Topological Quantum Field Theory and the Emergence of Physical Space–Time from Geometry. New Insights into the Interactions Between Geometry and Physics

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1 Introduction

This paper addresses various topics and different issues related essentially to general relativity theory and quantum field theories, and, more generally, to the interactions between geometry and physics. It aims at presenting recent works and discussing new ideas and results from these topics. It focuses on the subject of the geometric and topological structures and invariants which enriched in a remarkable way cosmology and quantum field theories in the last century, say, starting from Einstein's general relativity until string theory. In the last three decades, new and deep developments in this direction have emerged from cosmology and theoretical physics.

The general goal of the paper is to examine some striking aspects of the role of geometrical and topological concepts and methods in the developments of theoretical physics, especially in cosmology, quantum field theory, string theory, quantum gravity and non-commutative geometry, and then to show the great significance of these concepts and methods for a better understanding of our universe and the physical world at the very small scale. From the beginning we would like to stress the crucial fact that many physical phenomena appear to be related to deep geometrical and topological invariants (Atiyah, 1988) and furthermore that they are effect which emerge, in a sense, from the geometric structure of space–time (Connes & Chamseddine, 2006; Vafa, 1998).

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2 Einstein's General Relativity and the Interaction Between Curvature and Matter

The first good example we would like to mention of this new point of view, which however rely upon ideas advocated by Riemann, Clifford and Poincaré, is that of general relativity, which showed that gravity was an effect of the space–time curvature (Boi, 2004, 2006b; Penrose, 2004; Regge, 1992). More precisely, with the general relativity theory, actual (physical) geometry enters the picture of Minkowski space–time (which, mathematically speaking, is a manifold with a Lorentz metric, i.e., a non-degenerate pseudo-Riemannian metric of signature $+ \ldots + -$; $\mathbf{R}^n$ with metric $(dx^1)^2 + \ldots + (dx^{n-1})^2 - (dt)^2$) by assuming the world-history of each particle is a geodesic and that the Ricci curvature of the metric reflects the structure of matter and energy present at each point. The Einstein field equations,

$$R_{\mu\upsilon} - 1/2 g_{\mu\upsilon} R + \Lambda g_{\mu\upsilon} = 8\pi G T_{\mu\upsilon}$$

states that mass and pressure warp space–time. These equations relate the metric to matter distribution. Thus, according to the general theory of relativity, the gravitational force has to be reinterpreted as the curvature of space–time in the proximity of a massive object. When the energy is very concentrated, then the deformation of space–time may change sufficiently its topological structure and not only its metric (Baez & Muniain, 1994; Boi, 2004a; Regge, 1992). Let us stress that general relativity related two fundamental concepts which had, till then, been considered as entirely independent, namely, the concepts of space and time, on the one hand, and the concepts of matter and motion, on the other. Indeed, the geometry of space–time is not given a priori, for, in some sense, depends on the underlying physical structure of space-time. General relativity theory predicts at least three fundamental phenomena of the physical reality: (i) the gravitational waves; (ii) the black holes; (iii) the expanding of the Universe.

One of the most important ideas of general relativity was that space–time, not space, was the fundamental intrinsic object and that its structure was to be determined by physical phenomena. Einstein's main discoveries were as follow: (i) Spacetime is a pseudo- Riemannian manifold, i.e., its metric ds^2 is not Euclidean but has the signature (+,−,−,−) at each point. In presence of matter (the gravitational field), general relativity, based on the geometric concepts discovered by Riemann (see Riemann, 1854; and Boi 2019a), replaces the flat (pseudo) metric of Poincaré, Einstein (special relativity) and Minkowski, $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$, by a curved spacetime metric whose components form the gravitational potential $g_{\mu\upsilon}$, $ds^2 = g_{\mu\upsilon}\, dx^\mu\, dx^\upsilon$. (ii) Gravitation is just the physical manifestation of the curvature of spacetime (as foreseen by Clifford in 1876, see (Clifford, 1876)). (iii) Light travels along geodesics. Another point should, however, be added. (iv) The metric of (flat) space–time is not Euclidean but has the form $ds^2 = dx^2 - dx^2 - dx^2 - dx^2$ at each point. This is what nowadays is called a Lorentzian structure. However, even in the absence of matter, the geometry of space–time could not be asserted to be flat but only Ricci flat, i.e., that its Ricci tensor, which can be calculated from the Riemannian curvature tensor, is 0 (Penrose, 2004; Regge, 1992).

3 Quantum Mechanics and the Idea of Non-Commutativity

The next essential advance in twenty-century physics has been quantum mechanics. Let us summarize some fundamental idea of this theory (Cao, 1997; Heisenberg, 1930). In quantum mechanics and relativistic quantum field theory formulated by W. Heisenberg, P. Jordan, W. Pauli, P. Dirac and E. Wigner, the position and velocity of a particle (at the subatomic scale) are non-commuting operators acting on a Hilbert space, and classical notions such as "the trajectory of a particle" do not apply. In the 19th and early twentieth century physics, many aspects of nature were described in terms of fields—the electric and magnetic fields, and the gravitational field. So, since fields interacts with particles, to give an internally coherent account of nature, the quantum concepts must be applied to fields as well as to particles. When this is done, quantities such as the components of the electric field at different points in space–time become non-commutative. When one constructs a Hilbert space in which these operators act, one finds many surprises. The distinction between fields and particles break down, since the Hilbert space of a quantum field is constructed in terms of particle-like excitations. Conventional particles such as electrons are reinterpreted as arising from the quantization of a field. In the process, one finds the prediction of "antimatter": for every particle there must be a corresponding antiparticle, with the same mass and opposite electric charge (Coleman, 1985).

The quantum field theories (QFT's) that have proved to be very important in describing elementary particle physics are gauge theories (Zeidler, 2011). The classical example of gauge theory is the theory of electromagnetism. The gauge group is the Abelian group $U(1)$. If the (physical) potential A denotes the $U(1)$ gauge connection, which locally can be regarded, mathematically speaking, as a one-form on space–time, then the curvature or electromagnetic field tensor is the two-form $F = dA$, and Maxwell's equation read: $0 = dF = d^{*}F$. Here * is the Hodge duality operator.

4 Gauge Theories: From H. Weyl to Yang-Mills

The second main step of the geometrization of physics in the twenty-century has been gauge theory, thanks to which several new deep geometrical and topological structures have emerged (Bourguignon & Lawson, 1982; Boi, 2011). Gauge theory is a quantum field theory obeying to the geometrical principle of local gauge invariance. Gauge theory was introduced by Hermann Weyl in 1918 as an attempt to unify general relativity with electromagnetism (Weyl, 1918, 1929). However, the theory of Weyl failed because of lacking of an appropriate quantum physics framework.

Gauge idea rebirths with the formulation of non-Abelian Yang-Mills theory in 1954 by Yang and Mills (Boi, 2019b; Yang & Mills, 1954). This new theory stems from the recognition of the structural similarity, from the mathematical viewpoint, of non-Abelian gauge (quantum) fields with general relativity and the understanding that both are connections (Yang, 1977; Bourguignon & Lawson, 1982). This last, defined over a fiber bundle and possessing a curvature, is a very deep geometrical concept introduced by Weyl and Cartan, which generalize the concept of parallel transport of Levi–Civita to a new mathematical object: that of a non-point-like space or manifold in which precisely the points are replaced by the fibers (Boi, 2004a).

The very idea of Yang and Mills consists in suggesting a new program of geometrization of physics, this time applied to the physical forces supporting the quantum world. They proposed that the strong nuclear interactions be described by a (quantum) field theory in the same manner than electromagnetism, which is exactly local gauge invariant, as it is general relativity. More precisely, they postulated that the local gauge was the *SU(2)* isotopic spin-group or *SU(2)* isotopic spin-connection on which the non-Abelian group (a compact Lie group[1]) acts. This idea was "revolutionary" because it changed the very concept of "identity" of what has been ever assumed to be an "elementary particle". The novel idea that the isotopic spin connection, and therefore the potential A_μ (where, in order to relate the phases function $\lambda(x_i)$ at different points, the familiar gauge transformation for A_μ was written in terms of the phase change: $A_\mu \to A_\mu - 1/e\, \partial_\mu \lambda$) acts like the *SU(2)* symmetry group is the most important result of Yang-Mills theory. The concept of isotopic-spin connection lies at the heart of local gauge theory. It shows explicitly how the gauge symmetry group is built into the dynamics of the interaction between particles and fields (Atiyah, 1990; Yang, 1977). Moreover, some of the important physical characteristics of the field can be deduced directly from the connection (the potential), which can be viewed as a linear combination of the generators of the *SU(2)* group. We can, in fact, associate this formal operation with real physical processes.

Let's add few specifications on the mathematical structure of gauge theory (for a more comprehensive exposition, see Bourguignon & Lawson, 1982; Manin, 1988; Zeidler, 2011). Yang-Mills or non-Abelian gauge theory can, at the classical level, be described similarly to the "classical" Abelian gauge theory, with *U(1)* (see above) replaced by a more general compact gauge group *G*. The definition of curvature must be modified to $F = dA + A \wedge A$, and Yang-Mills equations: $0 = dAF = dA{*}F$, where dA is the gauge-covariant extension of the exterior algebra derivative. These equations can be derived from the Yang-Mills Lagrangian

$$\mathcal{L} = 1/4g^2 \int Tr\, \mathrm{Tr} F \wedge *F,$$

where Tr denotes an invariant quadratic form on the Lie algebra of *G*. The Yang-Mills equations are non-linear, so, in contrast to the Maxwell equations, but like

[1] *Finite groups* are spacial cases of *compact Lie groups*. For example, the rotation group SO(3) of the three-dimensional Euclidean space or the gauge group U(1) × SU(2) × SU(3) of the Standard Model in elementary particle physics are compact Lie groups.

the Einstein equations for the gravitational field, they are not explicitly solvable in general. But they have certain properties in common with the Maxwell equations and, in particular, they describe at the classical level massless waves that travel at the speed of light.

The first (classical) Yang-Mills theory corresponds to the quantum version of Maxwell theory—known as Quantum Electrodynamics—, which gave a very accurate account of the quantum behaviour of electromagnetic fields and forces. The non-Abelian gauge theory were introduced for describing the other forces in nature, notably the weak force (responsible among other things for certain forms of radioactivity) and the strong or nuclear force (responsible among other things for the binding of protons and neutrons into nuclei). For the weak force, we have now the Weinberg-Salam-Glashow electroweak theory with gauge group: $H = SU(2) \times U(1)$.

The masslessness of classical Yang-Mills waves was avoided by elaborating the theory with an additional "Higgs field". This is a scalar field, transforming in a two-dimensional representation of *H,* whose non-zero and approximately constant value in the vacuum state reduces the structure group from *H* to *U(1)* sub-group (diagonally embedded in $SU(2) \times U(1)$). This theory describes both the electromagnetic and weak forces, in a more or less unified way; because of the reduction of the structure group to *U(1),* the long-range fields are those of electromagnetism only, in accord with what we see in nature.

To sum up what we said about gauge theory, let's stress that Yang and Mills showed for the first time that local gauge symmetry was a powerful fundamental principle that provided new insights into the newly discovered "internal" quantum numbers like isotopic spin. In their theory, isotopic spin was not just a label for the charge states of particles, but it was crucially involved in determining the fundamental forms of the interaction between these particles. The most important philosophical point is that in the gauge theories of quantum fields, symmetries of nature determine the properties of forces; therefore, it is allowed to say that mathematical groups and invariants are at the origin of the dynamics of physical forces.

Let's add that in the search for a non-linear generalization of Maxwell's equations to explain elementary particles, there are various symmetry properties one would require. These are: (i) External (spatial–temporal) symmetries invariant under the Lorentz and Poincare's groups and under the conformal group if one is taking the rest-mass to be zero; (ii) Internal (physical) symmetries invariant under the non-Abelian groups like *SU(2)* or *SU(3)* to account for the known feature of weak and strong interactions, respectively; (iii) Covariance or its supersymmetric coupling by working on a complex topological space–time.

5 String Theory and the Supersymmetric Picture of the Quantum World

The next fundamental step in the geometrization of physics has been realized by string theory, a quantum field theory that tries to unify in a coherent picture general relativity and quantum mechanics at a deeper level than that of the Standard Model of particle physics (Witten, 1995). String theory entails beautiful geometrical and topological new structures, more rich and powerful with respect to those developed before by the other quantum field theories. It is yet theoretically incomplete and hitherto physically untested (Marino, 2005; Vafa, 1998).

It is worth of recalling that originally string program go back, in a sense, to the ideas putted forward by the German mathematician Bernard Riemann about hundred-fifth years early. According to him, one can make two fundamental assumptions. (i) First, on a given n- dimensional manifolds there are many possible metric structures (i.e., many different functions for measuring the distance between any pair of infinitesimally near points), so that the problem of which structure is the one appropriate for physical space required empirical methods for its solution. In other words, Riemann stated explicitly in 1854 (Riemann, 1854) that the question of the geometry of physical space does not make sense independently of physical phenomena. And (ii) space does not exist independently of phenomena and its structure depends on the extent to what we can observe and what happens in the physical world. From the previous follows, say, a corollary even more insightful: in its infinitely small parts (nowadays we would say at the quantum level) space may not be accurately described even by the geometrical notions of Riemannian geometry (Ashtekar & Lewandowski, 2004).

This last idea, which is hinted in Riemann's statement (ii), remain dormant until the search for a unified field theory at the quantum level forced the physicists to reconsider the structure of space–time at extremely small distances. One of the ideas to which their efforts led them was that the geometry of spacetime was supersymmetric with the usual coordinates supplemented by several anticommuting (fermionic) ones. This is a model that reflects the highly fuzzy structure of spacetime in small regions (at the quantum scale 10^{-33} cm) where one can pass back and forth between bosonic and fermionic particles. Modern string theory (i.e., superstring theory) takes Riemann's vision even further, and replaces the points of spacetime by strings, thereby making the geometry even more non-commutative (see Connes, 1994, 1996; and Landi, 1999).

Let's address briefly some conceptual aspects and issues of superstring theory. Superstring theory relies on the two ideas of supersymmetry and spacetime structure of eleven dimensions. Supersymmetry require that for each known particle having integer spin 0, 1, 2, and so on, measured in quantum units—there is a particle with the same mass but half-integer spin (1/2, 3/2, 5/2 and so on), and vice-versa. Supersymmetry transforms the coordinate of space and time such that the laws of physics are the same for all observers. Einstein's general theory of relativity derives from this condition, and so supersymmetry implies gravity. In fact, supersymmetry predicts "supergravity", in which a particle with a spin of 2—the graviton—transmits gravitational interactions and has as a partner a graviton, with spin of 3/2.

Superstring is based on the fundamental notion of T-duality, which relates two kinds of particles that arise when a string loop around a compact (spatial) dimension. One kind, call them "vibrating particles", is analogous to those predicted by Kaluza and Klein and comes from vibrations of the loop of the string. Such particles are energetic if the circle is small. In addition, the string can wind many times around the circle, its energy become higher the more times it wraps around and the larger the circle. Moreover, each energy level represents a new particle—call them "winding particle". T-duality states that the winding particles for a circle of radius R are the same as the "vibrating particles" for a circle of radius $1/R$, and vice-versa. So, to a physicist, the two sets of particles are indistinguishable: a fat compact dimension may yield apparently the same particles as thin one.

String theory, if correct, entails a radical change in our concepts of spacetime. That is what one would expect of a theory that reconciles general relativity with quantum mechanics. The answer involves duality again. A vibrating string is described by an auxiliary two-dimensional field theory, whose Lagrangian is roughly

$$L = 1/2 \int d\tau\, d\sigma\, (\partial X/\partial \tau)^2 + (\partial X/\partial \sigma)^2.$$

Here, $X(\tau, \sigma)$ is the position of the string at proper time τ, at a coordinate σ along the string. In string theory, the auxiliary two-dimensional field theory plays a more fundamental role than spacetime, and spacetime exists only to the extent that it can be reconstructed from the two-dimensional theory. In other words, duality symmetries of the two-dimensional field theory put a basic restriction on the validity of the classical notion of spacetime.

All the attempts mentioned, which are aimed at solving one of the central problems in twentieth-century physics, i.e.: how to combine gravity and the other forces into a unitary theoretical explanation of the physical word, essentially depend on the possibility of building a new geometrical framework conceptually richer than Riemannian geometry. In fact, as we saw, it plays a fundamental role in non-Abelian gauge theories and in superstring theory, thanks to which a great variety of new mathematical structure has emerged. A very interesting hypothesis is that the global topological properties of the manifold's model of spacetime play a major role in quantum field theory and that, consequently, several physical quantum effects arise from the non-local metrical and topological structures of these manifold (Isham, 1988; Labastida & Lozano, 1989). Thus, the unification of general relativity and quantum theory requires some fundamental breakthrough in our understanding of the relationship between spacetime and quantum processes (Penrose, 2004). In particular the superstring theory, but also, in a different manner, loop quantum gravity, lead to the guess that the usual structure of spacetime at the quantum scale must be dropped out from physical thought (Carfora, 2011). Non-Abelian gauge theories satisfy the basic physical requirements pertaining to the symmetries of particle physics because they are geometric in character. They profoundly elucidate the fundamental role played by bundles, connections and curvature in explaining the essential laws of nature. Kaluza-Klein theories and more remarkably superstring theory showed that spacetime symmetries and internal (quantum) symmetries might be unified through the introduction of new structures of space with a different topology. This essentially means that "hidden" symmetries of fundamental physics can be related to the phenomenon of topological change of certain class of (presumably) non-smooth manifolds (Atiyah, 1990). This entails a number of extremely important mathematical and physical consequences, which partly are discussed in this paper.

6 New Developments and Conceptual Issues in Quantum Field Theory

Let us now address some of the recent most fundamental developments in mathematical and theoretical physics, and in particular, the fact that these developments point forwards the search for a new scheme of spacetime structure at the quantum scale. Quantum mechanics culminated in the "standard model" of particle interactions, which is a quantum field theory. The fundamental ingredients of nature that appear in the underlying equations are fields: the familiar electromagnetic field, and some twenty or so other fields. The so-called elementary particles, like photons and quarks and electrons, are "quanta" of the fields-bundles of the field's energy and momentum. The properties of these fields and their interactions are largely dictated by principles of symmetry, including Einstein's special principle of relativity, together with a principle of "renormalizability", which dictates that the fields can only interact with each other in certain specially ways. The standard model has passed every test that can be imposed with existing experimental facilities. However, many unsolved problems and open questions remain. We do not know why it obeys certain symmetries and not others, or why it contains six types of quarks, and not more or less. Finally, gravitation cannot be brought into the quantum field theoretic framework of the standard model, because gravitational interactions do not satisfy the principles of renormalizability that governs the other interactions. This constitutes at present one the most fundamental and challenging issues of researches in theoretical physics and mathematics. Both topological quantum field theories and non-commutative geometry dedicate much effort to find out a solution to the very hard and key problem of the renormalization of the standard model. This problem might be answered, following different paths, by the Witten's topological string approach (Witten, 1988) and the Connes's non-commutative approach (Connes, 1996).

The not-yet-achieved incorporation of the fundamental ideas of a dynamical space–time geometry into a quantum theory of matter is one of the central open problems of contemporary physics, whose solution may well require another radical change in the physicist's conception of nature and space–time. We think that a real understanding of the cosmological questions and of the nature of elementary particles can ever been achieved without a simultaneous deeper understanding of the nature of space–time itself. It is well-known that quantum mechanics taught us that the classical notions of the position and velocity of a particle were only approximations of the truth. Notably, it is not clear whether the Riemannian geometry—even in a revised and generalized form—is adequate for the description of the small-scale structure of space–time (Isham, 1988; Penrose, 2004). The Planck length $lP = (G\hbar/c^3)$ 1/2$\sim 10^{-33}$ cm is considered as a natural lower limit for the precision at which coordinates of an event in space–time make sense. Nevertheless, not only does quantum mechanics have some striking geometrical characters, but its description of the world also reveals a wealth of deep underlying mysteries—even bordering on paradox—which cannot arise merely from an inadequate human understanding of the implications of the theory's mathematical formalism. Instead, at some level, there must be a deviation from purely unitary evolution, so that state-vector reduction can become a real phenomenon (Ashtekar & Lewandowski, 2004). Moreover, because of the (mysterious) non-local nature of quantum entanglement, whatever the nature of this revolution might be, the final theory that will emerge must have a fundamentally non-local character. In effect, according to certain mathematical-physical theories, such as topological quantum field theories and especially superstring theory, the local information of the space–time fields and of the other fields is stored in global (topological) structures of space–time (Boi, 2004).

7 Non-Commutative Geometry and the Quantum Fields

This is also truth for non-commutative geometry, where the quantum field equations are calculated for the full set of internal space metric fluctuations allowed by the non-commutative geometry axioms in the spectral triple formulation of the standard model (Connes & Chamseddine, 2006). These calculations have been given both from the perspective of the spectral triple and from the perspective of Fredholm module.[2] It has been showed that studying these Fredholm modules using algebraic *K* theory and *K* homology leads to a suggested non-commutative version of Morse theory—a well-known tool for studying the topology of manifolds—which is applied to the finite spectral action. According to the spectral action principle, which has been introduced ten years ago by Connes and Chamseddine, the standard model of particle physics is formulated with a product (whose image is called the total space) of two spectral triples—one that represents the Euclidean space–time manifold and the other the zero-dimensional internal space of particles charges. The space–time coordinate functions remain commutative but the internal space is a non-commutative "manifold". The spectral action principle is an important step towards the unification

[2] Recall that if A is an involutive algebra over the complex numbers **C**, then a *Fredholm module* over A consists of an involutive representation of A on a Hilbert space H, together with a self-adjoint operator F, of square 1 and such that the commutator $[F, a]$ is a compact operator for all $a \in A$.

of gravity with particle physics; the Einstein-Hilbert action plus Weinberg-Glashow-Salam theory all result from a calculation of the eigenvalues of the Dirac operator on the total space and since the Dirac operator encodes the metric, the spectral action principle is a purely geometrical theory (Connes, 1995).

Formally, a spectral triple (A, H, D) provides the analog of a Riemannian spin manifold to non-commutative geometry (here we follow closely Connes and Chamseddine, 1996). It consists of an involutive, non-necessarily commutative algebra A, a Hilbert space H: a finitely generated projective module on which the algebra is represented, and a Dirac operator D that gives a notion of distance, and from which is built a differential algebra. A very important technical point is that the geometry of any closed (even dimensional) Riemannian spin manifold can be fully described by a (real and even) spectral triple and a non-commutative geometry is essentially the same structure but with the generalization that the algebra of coordinates is allowed to be non-commuting. For the standard model the internal Hilbert space is $H = H \oplus H \oplus H^{\,C} \oplus HC$, where $LRLR$

$$H = (\mathrm{C2} \otimes \mathrm{CN} \otimes \mathrm{C3}) \oplus (\mathrm{C2} \otimes \mathrm{CN}),\ L$$

$$H = ((\mathrm{C} \oplus \mathrm{C}) \otimes \mathrm{CN} \otimes \mathrm{C3}) \oplus (\mathrm{C} \otimes \mathrm{CN}),\ R$$

and whose basis is labeled by the elementary fermions and their antiparticles. The symbol c is used to indicate the section represented by the antiparticles. The even triple has the Z/2-grading operator χ, the chirality (eigenvalues $+1$ or -1). In either case of HL and HR, the first direct summand is the quarks and the second the leptons. N Stands for the numbers of generations. For example, the left-handed up and down quarks form an isospin doublet and their right-handed counterparts are singlets and there are three colors for quarks and none for leptons. The charges on the particles are identified by the faithful representation of the algebra on the Hilbert space. In the definition of H above we see a second Z/2-grading that splits the Hilbert space into two orthogonal subspaces for particles and antiparticles: $H^{+} \oplus H^{-}$ or $H \oplus HC$. This is called $S0$ reality and is not an axiom but applies to the standard model as it excludes Majorana masses. The $S0$ reality grading operator ε satisfies:

$$[D, \varepsilon] = 0, \quad [J, \varepsilon]^{+} = 0, \quad \varepsilon* = \varepsilon, \quad \varepsilon^2 = 1.$$

8 The "ontology" of Newtonian Physics and Quantum Field Theory

Let us now address the important point concerning the differences between the "ontology" of classical physics and that of quantum physics. (Here this term stands for the nature and the kind of properties ascribed to the most fundamental physical entities from which a specific theory is built up and also to the mathematical objects by means of which one construct a definite space–time theory or model). One may affirm that Newtonian physics had a clear ontology: the world consisted of massive particles situated in Euclidean space. In that sense, the nature of space played a fundamental role. In the mathematical developments of Newtonian mechanics, however, the role of space is not clear. There is not much difference between the description of two particles moving in $\mathbf{R}^3$ and that of a single particle moving in $\mathbf{R}^6$, nor between that of a pivoted rigid body and that of a point moving on the group-manifold SO_3. In quantum mechanics the idea of space is even more elusive, for there seems to be no ontology, and, whatever wave-functions are, they are certainly not functions defined in space. Still, for about seventy years we have known that elementary particles must be described not by quantum mechanics but by quantum field theory, and in the field theory the role of space is quite different. Although it is an important fact that quantum field theory cannot be reconciled with general relativity, one could emphasize that the two theories have a virtual feature in common, for in both of them the points of space play a central and objective dynamical role. In quantum field theory two electrons are not described by a wave-function on $\mathbf{R}^6$; instead they constitute a state of a field in $\mathbf{R}^3$ which is excited in the neighborhood of two points. The points of space *index* the observables in the theory. The mathematics of quantum field theory is an attempt to describe the nature of space, but it proposes to look at space in a completely different way (Manin, 1988; Zeidler, 2011).

Like quantum field theory, Penrose's twistor theory is a radical attempt to get rid of space as a primary concept (Penrose, 1977). The Connes's program of noncommutative geometry amounts to a huge generalization of the classical notion of a manifold (Connes, 1994). Finally, string theory proposed a scheme for making space as an approximation to some more general kind of structure. One striking difference (maybe the essential one) between general relativity and quantum mechanics lie in the fact that, whereas in general relativity it seems impossible to separate the postulate of (continuous) space–time localization of events and the theory of gravitation from the (inner) geometric structure of space–time, on the other hand, it is precisely this postulate of the indistinguishability of the physical fields from the space–time geometry that got lost in quantum mechanics. It is particularly contradicted by the Bohr principle of complementarity and the Heisenberg uncertainty relations, which states the impossibility of knowing simultaneously the exact position and velocity of particles (electrons). These relations are indeed based on a model in which the electron jumps quickly from one orbit to another, radiating all energy thus liberated in the form of a global package, a *quantum* of light.

9 What It Could Be a Quantum Geometry of Space–time?

Many attempts have been made, starting from the sixties, to understand what kind of geometry and topology and therefore what kind of space–time model could be truly appropriate to describe the behavior of physical space both at the very large and quantum levels (Isham, 1988; Penrose, 2004). Among them, the most attractive and promising ones seem to be string theory, non-commutative geometry and loop quantum gravity (Ashtekar & Lewandowski, 2004; Carfora, 2011). The nature of quantum geometry is the central issue of non-perturbative quantum gravity. Is the familiar continuum picture then only an approximation? If so, what are the 'atoms' of geometry? What are its fundamental excitations? Is there a discrete underlying structure? If so, how does the continuum picture arise from this fundamental discreteness? By a quantized geometry, it is meant (Baez & Muniain, 1994) that there exist physical quantities which can take on continuous values classically but are such that the corresponding quantum operators have a discrete spectrum. In the resulting quantum geometry, Riemannian geometry can then emerge only as an approximation on a large scale. This topic can be discussed either from the perspective of topological quantum field theory and superstring theory or from that of non-commutative geometry.

The most attractive feature of non-commutative geometry is that it develops a new notion of geometric space where points do not play the central role, thus giving much more freedom for describing the subatomic-scale nature of spacetime. The theory proposed a framework which is sufficiently general to treat discrete space, Riemannian manifolds, configurations spaces of quantum field theory, and the duals of discrete groups which are not necessarily commutative. The development of a non-commutative geometry has been recently one of the most important attempts to unify (mathematically) quantum field theory with gravitation. In addition, its physical implications have found lately a confirmation in that it predicted a physical model for coupling gravity with matter (Connes, 1996).

The other fundamental change in our conception of spacetime and physics comes from superstring theory. Indeed, recent developments in theoretical physics suggest that a new kind of quantum geometry may enter physics, and that spacetime itself may be reinterpreted as an approximate, derived concept that one can extract from a two-dimensional field theory (Katz & Vafa, 1997; Witten, 1995). Intuitively, strings are viewed as one-dimensional objects whose modes of vibration represent the elementary particles. In addition, in string theory the one-dimensional trajectory (world-line) of a particle in space–time is replaced by a two-dimensional orbit (world-tube) of the string. The main conceptual point of the string program is that it entails some revolutionary ideas about our conception of space and space–time. Indeed, space is not more thought as formed up of points-like elements and therefore the particles not either. Space as well is endowed with a point-less structure. Instead of point-like elements, the space seems to be filled out of other kinds of geometrical objects, richer and more complex, like knots of many types, Riemannian surfaces, topological (unconventional) objects, and so on. The most interesting point is that space must be considered as a dynamical thing, which may change with respect to its metrical and topological properties (Boi, 2009b; Vafa, 1998). The main physical aspect of string theory is that all particles which we previously thought of as elementary, that is, as little points without any structure in them, turn out in fact not to be points at all but basically little loops of string which move through space, oscillating around it. We have thus that the different physical properties of matter emerge somehow from the different structural and dynamical patterns of these strings and loops in space. For example, the electric charge might be seen as a quality of the motion of the string rather than something which is just added on to a particle as fundamental object.

The idea of replacing point particles by strings sounds so naïve that it may be hard to believe that it is truly fundamental. But in fact, this naïve-sounding step is probably as basic as introducing the complex numbers in mathematics. If the real and complex numbers are regarded as real vector spaces, one has $\dim_{\boldsymbol{R}}(\mathbf{R}) = 1$, $\dim_{\boldsymbol{R}}(\mathbf{C}) = 2$. The orbit of a point particle in space–time is one-dimensional and should be regarded as a real manifold, while the orbit of a string in space–time is two-dimensional (over the reals) and should be regarded as a complex Riemann surface. Physics without strings is somehow analogous to mathematics without complex numbers.

10 New Insights Into the Nature of Space–time

We now outline some new ideas relating to the structure of space–time in the most recent physical theories, to start with general relativity. (i) The geometric structure of space–time gives rise to the dynamics of this same space–time, and in particular of the gravitational field. (ii) Even the other (fermionic and bosonic) fields describing matter and its electroweak and strong interactions seems to emerge as dynamical effects from the topological (global) structure of space–time. Conversely, the space–time itself must be henceforth thought of, in some sense, as a derived (changing) object whose metric and topological structures may be subject, to some extent, to the quantum fluctuations of these same fields. For example, one of the predictions of T-duality in string theory is that geometry and topology are *ambiguous* at the string length $lS = \sqrt{\alpha}$'. Furthermore, space is ambiguous at the Planck length $lP \ll lS$. Another more complicated and richer example of T-duality is the mirror symmetry and topology change in Calabi-Yau spaces. There are different types of dualities that play an important role in the recent developments of theoretical physics. One conclusion is, thus, that spacetime is likely to be an emergent, approximate, classical concept. The challenge is to have emergent spacetime, while preserving some locality (macroscopic locality, causality, analyticity, etc.). (iii) The recent developments of theoretical physics enable us to think that the discrete and continuous character of the laws of physics are but special cases according with each other in the framework of a new unitary mathematical-physical theory. With the theory of supergravity, and still more with string theory, we get a consistent theoretical framework which is finite and which simultaneously incorporate both quantum gravity and chiral supersymmetric gauge theories in a natural fashion. Supergravity generalizes a gauge theory proposed by H. Weyl in 1923 in order to unify the Einstein's theory of gravitation with the electromagnetic theory, and another by Kaluza and Klein in the 1920s, in which they suggested to further unify the concepts of internal and space–time symmetries by reducing the former to the latter through the introduction of some extra dimension of space, more precisely, a fifth (space-like) dimension, which has the topology of a circle. (iv) The physical (dynamical) and space–time symmetries dictate, at different extents, the various forces of nature and the interactions between particles. This is a very general principle and it is the crucial idea at the heart of quantum field theories. In fact, all physical phenomena seem to be founded upon such principle (Coleman, 1985). However, at a deeper level, one is increasingly led to believe that, beside symmetries (including, space–time, physical, broken symmetries, and maybe other "hidden" symmetries), topological structures and invariants might have an even more important role in determining physical phenomena at the very large and extremely small scales (Atiyah, 1989).

11 Topological Quantum Field Theory

Topological quantum field (TQFT) emerged in the eighties as a new relation between mathematics and physics. The relation connected some of the most advanced ideas in the two fields. The nineties have been characterized by its development, originating unexpected results in topology and testing some of the most fundamental ideas in quantum field theory and string theory. The first TQFT was formulated by Witten in 1988 (Witten, 1988). He constructed the theory now known as Donaldson-Witten theory, which constitutes a quantum field theory representation of the Donaldson invariants of four-manifolds (1983–84) (Donaldson, 1983). His work was strongly influenced by M. Atiyah. In 1988 Witten formulated also another two-dimensional TQFTs which have been widely studied during the last three decades: topological sigma models in two dimensions and Chern-Simons gauge theory in three dimensions (Marino, 2005). These theories are related, respectively, to Gromov invariants (Gromov, 1985), and to knot and link invariants as the Jones polynomial and its generalizations (Atiyah, 1988; Thurston, 1997; Turaev, 1994). TQFT has provided an entirely new approach to study topological invariants. Being a quantum field theory, TQFT can be analyzed from different point of view. The richness inherent to quantum field theory can be exploited to obtain different perspectives on the topological invariants involved in TQFT. This line of thought has shown to be very fruitful in the last two decades and new topological invariants as well new relations between them have been obtained.

TQFT have been studied from both, perturbative and non-perturbative points of view. In the case of Chern-Simons gauge theory, non-perturbative methods have been applied to obtain properties of knot and link invariants, as well as general procedures for their computation. Perturbative methods have also been studied for this theory providing integral representations for Vassiliev invariants. In Donaldson-Witten theory perturbative methods have proved its relation to Donaldson invariants (Donaldson, 1990). Non-perturbative methods have been applied after the work by Seiberg and Witten on $N = 2$ supersymmetric Yang-Mills theory. The outcome of this application is a totally unexpected relation between Donaldson invariants and a new set of topological invariants called Seiberg-Witten invariants.

Donaldson-Witten theory is a TQFT of cohomological type. TQFTs of this type can be formulated in a variety of frameworks. The most geometric one corresponds to the Mathai-Quillen formalism. In this formalism a TQFT is constructed out of a moduli problem. Topological invariants are then defined as integrals of a certain Euler class (or wedge products of the Euler class with other forms) over the resulting moduli space. A different framework is the one based on the twisting of $N = 2$ supersymmetry. In this case, information on the physical theory can be used in the TQFT. Indeed, it has been in this framework where Seiberg-Witten invariants have shown up. After Seiberg and Witten worked out the low energy effective action of $N = 2$ supersymmetric Yang-Mills theory, it became clear that a twisted version of this effective action could lead to topological invariants related to Donaldson invariants. The twisted action revealed a new moduli space, the moduli space of Abelian monopoles (Witten, 1994). Its geometric structure has been derived in the context of the Mathai-Quillen formalism. Invariants associated to this moduli space should be related to Donaldson invariants. This turned out to be the case. The relevant invariants for the case of *SU(2)* as gauge group are the Seiberg-Witten invariants.

Donaldson-Witten theory has been generalized after studying its coupling to topological matter fields. The resulting theory can be regarded as a twisted form of $N = 2$ supersymmetric Yang-Mills theory coupled to hypermultiplets, or, in the context of the Mathai-Quillen formalism, as the TQFT associated to the moduli space of non-Abelian monopoles. Perturbative and non-perturbative methods have been applied to this theory for the case of *SU(2)* as gauge group and one hypermultiplet of matter in the fundamental representation. In this case, again, it turns out that the generalized Donaldson invariants can be written in terms of Seiberg-Witten invariants. One would expect that in general the invariants associated to non-Abelian monopoles could be expressed in terms of some other simpler invariants, being Seiberg-Witten invariants just the first subset of the full set of invariants.
The present situation in three and four dimensions relative to Chern-Simons gauge theory and Donaldson-Witten theory, respectively, can be described as follows.

These theories share some common features. Their topological invariants are labeled with group-theoretical data: Wilson lines for different representations and gauge groups (Jones polynomials and its generalizations), and non-Abelian monopoles for different representations and gauge groups (generalized Donaldson polynomials); these invariants can be written in terms of topological invariants which are independent of the group and representation chosen: Vassiliev invariants and Seiberg-Witten invariants. This structure leads to the idea of universality classes of topological invariants. In this respect Vassiliev invariants constitute a class in the sense that all Chern-Simons or quantum group knot invariants for semi-simple groups can be expressed in terms of them. Similarly, Seiberg-Witten invariants constitute another class since generalized Donaldson invariants associated to several moduli spaces can be written in terms of them. This certainly holds for the two cases described above but presumably it holds for other groups. It is very likely that Seiberg-Witten invariants are the first set of a series of invariants, each defining a universality class.

12 Concluding Remarks

We stressed the crucial fact that many physical phenomena, at the quantum and at the cosmological level as well, appear to be deeply related to some geometrical and topological invariants, and furthermore that these phenomena are effects which emerge, in a sense, from the geometric and topological structure of space–time (Atiyah, 1990). The first good example of this new point of view, which actually rely upon ideas advocated by Riemann and Clifford, is that of general relativity, which showed that gravity is a manifestation of the curvature of space–time. The Einstein's field equations relate the metric to matter distribution. Thus, according to the general theory of relativity, the gravitational force has to be reinterpreted as the curvature of space–time in the proximity of a massive object. When the energy is very concentrated, then the deformation of space–time may change sufficiently its topological structure.

Topological quantum field theory (TQFT) appear as a very rich and promising research program in theoretical physics. Two conceptual points appear to be very significant, and likely promising for physics, in TQFT. (i) The first is the assumption of an effective correlation between knots and link invariants and the physical observables and states of quantum field theories and gauge theories. (ii) The second is, on the one hand, the idea of the fuzziness of physical space–time and of its emergence from the dynamical fluctuations of its metrical structure, on the other, the idea of the geometric and topological nature of physical phenomena at different scales.

More precisely, the main ideas we have addressed in this paper are the following:

(1) The geometric and topological deformations and invariants could generate the dynamics of space and time, of the quantum field and the gravitational field as well. For example, in string theory, the picture is that the different physical properties of matter are linked to the different topological configurations of strings and loops moving through space and oscillating around it. For instance, the electric charge might be seen as a quality of the motion of the string rather than something which is just added on to a particle as a fundamental object.

(2) The fermionic and bosonic fields composing matter and its electroweak and strong interactions seems to emerge as dynamical effects from the topological (global) deformations of the varying structure of space–time. Conversely, the space itself must be henceforth thought of, in some precise sense, as a derived and changing object whose metric and topological structures may be subject, to some extent, to the quantum fluctuations of these same fields. We already gave two very significative examples illustrating these facts, both relating to T-duality in string theory: the first predict the ambiguous character of geometry and topology at the string length scale; the second concerns mirror symmetry and topological change in Calabi-Yau spaces. After Riemann's revolution in the geometric vision of physical space, which goes very far beyond the discovering of what we now call "Riemannian geometry", for he has not only the idea that the distribution of matter in the universe depends upon the variation of curvature of space–time, but also the vision of a geometry for the microscopic (quantum) physical world as a dynamical and fluctuating object, the next revolution should

be to think that space–time might be an emergent, approximate, non-classical concept. The challenge is to prove the validity of the emergent global nature of space–time while preserving some locality (macroscopic locality, causality, analyticity, etc.) On of the most remarkable constituents of quantum geometry might be knots and other tangled structures. If different aspects of the link between the Jones polynomial and mathematical physics have been intensively studied in the last three decades and are quite well-known, the relationship between knots and quantum physics remain still almost unexplored. Recently, Witten suggested that, in quantum physics, a knot may be regarded as the orbit in space–time of a charged particle. One way of calculating the Jones polynomial in quantum theory involves using Chern-Simons function for gauge fields. But to use the Chern-Simons function, the knot must be a path in a space–time of three dimensions rather than the four dimensions of the real world.

(3) The recent developments of theoretical physics enable us to think that the discrete and continuous character of the laws of physics are but special situations according with each other in the context of a new unitary mathematical-physical theory. With the theory of supergravity, and still more with superstring theory, we get a consistent theoretical framework which is finite and which simultaneously incorporate both quantum gravity and chiral supersymmetric gauge theories in a natural fashion. Supergravity generalizes a gauge theory proposed by H. Weyl in 1923 in order to unify the Einstein's theory of gravitation with the electromagnetic theory, and another by Kaluza and Klein in the 1920s, in which they suggested to further unify the concepts of *internal* and *space–time* symmetries by the former to the latter through the introduction of some extra dimension of space, more precisely, a fifth (space-like) dimension, which has the topology of a circle.

(4) The physical ("internal") and space–time ("external") symmetries, which we tend to consider both dynamical because they can equally produce some physical effects, dictate, at different extents, the various forces of nature and the interactions between particles. This is a very general and meaningful principle and it is the crucial idea setting at the core of gauge quantum field theories. In fact, the most physical phenomena at different scales seem to be founded upon such principle. However, at a deeper level, one is increasingly led to believe that, beside symmetries—including space–time, physical and broken symmetries, and maybe other "hidden" symmetries –, topological deformations and invariants might have an even more important role in determining the dynamics of physical phenomena at the extremely small and very large scales. This is essentially related with the phenomenon of topological changes. It is much conceivable to think, on the one hand, that it can exist a deep link between symmetries and topological changes, and, on the other, that topological deformation be a new dynamical variable not depending on physical parameters but which may produce important physical effects as well.

All the previous aspects and ideas play an important role in the TQFT. Topological quantum field theory is a third sort of idealization of the physical world, besides general relativity and quantum field theory, which is attractative and deep from the mathematical and philosophical point of view as well. It is a background-free quantum field theory with no local degrees of freedom. The interesting thing is the presence of 'global' degrees of freedom (Baez & Muniain, 1994; Turaev, 1994).[3] Two spaces-times which are locally indistinguishable, since locally both look like the same model of space–time, can, hovewer, be distinguished globally, for example, by measuring the volume of the whole space–time or studying the behavior of geodesics that wrap around a 3-dimensional torus.

An axiomatic approach to topological quantum field theory was proposed by Atiyah (Atiyah, 1990). An important feature of TQFTs is that they do not presume a fixed topology for space or space–time. In other words, when dealing with an n-dimensional TQFT, we are free to choose any $(n-1)$-dimensional manifold to represent space at a given time. Moreover given two such manifolds, say S and S', we are free to choose any n-dimensional manifold M to represent the portion of spacetime between S and S'. For his construction, Atiyah used the notion of cobordism, introduced by R. Thom in the 1950s (Thom, 1954), and he developped a formalism in which he found that cobordism construction obeys to the algebraic properties of associativity (of manifolds), the non-commutativity of the composition of cobordism (this is related with the famous non-commutativity of observable in quantum theory) and an identity cobordism. The operations are dynamical in the sense that they formalize the notion of "passage of time" (temporal evolution) in a context where the topology of space–time is arbitrary and there is no background fixed metric. Atiyah's axioms relate this notion to quantum theory as follows. First, a TQFT must assign a Hilbert Space $Z(S)$ to each $(n–1)$-dimensional manifold S. Vectors in this Hilbert space represent possible states of the universe given that space is the manifold S. Second, the TQFT must assign a linear operator $Z(M)$: $Z(S) \rightarrow Z(S')$ to each n-dimensional cobordism M: $S \rightarrow S'$. This operator describes how states change given that the portion of space–time between S and S' is the manifold M. In other words, if space is initially the manifold S and the state of the universe is ψ, after the passage of time corresponding to M the state of the universe will be $Z(M)\,\psi$.

Baez and Muniain (1994) emphasized that the analogy between differantial topology and quantum theory "is exactly the sort of clue we should pursue for

[3] A good example is quantum gravity in 3-dimensional space–time. Classicaly, Einstein's equations predict qualitatively very different phenomena depending on the dimension of space–time. If space–time has 4 or more dimensions, Einstein's equations imply that the metric has local degrees of freedom. In other words, the curvature of space–time at a given point is not completely determined by the flow of energy and momentum through that point: it is an independent variable in its own right. For example, even in the vacuum, where the energy–momentum tensor vanishes, localized ripples of curvature can propagate in the form of gravitational radiation. In 3-dimensional space–time, hovewer, Einstein's equations suffice to completely determine the curvature at a given point of space-tume in terms of the flow of energy and momentum through that point. We thus say that metric has no local degrees of freedom. In particular, in the vacuum the metric is flat, so every small patch of empty space–time looks exactley like every other.

a deeper understanding of quantum gravity. At first glance, general relativity and quantum theory look very different mathematically: one deals with space and space-time, the other with Hilbert spaces and operators. (…) Topological quantum field theory suggests that perhaps they are not so different after all! Even better, it suggests a concrete program of synthesizing the two, which many mathematical physicists are currently pursuing. Sometimes this goes by the name of 'quantum topology'".

It seems likely that differential topology and quantum theory must merge if we are to understand background-free quantum field theories. In classical (Newtonian) physics, one treat space as a background on which states of the world are posed, and, similarly, one treat spacetime as a background on which the process of change occurs. But it could be that these be idealizations which we must overcome in a background-free theory, i.e. a theory with global degrees of liberty given by topological change. As Baez and Muniain pointed out, the concepts of 'space' and 'state' are, in fact, two aspects of a unified whole, and likewise for the concepts of 'spacetime' and 'process'. This fact might open new and significant perspectives for the mathematical and philosophical understanting of the physical world.

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