Vibration Analysis of Laminated Composite Beams Using a Novel Two-Variable Model with Various Boundary Conditions

Quoc-Cuong Le[1](✉), Trung-Kien Nguyen[2], and Ba-Duy Nguyen[3]

[1] Institute of Engineering and Technology, Thu Dau Mot University, 06 Tran Van On Street, Thu Dau Mot City, Vietnam
cuonglq@tdmu.edu.vn

[2] Faculty of Civil Engineering, Ho Chi Minh City University of Technology and Education, 1 Vo Van Ngan Street, Thu Duc City, Vietnam

[3] Faculty of Architecture, Thu Dau Mot University, 06 Tran Van On Street, Thu Dau Mot City, Vietnam

Abstract. The kinematics of the beam having only two variables are increased in a hybrid form under polynomial, and trigonometric series in thickness and axial directions, respectively. Lagrange's equations are then used to derive characteristic equations of the beams. Numerical results for laminated composite beams are equalled with previous studies and are used to investigate the effects of length-to-depth ratio, fibre angles and material anisotropy on the vibration of laminated composite beams.

Keywords: Laminated composite beams · Vibration · Elasticity solution

1 Introduction

Laminated composite materials are created by assembling multiple layers of fibrous materials to achieve the superior engineering properties such as bending stiffness, strength to weight ratio and thermal performance.

As a result, laminate composite has been widely applied in aerospace engineering, mechanical engineering as well as construction technology. In order to maximise the potential advantage of this multilayered material, numerous studies and computation modelling have been conducted to fine-tune the static and dynamic behaviours of laminated composite beams. Various beam theories have been developed in order to predict accurately their structural responses and capture anisotropy of laminated composite materials.

Classical beam theory (CBT) is the simplest one in analysing responses of laminated composite beams. Nonetheless, this theory underestimates deflections and overestimates natural frequencies of the beams due to neglecting effects of transverse shear deformation.

In order to account for this effect, thanks to its simplicity in formulation and programming, the first-order shear deformation beam theory (FSBT) is commonly used by

N. V. Khang et al. (Eds.): ASIAN MMS 2021, MMS 113, pp. 793–804, 2022.
https://doi.org/10.1007/978-3-030-91892-7_76

researchers and commercial soft wares for the analysis of laminated composite beams [1, 2]. However, in this theory, the inadequate distribution of transverse shear stress in the beam thickness requires a shear correction factor to calculate the shear force.

This adverse in practice could be overcome by using higher-order deformation beam theory (HSBT) [2, 3] or Quasi-3D beam theory (Quasi-3D) [4, 5] owing to the higher-order variation of axial displacement or both axial and transverse displacements, respectively. In such an approach, stresses of the beam can be directly computed from constitutive equations without shear coefficient requirement.

Many higher-order shear deformation theories have been developed with different approaches in which its kinematics could be expressed in terms of polynomial [6, 7], trigonometric [8, 9], exponential ones [10], hyperbolic [11, 12] and hybrid higher-order shear functions [13].

A literature review shows that a vast number of researches on development HSBT and Quasi-3D have been developed, however the accuracy of these theories strictly depends on the choice of shear functions and number of variables defining the problem. The development of new beam theories as well as suitable solution methods is a complicated problem and needs to be studied further.

The purpose of this paper is to develop a two-directional elasticity solution for vibration analysis of laminated composite beams. Based on the elasticity equations, the proposed theory only requires two unknowns in which the axial and transverse displacements are approximated in series terms in its two in-plane directions for different boundary conditions and Lagrange's equations are used to derive characteristic equations.

Numerical results are presented to investigate the effects of length-to-depth ratio, material anisotropy, Poisson's ratio and fiber angles of laminated composite beams.

2 Theoretical Formulation

Considering a laminated composite beam with rectangular section *b* x *h* and length *L* as shown in Fig. 1, the beam is composed of n layers of orthotropic materials.

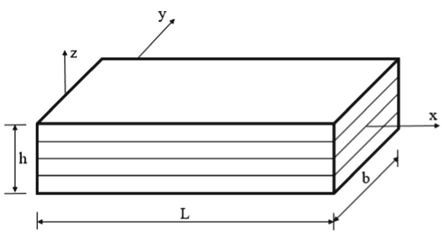

Fig. 1. Geometry of laminated composite beams.

2.1 Kinematic, Strain and Stress Relations

Denoting u and w are axial and transverse displacements at location (x, z) of the beam. The linear displacement-strain relations of the beam are given by:

$$\varepsilon_x = u_{,x} \tag{1}$$

$$\varepsilon_z = w_{,z} \tag{2}$$

$$\gamma_{xz} = u_{,z} + w_{,x} \tag{3}$$

where the comma indicates partial differentiation with respect to the coordinate subscript that follows. Based on an assumption of the plan stress in the plane $(x,\ z)$ of the beam, i.e. $\sigma_y = \sigma_{yz} = \sigma_{xy} = 0$, the elastic constitutive equation in the global coordinate system is expressed by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \overline{\overline{C}}_{11} & \overline{\overline{C}}_{13} & 0 \\ \overline{\overline{C}}_{13} & \overline{\overline{C}}_{33} & 0 \\ 0 & 0 & \overline{\overline{C}}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

where $\overline{\overline{C}}_{11}$, $\overline{\overline{C}}_{13}$ and $\overline{\overline{C}}_{55}$ are the reduced in-plane and out of plane elastic stiffness coefficients of the laminated composite beam in the global coordinates (see [5] for more details).

2.2 Variational Formulation

Lagrangian function is used to derive the equations of motion:

$$\Pi = U - K \tag{5}$$

where U, and K denote the strain energy, and kinetic energy, respectively.

The strain energy U of a system is given by:

$$\begin{aligned} U &= \frac{1}{2}\int_V (\sigma_x \varepsilon_x + \sigma_z \varepsilon_z + \sigma_{xz}\gamma_{xz}) dV \\ &= \frac{1}{2}\int_V \left[\overline{\overline{C}}_{11} u_{,x}^2 + 2\overline{\overline{C}}_{13} u_{,x} w_{,z} + \overline{\overline{C}}_{33} w_{,x}^2 + \overline{\overline{C}}_{55}\left(u_{,z}^2 + 2u_{,z}w_{,x} + w_{,x}^2\right)\right] dV \end{aligned} \tag{6}$$

The kinetic energy K is obtained as:

$$K = \frac{1}{2}\int_V \rho\left(\dot{u}^2 + \dot{w}^2\right) dV \tag{7}$$

where the differentiation with respect to the time t is denoted by dot-superscript convention; $\rho(z)$ is the mass density of each layer.

By substituting Eq. (6) and (7) into Eq. (5), Lagrangian function is explicitly expressed as:

$$\begin{aligned} \Pi = &\frac{1}{2}\int_V \left[\overline{\overline{C}}_{11} u_{,x}^2 + 2\overline{\overline{C}}_{13} u_{,x} w_{,z} + \overline{\overline{C}}_{33} w_{,x}^2 + \overline{\overline{C}}_{55}\left(u_{,z}^2 + 2u_{,z}w_{,x} + w_{,x}^2\right)\right] dV \\ &- \frac{1}{2}\int_V \rho\left(\dot{u}^2 + \dot{w}^2\right) dV \end{aligned} \tag{8}$$

2.3 Two-Directional Ritz Solution

Based on the Ritz method, the axial and transverse displacements at location (x, z) of the beam can be generally approximated in the following forms:

$$u(x, z, t) = \sum_{r=1}^{R} \sum_{s=1}^{S} \psi_{rs}(x, z) u_{rs} \tag{9}$$

$$w(x, z, t) = \sum_{r=1}^{R} \sum_{s=1}^{S} \varphi_{rs}(x, z) w_{rs} \tag{10}$$

where u_{rs},w_{rs} are unknown displacement values to be determined; $\psi_{rs}(x, z)$, $\varphi_{rs}(x, z)$ are the two-directional shape functions which are composed of admissible hybrid exponential-trigonometric function in the x-axis and polynomial function in the z-axis are given in Table 1.

Table 1. Shape functions and essentials BCs Beams.

BCs	$\psi_{rs}(x, z)$	$\varphi_{rs}(x, z)$	$x = 0$	$x = L$
S-S	$\cos\left(\frac{\pi x}{L}\right) e^{-rx/L} z^{s-1}$	$\sin\left(\frac{\pi x}{L}\right) e^{-rx/L} z^{s-1}$	$w = 0$	$w = 0$
C-F	$\sin\left(\frac{\pi x}{2L}\right) e^{-rx/L} z^{s-1}$	$\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right) e^{-rx/L} z^{s-1}$	$u = 0$, $w = 0$, $w_{,x} = 0$	
C-C	$\sin\left(\frac{\pi x}{L}\right) e^{-rx/L} z^{s-1}$	$\sin^2\left(\frac{\pi x}{L}\right) e^{-rx/L} z^{s-1}$	$u = 0$, $w = 0$, $w_{,x} = 0$	$u = 0$, $w = 0$, $w_{,x} = 0$

The governing equations of motion can be obtained by substituting Eq. (9), (10) into Eq. (8) and using Lagrange's equations:

$$\frac{\partial \Pi}{\partial q_{rs}} - \frac{d}{dt} \frac{\partial \Pi}{\partial \dot{q}_{rs}} = 0 \tag{11}$$

with q_{rs} representing the values of (u_{rs}, w_{rs}), that leads to:

$$\left(\begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} \\ {}^{T}\boldsymbol{K}^{12} & \boldsymbol{K}^{22} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathrm{M}^{11} & 0 \\ 0 & \mathrm{M}^{22} \end{bmatrix} \right) \begin{Bmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{12}$$

where the components of the stiffness matrix **K** and the mass matrix **M** are given by:

$$\begin{aligned}
K_{rspq}^{11} &= \int_0^L \int_{-h/2}^{h/2} \overline{\overline{C}}_{11}\psi_{rs,x}\psi_{pq,x} bdxdz + \int_0^L \int_{-h/2}^{h/2} \overline{\overline{C}}_{55}\psi_{rs,z}\psi_{pq,z} bdxdz \\
K_{rspq}^{12} &= \int_0^L \int_{-h/2}^{h/2} \overline{\overline{C}}_{13}\psi_{rs,x}\varphi_{pq,z} bdxdz + \int_0^L \int_{-h/2}^{h/2} \overline{\overline{C}}_{55}\psi_{rs,z}\varphi_{pq,x} bdxdz, \\
K_{rspq}^{22} &= \int_0^L \int_{-h/2}^{h/2} \overline{\overline{C}}_{33}\varphi_{rs,z}\varphi_{pq,z} bdxdz + \int_0^L \int_{-h/2}^{h/2} \overline{\overline{C}}_{55}\varphi_{rs,x}\varphi_{pq,x} bdxdz, \\
M_{rspq}^{11} &= \int_0^L \int_{-h/2}^{h/2} \rho\psi_{rs}\psi_{pq} bdxdz,\ M_{rspq}^{22} = \int_0^L \int_{-h/2}^{h/2} \rho\varphi_{rs}\varphi_{pq} bdxdz
\end{aligned} \tag{13}$$

Finally, the vibration responses of the laminated composite beams can be determined by solving Eq. (12). It should be noted that the Eq. (12) does not consider damping materials, the investigation of vibration frequencies of damping materials [14] is also a very interesting problem and will be the future development of this paper.

3 Numerical Results

In this section, convergence and verification studies are carried out to demonstrate the accuracy of the present study. For vibration analysis, laminates are assumed to be equal thicknesses and made of the same orthotropic materials whose properties are given in Table 2. For convenience, the following non-dimensional terms are used:

$$\overline{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho}{E_2}} \text{ for Material I (MAT I) and } \overline{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho}{E_1}} \text{ for MAT II} \tag{14}$$

Table 2. Material properties of composite beams.

Material properties	MAT I [15]	MAT II [16]
E_1 (GPa)	$E_1/E_2 =$ open	144.8
$E_2 = E_3$(GPa)	-	9.65
$G_{12}= G_{13}$ (GPa)	0.6$E2$	4.14
G_{23} (GPa)	0.5E_2	3.45
$\nu_{12}= \nu_{13}= \nu_{23}$	0.25	0.3
ρ (kg/m^3)	-	-
L (m)	$L/h =$ open	$L/h = 15$
h (m)	-	-
b (m)	-	-

The composite beams (MAT I, $0^0/90^0/0^0$, $L/h = 5$, $E_1/E_2 = 40$) with different BCs are considered to evaluate the convergence. The non-dimensional fundamental frequencies with respect to the number of series in $x -$ direction (R) and $z -$ direction are given in Table 3. It can be seen that the responses converge quickly in x-direction and number of series in this direction $R = 10$ can be the point of convergence of the fundamental frequencies for the boundary conditions, whereas the frequency of vibration tends to decrease with increasing number of series in z-direction, and the beam tends to become softer. As an example for further verification, $R = 10$ and $S = 4$ will be chosen in the following examples.

Table 3. Convergence studies for normalized fundamental frequencies of $0^0/90^0/0^0$ laminated composite beams (MAT I, $L/h = 5$, $E_1/E_2 = 40$).

BC	Number of series (S)	Number of series (R)					
		2	4	6	8	10	12
S-S	1	11.8886	11.8251	11.8245	11.8245	11.8245	11.8245
	2	9.9289	9.8192	9.8178	9.8178	9.8178	9.8178
	3	9.9258	9.8160	9.8146	9.8146	9.8146	9.8146
	4	9.3105	9.2046	9.2033	9.2033	9.2033	9.2033
	5	9.3095	9.2036	9.2023	9.2022	9.2022	9.2022
	6	9.3088	9.2032	9.2019	9.2019	9.2019	9.2019
	7	9.3088	9.2032	9.2019	9.2019	9.2019	9.2019
C-F	1	6.2477	6.0473	5.9839	5.9562	5.9420	5.9355
	2	4.5409	4.4005	4.3737	4.3638	4.3589	4.3572
	3	4.5401	4.3993	4.3723	4.3624	4.3572	4.3547
	4	4.3294	4.2047	4.1808	4.1719	4.1676	4.1656
	5	4.3292	4.2045	4.1804	4.1716	4.1672	4.1655
	6	4.3273	4.2011	4.1770	4.1681	4.1636	4.1625
	7	4.3273	4.2011	4.1770	4.1682	4.1639	4.1612
C-C	1	13.8049	12.6527	12.2761	12.1087	12.0330	12.0311
	2	13.2193	12.0773	11.7457	11.5999	11.5358	11.5326
	3	13.2185	12.0765	11.7449	11.5990	11.5338	11.5287
	4	12.5178	11.4420	11.1473	11.0216	10.9655	10.9639
	5	12.5168	11.4406	11.1455	11.0195	10.9625	10.9604
	6	12.4768	11.4031	11.1073	10.9811	10.9258	10.9243
	7	12.4767	11.4030	11.1072	10.9809	10.9252	10.9243

Vibration behaviors of cross-ply laminated composite beams are investigated in Table 4 which presents changes of the non-dimensional fundamental frequencies with S-S, C-F and C-C boundary conditions, span-to-thickness ratio $L/h = 5,\ 10,\ 50$ of the 0°/90°/0° and 0°/90° laminated composite beams. The solutions are computed with MAT I and $E1/E2 = 40$. The accuracy of the solutions is tested by verification with those derived from HSBTs (Nguyen et al. [3], Nguyen et al. [5], Khdeir et al. [17], Vo, T.P et al. [18], Murthy et al. [19]) and Quasi-3Ds (Nguyen et al. [5], Mantari et al. [20], Matsunaga [21]). It can be seen that the present solutions comply with those from the Quasi-3Ds, however there are slight deviations between them for the thick beams ($L/h = 5$) and for C-F and C-C boundary conditions. The softer characteristic of the present beam model is again found for all solutions in comparison with the HSBTs and Quasi-3Ds.

Table 4. Non-dimensional fundamental frequencies of $0^0/90^0/0^0$ and $0^0/90^0$ laminated composite beams (MAT I,$E_1/E_2 = 40$).

BCs	Theory	$0^0/90^0/0^0$				$0^0/90^0$		
		$L/h = 5$	10	50		$L/h = 5$	10	50
S-S	HSBT [5]	9.206	13.607	17.449		6.125	6.940	7.297
	HSBT [3]	9.208	13.614	17.462		6.128	6.945	7.302
	HSBT [17]	9.208	13.614	-		6.128	6.945	-
	HSBT [18]	9.206	13.607	17.449		6.058	6.909	7.296
	HSBT [19]	9.207	13.611	-		6.045	6.908	-
	Quasi-3D [5]	9.208	13.610	17.449		6.140	6.948	7.297
	Quasi-3D [20]	9.208	13.610	-		6.109	6.913	-
	Quasi-3D [21]	9.200	13.608	-		5.662	6.756	-
	Present	9.203	13.610	17.449		5.831	6.833	7.292
C-F	HSBT [5]	4.230	5.490	6.262		2.381	2.541	2.603
	HSBT [3]	4.234	5.498	6.267		2.383	2.543	2.605
	HSBT [17]	4.234	5.495	-		2.386	2.544	-
	HSBT [19]	4.230	5.491	-		2.378	2.541	-
	Quasi-3D [5]	4.223	5.491	6.262		2.382	2.543	2.604
	Quasi-3D [20]	4.221	5.490	-		2.375	2.532	-
	Present	4.168	5.478	6.262		2.314	2.522	2.603
C-C	HSBT [5]	11.601	19.707	37.629		10.019	13.653	16.414
	HSBT [3]	11.607	19.728	37.679		10.027	13.670	16.429
	HSBT [17]	11.603	19.712	-		10.026	13.660	-
	HSBT [19]	11.602	19.719	-		10.011	13.657	-
	Quasi-3D [5]	11.499	19.672	37.633		9.944	13.664	16.432
	Quasi-3D [20]	11.486	19.652	-		9.974	13.628	-
	Present	10.966	19.454	37.652		8.8361	12.913	16.388

In order to verify the vibration behaviors of present theory further, TableS 5 and Table 6 report non-dimensional fundamental frequencies with respect to the fiber angle. The results are obtained with three boundary conditions, two lay-ups 0°/θ°/0° and 0°/θ°, material MAT I, $E1/E2 = 40$ and $L/h = 5$. It is worth noticing that the solutions are compared with those of Nguyen et al. [5] based on a Quasi-3D theory. It is observed that there are significant differences between two models for C-C boundary conditions, whereas the solution field of the theories is in agreement for S-S and C-F boundary conditions.

The unsymmetrical (45^0/-45^0/45^0/-45^0) and (30^0/-50^0/50^0/-30^0) composite beams (MAT II) with various BCs are considered. The results of fundamental frequencies are given in Fig. 2 and Fig. 3. A good agreement between the present solutions and previous studies (Nguyen et al. [5], Chen et al. [22], Chandrashekhara et al. [23]) is again found.

Table 5. Non-dimensional fundamental frequencies of $0^0/\theta$ $/0^0$ and $0^0/\theta$ laminated composite beams (MAT I, $E_1/E_2 = 40$, $L/h = 5$).

Lay-up	BCs	Theory	Fiber angle (θ)						
			00	15^0	30^0	45^0	60^0	75^0	90^0
$0^0/\theta$ $/0^0$	S-S	HSBT [5]	9.5498	9.5165	9.4487	9.3630	9.2831	9.2279	9.2083
		Present	9.5360	9.5008	9.4354	9.3531	9.2759	9.2223	9.2033
	C-F	HSBT [5]	4.3628	4.3307	4.3047	4.2754	4.2484	4.2297	4.2231
		Present	4.2988	4.2707	4.2464	4.2182	4.1919	4.1740	4.1676
	C-C	HSBT [5]	12.0240	11.9365	11.8341	11.7130	11.6020	11.5260	11.4992
		Present	11.3751	11.3341	11.2621	11.1521	11.0571	10.9886	10.9655
$0^0/\theta$	S-S	HSBT [5]	9.5498	7.9829	6.8336	6.3948	6.2215	6.1561	6.1400
		Present	9.5360	7.8827	6.6059	6.1104	5.9178	5.8478	5.8314
	C-F	HSBT [5]	4.3628	3.3266	2.7077	2.4964	2.4173	2.3887	2.3819
		Present	4.2987	3.2836	2.6501	2.4314	2.3498	2.3208	2.3140
	C-C	HSBT [5]	12.0240	11.0882	10.4823	10.1844	10.0347	9.9640	9.9435
		Present	11.3769	10.5547	9.6617	9.1718	8.9404	8.8557	8.8300

Finally, the symmetric $(\theta/-\theta)_s$ composite beams (MAT II) are considered. The effects of the fiber angle on the natural frequencies is illustrated in Table 7. It can be seen that the present natural frequencies are closer to those of HSBT (Nguyen et al. [5] and Aydogdu [24]) smaller than those of HSBT (Nguyen et al. [3]), which neglected the Poisson's effect, especially for $10^0 \leq \theta \leq 60^0$. This phenomenon can be explained by

Table 6. Non-dimensional fundamental frequencies of $0^0/\theta/0^0$ and $0^0/\theta$ laminated composite beams (MAT I, $E_1/E_2 = 40$, $L/h = 10$).

Lay-up	BCs	Theory	Fiber angle (θ)						
			00	15^0	30^0	45^0	60^0	75^0	90^0
$0^0/\theta$ /0^0	S-S	HSBT [5]	13.9976	13.8822	13.8130	13.7400	13.6729	13.6264	13.6099
		Present	13.9968	13.8802	13.8118	13.7395	13.6728	13.6264	13.6098
	C-F	HSBT [5]	5.6259	5.5622	5.5403	5.5220	5.5059	5.4948	5.4909
		Present	5.6111	5.5484	5.5268	5.5087	5.4927	5.4816	5.4776
	C-C	HSBT [5]	20.4355	20.3428	20.1923	20.0062	19.8335	19.7144	19.6723
		Present	20.1538	20.0769	19.9452	19.7694	19.6065	19.4931	19.4541
$0^0/\theta$	S-S	HSBT [5]	13.9976	10.0656	7.9772	7.3028	7.0561	6.9682	6.9475
		Present	13.9968	10.0200	7.8862	7.1945	6.9425	6.8536	6.8330
	C-F	HSBT [5]	5.6259	3.7996	2.9428	2.6785	2.5837	2.5505	2.5428
		Present	5.6110	3.7877	2.9253	2.6588	2.5633	2.5300	2.5224
	C-C	HSBT [5]	20.4355	17.3592	15.0934	14.2004	13.8389	13.6989	13.6637
		Present	20.1626	17.0377	14.5214	13.4923	13.0949	12.9442	12.9133

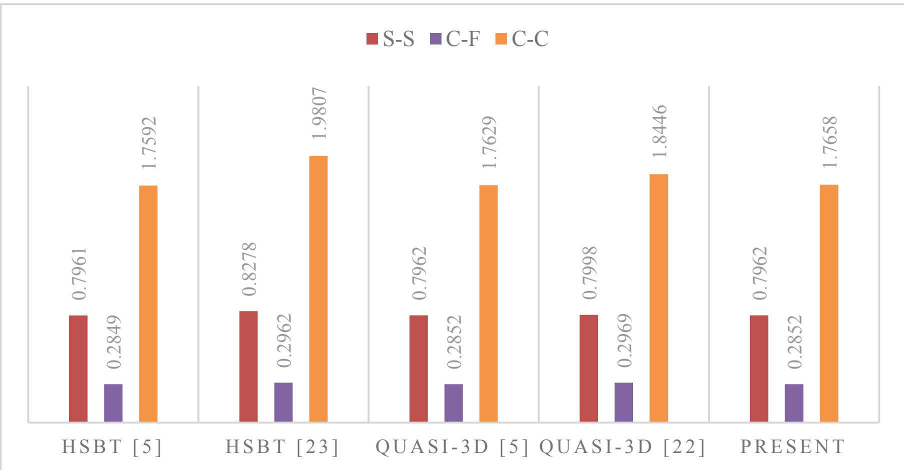

Fig. 2. Non-dimensional fundamental frequencies of 45^0/-45^0/45^0/-45^0 laminated composite beams (MAT II).

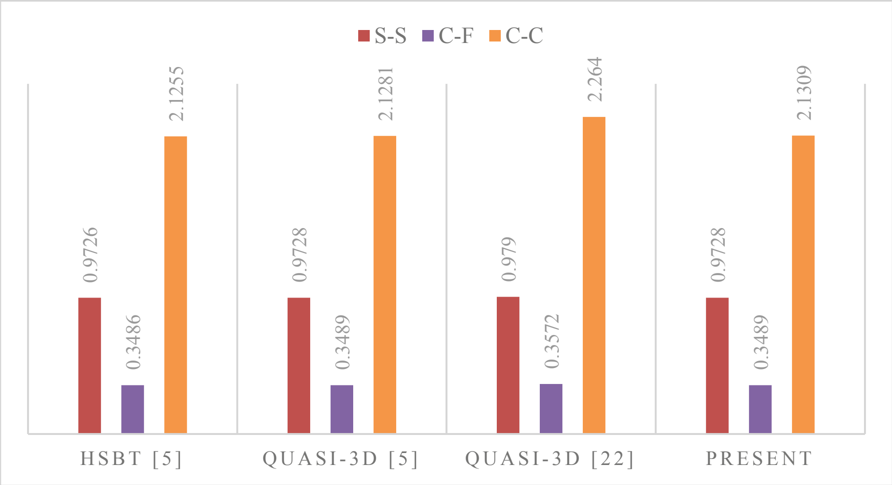

Fig. 3. Non-dimensional fundamental frequencies of $30^0/-50^0/50^0/-30^0$ laminated composite beams (MAT II).

Table 7. Non-dimensional fundamental frequencies of $(\theta/-\theta)_s$ laminated composite beams (MAT II).

BCs	Theory	Fiber angle (θ)						
		00	15^0	30^0	45^0	60^0	75^0	90^0
S-S	HSBT [5]	2.649	1.579	0.999	0.796	0.731	0.725	0.729
	HSBT [24]	2.651	1.896	1.141	0.804	0.736	0.725	0.729
	HSBT [3]	2.656	2.511	2.103	1.537	1.012	0.761	0.732
	Quasi-3D [5]	2.650	1.580	0.999	0.796	0.731	0.725	0.730
	Present	2.650	1.580	0.999	0.796	0.731	0.725	0.730
C-F	HSBT [5]	0.980	0.570	0.358	0.285	0.261	0.259	0.261
	HSBT [24]	0.981	0.676	0.414	0.288	0.262	0.258	0.260
	HSBT [3]	0.983	0.926	0.768	0.555	0.363	0.272	0.262
	Quasi-3D [5]	0.980	0.571	0.358	0.285	0.262	0.260	0.262
	Present	0.980	0.571	0.358	0.285	0.262	0.260	0.262
C-C	HSBT [5]	4.897	3.288	2.180	1.759	1.620	1.605	1.615
	HSBT [24]	4.973	4.294	2.195	1.929	1.669	1.612	1.619
	HSBT [3]	4.912	4.717	4.131	3.197	2.202	1.683	1.621
	Quasi-3D [5]	4.901	3.290	2.183	1.762	1.626	1.614	1.625
	Present	4.898	3.295	2.186	1.765	1.630	1.619	1.631

the fact that Poisson's effect is incorporated in the constitutive equations by assuming $\sigma_y = \sigma_{xy} = \sigma_{yz} = 0$. It means that the strains $(\varepsilon_y,\ \gamma_{yz}, \gamma_{xy})$ are nonzero and this causes the beams more flexible. This indicated that the Poisson's effect is quite significant to composite beams with arbitrary lay-ups, and neglecting this effect is only suitable for cross-ply composite beams.

4 Conclusions

The authors proposed a new two-unknown model for vibration analysis of laminated composite beams. The axial and transverse displacements of the beam are expanded in a hybrid form under polynomial, and trigonometric series. Lagrange's equations are used to derive characteristic equations of the beams. Numerical results for laminated composite beams with different boundary conditions are compared with previous studies and investigate effects of length-to-depth ratio, material anisotropy, Poisson's ratio and fiber angles on the natural frequencies of laminated composite beams. The obtained results show that the normal strain effects are significant for un-symmetric and thick beams. The Poisson's effect is also important for composite beams with arbitrary lay-ups, and thus omitting this effects is only suitable for the cross-ply ones. The present model is found to be appropriate for vibration analyses of laminated composite beams.

Acknowledgments. This research is funded by Thu Dau Mot University (TDMU).

References

1. Komijani, M., et al.: An analytical approach for thermal stability analysis of two-layer Timoshenko beams. Int. J. Struct. Stab. Dyn. **13**(08), 1350036 (2013)
2. Nguyen, T.-K., et al.: A novel unified model for laminated composite beams. Compos. Struct. **238**, 111943 (2020)
3. Nguyen, T.-K., et al.: Trigonometric-series solution for analysis of laminated composite beams. Compos. Struct. **160**, 142–151 (2017)
4. Mantari, J., Canales, F.: Finite element formulation of laminated beams with capability to model the thickness expansion. Compos. B Eng. **101**, 107–115 (2016)
5. Nguyen, N.-D., et al.: Ritz-based analytical solutions for bending, buckling and vibration behavior of laminated composite beams. Int. J. Struct. Stab. Dyn. **18**(11), 1850130 (2018)
6. Murin, J., et al.: Modal analysis of the FGM beams with effect of the shear correction function. Compos. B Eng. **45**(1), 1575–1582 (2013)
7. Nguyen, T.N., Thai, C.H., Nguyen-Xuan, H.: On the general framework of high order shear deformation theories for laminated composite plate structures: a novel unified approach. Int. J. Mech. Sci. **110**, 242–255 (2016)
8. Thai, C.H., et al.: Isogeometric analysis of laminated composite and sandwich plates using a new inverse trigonometric shear deformation theory. Eur. J. Mech. A/Solids **43**, 89–108 (2014)
9. Mantari, J., Oktem, A., Soares, C.G.: A new trigonometric shear deformation theory for isotropic, laminated composite and sandwich plates. Int. J. Solids Struct. **49**(1), 43–53 (2012)
10. Aydoğdu, M.: A new shear deformation theory for laminated composite plates. Compos. Struct. **89**, 94–101 (2009)

11. El Meiche, N., et al.: A new hyperbolic shear deformation theory for buckling and vibration of functionally graded sandwich plate. Int. J. Mech. Sci. **53**(4), 237–247 (2011)
12. Akavci, S., Tanrikulu, A.: Buckling and free vibration analyses of laminated composite plates by using two new hyperbolic shear-deformation theories. Mech. Compos. Mater. **44**(2), 145 (2008)
13. Mantari, J., Oktem, A., Soares, C.G.: A new higher order shear deformation theory for sandwich and composite laminated plates. Compos. B Eng. **43**(3), 1489–1499 (2012)
14. Mohammadian, M.: Nonlinear free vibration of damped and undamped bi-directional functionally graded beams using a cubic-quintic nonlinear model. Compos. Struct. **255**, 112866 (2021). https://doi.org/10.1016/j.compstruct.2020.112866
15. Aydogdu, M.: Vibration analysis of cross-ply laminated beams with general boundary conditions by Ritz method. Int. J. Mech. Sci. **47**(11), 1740–1755 (2005)
16. Chandrashekhara, K., Krishnamurthy, K., Roy, S.: Free vibration of composite beams including rotary inertia and shear deformation. Compos. Struct. **14**(4), 269–279 (1990)
17. Khdeir, A., Reddy, J.: Free vibration of cross-ply laminated beams with arbitrary boundary conditions. Int. J. Eng. Sci. **32**(12), 1971–1980 (1994)
18. Vo, T.P., Thai, H.-T.: Vibration and buckling of composite beams using refined shear deformation theory. Int. J. Mech. Sci. **62**(1), 67–76 (2012)
19. Murthy, M., et al.: A refined higher order finite element for asymmetric composite beams. Compos. Struct. **67**(1), 27–35 (2005)
20. Mantari, J., Canales, F.: Free vibration and buckling of laminated beams via hybrid Ritz solution for various penalized boundary conditions. Compos. Struct. **152**, 306–315 (2016)
21. Matsunaga, H.: Vibration and buckling of multilayered composite beams according to higher order deformation theories. J. Sound Vib. **246**(1), 47–62 (2001)
22. Chen, W., Lv, C., Bian, Z.: Free vibration analysis of generally laminated beams via state-space-based differential quadrature. Compos. Struct. **63**(3–4), 417–425 (2004)
23. Chandrashekhara, K., Bangera, K.M.: Free vibration of composite beams using a refined shear flexible beam element. Comput. Struct. **43**(4), 719–727 (1992)
24. Aydogdu, M.: Free vibration analysis of angle-ply laminated beams with general boundary conditions. J. Reinf. Plast. Compos. **25**(15), 1571–1583 (2006)