}{1-\alpha}$$

$$d(u, T(u)) \leq \varepsilon \Longrightarrow d(u, z) \leq \varepsilon + \frac{\alpha\varepsilon}{1-\alpha} = \frac{\varepsilon}{1-\alpha}$$

and

$$u, v \in L_c \Longrightarrow d(u, v) \leq d(u, z) + d(v, z) \leq 2\frac{\varepsilon}{1-\alpha}.$$

■

Remark 4.5 By taking $y = T(x)$ in $(*)$ one has

$$d(T(x), T^2(x)) \leq \alpha \operatorname{diam}(O(x, T(x))) = \alpha \operatorname{diam}(O(x)) \quad \text{for all } x \in X$$

and this quickly leads to

$$\operatorname{diam}(O(T(x))) \leq \alpha \operatorname{diam}(O(x)) \quad \text{for all } x \in X.$$

This can be rewritten as

$$\operatorname{diam}(O(x)) \leq (1-\alpha)^{-1}[\operatorname{diam}(O(x)) - \operatorname{diam}(O(T(x)))] \quad \text{for all } x \in X.$$

Since $d(x, T(x)) \leq \operatorname{diam}(O(x))$, if the mapping $\varphi \colon X \longrightarrow \mathbb{R}$ defined by setting $\varphi(x) = \operatorname{diam}(O(x))$ is lower semicontinuous then this condition, which is much weaker than $(*)$, assure that T has at least one fixed point by Caristi's theorem.

A natural question was addressed by Kirk and Saliga in [112].

Does the conclusion of Theorem 4.6 remain valid under the weaker assumption of Theorem 4.5? A partial answer to this question was given by Akkouchi in [2].

Let Φ be the set of continuous functions $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is nondecreasing on $\mathbb{R}^+$ and such that the mapping $x \mapsto x - \phi(x)$ from $[0, +\infty[$ onto $[0, +\infty[$ is strictly increasing. We notice that Φ contains strictly the set Φ_1 of continuous nondecreasing functions $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ and satisfying, $\psi(s) < \alpha s, \forall s > 0$, for some given $\alpha \in [0, 1[$. Akkouchi [2] gave the following example.

Example 4.1 Let $a > 0$ be a given number. For each $s > 0$, we set $\phi_a(s) := \dfrac{as}{s+a}$ and $\theta_a(s) := s - \phi_a(s) = \dfrac{s^2}{s+a}$. Then $\phi_a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing and $\phi_a(s) < s, \forall s > 0$. Moreover θ_a is a strictly increasing bijective mapping from $\mathbb{R}^+$ onto itself. Let us denote its inverse by ψ_a. Then ψ_a is given by $\psi_a(s) = \dfrac{s + \sqrt{s^2 + 4as}}{2}$, for all $s > 0$. It is easy to verify that $\phi_a \notin \Phi_1$. Indeed, there exists no number α in $[0, 1[$ such that $\phi_a(s) < \alpha s, \forall s > 0$.

The following is due to Akkouchi [2].

Theorem 4.7 *Let X be a complete metric space and suppose $T\colon X \longrightarrow X$ has bounded orbits and satisfies the following condition:*

$$d(T(x), T(y)) \leq \phi(\text{diam}(O(x, y))) \quad \textit{for all } x, y \in X,$$

where $\phi \in \Phi$. Then

1. *T has a unique fixed point $z \in X$ and $\lim_{k\longrightarrow\infty} T^k(x) = z$ for each $x \in X$.*
2. $\lim_{c\longrightarrow 0^+} \text{diam}(L_c) = 0$.
3. *For each sequence $(x_n)_n \subseteq X$, $\lim_{n\longrightarrow\infty} d(x_n, T(x_n)) = 0$ if and only if $\lim_{n\longrightarrow\infty} x_n = z$.*
4. *The map $F: x \mapsto d(x, T(x))$ is an* r.g.i. *on X.*

Proof Let $\phi \in \Phi$ and let ψ denote the inverse of the strictly increasing mapping $s \mapsto s - \phi(s)$ on the interval $[0, +\infty[$. Then 1. is a consequence of Theorem 4.5. To prove 2. and 3., we shall prove the following property:

$$\forall \varepsilon > 0, \forall u \in X,\ d(u, T(u)) \leq \varepsilon \implies d(u, z) \leq \varepsilon + \psi(\varepsilon). \tag{P}$$

Let $\varepsilon > 0$ and suppose $d(u, T(u)) \leq \varepsilon$. Then since $T(z) = z$,

$$d(u, z) \leq d(u, T(u)) + d(T(u), T(z)) \leq \psi(\text{diam}(O(u) \cup \{z\})).$$

The proof of the property (P) will be finished by showing that $\phi(\text{diam}(O(u) \cup \{z\})) \leq \psi(\varepsilon)$. To simplify the notations, we set $\tau := \text{diam}(O(u) \cup \{z\})$. We consider two cases.

(*a*) $\operatorname{diam}(O(u) \cup \{z\}) = \sup_p d(T^p(u), z)$. In this case let $\rho > 0$ be arbitrary and choose p_ρ so that $\sup_p d(T^p(u), z) \leq d(T^{p_\rho}(u), z) + \rho$. Then if $p_\rho = 0$, we have

$$\begin{aligned}\operatorname{diam}(O(u) \cup \{z\}) &\leq d(u, z) + \rho \\ &\leq d(u, T(u)) + d(T(u), T(z)) + \rho \\ &\leq \varepsilon + \phi(\operatorname{diam}(O(u) \cup \{z\})) + \rho,\end{aligned}$$

from which we get $\tau - \phi(\tau) \leq \varepsilon + \rho$. On the other hand, if $p_\rho \geq 1$,

$$\begin{aligned}\operatorname{diam}(O(u) \cup \{z\}) &\leq d(T^{p_\rho}(u), T(z)) + \rho \\ &\leq \phi(\operatorname{diam}(O(T^{p_\rho - 1}(u)) \cup \{z\})) + \rho \\ &\leq \phi(\operatorname{diam}(O(u) \cup \{z\})) + \rho.\end{aligned}$$

Hence, we get $\tau - \phi(\tau) \leq \rho$. Therefore, in the two cases, we obtain $\tau - \phi(\tau) \leq \varepsilon + \rho$, from which (since $\rho > 0$ is arbitrary) $\tau - \phi(\tau) \leq \varepsilon$. Since by assumption the function $t \mapsto t - \phi(t)$ is strictly increasing on $[0, +\infty[$ having ψ as inverse we obtain $\tau \leq \psi(\varepsilon)$. It follows that $\phi(\tau) \leq \phi \circ \psi(\varepsilon) \leq \psi(\varepsilon)$.

(*b*) $\operatorname{diam}(O(u) \cup \{z\}) = \sup_p d(T^p(u), u)$. Since $\lim_p d(T^p(u), u) = d(z, u)$, if one has $\sup_p d(T^p(u), u) = \lim_p d(T^p(u), u)$ then

$$\begin{aligned}\operatorname{diam}(O(u) \cup \{z\}) = d(z, u) &\leq d(u, T(u)) + d(T(u), T(z)) \\ &\leq \varepsilon + \phi(\operatorname{diam}(O(u) \cup \{z\})).\end{aligned}$$

Thus we get $\tau - \phi(\tau) \leq \varepsilon$, which gives as before $\phi(\tau) \leq \psi(\varepsilon)$. Hence we may assume there exists $q \leq 1$ such that $\operatorname{diam}(O(u) \cup \{z\}) = \operatorname{diam}(O(u)) = d(T^q(u), u)$. In this case we have

$$\begin{aligned}\operatorname{diam}(O(u) \cup \{z\}) &= \operatorname{diam}(O(u)) = d(T^q(u), u) \\ &\leq d(u, T(u)) + d(T(u), T^q(u)) \\ &\leq \phi(\operatorname{diam}(O(T^{q-1}(u)) \cup \{u\})) + \varepsilon \\ &\leq \phi(\operatorname{diam}(O(u)) + \varepsilon.\end{aligned}$$

Thus the number $\tau = \operatorname{diam}(O(u) \cup \{z\})$ satisfies $\tau - \phi(\tau) \leq \psi(\varepsilon)$. It follows as before that $\phi(\tau) \leq \psi(\varepsilon)$.

Therefore, taking all cases into account, we have

$$d(u, T(u)) \leq \varepsilon \implies d(u, z) \leq \varepsilon + \psi(\varepsilon).$$

Thus we have proved the property (P). We deduce

$$u, v \in L_\varepsilon \implies d(u, v) \leq d(u, z) + d(v, z) \leq 2(\varepsilon + \psi(\varepsilon)),$$

and since $\lim_{\varepsilon \to 0^+} \psi(\varepsilon) = 0$ this proves 2. and 3.. To prove 4., we use Proposition 4.1 and the property (P). This completes the proof. ■

Remark 4.6

(a) To obtain the results of Theorem 4.7, we need only to suppose that $s \mapsto s - \phi(s)$ is strictly increasing on some given interval of the type $[0, \beta]$ (with $\beta > 0$).
(b) It is easy to see that all the conclusions of Theorem 4.7 are valid for every nondecreasing continuous function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ for which there exists a number $\beta > 0$ and a positive function ψ defined on $]0, \beta]$ verifying:

  (i) $\lim_{t \to 0} \psi(t) = 0$ and
  (ii) $\forall t \in]0, \beta], \forall s > 0, s - \phi(s) \leq t \implies s \leq \psi(t)$.

The following is a variant of Theorem 4.1 where proved even more [112].

Theorem 4.8 *Let X be a complete metric space and let $T\colon X \longrightarrow \mathbb{R}$ be an* r.g.i. *for which $\inf_X(T) = c_0 \geq 0$. If $\lim_{c \longrightarrow (c_0)^+} \mu(L_c) = 0$, then the set L_{c_0} of global minimum points of T is nonempty and compact, and $\lim_{c \longrightarrow (c_0)^+} H(\overline{L_c}, L_{c_0}) = 0$. Moreover, if $(x_n)_n$ is a sequence in X for which $\lim_{n \longrightarrow \infty} T(x_n) = c_0$, then* $\lim_{n \longrightarrow \infty} \text{dist}(x_n, L_{c_0}) = 0$.

Proof Suppose T is an r.g.i. on X and let $(c_n)_n$ be a sequence of numbers for which $c_n > c_0$ and $\lim_{n \longrightarrow \infty} c_n = c_0$. In view of Proposition 4.2 $Y = \bigcap_n \overline{L}_{c_n}$ is nonempty and compact. Let $x \in Y$ and $n \in \mathbb{N}$. Since $x \in \overline{L}_{c_n}$ there exists $x_n \in L_{c_n}$ such that $d(x_n, x) \leq \frac{1}{n}$. Therefore $\lim_{n \longrightarrow \infty} x_n = x$ while

$$c_0 \leq \lim_{n \longrightarrow \infty} T(x_n) \leq \lim_{n \longrightarrow \infty} c_n = c_0.$$

By Proposition 4.1, $T(x) = \inf_X(T)$. Thus $x \in \bigcap_n \overline{L}_{c_n}$, hence $Y = \bigcap_n \overline{L}_{c_n}$, and again by Proposition 4.2 $\lim_{c \longrightarrow (c_0)^+} H(\overline{L}_{c_n}, L_{c_0}) = 0$. Now suppose $(x_n)_n \subseteq X$ satisfies $\lim_{n \longrightarrow \infty} T(x_n) = c_0$, and suppose there exists a subsequence $(y_n)_n$ of $(x_n)_n$ and a number $\rho > 0$ such that $\text{dist}(y_n, L_{c_0}) \geq \rho$. Then the condition

$$\lim_{c \longrightarrow (c_0)^+} \mu(L_c) = 0$$

implies that

$$\lim_{n \longrightarrow \infty} \mu(\{y_n, y_{n+1}, \dots\}) = 0$$

and thus $(y_n)_n$ has a subsequence which converges to $y \in X$. Since T is an r.g.i. on X this in turn implies $T(y) = c_0$, i.e., $y \in L_{c_0}$. Clearly this contradicts $\mathrm{dist}(y_n, L_{c_0}) \geq \rho$. ∎

Situations in which measures of noncompactness arise in the study of fixed point theory usually involve the study of either condensing mappings or k-set contractions. Continuity is always implicit in the definitions of these classes of mappings.

As before we let $F(x) = d(x, T(x)), x \in X$, and $L_c = \{x \in X : F(x) \leq c\}$. Now L_0 will denote the fixed point set of T. Also for a subset Y of X we use the notation

$$N_\xi(Y) = \{x \in X : d(x, y) \leq \xi \quad \text{for some } y \in Y\}.$$

It is natural to ask if in many instances it suffices to replace the continuity assumption with the weaker r.g.i. condition.

An affirmative answer in the following setting was given by Kirk and Saliga [112].

Theorem 4.9 *Let X be a complete metric space and let $T : X \longrightarrow X$ satisfy:*

1. $d(T(x), T^2(x)) \leq d(x, T(x))$ *for all* $x \in X$,
2. $\mu(T(L_c)) \leq k\mu(L_c)$ *for some* $k < 1$ *and all* $c > \inf_X(F)$,
3. F *is an* r.g.i. *on* X.

Then the set L_{c_0} of global minimum of F is nonempty and compact. Moreover, if $\inf_X(F) = 0$ and if $(x_n)_n \subseteq X$ satisfies $\lim_{n \longrightarrow \infty} d(x_n, T(x_n)) = 0$, then $\lim_{n \longrightarrow \infty} \mathrm{dist}(x_n, L_0) = 0$.

Proof Let $c > \inf_X(F)$. Then 2. implies $\lim_{n \longrightarrow \infty} \mu(T^n(L_c)) \leq \lim_{n \longrightarrow \infty} k^n \mu(L_c) = 0$. Since 1. implies $T(L_c) \subseteq L_c$. Proposition 4.2 implies

$$Y_c = \bigcap_n \overline{T^n(L_c)}$$

is nonempty and compact. Moreover $\overline{Y_c} \subseteq \overline{L_c}$. Now let $(c_n)_n$ be a sequence for which $c_n \longrightarrow (\inf_X(F))^+$ and let

$$Y = \bigcap \overline{Y_{c_n}}.$$

Then Y is nonempty and compact, and for each $y \in Y$ and $n \in \mathbb{N}$ there exists $x_n \in L_{c_n}$, such that $d(y, x_n) \leq \frac{1}{n}$. We now have $\lim_{n \longrightarrow \infty} x_n = y$ and $\lim_{n \longrightarrow \infty} F(x_n) = \inf_X(F)$. By Proposition 4.1, $y = \inf_X(F)$ and this proves that the set of global minimum points contains Y. The fact that this set is also compact follows from 2. and the fact that it is mapped into itself by T. In view of Theorem 4.8, for the final conclusion we need only show that $\inf_X(F) = 0$ implies that

$$\lim_{c \longrightarrow 0^+} \mu(L_c) = 0.$$

Assume $\lim_{c \longrightarrow 0^+} \mu(L_c) = r \geq 0$. By 1.$T : L_c \longrightarrow L_c$ for each $c > 0$ and by 2.:

$$\mu(T(L_c)) \leq k\mu(L_c),$$

whence $\lim_{c \longrightarrow 0^+} \mu(T(L_c)) \leq kr$. Now let $c > 0$ and suppose $\mu(T(L_c)) = d$. This means that for any $d' > d$ there exists a finite collection $\{Y_i\}$ of subsets of X, each having diameter $\leq d'$ and such that

$$T(L_c) \subseteq \bigcup_i Y_i.$$

If $x \in L_c$, then $d(x, T(x)) \leq c$, and since $T(x) \in Y_i$ for some i it follows that $x \in N_c(Y_i)$, i.e.,

$$L_c \subseteq \bigcup_i N_c(Y_i).$$

This in turn implies $\mu(L_c) \leq d' + c$, and since $d' > d$ is arbitrary,

$$\mu(L_c) \leq \mu(T(L_c)) + c \leq k\mu(L_c) + c.$$

Letting $c \longrightarrow 0^+$ we obtain $r \leq kr$ and this is clearly a contradiction if $r > 0$. ■

Corollary 4.2 *Let (X, d) be a complete metric space and suppose $T : X \longrightarrow X$ satisfies:*

1. $d(T(x), T^2(x)) \leq \alpha d(x, T(x))$ *for some* $\alpha \in (0, 1)$ *and all* $x \in X$,
2. $\mu(T(L_c)) \leq k\mu(L_c)$ *for some* $k < 1$ *and all* $c > 0$,
3. F *is an* r.g.i. *on* X.

Then the fixed point set $\text{Fix}(T)$ *of* T *is nonempty and compact. Moreover if* $(x_n)_n \subseteq X$ *satisfies* $\lim_{n \longrightarrow \infty} d(x_n, T(x_n)) = 0$, *then* $\lim_{n \longrightarrow \infty} \text{dist}(x_n, \text{Fix}(T)) = 0$.

Proof Condition 1. implies that $(T^n(x))_n$ is a Cauchy sequence for each $x \in X$, and in particular that $\inf_X(F) = 0$. ■

Meaningful applications of the precedent results would likely arise in a Banach space context and in such a setting more can be said.

In [112], some interesting, be they only illustrative, results have been produced.

In the ensuing statements we always take F to be $\|I - T\|$. The significance of these results again lies in the fact that continuity is not assumed.

Theorem 4.10 *Suppose K is a bounded closed convex subset of a Banach space and suppose $T : K \longrightarrow K$ satisfies*

1. $\inf_C(F) = 0$ *for any nonempty closed convex T-invariant subset C of K,*
2. $\mu(T(A)) < \mu(A)$ *for all $A \subseteq K$ for which $\mu(A) > 0$*
3. *F is an* r.g.i. *on K.*

Then the fixed point set $\mathrm{Fix}(T)$ *of T is nonempty and compact.*

Proof By a standard argument [196] it is possible to construct a nonempty closed convex subset $C \subseteq K$ for which $\overline{\mathrm{conv}}(T(C)) = C$. Since $\mu(\overline{\mathrm{conv}}(T(C))) = \mu(T(C))$, this implies $\mu(T(C)) = \mu(C)$ so in view of 2. C must be compact. 1. and the fact that F is an r.g.i. on K imply $\mathrm{Fix}(T) \cap C \neq \emptyset$. Condition 2. and the fact that F is an r.g.i. implies $\mathrm{Fix}(T)$ is compact. ■

Remark 4.7 The assumption $\inf_K(F) = 0$ is strong, especially in the absence of conditions which at the same time imply continuity of T. However there is a relatively simple condition which simultaneously yields both this fact and second assumption of 1. of Theorem 4.9.

For a convex subset K of a Banach space and $x, y \in K$ let $[x, y]$ denote the segment joining x and y, that is $[x, y] = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$.

A mapping $T : K \longrightarrow K$ is called directionally nonexpansive if $\|T(x) - T(m)\|$ for each $x \in K$ and $m \in [x, T(x)]$. If there exists $\alpha \in (0, 1)$ such that this inequality holds for $m = (1-\alpha)x+\alpha T(x)$ then we say that T is uniformly locally directionally nonexpansive.

The following is a special case of a result proved in [108].

Proposition 4.3 *Let K be bounded convex subset of a Banach space and suppose $T : K \longrightarrow K$ is uniformly locally directionally nonexpansive. Then $f = \dfrac{1}{2}(I + T)$ is asymptotically regular. In particular*

$$\inf_K \|x - T(x)\| = 0.$$

Combining this fact with Theorem 4.9 we have the following:

Theorem 4.11 *Let K be a bounded closed convex subset of a Banach space and suppose $T: K \longrightarrow K$ satisfies:*

1. *T is directionally nonexpansive on K,*
2. *$\mu(T(L_c)) \leq k\mu(L_c)$ for some $k < 1$ and all $c > 0$,*
3. *F is an* r.g.i. *on K.*

Then the fixed point set $\mathrm{Fix}(T)$ *of T is nonempty and compact. Moreover if $(x_n)_n \subseteq K$ satisfies $\lim_{n\longrightarrow\infty} \|x_n - T(x_n)\| = 0$, then $\lim_{n\longrightarrow\infty} \mathrm{dist}(x_n, \mathrm{Fix}(T)) = 0$.*

Proof Since 1. implies both $\inf_K(F) = 0$ and

$$\|T(x) - T^2(x)\| \leq \|x - T(x)\| \qquad \text{for each } x \in K,$$

the conclusion is immediate from Theorem 4.9. ∎

The following is a corollary of Theorem 4.10. (Of course if T is continuous this reduces to a special case of Sadovskii's theorem).

Corollary 4.3 *Let K be a bounded closed convex subset of a Banach space and suppose $T: K \longrightarrow K$ satisfies:*

1. *T is uniformly locally directionally nonexpansive on K,*
2. *$\mu(T(A)) < \mu(A)$ for all $A \subseteq K$ for which $\mu(A) > 0$*
3. *F is an* r.g.i. *on K.*

Then the fixed point set $\mathrm{Fix}(T)$ *of T is nonempty and compact.*

We now take up a simple application of Theorem 4.2. For this theorem we assume $\theta: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is any function for which $t \longrightarrow 0^+$ implies $\theta(t) \longrightarrow 0$.

Theorem 4.12 *Let K be a bounded closed convex subset of a Banach space and suppose $T: K \longrightarrow K$ satisfies:*

1. *T is uniformly locally directionally nonexpansive on K,*
2. *$\|T(x) - T(y)\| \leq \theta(\max\{\|T(x) - x)\|, \|y - T(y)\|\})$ for each $x, y \in K$.*

Then T has a unique fixed point $x_0 \in K$ if and only if F is an r.g.i. *on K.*

Proof By Proposition 4.3 $\inf_K(F) = 0$. Let $c > 0$ and let $x, y \in L_c$. Then by 2.,

$$\|x - y\| \leq \|T(x) - T(y)\| + 2c \leq \theta(\max\{\|T(x) - x)\|, \|y - T(y)\|\}) + 2c \longrightarrow 0$$

as $c \longrightarrow 0^+$. Thus $\lim_{c\longrightarrow 0^+} \mathrm{diam}(L_c) = 0$. ∎

Remark 4.8 We remark that the properties of mappings play the dominant role in the preceding discussion, but it is also true that the geometry of the underlying space may be a factor. In 1979, Moreau, [134] proved that if C is a closed subset of a Hilbert space H and if $T: C \longrightarrow C$ is a nonexpansive mapping whose fixed point set $\text{Fix}(T)$ has nonempty interior, then for every $x \in C$ the Picard iterates of T converge strongly to a point of $\text{Fix}(T)$. Subsequently, Beauzamy observed that this result also holds in a uniformly convex space [91] and in [114] it is shown that this fact extends even to reflexive locally uniformly convex spaces.

It was observed in [114] that in the uniformly convex case the nonexpansive assumption can be weakened. Essentially, it was shown that part of the analysis does not require the full force of nonexpansiveness, only the existence of at least one fixed point together with nonexpansiveness about each fixed point is sufficient.

Definition 4.3 A mapping $T: X \to X$ where X is a Banach space, is called quasi-nonexpansive provided T has at least fixed point in X (that is, $\text{Fix}(T)$ is nonempty), and if $p \in \text{Fix}(T)$, then

$$\|T(x) - p\| \leq \|x - p\| \text{ holds for all } x \in X.$$

This concept, which Dotson [47] has labeled quasi-nonexpansive, was essentially introduced, along, with some other related ideas, by Diaz and Metcalf [43]. It is clear that a nonexpansive mapping with at least one fixed point is quasi-nonexpansive. A linear quasi-nonexpansive mapping on a Banach space is nonexpansive on that space. But there exist continuous and discontinuous nonlinear quasi-nonexpansive mappings that are not nonexpansive. Dotson [47] gave the following example, which is continuous quasi-nonexpansive but not nonexpansive.

Example 4.2 The mapping $T: \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} \dfrac{x}{2} \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

is quasi-nonexpansive but not nonexpansive.

Following the approach in [114], the following is given in [112].

Theorem 4.13 *Let C be a closed subset of a uniformly convex Banach space and suppose $T: C \longrightarrow C$ is a mapping for which* $\text{intFix}(T) \neq \emptyset$*, and suppose also*:

1. *T is quasi-nonexpansive,*
2. *F is an* r.g.i. *on C.*

Then for each $x \in C$ the Picard sequence $(T^n(x))_n$ converges to a point of $\text{Fix}(T)$.

To prove the previous theorem we recall the following fact, due independently to Edelstein [53] and Steckin [184].

Proposition 4.4 *Let X be a uniformly convex space. Then for each $d > 0$ and for each $c, c' \in X$ satisfying $0 < \|c - c'\| = r < d$,*

$$\lim_{c \longrightarrow 0^+} \operatorname{diam}(B(c, d - r + \varepsilon) \cap (X \backslash B(c', d))) = 0.$$

Moreover, the convergence is uniform for all such c, c' lying in any bounded subset of X.

Proof of Theorem 4.13 Let $x \in C$ and $p \in \operatorname{intFix}(T)$. By 1.

$$d = \lim_{n \longrightarrow \infty} \|p - T^n(x)\|$$

always exists, and since $p \in \operatorname{intfix}(T)$, if $d = 0$ then $T^n(x) = p$ for some n and there is nothing to prove. Otherwise there exists $r > 0$, with $r < d$, and $q \in \operatorname{Fix}(T)$ such that $B(q, r) \subseteq \operatorname{Fix}(T)$. For each $n \in \mathbb{N}$ choose $q_n \in \operatorname{Fix}(T)$ so that $\|p - q_n\| = r$ and so that

$$\|p - q_n\| + \|q_n - T^n(x)\| = \|p - T^n(x)\|.$$

It follows that $\lim_{n \longrightarrow \infty} \|q_n - T^n(x)\| = d - r$. Let $\varepsilon > 0$. Then for n sufficiently large $T^n(x) \in B(q_n, d - r + \varepsilon)$. On the other hand, $T^n(x) \in \overline{X \backslash B(p, d)}$ for all $n \in \mathbb{N}$. By Proposition 4.4,

$$\lim_{\varepsilon \longrightarrow 0^+} \operatorname{diam}(B(q_n, d - r + \varepsilon) \cap (\overline{X \backslash B(p, d)})) = 0.$$

This implies $(T^n(x))_n$ is a Cauchy sequence, so there exist $z \in C$ such that $\lim_{n \longrightarrow \infty} T^n(x) = z$. At the same time $\lim_{n \longrightarrow \infty} F(T^n(x)) = 0$. Since F is an r.g.i., $T(z) = z$. ∎

Another geometric property proposed and studied by Rolewicz in [170, 171] called property (β).

Definition 4.4 Let $(X, \|.\|)$ be a Banach space and B_X its closed unit ball. Given $x \in X \setminus B_X$, the drop generated by x is the set $D(x, B_X) := \operatorname{conv}(\{x\} \cup B_x)$. Denote by $R(x, B_x) := D(x, B_X) \setminus B_X$.

The following is due to Rolewicz in [170].

Theorem 4.14 *A Banach space $(X, \|.\|)$ is uniformly convex if for any $\varepsilon > 0$ there is $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies that* $\operatorname{diam}(R(x, B_x)) < \varepsilon$.

Related to the previous result, Rolewicz introduced in [171] the following property.

Definition 4.5 A Banach space $(X, \|.\|)$ is said to have the property (β) if for any $\varepsilon > 0$ there is $\delta > 0$ so that $1 < \|x\| < 1 + \delta$ implies that $\mu(R(x, B_x)) < \varepsilon$.

Any uniformly convex space has property (β). [171]. In [110] Kirk proved the following result.

Proposition 4.5 *Suppose X is a Banach space with property (β). Then for each $d > 0$ and for each $c, c' \in X$ satisfying $0 < \|c - c'\| = r < d$,*

$$\lim_{c \longrightarrow 0^+} \mu(B(c, d - r + \varepsilon) \cap (X \backslash B(c', d))) = 0.$$

Moreover, the convergence is uniform for all such c, c' lying in any bounded subset of X.

The following convergence result is given in [112].

Theorem 4.15 *Let C be a closed convex subset of a Banach space which has the property (β). Suppose $T : C \longrightarrow C$ is a mapping for which* $\text{intFix}(T) \neq \emptyset$*, and suppose also*:

1. *T is quasi-nonexpansive,*
2. *F is an* r.g.i. *on C.*

Let $f = \dfrac{1}{2}(I + T)$. Then for each $x \in C$ the Picard sequence $(f^n(x))_n$ converges to a point of $\text{Fix}(T)$.

Proof We follow step by step the proof of Theorem 4.13 by replacing T with f and 'diam' with μ. Notice in particular that $\text{Fix}(f) = \text{Fix}(T)$ and that if 1. holds for T then it is also holds for f. Thus

$$\lim_{\varepsilon \longrightarrow 0^+} \text{diam}(B(q_n, d - r + \varepsilon) \cap (\overline{X \backslash B(p, d)})) = 0$$

implies that $(f^n(x))_n$ has a subsequence which converges to a point $z \in C$. Since f is asymptotically regular $\lim_{n \longrightarrow \infty} F(f^n(x)) = 0$, and since T is an r.g.i., $z \in \text{Fix}(T)$. ∎

Definition 4.1 is formulated in a topological space and this raises the question of whether there might be applications in a broader context. The fact that the weak topology often plays a key role in fixed point theoretic considerations in functional analysis suggests the following definition [112].

Definition 4.6 Let K be a subset of a Banach space X and let $T : X \longrightarrow \mathbb{R}$. Then T is said to be a weak regular-global-inf (weak r.g.i) at $x \in K$ if $T(x) > \inf_K(T)$ implies there exist $\varepsilon > 0$ such that $\varepsilon < T(x) - \inf_K(T)$ and a weak neighborhood N_x of x such that $T(y) > T(x) - \varepsilon$ for each $y \in N_x$. If this condition holds for each $x \in K$, then T is said to be a weak r.g.i. on K.

If the weak topology on K is metrizable, for example if X' is separable and K is weakly compact, then the analogue of Proposition 4.1 carries over [112].

Proposition 4.6 *Let K be a weakly compact subset of a separable Banach space X and let $T: K \longrightarrow \mathbb{R}$. Then T is a weak* r.g.i *on K if and only if for any sequence $(x_n)_n \subset K$, the conditions*

$$\lim_{n\longrightarrow\infty} T(x_n) = \inf_K(T) \quad \textit{and} \quad x_n \rightharpoonup x$$

imply $T(x) = \inf_K(T)$.

The following classical result in the theory of nonexpansive mappings. It was first explicitly formulated by Browder [31, 67] based on ideas of Göhde.

Theorem 4.16 *Let K be a closed convex subset of a uniformly convex Banach space $X$and let $T: K \longrightarrow X$ be nonexpansive. Then the mapping $(I - T)$ is demiclosed on K. In particular, if $\lim_{n\longrightarrow\infty} \|T(x_n) - x_n\| = 0$ and if $x_n \rightharpoonup x$, then $T(x) = x$.*

Thus under the assumptions of the above theorem $(I - T)$ is a weak-r.g.i. on K.

4.2 Synthetic Approaches to Problems of Fixed Points in Convex Metric Spaces

We give some fixed point results for mappings without a continuity condition on Takahashi convex metric spaces as an application of synthetic approaches to fixed point problems.

In the absence of linear structure, the concept of convexity can be introduced in an abstract form. In metric spaces, at first, it was done by Menger. Then Takahashi [186] introduced a new concept of convexity in metric spaces.

Definition 4.7 Let (X, d) be a metric space and I a closed unit interval. A mapping $W: X \times X \times I \longrightarrow X$ is said to be the convex structure on X if for all $x, y, u \in X, \lambda \in I$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

X together with a convex structure is called a Takahashi convex metric space (X, d, W) or abbreviated TCS.

Example 4.3 Any convex subset of a normed space is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$

Definition 4.8 ([186]) Let (X, d, W) be a TCS. A nonempty subset K of X is said to be convex if and only if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Proposition 4.7 *Let* (X, d, W) *be a* TCS. *If* $x, y \in X$ *and* $\lambda \in I$, *then*

1. $W(x, y, 1) = x$ *and* $W(x, y, 0) = y$,
2. $W(x, x, \lambda) = x$,
3. $d(x, W(x, y, \lambda)) = (1-\lambda)d(x, y)$ *and* $d(y, W(x, y, \lambda)) = \lambda d(x, y)$,
4. *balls (either open or closed) in* X *are convex,*
5. *intersections of convex subsets of* X *are convex.*

For fixed $x, y \in X$ let $[x, y] = \{W(x, y, \lambda), \lambda \in I\}$.

Definition 4.9 A TCS (X, d, W) has property *(P)* if for every $x_1, x_2, y_1, y_2 \in X, \lambda \in I$,

$$d(W(x_1, x_2, \lambda), W(y_1, y_2, \lambda)) \leq \lambda d(x_1, y_1) + (1-\lambda)d(x_2, y_2)$$

Obviously, in a normed space, the last inequality is always satisfied.

Example 4.4 ([186]) Let (X, d) be a linear metric space with the following properties:

1. $d(x, y) = d(x - y, 0)$, for all $x, y \in X$,
2. $d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x, 0) + (1-\lambda)d(y, 0)$ for all $x, y \in X$ and $\lambda \in I$.

For $W(x, y, \lambda) = \lambda x + (1-\lambda)y, x, y \in X$ and $\lambda \in I$, (X, d, W) is TCS with property *(P)*.

Remark 4.9 Property *(P)* implies that the convex structure W is continuous at least in first two variables which gives that the closure of a convex set is convex.

Definition 4.10 A TCS (X, d, W) has property *(Q)* if for any finite subset $A \subseteq X$ $\text{conv}A$ is a compact set.

Example 4.5 ([186]) Let K be a compact convex subset of a Banach space and let X be the set of all nonexpansive mappings on K into itself. Define a metric on X by $d(A, B) = \sup_{x \in K} \|A(x) - B(x)\|$, $A, B \in X$ and $W : X \times X \times I \longrightarrow X$ by $W(A, B, \lambda)(x) = \lambda A(x) + (1-\lambda)B(x)$, for $x \in K$ and $\lambda \in I$. Then (X, d, W) is a compact TCS, so X is with property *(Q)*. The property *(P)* is also satisfied.

Remark 4.10 Talman in [187] introduced a new notion of convex structure for metric space based on Takahashi notion the so called strong convex structure (SCS for short). In SCS condition *(Q)* is always satisfied so it seems to be natural.

Any TCS satisfying (P) and (Q) has the next important property [187].

Proposition 4.8 *Let* (X, d, W) *be a* TCS *with properties* (P) *and* (Q). *Then for any bounded subset* $A \subseteq X$

$$\mu(\text{conv}(A)) = \mu(A).$$

The following fixed point result for mappings without continuity condition on Takahashi convex metric space is proved in [63].

Theorem 4.17 *Let* (X, d, W) *be a* TCS *with properties* (P) *and* (Q)*,* K *a closed convex bounded subset of* X*, and* $T: K \longrightarrow K$ *a mapping satisfying the following*:

1. $\inf_C(F) = 0$ *for any nonempty closed convex* T*-invariant subset* C *of* K*, where* $F(x) = d(x, T(x)), x \in K$*,*
2. $\mu(T(A)) < \mu(A)$ *for all* $A \subseteq K$ *for which* $\mu(A) > 0$*,*
3. F *is* r.g.i. *on* K*.*

Then the fixed point set $\mathrm{Fix}(T)$ *of* T *is nonempty and compact.*

Proof Choose a point $m \in K$. Let σ denote the family of all closed convex subsets A of K for which $m \in A$ and $T(A) \subseteq A$. Since $K \in \sigma, \sigma \neq \emptyset$. Let

$$B = \bigcap_{A \in \sigma} A, \quad C = \overline{\mathrm{conv}(T(B) \cup \{m\})}. \tag{4.2}$$

Convex structure W has property *(P)* so C is a convex set as a closure of convex set. We are going to prove that $B = C$. Since B is a closed convex set containing $T(B)$ and $\{m\}$, $C \subset B$. This implies that $T(C) \subseteq T(B) \subseteq C$ so $C \in \sigma$ and hence $B \subseteq C$. The last two statements clearly force $B = C$. The properties of measure μ and Proposition 4.8 imply that

$$\mu(B) = \mu(\overline{\mathrm{conv}(T(B) \cup \{m\})}) = \mu(B), \tag{4.3}$$

so in view of 2. B must be compact. Now, Proposition 4.10 ensures that T has a fixed point on B so $\mathrm{Fix}(T)$ is nonempty. Condition 2. implies that $\mathrm{Fix}(T)$ is totally bounded. Since F is r.g.i. $\mathrm{Fix}(T)$ has to be closed. Finally, we conclude that $\mathrm{Fix}(T)$ is compact. ■

The assumption $\inf_K(F) = 0$ is strong, especially in the absence of conditions which at the same time imply continuity. Some sufficient conditions which are easier to check and more suitable for application are given in [63].

We recall some well-known definitions.

Definition 4.11 The mapping $T: K \longrightarrow K$ is called directionally nonexpansive if we have $d(T(x), T(y)) \leq d(x, y)$ for all $x \in K$ and $y \in [x, T(x)]$. If there exists $\alpha \in (0, 1)$ such that this inequality holds for $y = W(T(x), x, \alpha)$, then we say that T is uniformly locally directionally nonexpansive.

The following is given in [63] and its elaborate proof is taken from [108].

Proposition 4.9 *Let* (X, d, W) *be a* TCS *with property* (P)*,* K *a closed convex bounded subset of* X*, and* $T: K \longrightarrow K$ *a uniformly locally directionally nonexpansive. Let* $T_\alpha x = W(T(x), x, \alpha)$*. For the fixed* $x_0 \in K$*, sequences* $(x_n)_n$

and $(y_n)_n$ *are defined as follows*:

$$x_{n+1} = T_\alpha x_n, \quad y_n = T(x_n), \ n = 0, 1, 2, \ldots \tag{4.4}$$

Then for each $i, n \in \mathbb{N}$

$$d(y_{i+n}, x_i) \geq (1-\alpha)^{-n}(d(y_{i+n}, x_{i+n}) - d(y_i, x_i)) + (1+n\alpha)d(y_i, x_i), \tag{4.5}$$

$$\lim_{n \longrightarrow \infty} d(T(x_n), x_n) = 0. \tag{4.6}$$

Proof We prove (4.5) by induction on n. For $n = 0$ inequality (4.5) is trivial. Assume that (4.5) holds for given n and all i. In order to prove that (4.5) holds for $n+1$, we proceed as follows: replacing i with $i+1$ in (4.5) yields

$$d(y_{i+1+n}, x_{i+1}) \geq (1-\alpha)^{-n}(d(y_{i+n+1}, x_{i+n+1}) - d(y_{i+1}, x_{i+1})) \tag{4.7}$$
$$+(1+n\alpha)d(y_{i+1}, x_{i+1}).$$

Also

$$\begin{aligned} d(y_{i+1+n}, x_{i+1}) &\leq d(y_{i+n+1}, W(y_{i+n+1}, x_i, \alpha)) \\ &+ d(W(y_{i+n+1}, x_i, \alpha), W(T(x_i), x_i, \alpha)) \\ &\leq (1-\alpha)d(y_{i+n+1}, x_i) + \alpha d(y_{i+n+1}, T(x_i)) \leq (1-\alpha)d(y_{i+n+1}, x_i) \\ &+ \alpha \sum_{k=0}^{n} d(T(x_{i+1+k}), T(x_{i+k})) \\ &\leq (1-\alpha)d(y_{i+n+1}, x_i) + \alpha \sum_{k=0}^{n} d(x_{i+1+k}, x_{i+k}) \end{aligned} \tag{4.8}$$

since $x_{i+1+k} = W(T(x_{i+k}), x_{i+k}, \alpha)$ and T is uniformly locally directionally nonexpansive. Combining (4.7) and (4.8)

$$d(y_{i+1+n}, x_i) \geq (1-\alpha)^{-(n+1)}(d(y_{i+n+1}, x_{i+n+1}) - d(y_{i+1}, x_{i+1})) \tag{4.9}$$
$$+ (1-\alpha)^{-1}(1+n\alpha)d(y_{i+1}, x_{i+1}) - \alpha(1-\alpha)^{-1} \sum_{k=0}^{n} d(x_{i+1+k}, x_{i+k}).$$

By Proposition 4.7 3.,

$$d(x_{i+1+k}, x_{i+k}) = d(W(T(x_{i+k}), x_{i+k}, \alpha), x_{i+k}) = \alpha d(y_{i+k}, x_{i+k}), \tag{4.10}$$

so

$$d(y_{i+1+n}, x_i) \geq (1-\alpha)^{-(n+1)}(d(y_{i+n+1}, x_{i+n+1}) - d(y_{i+1}, x_{i+1})) \tag{4.11}$$

$$+(1-\alpha)^{-1}(1+n\alpha)d(y_{i+1}, x_{i+1}) - \alpha^2(1-\alpha)^{-1}\sum_{k=0}^{n} d(y_{i+k}, x_{i+k}).$$

On the other hand,

$$\begin{aligned} d(y_n, x_n) &= d(T(x_n), W(T(x_{n-1}), x_{n-1}, \alpha)) \leq d(T(x_n), T(x_{n-1})) \\ &+ d(T(x_{n-1}), W(T(x_{n-1}), x_{n-1}, \alpha)) \\ &\leq d(x_n, x_{n-1}) + (1-\alpha)d(T(x_{n-1}), x_{n-1}) \\ &= \alpha d(y_{n-1}, x_{n-1}) + (1-\alpha)d(y_{n-1}, x_{n-1}) \\ &= d(y_{n-1}, x_{n-1}) \end{aligned} \tag{4.12}$$

for any $n \in \mathbb{N}$, meaning that $(d(y_n, x_n))_n$ is a decreasing sequence. Now, using inequality $(1+n\alpha) - (1-\alpha)^{-n} \leq 0$, we have that

$$\begin{aligned} d(y_{i+n+1}, x_i) &\geq (1-\alpha)^{-(n+1)}(d(y_{i+n+1}, x_{i+n+1}) - d(y_{i+1}, x_{i+1})) \\ &+ (1-\alpha)^{-1}(1+n\alpha)d(y_{i+1}, x_{i+1}) - \alpha^{-2}(1-\alpha)^{-1}(n+1)d(y_i, x_i) \\ &= (1-\alpha)^{-(n+1)}(d(y_{i+n+1}, x_{i+n+1}) - d(y_i, x_i)) \\ &+ ((1-\alpha)^{-1}(1+n\alpha) - (1-\alpha)^{-(n+1)})d(y_{i+1}, x_{i+1}) \\ &+ ((1-\alpha)^{-(n+1)} - \alpha^{-2}(1-\alpha)^{-1}(n+1))d(y_i, x_i) \\ &\geq (1-\alpha)^{-(n+1)}(d(y_{i+n+1}, x_{i+n+1}) - d(y_i, x_i)) \\ &+ ((1-\alpha)^{-1}(1+n\alpha) - (1-\alpha)^{-(n+1)})d(y_i, x_i) \\ &+ ((1-\alpha)^{-(n+1)} - \alpha^{-2}(1-\alpha)^{-1}(n+1))d(y_i, x_i) \\ &= (1-\alpha)^{-(n+1)}(d(y_{i+n+1}, x_{i+n+1}) - d(y_i, x_i)) + (1+(n+1)\alpha)d(y_i, x_i). \end{aligned} \tag{4.13}$$

Thus (4.5) holds for $n+1$, completing the proof of the inequality. Further, the sequence $(d(y_n, x_n))_n$ is decreasing, so there exists $\lim\limits_{n \longrightarrow \infty} d(y_n, x_n) = r \geq 0$. Let us suppose that $r > 0$. Select a positive integer k such that

$$0 \leq d(y_k, x_k) - d(y_{k+n_0}, x_{k+n_0}) < \varepsilon.$$

Using (4.5), we obtain

$$d + r \le r(1 + \alpha n_0) \le (1 + \alpha n_0) d(y_k, x_k)$$

$$\le d(y_{k+n_0}, x_k) + \varepsilon(1-\alpha)^{-n_0} < d + r.$$

By the last contradiction we conclude that $r = 0$ and $\lim\limits_{n\longrightarrow\infty} d(y_n, x_n) = d(T(x_n), x_n) = 0$ what we had to prove. ■

Combining the last result with Theorem 4.17 we have the following consequence.

Corollary 4.4 *Let K be a closed convex bounded subset of complete* TCS (X, d, W) *with properties* (P) *and* (Q) *and let $T : K \longrightarrow K$ satisfy the following*:

1. *T is uniformly locally directionally nonexpansive on K,*
2. *$\mu(T(A)) < \mu(A)$ for all $A \subseteq K$ for which $\mu(A) > 0$,*
3. *F is* r.g.i. *on K.*

Then the fixed set $\text{Fix}(T)$ *of T is nonempty and compact.*

Moreover, using Proposition 4.9 we also get some other fixed point results.

Corollary 4.5 *Let K be a closed convex bounded subset of complete* TCS (X, d, W) *with properties* (P) *and* (Q) *and let $T : K \longrightarrow K$ satisfy the following*:

1. *T is uniformly locally directionally nonexpansive on K,*
2. *$d(T(x), T(y)) \le \theta(\max\{d(x, T(x)), d(y, T(y))\})$, where $\theta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is any function for which $\lim\limits_{t\longrightarrow 0^+} \theta(t) = 0$.*

Then T has a unique fixed point $x_0 \in K$ if and only if F is r.g.i. *on K.*

Proof Proposition 4.9 gives $\inf\limits_K(F) = 0$ one can prove that $\lim\limits_{c\longrightarrow 0^+} \text{diam}(L_c) = 0$. By Theorem 4.2, T has a unique fixed point if and only if F is r.g.i. on K. ■

Theorem 4.18 *Let K be a closed convex bounded subset of complete* TCS (X, d, W) *with properties* (P) *and* (Q) *and let $T : K \longrightarrow K$ satisfy the following*:

1. *T is directionally nonexpansive on K,*
2. *$\mu(T(L_c)) \le k\mu(L_c)$, for some $k < 1$ and all $c > 0$,*
3. *F is* r.g.i. *on K.*

Then the fixed set $\text{Fix}(T)$ *of T is nonempty and compact. Moreover, if $(x_n)_n \subseteq K$ satisfies $\lim\limits_{n\longrightarrow\infty} d(T(x_n), x_n) = 0$, then $\lim\limits_{n\longrightarrow\infty} d(\text{Fix}(T), x_n) = 0$.*

Proof By Proposition 4.9, $\inf\limits_K(F) = 0$. Since 1. implies that

$$d(T(x), T^2(x)) \le d(x, T(x)), \ \forall x \in K,$$

the conclusion follows immediately from Theorem 4.9. ■

Next, we recall the concept of weakly quasi-nonexpansive mappings with respect to a sequence introduced by Ahmed and Zeyada in [1].

Definition 4.12 Let (X, d) be a metric space and let $(x_n)_n$ be a sequence in $Y \subseteq X$. Assume that $T: Y \longrightarrow X$ is a mapping with $\text{Fix}(T) \neq \emptyset$ satisfying $\lim_{n\to\infty} d(x_n, \text{Fix}(T)) = 0$. Thus, for a given $\varepsilon > 0$ there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $d(x_n, \text{Fix}(T)) < \varepsilon$ for all $n \geq n_1(\varepsilon)$. Mapping T is called weakly quasi-nonexpansive mapping with respect to $(x_n)_n \subseteq Y$ if for every $\varepsilon > 0$ there exists $p(\varepsilon) \in \text{Fix}(T)$ such that for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$, $d(x_n, p(\varepsilon)) < \varepsilon$.

Theorem 4.19 *Let K be a closed convex bounded subset of complete* TCS (X, d, W) *with properties* (P) *and* (Q) *and let $T: K \longrightarrow K$ satisfy the following*:

1. *T is directionally nonexpansive on K,*
2. *$\mu(T(L_c)) \leq k\mu(L_c)$, for some $k < 1$ and all $c > 0$,*
3. *F is* r.g.i. *on K.*
4. *$(x_n)_n \subseteq K$ satisfies $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and T is weakly quasi-nonexpansive with respect to $(x_n)_n$.*

Then $(x_n)_n$ converges to a point in $\text{Fix}(T)$.

Proof Our assertion is a consequence of Theorem 4.18. ■

Using Proposition 4.9, the next corollary holds.

Corollary 4.6 *Let K be a closed convex bounded subset of complete* TCS (X, d, W) *with properties* (P) *and* (Q) *and let $T: K \longrightarrow K$ satisfy the following*:

1. *T is directionally nonexpansive on K,*
2. *$\mu(T(L_c)) \leq k\mu(L_c)$, for some $k < 1$ and all $c > 0$,*
3. *F is* r.g.i. *on K.*
4. *T is weakly quasi-nonexpansive with respect to sequence $x_n = T_\alpha^n(x_0), n \in \mathbb{N}, x_0 \in K, \alpha \in (0, 1)$.*

Then $(x_n)_n$ converges to a point in $\text{Fix}(T)$.

4.3 Approximation of Fixed Points by Means of Functions Convergent with Continuity

In this section, we show that some results on fixed point theorems in particular for nonexpansive mappings can be obtained based on convergence with continuity (by sequence).

For results in this section we refer to [6].

Definition 4.13 Let X and Y two metric spaces. Let $(f_n)_n$ be a sequence of functions from X into Y.

The sequence $(f_n)_n$ converges with continuity at f in an element x of X if and only if for each sequence in X, $(x_n)_n$ convergent at x, we have $\lim_{n \longrightarrow \infty} f_n(x_n) = f(x)$.

It is said that $(f_n)_n$ at f in X if and only if converges with continuity in each element of X.

Remark 4.11 Given two metric spaces X, Y there are non-continuous functions from X into Y if, and only if, what follows is true: X is not discrete from the topological point of view and Y has strictly cardinality than one.

Theorem 4.20 *If there is a non-continuous function g from the metric space X into the metric space Y, then we have:*

1. *The uniform convergence at a function f of a sequence of functions $(f_n)_n$ from X into Y on compact subsets of X does not imply that $(f_n)_n$ converges with continuity.*
2. *The convergence with continuity is not topological on the set of functions from X into Y (i.e., there is not topology with respect to which the convergence with continuity is the convergence).*

Proof

1. We observe that the convergence with continuity at a function h implies the point convergence at h. As $(f_n)_n$ point converges at f and $(f_n)_n$ does not converge with continuity at f, therefore $(f_n)_n$ does not converge with continuity. In fact if $(f_n)_n$ converges with continuity at a function h, this means that $(f_n)_n$ point converges at h and therefore $h = f$ (for the uniqueness of the limit point convergence for functions at values in a Hausdorff space), therefore $(f_n)_n$ converge with continuity at f, which is absurd. Thus $(f_n)_n$ does not converge with continuity and 1. is proved.
2. Take $f = g$ and $f_n = g$ for each $n \in \mathbb{N}$. A sequence of functions all equal to g converges uniformly at g in all the subsets of X, and in particular on the compact subsets, therefore $(f_n)_n$ converges uniformly in all the compact subsets of X at $f = g$. The fact that g is not continuous means that there exists in X at least one sequence $(x_n)_n$ convergent at an element $x \in X$, so that $(g(x_n))_n$ does not converge in Y at $g(x)$, i.e., $(f_n(x_n))_n$ does not converge at $f(x)$. This means in particular that $(f_n)_n$ does not converge with continuity at f in x, therefore it does not converge with continuity at f in X. Therefore, there is a constant sequence which does not converge with continuity and thus the convergence with continuity is not topological on the set of functions from X to Y.

■

Proposition 4.10 *Let g be a continuous function of X in X'. If the sequence $(f_n)_n$ converges with continuity at f in X', the sequence $(f_n \circ g)_n$ converges with continuity at $f \circ g$ in X.*

Proof $\lim_{n\longrightarrow\infty} x_n = x \implies \lim_{n\longrightarrow\infty} g(x_n) = g(x) \implies \lim_{n\longrightarrow\infty} f_n \circ g(x_n) = \lim_{n\longrightarrow\infty} f_n(g(x_n)) = f(g(x)) = f \circ g(x)$. ■

From Proposition 4.10 there comes immediately:

Proposition 4.11 *Let g be a continuous function of X in X. If the sequence of functions $(f_n)_n$ converges with continuity at the identity function, the sequence $(f_n \circ g)_n$ converges with continuity at g in X.*

Lemma 4.1 *If a sequence $(f_n)_n$ converges with continuity at f in an element x of a metric space X and $(x_m)_m$ is any sequence in X that tends to x, then*

$$\lim_{n,m\longrightarrow\infty} f_n(x_m) = f(x), \text{ i.e., } \forall\, \varepsilon > 0\ \exists\, k\ \forall\, n, m > k\, |f_n(x_m) - f(x)| < \varepsilon$$

(obviously the opposite is also true).

Proof Let us suppose that the thesis is untrue, there will thus exist a sequence $(x_m)_m$ tending to x, so that $\forall\ k\ \exists\ n, m > k\, |f_n(x_m) - f(x)| \geq \varepsilon$, and therefore there will exist sequences $(n_h)_h$ and $(m_h)_h$ so that $n_{h+1}, m_{h+1} > n_h$ and $\left|f_{n_h}(x_{m_h}) - f(x)\right| \geq \varepsilon$.

The sequence $(y_p)_p$ is therefore defined as follows: if k exists, so that $p = n_k$, we have $y_p = x_{m_k}$, otherwise $y_p = x$. By definition $(y_p)_p$ tends to x, while $(f_n(y_n))$ does not tend to $f(x)$ in so far as it has a subsequence whose terms do not belong to a neighborhood of $f(x)$, i, contrast to the hypothesis that $(f_n)_n$ converges with continuity at f. ■

Note in the above paragraph, when speaking of convergence with continuity, it has never been assumed that the function f and the functions of the sequence $(f_n)_n$ are continuous. The following proposition shows that f is necessarily continuous (even if the f_n are not continuous).

One also points out that, to be sure of the continuity of f at a point x, it is not sufficient to assume that $(f_n)_n$ converges with continuity at f in the single point x.

Proposition 4.12 *If the sequence of functions $(f_n)_n$ converges with continuity at function f in the metric space X, function f is continuous in X.*

Proof We shall prove that for each sequence $(x_n)_n$ tending to any $x \in X$, $(f(x_n))_n$ tends to $f(x)$ (which is the continuity of f). By Lemma 4.1 we have $\forall\ \varepsilon > 0\ \exists\ k\ \forall\ n, m > k\, |f_m(x_n) - f(x)| < \varepsilon$, and carrying out the limit on m, taking into account the convergence with continuity of $(f_n)_n$ at f in x_n, we have $\forall\, \varepsilon > 0\ \exists\ k\ \forall n > k\ |f(x_n) - f(x)| \leq \varepsilon$, i.e., $(f(x_n))_n$ tends to $f(x)$. ■

Theorem 4.21 *Let T be a continuous selfmapping in metric space X. Let $(T_n)_n$ be a sequence of selfmappings in X which converges with continuity at T in X. It is assumed that there exists a sequence $(x_n)_n$ so that:*

1. *for each $n \in \mathbb{N}$, x_n is a fixed point of T_n.*
2. *$(x_n)_n$ has a subsequence converging at an element $x \in X$.*

Thus x is a fixed point of the selfmapping of T.

Proof Let $(x_{n_j})_j$ be a subsequence of $(x_n)_n$ which converges at x. Thus $x = \lim_{j \longrightarrow \infty} x_{n_j} = \lim_{j \longrightarrow \infty} T_{n_j}(x_{n_j}) = T(x)$. ∎

Clearly, if it is assumed that space X is compact, all sequences $(x_n)_n$ have a convergent subsequence.

The following Lemma and Theorems come from Theorem 4.21.

Lemma 4.2 *Let X be a compact metric space with a distance $d(.,.)$ and T be a continuous selfmapping in X. Let G be a family of selfmappings in X, so that there exists a sequence of functions in G which converges with continuity at the identity function and so that, if $g \in G$ then $g \circ T$ has a fixed point in X.*

Proof Let $(g_n)_n$ be a sequence of functions of G convergent with continuity at the identity function. By Proposition 4.11, the sequence $(g_n \circ T)_n$ converges with continuity at T. Let x_n be a fixed point of $g_n \circ T$ (which exists because g_n belong to G), as X is compact $(x_n)_n$ has a convergent subsequence. The result comes from Theorem 4.21. ∎

Lemma 4.3 (Guseman and Peters) *Let X ba compact metric space with a distance $d(.,.)$ and T a continuous selfmapping in X. Let G be a family of selfmappings in X, so that there exists a sequence of functions in G which converges uniformly at the identity function and so that, if $g \in G$ then $g \circ T$ has a fixed point in X. Thus T has a fixed point in X [74].*

Proof Let $(g_n)_n$ be a sequence of functions in G converging uniformly at the identity function. $(g_n)_n$ converges with continuity at the identity function, since for each sequence $(x_n)_n$ in X converging at anyone element $x \in X$, we have:

$$d(g_n(x_n), x) \leq d(g_n(x_n), x_n) + d(x_n, x).$$

The result comes from Lemma 4.2. ∎

Theorem 4.22 *Let X ba compact metric space with a distance $d(.,.)$. If the identity function is the limit point of a sequence $(T_n)_n$ of weakly contractive selfmappings in X ($d(T_n(x), T_n(y)) < d(x, y)$ per $x \neq y, x, y \in X$), thus each nonexpansive selfmapping in X has a fixed point.*

Proof From the compactness of X and from the fact that $(T_n)_n$ is a sequence of weakly contractive selfmappings which converge punctually at the identity, it comes that $(T_n)_n$ converges uniformly at the identity. In fact, let $\varepsilon > 0$, and let $R_{\frac{\varepsilon}{3}}$ be the set of spherical neighborhoods of the amplitude $R_{\frac{\varepsilon}{3}}$ having as their centre anyone element of X and let $S_{\frac{\varepsilon}{3}}$ be a finite undercover of $R_{\frac{\varepsilon}{3}}$, let $x_1, \ldots, x_m$ be the centers of the spherical neighborhoods of $S_{\frac{\varepsilon}{3}}$, thus we have, on the basis of the hypotheses made, according to the way in which $x_1, \ldots, x_m$ have been defined and by the triangular inequality, we have:

$$\forall x \in X \, \exists i \in \{1, \ldots, m\} d(T_n(x), x) \leq d(T_n(x), T_n(x_i)) + d(x_i, T_n(x_i)) + d(x_i, x)$$

$$2d(x_i, x) + d(x_i, T_n(x_i)) \leq 2\frac{\varepsilon}{3} + d(x_i, T_n(x_i)).$$

Now, since it can only take on a finite number of values and since we are considering the point convergence, we definitely have (with respect to n), $\forall i \in \{1, \ldots, m\}\, d(x_i, T_n(x_i)) \leq \frac{\varepsilon}{3}$. Therefore, for each $x \in X$, definitely with the respect to n, we have $d(T_n(x), x) < 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, i.e., $(T_n)_n$ converges uniformly at identity. By Edelstein's theorem [51], and because of the fact that the composition of a weakly contractive selfmappings and of a nonexpansive selfmapping is weakly contractive, it comes that the functions $T_n \circ t$ (where T_n is a term of the sequence and t is a nonexpansive selfmapping) have a fixed point. Thus the result comes from Lemma 4.3. ■

Theorem 4.22 is simply Smart's theorem [181].

Theorem 4.23 (Smart's Theorem) *If in a compact metric space the identity selfmapping can be approximated uniformly to weakly contractive selfmappings, then each nonexpansive selfmapping has a fixed point.*

Remark 4.12 It should be observed that the expression of Theorem 4.23 point convergence can be substituted for uniform convergence (i.e., the first part of the proof of Theorem 4.22).

Smart's theorem according to which one should be able to approximate the identity function employing weakly contractive selfmappings, constitutes a first step in the weakening of the classical hypothesis of convexity in normed spaces. In these spaces in fact Smart's hypotheses by "starshaped sets".

Definition 4.14 A subset S of a vectorial space is said to be if there exists an element $p \in S$, so that $\forall x \in S \, \forall t \in [0, 1], (1 - t)p + tx \in S$, and such a p is called a star centre of S.

Lemma 4.4 *Let X be a normed space and $d(.,.)$ the distance that comes from the norm. If S is a starshaped subset of X, each nonexpansive selfmapping in S will have at least one fixed point.*

Proof The identity is uniformly approximate by the sequence $(f_n)_n$ given by $f_n(x) = x + \frac{1}{n}(p - x)$ where p is a star centre of S. (The convergence is uniform because S, being compact, is limited). The $f_n's$ are weakly contractive, in fact

$$d(f_n(x), f_n(y)) = \|x + \frac{1}{n}(p - x) - (y + \frac{1}{n}(p - y))\| =$$

$$\|(1 - \frac{1}{n})(x - y)\| = (1 - \frac{1}{n})\|x - y\| = (1 - \frac{1}{n})d(x, y) < d(x, y).$$

So the result comes from `Smart's` theorem. ■

We shall now express a theorem whose proof comes directly from Theorem 4.22.

Theorem 4.24 *Let X be a compact metric space with a distance $d(.,.)$ and let a function f be from* $[0, 1) \times X$ *in X so that:*

1. $\lim_{t \longrightarrow 1} f(t, x) = x \ \forall x \in X,$
2. $d(f(t, x), f(t, y)) < d(x, y)$ *per* $x \neq y, x, y \in X, t \in [0, 1).$

Thus every nonexpansive selfmapping has a fixed point.

Theorem 4.25 *Let X be a vectorial and metric space with a distance $d(.,.)$ invariable for translation and such as to make $d(tx, ty) = td(x, y)$. Let S be a starshaped and compact subset of X. Thus every nonexpansive selfmapping in S has a fixed point.*

Proof A function f can be defined so as to satisfy the hypotheses of Theorem 4.24, as follows: $f(t, x) = tx + (1 - t)P$, where P is a fixed star centre. Note that 2. comes from the invariability by translation of the distance. ■

Chapter 5
Approximate Fixed Points in Topological Vector Spaces

In this chapter, we study problems concerning approximate fixed point property on an ambient space with different topologies.

5.1 The KKM Principle and Almost Fixed Points

A multifunction $T: X \to 2^Y$ is a map with the values $T(x) \subseteq Y$ for $x \in X$ and the fibers $T^{-1}(y) = \{x \in X : y \in T(x)\}$ for $y \in Y$. For topological spaces X and Y a multifunction T is upper semicontinuous (u.s.c.) if for each closed set $Z \subseteq Y$, $T^{-1}(Z) = \{x \in X: T(x) \cap Z \neq \emptyset\}$ is closed in X, lower semicontinuous (l.s.c.) if for each open set $Z \subseteq Y$, $T^{-1}(Z) = \{x \in X: T(x) \cap Z \neq \emptyset\}$ is open in X, and compact if $T(X) = \bigcup\{T(x): x \in X\}$ is contained in a compact subset Y. T is said to be closed if its graph $\mathrm{Gr}(T) = \{(x, y): x \in X, y \in T(x)\}$ is closed in $X \times Y$.

If $T: X \to 2^Y$ is u.s.c. with compact values, then T is closed. The converse is true whenever Y is compact.

We have the following Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem [117].

Theorem 5.1 (KKM Principle) *Let Y be the set of vertices of a simplex S and $T: Y \to 2^S$ a multifunction with closed values such that*

$$\mathrm{conv} Z \subseteq T(Z) \text{ for each } Z \subseteq Y. \tag{5.1}$$

Then

$$\bigcap_{z \in Z} T(Z) \neq \emptyset.$$

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The following easily follows from the KKM principle [56, 117].

Lemma 5.1 *Let Y be a subset of a topological vector space, Z a nonempty subset of Y such that* $\text{conv}Z \subseteq Y$*, and* $T\colon Z \to 2^Y$ *a* KKM *map with closed (resp. open) values. Then* $\{T(z)\}_{z\in Z}$ *has the finite intersection property.*

Note that a multifunction $T\colon Z \to 2^Y$ is called a KKM map of

$$\text{conv}(N) \subseteq T(N) \text{ for each finite subset } N of Z.$$

In [3], Alexandroff and Pasynkoff gave an elementary proof of the essentiality of the identity map of the boundary of a simplex by using a variant of the KKM theorem. From Lemma 5.1, the following generalization of the Alexandroff-Pasynkoff theorem is given in [145].

Theorem 5.2 *Let Y be a subset of a topological vector space, $(Z_i)_{0\leq i\leq n}$ a family of $(n+1)$ closed (respectively, open) subsets covering Y, and $(x_i)_{0\leq i\leq n}$ a family of Y such that* $\text{conv}(\{x_i, i = 0, \cdots, n\}) \subseteq X$ *and* $\text{conv}(\{x_0, \cdots, \widehat{x_i}, \cdots, x_n\}) \subseteq Z_i$ *for each $i = 0, \cdots, n$. Then $\bigcap_{i=0}^{n} Z_i \neq \emptyset$.*

Proof Let $S := \{x_i, i = 0, \cdots, n\}$ and let $W_0 := \text{conv}(\{x_0, \cdots, x_{n-1}\}) \subseteq Z_n$ and $W_i := \text{conv}(\{x_0, \cdots, \widehat{x_{i-1}}, \cdots, x_n\}) \subseteq Z_{i-1}$ for $1 \leq i \leq n$. Let $T\colon S \to 2^Y$ be a map defined by $T(x_0) = Z_n$ and $T(x_i) = Z_{i-1}$ for $1 \leq i \leq n$. Now we show that T satisfies the requirement of Lemma 5.1. Note that

$$\text{conv}(\{x_0, \cdots, x_n\}) \subseteq Y = \bigcup_{i=0}^{n} Z_i = T(S).$$

Moreover, for any proper subset $\{x_{i_0}, \cdots, x_{i_k}\}$, $(0 \leq k < n, 0 \leq i_0 < \cdots < i_k \leq n)$ of S, we immediately have $\text{conv}(\{x_{i_0}, \cdots, x_{i_k}\}) \subseteq W_{i_j} \subseteq Z_{i_j-1} = T(x_{i_j})$ for some $j, 0 \leq j \leq k$, (with the convention $i_j = 0$ if and only if $i_j - 1 \equiv n$) and hence

$$\text{conv}(\{x_{i_0}, \cdots, x_{i_k}\}) \subseteq \bigcup_{j=0}^{k} T(x_{i_j}).$$

Consequently, condition (5.1) is satisfied. Now, the conclusion follows from Lemma 5.1. ■

It is well-known that the Alexandroff-Pasynkoff theorem implies the Brouwer theorem [147]. Therefore, Theorem 5.1 is also equivalent to the KKM theorem.

The following concept is well known [83].

Definition 5.1 Let X be a vector space endowed with a linear topology τ. A nonempty subset Y of X is said to be almost convex if, for any neighborhood V of

θ and for any finite set $\{x_1, \cdots, x_n\} \subseteq Y$, there exists a finite set $\{z_1, \cdots, z_n\} \subseteq Y$ such that

$$\text{conv}(\{z_1, \cdots, z_n\}) \subseteq Y \text{ and } z_i - x_i \in V \text{ for all } i = 1, \cdots, n.$$

We give some examples of almost convex sets:

(1) Any convex subset is almost convex.
(2) If we delete a certain subset of the boundary of a closed convex set, then we obtain an almost convex set.
(3) Let $C[0, 1]$ be the Banach space of all continuous real functions defined on the unit interval [0, 1] and $P[0, 1]$ its dense subset consisting of all polynomials. Then any subset of $C[0, 1]$ containing $P[0, 1]$ is almost convex. In general, by the various forms of the Stone-Weierstrass approximation theorem, we have a lot of examples of almost convex sets.

Proposition 5.1 *Let (X, τ) be a topological vector space. If $Y \subseteq X$ is an almost convex set, then Y is the dense subset of* $\text{conv}(Y)$.

Corollary 5.1 *Let (X, τ) be a topological vector space and $Y \subseteq X$ is an almost convex set. Then the closure of Y is a convex set.*

Definition 5.2 For a subset Y of a topological vector space X, a multifunction $T: Y \to 2^X$ is said to have (convexly) almost fixed point property if for any (convex) neighborhood V of the origin θ, there exists an $x_V \in Y$ such that $T(x_V) \cap (x_V + V) \neq \emptyset$.

Theorem 5.3 *Let Y be a subset of a topological vector space, and $T: Y \to 2^Y$ a closed compact multifunction. Then the following are equivalent:*

(i) T has a fixed point.
(ii) T has the almost fixed point property.

Proof $(i) \Rightarrow (ii)$: Clear.

$(ii) \Rightarrow (i)$: For each neighborhood U of θ, then exist $x_U, y_U \in Y$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(Y)$ is relatively compact, we can choose a subnet of the net (y_U) with a cluster point $x_0 \in \overline{T(Y)}$. Since X is Hausdorff, the corresponding subnet of (x_U) also has the cluster point x_0. Because the graph of T is closed in $Y \times \overline{T(Y)}$, we have $x_0 \in T(x_0)$. This completes our proof. ■

Theorem 5.4 *Let Y be a subset of a locally convex topological vector space, and $T: Y \to 2^Y$ a closed compact multifunction. Then the following are equivalent:*

(i) T has a fixed point.
(ii) T has the convexly almost fixed point property.

Proof In a locally convex topological vector space, the convexly almost fixed point property is equivalent to the almost fixed point property. ■

Remark 5.1

(1) If T is not compact, then $(ii) \not\Rightarrow (i)$. For example, let $X = \mathbb{R}$ and $T(x) := \left\{\dfrac{x-1}{x}\right\}$ if $x \neq 0$, $T(0) := \{1, -1\}$.

(2) If T is not closed, then $(ii) \not\Rightarrow (i)$. For example, let $X = [0, 1]$ and $T(x) := \{\frac{1}{2}\}$ if $x \neq \frac{1}{2}$, $T(\frac{1}{2}) := \{0, 1\}$.

From Lemma 5.1, Park gave the following very general almost fixed point theorem [145].

Theorem 5.5 *Let Y be a subset of a topological vector space X and Z an almost convex dense of Y. Let $T : Y \to 2^X$ be a lower (respectively, upper) semicontinuous multifunction such that $T(z)$ is convex for all $z \in Z$. If there is a precompact subset K of Y such that $T(z) \cap K \neq \emptyset$ for each $z \in Z$, then for any convex neighborhood U of the origin θ of X, there exists a point $x_U \in Z$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Proof There exists a symmetric neighborhood V of θ such that $\overline{V} + \overline{V} \subseteq U$. Since K is precompact in X, there exists a finite subset $\{x_0, \cdots, x_n\} \subseteq K$ such that $K \subseteq \bigcup_{i=0}^{n}(x_i + V)$. Moreover, since Z is almost convex and dense in X, there exists a finite subset $S = \{z_0, \cdots, z_n\}$ of Z such that $z_i - x_i \in V$ for each $i = 0, \cdots, n$, and $W = \mathrm{conv}(\{z_0, \cdots, z_n\}) \subseteq Z$.

If T is lower semicontinuous, for each i, let

$$F(z_i) := \Big\{w \in W : T(w) \cap (x_i + V) = \emptyset\Big\},$$

which is closed in W. Moreover, we have

$$\bigcap_{i=0}^{n} F(z_i) = \Big\{w \in W : T(w) \bigcap \bigcup_{i=0}^{n}(x_i + V) = \emptyset\Big\} = \emptyset.$$

since $\emptyset \neq T(w) \cap K \subseteq T(w) \bigcap \bigcup_{i=0}^{n}(x_i + V)$ for each $w \in Z$.

If T is upper semicontinuous, for each i, let

$$F(z_i) := \Big\{w \in W : T(w) \cap (x_i + \overline{V}) = \emptyset\Big\},$$

which is open in W. Moreover, we have $\bigcap_{i=0}^{n} F(z_i) = \emptyset$ as in the above.

Now we apply Lemma 5.1 replacing (Y, Z) by $(W, \{z_0, \cdots, z_n\})$. Since the conclusion of Lemma 5.1 does not hold, in any case, condition (5.1) is violated.

Hence, there exist a subset $N := \{z_{i_0}, \cdots, z_{i_k}\} \in \langle S \rangle$ and an $x_U \in \mathrm{conv}(N) \subseteq Z$ such that $x_U \notin F(N)$ or $T(x_U) \cap (x_{i_j} + \overline{V}) \neq \emptyset$ for all $j = 0 \cdots, k$. Note that

$$x_{i_j} + \overline{V} = x_{i_j} - z_{i_j} + z_{i_j} + \overline{V} \subseteq z_{i_j} + V + \overline{V} \subseteq z_{i_j} + U. \tag{5.2}$$

Let L be a subspace of X generated by S and

$$M := \{z \in L \colon T(x_U) \cap (z + U) \neq \emptyset\}.$$

From (5.2) we get $T(x_U) \cap (z_{i_j} + U) \neq \emptyset$ and hence $z_{i_j} \in M$ for all $j = 0 \cdots, k$. Since L, $T(x_U)$, and U are all convex, it is easily checked that M is convex. Therefore, $x_U \in M$ and, by definition of M, we get $T(x_U) \cap (x_U + U) \neq \emptyset$. ■

In the case $Z = Y$, Theorem 5.5 reduces to the following:

Corollary 5.2 *Let Y be a convex subset of a topological vector space X. Let $T \colon Y \to 2^X$ be a lower (respectively, upper) semicontinuous multifunction such that $T(x)$ is convex for all $x \in Y$. If there is a precompact subset K of Y such that $T(x) \cap K \neq \emptyset$ for each $x \in Y$, then T has the convexly almost fixed point property.*

Ky Fan [58] obtained Corollary 5.2 for a locally convex topological vector space X and for lower semicontinuous multifunction $T \colon Y \to 2^X$. Lassonde [124] obtained Corollary 5.2 for a compact upper semicontinuous multifunction $T \colon Y \to 2^Y$ having nonempty convex values.

The following fixed point result is due to Park and Tan [148] and extends the Himmelberg-Idzik theorem and many other fixed point results in the analytical fixed point theory.

Corollary 5.3 *Let Y be a subset of a locally convex topological vector space X and Z an almost convex dense subset of Y. Let $T \colon Y \to 2^Y$ be a compact upper semicontinuous multifunction with closed values such that $T(z)$ is nonempty convex for all $z \in Z$. Then T has a fixed point $x_0 \in X$, that is, $x_0 \in T(x_0)$.*

Proof By Theorem 5.5, for each neighborhood U of θ, there exist $x_U, y_U \in Y$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(Y)$ is relatively compact, we may assume that the net (y_U) converges to some $x_0 \in \overline{T(Y)} \subseteq Y$. Since X is Hausdorff, the net (x_U) also converges to x_0. Because T is upper semicontinuous with closed values, the graph of T is closed in $Y \times T(Y)$ and hence we have $x_0 \in T(x_0)$. ■

From Theorem 5.5, we have the following almost fixed point result [145].

Corollary 5.4 *Let Y be a subset of a topological vector space X and Z an almost convex dense subset of Y. Let $T \colon Y \to 2^X$ be a multifunction such that*

(1) *$T^-(y)$ is open for each $y \in X$, and*
(2) *$T(z)$ is convex for each $z \in Z$.*

If there is a precompact subset K of Y such that $T(z) \cap K \neq \emptyset$ for each $z \in Z$, then for any convex neighborhood U of the origin θ of X, there exists a point $x_U \in Z$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Proof Since T is lower semicontinuous, Corollary 5.4 follows immediately from Theorem 5.5. ■

When $X = Y$, Corollary 5.4 reduces to the following:

Corollary 5.5 *Let Y be a convex subset of a topological vector space X, and $T: Y \to 2^Y$ be a multifunction such that*

(1) *$T(x)$ is nonempty and convex for each $x \in Y$,*
(2) *$T^-(y)$ is open for each $y \in Y$, and*
(3) *$T(X)$ is contained in a compact subset K of Y.*

Then T has the convexly almost fixed point property.

Ben-El-Mechaiekh [17, 20] obtained that, if X is further to be locally convex in Corollary 5.5, then T has a fixed point, and conjectured that, under the hypotheses of Corollary 5.5, T would have a fixed point, so Corollary 5.5 is a partial solution.

We need the following definitions and examples with great importance in fixed point theory.

A subset K of a topological vector space X is said of the Zima type, by Hadžić [76], if for each neighborhood U of $\theta \in X$ there exists a neighborhood V of $\theta \in X$ such that $\operatorname{conv}(V \cap (K - K)) \subseteq U$.

A set $Y \subseteq X$ is said to be convexly totally bounded, by Idzik [88], if for every neighborhood V of $\theta \in X$ there exist a finite subset $\{x_i : i \in I\} \subseteq Y$ and a finite family of convex sets $\{Z_i : i \in I\}$ such that $Z_i \subseteq V$ for each $i \in I$ and $Y \subseteq \bigcup\{x_i + Z_i : i \in I\}$. Note that $\{x_i : i \in I\}$ can be chosen in X [89].

Idzik [88] gave examples of c.t.b. sets:

(1) Every compact set in a locally convex topological vector space.
(2) Any compact set in a topological vector space which is locally convex or is of the Zima type.

Other examples of c.t.b. sets were given in [41]:

(3) Every compact convex subset of $X = l_p$, $0 < p < 1$.
(4) More generally, every compact convex subset of a topological vector space X on which its topological dual X' separates points.

The well-known Schauder conjecture is as follows:

every continuous function, from a compact convex subset in a topological vector space into itself, would have a fixed point.

One of the most general partial solutions is due to Idzik using the concept of c.t.b. sets.

Theorem 5.6 *Let Y be a convex subset of a topological vector space X and $T: Y \to 2^Y$ be a Kakutani map (that is,* u.s.c. *with nonempty compact convex* *values). If $\overline{T(Y)}$ is a compact* c.t.b. *subset of Y, then there exists an $x \in Y$ such that $x \in T(x)$.*

Further, Idzik [88] raised the following question:

Is every compact convex subset of a topological vector space convexly totally bounded?

A positive answer to this question would resolve the Schauder conjecture. However, Idzik's problem was resolved negatively by the following [41].

Theorem 5.7 *For $0 \leq p < 1$, the space $L_p(\mu)$, where μ denotes the Lebesgue measure on $[0, 1]$, contains compact convex subsets which are not c.t.b.*

Moreover, Weber introduced the following definition of strongly convexly totally bounded sets [193, 194].

Definition 5.3 A subset Y of a topological vector space X is said to be strongly convexly totally bounded (s.c.t.b.) if for every neighborhood V of $\theta \in X$ there exist a convex subset Z of V and a finite subset N of Z such that $Y \subseteq N + Z$.

The following is known [193].

Theorem 5.8 *Let Y be a compact convex subset of a topological vector space (X, τ) and $Z = \mathrm{span} Y$. Then the following conditions are equivalent:*

(1) *Y is* s.c.t.b.
(2) *Y is of Zima type.*
(3) *Y is locally convex.*
(4) *Y is affinely embeddable in a locally convex topological vector space.*
(5) *X admits a Hausdorff locally convex linear topology $\sigma = \sigma(X, X')$, which induces on Z a finer topology than τ such that $\sigma|_Y = \tau|_Y$.*

Further, Weber [193] raised the following question:

Is every convex c.t.b. set s.c.t.b.?

The following almost fixed point results for multifunctions having totally bounded ranges were established by Park [146], where the closures of the ranges satisfy more restrictive conditions than that of c.t.b. sets.

Theorem 5.9 *Let Y be a convex subset of a topological vector space X and $T : Y \to 2^X$ a* u.s.c. *multifunction with convex values. If there is an* s.c.t.b. *subset Z of Y such that $T(x) \cap Z \neq \emptyset$ for each $x \in Y$, then T has the almost fixed point property.*

Proof For any neighborhood V of $\theta \in X$, choose a symmetric open neighborhood U of θ such that $\overline{U} \subseteq V$. Since Z is s.c.t.b. in X, there exist a finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq Z \subseteq Y$ and a convex subset $W \subseteq U$ such that $Z \subseteq \bigcup_{i=1}^{n}(x_i + W)$.

For each i, let

$$F(x_i) := \{x \in Y : T(x) \cap (x_i + \overline{W}) = \emptyset\}.$$

Then each $F(x_i)$ is open since T is u.s.c. Moreover, we have

$$\bigcap_{i}^{n} F(x_i) = \left\{ x \in Y : T(x) \cap \bigcup_{i}^{n} (x_i + \overline{W}) = \emptyset \right\} = \emptyset$$

since $\emptyset \neq T(x) \cap Z \subseteq T(x) \cap \bigcup_{i=1}^{n} (x_i + \overline{W})$ for each $x \in Y$.

Now we apply Lemma 5.1 with $\{x_1, \cdots, x_n\}$. Since the conclusion of Lemma 5.1 does not hold, in any case, condition (5.1) is violated. Hence, there exist a subset $N := \{x_{i_1}, \cdots, x_{i_k}\} \subseteq \{x_1, \cdots, x_n\}$ and an $x_V \in \mathrm{conv}(N) \subseteq Y$ such that $x_V \notin F(N)$ or $T(x_V) \cap (x_{i_j} + \overline{W}) \neq \emptyset$ for all $j = 1 \cdots, k$. Let L be the subspace of X generated by $\{x_1, \cdots, x_n\}$, and

$$M := \{y \in L : T(x_V) \cap (y + \overline{W}) \neq \emptyset\}.$$

Note that $N \subseteq M$. Since L, $T(x_V)$, and $\overline{W}$ are all convex, it is easily checked that M is convex. Therefore, $x_V \in \mathrm{conv}N \subseteq \mathrm{conv}M = M$ and, by definition of M, we get $T(x_V) \cap (x_V + \overline{W}) \neq \emptyset$. This shows that $T(x_V) \cap (x_V + V) \neq \emptyset$. This completes our proof. ■

Corollary 5.6 *Let Y be a compact convex subset of a topological vector space X satisfying one of the conditions* (1)-(5) *of Theorem 5.8. Let $T : Y \to 2^X$ a* u.s.c. *multifunction with convex values such that $T(x) \cap Y \neq \emptyset$ for each $x \in Y$, then T has the almost fixed point property.*

Let X be a convex subset of a vector space and Y a topological space. Motivated by earlier works, Chang and Yen [37] defined the following:

$T \in \mathrm{KKM}(X, Y) \Leftrightarrow T : X \to 2^Y$ is a multifunction such that the family $\{S(x) : x \in X\}$ has the finite intersection property whenever $S : X \to 2^Y$ has closed values and $T(\mathrm{conv}(Z)) \subseteq S(Z)$ for each nonempty finite subset Z of X.

The following is another almost fixed point result.

Theorem 5.10 *Let Y be a convex subset of a topological vector space X and $T \in$* $\mathrm{KKM}(Y, \overline{Y})$. *If $\overline{T(Y)}$ is totally bounded, then T has the convexly almost fixed point property.*

Proof For any convex neighborhood V of $\theta \in X$, we have an open convex neighborhood Ω of θ such that $\Omega \subseteq V$ and a nonempty finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq \overline{T(Y)}$ such that $\overline{T(Y)} \subseteq \bigcup_{i=1}^{n} (x_i + \Omega)$. We may assume that $\{x_1, x_2, \cdots, x_n\} \subseteq Y$. In fact, let U be a symmetric neighborhood of $\theta \in X$ such that $U + U \subseteq \Omega$. Suppose that $\{y_1, y_2, \cdots, y_n\} \subseteq \overline{T(Y)} \subseteq \overline{Y}$ and W is an open convex neighborhood of θ such that $\overline{T(Y)} \subseteq \bigcup_{i=1}^{n}(y_i + W)$ and $W \subseteq U$. Since $\{y_i + W : i = 1, \cdots, n\}$ is an open cover of $\overline{T(Y)} \subseteq \overline{Y}$, we have $(y_i + W) \cap Y \neq \emptyset$ for each i. Choose an $x_i \in (y_i + W) \cap Y$ for each i. Then $\overline{T(Y)} \subseteq \bigcup_{i=1}^{n}(x_i + (y_i - x_i) + W)$ and the open convex set $y_i - x_i + W \subseteq -W + W \subseteq U + U \subseteq \Omega$. Then, $\overline{T(Y)} \subseteq \bigcup_{i=1}^{n}(x_i + \Omega)$ and $x_i \in Y$ for each i. Let us define a multifunction $F : Y \to 2^{\overline{Y}}$ by

$$F(x) := \overline{T(Y)} \setminus (x + \Omega) \quad \text{for each } x \in Y.$$

Then F is closed-valued and

$$\bigcap_{i=1}^{n} F(x_i) = \overline{T(Y)} \setminus \bigcup_{i=1}^{n}(x_i + \Omega) = \emptyset.$$

Since $T \in \mathrm{KKM}(Y, \overline{Y})$ and $\{F(x) : x \in Y\}$ does not have the finite intersection property, we have $T(\mathrm{conv}Z) \not\subseteq F(Z)$ for a nonempty finite subset $Z \subseteq Y$. Therefore, there exist $x_V \in \mathrm{conv}Z \subseteq Y$ and $y_V \in T(x_V) \subseteq \overline{T(Y)}$ such that

$$y_V \notin F(z) = \overline{T(Y)} \setminus (z + \Omega) \quad \text{for all } z \in Z.$$

Therefore, $y_V \in z+\Omega$ for all $z \in Z$. Let $Z := \{z_1, z_2, \cdots, z_m\}$ and $x_V := \sum_{j=1}^{m} \lambda_j z_j$, where $\lambda_j \geq 0$ and $\sum_{j=1}^{m} \lambda_j = 1$. Then $y_V = (\sum_{j=1}^{m} \lambda_j) y_V \in \sum_{j=1}^{m} \lambda_j z_j + \sum_{j=1}^{m} \lambda_j \Omega \subseteq x_V + \Omega \subseteq x_V + V$. Therefore, $y_V \in T(x_V) \cap (x_V + V) \neq \emptyset$. This completes our proof. ■

Note that a particular form of Theorem 5.10 was obtained by Chang and Yen [37].

From Theorem 5.10, we have the following corollary.

Corollary 5.7 *Let Y be a convex subset of a locally convex topological vector space X and $T : Y \to \overline{Y}$ a continuous map such that $\overline{T(Y)}$ is totally bounded. Then T has the almost fixed point property.*

Proof Note that a continuous map $T : Y \to \overline{Y}$ belongs to $\mathrm{KKM}(Y, \overline{Y})$. ■

We remark that we can derive Corollary 5.7 directly from the KKM principle just by following the proof of Theorem 5.9.

Corollary 5.8 *Let Y be a convex subset of a locally convex topological vector space X and $T: Y \to Y$ a continuous map. If Y is totally bounded, then T has the almost fixed point property.*

Proof Note that $T: Y \to Y$ can be regarded as $T: Y \to \overline{Y}$. Since $T(Y) \subseteq Y$ is totally bounded, so is $\overline{T(Y)}$. Now, the conclusion follows from Corollary 5.7. ■

Theorem 5.9 can be applied to obtain fixed point theorems [146].

Lemma 5.2 *Let Y be a subset of a topological vector space X, K a compact subset Y, and $T: Y \to 2^Y$ a closed map with nonempty values. If, for any neighborhood V of $\theta \in X$, there exists an $x_V \in Y$ such that $K \cap T(x_V) \cap (x_V + V) \neq \emptyset$, then T has a fixed point $x_0 \in K$, that is, $x_0 \in T(x_0)$.*

Proof For each neighborhood V of $\theta \in X$, there exists a $y_V \in K$ such that $y_V \in T(x_V) \cap (x_V + V)$. Since K is compact, we may assume that the net (y_V) converges to some $x_0 \in K \subseteq Y$. Since Y is Hausdorff and $y_V \in x_V + V$, the net (x_V) also converges to x_0. Since $(x_V, y_V) \in \mathrm{Gr}(T)$ and $\mathrm{Gr}(T)$ is closed, we have $(x_0, x_0) \in \mathrm{Gr}(T)$. This completes our proof. ■

We have the following from Theorem 5.9 and Lemma 5.2:

Theorem 5.11 *Let Y be a subset of a topological vector space X and $T: Y \to 2^Y$ a compact* u.s.c. *multifunction with nonempty closed values. If $\overline{T(Y)}$ is an* s.c.t.b. *subset of Y, then T has a fixed point $x_0 \in Y$.*

Proof Note that T is closed and $K := \overline{T(Y)}$ is a compact subset of Y. By Theorem 5.3, any closed compact multifunction having the almost fixed point property has a fixed point. ■

Note that Theorem 5.11 is a particular case of Theorem 5.6 of Idzik. However, its proof is based on the KKM principle only, and is more easily accessible.

Corollary 5.9 *Let Y be a compact convex subset of a topological vector space satisfying one of the conditions* (1)–(5) *of Theorem 5.8. Then any Kakutani map $T: Y \to 2^Y$ has a fixed point.*

The following is proved in [105].

Theorem 5.12 *Let Y be a nonempty subset of a topological vector space X, V a convex neighborhood of θ in X, and $T: Y \to 2^X$ a multifunction with convex values. Suppose that there is a finite subset $Z := \{x_1, x_2, \cdots, x_n\}$ of Y such that* $\mathrm{conv} Z \subseteq Y$ *and $T(Y) \subseteq \bigcup_{i=1}^{n} (x_i + V)$. If one of the following conditions is satisfied:*

1. T is upper semicontinuous and V is closed,
2. T is lower semicontinuous and V is open,

then T has a V-fixed point, that is, there exists a point $x_0 \in Y$ such that $T(x_0) \cap (x_0 + V) \neq \emptyset$.

Proof We will prove the result only for the case 1. A similar argument establishes the result for the case 2. Suppose that $T: Y \to 2^X$ is an upper semicontinuous multifunction with convex values and V is a convex closed neighborhood of θ in X. Define a multifunction $F: Y \to 2^X$ by

$$F(x_i) = \Big\{x \in X : T(x) \cap (x_i + V) = \emptyset\Big\} \text{ for each } x_i \in Z.$$

Then F has open values in X since T is upper semicontinuous. Note that

$$\bigcap_{i=1}^{n} F(x_i) = \Big\{x \in X : T(x) \bigcap \bigcup_{i=1}^{n} (x_i + V) = \emptyset\Big\} = \emptyset.$$

By Lemma 5.1, $F: Y \to 2^X$ is not a KKM map, that is, there is a finite subset $A := \{y_1, y_2, \cdots, y_n\}$ of Z such that $\text{conv}(Z) \nsubseteq F(A)$. Hence there is an $x_0 \in \text{conv}(Z)$ such that $x_0 \notin F(y_i)$ or $T(x_0) \cap (y_j + V) \neq \emptyset$ for all $j = 1, 2, \cdots, m$. Let $x_0 = \sum_{j=1}^{m} r_j y_j$ with $0 \le r_j \le 1$ and $\sum_{j=1}^{m} r_j = 1$. Since $z_j \in T(x_0) \cap (y_j + V)$ for some z_j, $j = 1, 2, \cdots, m$ and the sets $T(x_0)$ and V are convex, we conclude that

$$z_0 := \sum_{j=1}^{m} r_j z_j \in T(x_0) \cap (x_0 + V), \text{ that is } T(x_0) \cap (x_0 + V) \neq \emptyset.$$

This completes the proof. ■

Corollary 5.10 *Let Y be a nonempty subset of a topological vector space X, V a convex open (or closed) neighborhood of θ in X, and $T: Y \to X$ a continuous map. If there is a finite subset $Z := \{x_1, x_2, \cdots, x_n\}$ of X such that* $\text{conv}(Z) \subseteq Y$ *and $T(Y) \subseteq \bigcup_{i=1}^{n} (x_i + V)$, then T has a V-fixed point $x_0 \in X$, that is, $T(x_0) \in x_0 + V$.*

Corollary 5.11 *Let Y be a nonempty convex subset of a topological vector space X and $T: Y \to Y$ a continuous map. If Y is totally bounded, then T has the convexly almost fixed point property.*

Proof Since X is totally bounded and $T(X) \subseteq X$, the conclusion follows immediately from Corollary 5.10. ■

The following is given in [105]

Theorem 5.13 *Let Y be a nonempty subset of a topological vector space X and $T: Y \to 2^{\overline{Y}}$ an upper semicontinuous multifunction with convex values such that $T(Y)$ is totally bounded. Then for any convex closed neighborhood U of θ in X, T has a U-fixed point $x_U \in X$.*

Similarly, if T is lower semicontinuous, then T has a U-fixed point for any convex open neighborhood U of θ in X.

Proof By symmetry, it suffices to show the result for the upper semicontinuous multifunction T. Let $T: Y \to 2^{\overline{Y}}$ an upper semicontinuous multifunction with convex values and U a convex closed neighborhood of θ in X. Then there exists a neighborhood V of θ in X such that $V + V \subseteq U$. Since $T(Y)$ is totally bounded, there is a finite subset $\{y_1, y_2, \cdots, y_n\}$ of $T(Y)$ such that $T(Y) \subseteq \bigcup_{i=1}^{n}(y_i + V)$. For each $i \in \{1, 2, \cdots, n\}$ we can choose an $x_i \in X$ such that $y_i - x_i \in V$. From this it follows that

$$T(Y) \subseteq \bigcup_{i=1}^{n}(y_i + V) \subseteq \bigcup_{i=1}^{n}(x_i + V + V) \subseteq \bigcup_{i=1}^{n}(x_i + U).$$

By Theorem 5.12, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes the proof. ■

Corollary 5.12 *Let Y be a nonempty convex subset of a topological vector space X and $T: Y \to \overline{Y}$ a continuous map such that $T(Y)$ is totally bounded. Then T has the convexly almost fixed point property.*

If X is a locally convex or a metrizable topological vector space whose balls are convex, then Corollaries 5.11 and 5.12 hold for any neighborhood U of θ in X. Note that Corollaries 5.11 and 5.12 generalize Theorem 2.17.

Corollary 5.12 does not guarantee the existence of fixed points of T as illustrated in the following example [105].

Example 5.1 Let $Y = \{(x, y): x^2 + y^2 < 1\}$ be the open unit disk in $\mathbb{R}^2$ and $T: Y \to \overline{Y}$ defined by $T(x, y) := (x, \sqrt{1 - x^2})$ for each $(x, y) \in Y$. Then the continuous map T has no fixed point.

However, the following celebrated Himmelberg fixed point theorem [83] is deduced from Theorem 5.13.

Theorem 5.14 *Let Y be a nonempty convex subset of a locally convex topological vector space X and $T: Y \to 2^Y$ a compact upper semicontinuous multifunction with convex closed values. Then T has a fixed point $x_0 \in Y$, that is, $x_0 \in T(x_0)$.*

Proof For any closed neighborhood U of θ in X, by Theorem 5.13, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$, say $y_U \in T(x_U) \cap (x_U + U)$. Since T is compact and $y_U \in \overline{T(X)} \subseteq X$, we may suppose that the net (y_U) converges to some point $x_0 \in X$. By the Hausdorffness of X, the net (x_U) also converges to x_0. Since the graph of T is closed, we have $x_0 \in T(x_0)$. This completes the proof. ■

The following interesting corollaries are worth mentioning [105].

Corollary 5.13 *Let Y be a nonempty convex subset of a locally convex topological vector space X and $T: Y \to Y$ a compact continuous map. Then T has a fixed point.*

Corollary 5.13 was due to Hukuhara [86] with different proof, and includes fixed point theorem due to Brouwer (for an n-simplex Y), Schauder (for a normed vector space X), and Tychonoff (for a compact convex subset Y).

We have one more

Corollary 5.14 *Let Y be a nonempty convex subset of a metrizable topological vector space X whose balls are convex and $T: Y \to Y$ a compact continuous map. Then T has a fixed point.*

If X itself is compact, then Corollary 5.14 reduces to a result of Rassias [162].

Note that, since the KKM principle is equivalent to the Brouwer fixed point theorem, each of Corollaries 5.10, 5.11, 5.12, 5.13, and 5.14 is also equivalent to the Brouwer theorem.

As an application of Corollary 5.10, the following almost fixed point theorem in a normed space is proved in [105].

Theorem 5.15 *Let Y be a convex subset of a normed vector space X and $T: Y \to N_\varepsilon(Y) := \{y \in X : \inf\{\|x - y\| : x \in Y\} \le \varepsilon\}$ a continuous map which has totally bounded range, where ε is a positive real number. Then $\inf\{\|x - T(x)\| : x \in Y\} \le \varepsilon$.*

Proof For any natural n, $V_n := \{z \in X : \|z\| \le \varepsilon + \frac{1}{n}\}$. Since $T(Y)$ is covered by the family $\{x + V_n : x \in Y\}$ and totally bounded, there exists a finite subset $\{x_1, x_2, \cdots, x_k\}$ of Y such that $T(Y) \subseteq \bigcup_{i=1}^{k}(x_i + V_n)$. Therefore, by Corollary 5.10, T has V_n-fixed point $x_0^n \in Y$, that is, $T(x_0^n) \in x_0^n + V_n$ or $\|x_0^n - T(x_0^n)\| \le \varepsilon + \frac{1}{n}$. Therefore, we have the conclusion. ∎

Note that Kirk [109] obtained Theorem 5.15 for the case when Y is a closed convex subset of a Banach space.

We give a few definitions.

For topological spaces X and Y, an admissible class $\mathcal{U}_c^k(Y, Z)$ of maps $F: Y \to 2^Z$ is one such that, for each F and each nonempty compact subset K of X, there exists a map $G \in \mathcal{U}_c(Y, Z)$ satisfying $G(x) \subseteq F(x)$ for all $x \in K$, where $\mathcal{U}_c$ consists of finite compositions of maps in a class $\mathcal{U}$ of maps satisfying the following properties:

(i) $\mathcal{U}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions,

(ii) each $T \in \mathcal{U}_c$ is upper semicontinuous (u.s.c.) with nonempty compact values, and

(iii) for any polytope P, each $T \in \mathcal{U}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

The better admissible class $\mathcal{B}$ of multifunctions defined from a convex set Y to a topological space Z is defined as follows:

$F \in \mathcal{B}(Y, Z) \Leftrightarrow F: Y \to 2^Z$ is a multifunction such that for any polytope P in Y and any continuous map $f: F(P) \to P$, $f \circ (F|_P) \to 2^P$ has a fixed point.

Subclasses of $\mathcal{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps (u.s.c. with nonempty compact convex values and codomains are convex spaces), the Aronszajn multifunctions $\mathbb{M}$ (u.s.c. with R_δ values) [73], the acyclic multifunctions $\mathbb{V}$ (u.s.c. with compact acyclic values), the powers multifunctions $\mathbb{V}_c$ (finite compositions of acyclic multifunctions), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed) [73], the approachable multifunctions $\mathbb{A}$ (whose domains and codomains are uniform spaces) [18, 19], admissible multifunctions of Górniewicz [70], σ-selectional multifunctions of Haddad and Lasry, permissible multifunctions of Dzedzej [50], the class $\mathbb{K}_c^+$ of Lassonde[125], the class $\mathbb{V}_c^+$ of Park, Singh, and Watson [149], and approximable multifunctions of Ben-El-Mechaiekh and Idzik, and many others.

These subclasses are all examples of the admissible class $\mathcal{U}_c^k$. Some examples of multifunctions in $\mathcal{B}$ not belonging to $\mathcal{U}_c^k$ are known.

The following is known [143, 144].

Lemma 5.3 *Let Y be a convex subset of a topological vector space X and Z a Hausdorff space. Then*

(1) $\mathcal{U}_c^k(Y, Z) \subseteq \mathrm{KKM}(Y, Z)$, *and*
(2) *in the class of closed compact multifunctions, two subclasses $\mathcal{B}(Y, Z)$ and $\mathrm{KKM}(Y, Z)$ coincide.*

It should be noted that there are only a few examples of multifunctions in $\mathrm{KKM}(Y, Z)$ which are not in $\mathcal{U}_c^k$ or $\mathcal{B}$ [37].

The following fixed point result is given in [144, 146].

Theorem 5.16 *Let Y be a convex subset of a locally convex topological vector space X and $T \in \mathcal{B}(Y, Y)$. If T is closed and compact, then T has a fixed point.*

Proof By Theorem 5.3, any closed compact multifunction having the almost fixed point property has a fixed point. Therefore, Theorem 5.16 follows from Theorem 5.10 and Lemma 5.3. ■

Comparing Theorem 5.16 with Theorem 5.6, a fixed point theorem is given for a much more general class of multifunctions under a more restrictive condition on the space itself than Idzik's.

Theorem 5.16 contains fixed point theorems due to Himmelberg [83], Lassonde [123], Park [142], Park et al. [149], Chang and Yen [37], and many others. Consequently, different proofs of those known results are given.

Given Theorem 5.8, we have the following [146].

Corollary 5.15 *Let Y be a compact convex subset of a topological vector space and Let $T \in \mathcal{B}(Y, Y)$ be a closed multifunction. Then T has a fixed point if one of the following equivalent conditions hold*:

(1) *Y is* s.c.t.b.
(2) *Y is of Zima type.*
(3) *Y is locally convex.*

The following is a generalized version of Fort's theorem for better admissible multifunctions on balls of a normed vector space [105].

Theorem 5.17 *Let X be a normed vector space and $B = \{x \in X : \|x\| < d\}$ for some $d > 0$. Let $T \in \mathcal{B}(B, \overline{B})$ be a closed multifunction. If T maps each smaller concentric ball to a compact set in $\overline{B}$, then for any $\varepsilon > 0$, there exists an $x_0 \in B$ and a $y_0 \in T(x_0)$ such that $\|x_0 - y_0\| \leq \varepsilon$.*

Proof Let $\varepsilon > 0$ be given. We may assume $\varepsilon < d$. Let $C = \{x \in B : \|x\| \leq d - \varepsilon\}$, and define a retraction $r : B \to C$ by

$$r(x) = \begin{cases} \dfrac{(d-\varepsilon)x}{\|x\|} & \text{for } x \in \overline{B} \setminus C, \\ x & \text{for } x \in C. \end{cases}$$

Then $G := r \circ T|_C \in \mathcal{B}(C, C)$ and G is compact and closed since r is continuous and $T|_C$ is upper semicontinuous. Therefore, by Theorem 5.16, there exists a point $x_0 \in C \subseteq B$ such that $x_0 \in G(x_0) = r \circ T(x_0)$. Hence, there exists a $y_0 \in T(x_0) \subseteq \overline{B}$ such that $r(y_0) = x_0$. Since $\|r(x) - x\| \leq \varepsilon$ for all $x \in \overline{B}$ and $y_0 \in \overline{B}$, we have $\|x_0 - y_0\| = \|r(y_0) - y_0\| \leq \varepsilon$. This completes the proof. ■

Corollary 5.16 *Let $B^n = \{x \in \mathbb{R}^n : \|x\| < d\}$ for some $d > 0$, and let $T : B^n \to \overline{B^n}$ be continuous. Then for each $\varepsilon > 0$, there exists a point $x \in B^n$ such that $\|x - T(x)\| \leq \varepsilon$.*

Remark 5.2

(1) For $n = 2$ and $T : B^2 \to B^2$, Corollary 5.16 is Fort's theorem [60] obtained with different proof.
(2) For $T : B^n \to B^n$, Corollary 5.16 is obtained by van der Walt [191] who applied his result to show that the Euclidean plane $\mathbb{R}^2$ has the almost fixed point property with respect to continuous maps and finite covers by convex open sets (that is, for every continuous $T : \mathbb{R}^2 \to \mathbb{R}^2$ and a finite cover α of $\mathbb{R}^2$ by convex open sets, there exists a member $U \in \alpha$ such that $U \cap T(U) \neq \emptyset$). This fact was extended by Hazewinkel and van de Vel [80] to any locally convex topological vector space instead of $\mathbb{R}^2$. Some related results can be seen in [88].

5.2 Approximate Fixed Point Sequences

Definition 5.4 Let Y be a nonempty subset of a Hausdorff topological vector space (X, τ) and $T: Y \to \overline{Y}$ be a mapping. A sequence $(x_n)_n$ in Y is called a τ-approximate fixed point sequence for T if $x_n - T(x_n) \xrightarrow{\tau} \theta$, as $n \longrightarrow \infty$.

Definition 5.5 Let Y be a nonempty subset of a Hausdorff topological vector space (X, τ). We will say that Y has the τ-approximate fixed point property if, whenever we take another Hausdorff vector topology σ in X, then every sequentially continuous mapping $T: (Y, \sigma) \to (\overline{Y}, \tau)$ has a τ-approximate fixed point sequence.

For the sake of simplicity, we shall use the term "τ-afp property" to refer to sets with this property.

Definition 5.6 A topological Hausdorff vector space (X, τ) is said to have the τ-afp property if every compact convex subset Y of X has the τ-afp property.

Remark 5.3 We note, however, that it is not immediately clear what happens if $\sigma \neq \tau$. In this context, it is worthwhile to remark that τ-convergence in Definition 5.5 is the most natural way to approximate fixed points for T. The reason for this is that there are situations where σ is finer than τ and T has no σ-approximate fixed points.

5.2.1 *On Lipschitz and Approximate Lipschitz Fixed Point Properties*

In [115], Klee proved that a noncompact convex set in a normed space lacks the fixed point property for continuous maps. In [126], Lin and Sternfeld asked if this result remains true for Lipschitz mappings. They introduced the following

Definition 5.7

- Let (X_1, d_1) and (X_2, d_2) be metric spaces. A function $T: X_1 \to X_2$ is a Lipschitz map if

$$\|T\|_L = \sup\left\{\frac{d_2(T(x), T(y))}{d_1(x, y)} : x, y \in X_1\right\} < \infty.$$

- (X_1, d_1) has the Lipschitz fixed point property (L.f.p.p.) if every Lipschitz self map of X_1 has a fixed point.
- (X_1, d_1) is said to have the approximate Lipschitz fixed point property (approx. L.f.p.p.) if every Lipschitz self map of X_1 $\inf\{d_1(x, T(x)): x \in X_1\} = 0$.

Notations
Let $(e_n)_{n\geq 1}$ the canonical base of l_∞ and set

$$\Delta_n = \text{conv}(\{\theta, e_n, e_{n+1}\}), n \geq 1, \quad \text{and } \Delta = \bigcup_{n=1}^{\infty} \Delta_n. \tag{5.3}$$

Definition 5.8

- A metric space Y is a Lipschitz absolute retract (L.A.R.) if whenever a metric space X contains Y as a closed set, there exists a Lipschitz retraction $r\colon X \to Y$.
- A mapping $h\colon (X_1, d_1) \to (X_2, d_2)$ is a Lipschitz equivalence if h is Lipschitz, one-to-one, and h^{-1} is Lipschitz. If there exists a Lipschitz equivalence of X_1 onto X_2 then X_1 and X_2 are said to be Lipschitz equivalent.
- Two metric functions d and ρ on a set are equivalent if the identity map $id\colon (X, d) \to (X, \rho)$ is a Lipschitz equivalence.

The following is given in [126].

Lemma 5.4 *Let (X, d) be a metric space, and let Y be a Lipschitz retract of X. If Y lacks the* L.f.p.p. *(*approx. L.f.p.p.*) then so does X.*

Proof We prove for the approx. L.f.p.p. Let $r\colon X \to Y$ be a retraction, and let $T\colon Y \to Y$ be a Lipschitz map with $\inf\{d(x, T(x))\colon x \in Y\} = \alpha > 0$. Let $G\colon X \to X$ be defined by $G = T \circ r$, and $\varepsilon = \dfrac{\alpha}{(\|G\|_L + 2)}$. (Note that $\|G\|_L \leq \|T\|_L \|r\|_L$). Let $x \in X \setminus Y$. If $\text{dist}(x, Y) \geq \varepsilon$ then $d(x, G(x)) \geq \text{dist}(x, Y) \geq \varepsilon$. If $\text{dist}(x, Y) < \varepsilon$ let $y \in Y$ be such that $d(x, y) < \varepsilon$, and then

$$d(x, G(x)) \geq d(y, G(y)) - d(x, y) - d(G(x), G(y)) \geq \alpha - \varepsilon - \|G\|_L \varepsilon = \varepsilon.$$

Hence $d(x, G(x)) \geq \varepsilon$ for all $x \in X$. ■

Definition 5.9 A metric space X is a Lipschitz absolute extensor (L.A.E.) if for every metric space W, a closed subset Z of W, and a Lipschitz map $T\colon Z \to X$, T admits a Lipschitz extension $\widetilde{T}\colon W \to X$. If there exists a $\lambda \geq 1$ such that $\|\widetilde{T}\|_L \leq \lambda \|T\|_L$, then X is said to be a λ *L.A.E.*

We have the following example of a λ *L.A.E.* [131].

Proposition 5.2 $\mathbb{R}$ *is a* 1 L.A.E.

Proof Let $T\colon Z \to \mathbb{R}$ be a Lipschitz map, then $\widetilde{T}(w) = \sup\{T(z) - \|T\|_L d(z, w)\colon z \in Z\}$ is an extension of T with $\|\widetilde{T}\|_L = \|T\|_L$. ■

Corollary 5.17 *For every set Ξ, $l_\infty(\Xi)$ is a* 1 L.A.E.*, where $l_\infty(\Xi)$ denotes the Banach space of bounded real valued functions on Ξ with the norm $\|f\|_\infty =$* $\sup\{|f(x)|\colon x \in \Xi\}$.

Proof Apply Proposition 5.2 to each coordinate $T(x, .)$, $x \in \Xi$, of a Lipschitz map $T\colon Z \to l_\infty(\Xi)$. ■

The following Lipschitz version of a theorem by Hausdorff [79] is given in [126]. The proof follows that of Arens [8]. A local version is given in [128].

Theorem 5.18 *Let Y and W be two metric spaces, $Z \subseteq W$ closed, and $T: Z \to Y$ a Lipschitz map. There exists a metric space X which contains Y (isometrically) as a closed set, a Lipschitz extension $G: W \to X$ of T.*

Proof Note first that Y is isometric to a subset of $l_\infty(Y)$ ($x \mapsto d(x, .) - d(., x_0)$ is an isometry, where $x_0 \in Y$ is some fixed point). Set $C = l_\infty(Y) \times \mathbb{R}$, we realize $l_\infty(Y)$ in C as $l_\infty(Y) \times \{0\}$ and we may assume that $Y \subseteq l_\infty(Y) \times \{0\} \subseteq C$. So, in particular $T: Z \to l_\infty(Y) \times \{0\}$, and since this is a L.A.E. (by Corollary 5.17) T admits a Lipschitz extension $H: W \to l_\infty(Y) \times \{0\}$. Let $G: W \to C$ be defined by $G(w) = H(w) + (0, \operatorname{dist}(w, Z))$ and set $X = G(W) \cup Y$. One checks easily that Y is closed in X and that $G_{|Z} = T$. ∎

As a consequence of Theorem 5.18, we have the following results on L.A.R. and L.A.E. metric spaces [126].

Theorem 5.19 *A metric space X is a* L.A.R. *if and only if it is a* L.A.E.

Proof $L.A.E. \Longrightarrow L.A.R.$ Let Y be a L.A.E. and let X contain Y as a closed set. Then a Lipschitz extension $T: X \to Y$ of the identity mapping $id: Y \to Y$ is a retraction.

$L.A.R. \Longrightarrow L.A.E.$ Let Y be a L.A.R., let $Z \subseteq W$ be closed, and let $T: Z \to Y$ be a Lipschitz map. By Theorem 5.18, there exists a space X which contains Y as a closed set, and a Lipschitz extension $G: W \to X$ of T. Since Y is a L.A.R. there exists a Lipschitz retraction $r: X \to Y$. Then $\widetilde{T} = r \circ G: W \to Y$ is a Lipschitz extension of T. ∎

Corollary 5.18 *A retract of a* L.A.R. *is a* L.A.R.

Proof Let Y be a retract of a L.A.R. X with a Lipschitz retraction $r: X \to Y$. We prove the X is a L.A.E. Let $Z \subseteq W$ be closed, and $T: Z \to Y$ be given. Then also $T: Z \to X$, and since X is a L.A.E. there exists an extension $G: W \to X$ of T. It follows that $\widetilde{T} = r \circ G: W \to Y$ is a Lipschitz extension of T. ∎

Corollary 5.19 *If X is Lipschitz equivalent to a* L.A.R. *then it is a* L.A.R.

Proof This is trivial for a L.A.E., and hence follows from Theorem 5.19 ∎

The following properties of Δ are given in [126].

Proposition 5.3 *There exists a Lipschitz retraction $r: l_\infty \to \Delta$.*

Proof We consider l_∞ as a lattice with the natural order. Note that for $x \in \Delta$ and $y \in l_\infty$ $0 \le y \le x$ implies $y \in \Delta$. Let $e = (1, 1, 1, \cdots) \in l_\infty$, and $x \in l_\infty$. Set

$$E(x) = \{\varepsilon : \varepsilon \le 0, (x - \varepsilon e) \wedge 0 \in \Delta\}.$$

Clearly $\|x\| \in E(x)$. Let $\varepsilon: l_\infty \to \mathbb{R}^+$ be defined by $\varepsilon(x) = \inf E(x)$. Then ε is a Lipschitz map with $\|\varepsilon\|_L = 1$. Indeed, for x and y in l_∞, $x \le y + \|x - y\|$. Hence

$\varepsilon(y) + \|x - y\| \in E(x)$ and it follows that $\varepsilon(x) \leq \varepsilon(y) + \|x - y\|$, and by symmetry $|\varepsilon(x) - \varepsilon(y)| \leq \|x - y\|$. Let now $r : l_\infty \to \Delta$ be defined by $r(x) = (x - \varepsilon(x).e) \wedge 0$. Then r is a Lipschitz retraction and $\|r\|_L \leq 2$. ■

Proposition 5.4 *The space Δ as well as the spaces $\mathbb{R}^+$ and $(0, 1] = \{t \in \mathbb{R} : 0 < t \leq 1\}$ (with the metric induced from $\mathbb{R}$) are Lipschitz absolute retracts.*

Proof The fact that Δ is a L.A.R. follows from Corollary 5.17, Corollary 5.18 and Proposition 5.3 Since $\mathbb{R}^+$ is a retract of $\mathbb{R}$, it is a L.A.R., too. To prove that $(0, 1]$ is a L.A.R., we show that $(0, 1]$ is L.A.E. So let $Z \subseteq W$ closed and $T : Z \to (0, 1]$ be given. Then also $T : Z \to [0, 1]$ and since $[0, 1]$ a L.A.E., there exists a Lipschitz extension $G : W \to [0, 1]$. Then

$$\widetilde{T}(w) = G(w)\frac{1}{1 + \operatorname{dist}(w, Z)} : W \to (0, 1]$$

is a Lipschitz extension of T. ■

Proposition 5.5 *Let Y be a noncompact convex subset of a normed space X.*

1. *If Y is not totally bounded then it contains a closed set which is Lipschitz equivalent to either Δ or $\mathbb{R}^+$.*
   More precisely: If some bounded subset of Y is not totally bounded then Y contains a closed set which is Lipschitz equivalent to Δ, while if some ball $\{x \in Y : \|x - x_0\| \leq 1\}$ in Y is totally bounded (and Y itself is not) then Y contains a closed set which is Lipschitz equivalent to $\mathbb{R}^+$.
2. *If Y is totally bounded then it contains a closed set which is Lipschitz equivalent to $(0, 1]$.*

Proof Let Y be a noncompact convex subset of a normed space X. We distinguish between the following two cases: Case (i): Y is not totally bounded, and Case (ii): Y is totally bounded.

Cases (i). Here also we separate the proof into two cases.

Cases $(i)a$. Some bounded subset of Y is not totally bounded. In this case we may assume without loss of generality that $\theta \in Y$, and that $Y_1 = \{x \in Y : \|x\| \leq 1\}$ is not totally bounded. Hence, there exists some $r > 0$ such that Y_1 cannot be covered by many finitely many balls of radius $2r$. It follows that

1. For every finite-dimensional linear subspace Z of X, there exists some $x \in Y_1$ with $\operatorname{dist}(x, Z) \geq r$.

Indeed, if not then $Z + B_r(\theta) \supseteq Y_1$ and from the compactness of $\{y \in Z : \|y\| \leq 2\}$ it follows that finitely many balls of radius $2r$ cover Y_1.

Now we select inductively a sequence $(x_n)_{n\geq 1}$ in Y_1 as follows: let $x_1 \in Y_1$ be any element with $\|x_1\| \geq r$. Assume that $x_1, x_2, \cdots, x_n$ have been selected. Set $W = \operatorname{span}\{x_1, x_2, \cdots, x_n\}$, and apply 1. to find $x_{n+1} \in Y_1$ with $\operatorname{dist}(x_{n+1}, W) \geq r$.

For $n \geq 1$ set

$$\Delta'_n = \mathrm{conv}(\{\theta, x_n, x_{n+1}\}) \quad \Delta' = \bigcup_{n=1}^{\infty} \Delta'_n.$$

Then Δ' is a closed subset of Y and is Lipschitz equivalent to Δ by the map $H\colon \Delta \to \Delta'$ which is defined by $H(\theta) = \theta$, $H(e_n) = x_n, n \geq 1$, and H is linear on each Δ_n.

Case$(i)b$. Some ball $\{x \in Y\colon \|x - x_0\| \leq 1\}$ is totally bounded. Note that in this case Y must be unbounded. Again we assume $\theta \in Y$. Let $\widetilde{X}$ be the completion of X, and let $\overline{Y}$ be the closure of Y in $\widetilde{X}$. Set $\overline{Y_n} = \{x \in \overline{Y}\colon \|x\| \leq n\}$. Then $\overline{Y_1}$ is compact, and since $(n+1)^{-1}\overline{Y_{n+1}} \subseteq n^{-1}\overline{Y_n} \subseteq \overline{Y_1}$, and $n^{-1}\overline{Y_n}$ contains a unit vector for each n, $\bigcap_{n=1}^{\infty} n^{-1}\overline{Y_n}$ must contain some vector y_0 with $\|y_0\| = 1$. Then $ty_0 \in \overline{Y}$ for all $t \in \mathbb{R}^+$. For each $n \geq 1$ pick some $x_n \in Y$ with $\|ny_0 - x_n\| < (n+10)^{-1}$. Then $\bigcup_{n=1}^{\infty}[x_n, x_{n+1}]$ is a closed subset of Y is Lipschitz equivalent to $\mathbb{R}^+$.

Case (ii). Once again let $\overline{Y}$ denote the closure of Y in the completion of X. Pick $x_0 \in \overline{Y} \setminus Y$ and $x_1 \in Y$. For $n \geq 2$ select $x_n \in Y$ such that

$$\left\| \left(\left(1 - \frac{1}{n}\right) x_0 + \frac{1}{n} x_1 \right) - x_n \right\| < 2^{-(n+10)}.$$

Then $\bigcup_{n=1}^{\infty}[x_n, x_{n+1}]$ is a closed subset of Y which is Lipschitz equivalent to $(0, 1]$. ■

Proposition 5.6 Δ *lacks the* approx. L.f.p.p.

In [126], Lin and Sternfeld gave a characterization of noncompact convex sets in a normed space, having the approx. L.f.p.p.

Theorem 5.20 *Let Y be a noncompact convex set in a normed space.*

1. *If Y is not totally bounded then it lacks the* approx. L.f.p.p.
2. *If Y is totally bounded then it has the* approx. L.f.p.p.*, but lacks the* L.f.p.p.

Proof

1. Follows from Lemma 5.4 and Propositions 5.4, 5.5 and 5.6.
2. The second part follows from Lemma 5.4 and Propositions 5.4, 5.5 and 5.6.
   For the first part, Let Y be a noncompact totally bounded convex subset of a normed space X, and let $T\colon Y \to Y$ be a Lipschitz map. Let $\widetilde{X}$ denote the completion of X, and $\overline{Y}$ the closure of Y in $\widetilde{X}$. Then $\overline{Y}$ is compact, and T admits an extension $\widetilde{T}\colon \overline{Y} \to \overline{Y}$. By the Schauder fixed point theorem $\widetilde{T}$ has a fixed

point $x_0 \in \overline{Y}$. Let $(x_n)_n \subseteq Y$ be a sequence which converge to x_0. Then $(x_n)_n$ is an approximate fixed point for T, i.e., $\lim_{n\longrightarrow\infty} \|x_n - T(x_n)\| = 0$, and it follows that Y has the approx. L.f.p.p.

■

The closed unit ball in an infinite-dimensional normed space lacks the approx. L.f.p.p. [23]. For Banach spaces, we have this more general result.

Theorem 5.21 *A closed noncompact convex set in a Banach space lacks the* approx. L.f.p.p.

Combine Theorem 5.20 with the Schauder fixed point theorem and we obtain:

Theorem 5.22 *A convex set in a normed space has the* L.f.p.p. *if and only if it is compact.*

The precedent results bring out the necessity of considering weaker topologies ensuring the sequential approximation of fixed points where no stronger convergence can be expected.

When X is a Banach space, a nonexistence result was reported by Domínguez Benavides et al. [22].

Theorem 5.23 *Let Y be a closed convex of a Banach space X. If Y is not weakly compact, then there exists a closed convex subset Z of Y and a continuous affine map $T: Z \to Z$ such that* $\inf\{\liminf \|y - T^n(x)\|: x, y \in Z\} > 0$.

It is natural then to look for weak-approximating fixed point sequences instead of stronger ones.

Definition 5.10 Let $(X, \|.\|)$ be a Banach space and $Y \subseteq X$. A mapping $T: Y \to X$ is called demicontinuous if it maps strongly convergent sequences into weakly convergent sequences.

The next result is due to Moloney and Weng [133].

Proposition 5.7 *Let X be a Hilbert space, Y a closed ball and $T: Y \to Y$ a demicontinuous mapping. Then T admits a weak-approximate fixed point sequence, that is, a sequence $(x_n)_n \subseteq Y$ such that $(x_n - T(x_n))_n$ converges weakly to θ.*

To study the weak-approximate fixed property in Banach and abstract spaces, the following results have been given by Barroso [12].

Theorem 5.24 *Let Y be a compact convex subset of a topological vector space (X, τ). Assume that Y has an admissible function on X. Then Y has the τ-afp property.*

Proof Fix any $n \geq 1$. From 1. and the fact that $\rho(\theta) = 0$, it follows that if $x \in Y$ then the set $B_{\frac{1}{n}}(x) = \{y \in Y: \rho(y - x) < \dfrac{1}{n}\}$ is τ-open in Y with respect to the relative topology of X. Thus, the family $\{B_{\frac{1}{n}}(x): x \in Y\}$ is an open covering of the compact set Y. From compactness we can extract the finite sub covering, i.e., a finite subset $\Gamma_n = \{x_1, \cdots, x_{N_n}\}$ of Y such that

$$Y = \bigcup_{i=1}^{N_n} B_{\frac{1}{n}}(x_i).$$

Let $P_n \colon Y \to \mathrm{conv}(\Gamma_n) \subseteq \overline{\mathrm{conv}}(\Gamma_n)$ be the Schauder's projection associated to Γ_n and ρ, where $\overline{\mathrm{co}}(\Gamma_n)$ denotes the τ-closure of the convex hull of Γ_n. In view of Proposition 1.44 it follows that P_n is τ-continuous. Moreover, by using 2. and 3. we see that

$$\rho(P_n(x) - x) < \frac{1}{n},$$

for all $x \in Y$. Let now σ be another Hausdorff vector topology in X and $T \colon (Y, \sigma) \to (Y, \tau)$ a sequentially continuous mapping. Then the mapping

$$P_n \circ T \colon (\overline{\mathrm{conv}}(\Gamma_n), \sigma) \to (\overline{\mathrm{conv}}(\Gamma_n), \tau)$$

is also sequentially continuous. Let us denote by $\mathcal{G}_n$ the linear span of Γ_n. Observe that the linear operator $\Phi \colon \mathcal{G}_n \to \mathbb{E}$ defined by $\Phi(\sum \alpha_i x_i) = \sum \alpha_i e_i$ is an algebraic isomorphism, where $\{e_i\}$ denotes the canonical basis of the space $\mathbb{E} = (\mathbb{R}^{N_n}, \mathrm{eucld})$. Here the word "eucld" indicates the euclidean topology. Thus, if we denote by Φ_σ (resp. Φ_τ) the mapping Φ from $(\mathcal{G}_n, \sigma)$ (resp. $(\mathcal{G}_n, \tau)$) into $\mathbb{E}$ then, it follows that both these maps are linear homeomorphisms. Hence, setting $Z_n = \Phi_\tau(\overline{\mathrm{conv}}(\Gamma_n))$ we see that $Z_n = \Phi_\sigma(\overline{\mathrm{conv}}(\Gamma_n))$.

$$\begin{array}{ccc}
(\overline{\mathrm{conv}}(\Gamma_n), \sigma) & \xrightarrow{P_n \circ T} & (\overline{\mathrm{conv}}(\Gamma_n), \tau) \\
\uparrow \Phi_\sigma^{-1} & & \downarrow \Phi_\tau \\
(Z_n, \mathrm{eucld}) & \longrightarrow & (Z_n, \mathrm{eucld})
\end{array}$$

According to the above diagram, $\left[\Phi_\tau \circ (P_n \circ T) \circ \Phi_\sigma^{-1}\right]$ is a sequentially continuous mapping from (Z_n, eucld) into itself. Since Z_n is convex and compact with respect to the eucld-topology, it follows from Brouwer's fixed point theorem that

$$\left[\Phi_\tau \circ (P_n \circ T) \circ \Phi_\sigma^{-1}\right](z_n) = z_n,$$

for some $z_n \in Z_n$. Thus $(P_n \circ T)(u_n) = u_n$, where $u_n = \Phi^{-1}(z_n)$. It follows that

$$\rho(u_n - T(u_n)) < \frac{1}{n},$$

for all $n \geq 1$. Using now Proposition 1.7 we can conclude that Y is sequentially compact, for τ is finer than τ_ρ on Y, the metric topology induced by ρ. Thus, we may assume (by passing to a subsequence if necessary) that $u_n \xrightarrow{\tau} x$ and $T(u_n) \xrightarrow{\tau} y$, for some $x, y \in Y$. Hence, in view of 1., we get

$$\rho(x - y) = 0,$$

and so $x = y$ by 4.. This shows that $u_n - T(u_n) \xrightarrow{\tau} \theta$ and concludes the proof. ■

Next, some of the theoretical implications of Theorem 5.24 are given [12].

Corollary 5.20 *Let Y be a compact convex subset of a topological vector space (X, τ) and $\mathcal{F} = \{\rho_n : n \in \mathbb{N}\}$ a countable family of seminorms on X which separate points of $Y - Y$ and such that the topology Γ generated by $\mathcal{F}$ is coarser than τ in Y. Then Y has the τ-afp property.*

Proof By Proposition 1.45, Y has an admissible function on X. Theorem 5.24 now implies that Y has the τ-afp property. ■

Corollary 5.21 *Every (weakly) compact convex subset Y of a Hausdorff locally convex space (X, τ) whose topological dual space X' is weak* separable has the (weak) τ-afp property.*

Proof Since X' is weak* separable, it follows that X' is total over X. Then, for a weak* dense sequence $(\phi_n)_n$ in X', it follows that

$$x \mapsto |\phi(x)|$$

yields a countable family $\mathcal{F}$ of τ-continuous (resp. weak continuous) seminorms on X which separates points. In this case, notice that the topology Γ determined by $\mathcal{F}$ is coarser than τ (resp. weak topology). Therefore, in view of Corollary 5.20, we conclude that every compact (resp. weakly compact) convex subset of X has the τ-approximate (resp. weak) fixed point property. ■

Remark 5.4 As a consequence of the preceding corollary it follows that every separable Banach space has the weak-afp property. Indeed, if X is a separable Banach space then by the Banach-Alaoglu theorem (Theorem 1.28), each dual ball $\overline{B_n(\theta)}$ centered at the origin with radius $n, n \geq 1$, is weak* compact metric space and hence a separable metric space. This implies that the dual X' is weak* separable.

In view of Lin-Sternfeld's theorem the existence of weak-approximate fixed point sequences.

Theorem 5.25 *Let Y be a weakly compact convex subset of a Banach space $(X, \|.\|)$. Then every demicontinuous mapping $T : Y \to Y$ has a weak-approximate fixed point sequence.*

Proof Without loss of generality, we may assume that T is fixed point free and X is not separable. Pick $y \in Y$ and denote by $\mathcal{O}(y) = \{T^n(y),\ n \in \mathbb{N}\}$ the orbit of y

under T. Now, we construct inductively a sequence $(Y_n)_n$ of closed convex subsets of Y as follows. We set $Y_0 = \overline{\text{conv}}\mathcal{O}(y)$ and if $n \geq 1$ we put $Y_{n+1} = \overline{\text{conv}}(T(Y_n))$, where the overline denotes the closure w.r.t the norm $\|.\|$. It is easily verified that

$$\mathcal{O}(T^{n+1}(y)) \subseteq T(Y_n) \subseteq Y_{n+1},$$

for all $n \geq 1$. We claim now that each Y_n is separable. This is evident if $n = 0$ since the closed linear span of $\mathcal{O}(y)$ is a separable Banach subspace of X. By induction on n, and by the fact that T is demicontinuous together with Mazur's theorem, we conclude that if $Y_n \subseteq \overline{\{x_k^n : k \geq 1\}}$ for some $\{x_k^n : k \geq 1\} \subseteq Y_n$, then $T(Y_n) \subseteq \overline{\text{conv}}(T(x_k^n) : k \geq 1)$. This completes the proof of our claim. As a consequence, if we set $Z_k = \bigcap_{n=k}^{\infty} Y_n$, then the following closed convex subset of Y

$$W = \overline{\bigcup_{k=0}^{\infty} Z_k},$$

must be separable too. Notice that, since Y is weakly compact, each Z_k is nonempty. Moreover, it is easy to see that $T(Z_k) \subseteq Z_{k+1}$, for all $k \geq 1$. Hence, using again the fact that T is demicontinuous, we see that W is invariant under T. Finally, since $W \subseteq \overline{\text{span}}(d_j : j \geq 1)$, for some dense sequence $(d_j)_{j\geq 1}$ in W, we reach the conclusion of theorem by means of Corollary 5.21. ■

Here, as a direct application of Theorem 5.25, we obtain the following fixed point result for continuous maps in general Banach spaces.

Corollary 5.22 *Let Y be a weakly compact convex subset of a Banach space $(X, \|.\|)$ and $T : Y \to Y$ a continuous mapping. Suppose that $(I - T)(Y)$ is sequentially weakly closed. Then T has fixed point.*

Proof By Theorem 5.25, there exists a sequence $(x_n)_n$ in Y such that $x_n - T(x_n) \rightharpoonup \theta$. By assumption, we get $\theta \in (I - T)(Y)$ and so $T(x) = x$, for some $x \in Y$. This completes the proof. ■

Definition 5.11 Let $(X, \|.\|)$ be a Banach space and $Y \subseteq X$. A mapping $T : Y \to X$ is called

(a) proper if the preimage of each compact set is compact.
(b) weakly proper if the preimage of each weakly compact set is weakly compact.

The next result yields some sufficient conditions for concluding a map is strongly continuous [13].

Proposition 5.8 *Let Y be a closed convex subset of a Banach space X and $T: Y \to Y$ a mapping such that T^m is compact for some integer $m \geq 2$. Then T^m is strongly continuous in any the following cases*:

(a) *T^m is proper and continuous. Moreover, if T is demicontinuous then it is sequentially weakly continuous.*
(b) *T^m is sequentially weakly continuous. This holds, in particular, when T is continuous and affine.*
(c) *T is an isometry.*

Proof Let be $x_n \rightharpoonup x$ for some $x \in Y$.

(a) As T^m is compact and proper, it follows that $\{x_n\}$ is relatively compact since $\overline{\{x_n\}} \subseteq T^{-m}(\overline{\{T^m(x_n)\}})$ and $T^{-m}(\overline{\{T^m(x_n)\}})$ is compact. Therefore, up to a subsequence, we can conclude that $x_n \to x$ and hence $T^m(x_n) \to T^m(x)$.
(b) Since T^m is sequentially weakly continuous, $T^m(x_n) \rightharpoonup T^m(x)$. By using the fact that T^m is compact and by passing to a subsequence if needed, we may assume that $T^m(x_n) \to y$ for some y in X. This implies that $y = T^m(x)$ and proves the result.
(c) This item is easily proved with the aid of the fact that T^m is compact.

■

Definition 5.12 Let Y be a nonempty subset of a Banach space X and let $T: Y \to Y$ be a continuous mapping. For $x \in Y$ let

$$\gamma^+(x) = \{T^k(x) : k = 1, 2, \cdots\}, \ T^0(x) = x$$

be the positive semiorbit of x and

$$\omega(x) = \{\omega \in X : \exists\, k_l \to \infty \text{ such that } T^{k_l}(x) \to \omega \text{ as } l \to \infty\}$$

the ω-limit set of x. A point $x \in Y$ is a k-periodic point of T $(k \geq 2)$ if $T^k(x) = x$, and $T^l(x) \neq x, l = 1, \cdots, k-1$. A set Z is called a k-cycle (of T) if $Z = \gamma^+(x)$ for some k-periodic point x of T.

The following definition will play an important role [178].

Definition 5.13 Let Y be a nonempty subset of a Banach space X and let $T: Y \to Y$ be a continuous mapping. A couple (Z_1, Z) will be called admissible (with respect to the mapping T) if

1. $\emptyset \neq Z_1 \subseteq Z \subseteq Y$,
2. the set Z_1 is compact, and
3. the set Z is convex and closed, $T(Z) \subseteq Z$.

Some properties of an admissible couple are collected in [178].

Lemma 5.5 *Let Y be a closed convex subset of a Banach space X and $T : Y \to Y$ be a continuous mapping. Then the following statements hold*:

1. *If (Z_1, Z) is an admissible couple and $T(Z_1) \subseteq Z_1$, then so is the couple (Z_0, Z) where $Z_0 = \bigcap_{k=0}^{\infty} T^k(Z_1)$ has the property $T(Z_0) = Z_0$.*
2. *If (Z_1, Z) is an admissible couple, then there exists the least convex closed set Z_2 such that If (Z_1, Z_2) is admissible.*

Proof

1. As it was already mentioned, Z_0 is a nonempty compact set such that $T(Z_0) \subseteq Z_0 \subseteq Z_1$. If $x \in Z_0$ is an arbitrary element, then there exists $y_k \in T^k(Z_1)$ such that $T(y_k) = x$ and by the compactness of Z_1 there exists a subsequence $y_l \in T^l(Z_1)$ which converges to $y \in Z_0$ as $l \to \infty$. $T(y) = x$ and hence $x \in T(Z_0)$.
2. Let

$$G = \left\{ W \in 2^X : Z_1 \subseteq W \subseteq Y,\ W \text{ is convex, closed and } T(W) \subseteq W \right\}.$$

Let $Z_2 = \bigcap_{W \in G} W$. Then Z_2 is the least element of G in the sense of the set inclusion.

■

Definition 5.14 The admissible couple (Z_1, Z_2) will be called minimal if Z_2 is the least convex closed set containing Z_1.

This fundamental lemma is stated in [178].

Lemma 5.6 *Let Y be a closed convex subset of a Banach space X and $T : Y \to Y$ be a continuous mapping. Let (Z_1, Z_2) be a minimal admissible couple. Then*:

1.

$$Z_2 = \overline{\bigcup_{k=1}^{\infty} W_k} \tag{5.4}$$

where $(W_k)_{k \geq 1}$ is a nondecreasing sequence of convex compact subsets of Z_2 which are defined by the relations

$$W_1 = \overline{\text{conv}(Z_1)}, \tag{5.5}$$

$$W_{k+1} = \overline{\text{conv}(W_k \cup T(W_k))}, \quad k = 1, 2, \cdots \tag{5.6}$$

2. *Z_2 is separable.*
3. *If $\bigcup_{k=1}^{\infty} W_k$ is not closed, then $\overline{\bigcup_{k=1}^{\infty} W_k} \setminus \bigcup_{k=1}^{\infty} W_k$ is a G_δ set.*
4. *If $Z_1 \subseteq T(Z_1)$, then $\overline{\text{conv}(Z_2)} = Z_2$.*

Proof

1. By Theorem 1.20, W_1 is convex and compact. Since $W_1 \subseteq Z_2$, we have that the compact set $W_1 \cup T(W_1) \subseteq Z_2$ and the set $W_2 = \overline{\text{conv}(W_1 \cup T(W_1))} \subseteq Z_2$ is convex and compact. By mathematical induction we get that the sequence $(W_k)_{k\geq 1}$ which is defined by (5.5) and (5.6) is a nondecreasing sequence of convex compact subsets of Z_2. Clearly $\bigcup_{k=1}^{\infty} W_k \subseteq Z_2$ and $\overline{\bigcup_{k=1}^{\infty} W_k}$ is a convex and closed subset of Z_2. Further, $T\left(\bigcup_{k=1}^{\infty} W_k\right) \subseteq \bigcup_{k=1}^{\infty} W_k$ and, from the continuity of T, we have $T\left(\overline{\bigcup_{k=1}^{\infty} W_k}\right) \subseteq \overline{\bigcup_{k=1}^{\infty} W_k}$. Hence $\left(Z_1, \overline{\bigcup_{k=1}^{\infty} W_k}\right)$ is an admissible couple and since $\overline{\bigcup_{k=1}^{\infty} W_k} \subseteq Z_2$, equality (5.4) follows.
2. Every compact metric space is separable, countable union of separable sets is separable and the closure of separable set in a metric space is separable.
3. $\overline{\bigcup_{k=1}^{\infty} W_k} \setminus \bigcup_{k=1}^{\infty} W_k \neq \emptyset$, then $\overline{\bigcup_{k=1}^{\infty} W_k} \setminus \bigcup_{k=1}^{\infty} W_k = \bigcap_{k=1}^{\infty}\left(\overline{\bigcap_{l=1}^{\infty} W_l} \setminus W_k\right)$ is a G_δ set.
4. Denote $Z_3 = \overline{\text{conv}(Z_2)}$. As $T(Z_3) \subseteq Z_3$, $Z_1 \subseteq Z_3$, we have that (Z_1, Z_3) is an admissible couple and thus, the minimality of (Z_1, Z_2) implies that $Z_3 = Z_2$.

■

Lemma 5.7 *Let Y be a closed convex subset of a Banach space X and $T : Y \to Y$ be a continuous mapping. Let $Z_1, \emptyset \neq Z_1 \subseteq Y$, be a compact set. Then there exists a closed convex subset Z of Y such that $Z_1 \subseteq Z$ and*

$$\overline{\text{conv}}(T(Z)) = Z. \tag{5.7}$$

Proof Let

$$G = \left\{W \in 2^X : Z_1 \subseteq W \subseteq Y,\ W \text{ is convex, closed and } T(W) \subseteq W\right\}$$

and a be the cardinal number of the set G. By the Cantor theorem , the cardinal number $2^a > a$. Let b be the initial ordinal number of power 2^a. Then we define a transfinite $\{W_\alpha\}$ of the type b with values in G in the following way:

$$W_0 = Y,$$

$$W_\alpha = \begin{cases} \overline{\text{conv}}(T(W_{\alpha-1})), & \text{if } \alpha - 1 \text{ exists,} \\ \bigcap_{\beta<\alpha} W_\beta, & \text{in the other case } (\alpha \text{ is a limit number}) \end{cases} \tag{5.8}$$

for $\alpha > 0$. The sequence $\{W_\alpha\}$ is non-increasing with respect to the set inclusion and we claim: There exists an ordinal number $\delta < b$ such that $W_\delta = W_{\delta+1}$ which on the basis of (5.8) means that $Z = W_\alpha$ satisfies (5.7).

If (5.7) were not true for any $Z = W_\delta$, then the sequence $\{W_\alpha\}$ would be injective and the cardinal number of G would be greater or equal to 2^a which, on the basis of the Cantor theorem, is a contradiction with the properties of cardinal numbers. ■

Lemma 5.8 *Let Y be a nonempty subset of a Banach space X and let $T: Y \to Y$. Then the following statements are true:*

1. *Each point of a k-cycle of T is a fixed point of T^k.*
2. *Each fixed point of T^k is either a fixed point of T or belongs to an l-cycle of T where l is a divisor of k.*

Proof Only the statement 2. will be proved. Let $x = T^k(x)$ and let $x \neq T(x)$. Consider the sequence $\{x, T(x), \cdots, T^{k-1}(x)\}$. Then two cases may occur. Either all terms $T^l(x), l = 1, \cdots, k-1$ are different from x and the sequence $\{x, T(x), \cdots, T^{k-1}(x)\}$ is injective and x belongs to a k-cycle of T, or there exists an l, $1 < l < k$ such that $x = T^l(x)$ and $x \neq T^m(x)$ for $m = 1, \cdots, l-1$. In this case x belongs to an l-cycle of T and with respect to the fact that $x = T^k(x)$ we must have that l is a divisor of k. ■

The following is an approximate fixed point result for the case when T is weakly proper [13].

Theorem 5.26 *Let Y be a bounded, closed and convex subset of a Banach space X and $T: Y \to Y$ a continuous map such that T^m is compact for some integer $m \geq 1$. Suppose that T is weakly proper. Then T has a weak-approximate fixed point sequence.*

Proof We can suppose that T is fixed point free. By the Schauder fixed point theorem we have $T^m(x) = x$ for some $x \in Y$. Then by Lemma 5.8 there exists a natural $k \geq 2$ such that $W = \{x, T(x), \cdots, T^{k-1}(x)\}$ is a k-cycle of T. In particular W is compact and $T(W) = W$. By Lemma 5.7, there exists a convex, closed set Z such that $W \subseteq Z \subseteq Y$, $\overline{\text{conv}}(T(Z)) = Z$ and $T(Z) \subseteq Z$. Moreover, Z is the least convex, closed set containing W. Since T is weakly proper, from the compactness of $\overline{T^m(Z)}$ and the fact that $\overline{T^{m-k}(Z)} \subseteq T^{-k}(\overline{T^m(Z)})$ for all $k = 1, \cdots, m$, we can conclude with the aid of Mazur's theorem that Z is weakly compact. The result follows now from Theorem 5.25. ■

5.2.2 On the $\sigma(X, Z)$-Approximate Fixed Point Property in Topological Vector Spaces

Let X be a topological vector space, X' its topological dual and Z a subset of X'. Some results concerning the $\sigma(X, Z)$-approximate fixed property for bounded, closed convex subsets Y of X are given.

Definition 5.15 Let Y be a nonempty subset of a topological vector space (X, τ), Z a subspace of its topological dual X' and $T: Y \to \overline{Y}$ be a continuous mapping. A sequence $(x_n)_n$ in Y is called a $\sigma(X, Z)$-approximate fixed point sequence for T if $(\phi(x_n - T(x_n))_n$ converges to zero for all $\phi \in Z$.

Definition 5.16 Let Y be a nonempty subset of a topological vector space (X, τ) and Z a subspace of X'. We will say that Y has the $\sigma(X, Z)$-approximate fixed point property if every continuous mapping $T: Y \to \overline{Y}$ has a $\sigma(X, Z)$-approximate fixed point sequence.

For the sake of simplicity, we shall use the term "$\sigma(X, Z)$-afp property" to refer to sets with this property.

Definition 5.17 Let (X, τ) be a topological vector space and Z a subspace of X'. X is said to have the $\sigma(X, Z)$-afp property if every bounded, closed convex subset Y of X has the $\sigma(X, Z)$-afp property.

Remark 5.5 When $Z = X'$ we simply write weak-afp property instead writing $\sigma(X, X')$. In a similar way, we can also define the $\sigma(X', Z)$-afp property for some subset Z of X.

The following lemma is given in [14]. Its proof avoids paracompactness used in [13] in the case of Banach spaces.

Lemma 5.9 *Let (X, τ) be a topological vector space, Z a subspace of its topological dual X', $\Gamma = \{\phi_1, \cdots, \phi_n\}$ a finite subset of Z, and Y a nonempty, bounded convex subset of X. For any mapping $T: Y \to \overline{Y}$ which is τ-to-$\sigma(X, Z)$ sequentially continuous, and any $\varepsilon > 0$, there is $y \in Y$ such that*

$$|\phi_i(y - T(y))| < \varepsilon, \quad \textit{for } i = 1, \cdots, n.$$

Proof Equip the space $\mathbb{R}^n$ with the max-norm $\|.\|_\infty$ and define the mapping Ψ from $\overline{Y}$ to $\mathbb{R}^n$ by

$$\Psi(x) = (\phi_1(x), \cdots, \phi_n(x)).$$

It is clear that Ψ is a continuous linear mapping. Since Y is bounded in X, $\overline{Y}$ is bounded as well and hence the set $\Psi(\overline{Y})$ is bounded in $\mathbb{R}^n$. It follows that $\Psi(\overline{Y})$ is totally bounded. Let $U = \prod_{i=1}^{n}(\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}) \subseteq \mathbb{R}^n$. (It is an open ball with respect to the max-norm.) There is a finite $Z \subseteq \Psi(Y)$ such that $\{z + U : z \in Z\}$ is an open cover of $\overline{\Psi(Y)}$. There is a finite $W \subseteq \Psi(Y)$ such that $\{z + U : z \in W\}$ is an open cover of $\overline{\Psi(Y)}$. Let y_z be a fixed element in $\Psi^{-1}\{z\}$ for $z \in W$. Set $L = \{y_z, z \in W\}$ and $K = \text{conv}(L)$. Then, by Theorem 1.20 4. K is a finite-dimensional compact convex subset of Y. Now for each $x \in K$, let $w_x = y_z$ be a fixed element such that $\Psi(T(x)) \in z + U$. Then

$$|\phi_i(w_x - T(x))| < \frac{\varepsilon}{2},$$

for $i = 1, \cdots, n$. Moreover, the restriction $T|_K$ is τ-to-$\sigma(X, Z)$ as K is metrizable. Further, Ψ is $\sigma(X, Z)$-continuous, hence the composed mapping $\Psi \circ T|_K$ is τ-continuous. Therefore we can, for each $x \in K$, choose a τ open neighborhood U_x of x (relatively in K) such that for any $\omega \in U_x$ and any $i = 1, \cdots, n$,

$$|\phi_i(T(\omega) - T(x))| < \frac{\varepsilon}{2}.$$

Then $\Lambda = \{U_x : x \in Y\}$ is an open cover of K. Since K is compact, there exists a locally finite partition of unity $\{\varphi_x : x \in K\}$ on Y dominated by $\{U_x : x \in K\}$. Then the mapping $F(\omega) = \sum_{x \in K} \varphi_x(\omega) w_x$ is a continuous function from $K$to K. By Brouwer's fixed point, it has a fixed point $y \in K$. If $\varphi_x(y) \neq 0$, then $y \in U_x$ and

$$|\phi_i(T(y) - T(x))| < \frac{\varepsilon}{2}, \quad \text{for } i = 1, \cdots, n.$$

Therefore, for $i = 1, \cdots, n$,

$$\begin{aligned} |\phi_i(y - T(y))| &= |\phi_i(F(y) - T(y))| \\ &\leq \sum_{x \in K} \varphi_x(y) |\phi_i(w_x - T(y))| \\ &\leq \sum_{x \in K} \varphi_x(y) (|\phi_i(w_x - T(y))| + |\phi_i(T(x) - T(y))|) < \varepsilon. \end{aligned}$$

The proof is complete. ∎

As a consequence of Lemma 5.9, the following approximate fixed point results in topological vector spaces are given in [14], where sets are not necessarily closed and maps are not necessarily continuous.

Proposition 5.9 *Let (X, τ) be a topological vector space, Z a subspace of its topological dual X', Y a nonempty, bounded convex subset of X and $T: Y \to \overline{Y}$ a τ-to-$\sigma(X, Z)$ sequentially continuous mapping. Then $\theta \in \overline{\{x - T(x): x \in Y\}}^{\sigma(X,Z)}$.*

if X is a topological vector space with separable strong dual:

Proposition 5.10 *Let (X, τ) be a topological vector space, Z a subspace of its topological dual X', and Y a nonempty, bounded convex subset of X. Assume that $T: Y \to \overline{Y}$ which is τ-to-$\sigma(X, Z)$ sequentially continuous. If, Z is separable in the strong topology (i.e., the topology of uniform convergence on τ-bounded subsets of X), then there is a sequence $(x_n)_n$ in Y such that $x_n - T(x_n)$ converge to θ in the topology $\sigma(X, Z)$ [14].*

Proof Let $(\phi_i)_i$ be a strongly dense sequence in Z. By Lemma 5.9 we can find for any $n \in \mathbb{N}$ a point x_n in Y so that

$$|\phi_i(x_n - T(x_n))| < \frac{1}{n}, \quad \text{for } i = 1, \cdots, n.$$

Then for all integer $i \geq 1$, $|\phi_i(x_n - T(x_n))| \to 0$ as $n \to \infty$. The denseness of $(\phi_i)_i$ in the strong topology on Z implies $x_n - T(x_n) \to 0$ with regarding the topology $\sigma(X, Z)$. This completes the proof. ∎

As an immediate consequence we get easy proof of a well-known result of Fan [57].

Corollary 5.23 *Let (X, τ) be a topological vector space such that its topological dual X' separates the points of X. (This is satisfied, for example, if (X, τ) is locally convex.) Let $Y \subseteq X$ be a nonempty compact convex set. Then each continuous mapping $T: Y \to Y$ has a fixed point.*

Proof Set $W = \{x - T(x): x \in Y\}$. Then W is compact as the image of Y by the continuous map $x \mapsto x - T(x)$. So, W is also weakly compact. Since X' separates points of X, the weak topology is Hausdorff and hence W is weakly closed. By Proposition 5.9 θ belongs to the weak closure of W, hence $\theta \in W$, i.e., T has a fixed point. ∎

In the following theorem, we collect some situations in which Proposition 5.10 can be applied [14].

Theorem 5.27 *Let X be a normed space. Then the following statements hold true:*

- (i) *Assume that the completion of X is an Asplund space. Let τ be a complete metrizable locally convex topology on X compatible with the duality. Then (X, τ) has the weak-afp property.*
- (ii) *If $(X', \sigma(X', X))$ is $\aleph_0$-monolithic (i.e., each separable subset of $(X', \sigma(X', X))$ has countable network), then $(X', \|.\|)$ has the $\sigma(X', X)$-afp property.*

Proof

(i) Let $Y \subseteq X$ be a nonempty bounded closed convex set and $T\colon Y \to Y$ a τ-continuous mapping. Let $a \in Y$ and set $W_1 = \{a\}$. If W_n is defined, then let

$$W_{n+1} = \overline{\text{conv}}\,(W_n \cup T(W_n))\,.$$

Then W_{n+1} is compact. So by Theorem 1.20 3., the set

$$W = \overline{\bigcup_{n=1}^{\infty} W_n}$$

is τ-separable and $T(W) \subseteq W$. W is clearly norm-separable. Indeed, Let $S \subseteq W$ be a countable τ-dense set. Denote by S' the norm-closed convex hull of S. Then S' is norm-separable. Moreover, as it is a closed convex set, it is also weakly closed by Theorem 1.25. Hence it is τ-closed as well, so in particular $W \subseteq S'$. It follows that W is norm-separable. Therefore the closed linear span of W is norm-separable as well. So, we can without loss of generality suppose that X is separable. By our assumption X' is separable, we can conclude by Proposition 5.10.

(ii) It is enough to show that each nonempty separable closed convex bounded subset of X' has the $\sigma(X', X)$-afp property. Let $Z \subseteq X'$ be such a set. Set $Y = Z_\perp$ and denote by W the quotient space X/Y. Denote by q the canonical quotient map $q\colon X \to W$. The adjoint map $q^*\colon W^{'} \to X'$ is an isometric injection which is, moreover, weak*-to-weak* homeomorphism. The image $q^*(W')$ is equal to $Y^\perp = (Z_\perp)^\perp$, which is (by the bipolar theorem) the weak* closed linear span of Z. It follows that $q^*(W')$ is weak* separable, hence the weak* topology of W' has countable network. Therefore the dual ball $(B_{W'}, \sigma(X', X))$ is metrizable, thus W is separable. By Proposition 5.10 we get that W' has the $\sigma(W', W)$-afp property. As q^* is both an isometry and weak*-to-weak* homeomorphism, we get that Z has the $\sigma(X', X)$-afp property.

■

As a consequence we get

Corollary 5.24 *Let X be an Asplund Banach space. Then X has the weak-afp property [13].*

Remark 5.6 There is a difference between approximation in the norm and in the weak topology of a Banach space X. Let $Y \subseteq X$ be a nonempty closed bounded set and $T\colon Y \to Y$ a continuous mapping. For approximation in the norm, we have the equivalence of the following three conditions:

(i) There is a sequence $(x_n)_n$ in Y such that $x_n - T(x_n) \longrightarrow \theta$.
(ii) The point θ is in the norm-closure of the set $\{x - T(x)\colon x \in Y\}$.
(iii) $\inf\{\|x - T(x)\colon x \in Y\|\} = 0$.

These three statements are trivially equivalent (by properties of metric spaces) and are rather strong. For the weak topology, the situation is different. First, there is no analogue of the third condition. Secondly, the analogue of the second one is satisfied by Proposition 5.9. But the analogue of the first one is not satisfied always, as the weak topology is not in general described by sequences.

We give two instructive examples.

Definition 5.18 A Banach space X is called

(i) weakly compactly generated if there is a weakly compact subset $Y \subseteq X$ whose linear span is dense in X.
(ii) weakly Lindelöf determined provided there is $Y \subseteq X$ with dense linear span such that for each $\phi \in X'$ there are only countably many $x \in Y$ with $\phi(x) \neq 0$.

Basic properties of these classes of Banach spaces and complements on these notions can be found in [55, 100].

Examples 5.1

1. Every reflexive Banach space is weakly compactly generated by its closed unit ball.
2. Any separable Banach space is weakly compactly generated.
3. Any weakly compactly generated Banach space is weakly Lindelöf determined.

Proposition 5.11 *Let X be a Banach space. Then X' has the $\sigma(X', X)$-afp property in the following cases*

- *X is separable.*
- *X is weakly compactly generated. In particular, $X = c_0(\Gamma)$ or $X = L_1(\mu)$ for σ-finite measure μ.*
- *X is weakly Lindelöf determined.*

Proof Let X be weakly Lindelöf determined. Then any bounded separable subset of $(X', \sigma(X', X))$ is metrizable. Therefore $(X', \sigma(X', X))$ is $\aleph_0$-monolithic. ■

5.2.3 *The Weak Approximate Fixed Point Property in Metrizable Locally Convex Spaces and l_1-Sequences*

Using the slight generalization of Rosenthal's l_1-theorem and the Fréchet-Urysohn property of the space $(\overline{Y - Y}^{\sigma(X,X')}, \sigma(X, X'))$, the following is proved in [14].

Proposition 5.12 *Let (X, τ) be a metrizable locally convex space, $Y \subseteq X$ a nonempty convex bounded set which does not contains any l_1-sequence. Then Y has the weak-afp property.*

Proof Let $T : Y \to Y$ be a τ-to-weak continuous mapping. First let us find a nonempty separable convex $W \subseteq Y$ with $T(W) \subseteq \overline{W}$. To do that fix $x_0 \in Y$

and set $W_0 = \{x_0\}$. Suppose that $W_n \subseteq Y$ is a nonempty separable convex set. Then $T(W_n)$ is a weakly separable subset of $\overline{Y}$. As weakly separable sets are separable, we can find $S_n \subseteq T(W_n)$ a countable τ-dense set. As τ is metrizable and $S_n \subseteq \overline{Y}$, there is a countable set $Q_n \subseteq Y$ with $S_n \subseteq \overline{Q_n}$. Set $W_{n+1} = \text{conv}(W_n \cup Q_n)$. Then W_{n+1} is a separable convex subset of Y containing W_n. Finally, Set $W = \bigcup_{n=0}^{\infty} W_n$. Then W is a nonempty convex separable subset of Y and, moreover,

$$T(W) = \bigcup_{n=0}^{\infty} T(W_n) \subseteq \bigcup_{n=0}^{\infty} \overline{S_n} \subseteq \bigcup_{n=0}^{\infty} \overline{Q_n} \subseteq \bigcup_{n=0}^{\infty} \overline{W_{n+1}} \subseteq \overline{W}.$$

From Proposition 1.40 we get $\theta \in \overline{\{x - T(x) \colon x \in W\}}^{\sigma(X,X')}$. Thus, according to Proposition 5.10, there exists a sequence $(x_n)_n$ in W so that $x_n - T(x_n) \longrightarrow \theta$ in the weak topology. This proves the result. ∎

The following characterizes the heredity of the $\sigma(X, X')$-afp property [14].

Theorem 5.28 *Let X be a metrizable locally convex space and Y a nonempty closed convex bounded subset of X. Then the following assertions are equivalent.*

1. *Each nonempty closed convex subset of Y has the weak-afp property.*
2. *Y contains no sequence equivalent to the standard basis of l_1.*

Proof 1. $\Rightarrow$ 2. Let us suppose by contradiction that 2. Fix an l_1-sequence $(x_n)_n$ in Y, and denote by W the closed convex hull and by Z the closed linear span of the set $\{x_n : n \in \mathbb{N}\}$. Let $T_0 \colon l_1^0 \to X$ be defined by (1.17). By our assumption T_0 is an isomorphism of l_1^0 onto $T_0(l_1^0)$. Denote by S_0 its inverse. Then S_0 is an isomorphism of $T_0(l_1^0)$ onto l_1^0. In particular, S_0 maps Cauchy sequences to Cauchy sequences. Thus S_0 can be uniquely extended to a continuous linear mapping $S \colon \overline{T_0(l_1^0)} \to l_1$. Note that $\overline{T_0(l_1^0)} = Z$ and that S is an isomorphism of Z onto $S(Z) \subseteq l_1$. As S is linear, it is also a weak-to-weak homeomorphism.

We claim that the set W does not have the weak-afp property. Suppose on the contrary that it has the weak-afp property. Then $S(W)$ has the weak-afp property as well. But then, by Schur's theorem, $S(W)$ has the afp property. By Theorem 5.20, we get $S(W)$ is totally bounded. But it cannot be the case as $S(W)$ contains the canonical basis of l_1. This completes the proof.

2. $\Rightarrow$ 1. Follows from Proposition 5.12. ∎

Corollary 5.25 *Let X be a metrizable locally convex space not containing any l_1-sequence. Then X has the weak-afp property.*

We have the following illustrative examples on the weak-afp property and the $\sigma(X', X)$-afp property [14].

Proposition 5.13

1. *If $X = c_0$ endowed with $\|.\|_\infty$, then $X' = l_1$ has the $\sigma(X', X)$-afp property, but does not have the weak-afp property.*
2. *If $X = l_\infty$, then X' does not have the $\sigma(X', X)$-afp property.*

Proof As c_0 is separable, by Proposition 5.11, c_0' has the $\sigma(X', X)$-afp property. Further by Theorem 5.28, l_1 does not have the weak-afp property.

1. As c_0 is separable, by Proposition 5.11, c_0' has the $\sigma(X', X)$-afp property. Further by Theorem 5.28 does not have the weak-afp property.
2. The space l_∞ is a Grothendieck space. So, if X' had the $\sigma(X', X)$-afp property, then it would have also the weak-afp property. But it is not the case as X' contains an isometric copy of l_1.

■

5.2.4 The Weak Approximate Fixed Point Property in Non-metrizable Locally Convex Spaces

In the previous subsection, the metrizability assumption was used several times to obtain the weak-afp property. It is natural to ask if this assumption is necessary.

The following example given in [14] illustrates that the assumption of metrizability cannot be dropped in the statement of Theorem 1.37.

Example 5.2 Let $X = (l_1, \sigma(l_1, l_1'))$. Let $(e_n)_{n\geq 1}$ denote the canonical basic sequence. By Remark 1.28, X contains no l_1-sequence and the sequence $(e_n)_{n\geq 1}$ contains neither a weakly Cauchy subsequence nor a subsequence which is an l_1-sequence. But X does not have the weak-afp property.

Proof Let Z be the closed convex hull of $\{e_n : n \geq 1\}$. As Z is contained in the positive cone of l_1, by Proposition 1.39, the norm and weak topologies coincide on Z. Thus Z has the weak-afp property in X if and only if it has the weak-afp property in $(l_1, \|.\|_1)$. But it does not have the weak-afp property in $(l_1, \|.\|_1)$ as it contains an l_1-sequence when considered in the norm topology. ■

In [14], Barroso, Kalenda, and Lin raised the following open questions:
Let X be a Hausdorff locally convex space.

- Is it true that each bounded sequence in X has a weakly Cauchy subsequence if and only if each bounded separable subset is Fréchet-Urysohn in the weak topology?
- Is it true that X has the weak-afp property if and only if each bounded sequence in X has a weakly Cauchy subsequence?

An affirmative answer to both questions is given in [14] where X admits a locally convex topology compatible with the duality.

Theorem 5.29 *Let (X, τ) be a Hausdorff locally convex space such there is a metrizable locally convex topology on X compatible with the duality. Let Y be a nonempty closed convex bounded subset of X. The following assertions are equivalent.*

1. *Each nonempty closed convex subset of Y has the weak-afp property.*
2. *Each sequence in Y has a weakly Cauchy subsequence.*

Proof Let ρ be a metrizable locally convex topology on X compatible with the duality. As any metrizable locally convex topology is Mackey, we get $\sigma(X, X') \subseteq \tau \subseteq \rho$.

2. $\Rightarrow$ 1. Let $Z \subseteq Y$ be a nonempty closed convex subset of Y and $T: Z \to Z$ a continuous map. We find a nonempty τ-separable closed convex set $W \subseteq Z$ with $T(W) \subseteq W$ (see the proof of Theorem 5.27). By Proposition 5.9 we get that θ belongs to the weak closure of $\{x - T(x): x \in W\}$. Further, as W is τ-separable, it is also ρ-separable. By Theorem 1.37 W contains no l_1-sequence in (X, ρ), hence by Proposition 1.40 the weak closure of $W - W$ is Fréchet-Urysohn in the weak topology, hence there is a sequence $(x_n)_n$ in W such that $x_n - T(x_n)$ weakly converge to θ.

1. $\Rightarrow$ 2. Suppose that 2. does not hold, i.e., that there is a sequence in Y having no weakly Cauchy subsequence. By Theorem 1.37 there is a sequence $(x_n)_n$ in Y which is an l_1-sequence in (X, ρ). Let Z be the closed convex hull of $\{x_n: n \in \mathbb{N}\}$ and W be the closed linear spans with respect to the topology τ, ρ or $\sigma(X, X')$ as all these topologies have the same dual. By the proof of the implication 1. $\Rightarrow$ 2. of Theorem 5.28 there is a linear mapping $G: W \to l_1$ which is an isomorphism of (W, ρ) onto $G(W)$ and, moreover, (Z, ρ) does not have the weak-afp property. We claim that (Z, τ) does not have the weak-afp property. This will be done if we show that the topologies ρ and τ coincide on Y. To do that we recall that G is a ρ-to-norm isomorphism and weak-to-weak homeomorphism of Z onto $G(Z)$ and, moreover, $G(Z)$ is contained in the positive cone of l_1 the norm and weak topologies coincide. It follows that ρ and $\sigma(X, X')$ coincide on Z. As $\sigma(X, X') \subseteq \tau \subseteq \rho$, the proof is complete. ■

In [14], Barroso et al. conjectured that at least the first question has negative answer. They gave a candidate for a counterexample the space $(X, \sigma(X', X))$ where X is one of the Johnson-Lindenstrauss spaces constructed in [94].

5.3 Approximate Fixed Point Nets

For fixed point results, if one relaxes the compactness assumption, the assumption of continuity must be strengthened. Another possibility is to relax the continuity assumption.

In [9], attention is paid to sequential continuity and the following result is proved.

Theorem 5.30 *Let Y be a nonempty weakly compact subset of a metrizable locally convex space X. If $T: Y \to Y$ is weakly sequentially continuous, then T has a fixed point.*

It is natural to ask whether the metrizability condition can be dropped?

The sequential continuity is a very weak condition. Sequential continuity does not help in obtaining approximate fixed point sequences even for self maps of compact convex sets. So, sequential continuity is too weak to ensure the existence of fixed points.

To illustrate this fact, the following example is given in [15].

Example 5.3 There is a Hausdorff locally convex space X equipped with its weak topology, a nonempty compact convex subset $Y \subseteq X$, and a sequentially continuous map $T: Y \to Y$ with no approximate fixed point sequence.

Proof Let $X = (l'_\infty, \sigma(l'_\infty, l_\infty))$ and $Y = \{\mu \in X: \mu \geq 0 \text{ and } \|\mu\| \leq 1\}$. Then X is a locally convex space, the topology is its weak one, and Y is a nonempty convex compact subset of X. It remains to construct the function T.

The space l'_∞ can be canonically identified with the space $M(\beta\mathbb{N})$ of signed Radon measures on the compact space $\beta\mathbb{N}$ (Cech-Stone compactification of natural numbers). Let $P: M(\beta\mathbb{N}) \to M(\beta\mathbb{N})$ be defined by

$$P(\mu) = \sum_{n=1}^{\infty} \mu(\{n\})\delta_n, \quad \mu \in M(\beta\mathbb{N}),$$

where δ_x denotes the Dirac measure supported by x. Then P is a bounded linear operator. We set $Y_0 = P(Y)$. Then $Y_0 \subseteq Y$ and Y_0 is a convex subset of l'_∞ which is not totally bounded in the norm. Hence, by Theorem 5.20, there is a Lipschitz map $G: Y_0 \to Y_0$ without an approximate fixed point sequence (with respect to the norm).

Set $T = G \circ P|_Y$. We claim that T is weak*-to-weak* sequentially continuous and has no approximate fixed point sequence in the weak* topology.

To show the first assertion, let $(\mu_n)_n$ be a sequence in Y weak* converging to some $\mu \in Y$. Since l_∞ is a Grothendieck space, $\mu_n \to \mu$ weakly in l'_∞. Since P is a bounded linear operator, it is also weak-to-weak continuous, hence $P(\mu_n) \to P(\mu)$ weakly in l'_∞. Since $P(l'_\infty)$ is isometric to the space l_1, by the Schur property we have $P(\mu_n) \to P(\mu)$ in the norm, so $G(P(\mu_n)) \to G(P(\mu))$ in the norm. We conclude that $T(\mu_n) \to T(\mu)$ in the norm, and hence is also in the weak* topology. This completes the proof that T is sequentially continuous.

Next, suppose that $(\mu_n)_n$ is an approximate fixed point sequence in Y. Then $\mu_n - T(\mu_n) \to \theta$ in the weak* topology. By the Grothendieck property of l_∞ we get that $\mu_n - T(\mu_n) \to \theta$ weakly in l'_∞. Since P is a bounded linear operator, we get $P(\mu_n) - P(T(\mu_n)) \to \theta$ weakly, so $P(\mu_n) - P(T(\mu_n)) \to \theta$ in the norm by the Schur theorem. Further,

$$P(\mu_n) - P(T(\mu_n)) = P(\mu_n) - T(\mu_n) = P(\mu_n) - G(P(\mu_n)),$$

so $(P(\mu_n))_n$ is an approximate fixed point sequence for G with respect to the norm. This is a contradiction, completing the proof. ■

Definition 5.19 Let Y be a nonempty subset of a Hausdorff topological vector space (X, τ) and $T\colon Y \to \overline{Y}$ be a mapping. A net $(x_\alpha)_\alpha$ in Y is called a τ-approximate fixed point net for T if $x_\alpha - T(x_\alpha) \xrightarrow{\tau} \theta$, or equivalently that $\theta \in \overline{\{x - T(x)\colon x \in Y\}}$.

The lemma given below is a slight generalization of a result of Fan [56].

Lemma 5.10 *Let Y be a subset of a topological vector space (X, τ), Z a nonempty finite subset of Y such that* $\operatorname{conv}(Z) \subseteq Y$ *and $F\colon Z \to 2^Y$ a multifunction with the following two properties.*

1. *$F(z)$ is sequentially closed in Y for all $z \in Z$.*
2. $\operatorname{conv}(W) \subseteq \bigcup_{z\in W} F(z)$ *for all $W \subseteq Z$.*

Then $\bigcap_{z\in Z} F(z) \neq \emptyset$.

The only generalization consists in assuming that the values $T(z)$ are sequentially closed in Y (not necessarily closed in X).

Lemma 5.11 *Let Y be an almost convex subset of a topological vector space (X, τ), let p be a continuous seminorm on X, and let $T\colon Y \to \overline{Y}$ be a τ-to-p sequentially continuous map such that $T(Y)$ is p-totally bounded . Then for each $\varepsilon > 0$ there is $x \in Y$ with $p(x - T(x)) < \varepsilon$ [15].*

Proof Let $\varepsilon > 0$ be arbitrary. Since $T(Y)$ is p-totally bounded, and $T(Y) \subseteq \overline{Y}$, there is a finite set $\{x_1, \cdots, x_n\} \subseteq Y$ such that for any $x \in T(Y)$ there is some $i \in \{1, \cdots, n\}$ with $p(x - x_i) < \frac{\varepsilon}{2}$. Since Y is almost convex, we can also find a finite set $\{z_1, \cdots, z_n\} \subseteq Y$ so that $p(z_i - x_i) < \frac{\varepsilon}{2}$ for each $i = 1, \cdots, n$ and $\operatorname{conv}(\{z_1, \cdots, z_n\}) \subseteq Y$. Now, set $Z = \{z_1, \cdots, z_n\}$, and define a multifunction $F\colon Z \to 2^Y$ by putting, for each i,

$$F(z_i) = \left\{x \in Y\colon\, p(T(x) - x_i) \geq \frac{\varepsilon}{2}\right\}.$$

Since p is continuous and T is sequentially continuous, each $T(z_i)$ is sequentially closed in Y. Moreover, we have

$$\bigcap_{i=1}^{n} F(z_i) = \emptyset.$$

This follows from the choice of $x_1, \cdots, x_n$. By Lemma 5.10 applied to F, Z and Y, we conclude that there exist a subset $\{z_{k_1}, \cdots, z_{k_m}\}$ of Z and an $x \in$

$\text{conv}(\{z_{k_1}, \cdots, z_{k_m}\})$ such that $x \notin \bigcup_{j=1}^{m} F(z_{k_j})$. Hence $p(T(x) - x_{k_j}) < \frac{\varepsilon}{2}$ for all $j = 1, \cdots, m$, so by the triangle inequality,

$$p(T(x) - z_{k_j}) < \varepsilon \ \text{ for all } \ j = 1, \cdots, m.$$

Since $x \in \text{conv}(\{z_{k_1}, \cdots, z_{k_m}\})$, we get, by using again the triangle inequality,

$$p(x - T(x)) < \varepsilon.$$

This completes the proof. ■

As a consequence of the above lemma, the following generalization of Propositions 5.9 and 5.10 is given in [15] and shows that for approximate fixed points, sequential continuity is strong enough.

Theorem 5.31 *Let Y be an almost convex subset of a topological vector space (X, τ), let σ be a weaker locally convex topology on X, and let $T : Y \to \overline{Y}$ be a τ-to-σ sequentially continuous map such that $T(Y)$ is σ-totally bounded. Then T has an approximate fixed point net.*

As a consequence of the above theorem, the following optimal extension of Corollary 5.7 to sequential continuous mapping is given [15].

Corollary 5.26 *Let Y be a convex subset of a locally convex space X, and let $T : Y \to \overline{Y}$ be a sequentially continuous map such that $T(Y)$ is totally bounded. Then T has an approximate fixed point net.*

Remark 5.7 When X is metrizable (or, more generally, *Fréchet-Urysohn*), we even get an approximate fixed point sequence. In general, an approximate fixed point sequence need not exist even if Y is compact (see Example 5.3) or if T is continuous (this follows from Remark 1.28 and Example 5.2 if we observe that in weak topology any bounded set is totally bounded).

The following interesting Propositions on approximate fixed point sequences and fixed points of affine maps are worth mentioning [15].

Proposition 5.14 *Let τ be a linear topology on the vector space X, $Y \subseteq X$ a nonempty bounded convex set, and $T : Y \to Y$ an affine selfmap. Then the mapping T has an approximate fixed point sequence.*

Proof Fix any $y_1 \in Y$ and define inductively the sequence $(y_n)_n$ by setting $y_{n+1} = T(y_n)$. Set

$$x_n = \frac{y_1 + \cdots + y_n}{n}.$$

Then

$$x_n - T(x_n) = \frac{y_1 - y_{n+1}}{n} \longrightarrow \theta$$

as Y is bounded. ■

Remark 5.8 In Proposition 5.14 no continuity property of T is assumed, and we obtain a generalization of a result in [62] where the result is proved for nets.

Proposition 5.15 *Let X be a topological vector space, $Y \subseteq X$ a nonempty bounded convex set, and $T: Y \to Y$ an affine selfmap. If Y is countably compact, and T is continuous, then T has a fixed point.*

Proof Let $(x_n)_n$ be an approximate fixed point sequence given by Proposition 5.14. Since Y is countably compact, there is some $x \in Y$ which is a cluster point of $(x_n)_n$, hence there is some subnet $(x_\alpha)_\alpha$ of $(x_n)_n$ which converges to x. By continuity of T we get $T(x_\alpha) \longrightarrow T(x)$. However, $(x_\alpha - T(x_\alpha))_\alpha$ is a subnet of $(x_n - T(x_n))_n$, hence $x_\alpha - T(x_\alpha) \longrightarrow \theta$. So, $x = T(x)$. ■

The following Proposition shows that if the metrizability condition is dropped, then in some very special cases sequential continuous maps have fixed points.

Proposition 5.16 *Let X be a topological vector space, $Y \subseteq X$ a nonempty bounded convex set, and $T: Y \to Y$ an affine selfmap. If Y is sequentially compact, and T is sequential continuous, then T has a fixed point.*

Proof The proof can be done in the same way as that of Proposition 5.15, we only use sequential compactness to extract a subsequence $(x_{n_k})_k$ converging to some $x \in Y$ and then we use sequential continuity to deduce that $T(x_{n_k}) \longrightarrow T(x)$. ■

In Proposition 5.16, the assumption of sequential compactness cannot be replaced by compactness as witnessed by the following example [15].

Example 5.4 There is a Hausdorff locally convex space X equipped with its weak topology, a nonempty compact convex subset $Y \subseteq X$, and an affine sequentially continuous function $T: Y \to Y$ with no fixed point.

Proof Let $X = (l'_\infty, \sigma(l'_\infty, l_\infty))$. We can regard X as signed Radon measures on $\beta\mathbb{N}$. Let Y be the subset of X consisting of probability measures. Then Y is compact and convex. Now, pick a decomposition $\{Z_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into infinite disjoint subsets. Next, we shall use this decomposition to define a sequence $(k_m)_m$ of natural numbers as follows.

(i) $k_1 \geq 2$ and $k_1 \notin Z_1$.

(ii) $k_{m+1} > k_m$, and $k_{m+1} \notin \bigcup_{i=1}^{m+1} Z_i$ for each $m \in \mathbb{N}$.

Let us define now a linear map $T: Y \to Y$ by the formula

$$T(\mu) = \mu(\beta\mathbb{N} \setminus \mathbb{N}).\delta_1 + \sum_{m=1}^{\infty} \mu(Z_m).\delta_{k_m},$$

where δ_x denotes the Dirac measure supported by x. Then T is a linear mapping which is norm-to-norm continuous, and hence weak-to-weak continuous on l'_∞. As l_∞ is a Grothendieck space, T is weak*-to-weak* sequentially continuous. In other words, it is sequentially continuous when considered from X to X. Further it is obvious that $T(Y) \subseteq Y$.

Finally, T has no fixed point in Y. Indeed, suppose that $\mu \in Y$ is a fixed point, i.e., $T(\mu) = \mu$. Since $T(\mu) = \mu$ is supported by the $\mathbb{N}$, we have

$$\mu(\{1\}) = T(\mu)(\{1\}) = \mu(\beta\mathbb{N} \setminus \mathbb{N}) = T(\mu)(\beta\mathbb{N} \setminus \mathbb{N}) = 0.$$

Hence μ is supported by the set $\{k_m : m \in \mathbb{N}\}$. Since μ is a probability measure, we can find the minimal m such that $\mu(\{k_m\}) \neq 0$. However,

$$\mu(\{k_m\}) = T(\mu)(\{k_m\}) = \mu(Z_m) = 0,$$

as $k_l \notin Z_m$ for $l \geq m$ by condition (ii). This is a contradiction. ■

It seems not to be clear whether the assumption that T is affine is essential in the statement of Propositions 5.15 and 5.16. The proofs given works provided T admits an approximate fixed point sequence. However, the best thing we can obtain is an approximate fixed point by Corollary 5.26. Indeed, countably compact sets in topological vector spaces are necessarily totally bounded [15].

Proposition 5.17 *Let (X, τ) be a topological vector space, and let $Y \subseteq X$ be a relatively countably compact subset. Then Y is totally bounded.*

Proof The proof will be done by contradiction. Suppose that Y is not totally bounded. This means that there is V a balanced neighborhood of zero, such that Y cannot be covered by finitely many translates of V. We can then construct by induction a sequence $(x_n)_n$ in Y such that for each $n \in \mathbb{N}$ we have

$$x_{n+1} \notin \{x_1, \cdots, x_n\} + V.$$

Then the set $Z = \{x_n, n \in \mathbb{N}\}$ is a closed discrete subset of X. Indeed, let W be a balanced neighborhood of θ such that $W + W \subseteq V$. Then for any $x \in X$ the set $x + W$ contains at most one element of Z. Indeed, suppose that $m < n$ and $\{x_m, x_n\} \subseteq x + W$, thus

$$x_n \in x + W \subseteq x_m + W + W \subseteq x_m + V,$$

a contradiction. It follows that Z is an infinite subset of Y without an accumulation point in X. Therefore Y is not relatively countably compact. ■

We have the following strengthening of Theorem 5.20 and a consequence of Corollary 5.11.

Corollary 5.27 *Let Y be a totally bounded convex subset of a normed space X. Then every continuous map $T : Y \to Y$ admits an approximate fixed point sequence.*

In [15], Barroso et al. raised the following open question:

Let X be a Hausdorff locally convex space, $Y \subseteq X$ a convex set, and $T : Y \to Y$ a mapping. Suppose that one of the following two conditions is satisfied.

- Y is countably compact and T is continuous.
- Y is sequentially compact and T is sequentially continuous.

Does T necessarily admit a fixed point?

It follows from Proposition 5.17 and Corollary 5.26 that T has an approximate fixed point net provided Y is countably compact and T is sequentially continuous. However, Example 5.3 illustrates that T need not have an approximate fixed point sequence even if Y is compact. And, Example 5.4 illustrates that even if T admits an approximate fixed point sequence, it need not have a fixed point. It follows that the above question is natural, as in the quoted examples the respective sets are not sequentially compact and the respective maps are not continuous.

The proof of Proposition 5.15 shows that if we are able to construct an approximate fixed point sequence, then we get the positive answer to the above question. However, there is no idea how to do that.

Some special cases when the answers are positive were given in [15]:

(1) X is angelic: the countably compact sets are compact and sequentially continuous maps on them are continuous.
(2) The countable subsets of Y have compact closures in Y.

The following theorem extends the assertion 1. of Theorem 5.20 to locally convex spaces [15]. It shows that any nonempty bounded convex of a locally convex space that is not totally bounded admits a uniformly continuous self map without an approximate fixed point net.

Theorem 5.32 *Let (X, τ) be a locally convex space and $Y \subseteq X$ a bounded convex set. Let p_0 be a continuous semi-norm such that Y is not p_0-totally bounded. Then there is a mapping $T : Y \to Y$ with the following properties.*

(i) T admits no approximate fixed point net.
(ii) For any continuous semi-norm p the mapping T is p_0-to-p Lipschitz.

Proof Without loss of generality, we can suppose that $\theta \in Y$. It follows that there exists $\delta > 0$ such that for any finite-dimensional subspace $F \subseteq X$ there is some $x \in Y$ with $\text{dist}_{p_0}(x, F) > \delta$. So, we can construct by induction a sequence $(x_n)_n$ in Y such that

(a) $p_0(x_1) > \delta$,
(b) $\text{dist}_{p_0}(x_{n+1}, \text{span}\{x_1, \cdots, x_n\}) > \delta$ for any $n \in \mathbb{N}$.

Let p be a continuous seminorm on X such that $p \geq p_0$. Since Y is bounded, there is $M > 0$ such that $p(x_n) \leq M$ for each $n \in \mathbb{N}$. Without loss of generality, we can suppose that $M > \delta$. Let $k_1 < k_2 < k_3 < k_4$ be natural numbers, and let α_i be scalars for $i \in \{1, \cdots, 4\}$. Then clearly

$$p\left(\sum_{i=1}^{4} \alpha_i x_{k_i}\right) \leq M \sum_{i=1}^{4} |\alpha_i|. \tag{5.9}$$

Let

$$c_i = \frac{1}{2^{2i+1}} \left(\frac{\delta}{M}\right)^{i-1} \quad \text{for } i = 1, \cdots, 4.$$

Then $\sum_{i=1}^{4} c_i < 1$, and hence there is some $i \in \{1, \cdots, 4\}$ such that $|\alpha_i| \geq c_i \sum_{i=1}^{4} |\alpha_i|$. Let i_0 be the greatest such i. Then

$$\begin{aligned} p\left(\sum_{i=1}^{4} \alpha_i x_{k_i}\right) &\geq p\left(\sum_{i=1}^{i_0} \alpha_i x_{k_i}\right) - p\left(\sum_{i=i_0+1}^{4} \alpha_i x_{k_i}\right) \geq \delta|\alpha_{i_0}| - M \sum_{i=i_0+1}^{4} |\alpha_i| \\ &\geq \left(\delta c_{i_0} - M \sum_{i=i_0+1}^{4} c_i\right) \sum_{i=1}^{4} |\alpha_i| = \delta\left(c_{i_0} - \frac{M}{\delta}\right) \sum_{i=1}^{4} |\alpha_i| \\ &\geq \frac{1}{2}\delta c_{i_0} \sum_{i=1}^{4} |\alpha_i| \geq \frac{1}{32} \cdot \frac{\delta^4}{M^3} \sum_{i=1}^{4} |\alpha_i|. \end{aligned}$$

The first inequality follows from the triangle inequality, and the second one follows from condition (**b**) above, using the fact that $p \geq p_0$ and from the choice of M. The third one follows from the choice of i_0, the next equality is obvious. The last two inequalities follow from the choice of the constants $c_1, \cdots, c_4$. So, for $m = \frac{1}{32} \cdot \frac{\delta^4}{M^3}$ we get

$$m \sum_{i=1}^{4} |\alpha_i| \leq p\left(\sum_{i=1}^{4} \alpha_i x_{k_i}\right). \tag{5.10}$$

Set

$$\Delta'_n = \operatorname{conv}(\{\theta, x_n, x_{n+1}\}) \quad \Delta' = \bigcup_{n=1}^{\infty} \Delta'_n.$$

Then $\Delta' \subseteq Y$. Moreover, by (5.9) and (5.10), we get that (Δ', p_0) is bi-Lipschitz isomorphic to Δ defined by(5.3) considered with the metric inherited from l_1 and for any continuous seminorm p on X satisfying $p \geq p_0$, the identity on Δ' is p_0-to-p bi-Lipschitz.

Let p be a continuous seminorm. Set $p_1 = p + p_0$. Then p_1 is a continuous seminorm satisfying $p_1 \geq p_0$. So, the identity on Δ' is p_0-to-p_1 Lipschitz. Hence, it is a fortiori p_0-to-p Lipschitz. It follows that the identity on Δ' is p_0-to-τ uniformly continuous. Since p_0 is a continuous seminorm, the identity is clearly τ-to-p_0 uniformly continuous. Let $J\colon (X, \tau) \to (X, p_0)$ be the identity mapping. Further, let Z be the quotient $(X, p_0)/p_0^{-1}(0)$. Then Z is a normed space. Let $q\colon (X, p_0) \to Z$ denote the quotient mapping. Then $q \circ J$ is a continuous linear mapping, hence it is uniformly continuous. Moreover, J is a uniform homeomorphism of Δ' onto $J(\Delta')$ and q is an isometry of $J(\Delta')$ onto $q(J(\Delta'))$. It follows that $q(J(\Delta'))$ is bi-Lipschitz isomorphic to Δ. So, by Proposition 5.4 there is a Lipschitz retraction $r_0\colon Z \to q(J(\Delta'))$. Then

$$r = (q \circ J_{|\Delta'})^{-1} \circ r_0 \circ q \circ J$$

is a uniformly continuous retraction of (X, τ) onto Δ'. Moreover, r is p_0-to-p_0 Lipschitz as $q \circ J$ has this property, r_0 is Lipschitz, and $q \circ J_{|\Delta'}$ is an isometry of (Δ', p_0) onto Z.

Since (Δ', p_0) is bi-Lipschitz isomorphic to Δ, by Proposition 5.6 there is a Lipschitz map $G_0\colon (\Delta', p_0) \to (\Delta', p_0)$ without an approximate fixed point net with respect to p_0. Further, set

$$T = G_0 \circ r_{|Y}.$$

Then T is a selfmap of Y which is p_0-to-p_0 Lipschitz. Moreover, T is p_0-to-p Lipschitz for any continuous semi-norm p (as $T(Y) \subseteq \Delta'$). Further, it has no approximate fixed point net. To see this, it is enough to find an $\varepsilon > 0$ such that $p_0(x - T(x)) \geq \varepsilon$ for each $x \in Y$. Let L be the p_0-to-p_0 Lipschitz constant of T. Let $\eta = \inf\limits_{x \in q(J(Y))} \|x - G_0(x)\|$. Set

$$\varepsilon = \frac{\eta}{L+2}.$$

Fix an arbitrary $x \in Y$. If $\mathrm{dist}_{p_0}(x, \Delta') \geq \varepsilon$, then $p_0(x - T(x)) \geq \varepsilon$ as $T(x) \in \Delta'$. If $\mathrm{dist}_{p_0}(x, \Delta') < \varepsilon$, find $y \in \Delta'$ with $p_0(x - y) < \varepsilon$. Then

$$\begin{aligned} p_0(x - T(x)) &\geq p_0(y - T(y)) - p_0(x - y) - p_0(T(x) - T(y)) \\ &\geq \eta - (1 + L)p_0(x - y) \\ &> (L + 2)\varepsilon - (1 + L)\varepsilon = \varepsilon. \end{aligned}$$

This completes the proof. ∎

Remark 5.9 Theorem 5.32 proves that the result of Corollary 5.26 is the best possible in a sense: the assumption of total boundedness is essential even for uniformly continuous maps.

5.4 Applications to Asymptotic Fixed Point Theory

Fixed point results for iterates T^m for m sufficiently large are intrinsically related to the problem for finding periodic solutions of ordinary differential equations, differential-difference equations, and functional differential equations [77, 95–97], and [195]. Jones [96] introduced the term "asymptotic fixed point theorems" to describe such results.

The asymptotic fixed point property concerns the possibility of getting fixed point results for continuous maps by imposing conditions in some of its iterates.

A long-standing conjecture in the fixed point theory probably due to Browder and which was formulated by Nussbaum in 1972 in [138] reads as follows:

Conjecture Let $(X, \|.\|)$ be a Banach space, $Y \subseteq X$ a nonempty closed bounded convex set and $T : Y \to Y$ be a continuous mapping such that T^m is compact for some $m \in \mathbb{N}$. Then T has a fixed point.

Nussbaum [137–140] proved this conjecture with the additional assumption that T restricted to an appropriate open set is continuously Fréchet differentiable and using algebraic topology methods. Similarly, Browder in [30] proved the above conjecture under the assumption that Y is a compact absolute neighborhood retract and $T^n(Y)$ is homologically trivial in Y. Deimling in [42] recalled the above conjecture. In [71] Górniewicz and Rozploch-Nowakowska with using algebraic topology, proved the above conjecture by adding assumption that T is locally compact. Steinlein [185], Hale, and Lopes [78] gave related results to the above conjecture.

A partial solution of the above conjecture is the following [12].

Proposition 5.18 *Let Y be a weakly compact convex subset of a Banach space $(X, \|.\|)$ and $T : Y \to Y$ be a demicontinuous mapping. Suppose that T^m is strongly continuous for some $m \in \mathbb{N}$. Then T has a fixed point.*

Proof Let $(x_n)_n$ be a weak-approximate fixed point sequence for T, for example the one given in Theorem 5.25. From Eberlein's theorem we conclude that up to a subsequence, still denoted by $(x_n)_n$, $x_n \rightharpoonup x$ in Y for some $x \in Y$. In particular, $T(x_n) \rightharpoonup x$. Since T^m is strongly continuous, this implies that $T(y) = y$ where $y = T^m(x)$. ∎

The following is an additional contribution to asymptotic fixed point theory [13].

Theorem 5.33 *Let X be an Asplund space, Y a bounded, closed convex subset of X and $T : Y \to Y$ a continuous map such that T^m is strongly continuous for some integer $m \geq 1$. Assume that T is weakly completely continuous, that is, it maps* *weakly Cauchy sequences into weakly convergent sequences. Then T has a fixed point.*

Proof Without loss of generality, we may assume that X' is separable. Let $(x_n)_n$ be a sequence in Y such that $(x_n - T(x_n))_n$ converges to θ weakly (for example, that given by Theorem 5.27). It follows from Rosenthal's theorem that $(x_n)_n$ has a weak Cauchy subsequence, say $(x_{n_k})_k$. Since T is weakly completely continuous, by passing to a subsequence if needed, we can assume that $T(x_{n_k}) \rightharpoonup x \in Y$. In particular, we have $x_{n_k} \rightharpoonup x$. As T^m is strongly continuous, it follows that $T^m(x_{n_k}) \to T^m(x)$. Similarly, since $T(x_{n_k}) \rightharpoonup x$, we have $T^m(T(x_{n_k})) \to T^m(x)$. Thus we have $T(T^m(x)) = T^m(x)$ and the proof is complete. ∎

Remark 5.10 Alspach [4] constructed an example of a weakly convex subset of $L_1[0,1]$ which admits a fixed point free isometry $T: Y \to Y$. In particular, since every isometry is proper, Alspach's example in conjunction with Proposition 5.8 show that the assumption T^m is strongly continuous in the previous theorem cannot be dropped.

The following asymptotic fixed point results are due to Šeda [178].

Theorem 5.34 *Let Y be a closed convex subset of a Banach space X and $T: Y \to Y$ be a continuous mapping, T is proper and there exists an integer $n \geq 2$ such that T^n is compact. Then T has a fixed point.*

Proof By the Schauder fixed point theorem, the assumption T^n is compact implies that there exists a point $y \in Y$ such that $y = T^n(y)$. Then Lemma 5.8 gives that either y is a fixed point of T, or there is a natural $l \geq 2$ such that $C = \{y, T(y), \cdots, T^{l-1}(y)\}$ is an l-cycle of T whereby $T(C) = C$. Suppose that the latter case is true. Then there exists a unique minimal admissible couple (C, Z_2). In view of Lemma 5.6, Z_2 satisfies (5.7). As T is proper, from the compactness of $\overline{T^n(Z_2)}$ it follows that $T^{-1}(\overline{T^n(Z_2)})$ as well as $\overline{T^{n-1}(Z_2)}$ are compact. Proceeding in this way, step by step we get that $\overline{T^{n-2}(Z_2)}, \cdots, \overline{T(Z_2)}$ are compact and hence Z_2 is compact, too. ∎

Theorem 5.35 *Let Y be a closed convex subset of a Banach space X and $T: Y \to Y$ be a continuous mapping and there exists an integer $n \geq 2$ such that T^n is compact. Then in the compact set $Z = \overline{T^n(Y)}$ either T has a fixed point or for each prime number $p \geq n$ there exists a p-cycle of T. Moreover, each cycle of T lies in Z.*

Proof As $W = \overline{\mathrm{conv}(Z)}$ is a convex compact subset of Y, and for each $k \geq n$ $T^k(W) \subseteq Z$, there exists a fixed point $x_k \in Z$ of T^k. If T has no fixed point, then x_k belongs to an l-cycle of T where l is a divisor of k. In case $k = p$, l is p. Further $T(Z) \subseteq Z$ and hence, together with x_p, all elements of this p-cycle of T belong to Z. The last statement follows from the fact that in each cycle of T there is an element in Z. ∎

5.4.1 Existence of Limit-Weak Solutions for Differential Equations

We consider the problem of finding limiting-weak solutions for a class of ordinary differential equations in reflexive Banach spaces. Such equations are closely related to Peano's theorem in infinite dimensional spaces.

We are concerned with the following vector-valued differential equations:

$$\begin{cases} u_t = f(t, u), & \text{in } X, \\ u(0) = u_0 \in X, \end{cases} \tag{5.11}$$

where $t \in I = [0, \gamma]$, $\alpha > 0$, X is a reflexive Banach space and $f : I \times X \to X$. Here, the field f is assumed to be a Carathéodory mapping, that is,

(H_1) for all $t \in I$, $f(t, .) : X \to X$ is continuous.
(H_2) for all $x \in X$, $f(t, .) : X \to X$ is measurable.

In [12], Barroso explored a new approach to (5.11). The basic idea is to weaken the notion of solution in a way that allows us to derive general existence results even without having additional conditions of continuity other than (H_1). To this aim, the developed theory on weak approximate fixed points for continuous mappings will be invoked.

The following notion of weak-approximate solution for (5.11) is due to Barroso [12].

Definition 5.20 (Limiting Weak Solutions) We say that an X-valued function $u : I \to X$ is a limiting-weak solution to the problem (5.11) if $u \in C(I, X)$ and there exists a sequence $(u_n)_n$ in $C(I, X)$ such that

1. $u_n \rightharpoonup u$ in $C(I, X)$,
2. for each $t \in I$,

$$u_0 + \int_0^t f(s, u_n(s))ds \rightharpoonup u(t), \text{ in } X,$$

3. and, u is almost everywhere strongly differentiable in I.

where the above integral is understood in Bochner sense.

To get an existence result of limiting-weak solutions to (5.11), Barroso [12] proved the following:

Theorem 5.36 *Let* $(X, \|.\|)$ *be a reflexive Banach space and* $f : I \times X \to X$ *be a Carathéodory mapping satisfying*

$$\|f(s, x)\| \leq \alpha(s)\varphi(\|x\|), \textit{for a. e.s} \in I, \textit{ and all } x \in X, \tag{5.12}$$

where $\alpha \in L_p[0, \gamma]$ *for some* $1 < p < \infty$, *and* $\varphi\colon [0, \infty) \to (0, \infty)$ *is nondecreasing continuous function such that*

$$\int_0^\gamma \alpha(s)ds < \int_0^\infty \frac{ds}{\varphi(s)}.$$

Then (5.11) *has a limiting-weak solution.*

For the proof of the above theorem, we will rely on the following weak compactness result of Dunford [49].

Theorem 5.37 *Let* (Ω, Σ, μ) *be a finite measure space and* X *be a Banach space such that* X *and* X' *have the Radon-Nikodým property. A subset* Y *of* $L_1(\mu, X)$ *is relatively weakly compact if*

1. *Y is bounded,*
2. *Y is uniformly integrable, and*
3. *for each* $\Lambda \in \Sigma$, *the set* $\left\{\int_\Lambda u d\mu\colon u \in Y\right\}$ *is relatively weakly compact.*

Proof Let us consider (Ω, Σ, μ) the usual Lebesgue measure space on I and denote by $L_1(\mu, X)$ the standard Banach space of all equivalence classes of X-valued Bochner integrable functions u defined on I equipped with its usual norm $\|.\|_1$. In what follows we shall use the following notations

$$\mathcal{A} = \Big\{u \in L_1(\mu, X)\colon \|u(t)\| \le b(t) \text{ for a.e. } t \in I\Big\},$$

$$\mathcal{B} = \Big\{u \in L_1(\mu, X)\colon \|v(t)\| \le \alpha(t)\varphi(b(t)) \text{ for a.e. } t \in I\Big\},$$

where

$$b(t) = J^{-1}\left(\int_0^t \alpha(s)ds\right) \text{ and } J(z) = \int_{\|u_0\|}^z \frac{1}{\varphi(s)}ds.$$

A straightforward computation shows that both $\mathcal{A}$ and $\mathcal{B}$ are convex. Also, as is readily seen, $\mathcal{A}$ is closed in $L_1(\mu, X)$. Moreover, since X is a reflexive space, we can apply Theorem 5.37 to conclude that $\mathcal{B}$ is a relatively weakly compact set in $L_1(\mu, X)$. Let us consider now the set

$$Y = \left\{u \in \mathcal{A}\colon u(t) = u_0 + \int_0^t \bar{u}(s)ds \text{ for a.e. } t \in I, \text{ and some } \bar{u} \in \mathcal{B}\right\}.$$

It is easy to see that Y is nonempty and convex. We claim now that Y is closed. Indeed, let $(u_n)_n$ be a sequence in Y such that $u_n \longrightarrow u$ in $L_1(\mu, X)$. Then

$$u_n(t) \longrightarrow u(t) \text{ in } X$$

for a.e. $t \in I$. In particular, $u \in \mathcal{A}$. On the other hand, since $\mathcal{B}$ is sequentially weakly compact and

$$u_n(t) = u_0 + \int_0^t \bar{u}_n(s)ds \text{ for a.e. } t \in I,$$

with $\bar{u}_n \in \mathcal{B}, n \geq 1$, we may assume that $(\overline{u}_n)_n$ converges weakly to some $\overline{u} \in L_1(\mu, X)$. Then, by fixing $\phi \in X'$ and taking into account that each $\int_0^t \phi(.)ds$ defines a bounded linear functional on $L_1(\mu, X)$, it follows that

$$\phi(u(t) - u_0) = \lim_{n\to\infty} \phi\left(\int_0^t \bar{u}_n(s)ds\right) = \lim_{n\to\infty} \int_0^t \phi(\bar{u}_n(s))ds = \int_0^t \phi(\bar{u}(s))ds,$$

for a.e. $t \in I$. Hence $\phi(u(t) - u_0) = \phi\left(\int_0^t \bar{u}(s)ds\right)$ for a.e. $t \in I$. This implies that

$$u(t) = u_0 + \int_0^t \bar{u}(s)ds \text{ for a.e. } t \in I,$$

since ϕ was arbitrary. It remains to show that $\overline{u} \in \mathcal{B}$. To this end, it suffices to apply Mazur's theorem since $\mathcal{B}$ is closed in $L_1(I, X)$ and $\overline{u}_n \rightharpoonup u$ in $L_1(I, X)$. This concludes the proof that Y is closed.

Thus, by applying once more Theorem 5.37, we reach the conclusion that Y is weakly compact in $L_1(I, X)$. Let us define now a mapping $F : Y \to Y$ by

$$F(u)(t) = u_0 + \int_0^t f(s, u(s))ds.$$

From now on, our strategy will be to obtain a weak-approximate fixed point sequence for F in $L_1(I, X)$ and then deduce that it is itself a weak-approximation of fixed points for F in $W^{1,p}(I, X)$, the Sobolev space consisting of all $u \in L_p(I, X)$ such that u' exists in the weak sense and belongs to $L_p(I, X)$. After this we will use the fact that the embedding $W^{1,p}(I, X) \hookrightarrow C(I, X)$ is continuous to recover the corresponding weak convergence in $C(I, X)$.

By using (H_1)-(H_2), we see that F is well-defined and that it is continuous with respect to the norm-topology of $L_1(I, X)$. The last assertion follows easily from Lebesgue's theorem on dominated convergence. According to Theorem 5.25, there exists a sequence $(u_n)_n$ in Y so that $u_n - F(u_n) \rightharpoonup \theta$ in $L_1(I, X)$. Observe that $u_n \in C(I, X)$ for all $n \geq 1$. Moreover, up to subsequences, we may assume that both $(u_n)_n$ and $(F(u_n))_n$ converge in the weak topology of $L_1(I, X)$ to some $u \in Y$. We claim now that $u_n - F(u_n) \rightharpoonup \theta$ in $C(I, X)$. Before proving this, we get a priori L_p-estimates for arbitrary functions $u \in Y$.

Fix any $u \in Y$:

(1) Using (H_2) we have

$$\begin{aligned}\|F(u)\|_{L_p} &\leq \|u_0\||I|^{\frac{1}{p}} + \left\{ \int_0^\gamma \left\| \int_0^t f(s, u(s))ds \right\|^p dt \right\}^{\frac{1}{p}} \\ &\leq \|u_0\||I|^{\frac{1}{p}} + \left\{ \int_0^\gamma \left(\int_0^t \|f(s, u(s))\|ds \right)^p dt \right\}^{\frac{1}{p}} \\ &\leq \|u_0\||I|^{\frac{1}{p}} + \left\{ \int_0^\gamma \left(\int_0^t \alpha(s)\varphi(b(s))ds \right)^p dt \right\}^{\frac{1}{p}} \\ &\leq \|u_0\||I|^{\frac{1}{p}} + \|\alpha\|_{L_1[0,\gamma]}\varphi(\|b\|_\infty)\gamma^{\frac{1}{p}},\end{aligned}$$

where $\|b\|_\infty$ denotes the supremum norm of b on I.

(2) Analogously, one can shows that

$$\|u_0\|_{L_p}|I|^{\frac{1}{p}} + \|\alpha\|_{L_1[0,\gamma]}\varphi(\|b\|_\infty)\gamma^{\frac{1}{p}}.$$

(3) It follows now from (H_2) and the L_p-assumption on α that

$$\|\partial_t F(u)\|_{L_p} \leq \varphi(\|b\|_\infty)\|\alpha\|_{L_p[0,\gamma]},$$

and

$$\|\partial_t u\|_{L_p} \leq \varphi(\|b\|_\infty)\|\alpha\|_{L_p[0,\gamma]}.$$

In consequence, the above estimates show that both $(u_n)_n$ and $(F(u_n))_n$ are bounded sequences in $W^{1,p}(I, X)$. In view of the reflexivity of $W^{1,p}(I, X)$, by passing to a subsequence, if necessary, we can find $v, w \in W^{1,p}(I, X)$ such that $u_n \rightharpoonup v$ and $F(u_n) \rightharpoonup w$ in $W^{1,p}(I, X)$. In particular, $u = v = w$ since the embedding $W^{1,p}(I, X) \hookrightarrow L_1(I, X)$ is continuous. Thus $u_n - F(u_n) \rightharpoonup \theta$ in $W^{1,p}(I, X)$. On the other hand, using now the fact the embedding $W^{1,p}(I, X) \hookrightarrow C(I, X)$ is also continuous, it follows that

$$u_n - F(u_n) \rightharpoonup \theta \text{ in } C(I, X), \text{ and}$$

$$u_n \rightharpoonup u \text{ in } C(I, X).$$

Therefore

$$u_0 + \int_0^t f(s, u_n(s))ds \rightharpoonup u(t) \text{ in } X, \tag{5.13}$$

for all $t \in I$, which proves 1. and 2. of Definition 5.20. It remains to prove the optimal regularity of the limiting-weak solution u. To this end, we may apply again Theorem 5.37 to conclude that

$$Z = \{f(., u_n(.)) : n \in \mathbb{N}\}$$

is relatively weakly compact in $L_1(I, X)$. Hence, by passing to a subsequence if necessary, we get

$$\int_0^t f(s, u_n(s))ds \rightharpoonup \int_0^t v(s)ds \ \text{ in } X, \tag{5.14}$$

for all $t \in I$ and some $v \in L_1(I, X)$. Combining (5.13) and (5.14) it follows that

$$u(t) = u_0 + \int_0^t v(s)ds,$$

for all $t \in I$. Hence, following the same arguments as [188], one can prove that u is almost everywhere strongly differentiable in I. This completes the proof of Theorem 5.36.

■

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