Chapter 15
Collective Games on Hypergraphs

Unai Alvarez-Rodriguez, Federico Battiston, Guilherme Ferraz de Arruda, Yamir Moreno, Matjaž Perc, and Vito Latora

Abstract Human activities often require simultaneous decision-making of individuals in groups. These processes cannot be coherently addressed by means of networks, as networks only allow for pairwise interactions. Here, we propose a general implementation for collective games in which higher-order interactions are encoded on hypergraphs. We employ it for the study of the public goods game by first validating the analytical expression of the replicator dynamics in uniform and heterogeneous populations, and then by introducing a procedure for retrieving empirical synergistic effects of group interactions from real datasets.

U. Alvarez-Rodriguez (✉)
Data Analytics Group, University of Zurich, Zurich, Switzerland

F. Battiston
Department of Network and Data Science, Central European University, Vienna, Austria

Department of Anthropology, University of Zurich, Zurich, Switzerland

G. Ferraz de Arruda · Y. Moreno
ISI Foundation, Turin, Italy

Y. Moreno
Institute for Biocomputation and Physics of Complex Systems, University of Zaragoza, Zaragoza 50008, Spain

Department of Theoretical Physics, University of Zaragoza, Zaragoza 50009, Spain

M. Perc
Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia

Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

Complexity Science Hub Vienna, Josefstädterstraße 39, 1080 Vienna, Austria

V. Latora
School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK

Dipartimento di Fisica ed Astronomia, Università di Catania and INFN, 95123 Catania, Italy

The Alan Turing Institute, The British Library, London NW1 2DB, UK

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15.1 Introduction

Cooperation has been detected to be a key element in the explanation of the evolutionary success of our species [1, 2]. The problem of understanding the emergence of cooperation lies at the boundary of a plethora of scientific disciplines [3–9]. Previous works have been able to explain how cooperation in a population is sustained by the structure of the social network encoding the interactions amongst individuals [10, 11]. Evolutionary game theory is the branch of applied mathematics providing a theoretical framework to address these questions, enabling quantitative statements about the conditions that give rise to stable cooperation [12–14]. In this context, a social dilemma is a scenario in which defection results in a higher payoff for individuals while cooperation entails a higher payoff for the collective of players [15]. The challenge presented by social dilemmas is solved by network reciprocity [16], a property by which groups of nodes are strongly connected between them and weakly connected to nodes outside the group, and thus protected from defector invasion. This feature may be observed in structures of different nature, such as networks with a heterogeneous degree distribution [17–19], networks with community structures [20, 21], and even in multilayered systems [22–29].

These advances in evolutionary game theory have been restricted to the realm of two-player games, as standard networks are not suitable for encoding group interactions. In order to bypass this limitation, it was proposed to infer higher-order interactions from the dyadic structure, by assuming that every node would act as a vehicle for a group-game involving all of his neighbours [30, 31]. However, such approach is structurally ambiguous, and thus incompatible with the well known mechanisms favouring cooperation [32–39]. To overcome the constraints of traditional graphs, higher-order interactions have been suggested as the natural way to encode non-diadic relationships [40, 41]. In particular, it has recently been shown that hypergraphs are a natural solution to formalise n-player games [42]. In the following we elaborate on this idea by explaining how to implement n-player games on hypergraphs and applying this to the study of the public goods game (PGG) [43, 44] for uniform and heterogeneous structures.

The PGG is an n-player game of two strategies where at each round of the game participants are requested to contribute to a common pool with a token of value c. We shall call cooperators C those players who do contribute and defectors D those that do not contribute. The collected amount is multiplied by the synergy factor R and the outcome is equally split between all the participants [8]. It is standard to assign a fixed value of $c = 1$ to the token, and to describe the payoffs in terms of the reduced synergy factor $r = R/g$ where g is the number of players. In a round with w_C cooperators the payoff for the defectors is $\pi_D = rw_C$, while the payoff of the cooperators is given by $\pi_C = rw_C - 1$.

15.2 Collective Games on Higher-Order Networks

The goal of evolutionary dynamics is to predict the number of cooperators and defectors in a population undergoing a continuous iteration of the game combined with the adaptation of strategies. A game implementation is a set of rules that determines how this process occurs. In the hypergraph implementation (HI) each round of the PGG is hosted by a hyperlink $l \in \mathcal{L}$ of the hypergraph $H(\mathcal{N}, \mathcal{L})$ representing the system. The evolutionary process is a concatenation of micro-steps: At each micro-step a hyperlink $l \in \mathcal{L}$ and one of its nodes $n \in l$ are randomly selected. All the nodes present in the hyperlink play a round of the game for each hyperlink they are part of and accumulate their payoffs. The payoffs are then normalised and compared to select the node with the highest payoff per game ratio. Only then node n copies the winning strategy with a probability proportional to the payoff difference $\frac{1}{\Delta}[(\max_l \pi_j) - \pi_i]$. Here Δ is the normalisation factor that accounts for the maximal payoff difference between two nodes. For a system with $|\mathcal{N}| = N$ nodes, N micro-steps add up to a step, in which every node has the opportunity to change its strategy at least once. In Fig. 15.1 we graphically explain HI, and we compare it with the original network implementation [30] or graph implementation (GI).

In the next subsections we analyse in depth the hypergraph implementation on different families of hypergraphs. Thorough this chapter we adopt the fixed cost per game perspective, where players contribute with a token to every round they play.

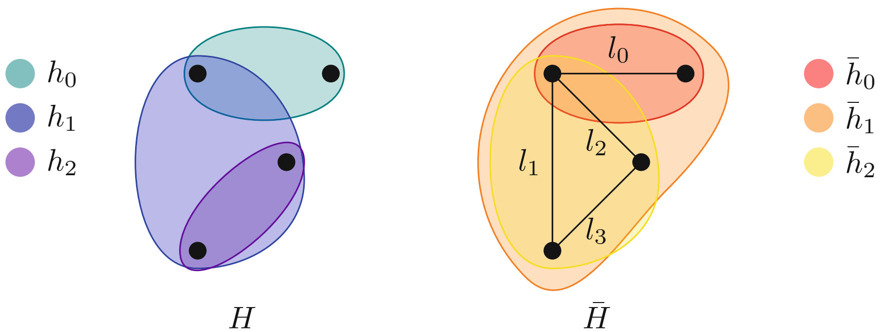

Fig. 15.1 Collective games on Hypergraphs. We graphically argument that the hypergraph implementation provides a reliable alternative to the ambiguous graph implementation for collective games. The hypergraph on the left H accounts for the real higher-order connections represented by hyperlinks h_i. The network with links l_i is inferred by linking all the nodes that are part of a common hyperlink in the original structure. The hypergraph on the right $\bar{H}$ with hyperlinks $\bar{h}_i$ is obtained as a product of the graph implementation, which imposes a group interaction between a node and its first neighbours. The difference between the real groups h_i and the ones imposed by the graph implementation $\bar{h}_i$ shows the inconsistency of the dyadic approach. Adapter from Ref. [42]

15.2.1 Uniform Hypergraphs

Uniform Hypergraphs are a subset of all the possible hypergraphs in which all the hyperlinks have the same cardinality. This means in our context that all the rounds of the PGG will have the same number of players g. The system may be described by using the replicators approximation, in which the relevant properties of the evolutionary dynamics are captured by the fraction of population selecting each strategy. We use x_C and x_D for the fraction of cooperators and defectors respectively. We first compute the average payoffs for cooperators π_C and defectors π_D for a generic order g by counting all the possible configurations of strategies

$$\pi_D = \sum_{i=0}^{g-1} \binom{g-1}{i} x_D^{g-1-i} x_C^i i r$$
$$\pi_C = \sum_{i=0}^{g-1} \binom{g-1}{i} x_D^{g-1-i} x_C^i ((i+1)r - 1)$$

and obtain the average payoff difference as $\pi_D - \pi_C = 1 - r$. We also compute Δ, the maximal payoff difference

$$\Delta = \begin{cases} r(g-2) + 1 & \text{if } r < 1 \\ gr - 1 & \text{if } r > 1 \end{cases}$$

The time evolution equation can be derived by counting all the possible combinations leading to a strategy change: for every group in which at least two strategies are present one has to consider the probabilities that a cooperator defector pair is involved in a potential strategy change combined with the probability of the strategy change actually occurring given the payoff difference. Although the equation for cooperators is not presented here, its formulation is analogue to that of defectors.

$$\begin{aligned}\dot{x}_D &= \sum_{i=0}^{g-2} \binom{g}{1+i} x_D^{g-1-i} x_C^{1+i} \frac{(g-1-i)(1+i)}{g(g-1)} Q\Big[\theta(\pi_C - \pi_D) + \theta(\pi_D - \pi_C)\Big] \\ &= Q x_D x_C\end{aligned}$$

where Q is the normalized payoff difference, $Q \equiv (\pi_D - \pi_C)/\Delta$. From this equation we observe that the dynamics has two absorbing states $x_D = 0, x_C = 1$ and $x_D = 1, x_C = 0$ and a phase transition at $r = 1$. Therefore one should expect cooperators emerging in uniform hypergraphs only if $R > g$ holds.

We test the replicators prediction on synthetically designed uniform random hypergraphs (URH). For a fixed order g, these hypergraphs are composed of L independent hyperlinks created by randomly selecting L g-tuples of different nodes. We run $T = 10^4$ steps of the game in a system with $N = 1000$ nodes for $g = 2, 3, 4, 5$. The number of hyperlinks L is tuned to exceed the critical threshold guaranteeing a

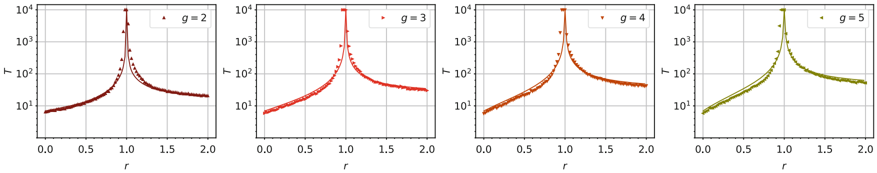

Fig. 15.2 Uniform Hypergraphs. Relaxation time as a function of the reduced synergy factor on uniform random hypergraphs with $N = 1000$ nodes and $L = 5L_c$ hyperlinks for groups of different sizes g. The triangles report the numerical simulations and the continuous line accounts for the replicators prediction. Adapted from Ref. [42]

connected hypergraph L_c, $L = 5L_c$, with $L_c = \frac{N}{g} \ln N$. The plot in Fig. 15.2 reports the relaxation time for an initial population in which cooperators and defectors are evenly distributed. A good agreement between the theory and the numerical simulations is observed.

The replicators equation is derived by assuming that all nodes are indistinguishable and potentially connected to each other, and therefore an increasingly alike behaviour to that of the replicator is expected when density is increased in URHs. However, real-world scenarios seldom provide these conditions, as structures emerging from optimisation processes tend to display strong heterogeneities and low densities. Hence, further analysis of the limits of the replicators approximation is needed to understand its applicability range.

Let us start by introducing the hyperdegree of a node as an additional degree of freedom in our model. We have $p(k)$ for the probability of a node having hyperdegree k, and $p(D|k)$ or $p(C|k)$ for the probabilities that a node with hyperdegree k is either a defector or a cooperator. We can recover the fraction of defectors and cooperators by adding the contributions from all the possible hyperdegrees $\mathcal{K}$.

$$\begin{aligned} x_D &= \sum_{k \in \mathcal{K}} p(k) p(D|k) \\ x_C &= \sum_{k \in \mathcal{K}} p(k) p(C|k) \end{aligned} \tag{15.1}$$

From the rest of this section we will use p_{Dk} and p_{Ck} to lighten the notation. Following the procedure explained in Eq. (15.1) one may derive the time evolution of the hyperdegree restricted variables by adding all the possible channels that lead to a strategy change. In essence, the population of defectors with degree k can only increase if a cooperator of degree k changes its strategy, or if a defector of degree k becomes a cooperator. Mathematically this can be expressed as

$$\dot{x}_{Dk} = -x_{Dk} \sum_{k' \in \mathcal{K}} p(k'|k) p(C|k') \sum_{k'' \in \mathcal{K}^{g-2}} p(k''|kk') \sum_{x \in \mathcal{X}^{g-2}} p(x|k'') \times \frac{(\pi_{Ck'} - \pi_{Dk}) \theta(\pi_{Ck'} - \pi_{Dk})}{\Delta} + x_{Ck} \sum_{k' \in \mathcal{K}} p(k'|k) p(D|k') \sum_{k'' \in \mathcal{K}^{g-2}} p(k''|kk') \sum_{x \in \mathcal{X}^{g-2}} p(x|k'') \times \frac{(\pi_{Dk'} - \pi_{Ck}) \theta(\pi_{Dk'} - \pi_{Ck})}{\Delta} \quad (15.2)$$

While different, the terms associated to each of the channels respect the same principle: In the first summation we consider all the possibilities of a neighbouring node being a cooperator with a given degree k'. In the second summation we account for the hyperdegrees of the rest of $g-2$ nodes in the group, conditioned to the hyperdegrees of the defector-cooperator pair. In the third summation we include all the possible strategy selections by these nodes. The last element in the product includes the normalized average payoff difference between a defector with degree k and a cooperator with degree k'. We shall compute these to move forward. We have

$$\pi_{Dk} = \sum_{k'' \in \mathcal{K}^{g-1}} p(k''|k) \sum_{x'' \in \mathcal{X}^{g-1}} p(x''|k'')(nr)$$

$$\pi_{Ck'} = \sum_{k'' \in \mathcal{K}^{g-1}} p(k''|k') \sum_{x'' \in \mathcal{X}^{g-1}} p(x''|k'')((n+1)r - 1)$$

where n is a function of x'' accounting for the number of cooperators. The first term in each expression, $p(k''|k)$ and $p(k''|k')$ represent the hyperdegree-hyperdegree correlations, i.e., how likely is that a node with hyperdegree k or k' is part of a group of $g-1$ with a given combination of hyperdegrees. We notice that if we make these probabilities node independent, i.e., $p(k''|k) = p(k''|k') = p(k'')$ we can simplify the expression for the average payoff difference, and recover the result of the original replicators approach $\pi_{Dk} - \pi_{Ck'} = 1 - r$. By introducing this result on Eq. (15.2) one obtains the more simplified

$$\dot{x}_{Dk} = Q\theta(r-1)x_{Dk}x_C + Q\theta(1-r)x_{Ck}x_D \quad (15.3)$$

where Q is the normalized average payoff difference, and the absence of hyperdegree-hyperdegree correlations is used again. This expression is then combined with Eq. (15.1) to yield a final formula for the time evolution of cooperators and defectors that exactly coincides with the one derived above without considering the hyperdegrees.

$$\dot{x}_D = Q\theta(r-1)x_D x_C + Q\theta(1-r)x_C x_D = Q x_D x_C$$

This result establishes a precise boundary between the hypergraphs that are suited to be described with a replicators approach and those that are not. We point out that these derivations corroborate the intuition behind the indistinguishability of the nodes, as the absence of hyperdegree-hyperdegree correlations implies that not only the nodes but also their neighbourhoods are equal.

15.2.2 Heterogeneous Hypergraphs

Heterogeneous Hypergraphs are the next step in the path towards the application of HI to more realistic scenarios. In these, hyperlink orders are not fixed, and therefore effects arising from the mixture of group sizes are expected. A particularly interesting question is to understand the phenomenology driven by a group-size dependent synergy factor. We start by providing a description for heterogeneous hypergraphs in terms of the abundance of hyperlinks of order g. We say that $\mathbf{p} = \{p^g\}_{g=g^-}^{g^+}$ of elements $p^g = k^g/k$ contains the fraction of hyperlinks of order g one node is part of, where g^- and g^+ are the minimal and maximal orders respectively, and $g \in \mathcal{G} = \{g^-, g^- + 1, ..., g^+ - 1, g^+\}$. We are interested in describing the dynamics for synergy factors modelled by non-linear functions of the group size, $R(g) = \alpha g^\beta$ with $\alpha, \beta \geq 0$. Given that g takes only discrete values, we find more convenient to work with $\mathbf{r} = \{r^g\}_{g=g^-}^{g^+}$ of elements $r^g = \alpha g^{\beta-1}$. With $\beta = 1$, we would factor out the dependence of g in the reduced synergy factor, and therefore recover the uniform case. The average payoff difference can be computed by adding the contribution of all the group sizes

$$\pi_D - \pi_C = \sum_{g \in \mathcal{G}} p^g(1 - r^g) = 1 - \alpha \sum_{g \in \mathcal{G}} p^g g^{\beta-1}$$

We use Q again as $Q = (\pi_D - \pi_C)/\Delta$ to represent the normalized payoff difference, in terms of which Eq. (15.1) is obtained for the dynamics. Therefore, the condition $Q = 0$ yields the critical value α_c

$$\alpha_c = \frac{1}{\sum_{g \in \mathcal{G}} p^g g^{\beta-1}}$$

We validate our predictions on a series on numerical experiments with synthetic heterogeneous hypergraphs. We consider hypergraphs whose $\mathbf{p}$ is restricted to $p^g = n^g/4$ with $n^g \in \{0, 1, 2, 3, 4\} \forall g \in \mathcal{G}$ with $g^- = 2$ and $g^+ = 5$. We construct a hypergraph with $N = 1000$ and $L = 2L_c$ for each of the possible values of $\mathbf{p}$ fulfilling the aforementioned condition, and collect all of them in an hypergraph ensemble $\mathcal{H}$ with $|\mathcal{H}| = 35$. We then run $T = 10^4$ steps of the evolutionary dynamics and obtain the asymptotic fraction of cooperators as a function of α for $\beta \in \{0, 1, 2, 3\}$ for all the hypergraphs in $\mathcal{H}$. The simulations in Fig. 15.3 (from [42]) display a good agreement between our prediction of the critical point and the empirical transition between cooperators and defectors.

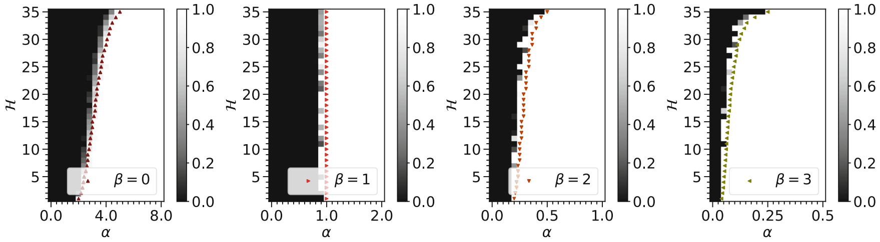

Fig. 15.3 Heterogeneous Hypergraphs. Fraction of cooperators x_C after $T = 10^4$ time steps as a function of α for each hypergraph in $\mathcal{H}$ with $N = 1000$ nodes. The panels correspond to values of $\beta = 0, 1, 2, 3$ from left to right. The coloured triangles mark the analytical phase transition yield by the replicators approximation. Adapted from Ref. [42]

15.2.3 Synergy Factor on Real Games

The advances in uniform and heterogeneous hypergraphs pave the way towards the application of evolutionary dynamics to explain real-life systems. In particular, a challenging question is to understand how group size influences the performance of teams of cooperating individuals. In this section we introduce a procedure based on a series of assumptions to extract the synergy factor from datasets of interacting individuals and apply it to study the bibliographic dataset of the American Physical Society (APS).

Our technique is grounded on the hypothesis that the structure of the hypergraph is the outcome of an optimisation process, in which the players have selected their connections to maximise their payoff. Therefore, one should expect a one to one correlation between the hyperdegree distribution **p** and the group size dependent synergy factor r^g. Based on this idea, we argue that the system has to be constrained by two conditions: **r** has to be proportional to **p** and the system has to be at equilibrium, and thus defectors and cooperators have the same average payoff. The combination of $r^g = zp^g$ and $\sum_{g \in \mathcal{G}} p^g(1 - r^g) = 0$ yields

$$r^g = \frac{p^g}{\sum_{g \in \mathcal{G}} (p^g)^2} \tag{15.4}$$

In a random graph with no hyperdegree heterogeneities the average fraction of hyperdegrees $p^g = k^g/k$ can be obtained from the total number of hyperlinks at each order as $k^g = gL^g/N$ as long as hyperlinks are uniformly distributed. Under these conditions the synergy factor can be extracted as a function of the total number of groups at each order L^g. We employ this technique to retrieve the synergy factor of 13 different APS journals with a total of 577886 papers. Authors and papers are respectively represented as nodes and hyperlinks of 13 different hypergraphs from where L^g is measured as the total number of papers produced by a given number of authors.

We have now an algorithm for detecting the synergy factor, but we are also interested in explaining its origin. With that goal in mind we propose to model the group

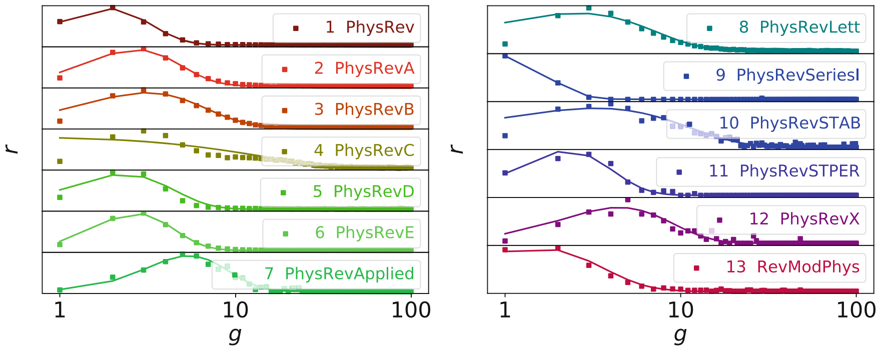

Fig. 15.4 Synergy factor on real games. We show the reduced synergy factor as a function of the group size extracted from the bibliographic data of the APS for a selection of 13 journals. The dots represent the experimental values while the continuous line corresponds to the model at Eq. (15.5) explaining the synergy factor as the combination of positive and negative group effects. Adapted from Ref. [42]

size dependence of the synergy factor as the product of two opposite contributions, a first one increasing with g, and a second one decreasing with g

$$f(g, \alpha, \beta, \gamma) = \alpha g^{\beta} e^{-\gamma(g-1)} \tag{15.5}$$

From these three parameters α has a fixed value given by normalisation, and therefore we are left with β and γ, both larger than zero. Due to their functional dependence on g, β accounts for the positive aspects of group interactions while γ represents an exponential dumping and therefore is associated to lower synergy factors in larger groups. For the particular dataset we are studying, these opposite terms have specific meanings in the production of scientific manuscripts: One could associate the benefits (β) with a multiplication in the amount and depth of ideas and discussions preceding the manuscript preparation when increasing the group size. Analogously, the costs (γ) may be associated with difficulties in coordination when recruiting additional authors for a paper. This interpretation is also compatible with the shape of r^g, whose maximum is predicted to be at β/γ.

In Fig. 15.4 (from [42]) the empirically derived profiles of the reduced synergy factor for the APS dataset are shown, as well as the curve that better fits such profile according to the model in Eq. (15.5). The parameters are reported in Table 15.1.

15.3 Discussion

In this chapter we have presented the formalism introduced in [42] for studying the evolutionary dynamics of systems with explicit higher-order interactions. We have derived the replicators approximation and showed that it successfully accounts

Table 15.1 Synergy factor on real games. Parameters of the study carried out for the APS bibliographic dataset. For each journal we indicate the total number of hyperlinks L, the average order $< g >$, the order with the highest synergy factor $g(\max r)$, the benefit and cost parameters β, γ and the distance between the approximation by Eq. (15.5) and the data d_r

Journal	L	$< g >$	$g(\max r)$	β	γ	d_r
PhysRev	47313	1.95	2	2.936	1.573	0.033
PhysRevA	70502	3.07	3	2.679	0.986	0.067
PhysRevB	171268	3.75	3	1.531	0.49	0.05
PhysRevC	36290	5.98	3	0.02	0.075	0.146
PhysRevD	74715	3.02	2	2.178	0.941	0.206
PhysRevE	50988	2.93	3	3.84	1.41	0.048
PhysRevApplied	327	5.39	5	3.356	0.62	0.09
PhysRevLett	113674	4.57	3	0.848	0.33	0.175
PhysRevSeriesI	1240	1.21	1	2.691	2.831	0.019
PhysRevSTAB	2393	5.52	4	0.566	0.173	0.127
PhysRevSTPER	484	2.42	3	2.75	1.21	0.078
PhysRevX	611	5.28	5	1.85	0.416	0.127
RevModPhys	3153	2.05	2	1.19	0.79	0.112

for the system's dynamics in uniform and heterogeneous hypergraphs as long as no hyperdegree-hyperdegree correlations are present. We have then discussed a proposal for extracting the synergy factor of real games and apply it to the analysis of the bibliographic dataset of the APS.

This new framework for higher-order interactions calls for a hypergraph adaptation of additional game characteristics that complement the PGG to facilitate the emergence of cooperation, such as image scoring [34–36], rewarding [45], and punishment [46–49]. Similarly, the hypergraph implementation motivates new research in the direction of understanding the influence of more complex structures, such us communities or multilayer organization, which were well characterized for games in standard networks [21–29]

All these assets will surely strengthen the already consistent and reliable hypergraph implementation of evolutionary dynamics for modelling the emergence of cooperation.

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