Tri-simplicial Contradiction: The "Pascalian 3D Simplex" for the Oppositional Tri-segment

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To Jean-Yves, Régis, and Hans
Three structuralist deep minds
Who changed forever my
Creative perception
Of mathematics

Abstract In this paper, we deal with the theory of the "oppositional poly-simplexes", producing the first complete analysis of the simplest of them: the oppositional tri-segment, the three-valued counterpart of the oppositional-geometrical red *contradiction* segment. The concept of poly-simplex has been proposed by us in 2009 (Moretti A, The geometry of logical opposition. PhD thesis, University of Neuchâtel, Switzerland, 2009), for generalizing the theory of the "oppositional *bi*-simplexes", which is the heart of our and Angot-Pellissier's "oppositional geometry" (Angot-Pellissier R, 2-opposition and the topological hexagon. In: [30], 2012; Moretti A, Geometry for modalities? Yes: through *n*-opposition theory. In: [27], 2004; Pellissier R, Logica Universalis 2:235–263, 2008). The latter is meant to be the general theory of structures like the logical hexagon (which is a bi-triangle). The *poly*-simplexes are the most straightforward way to turn any "geometry of oppositions" consequently *many*-valued, which is otherwise still a *desideratum* of the field. We start by recalling the general theoretical context: how the field was opened, around 2002, by a reflection on the foundations of paraconsistent negation (Béziau J-Y, Logical Investigations 10:218–233, 2003) and how from that has progressively emerged "oppositional geometry", the theory of the "oppositional structures", enabling to model the "oppositional complexity" of "oppositional phenomena". After recalling how emerged the idea of *poly*-simplex, we explain why time seems to have now come to explore them for real, since we have two powerful new tools: (1) Angot-Pellissier's sheaf-theoretical technique

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J.-Y. Beziau, I. Vandoulakis (eds.), *The Exoteric Square of Opposition*, Studies in Universal Logic, https://doi.org/10.1007/978-3-030-90823-2_16

(Angot-Pellissier R, Many-valued logical hexagons in a 3-oppositional trisimplex. In: this volume, 2022; Angot-Pellissier R, Many-valued logical hexagons in a 3-oppositional quadrisimplex. Draft, January 2014) for producing the many-valued oppositional *vertices* of the poly-simplexes (and evaluating their *edges*) (2) and a generalization of “Pascal’s triangle”, turned here into a more general “Pascalian ND simplex”, whose suited “horizontal (N-1)D sections” provide a much needed and very powerful numerical-geometrical “roadmap”, for constructing and exploring *any* arbitrary oppositional poly-simplex (Sect. 1). After unfolding successfully the structure of the tri-segment (Sects. 2 and 3), we make an unexpected detour (Sect. 4) by Smessaert and Demey’s “logical geometry” (Smessaert H. and Demey L., Journal of Logic, Language and Information 23:527–565, 2014), composed of two “twin geometries”: one for “opposition” and another for “implication”. We thus develop the “implication geometry” of the tri-segment and so discover that what these authors take for a “bricolage” (called by them “Aristotelian geometry”) is in fact, when considered as general “Aristotelian *combination*” in opposition*al* spaces higher than the bi-simplicial one (the one in which they remain tacitly but constantly), the mathematically optimal way, bottom-*up*, for exploring methodically the poly-simplicial space. We end (Sect. 5) by considering *applications of this tri-segment* resulted from such a tri-simplicial diffraction of the bi-simplicial contradiction segment (which adds to it paracomplete, i.e., intuitionist, and paraconsistent, i.e., co-intuitionist, features) in many-valued logics, paraconsistent logics, quantum logic, dialectics, and psychoanalysis. In particular, we show that the tri-segment, by its paracomplete substructure, models, better than did anything before it, “Lacan’s square”.

Keywords Pascal’s triangle · Pascal’s simplex · Multinomial theorem · Oppositional bi-simplex · Oppositional poly-simplex · Oppositional geometry · Logical geometry · Logical hexagon · Contradiction · Negation · Classical negation · Many-valued logics · Topos-theory · Sheaf theory · Paraconsistent logics · Paracomplete logics · Intuitionism · Co-intuitionism · Hegelian logics · Dialectics · Quantum logics · Psychoanalysis · Lacan’s square · Square of sexuation

Mathematics Subject Classification (2000) Primary 18N50, Secondary 11B65, 03B50 18F20, 03B53, 03A05

1 The Context of This Study

Before studying the oppositional tri-segment (Sects. 2, 3, and 4) and its applications (Sect. 5), it will be useful to recall the context where this new strange mathematical concept has arisen in 2009. This obliges us to say something, in the following Sect. 1.1, about the genesis of “oppositional geometry”. In Sect. 1.2 we will present oppositional geometry and in particular its link with the concept of “bi-simplex” (2004). Starting from Sect. 1.3, we will recall the concept of “poly-simplex” (2009). Finally, in Sect. 1.6 we will introduce the original idea of “tri-segment” (2009), meaning by that the simplest oppositional poly-simplex (poly≥3).

1.1 The Controversy on the Foundations of Logical Negation (2003)

Starting in 1995, some logical inquiries on the philosophical foundations of “paraconsistent logics” (the logic of “nontrivial inconsistency”; cf. [21, 35, 106, 117–118]) by Slater, Priest, Restall, Paoli, Béziau, and several other logicians and philosophers bore progressively to the front a quite ancient and almost forgotten *structure*: the “*square* of opposition” (a.k.a. “logical square”, “Aristotle’s square”, etc.). This old structure (second century) condenses some of the main concepts of Aristotle’s (384–322 BC) theory (and logic) of “opposition” (Fig. 1).

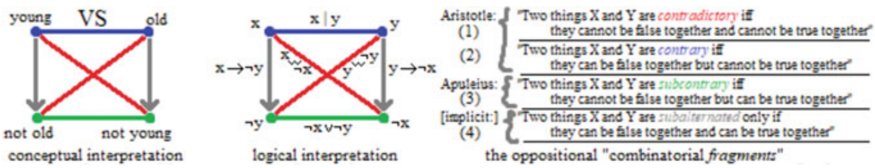


Fig. 1 The “square of opposition” (or logical square, or Aristotle’s square) and some of its fundamental properties

What happened is that some scholars (and *in primis* Slater in 1995, [132]) used *abstractly* this square (i.e., without drawing it but by recalling its definitions – Fig. 1) in order to show that what paraconsistent logic (since at least 1948, with N.C.A. da Costa and his Brazilian school of paraconsistent logic, [46]) argued to be, namely, a logic provided with a new kind of “negation” operator (i.e., a “paraconsistent *negation*”, a negation “~” capable of having nontrivially “A&~A”), is in fact something much less interesting than promised (by paraconsistent logicians): because “paraconsistent negation” truly speaking reveals to be something not even deserving the name “negation”. So, paraconsistent logic and its paraconsistent negation would be, according to Slater (and his many followers on this point), a brutal deceit. More precisely, the claim was that, formulated in the ancient but fundamental (i.e., “transcendental”) language of the old square (Aristotle’s language), a conceptual language which more or less gave birth to “logic” and has been conserved (as a primitive but sound fragment) by the successive developments of logic, paraconsistent negation, seen as a binary relation, is neither a relation of “contradiction” nor a relation of “contrariety” (both being indeed two currently serious forms of negation, namely, the classical and the “intuitionist” ones). It is only (and disappointingly) a relation of “subcontrariety” (i.e., some kind, so to say, of *inverse* of contrariety, some kind of *anti*-contrariety, i.e., some form not of incompatibility but, on the contrary, of close “collaboration”!). Put more crudely, Slater’s argument pretends to show that paraconsistent “negation” is in fact no other than “inclusive disjunction” (i.e., the well-known – and not negation-like! – “∨” operator of propositional logic), given that subcontrariety has precisely that meaning (cf. Fig. 1) when translated, as it can, into “propositional logic” (the logic of the “binary connective” relations). Accordingly, at least some of the *many* attempts (which I reviewed in 2010 [96]), by paraconsistent logicians, to resist Slater’s devastating criticism against the very idea of their working field, paraconsistent logic, could have involved a renewed study of the logical square. But, in fact, that happened to be done almost uniquely by the paraconsistent logician and philosopher Jean-Yves Béziau (2003, [24], as a complement to [26]), who tried to answer frontally Slater’s objection (i) by adopting its main argument (“paraconsistent negation *is* the square’s relation of *subcontrariety*”) but (ii) by reconsidering radically the very idea of logical square (and this strategy, as we will see, is *a posteriori*, without exaggeration, a bit of a stroke of genius). Béziau claimed, for short, (1) that Slater was right in his advocation of the old square concerning paraconsistency (2) *but* that “subcontrariety” is in fact not a disappointing (or marginal) but, on the contrary, *a very important relation*, indeed, such that – quite contrary to Slater’s claim – it deserves *plainly* being considered *a very interesting “new” kind of nonclassical “negation”*! In a nutshell, Béziau did this by resuming the totally forgotten concept of “logical hexagon” (1950, cf. [97]), which, when speaking about the logical square, *de jure* is a mathematically unavoidable reference (but *de facto* so much and so badly underestimated by logicians, even now, that it is almost unknown by them!). This is because this hexagonal *structure* shows (cf. Blanché [33]), with mathematical certainty, nothing less than the undisputable fact that the logical square is only a problematic and misleading *fragment* of the mathematically unproblematic (but still mysterious) logical hexagon (Fig. 2).

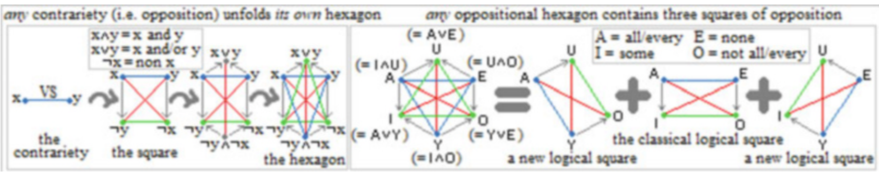


Fig. 2 From *any* “contrariety” emerges a “logical square”, and hence a “logical hexagon” containing three logical squares

Then Béziau showed, by a discovery he made while reasoning, against Slater, on the possible applications of the logical hexagon to contemporary “alethic-modal logic” (Sect. 1.2), that there are not *one* but in fact *three* logical hexagons, the classical one (under its alethic-modal reading, already proposed by Blanché around 1953), which expresses classical negation, and two new (Béziauian) logical hexagons, also for alethic-modal logic, which express, respectively, relatively to alethic-modal-logical operators, not the *classical*, but the *paracomplete* (i.e., intuitionist) and the *paraconsistent* (i.e., co-intuitionist) negation (the “co-” being an important concept of "category-theory", cf. Sect. 1.4, and the "para-" being an important concept of metalogic) (Fig. 3).

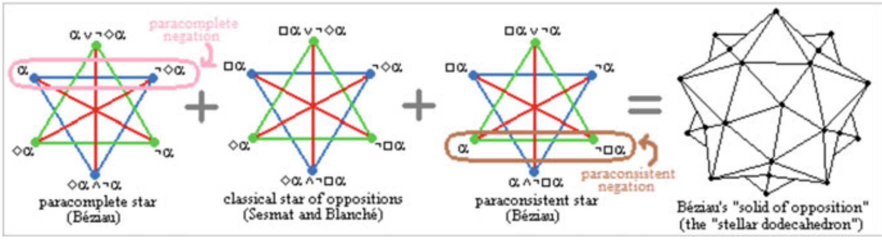


Fig. 3 Beziau's very original defense of the idea of "paraconsistent negation" (2003) against Slater's attack to it (1995)

By the proposal of that new global setting of opposition matters (i.e., notably by conjecturing, at the end of his very original – and not enough appreciated – 2003 paper, the existence of a new 3D "stellar dodecahedron" solid of opposition made by the 3D intersection of the three 2D "modal hexagons" (a 3D stellar solid, however, that Béziau was imprudent enough not to draw and check . . .) taking the place of Aristotle's 2D square, as the new *transcendental core* of the foundations of logic), Béziau argued he had proven the mathematical naturalness of "subcontrariety" and thus defeated in its very heart Slater's fundamental attack to the very idea of paraconsistent negation and paraconsistent logic (to this we will return on Sect. 5.3).

1.2 *The Mathematics of Opposition: There Is an Oppositional Geometry*

Béziau's 2003 paper [24] – where he introduced, as a promising conjecture, the idea of a 3D solid of opposition, larger (and more meaningful) than the old 2D square, although Béziau's very solid revealed soon mistaken in its details: not 12, but 14 vertices – unexpectedly opened, notably through Moretti (2004, [93]), Pellissier (2008) [111], and Smessaert (2009) [133], a whole new discipline devoted to the study of this kind of oppositional-geometrical solids (or polytopes): currently some (Smessaert and Demey, cf. [49, 51, 135]) call it "logical geometry", while others (ourselves and Angot-Pellissier, cf. [3, 98, 101]) call it "oppositional geometry". Whatever the name, the result, after more than 15 years of studies until now, is that there is (by now still small, but growing) a new branch (or at least a new theory) of mathematics, having "opposition" as its focus *object* of study. Epistemologically, it must be noticed that, still nowadays, this is a difficult bit to swallow for many established scholars: "opposition" was *reputed* – and so it tends to remain (mistakenly!) – to be more or less the "strict possession" of mathematical *logic* and of the "analytical philosophy" allegedly based on it. The latter, on that respect, is *reputed* to have fought victoriously against at least two tough competitors in the twentieth century: Hegelian-Marxist "dialectics" (which claimed also to be the *science* of "contradiction" and of "opposition"; Sect. 5.5) and transdisciplinary "structuralism" (which used quite often, together with instances of the mathematical concept of "group", duly modified instances of ... the square of opposition; cf. [97, 100], but also [126, 127]) and, as such, still nowadays, analytical philosophers and logicist logicians (there are!) do not want to "share their cake" (i.e., the concept of *opposition*). But by now, oppositional geometry, although still almost unknown, does indeed exist and keeps growing (as this paper will try to show, with its climax on Sect. 4.6), given its robust results and main mathematical *ideas*. As for its "body", it consists (at least so far) mainly of two infinite *series* of new mathematical structures: (1) *n*-oppositions (i.e., so to say, "*n*-oppositional *kernels*", the so-called bi-simplexes), or A*n* (Fig. 4), (2) and oppositional *closures* (one for each kernel of *n*-opposition; cf. Fig. 7 for a clear example of application of the *series* of the B*n*-structures). In fact, an important result of oppositional geometry of 2008 [111] is that *any* oppositional bi-simplex or A*n*-structure (which expresses, as such, the kernel of *n*-opposition) has its own "oppositional closure" or B*n*-structure (obtained by adding to it an "envelope"; cf. Figs. 49 and 51, which complete the kernel by adding some more vertices to it, forming its "cloud"); moreover, there also are "oppositional generators" or Γ-structures, which appeared first of all – but up to now still without a specific theorization – in modal logic (among others in Prior and Hamblin, Chellas, Hugues and Cresswell, and Popkorn; cf. Fig. 35 of [97]) but seem to have a much more general nature, still to be explored. The general character of these two series (the A*n* and B*n*) is granted mathematically by a demonstration provided by the mathematician Régis Angot-Pellissier in 2008 [111], which also gives a general mathematical handling method for oppositional geometry: this is called the "set-theoretical partition technique for *n*-oppositions" or "setting technique". Consequently, oppositional geometry is so to say ruled by this general set-theoretical method (the linguist and logician Hans Smessaert has provided in 2009 [133], independently, another one, based on "bit strings", roughly equivalent, later leading to what he and Lorenz Demey call since 2011 "logical geometry"; cf. [135]). This method allows putting into precise relation the oppositional generators (the Γ-structures, when there are), the oppositional closures (the B*n*-structures), and the oppositional kernels or *n*-oppositions (the A*n*-structures) (Fig. 5).

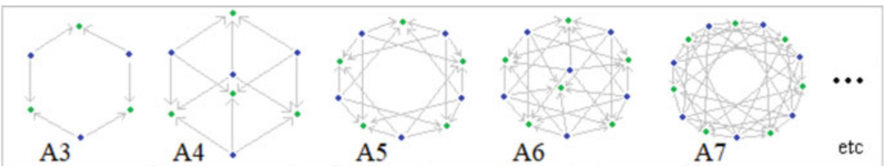


Fig. 4 The bi-simplexes of dim. $n-1$ (hexagon, cube...) are the "kernels" (A*n*) of the *n*-opposition structures

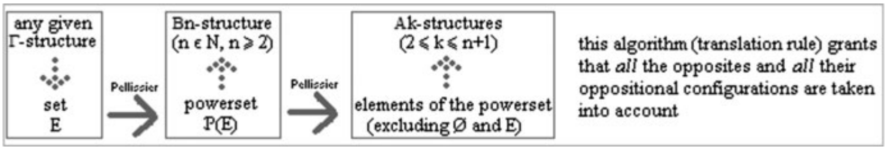


Fig. 5 Pellissier’s setting method for oppositional geometry: from generators (Γ), to closures (B), to kernels (A)

Oppositional geometry, through its abstract study of “oppositions” as such, therefore offers a tool for measuring (and studying) a transversal object which can be called “oppositional *complexity*” and which in fact can appear in *every* discipline (*oppositionality* is a universal mathematical property). Let us recall two quick examples of this.

First, different phenomena (whatever their field) sharing the same degree of oppositional complexity have the same oppositional-geometrical model (or “oppositional attractor”): for instance, the “oppositional tetrahexahedron” or B4 (i.e., the complete structure of 4-opposition – i.e., its *closure* – whose *kernel* is the “oppositional cube”, or bi-tetrahedron, or A4) formalizes, among many other things, propositional logic, first-order predicate logic, alethic-modal logic, and partial order-theory (Fig. 6).

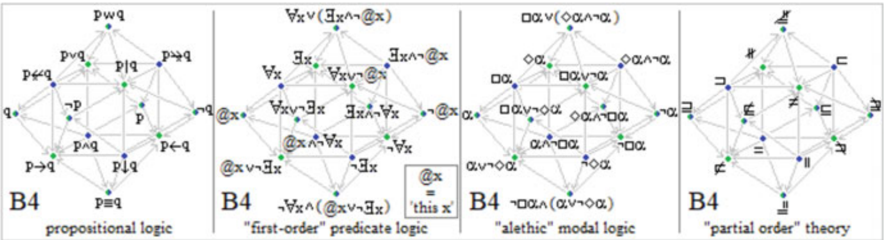


Fig. 6 The “oppositional tetrahexahedron” (B4), the attractor of 4-opposition, characterizes several different fields

But it also formalizes things much more informal, in whatever field (including humanities and art), like, for instance, some important gender issues (gender identity) and some conceptual frameworks related to sexual preference, just to give two “sexier” examples (Sect. 5.6). Remark, *en passant*, that the 3D opposition solid Béziau was *rightly* looking for (Sect. 1.1) is precisely the B4 (and more precisely the third B4-structure in Fig. 6). Béziau missed its precise structure (cf. Fig. 3) but was quite right in guessing its existence (and its three-dimensional nature).

Second, and conversely, *a same qualitative (or conceptual) phenomenon can admit (in different contexts possible for it) different degrees of oppositional complexity*. This is, for instance, the case with the mathematical object “order”, studied by the fundamental branch of mathematics called “order-theory” (or “lattice theory” [48]): in different although strictly related mathematical universes (viz., discrete order, total order, partial order, set-theoretical order . . .), this object (“order”) admits different degrees of oppositional complexity, and oppositional geometry, as a plastic tool for that, allows giving of this, at the same time, a precise *arithmetical* measure and *geometrical* description (the arithmetical measure of the oppositional complexity of "order" can be, respectively, 2, 3, 4, and 5, meaning oppositional-geometrically the B2-, B3-, B4-, and B5-structure, respectively; Fig. 7).

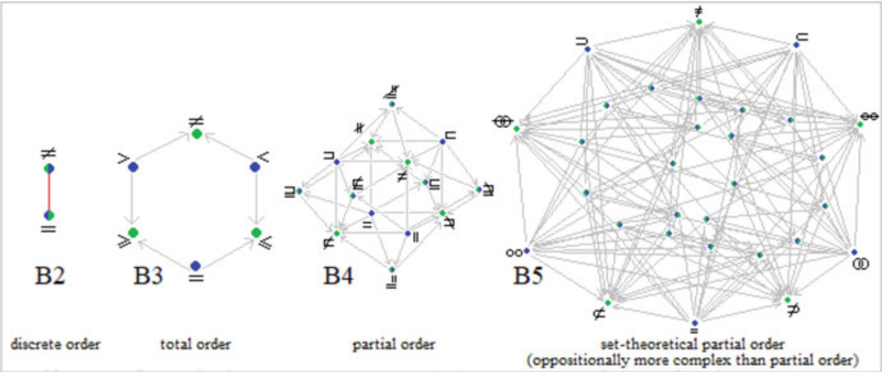


Fig. 7 Oppositional geometry, by the series of its closures (B*n*), allows *measuring* the complexity of different "orders"

It must be remarked that oppositional geometry seems to be, as any part of contemporary mathematics (this important *structuralist* feature was magistrally explained in 1968 by Piaget [112]), a converging point of several distinct mathematical "distant" areas: oppositional geometry is known to have in it important elements of graph theory [125], mathematical logic, modal logic, fractal geometry [109, 110], and knot-theory ([101]), and some facts (e.g., in the tetrahexahedron and higher similar closure structures) seem to suggest also the presence of relevant aspects of "differential topology" ([62] p. 52), to be inquired in the future (Fig. 8).

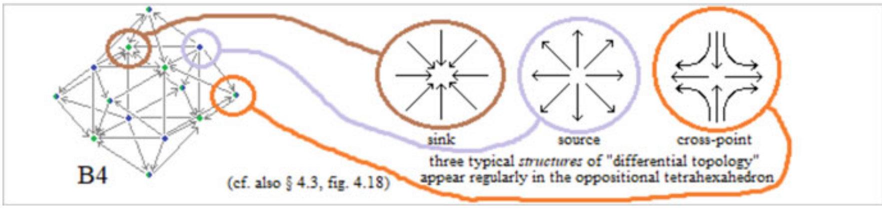


Fig. 8 There are several mathematical, "extra-logical" interesting properties in the oppositional tetrahexahedron (B4)

In our own so to say "anti-logicist" and structuralist way of understanding it (meaning by "structuralism" something like "taking seriously and therefore pushing to their still unknown limits any new combinatorial fragments"), oppositional geometry seems to be structured around the mathematical concept of "oppositional bi-simplex" (cf. [93]), and "simplexes" are *mathematical* objects usually absent from the vocabulary and the "ontological pantheon" of logic, since the latter tends to be independent from (and anterior to) "numbers", whereas simplexes are, so to say, precisely "geometrical numbers" (cf. [15]; and [45], p. 120) (Fig. 9).

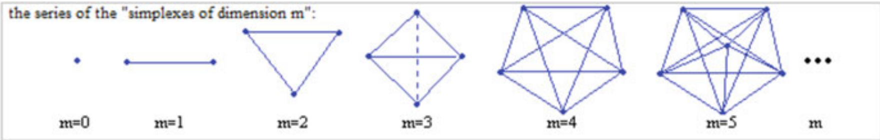


Fig. 9 "Simplexes" are so to say the geometrical counterpart of "numbers": their dimensionality grows into infinite

Historically, oppositional geometry seems to have popped out all of a sudden, unexpected, around 2004, and as such it still is "not very well accepted", as we said, for instance (and mainly), by analytical philosophy "leaders" and therefore "troops" – witness, paradigmatically, the otherwise unexplainable and unjustifiable lasting absence of *any* reference to the logical hexagon in, say, Terence Parsons' very famous, and since 1997 regularly updated (!!!), entry on the logical square in the very famous *Stanford Encyclopedia of Philosophy* [107]: remark that this valuable scholar was invited (and thereafter published) as speaker in the "First World Congress on the Square of Oppositions" (Montreux, 2007) where the logical hexagon (and the premises of oppositional geometry) was one of the main objects in explicit focus, so Parsons cannot claim to "ignore *de jure*" what he seemingly has *decided* to "ignore *de facto*". But, at the same time, despite this "analytical opposition to the geometry of oppositions" (Sect. 4.6), truly speaking, ongoing historical and epistemological studies constantly discover from time to time new elements of evidence that, although oppositional geometry has remained unveiled until very recently (i.e., 2004, with [93]), there have been, and since long, several premonitions of the existence of such a "geometry of oppositions" and even some true forerunners of it: several forgotten strange, isolated, and badly understood glimpses into oppositional-geometrical possibilities were given, among others, also by historically important thinkers like Aristotle, Apuleius, Llull, Buridanus, Lewis Carroll, De Morgan, Vasil'ev, Reichenbach, Prior, etc. (on all this cf., for instance, [97, 98, 100], but also [126, 127]).

Now, another chapter of this young and still open mathematical adventure is relative to the generalization trying to go from the aforementioned concept of bi-simplex to a new concept naturally built on top of that, namely, the concept of *oppositional poly-simplex*.

1.3 A General Extension of OG: The "Oppositional Poly-Simplexes"

In 2004, A. Costa-Leite (then a PhD student of Béziau, as soon myself) challenged oppositional geometry (then called by me "n-opposition theory", N.O.T.) to get rid, if it only could, of its founding but also constrictor notion of "bi-simplex". The (friendly but frank) reproach signified that, however (finitely) big and complex the blue (n-1)-dimensional simplex of n-contrariety and hence the geometrical n-opposition built on top of it (its "oppositional closure", the Bn of the An, Sect. 1.2, Fig. 7), it always consisted, in some sense, of four and only four oppositional main "colors" (conventionally – since the proposal of Béziau [24], generally adopted after him – blue, red, green, and black/gray). This led me to accept such radical challenge, trying to go from the concept of bi-simplex to something wider, namely, as I proposed, the more general (but at that time inexistent!) concept of oppositional "poly-simplex". In my 2009 PhD dissertation [94], in some sense I succeeded in responding this challenge (as for the key ideas for this, my intuition traces back to *at least* 2006, and I obtained the main results *no later than* in 2007). The original idea consisted of the following: (1) in trying to understand how the two oppositional "simplexes" (responsible of the four colors) so to say popped up with Aristotle's theory of opposition, and this led back to his elegant combinatorial definition (Sect. 1.1, Fig. 1 – a "combinatorial fragment"!), expressed by words, of "contrariety" and "contradiction" (completed by Apuleius with the tantamount elegant combinatorial definition – also a "combinatorial fragment"! – given by him or by the "Pseudo Apuleius", together with the square visual device, of "subcontrariety"; Fig. 1) (this was called "Aristotelian 2^2-semantics" and "Aristotelian 2^2-lattice") (Fig. 10); (2) in trying to see whether one could go beyond "bi-simpliciality" (Costa-Leite's challenge) so understood (i.e., taken as the "Aristotelian" game-theoretical "meta-level" generating the blue and the green simplexes and the red and the gray "links" between these two simplexes); and (3) that is (I proposed), in changing the number of the truth-values authorized in its generative "ask-answer" game-theoretical meta-level, observing that this kind of change seemingly generates, automatically, in an infinite variety of different possible ways, precisely extra "oppositional simplexes" (and interesting extra pairs of links between them). This was called "Aristotelian p^q-semantics" (with "q" as the number of possible different questions "Can two things ...?" and "p" as the number of possible answers "Yes/no/maybe/..." to each of these questions); it was first examined as "Aristotelian 3^2-semantics" and "3^2-lattice", which generate three instead of two simplexes (Fig. 11).

Each such Aristotelian p^q-semantics results in fact in a correlated "Aristotelian p^q-lattice", useful in so far as it gives, *a priori*, the "oppositional colors" (i.e., the possible *qualities* of opposition) of the mathematical universe under discussion. In the case where $q = 2$ remains unchanged, the variations of p generate Aristotelian 2D *square* lattices (presented as lozenges) which are bigger and bigger but remain two-dimensional. Differently, when what varies is "q", the lattice becomes increasingly many-dimensional and n-dimensionally hypercubic, a.k.a. "measure polytopic" ([45], p. 123). The q is a new, third dimension – along with p and with n

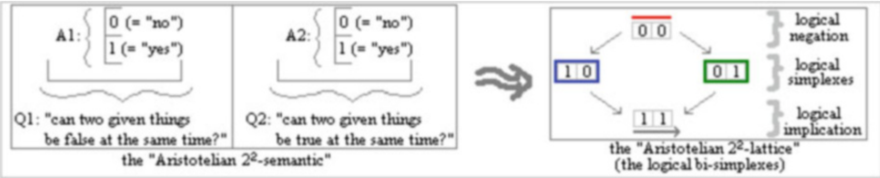


Fig. 10 The way proposed in 2009 in order to formalize Aristotle’s game-theoretical generator of opposition theory

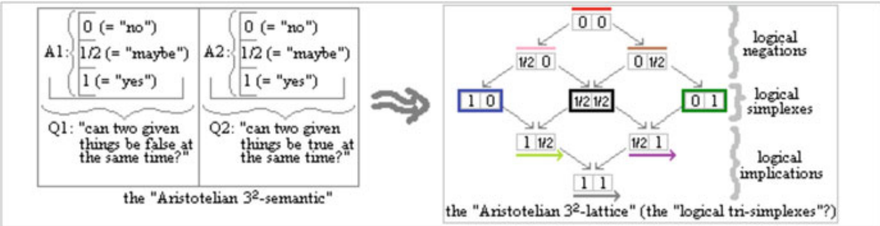


Fig. 11 The Aristotelian 3^2-semantic and its correlated 3^2-lattice giving the “kinds of opposition” of the tri-simplexes

- of *complexity growth* of the whole poly-simplicial *n*-opposition. So, in a nutshell, the changes in *p* and in *q* generate a mathematical 2D space (populated by *n*-dim measure polytopic lattices) of possible changes in the meta-level of the theory of opposition (Fig. 12).

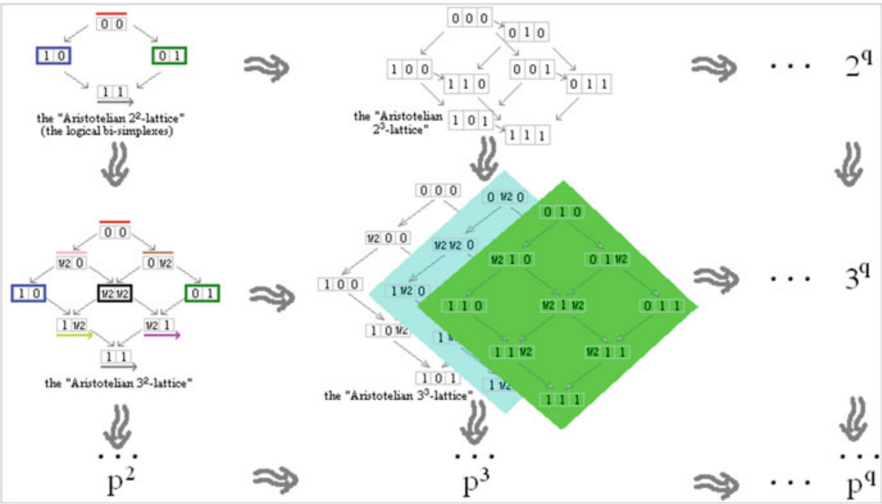


Fig. 12 An overview of the space of the possible Aristotelian p^q-lattices (for the general oppositional poly-simplexes) when “*p*” and/or “*q*” vary

Given that the growing complexity of the simplexes (indexed by the parameter "n") constitutes, as said, a third dimension, the whole space of such possible generalizations of the Aristotelian bi-simplicial case (generalizations abstractly conjectured by me in 2009) can be figured through an infinite "Aristotelian parallelepiped" (the front rectangular face of which corresponds (modulo a 3D diagonal rotation) to the 2D space of Fig. 12) (Fig. 13).

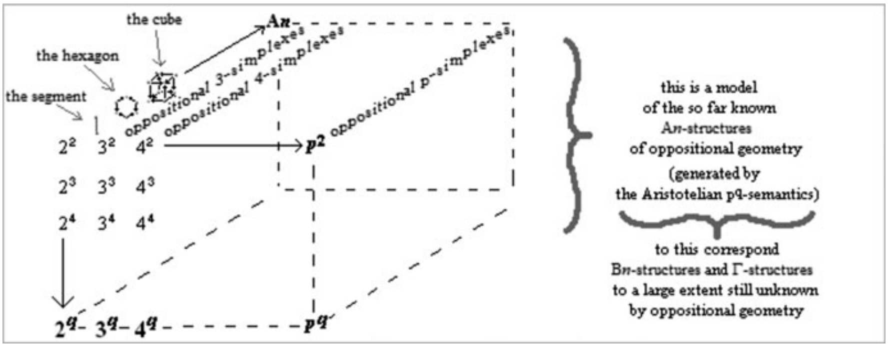


Fig. 13 The Aristotelian infinite parallelepiped for the oppositional poly-simplexes (and beyond)

Remark that in all what follows only $q = 2$ will be explored (i.e., only 2D, square-shaped Aristotelian p^2-lattices). This means that we will remain 2D at the level of the metatheory of opposition (leaving the exploration of the still intriguing "q" parameter – supposing it leads, as I believe, to something sound, meaningful, and tractable – for further studies, Sects. 4.1, 5.1 and 5.2).

It is by this "Aristotelian" method that it was proposed in 2009 to consider the existence of such a new family of mathematical structures relative to oppositions (integrating logical many-valuedness and giving birth to the structure of the poly-simplexes). The problem then, at this still very hypothetical and programmatic level, was that of seeing what concrete oppositional *geometry* could, if it could, result from this new research paradigm and program. So, for instance, one way to explore such still hypothetical oppositional poly-simplexes seemed to be fixing one simplex, for instance, the 2D simplex (i.e., the triangle, Sect. 1.2, Fig. 9), and studying the series of its infinitely growing oppositional poly-instances, namely, the (still conjectural) space of the oppositional poly-*triangles* (Fig. 14).

Another way in order to explore the still unknown space of the poly-simplexes (Fig. 13) seemed to be fixing instead the number "p" of simplexes considered (taking, for instance, $p = 3$, i.e., the *tri*-simplexes) and considering the increasing structural complexity of the series when the constitutive *simplex* (present in three different colors: blue, black, green) grows (from segment to triangle, to tetrahedron, to four-dim simplex, to five-dim simplex, etc., Sect. 1.2; Fig. 9): this was conjectured as the (hypothetical) space of the oppositional *tri*-simplexes (Fig. 15).

One of the many exciting parts of this new research line seemed to be the search for new oppositional solids. For instance, the tri-triangle seemed to be possibly

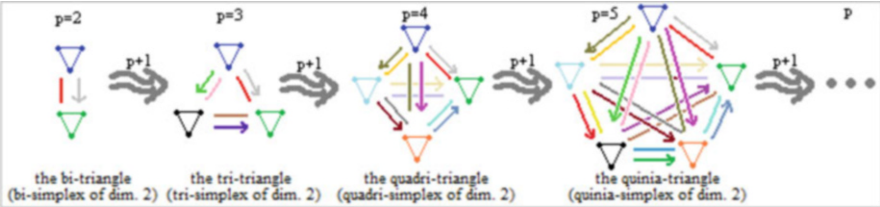


Fig. 14 The space of the oppositional poly-triangles: from the bi-triangle (i.e., the logical hexagon) to the *p*-triangle

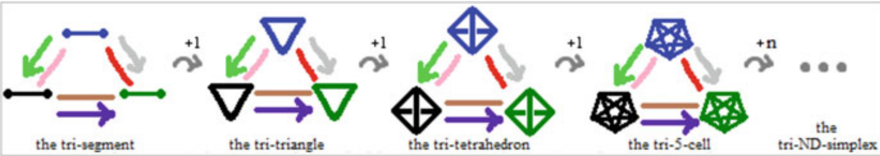


Fig. 15 The (hypothetical) space of the growing oppositional tri-simplexes (tri-segment, tri-triangle, tri-tetrahedron . . .)

represented by a compact 2D figure (provided one adopts curve lines, as in non-Euclidean geometry) (Fig. 16).

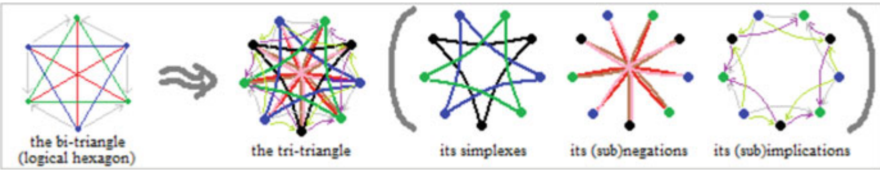


Fig. 16 From the "bi-triangle" (logical hexagon) to the "tri-triangle" (its simplexes, contradictions, and subalternations)

The idea seemed, if not worth the tiniest shadow of a postdoc (. . .), at least quite new, interesting, and promising, and it was proposed this could have (because of what recalled in Sect. 1.1) some non-negligible impact over the philosophy of the foundations of logic (especially with respect to nonclassical logic – paraconsistent and many-valued – as we argued in [96]).

However, the problem with this conceptual and visual conjecture of us of 2009 was that something quite important was still missing badly, namely, (1) some kind of mathematical proof (I am trained as – and I am! – a "continental philosopher"), or axiomatic construction, of the fact that these oppositional-geometrical poly-simplicial entities (of which we have seen at least the hypothetical conceptual idea, but also some possible concrete oppositional-geometrical shapes) are mathematically sound under every respect, and (2) some device (comparable to Angot-Pellissier's set-theoretical technique for generating the bi-simplicial oppositional closures, cf. Sect. 1.2, Figs. 5 and 7) in order to make, concretely, the "jungle" of the oppositional poly-simplexes mathematically real, testable, and applicable.

Tackling this problem leads us to the next Sect. 1.4 of this first chapter.

1.4 Angot-Pellissier's Sheaf-Theoretical Method for Poly-Simplexes

This needed mathematical method for the *poly*-simplexes happened to be excitedly announced and then shown on blackboard, in 2009 by Angot-Pellissier in a talk given in a four-people (!) workshop on the geometry of oppositions organized the day after my PhD defense. But it arrived in written version only 4 long years later (first in a draft in 2013 and then in a sequel draft in 2014, the first appearing only now [3], the second still unpublished [4]). The method mainly consists, so to say, in shifting from set-theory to sheaf-theory. For that recall, first of all, that sheaf-theory and topos-theory are important parts, or consequences, of "category-theory" [81, 83], which, in turn, so to say, has taken the place of "set-theory" as the main conceptual framework of general mathematics (this is called the "dynamic turn" of mathematical *structuralism*, cf. Avodey [7–9]). Recall also, secondly, that the "setting technique" for bi-simplicial oppositional geometry [111] consisted mainly in studying the possible *partitions* of a given set: the theory tells, among others, how to get to such a starting partitionable set ("Angot-Pellissier's set"), case by case (it is here that can play a role the Γ-structures, as we show successfully in a particular case study in [95]), and then how to study its *partitions*. Recall, thirdly, that a "sheaf", here, can be seen as a "topological diffraction" of this concept of "set", giving thus access to a more complex and powerful viewpoint over mathematical creativity (a set is then seen, retrospectively, as a particularly simple and "static" instance of sheaf, so to say a sheaf reduced to a point, whereas the latter, generally, has an extra topological richness [81]). *Consequently, Angot-Pellissier's new "sheafing technique"* [3] *for poly-simplicial oppositional geometry consists, mainly, in studying, instead of the partitions of a starting suited set, the possible sub-sheaves of a given starting suited sheaf (in fact, as we will see, a "numerical sheaf") and thereby in giving access to this sheaf's finer-grained "partitions" (the theory tells you how to determine this starting sheaf)*. And this starting sheaf takes into account both (1) the complexification of mathematical discourse that results (when talking about poly-simplexes) from the adoption of more than two truth-values in the Aristotelian game-theoretical algorithm (the Aristotelian p^2-semantics) generating the possible kinds of opposition relations (through the correlated Aristotelian p^2-lattice) (2) *and* the dimension of the simplex (i.e., whether it is a segment, a triangle, a tetrahedron, a five-cell, etc.).

More concretely, the first move of Angot-Pellissier's new method for oppositional poly-simplexes consists in translating in sheaf-theoretical terms the change in the number of truth-values. If "truth" (i.e., "1") is seen as a maximal starting set "X", and "false" (i.e., "0") as the (minimal) empty set "∅", the needed interpolated additional truth-values (remember the "*p*" parameter, Sect. 1.3) – like, for instance,

“$^1/_3$”, “$^1/_2$”, “$^2/_3$”, etc. – will be constructed as (or represented by) interpolated intermediate sets “U”, “V”, “Y”, etc., such that the first is a strict subset of “X” and that each of the following is a strict subset of the preceding ones, the empty set being by definition a strict subset of all. This constructs a suited “topos” (Fig. 17).

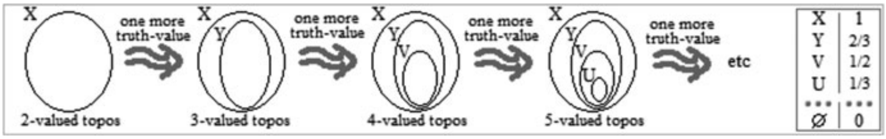


Fig. 17 The introduction of additional truth-values between “0” (false) and “1” (true) by means of topoi and sheaves

Then, another of the keys of this “sheafing method” is the redefinition in such sheaf-theoretical mathematically precise terms of the Aristotelian classical definitions of “contrariety” and subcontrariety (Sect. 1.1, Fig. 1), i.e., the two questions Q1 and Q2 of the Aristotelian p^2-semantics (cf. Sect. 1.3, Figs. 10 and 11). Provided with that, Angot-Pellissier can reconstruct, in a mathematically understandable and rigorous way, our intuitive and conjectural idea of Aristotelian p^2-semantics and p^2-lattices, by deriving any new *kind* of opposition as an articulation of two answers to the two questions relative to the sub-sheaves of the starting sheaf (we will give a step-by-step concrete example and illustration of this on Sect. 2.2). Angot-Pellissier’s study, in some sense, thus confirms our general conjecture of 2009 over the possibility of theorizing with mathematical rigor the oppositional poly-simplexes. As a paradigmatic example, he finds back in [3, 4], by his new technique, the two “Aristotelian lattices” (the 3^2- and the 4^2- ones) proposed by me for the tri-simplexes and the quadri-simplexes, respectively (which he applies to the simplex “triangle”) (Fig. 18).

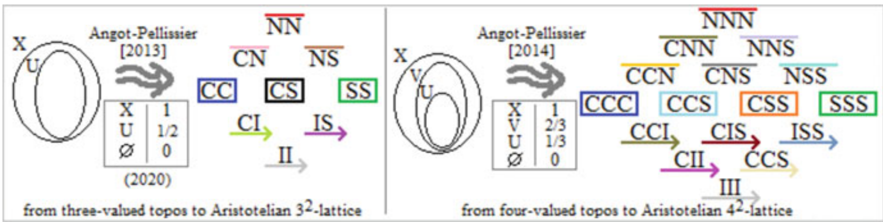


Fig. 18 Angot-Pellissier’s redefinition (through sheaf-theory) of the Aristotelian 3^2- and 4^2-lattices ([2013], [2014])

An important point – to which we will come back in Sect. 5.3 – is that Angot-Pellissier, by the way, confirms by a demonstration (by *topological* arguments) that the two left-right families of “infra-negations” (“CN” on one side and “NS” on the other side in the tri-triangle’s Aristotelian 3^2-lattice; “CNN”, “CCN”, and “CNS” on one side and “NNS”, “NSS”, and “CNS” on the other side in the quadri-triangle’s

Aristotelian 4^2-lattice) are mathematically such that the members of the first are "paracomplete" (i.e., intuitionist) negations, while the members of the second are "paraconsistent" (i.e., co-intuitionist – cf. Sect. 1.1, Fig. 3) negations (remark that "CNS" is member of both families: it is both paracomplete and paraconsistent, behavior logically called – João Marcos *docet* – "paranormality" [sic]).

Finally, by introducing one last element, namely, the *length* of the numerical sheaves in question, by expressing them as finite indexed strings of the form "$1_j 2_k 3_l 4_m \ldots$" (with j, k, l, m belonging to the set {∅, U, V, . . . , X} of the topos-theoretical truth-values), he expresses, in the sheaf's very structure (i.e., in its length, defined as numerical sheaf's string length), the dimensionality of the involved simplex of the studied poly-*simplex* (e.g., "$1_j 2_k 3_l$" is for triangles, "$1_j 2_k 3_l 4_m$" is for tetrahedra, etc.). Angot-Pellissier also finds back, by his new rigorous mathematical method (i.e., in a new way), the oppositional tri-triangle I had predicted and tentatively represented in 2009 (Sect. 1.3, Figs. 14 and 16) (Fig. 19).

Fig. 19 Angot-Pellissier's sheaf-theoretical construction of the oppositional tri-triangle [2013] (2020)

(Angot-Pellissier also offers in [3] a new, original global 2D representation of the tri-triangle, the "nonagon", which we omit reproducing here). In the same way, in his second draft study [4], Angot-Pellissier finds back, but constructed in mathematically more rigorous (and understandable) terms than I did in 2009 [94], the oppositional quadri-triangle (Fig. 20).

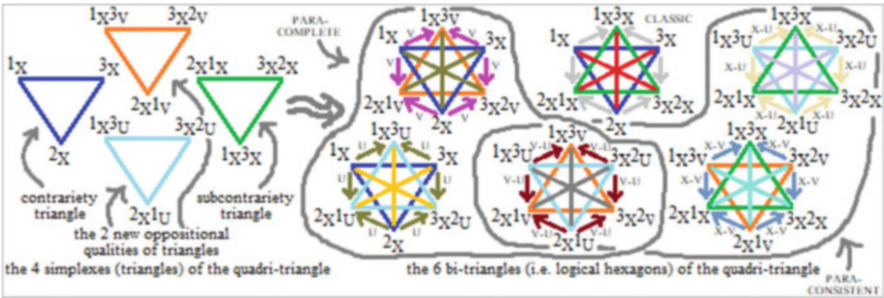


Fig. 20 Angot-Pellissier's sheaf-theoretical construction of the oppositional quadri-triangle [2014]

However, Angot-Pellissier in his two pioneering draft studies also introduces some strange and puzzling elements, which he just mentions, without discussing them much, both in his study on the tri-triangle [3] and in his study on the quadri-triangle [4]. Namely, a first element of puzzlement is that he mentions the existence, among the things generated by his new tools, of (at least) one extra triangle (lapidary judged irrelevant, because redundant) in the tri-triangle and two extra triangles (also judged irrelevant, again because redundant) in the quadri-triangle: he dismisses further discussion of this point, to us unexpected and puzzling – leaving the lucky reader (we have had the prepublication deep friendly privilege to be) rather confused – again, only mentioning that such extra triangles are oppositionally (i.e., combinatorially) equivalent to others he takes into account, and as such the redundant ones can/must (?) be neglected (Fig. 21).

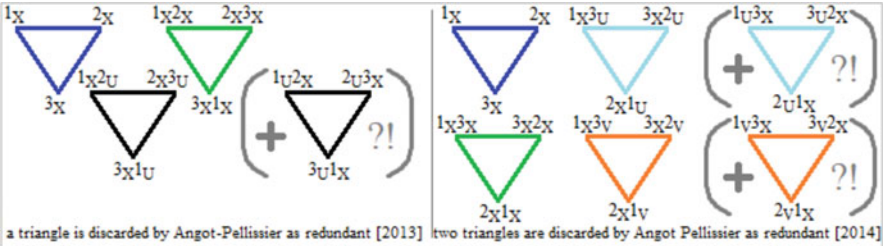


Fig. 21 Angot-Pellissier’s sheaf-theoretic method reveals the existence of some “equivalent” extra triangles

Whereas with his 2008 [111] “setting” technique Angot-Pellissier gave to oppositional geometry a powerful tool such that in some sense it possibly generated all (it thus helped fixing the boundaries of the oppositionally possible/thinkable), with his 2013 [3] “sheafing” technique Angot-Pellissier did not explain what kind of “totality” this new method could lead to, speaking about oppositional poly-simplexes. As we are going to see (Sect. 1.5), the present paper proposes a first clear answer to this crucial, until now open question. A second possibly puzzling aspect in his two otherwise absolutely groundbreaking draft papers, is that Angot-Pellissier still does not provide *concrete* examples of application of the two poly-simplexes he successfully studies (successfully from a needed, *purely mathematical*, but then *not* applied point of view). So, for short, the problem, if any, with Angot-Pellissier’s long waited for method, otherwise very promising and in fact quite exciting (for people interested in oppositional-geometrical research), was that (1) on one side it allowed finding more instances of poly-simplicial entities than what I predicted in 2009 (without this fact being further explained and explored by him) (2) but on the other side Angot-Pellissier himself seemingly explored (and explained) less entities than what his promising method seemed to make possible: he just selected things sufficient for confirming (as the friend he is) my analysis of 2009 (and this he did indeed successfully).

Tackling and solving these two residual problems (while benefiting decisively from the powerful and long waited for sheafing technique offered to us oppositional geometers) leads us to the next Sect. 1.5, possibly the most important, if any, of all this study.

1.5 Our Proposal of a "Pascalian" Extra Tool for the Poly-Simplexes

Remaining faithful to the structuralist methodology (or "ideology"! – as we explain in our theory of the "elementary structures of ideology", [102]) according to which oppositional geometry (which inquires oppositional *structures*), as any mathematical investigation, is a matter of *general mathematics* (and not of "essentially and primarily of *logic*"! – as might seem to suggest, for instance, the more sellable but misleading label "logical geometry", Sect. 4.6), we propose now to use, here, tools promising (and fit!) even if they usually are not used in "logic": repeating a gesture we dared in 2004 [93], when we successfully introduced "out of the blue", in the study of "oppositions", the mathematical n-dimensional concept of "simplex". Being a matter of "n-opposition" (with n any integer such that $n \geq 2$), oppositional geometry (which was called, at the beginning, "n-opposition theory", i.e. "N.O.T". in acronym, with n a *numerical* parameter) has inescapably to do with *numbers*, whereas "logic" (with the magnificent exception of "linear logic", which is precisely by no means – J.-Y. Girard [68–72] *docet* – a logicist tool!) normally doesn't. Recall that numbers – i.e., arithmetic – are so to say the structural "deadly threshold" of Gödel's complexity for formal systems (his famous second theorem of 1931, cf. [104]) and by that a proof of the deadly uselessness, and in fact harmfulness, of the logicist "ideology" (cf. [68]), consisting allegedly, but fruitlessly, in (keeping trying, over and over) "reducing things to logic" (logic seen – very mistakenly – as "the deepest element in mathematics"). Deepening this *fundamental relation of oppositional geometry to numbers*, we will turn now to a fundamental (and famous) *structure* of arithmetic and general "number-theory", namely, "Pascal's triangle". Remark, incidentally, that historically speaking this very important *mathematical structure* was already known in India (by the mathematician Pingala, in the second century BC) and, much later, in China (no later than in the fourteenth century). And after that, but long before its "discovery" by Blaise Pascal (1623–1662) in 1654, it had been rediscovered, independently, by the mathematicians Michael Stifel (1487–1567) in 1544 and Niccolò Tartaglia (1499–1557) in 1556. Now, as is well known even at school, "Pascal's" triangle (as it is now thus improperly called) is obtained very simply: starting from the top, with a "1", each lower integer (of the triangle) is the sum of the two integers above it (on its left and on its right, considering, by a typically structuralist move, that the absence of an integer means the presence of the number "0"...). Generated thus by a simple arithmetical algorithm, Pascal's triangle results in a *structure* which condensates in itself a huge number of really fundamental features of mathematical complexity, which potentially can be unfolded into infinite. Most famously, Pascal's triangle expresses, through the series of numbers in each of its horizontally left-right symmetric lines, each of the coefficients of Newton's (1642–1727) "binomial formula" for expressing the development of "(a + b)^n", whatever the natural number "n". And in fact, an important remark here (we will see why very soon) is that Pascal's triangle is so to say parallel to this Newtonian formula (a + b)^n for binomial powers (Fig. 22).

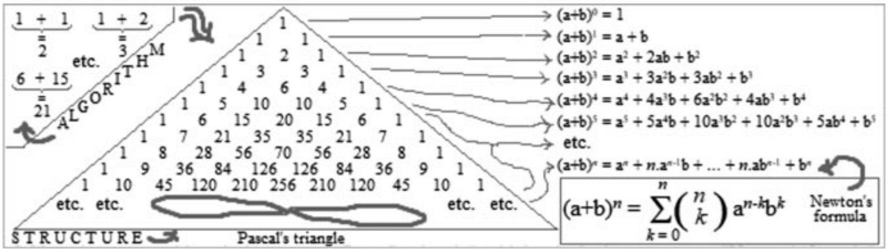


Fig. 22 An important arithmetical *structure*, "Pascal's triangle", with its algebraic counterpart, "Newton's formula"

This is perhaps the most famous application of Pascal's triangle, at least at school's level, although *by far* not the only one: for instance, reading it "diagonally" (and still top-bottom) also gives the series of the "polytopic numbers" (a.k.a. "figurate numbers", i.e., *simplicial* generalizations of the "triangular numbers", i.e., numbers – those studied by the Pythagorean – characterized by intrinsic geometrical properties). Moreover, Pascal's triangle is also known for having very strong *fractal* properties, among others, in the distribution of its numbers – be them prime numbers, or even numbers, or powers, etc. More concretely, it exhibits fractal patterns akin to "Sierpiński's gasket", cf. [109], p. 91–102, and [110], p. 85–96 (Fig. 23).

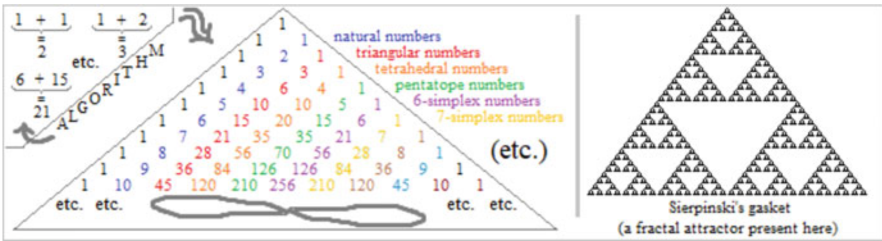


Fig. 23 Pascal's triangle also displays, "obliquely", the "polytopic numbers", and has many fractal properties

But then, what about oppositional geometry? As it happens, Pascal's triangle contains, among other *mathematical* treasures, nothing less than *all* the numerical features of the *closures* of the oppositional bi-simplexes (!!!): each of its horizontal lines, starting from the third from top, gives, line by line, the exact *full* "numericity" of the closure of one of the n-oppositions for $n \geq 2$ (with no possible exception, it is an isomorphism: no "gap" and no "glut", Sect. 3.5, Fig. 81). As such Pascal's triangle presents the two constitutive simplexes (blue and green) of any bi-simplex but also, in between them, their "cloud" (!): for short, again, *all* (bi-simplicial . . .) oppositional geometry is contained in Pascal's triangle (Fig. 24).

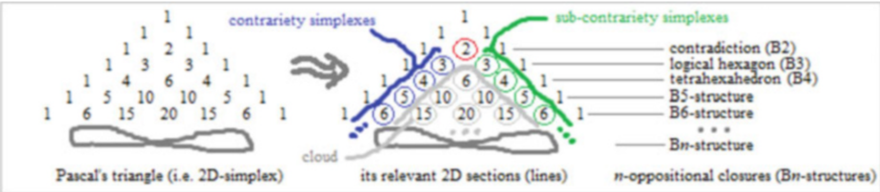


Fig. 24 Pascal's triangle gives, with its horizontal lines, *all* the numbers of (bi-simplicial) oppositional geometry!

Remark that retrospectively this is not totally surprising: since the n-oppositions are all the possible set partitions of any finite set of n elements ($n \geq 2$), as Angot-Pellissier has established in 2008 [111]. Now, it must be remarked that Newton's formula, mentioned above, also does this "oppositional job", starting from what corresponds in it to the third line from the top of Pascal's triangle (in bold the *nontrivial* oppositional elements):

$(a+b)^2 = a^2 + \mathbf{2ab} + b^2$	which are the numbers of 2-opposition (segment);
$(a+b)^3 = a^3 + \mathbf{3a^2b} + \mathbf{3ab^2} + b^3$	the numbers of 3-opposition (hexagon);
$(a+b)^4 = a^4 + \mathbf{4a^3b} + \mathbf{6a^2b^2} + \mathbf{4ab^3} + b^4$	the numbers of 4-opposition (tetrahexahedron);
etc.	etc.
$(a+b)^n = a^n + \boldsymbol{n.a^{n-1}b} + \ldots + \boldsymbol{n.ab^{n-1}} + b^n$	the numbers of n-opposition.

But this, although it might seem (to some good mathematical eye) *a posteriori* mathematically "natural", seems nonetheless absolutely remarkable! And as it happens, it gave us the idea (in 2018) of trying to generalize geometrically Pascal's triangle, so to obtain a general method for having an equivalent "oppositional numericity" for the oppositional *poly*-simplexes. With this very goal in mind, we propose here to introduce a new concept (new at least for us: we almost surely are rediscovering something known in contemporary – and maybe even in classical? – mathematics, as for the idea we propose here of *extending* n-*dimensionally Pascal's triangle*): something we propose to call, accordingly, the "Pascalian ND *simplex*", seen as a general geometrical-numerical *structure* such that Pascal's triangle is only a particular case of it, the case where, in "ND", $N = 2$ (*addendum*: in fact we found in Wikipedia, afterward, mention of the existence of "Pascal's simplex", which seems to be exactly what we are speaking about here – all what follows in this Sect. 1.5 has nevertheless been developed, or redeveloped, by us “out of nothing”). Let us see how to unfold this idea progressively. For a start, the very first step of this will be considering, after the well-known Pascalian “triangle” (seen as a “simplex of dimension 2”), a “tetrahedron” (seen as a “simplex of dimension 3”) that we will call “Pascalian” as well, for it is made so that it has a horizontal *triangular* equilateral “basis”, going downward (as the horizontally *segmental*, infinite “basis” at the infinite bottom of Pascal’s triangle goes endlessly downward): by construction this horizontal triangular infinite basis of the Pascalian tetrahedron dives, step by step, into growing infinite numerical depth and complexity. Each of the remaining three non-horizontal triangular faces of this Pascalian tetrahedron will be, like Pascal’s classical triangle taken as a whole, triangular and with a horizontal linear basis, step by step going down endlessly into infinitely more complex numericity (in fact, precisely the numericity of Pascal’s triangle) (Fig. 25).

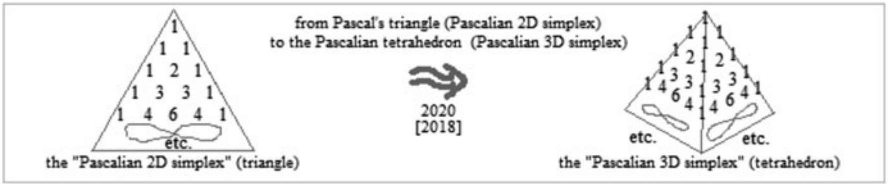


Fig. 25 Our proposal [2018]: from “Pascal’s triangle” (2D simplex) to a “Pascalian tetrahedron” (3D simplex)

Now, the crucial point is that this Pascalian tetrahedron has numbers even “inside” of it, determined by a suited analog of the simple algorithm generating the numbers of Pascal’s triangle. This “internal” algorithm explaining the internal numbers of the Pascalian tetrahedron can be seen, again, in (at least) two ways: (1) either as a graphical or (2) as an algebraic algorithm, the former being an extension of “Pascal’s” graphic algorithm (in a nutshell, each number in a layer – i.e., horizontal triangular “section” – is the sum of the *three* numbers above it, the absence of a number being counted as number “0”) and the latter being an extension of Newton’s formula, such that it calculates (for each horizontal *triangle*, instead of for each horizontal *line*) the powers of the sum of *three*, instead of *two*, addenda (i.e., it calculates “$(a + b + c)^n$” instead of “$(a + b)^n$”) (Fig. 26).

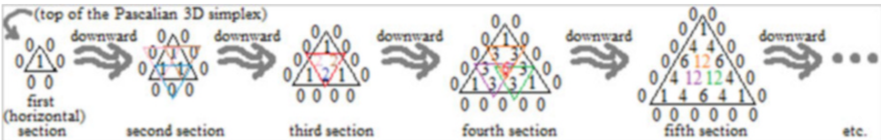


Fig. 26 The graphical 3D algorithm, extending the classical 2D one, for generating by sum the “internal numbers”

Consequently, what is immediately interesting for us, for the oppositional poly-simplexes, through this deepening move (from triangle to tetrahedron) is in fact *the generalization of the concept of "horizontal line of Pascal's triangle",* because this happens to yield the corresponding, duly changed concept of "horizontal sections" (of the Pascalian tetrahedron) for the oppositional tri-simplexes. The Pascalian 3D simplex (a tetrahedron) happens to have, in fact, an infinite number (one simultaneously intersecting a horizontal line in every of its three lateral triangular faces) of top-bottom growing 2D horizontal "sections", which are triangles, and *these triangular sections happen to be such that they are perfectly suited for exploring ... the tri-simplexes!* The correspondence can be matched by comparing the numbers given by these triangles with the numbers given by Angot-Pellissier's sheaf-technique. In fact the more complex is the *simplex* (of the studied oppositional *tri*-simplex), the deeper you will have to go down in the 3D Pascalian simplex (the tetrahedron) for finding the adequate horizontal triangular section. We will demonstrate and explain duly our *general* claim in another paper; here it will suffice to show that it holds at the levels we are interested in and works *perfectly* with the oppositional tri-segment we will thus inquire in Sects. 2 and 3 of this study (Fig. 27).

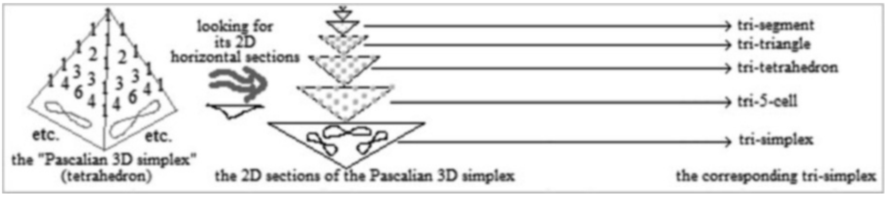


Fig. 27 The 2D simplicial sections (i.e., triangles) of the "Pascalian 3D simplex" (i.e., tetrahedron) map the tri-simplexes

Remark that, as it happens, the aforementioned Newton formula for binomial powers (i.e., the coefficients of the development of "$(a + b)^n$") still gives a parallel vision: as said, it simply becomes here $(a + b + c)^n$, i.e., what changes when going from the Pascalian 2D simplex (Pascal's triangle) to the Pascalian 3D simplex ("Pascal's tetrahedron") is, in its "Newtonian translation", the fact of having three addenda ("a", "b" and "c") instead of, classically, only two ("a" and "b"). This can be used for generating the same set of internal numbers: in fact, the formula generates all the numbers of any 2D section (triangle) of the Pascalian 3D simplex (tetrahedron).

But let us unfold this idea further. In the same way, if you now want to explore the oppositional *quadri*-simplexes (Sects. 1.3 and 1.4, Figs. 18 and 20), as we do in other working papers, you need a more complex Pascalian ND simplex: you need, no more no less, what we propose here to call the "Pascalian 4D simplex" (i.e., a 4D hyper-tetrahedron, or 4D "five-cell", i.e. a 4D "volume" delimited by five 3D tetrahedral “faces”, by analogy with the tetrahedron, whose 3D volume is delimited by four 2D triangles, and by analogy with the 2D triangle whose 2D “volume”, i.e. surface, is delimited by three 1D segments, its three sides, etc.). From this Pascalian 4D simplex, what you further need to consider for dealing with the oppositional *quadri*-simplexes are, again, its “horizontal sections”: but these horizontal sections (i.e., sections “parallel” to the 4D simplex’s infinite “horizontal” – by construction – 3D “base”) now are no more 2D triangles (as they were, in the previous case, for tri-simplexes) but 3D tetrahedra; they are the “3D horizontal sections of the 4D Pascalian simplex”. Incidentally, representing a 4D solid is notoriously not always straightforward [15, 45, 137]. But one way (among others) of doing it rather simply (when possible) consists, so to say, in resorting to “slicing” (i.e., in cutting this 4D solid into suited superposable slices, each of them being lower-dimensional, namely, each of them being a 3D tetrahedron, which can in turn be sliced, this time in terms of stacks of 2D triangles, as previously with the 2D sections of the Pascalian 3D simplex) (Fig. 28).

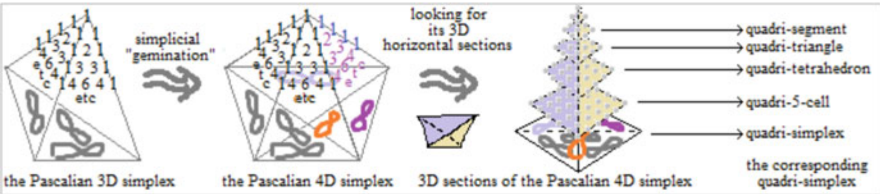


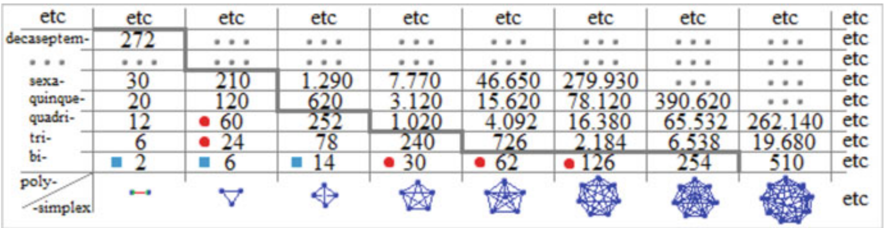
Fig. 28 The 3D simplicial sections (i.e., tetrahdra) of the “Pascalian 4D simplex” (i.e., 5-cell) map the quadri-simplexes

We will not study here its sections (we do it in other working papers on oppositional quadri-simplexes). Just remark that, as it happens, the 4D counterpart of Newton’s formula still exists here, and it is now the development of “$(a + b + c + d)^n$” (the addenda become *four*, instead of *three*).

Now, as we will prove it elsewhere, this “Pascalian” geometrical-arithmetical algorithm for the oppositional *poly*-simplexes goes, always working perfectly, into infinite! And this both in its geometrical-arithmetical *stricto sensu* Pascalian aspect and in its (corresponding) algebraic Newtonian aspect: on one side the Pascalian ND solid, with its (n-1)-D horizontal sections, and on the other side the generalized Newtonian formula $(a + b + \ldots + z)^n$, where, as in all previous cases, the N addenda between parentheses stay for the degree of the oppositional *poly*-simplex (i.e., the numerical value of the “poly-”), whereas the exponent “n” stays for the quality of the *simplex* ($n = 2$ for the segment, $n = 3$ for the triangle, $n = 4$ for the tetrahedron... n for the (n-1)-dimensional oppositional simplex). *The important, general result for oppositional geometry is that, in each case, equivalently, the numbers produced by such a generalized Pascalian ND simplex (or by the correlated generalized Newtonian formula) and those generated by Angot-Pellissier’s sheaf theoretic method for the oppositional poly-simplexes* [3, 4] *do match perfectly and* *give the "poly-simplicial oppositional closure"* (*addendum*: as it seems – and as we found, only afterward, in Wikipedia – these two correlated things we are speaking about are already known in mathematics, namely, as "Pascal's simplex" and as the "multinomial theorem"; but on the one side we rediscovered them by ourselves, and on the other we still found no exact bibliographic references for this).

Remark that, conversely, the clear understanding of the (very wide!) scope of Angot-Pellissier's until know mysterious sheafing method (i.e., understanding that, when duly followed – as we will explain on a precise case in Sect. 2.4 – i.e., with the "Pascalian ND simplex", it can lead to the oppositional closures of the poly-simplexes) shows that his own concrete analyses ([3] and [4], respectively) of the tri-triangle and quadri-triangle happen to show only a small fragment (and therefore not the oppositional *closure*) of the real structure to be brought to light in both cases (we will show this in future studies).

Remark also that given the aforementioned correlation between the oppositional poly-simplexes à la Angot-Pellissier and the generalized Pascalian ND simplex (and its correlated generalized Newtonian formula "$(a + b + c + \cdots)^n$"), the latter (Pascal and/or Newton), besides giving us an Ariadne thread (as we are going to see in the rest of this paper, starting from Sect. 2.4) for studying poly-simplexes in full rigor, allows us doing some quite useful preliminary action with respect to inquiring directly poly-oppositional structures (poly$\geq$3): having *a synoptic view* of the geometrical complexity of the poly-simplexes (provided one concedes, as it seems reasonable, that this complexity is somehow measured by the number of the vertices of each of these structures) (Fig. 29).



etc	etc	etc	etc	etc	etc	etc	etc	etc	etc
decaseptem-	272	...	...	...	...	...	...	...	etc
...	...	...	...	...	...	...	...	...	etc
sexa-	30	210	1.290	7.770	46.650	279.930	...	...	etc
quinque-	20	120	620	3.120	15.620	78.120	390.620	...	etc
quadri-	12	● 60	252	1.020	4.092	16.380	65.532	262.140	etc
tri-	6	● 24	78	240	726	2.184	6.538	19.680	etc
bi-	■ 2	■ 6	■ 14	● 30	● 62	● 126	254	510	etc
poly- / -simplex									etc

Fig. 29 Synoptic view of the complexity degree of the poly-simplicial oppositional geometry (number of vertices)

The structures with a blue square are those already studied, while the ones with a red disk are those which have been studied only partly (as, for instance, tri-triangles and quadri-triangles by Angot-Pellissier [3, 4] or the bi-simplicial closures B5-7 by myself [94, 95]). The figure highlights, by a diagonal cut, the left-bottom domain of the general poly-simplicial space having no more than around 270 vertices. So, the above synoptic view strongly suggests that *the easiest and most reasonable thing to do next*, thanks to our fresh two new tools, in order to explore the poly-simplicial space would be, at present, to study the oppositional "tri-segment" (characterized, as we are going to explain in Sect. 2.4, by the complexity degree 6). And this is precisely what we are going to do in the rest of this paper.

But before starting the inquiry on the tri-segment, let us now have one last preliminary retrospective look to the “old style” study of oppositional tri-*segments*: recalling what is known so far about them, and then (as we will inquire in Sect. 2) what we can try, from now on, to learn about them by a deeper and more accurate investigation.

1.6 Flashback: The Primitive Idea of Oppositional Tri-segment (2009)

As we saw, Angot-Pellissier’s two very important draft studies of 2013 and 2014 [3, 4] concerned poly-*triangles* (viz., the *tri*-triangle and the *quadri*-triangle). But in my 2009 PhD dissertation [94], I also took into account as simplexes (for the poly-simplexes), before triangles, segments. There the “poly-segments” were imagined, roughly, as being some kind of diffraction of the logical square, since the latter *seems* to be a 2-opposition and is precisely based on two simplexes, a blue and a green segment (we come back to this in Sect. 2.5). In the case of the first higher poly-segment (poly≥3), the tri-segment, it was imagined as made of the classical blue-green logical square, plus two interpolated new ones, a blue-black and a black-green squares – but of course in a way different from that in which three logical squares merge to form a logical hexagon (Sect. 1.1, Fig. 2). Therefore it was thought we could have (1) *a blue segment* of contrariety; (2) *a green segment* of subcontrariety; (3) and, interpolated (in the 3D space), *a black new segment* (interpolated) of a “pivotal” simplex. This was my guess as for having a “tri-segment”, the second element of the series of the poly-*segments* and the first element of the series of the *tri*-simplexes. It was seen as made basically out of three simplexes of dimension 2 (i.e., three segments): the two classical ones (the blue and the green horizontal segments of the logical square) and a new one (the third simplex), black (Fig. 30).

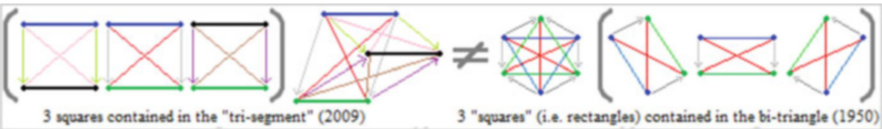


Fig. 30 How was imagined to be, in 2009, the hypothetical structure of “oppositional tri-segment”

Being committed to three truth-values (say: “0”, “½” and “1”, Sect. 1.3) one had to try to understand how three-valued logic can intervene here, if it does (we afford this in Sect. 3.6). Based on their definition through the Aristotelian 3^2-semantics and its correlated 3^2-lattice, the three-valued propositional connectives, allegedly embodied in this oppositional tri-segment, were defined (quite experimentally) by means of an "extensional definition" (Fig. 31).

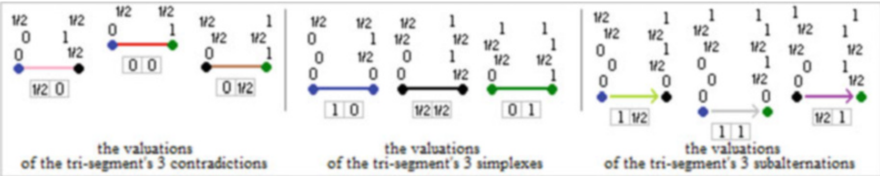


Fig. 31 2009 Conjecture over the possible valuations of the tri-segment's negations, simplexes, and implications

As for the global "valuations" we proposed something like the following three (Fig. 32).

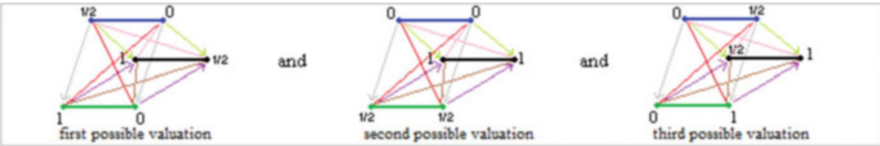


Fig. 32 The problem of the "valuation", with three truth-values, of the (still hypothetical!) tri-segment (2009)

All this, in order to think the "tri-segment", was intuitively inspired by the logical square.

But in 2012 Angot-Pellissier [1] demonstrated (or more precisely: he gave a deep mathematical explanation and clarification of a fact until then "known" too confusedly) that 2-opposition, until then not clearly elucidated in its oppositional-geometrical specificity, is in fact not a blue segment of two-contrariety (which would have *automatically* generated, by central symmetry of the contradictories, a correlated green segment of subcontrariety and therefore the logical square – and therefore the logical hexagon, Sect. 1.1, Fig. 2) but simply the red segment of (any) contradiction. "2-opposition", Angot-Pellissier demonstrated implacably, is tantamount the red segment of contradiction, with no trace of any other possible oppositional color (no blue segment!). This was strange with respect to the idea, otherwise good working, of bi-simplex, but was so. And the logical square, which exhibits a blue segment of contrariety, is therefore (as already established around 1950 by three different people!) only a fragment of the logical hexagon (i.e., 3-opposition, Sect. 1.1, Fig. 2). Angot-Pellissier clarified why contrariety, with its blue simplexes, can begin only with the triangle: *there is no contrariety smaller than 3-contrariety*, as in fact Aristotle knew already. This important and long needed clarifying result by Angot-Pellissier (on which we will come back shortly, from another viewpoint, in the introduction of ch.3) implied, among others, that *my* *proposal of a tri-segment was at least quite mistaken in the way I had tried it, if not totally unthinkable (as tri-segment) per se*.

So, having recalled the current context (state of the art) starting from this point, in the remaining of the present study we will try to come back to this issue of the classical (and “not complex”) segment of (bi-simplicial?) contradiction (seen, as we will explain, as being nevertheless rightfully a “bi-segment”, *cum grano salis*) and to the question of knowing whether it can admit (as proposed in 2009) a tri-simplicial counterpart: something like the oppositional tri-segment . . . As a very last preliminary remark, notice that this research results, therefore, in a voluntary provisory blindness (in this study) with respect to tri-*triangles*, whereas with bi-simplicial oppositional geometry important things clearly begin, so to say, precisely *only* with contrariety (blue) triangles. But that price once paid the expected non-negligible gain is that our present study will be simpler and therefore easier (as suggested in the synoptic view of Sect. 1.5, Fig. 29), which will be no luxury, as we will see (poly-simplexes are “wild”!). Again, the loss, of course, is that oppositional segments (and tri-segments as well) do express much less than oppositional triangles, as well as tri-triangles, and higher poly-simplexes (we study tri-triangles and higher poly-simplexes in other studies, already ongoing). Then what can be expressed by bi-oppositional segments (and by their hypothetical poly-simplicial diffractions)? (1) Neither the relation of two *independent* “atoms”, neither, alternatively, the mutual opposition (i.e., contrariety) of four things: both things need a 4-oppositional A4 bi-tetrahedron (and its B4 closure); (2) nor even the simple contrariety of *two* things (antonymy) (this needs a 3-oppositional A3 bi-triangle (which is its own B3 closure, Sect. 1.1, Fig. 2)); but (3) a 2-oppositional B2 (red) segment, and *a fortiori* its hypothetical poly-simplicial diffractions, can nevertheless express something not so trivial (Sect. 1.1): the *negation* of a given element (and in fact, as we are going to see, Sect. 3.5, also this element’s *affirmation*!) and, again, possibly the “diffractions” of this concept of negation/affirmation . . . (Fig. 33).

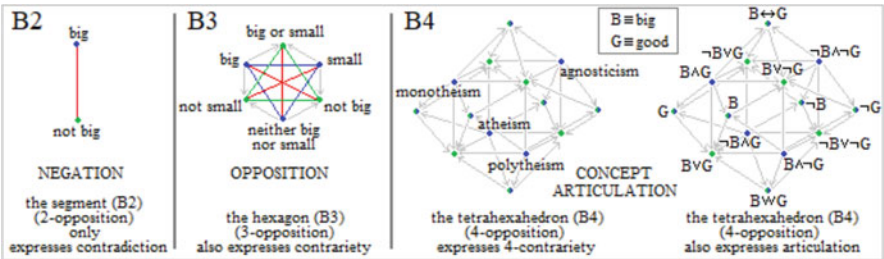


Fig. 33 Segment, triangle, tetrahedron: the increasing expressive power of the bi-*simplicial* oppositional structures

So let us now try to study, in the next two Sects. 2 and 3, this still mysterious structure anew, with the help of the two new tools which are Angot-Pellissier’s sheaf-theoretical technique (Sect. 1.4) and our own new concept of "Pascalian ND simplex" (Sect. 1.5).

2 Studying with These New Tools the Oppositional Tri-segment

In the previous Sect. 1, we recalled the main ingredients, historical, conceptual, and methodological of the context of our present inquiry on poly-simplexes. In this Sect. 2, we are going to study anew the simplest poly-simplex (poly≥3), i.e., the tri-segment (Sect. 1.6), reaching non-negligible new elements of knowledge.

2.1 Oppositional Sub-sheaves of the Tri-segment: Which Are Vertices?

In order to explore the concept of tri-simplex, we start by resorting to Angot-Pellissier's sheaf-theoretical technique (Sect. 1.4), but limiting it to the study of a smaller object than what he considered in his two seminal papers on the subject [3, 4]: *not triangles* (i.e., the tri-triangle and the quadri-triangle, in Angot-Pellissier) *but segments* (here: the *tri*-segment). How to do that?

Two things must be recalled: (1) in Angot-Pellissier's sheaf-theoretical method, one parameter is the *number of truth-values*; (2) the other is the *simplex* considered; more precisely Angot-Pellissier (i) considered *three* truth-values (in [3]) and *four* truth-values (in [4]); (ii) he considered triangles (in both [3, 4]); he thus remained inside the study of oppositional poly-*triangles*.

For us, the way of applying his sheaf-theoretical method to the study of oppositional tri-*segments* (for which he gives no hint) will then consist in the following: (1) we will consider *three* truth-values (because here we want to study *tri*-segments); (2) but we will not deal with triangles, but with *segments* (we are interested in tri-*segments*): therefore, we will not resort to a total numerical sheaf "$1_X 2_X 3_X$" (suited for triangles), but to a shorter total numerical sheaf "$1_X 2_X$" (suited for segments). The "job", then, will consist in working methodically with the sub-sheaves of this shorter total numerical sheaf (remembering Angot-Pellissier's 2008 lesson of [111]: "opposition, i.e., contrariety, is a partition of the true"). Accordingly, since we are dealing with three truth-values, the "topos" (i.e., the category-theory tool ruling many-valuedness) in our study will be the same as the one in Angot-Pellissier's first study [3], namely, one with three levels (three strictly nested sets): the total set "X", one strict *open* subset of it "U", and the empty set "∅" (as such contained in any other set). By construction these three elements are therefore strictly ordered: $X \supset U \supset \varnothing$ (Fig. 34).

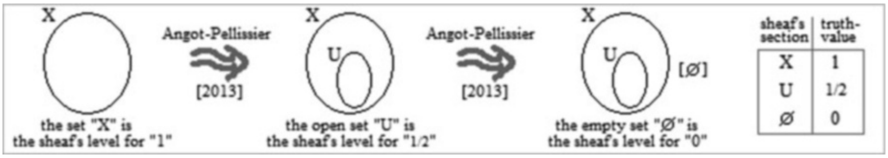


Fig. 34 Expressing, *via* the sheafing technique, the three-valuedness of the space of the oppositional *tri*-simplexes

Now, this, applied to the starting total numerical sheaf (for segments) “$1_X 2_X$”, will give that *all in all* there are, as total number of possible sub-sheaves of this total sheaf, $3^2 = 9$ possible terms (including here those particular sub-sheaves which are the total sheaf $1_X 2_X$ itself and the null sheaf $1_\varnothing 2_\varnothing$). Put into a lattice, they result in a familiar lozenge-shaped structure (Fig. 35).

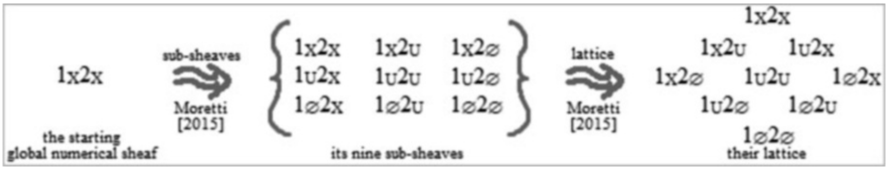


Fig. 35 Distribution of the nine sub-sheaves (i.e., the oppositional-geometrical vertices) of the tri-segment in a lattice

But one must beware: the lattice at the right hand of Fig. 35 looks like the Aristotelian 3^2-lattice of the tri-simplexes (Sect. 1.3, Fig. 11, and Sect. 1.4, Fig. 18), to which we come back in the Sect. 2.2, but truly speaking the lattice here is something totally different (and, as said in Sect. 1.4, this *totality* of the sub-sheaves is thinkable, thanks to Angot-Pellissier, but it must be remarked – in order to understand where we are going to in this study – that this is something that Angot-Pellissier did not study yet as such; notice in particular that the “extended indicial notation” here, admitting “∅” as an explicit index alongside with “X” and “U”, is ours – in Angot-Pellissier’s notation, our “$1_U 2_\varnothing$” is “1_U”, our “$1_\varnothing 2_X$” is “2_X”, etc.).

The next important step, as in the case of Angot-Pellissier’s set-theoretical method for the oppositional *bi*-simplexes (Sect. 1.2, Fig. 5), consists in keeping out from this set of possibilities those which are *trivial with respect to oppositional geometry*. In fact, the sub-sheaf $1_X 2_X$ is trivial: it is an analog of “T” (the “verum” and of the “universal set”). And the sub-sheaf $1_\varnothing 2_\varnothing$ is also trivial: it is an analog of “⊥” (the “falsum”, the null-element of logic, and of the “empty set”, the null-element of set-theory). In oppositional geometry, which – again – in a sense is a matter of *partitioning methodically* a “cake”, the “whole cake” and “no cake” are *trivial partitioning situations*, and as such, by construction, they are put outside the structural game. In fact in the bi-simplicial space, the equivalent of these tri-simplicial two points ($1_X 2_X$ and $1_\varnothing 2_\varnothing$) does exist, mathematically speaking, but, for any oppositional structure, they implode "by construction" to the symmetry center of that structure (this important result – ruling out the reproach otherwise recurrent, as in [87], against oppositional geometry of being "Boolean incomplete" – is due independently to Smessaert and Angot-Pellissier).

Let us remark that the sub-sheaf "1_U2_U" can be seen, of itself, as a bit mysterious of its own as well, so far: for, seen with our "extended indicial notation", it resembles quite much the two trivial sheaves 1_X2_X and $1_\varnothing 2_\varnothing$ (i.e., it bears on both digits of its numerical string, "1" and "2", the same index, viz., "U"), without however being trivial itself at least as long as we know (we will have to come back later, in Sect. 2.4, to this rather important and strange point). In any case, what seems to be sure so far is that in what follows the sub-sheaves 1_X2_X and $1_\varnothing 2_\varnothing$ must, by construction, be neglected (as "oppositionally trivial"). So our study will, from now on, concern no more than seven oppositional sub-sheaves (i.e., supposedly, seven oppositional-geometrical vertices) over the starting nine possible ones (Fig. 36).

Fig. 36 From the lattice of all the nine sub-sheaves of "1_X2_X" to the set of its seven presumed nontrivial sub-sheaves

Now that we have the entities supposed to be the vertices, what must be considered next, if we want to be able to study the oppositional *geometry* of this (i.e., of the tri-segment), is the "lines" uniting each possible pair of these seven vertices (including here the pairs made of a vertex with itself! – this is also something Angot-Pellissier did not in his two draft studies [3, 4]). These are *all* the possible (oppositional) relations between the (oppositional) vertices of the tri-segment, its "oppositional colors" (and this brings us to the oppositional *closure*).

2.2 The Oppositional Relations Between the Sub-sheaves: Edges!

The theory, outlined in my PhD [94] and confirmed and deepened mathematically by Angot-Pellissier's method [3], tells that for tri-simplexes, there are nine possible qualities of oppositional relations, which are, visually, nine "oppositional colors" (Sect. 1.3, Fig. 11, Sect. 1.4, Fig. 18). This is also true of tri-segments, which are a particular case of the general concept of tri-*simplex*. The nine relations predicted by the Aristotelian 3^2-lattice are the following (2009 style on the left, 2013 style on the middle, and 2020 style on the right of Fig. 37) (Fig. 37).

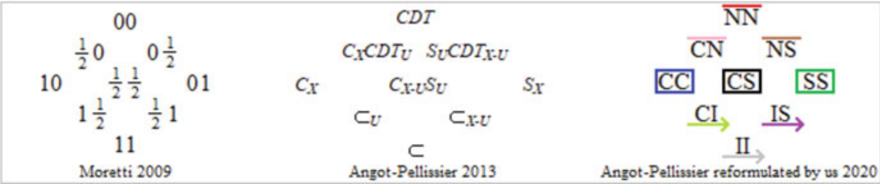


Fig. 37 The lattice of the nine “oppositional qualities” of the tri-simplexes (2009, 2013, 2020)

Now, and this is a major theoretical advance since 2009, Angot-Pellissier’s method not only allows *generating* mathematically *all* the vertices of the poly-simplexes as sub-sheaves of a starting total numerical sheaf (as we will see soon, the “all” is in fact clear only now thanks to the Pascalian method, Sect. 1.5 and 2.4), but it also allows *calculating*, for *any* given pair of such sub-sheaves, the *precise* kind of oppositional relation (among the predicted nine possible ones) that holds between the two elements of the considered pair. Combinatorially speaking, regarding only the geometrical side (i.e., making momentarily abstraction of colors), there are exactly 7!, i.e. $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$ such possible pairs (including the seven pairs of identical elements, which will be twisted segments, i.e., “curls”). In the following, we will give, on the one side, at least an example of calculation for each of the three main oppositional kinds (i.e. negations, simplexes, arrows) and then, on the other side, the complete list of the colors of each of the 28 vertex-vertex relations (of which 21 are edges properly said – i.e., edges – and 7 are reflexive “curls”).

Remark that I will use the notation I propose on the right side of Fig. 37 (equivalent to, but different from, Angot-Pellissier’s original one, in particular more precise relatively to the two nonclassical implications CI and IS, cf. [3]).

Let us start from negations (i.e., the red, pink, and brown oppositional relations). For instance, let us compare the vertices 1_X2_U and $1_{\varnothing}2_X$. We must ask for this pair of nontrivial sub-sheaves the two Angot-Pellissierian meta-questions, first at the lower sheaf-theoretical level U and then at the higher level X (of the three-valued topos we use): the respective four answers to these $2 + 2 = 4$ questions will give us, combined two by two, two literals, *x* and *u* (each taken among the set {N,C,S,I}, respectively, for “negation”, “contrariety”, “subcontrariety”, “implication”), and the noncommutative concatenation “*xu*” of these two literals will give us precisely the tri-simplicial oppositional quality of the relation (or segment) under examination (viz., one among the NN, CN, NS, CC, CS, SS, CI, IS, II). So (call this meta-question “Q1/U”) “Can these two sub-sheaves have a false *(inclusive) disjunction* at level U?” The answer (call it “A1/U”), here, is “0” (i.e., “No”, because their sections on U are, respectively, {1,2} and {2}, so the set-theoretical *union* $\cup$ of the two is {1,2}, which is the “*total* section”, which as such cannot be false). Further (second meta-question, “Q2/U”), “Can these two sub-sheaves have a true *conjunction* at level U?” The answer (“A2/U”), here, is “1” (i.e., “Yes”, because the set-theoretical *intersection* $\cap$ of their aforementioned respective sections on

U, i.e., {1,2} and {2}, is {2}, and therefore it is non-empty). Further ("Q1/X"), "Can these two sub-sheaves have a false *(inclusive) disjunction* at level X?" The answer ("A1/X"), here, is "0" (because their respective sections on X, i.e., {1} and {2}, have a *union* which is the *total* section {1,2}). Further ("Q2/X"), "Can these two sub-sheaves have a true *conjunction* at level X?" The answer ("A2/X"), here, is "0" (because their aforementioned respective sections on X, i.e., {1} and {2}, have an *intersection* which is empty). The last two answers (i.e., those at level X), "0" and "0", determine the first literal, "*x*" (the one standing, on top of the tri-simplicial oppositional quality we are determining, for the level X) as "N" (for "negation": recall that in the Aristotelian 3^2-semantics negation is precisely [0|0], cf. Fig. 10); the first two answers (i.e., those at level U), "0" and "1", determine the second literal, "*u*" (the one standing, at the bottom of the tri-simplicial oppositional quality we are now determining, for the level U) as "S" (for "subcontrariety": recall that subcontrariety is defined as [0|1], cf. Fig. 10). So, together these two ordered literals "N" and "S" give, concatenated (i.e., as the noncommutative string "*xu*"), "NS", which, as shown by the Aristotelian lattice on the right of Fig. 37, is the brown paraconsistent negation. The same kind of reasoning holds for (and only for) the further two (commutative) pairs of vertices, "1_X2_U and 1_U2_X" and "$1_X2_\varnothing$ and 1_U2_X" (thus, all in all, in the tri-segment there are three brown segments of paraconsistent negation NS). A similar reasoning establishes that the CN relation (the pink paracomplete negation) holds for (and only for) the three (commutative) pairs of vertices "$1_X2_\varnothing$ and $1_\varnothing2_U$", "$1_\varnothing2_U$ and $1_U2_\varnothing$", and "$1_U2_\varnothing$ and $1_\varnothing2_X$" (so there are three pink segments in the tri-segment). Finally, a similar reasoning establishes that the NN relation (the red classical negation) holds only between the two vertices "$1_X2_\varnothing$ and $1_\varnothing2_X$" (thus there is only one red segment in the tri-segment). So we have seen here $3 + 3 + 1 = 7$ over the 28 segments of the tri-segment.

Let us now see simplicial colors (among blue, black, and green). For instance, let us compare the vertices 1_X2_U and $1_\varnothing2_U$. So, as previously (Q1/U), "Can these two sub-sheaves have a false *(inclusive) disjunction* at level U?" The answer (A1/U) here is "0". Further (Q2/U), "Can these two sub-sheaves have a true *conjunction* at level U?" The answer (A2/U) here is "1". Further (Q1/X), "Can these two sub-sheaves have a false *(inclusive) disjunction* at level X?" The answer (A1/X) here is "1". Further (Q2/X), "Can these two sub-sheaves have a true *conjunction* at level X?" The answer (A2/X) here is "0". So, the last two answers (level X), "1" and "0", determine the first literal as "C" (contrariety); the first two answers (level U), "0" and "1", determine the second literal as "S" (for "subcontrariety"). Together these two literals give, concatenated, "CS", which is the black pivotal simplicial relation (a mixture of contrariety at level X-U and subcontrariety at level U, cf. Fig. 37). The same reasoning holds for (and only for) the further ten (commutative) pairs of vertices – "1_X2_U and $1_U2_\varnothing$", "$1_\varnothing2_U$ and 1_U2_X", and "$1_U2_\varnothing$ and 1_U2_X" – and all the remaining seven pairs containing at least one occurrence of the vertex "1_U2_U" (thus, all in all, there are $3 + 7 = 10$ black segments, one of which is in fact the black non-arrowed reflexive curl "1_U2_U and 1_U2_U"). Remark that neither the blue (i.e., "CC", contrariety) nor the green (i.e., "SS", subcontrariety) tri-simplicial oppositional relation (i.e., the two "simplicial colors" other than black) does emerge here as 1 of the 21 segments or 7 curls between the possible pairs of the 7 vertices. So we have seen here 10 over the 21 segments and 1 among the 7 curls of the tri-segment.

Finally, let us see implication arrows (we know they can be gray, light green, and violet). For instance, let us compare the commutative pair of vertices $1_U 2_\varnothing$ and $1_X 2_\varnothing$. So (Q1/U), "Can these two sub-sheaves have a false (inclusive) disjunction at level U?" The answer (A1/U) here is "1". Further (Q2/U), "Can these two sub-sheaves have a true conjunction at level U?" The answer (A2/U) here is "1". Further (Q1/X), "Can these two sub-sheaves have a false disjunction at level X?" The answer (A1/X) here is "1". Further (Q2/X), "Can these two sub-sheaves have a true conjunction at level X?" The answer (A2/X) here is "0". So, the last two answers (level X), "1" and "0", determine the first literal as "C"; the first two answers (level U), "1" and "1", determine the second literal as "I" (for "implication"). Together these two (orderly) literals give, concatenated, "CI", which is the light green paracomplete implication (a mixture of contrariety at level X-U and implication at level U). Notice that the direction of the arrow (which can even be two-sided) is determined, further, by *comparing the relevant sections* (for CI this is the sections at level U): the implication then goes from the shorter to the greater section: e.g., $1 \rightarrow 12$, $1 \longleftrightarrow 1$, $12 \leftarrow 2$, etc. (recall that since Angot-Pellissier's proposal in 2008 [111], "12" means "$1 \vee 2$", i.e., "either 1 or 2 or both is true"). In the case under examination the sections on U being, respectively, {1} and {1}, the light green relation CI takes the form of a biconditional. The same relation holds between the three (commutative) pairs "$1_\varnothing 2_U$ and $1_\varnothing 2_X$", "$1_\varnothing 2_U$ and $1_\varnothing 2_U$" (this is a curl), and "$1_U 2_\varnothing$ and $1_U 2_\varnothing$" (another curl). So, all in all in the tri-segment there are, in light green, two segments and two curls. A similar reasoning establishes that the IS relation (the violet paracomplete implication) holds, here also as biconditional, for the two pairs of vertices "$1_X 2_\varnothing$ and $1_X 2_U$" and "$1_\varnothing 2_X$ and $1_U 2_\varnothing$", as well as for the two reflexive pairs "$1_X 2_U$ and $1_X 2_U$" and "$1_U 2_X$ and $1_U 2_X$" (thus two violet arrowed curls). Finally, a similar reasoning establishes that the II relation (the gray classical implication) holds (only) for the two reflexive pairs of vertices "$1_X 2_\varnothing$ and $1_X 2_\varnothing$" and "$1_\varnothing 2_X$ and $1_\varnothing 2_X$" (thus two gray arrowed curls). So we have seen here, as for tri-simplicial implication arrows (of 3 colors), 4 over the 21 segments and 6 over the 7 curls. In sum, 7 (negations) + 11 (simplicial segments) + 10 (implications) = 28. *Le compte est bon*.

Incidentally, remark that to see more directly the link existing with the Aristotelian 3^2-semantics (i.e., with its terms like "[1|½]", etc.), you can read *vertically* the four numbers of the aforementioned *xu* code (obtained as in the examples just described), as in a 2×2 square matrix, where the two numbers of *x* are put on top of the two numbers of *u*: then, a left (resp. right) column of this 2×2 square matrix containing a same number "j" (j∈{0,1}) gives, as half of the Aristotelian code (relative to that column), "[j|" (resp. "|j]"), while a left (resp. right) column made of two different numbers "j" and "k" (j,k∈{0,1}, j ≠ k) gives "[½|" (resp. "|½]").

Now, further focusing on each of the seven vertices we determined (Sect. 2.1), the one after the other, and on the just calculated list of the precise quality of each of their 28 possible mutual two-terms relations, these results can be tentatively displayed in a synoptic way (i.e., in a unique picture), vertex after vertex, in a row, in the following way (Fig. 38).

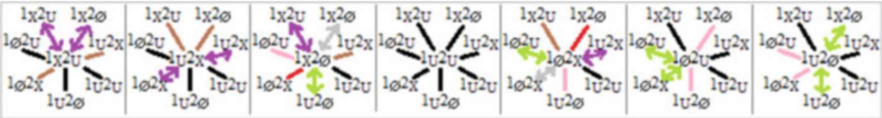


Fig. 38 Synoptical view of the kinds of tri-simplicial opposition relation each vertex has with any possible vertex

As we remarked, two over the nine possible tri-simplicial "colors" are in fact absent here: the blue and the green. This reduces the Aristotelian 3^2-lattice of the tri-*segments* (Fig. 39).

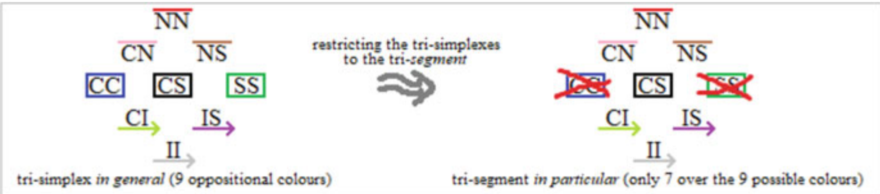


Fig. 39 The oppositional tri-*segment* has only seven of the nine possible colors of the oppositional tri-*simplexes*

But this absence (of two over the nine oppositional colors: blue and green) is normal: poly-*segments* are based on segments (and not on triangle, tetrahedron, five-cell or higher, cf. end of Sect. 1.6, Fig. 33); as already known by wordy reasonings by Aristotle and elucidated by Angot-Pellissier [1], contrariety and subcontrariety need triangles to emerge. So tri-segments are a very particular (and primitive) case of the tri-simplexes (the blue and the green colors will appear as soon as triangles do intervene, that is, in tri-triangles – as Angot-Pellissier's 2013 study [3] has in fact precisely confirmed). Remark that this establishes not only *that*, but also (in part) *why*, my model of tri-segment of 2009 (Sect. 1.6, Figs. 30, 31, and 32) was mistaken.

In our trip toward the global oppositional geometry of the tri-segment, several points come next. One is that of determining the "logical" meaning of each of the oppositional colors. Angot-Pellissier [3] has provided elements of answer to that. Recall that in the bi-simplexes the meanings of colors are clear and related to the connectives of propositional calculus (Sect. 1.1, Fig. 1), although it has taken time to understand – thanks Smessaert – two important additional things to be signaled here, namely, (1) the exact nature of the Aristotelian subalternation (to which we come in a few lines) and (2) the existence, in some sense, of one more oppositional "joker" color: orange for *the "no-relation" relation* – in some sense the structuralist null-element. Now, in the tri-simplexes what seems to be expected is – we will not discuss it here – that *three*-valued connectives (rather than two-valued) will intervene somehow in a similar way (we will try to come back to this in Sect. 5.2).

Before going on, we must recall a quite important point, Smessaert's lesson on subalternation (in some sense seemingly at the origin of his and Demey's idea of calling "*logical geometry*", cf. [49, 135], the field he and several others – including myself – are investigating since years, if not centuries). In 2009 (Sect. 1.3) I interpreted the "[1|1]" (which in some sense I created!), in the Aristotelian lattice, as meaning "logical implication" (i.e., the Aristotelian-Apuleian classical "subalternation"). I signaled that to do this a restriction of the combinatorial "yes yes" (i.e., "[1|1]") definition was necessary; otherwise, we would have had "logical equivalence" instead of "logical implication". This point remained strange and awkward (this sudden asymmetry in Aristotle and Apuleius' otherwise so elegant oppositional combinatorics). But Smessaert, later, showed, in his studies where he proposed the idea of a larger "*logical* geometry" [134, 135], that "[1|1]" is more properly to be understood as something more primitive than logical implication, namely, "noncontradiction", that is as a very general relation (of which "implication" is only a meaningful and useful *restriction*). To explain that point more deeply, he discovered that one must in fact consider a larger ask-answer game-theoretical semantics (than mine of 2009; Figs. 10 and 11), which, as it happens, generates not one but *two geometries*: "opposition geometry" (which is *more or less* "opposition*al* geometry", Sect. 1.2) and "implication geometry" (which is the new thing). And that the classical geometry of oppositions (i.e., opposition*al* geometry), called by Smessaert the new name "Aristotelian geometry", emerged, historically, as an unconscious composition of three over the four elements of "opposition geometry" (i.e., dropping precisely "noncontradiction") and one over the four elements of "implication geometry ("one-sided implication", more precisely "right-implication", taken to replace "noncontradiction") (Fig. 40).

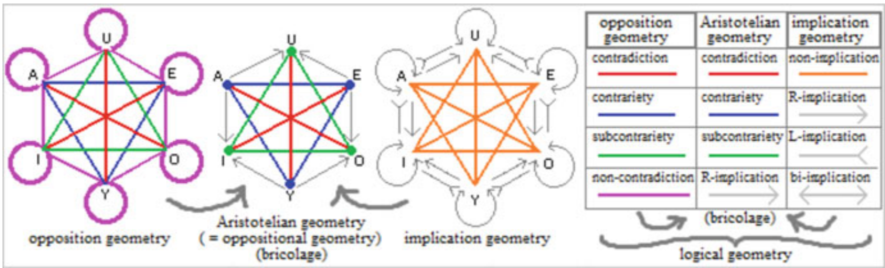


Fig. 40 Smessaert (2012): the logical hexagon (its vertices and edges) can support (at least) three different "geometries"

The discovery is important and clarifying. But, as for bi-simplexes, it does not seem to change much: one just has to be aware that in fact "I" (i.e., "[1|1]") is in fact noncontradiction and that it can be "enriched" with implication. And it justifies in no way Demey's repeated argument that high-dimensional solids (i.e., higher than 3D) (and therefore, *a fortiori*, poly-simplexes . . .) are not worth being investigated (which is a *logicist* move: logicism, among others, aims at reducing all numericity to the binary "0 and 1", so that it generally takes no pleasure in exploring mathematical depth, as, for instance, poly-simplicial depth . . .). In [100], where I solved the (rather difficult and unsolved since 1968) riddle of the nature of Greimas' problematic (and very famous) "semiotic square", I proposed to see Smessaert and Demey's two geometries as "*meta*-geometries", that is, as useful preconditions of the real thing: opposition*al* infinite complexity (infinite as for the dimensionality of the contrariety simplexes and as for the variability of the "poly" diffractions). *My point was, and up to now remains, that oppositional geometry is the real mathematical thing at stake and that it is by no means "a new chapter of logic" (and one that according to Demey and Smessaert should refrain from exploring higher dimensions!)*, as the label "logical geometry" strongly and recklessly suggests to a philosophically non-naïve (and non-cynical . . .) eye (Sect. 4.6). Again, practically this means for us that "[1|1]" needs *interpretation* (it is more abstract than just logical implication; it is the "noncontradiction" relation, logical implication being a particular case of the more general "noncontradiction" relation). Our way of dealing with it will be "Aristotelian" (in the sense of Smessaert and Demey; Fig. 40), and this will be done, as we just saw in this Sect. 2.2, relying on Angot-Pellissier's sheaf-theoretical method for the poly-simplexes (Sect. 1.4): in front of codes containing "I" (i.e., II, CI, or IS), we will see, by examining the relevant sheaf-sections, whether the simple colored line (fluo pink, light green, or violet) can be turned into a single-sided or a double-sided arrow (in fact we will come back to this important issue in Sect. 4). In the rest of this paper, we will take into account this while refusing as inappropriate and very misleading the academically fashionable label "*logical* geometry": (1) it loses the stress put on "opposition", as a mathematical concept largely independent from logic (Sect. 1.5!!!), (2) and it falls into "logicism" (which is a deadly constant in the history of the geometry of oppositions – cf. [97] and Sect. 4.6).

The first point waiting for us right now is that of finding the "most natural" geometries (in principle "tri-simplicial" instead of "bi-simplicial") of this set of seven vertices with the relations holding between any pair of them (including the curls of the reflexive pairs).

2.3 The Geometrical Problem: Having a Strange Pentadic Structure

Let us try to end our inquiry still without the help of the "Pascalian ND simplexes" (Sect. 1.5). As said (Sect. 2.1, Fig. 36), from the nine numerical sub-sheaves we eliminated the oppositional-geometrical analog of T and ⊥ (i.e., $1_X 2_X$ and $1_\varnothing 2_\varnothing$).

This leaves in our hands seven sub-sheaves, among which one (i.e., 1_U2_U), as said, seems mysterious (we saw in Sect. 2.2, Fig. 38, that the strangeness of 1_U2_U grows with its black non-arrow curl).

How can we try to find a good global geometrical expression of this reality, namely, something like *the* polygon or *the* solid (or polytope) of the tri-segment? How to display in the *n*-dimensional (2D? 3D? 4D?) oppositional-geometrical space the 7 vertices (with their 7 curls) and the 21 non-curl edges relating any pair of (nonidentical) vertices? (Fig. 41).

Fig. 41 How to display at best, in the oppositional-geometrical space, these seven non-trivial vertices of the tri-segment?

Given that in some sense the tri-segment can be seen as a transformation (tri-simplicial "oppositional *diffraction*") of the (red) segment (of contradiction), we can try to singularize this starting classical red segment $1_X2_\varnothing$—$1_\varnothing 2_X$ (of which the tri-segment seems to be a diffraction *and* a conservative extension), by putting it so to say on a "radial position" (i.e., visual-metaphorically, as the axle of a wheel). Remark that we can try to color this segment's vertices, but we do not know how to color the five others (there are no simplexes here): so we will use gray points. But then several different alternative dispositions of the five remaining sub-sheaves are possible, and, as it happens, it does not seem to be easy to find a particular configuration more convincing and "mathematically natural" than the other possible ones. Remark that since the self-relations (of any vertex to itself) happen, here, to resort not to only one (gray), as in the bi-simplicial space, but to three possible "arrow colors" (gray, light green, and violet and seemingly even the *non-arrow* black), we cannot go on keeping them implicit (as they usually are, *pace* Smessaert, in bi-simplicial oppositional "Aristotelian" geometry, where they usually are not drawn, since they are tautological): so to say following, at least partially, the suggestion of Smessaert inside his and Demey's aforementioned "logical geometry" (Sect. 2.2, Fig. 40), we will do better here by choosing to *represent them as well explicitly, as "curls"* (so, on this point our research on the tri-segment goes in the direction of logical geometry). Remark, again, that over the seven curls, six are implication arrows (two gray, two light green, and two violet), while one (the black CS of the 1_U2_U vertex ...) is not (CS contains no "I" in its code): as said, this adds more suspicion or puzzlement of us regarding the strangeness or, in the best case, the mathematical *singularity* (but then: why and how?), of the 1_U2_U nontrivial sub-sheaf and vertex (Sect. 2.1) (Fig. 42).

Trying to clarify this complex oppositional-geometrical (and chromatic) riddle, we can try to decompose it (*divide et impera*), hoping to lower its complexity, by

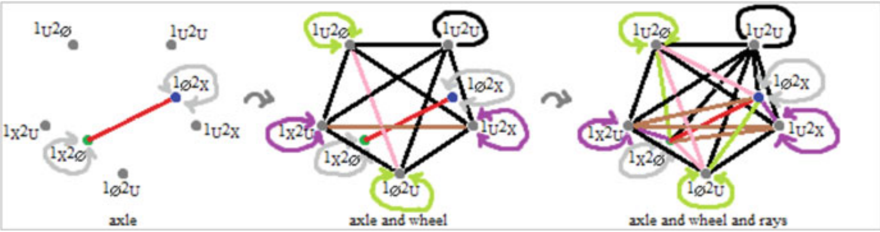


Fig. 42 Trying to represent the tri-segment like "axle and wheel", putting into light the red segment of contradiction

detaching so to say the pentadic circular "tire" from its radial "axle" (the latter can be cut in two halves) (Fig. 43).

Fig. 43 Trying to decompose into parts the oppositional-*geometrical* problem of finding *the* tri-segment

Remark that this representation, and more fundamentally this structure, whatever its tentative representation, might in some sense already seem exciting *per se*, for in some sense it seems to lead us to forms or patterns rather unexpected with respect to the usual standards of oppositional geometry (mainly by having here something *pentadic*, and so to say numerically seven-based, in an *a priori* non-heptadic and non-pentadic context of mathematical analysis: the tri(3)-segment(2)). This is mathematically surprising with respect to what known so far about bi-simplicial (i.e., two-valued) oppositional geometry, which tends to be reasonably simple and symmetric. In other words, we might be tempted to *accept* as (unexpectedly) "typical" of the still mysterious and maybe durably exotic universe of the "tri-segments" this intractable pentadic flavor, if one wants to represent the classical (red) contradiction segment as the starting term of a tri-simplicial "oppositional diffraction". But truly speaking, this structure in some sense remains hard to interpret, at this stage (the "logic – *cum grano salis*! – of tri-simplicial diffraction" is not at all clear here), and this can be seen as being no specifically good sign; notably we can remain puzzled about the particular status of the still mysterious 1_U2_U term: for it clearly seems to introduce a strong geometrical (and chromatic) disequilibrium that, supposing it is justified and meaningful – but then why? –

we do not yet understand by resorting to Angot-Pellissier’s sheafing technique. In other words, the so far reached structure seems to lack badly *symmetry*, and this seems to be notably influenced by the presence of a large number of black segments (which, again, converge on the still mysterious 1_U2_U sub-sheaf, Sect. 2.2, Fig. 38). Of course, we can try to invert geometrical representation priorities (in the hope of finding unseen, better oppositional-geometrical arrangements): for instance, by putting tentatively the red contradiction segment in positions other than axial (but then: why? and how?), so to let 1_U2_U take such a geometrically and chromatically (for short: oppositionally) prominent place. But no clear better alternative (to the pentadic “wheel and axle”) seems, even at that price, to emerge. On that respect we might say that apart when considering “seven” to be a hexagon with an extra inner point, or as a blue seven-contrariety simplex (or 7-opposition), seven seems, fundamentally, to be (at least as long as we know) no easy number to let emerge “from itself” a geometrical regular configuration provided with the kind of symmetries oppositional geometry got us used to (Fig. 44).

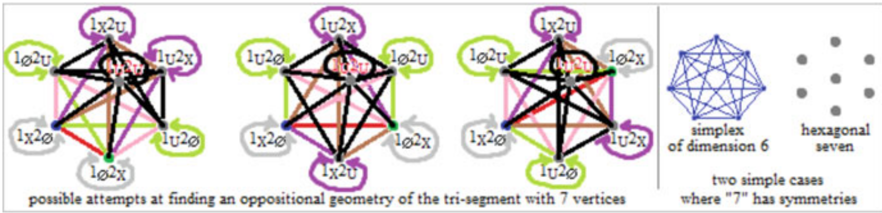


Fig. 44 The geometrical strangeness of the number seven: hard to unfold through spatial symmetries

So let us try to see, instead, in the next Sect. 2.4, whether applying to this oppositional-geometrically still mysterious putative tri-segment, so far a nut hard to crack, a totally different strategy, or better an extra piece of the puzzle, namely, the “Pascalian lens” (Sect. 1.5), can help us in finding some more fundamental (and helpful) order, regularity, and understanding of the still hypothetical oppositional tri-segment.

2.4 *The Pascalian 3D Simplex and Its “2D Section for Tri-segments”*

As we saw (Sect. 1.5, Fig. 24), Pascal’s triangle matches perfectly the oppositional bi-simplexes (and most importantly, their closures, the B*n*, Sect. 1.2, Fig. 7), and we claimed to have successfully defined (with a general proof that we will give elsewhere) a generalization of Pascal’s triangle to be called the “Pascalian ND *simplex*”, such that it matches, case by case, the oppositional *poly*-simplexes (Sect.

1.5). So, in order to see whether we can find a way of solving the not so easy puzzle of reaching (if possible) a harmonious oppositional-geometrical structure of the tri-segment (Sect. 2.3), let us now turn back to the Pascalian *3D* simplex (i.e., a 3D tetrahedron) for the *tri*-simplexes (Sect. 1.5, Fig. 27). Among its "2D horizontal sections" (i.e., the horizontal "arithmetical triangles" this arithmetical tetrahedron is made of, when sliced horizontally) for all the tri-simplexes, the one relative to the *segments*, which gives us therefore the oppositional numericity relative to the tri-*segments*, is the third, starting from the top (Fig. 45).

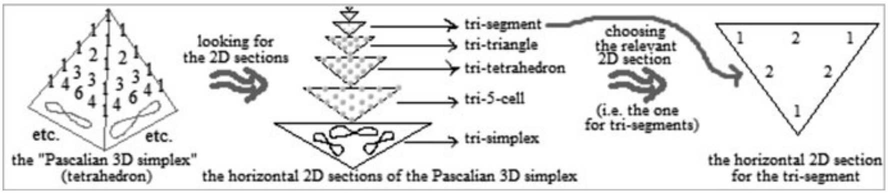


Fig. 45 The Pascalian 3D simplex (tetrahedron) for the tri-simplexes and its "2D section" (triangle) for tri-segments

How to use this "Pascalian horizontal 2D section" in order to reinterpret the useful, but so far puzzling, combinatorics of Angot-Pellissier? The idea is rather simple (and seemingly natural): one has to *distribute, as useful symbols on a roadmap*, the nine possible vertices determined through Angot-Pellissier's combinatory method, as being all the sub-sheaves of the relevant total numerical sheaf (Sect. 2.1, Fig. 35), in what we might call "Pascalian places", which happen to be qualitatively different. In this case (i.e., the tri-segment), which is rather simple (Sect. 1.5, Fig. 29), such "Pascalian" distinction essentially only runs between the three vertices (each bearing "1") on the one side and the three intermediate points (each bearing "2") on the other side. By construction, on one hand, the three Pascalian "1" correspond, respectively, to the Angot-Pellissierian sub-sheaves "$1_{\varnothing}2_{\varnothing}$", "$1_U2_U$" and "$1_X2_X$". But then, *this is big news: for, so to say, monsieur Pascal himself suggests us here nothing less than to consider the up to now problematic and mysterious numerical sub-sheaf "1_U2_U" as being an "extremum" that is something to be eliminated by (oppositional) construction together with the already known extrema "$1_{\varnothing}2_{\varnothing}$" and "$1_X2_X$"* (Sect. 2.1, Fig. 36). By construction, still, on the other hand, the three Pascalian "2" (on the Pascalian horizontal section, Fig. 45) correspond to pairs of the remaining six nontrivial sub-sheaves (the nontrivial oppositional-geometrical vertices of the tri-segment): more precisely, these three Pascalian "2" correspond, respectively, to "$1_X2_{\varnothing}$" and "$1_{\varnothing}2_X$" (on the horizontal upper side of the Pascalian section), "$1_U2_{\varnothing}$" and "$1_{\varnothing}2_U$" (on its oblique left side), and "1_X2_U" and "1_U2_X" (on its oblique right side) (Fig. 46).

Summing up, the Pascalian section helps us in solving neatly the puzzle of the term "1_U2_U": by suggesting us quite clearly to see it as being (mathematically) "on a same plane as the sub-sheaves $1_{\varnothing}2_{\varnothing}$ and 1_X2_X" and therefore to eliminate it (i.e., by

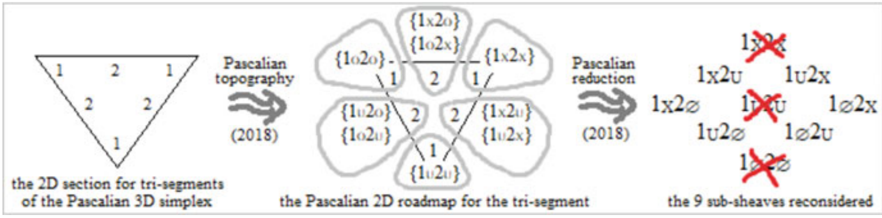


Fig. 46 The “Pascalian 2D section” as a roadmap from the nine sub-sheaves to the six vertices of the tri-segment

making “$1_U 2_U$” implode in a way similar to the previous – bi-simplicial – implosion of the sub-sheaves “$1_\varnothing 2_\varnothing$” and “$1_X 2_X$”). The result, in fact, is that not only two but in fact three over the nine sub-sheaves of the starting Angot-Pellissierian total numerical sheaf “$1_X 2_X$” are “extrema” and must so to say be eliminated by (oppositional) construction. There remain, consequently, six nontrivial numerical sub-sheaves. Furthermore, thanks to the roadmap embodied by the Pascalian 2D horizontal section (the horizontal triangle, Figs. 45 and 46) for segments, we see that these remaining six terms distribute themselves in three precise “places”, of two terms each, among which two (“$1_X 2_\varnothing$” and “$1_\varnothing 2_X$”) are the classic ones of bi-simplicial oppositional geometry.

But before going back, in Sect. 2.6 (and then in the next Sect. 3), to the puzzle of a global oppositional-geometrical representation of the tri-segment, let us have a quick look at the order-theoretical question possibly raised by our present “Pascalian” proposal of having not two but *three extrema* (for in some sense, order-theory deals exactly with the question of extrema, but in rigorous and systematic mathematical terms). Recall that the already mentioned order-theory (Sect. 1.2, Fig. 7) is a rich and important region of general mathematics, dealing with the most general order structures and lattice structures [48]. How can we posit ourselves safely in it relatively to our daring idea here of having not two but three extrema? Provided I am – alas – no expert on the field, if we try to think at least intuitively what can mean to have not two but three extrema, a *tentative* figuration which seems possible and hopefully helpful is maybe the following, where essentially we represent the Pascalian “2” as two “white spheres” (the three-colored spheres represent the three Pascalian extrema: “$1_X 2_X$ being green”, “$1_U 2_U$ being black”, and “$1_\varnothing 2_\varnothing$ being blue”) (Fig. 47).

The intuition we propose to follow with this 3D order-theoretical *tentative* model seemingly can be decomposed, more classically, in terms of three 2D order-theoretical models, suggesting that in some sense this fundamental “tri-polarity” remains submitted to a *binary* transitive order (T > Y > ⊥) (Fig. 48).

In our eyes, there seem to be at least three possibilities: (1) either there is transitivity, so that the triangular Pascalian shape (of the section) might seem illusory (in its claim of a strong ternary symmetry of what it depicts), (2) or somehow there is not transitivity, so that the triangular shape can hold on; (3) or

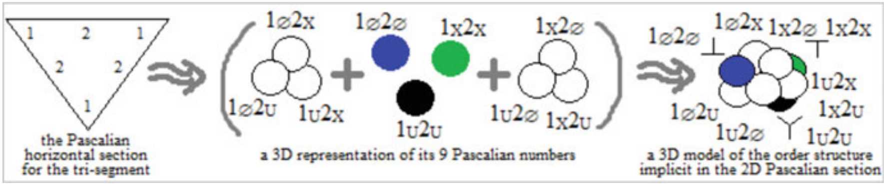


Fig. 47 A tentative 3D order-theoretical qualitative view of the Pascalian 2D section for the tri-segment

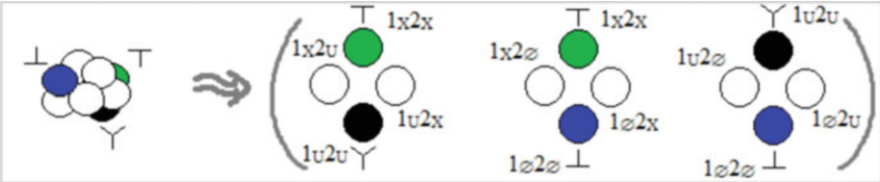


Fig. 48 Decomposing a *tentative* 3D order structure (for the tri-segment) into three more classical 2D components

(most probably), even if there is in some sense a binary order over (or embedded in) this (*per se*) ternary structure (which is the case as for truth-values as we take them here: 1 > ½ > 0), its ternarity as for its polarities "holds on". This question seems to be some order-theoretical counterpart of the question about the very idea of "oppositional tri-simplex" (in its intended innovating radicality). In our view, it could echo a famous point raised in 1975 by R. Suzsko [138] against the very idea of "many-valued logic" and the reactions (as, to mention one, [131]) this originated (we recalled and tried to discuss this in [96]). But in this paper we are not yet able to say more on it.

Let us now turn, in what follows, to our lasting problem and goal of finding *the* oppositional geometry of the tri-segment, but starting by a (last) preliminary important question on the possible structure of *oppositional colors of the six vertices* of the tri-segment.

2.5 A Point About "Points": The Oppositional Colors of the Six Vertices

Notwithstanding the undeniable material difficulty involved by this at the level of future black and white *printed* papers on the subject, and not forgetting the potential *real* pain and discrimination I thus will – alas – increase among color-blind readers (there are), I keep thinking, as the years pass, that *sua juxta principia* the *vertices* of the oppositional-geometrical structures do gain, as much as the *edges* clearly do, in getting colored: this gives to them some extra expressive power, which enhances oppositional-geometrical creative thinking. But how to proceed, thus getting more graph-theoretical (because of colors)? In a nutshell, the situation is currently the following: inside bi-simplicial oppositional geometry, coloring the vertices seems pretty fine, except for the 2-oppositional contradiction segment!

Let us recall more closely this problem. With the exception of the red segment of 2-oppositional contradiction (i.e., B2), the red segments of contradiction more generally – i.e. when they are part of an *n*-opposition strictly bigger than a 2-opposition – are by no means a problem as for coloring their two vertices. For short, (1) in the B3, each of the three red segments of contradiction it contains has one blue and one green vertex; (2) in the A4, we have the same good-functioning behavior for the four red segments of contradiction it contains; and (3) in the closure of the A4, namely, the B4, we have the A4, plus its “cloud”, made of six extra vertices, two by two centrally symmetric, such that they thus let emerge three extra red segments of contradiction (so B4 has 4 + 3 = 7): but this time each of these three extra red segments of contradiction will have at each of its two extremities a *blue-green* vertex. Put more abstractly, in the case of bi-simplicial (i.e., classical, two-valued) oppositional geometry the main use, never properly theorized, in coloring the vertices went, so far, as follows: (1) in the “oppositional kernel” (of an *n*-oppositional closure), the vertices of any simplex of contrariety are *blue* dots (related, two by two, by a blue line meaning the contrariety relation between them), and the vertices of the correlated simplex of subcontrariety are “oppositionally-dually” *green* dots (related, two by two, by a green line meaning the subcontrariety relation between them); (2) in the remaining, non-kernel part of the oppositional closure itself (and *de facto* this most of the time, so far, concerns the B4), the vertices are represented as *blue-green* dots (the mutual proportion of blue and green, in principle, can vary according to the typology of the cloud, as we study in [94], but as this enters seldom into play (so far), this point has not yet been afforded too systematically) (Fig. 49).

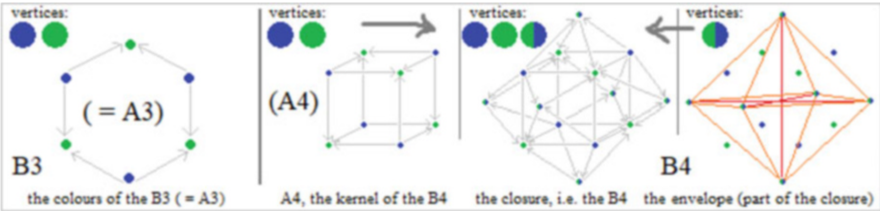


Fig. 49 By coloring the vertices one can distinguish “kernel” and “cloud” (or “envelope”) of a bi-simplicial B4

This color of its two vertices, importantly, tells something about the structure where this red segment of contradiction intervenes, that is, respectively, either (i) in the kernel (i.e., an A*n*-structure) or (ii) in the cloud of a kernel (the union of these two constituting a B*n*-structure).

Now, as said, the only problem is that, when facing the question of depicting the two vertices of *a red segment of 2-opposition* (and not the two vertices of a red segment which is *a fragment of a larger* n-*oppositional solid*), two options seem possible by analogy with what precedes: representing its two vertices either as: (i) one blue and the other green (by analogy with the oppositional kernels) or (ii) depicting them as both blue-green (by analogy with the clouds). The pros and cons of this choice, so far, are not too clear. But in my own researches so far, I tended to adopt resolutely the first option. But, as we are going to show, the study of the tri-simplexes (and higher) reveals, retrospectively, that *I made the bad choice*.

The Pascalian 2D simplex (Sect. 1.5, Fig. 24) can be read as strongly suggesting that, in its third line (top-down), corresponding to bi-simplicial – *cum grano salis*! – 2-opposition, in fact *the "2" does belong to the cloud!* In other words, the 2-oppositional contradiction segment, being pre-simplicial (because in oppositional geometry *contrariety* simplexes start with triangles, cf. Sect. 2.2, Fig. 39), "lives *inside* the classical blue-green cloud" (!). More precisely, the Pascalian 2D simplex suggests, for the bi-simplexes, that their "2" belongs, in fact, not to one of the two "simplicial diagonal lines" (either the blue on the left or the green on the right or maybe even to both . . .) but to the "cloud zone". More precisely, *"2" belongs to the vertical central line of the "pivotal elements of the clouds" of the n-oppositional closures* (with *n* an even number): 2, 6, 20, 70, 256 . . . (Fig. 50).

Fig. 50 Seeing, in the 2D Pascalian simplex, the 2-oppositional contradiction as an instance of bi-simplicial "cloud"

And this changes quite much as for our questioning about 2-oppositional contradictory vertices: it means, as for these vertices, that those of any B2 are, respectively, not the one "blue" and the other "green" but the one "blue-green" and the other "green-blue" (Fig. 51).

And in fact, one can and must read the same way also the more general (triangular) 2D section of the Pascalian 3D simplex (i.e., tetrahedron) for the tri-segment (Figs. 45 and 46). This means that the "Pascalian colors" of the tri-segment are in turn simple enough and must be represented in terms of (i) three pure colors (blue, black, green) for its three extrema ("$1_{\varnothing}2_{\varnothing}$", "$1_U 2_U$", "$1_X 2_X$") and (ii) the three mixed colors for the three pairs of vertices in between any pair of them: one Pascalian "2" is blue-green, another is blue-black, the third is black-green (for coloring newly the vertices of the simplexes, cf. Sect. 5.1, Fig. 125) (Fig. 52).

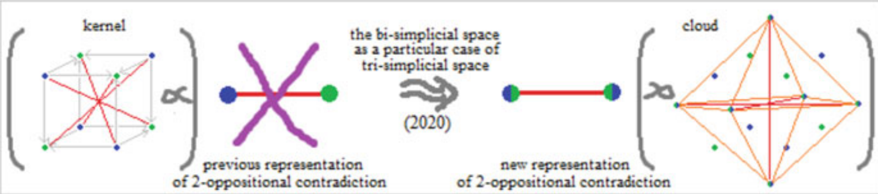


Fig. 51 Our study tells to change the way of representing the vertices of a 2-oppositional contradiction segment

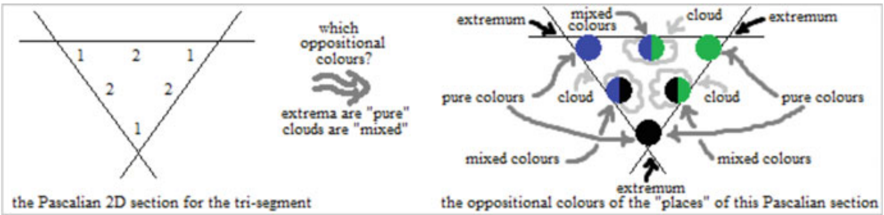


Fig. 52 The oppositional colors of the "Pascalian 2D section" for the tri-segment

So, not only the classical blue-green 2-oppositional red contradiction but each of the three "2" positions of the Pascalian 2D roadmap is in fact, in the same way, a pre-simplicial two-elements *cloud* (the simplexes, if any – and they appear for real in the tri-*triangle* – would be, for each of the three sides of the Pascalian 2D section, somewhere in between the extrema "1" and the cloud "2" positions), and *the points living in each of these three "2" cloud positions must therefore be seen as 2-oppositional negations and labeled, from the viewpoint of the oppositional colors of the extrema they live (as clouds) in between, by mixed-points* (i.e., *blue-green, green-black, and black-blue)*. But if the three extrema are (as they must) identified with the three oppositional "simplicial colors" of the tri-simplexes, then it seems natural to conceive *each* of the six remaining *numerical sub-sheaves* (Sect. 2.4, Fig. 46) as composed (in its two vertex-indices) of two colors: those of its two indices! Since each pair of sub-sheaves in each Pascalian "2" represents a tri-simplicial negation (including, as we said, the classical one), this also suggests us how these diagonal negations will look like with respect, so to say, to the general spatial architecture (Sect. 2.6) of the tri-segment (Fig. 53).

To sum up, this regular and understandable combinatorial behavior suggests, very strongly, a very interesting simple but powerful relation of this *Pascalian* "oppositional chromaticity" with the *Angot-Pellissierian* numerical sub-sheaves (i.e., the six vertices, Sect. 2.4, Fig. 46) corresponding to the Pascalian view. The general rule for tri-segments (generalizable, with extensions for the chromatic expression of the simplexes, Sect. 5.1, Fig. 125, to any poly-simplex), as carried usefully by the *indices* of the Angot-Pellissierian numerical sub-sheaves, seems to be straightforward: each vertex "1_J2_K" (with J, K $\in$ {∅, U, X}) will have, representing it faithfully, a dot of "type", so to say, "J-K" (as for its colors), and with

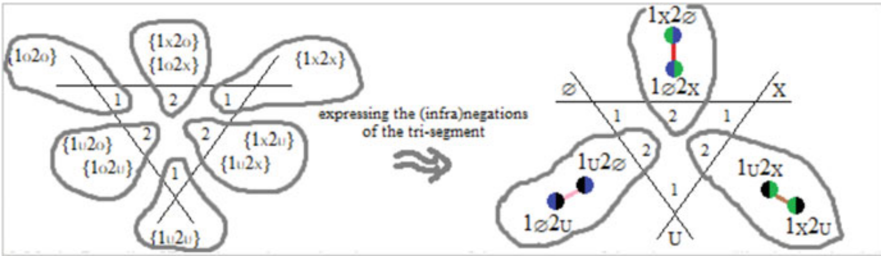


Fig. 53 The Pascalian 2D roadmap shows that the segments of the *negations* of the tri-segment "live in the clouds"

blue staying for ∅, black staying for U, and green staying for X. Here as well should become clearer the not negligible usefulness of our proposal of an "extended indicial notation" (with respect to Angot-Pellissier's original one of [3]) for his oppositional numerical sub-sheaves, i.e., the fact of systematically writing, for instance, "$1_X2_\varnothing$" instead of "1_X" or "$1_\varnothing 2_U$" instead of "2_U" (Sect. 1.5). So, the final "chromatic" reading of the Pascalian roadmap for tri-segments (but this is nicely generalizable, as we will show in other ongoing draft studies, to higher poly-simplexes) as for vertices seems to be the following (Fig. 54):

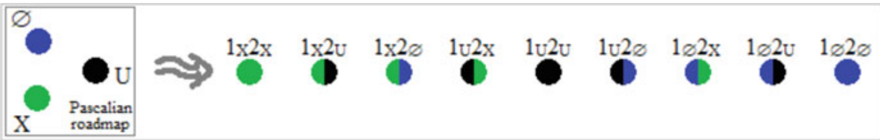


Fig. 54 Angot-Pellissier's numerical sub-sheaves (vertices) command by their full indices their "chromaticity"

With this last new tool (a theory of the oppositional chromaticity of poly-simplicial points), we can now at last come back to the main question left open at the end of Sect. 2.3, namely, the problem of finding a convincing oppositional *geometry* of the tri-segment.

2.6 Back to the Geometrical Quest of the Oppositional Structure

After having dealt with the problem of coloring the "points" (i.e., the vertices, Sect. 2.5), let us now turn to the oppositional geometry of the "lines" (the edges, i.e., the – oppositional – relations between the six nontrivial sub-sheaves of the tri-segment, Sect. 2.2). Let us come back now, with better "weapons", to the problem of expressing the global geometry of the tri-segment (Sect. 2.3). We know now that 1_U2_U is, as 1_X2_X and $1_\varnothing 2_\varnothing$, an extremum and as such must be put away (Sect. 2.4, Fig. 46). This makes at once disappear as well the seven black segments which linked this vertex to any of the possible seven (including itself, with a curved reflexive “curl segment” which was strange, being the only non-arrow curl). So we have now to arrange oppositionally-geometrically not seven but six oppositional sub-sheaves (i.e., vertices), taking into account (in order to let emerge a “geometry”) all the segments relying pairs of them (including, as said, reflexive pairs, which by construction yield curl arrows) which now are not $C^2{}_7 = 28$, but $C^2{}_6 = 6!/((6 - 2)!\,2!) = 21$. As a start, we know the relation between the two classical (i.e., bi-simplicial) vertices $1_X2_\varnothing$ e $1_\varnothing 2_X$ and we know (Sect. 2.5) that each of these vertices is |green-blue|, i.e., the first is green-blue, while the second is blue-green (Fig. 55).

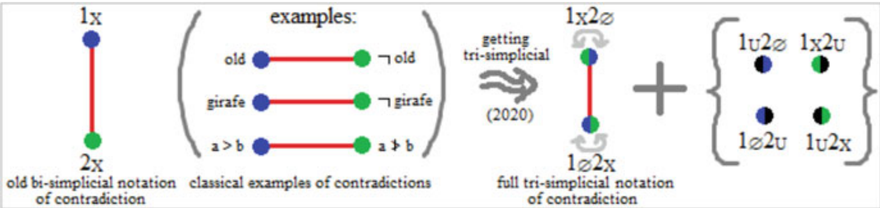


Fig. 55 The bi-simplicial starting point: the (red) segment of “ contradiction” (i.e., “classical propositional negation”)

How to posit the remaining four vertices (numerical sub-sheaves), of which we now know the oppositional color (Sect. 2.5), with respect to this starting pair? Let us rely, for a start, on intuitive *visual* symmetries of the symbols, namely, on those relating, for instance, 1_X2_U and 1_U2_X, and, in a similar way, $1_\varnothing 2_U$ and $1_U2_\varnothing$ (i.e., symmetries relative to the *indexes* of the numerical sub-sheaves): this is interesting since all the pairs of vertices with symmetric indexes (including $1_X2_\varnothing$ and $1_\varnothing 2_X$) happen to be related by kinds of negations (Sect. 2.2), so putting this into geometrical evidence (by construction), by *imposing central symmetry to these three pairs of indicially symmetric points, would keep something of the classical bi-simplicial interpretation of central symmetry as contradiction (we have here a conservative extension of bi-simplicial central symmetry)* (Fig. 56).

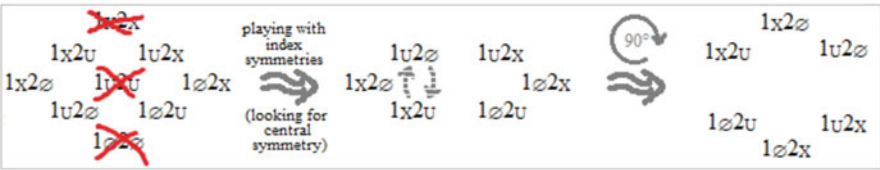


Fig. 56 Rearranging the six nontrivial sub-sheaves of the tri-segment, looking for its oppositional geometry

So let us try to put these four nonclassical vertices "around" the classical red segment (two on the left, two on the right of it); then let us draw progressively all the relevant colored segments (such as Angot-Pellissier's method of [3] has allowed us determining, cf. Sect. 2.2, Fig. 38). Thus doing, we witness at the end of the process, as one of the possible representations of the tri-segment, the emergence of a new kind of *hexagon* (Fig. 57).

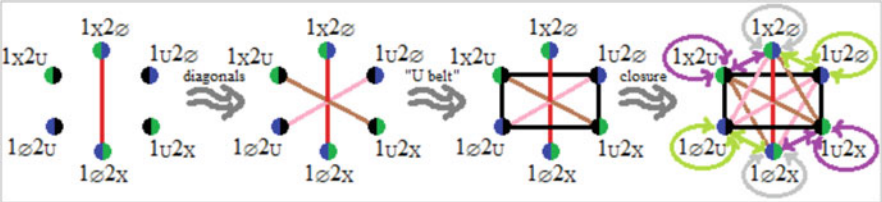


Fig. 57 Trying to let emerge, from its six vertices, a good oppositional geometry of the oppositional tri-segment

Now, there seem to be several good points with this possible representation of the tri-segment: (1) it highlights (by construction) the central position of the classical contradiction (red) segment; and (2) it imitates the bi-simplicial oppositional hexagon (i.e., the bi-triangle, Sect. 1.1, Fig. 2) (i) by putting (by construction with respect to the interpretation of central symmetry) as diagonals its "negations" (i.e., the red contradiction and the pink and the brown "infra"-contradictions) and (ii) by putting as hexagonal perimeter its "implications" (i.e., the light green and the violet "infra"-subalternations). Notice also that the final (oppositional) geometrical result toward which we provisorily tend can be seen both as a 2D and as a 3D oppositional figure (Sect. 3.3), namely, as a hexagon or as an octahedron (this was already the case with the classical bi-triangle, notably in Smessaert), and this 3D representation stresses a bit more an interesting feature of the previous hexagon: the presence of some kind of "horizontal (black) belt" (Fig. 58).

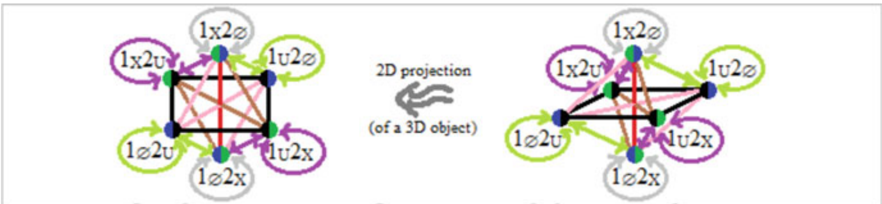


Fig. 58 The hexagon representation of the tri-segment seen as a 2D projection of an equivalent 3D octahedron

As with any 2D hexagon or 3D octahedron, one can read rather easily inside of it three (oppositionally) interesting substructures: respectively, three rectangles or (equivalently) three squares (in the hexagonal representation the components of the tri-segment are rectangles, whereas in the octahedral representation they are, equivalently, squares) (Fig. 59).

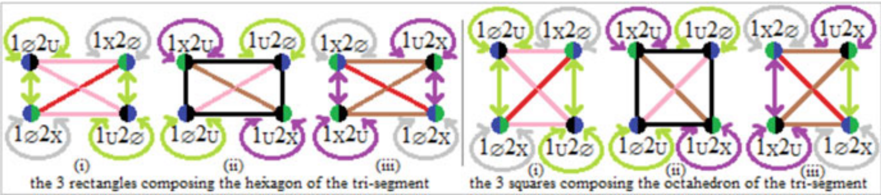


Fig. 59 Two different but equivalent views on some components of the oppositional geometry of the tri-segment

As said, our choice – among several possible other ones (Sect. 3.3) – of this particular presentation stresses the "logical square"-like expression, as diagonals, of three "negation segments" (red, pink, and brown) and the expression, as lateral vertical edges, of at least two "implication arrows" (respectively, light green and violet – in fact here biconditionals).

However, on one point at least one must beware: in this representation, differently with respect to the usual one for the bi-simplexes in oppositional geometry, central symmetry is not uniquely meaning "contradiction". And as it happens, the starting intuition of the poly-simplexes (emerged around 2006 and then presented in my PhD in 2009, [94]) consisted precisely in admitting the idea of having, in the tri-simplexes, several (in fact three) distinct symmetry centers (i.e., one for each of the three bi-simplexes composing a tri-simplex). Notice also that the comparison (which we afford in other papers) with other tri-simplexes, namely, the tri-triangle and the tri-tetrahedron, suggests that another "regular" geometrical representation of the tri-segment could be rather useful, namely, the one in terms of a 3D "trihedron" (Sect. 3.3, Fig. 71). In any case (be it hexagon, octahedron, or trihedron, Sect. 3.3), this result seems much more "regular" (and promising) than the one of Sect. 1.6, Fig. 30 (2009), on one side, and then the one of Sect. 2.3, Figs. 42 or 44, on the other side, for (1) it infirms, by correcting it strongly, my unfortunate tentative "trihedral" model of 2009 on that (Sect. 1.6); and (2) it avoids the seemingly intractable strangeness and the chromatic (and geometric) disharmony of the pentagonal-heptagonal seven-vertices model of Sect. 2.3.

Having reached, at the end of this Sect. 2, our main target (i.e., a *basic* but reliable oppositional *geometry* of the tri-segment), in the next Sect. 3, we will try to go deeper into detail with respect to the structure of the tri-segment, so to be able, notably, to start using it a little bit concretely (in Sect. 5).

3 More on the *Inner Geometry* of the Oppositional Tri-segment

At the end of the previous chapter, we reached an important point: a first, convincing approximation of the global oppositional-geometrical structure of the tri-segment, in terms of a hexagon, which is elegant and promising. But at the moment the possible functioning of this structure is not yet fully clear. Therefore, among other possible consequences, it is not yet clear which of its "parts" can be more meaningful and which ones should be seen as less interesting. As a consequence, praising complexity, we propose, in a partly *experimental* way (experimental mathematics, as defended by Mandelbrot [88]), to go in this chapter through the tri-segment's "inner jungle" in order to try to lay some possible milestones of its study, hopefully useful in the future. For the notion of "hybrid" oppositional structures and for that of "inner jungle", cf. our [101], important seminal elements of this are in Angot-Pellissier's [111], where he discovered and explained, inside the B4, what he theorized as being four equivalent instances of (previously unseen) "*weak* hexagons" in addition to the classical six instances of "strong" hexagons (until then simply known as "logical hexagons"). Our exploration will be gradual: from proximal (horizon), through fluent (circuits), and finally to global (representation optimality, inner jungle, semantic roles, and global valuation patterns).

3.1 There Are Three Possible Vertex Horizons Inside the Tri-segment

One interesting starting question is that of the possible "destinies" of any of the six terms inside the tri-segment, imagining that we "walk" from any of them toward the others: seeing things with a particular vertex's eyes (this was already in Sect. 2.2, Fig. 38). As it happens (this will be shown very soon by the emerging patterns), there seem to be exactly three pairs of such possible "proximal horizons". Let us see them. The first concerns the "classical pair" of vertices (i.e., the two extremities of the red segment): $1_X2_\varnothing$ and $1_\varnothing2_X$ (i.e., two terms centrally symmetric in the hexagonal or octahedral model of the tri-segment) (Fig. 60).

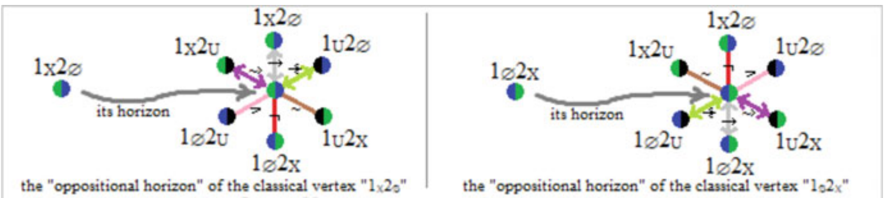


Fig. 60 Establishing the "oppositional horizon" of the classical vertices $1_X2_\varnothing$ and $1_\varnothing2_X$

The “vision” these two vertices have (i.e., their horizon) is exactly the same (i.e., the two are “chromatically congruent”), *modulo* (i.e., provided) a rotation of 180° of the two hexagonal patterns in their 2D plane (including, in this rotation of the global colored structure, even the colored vertices, but not the algebraic expression of the numerical sub-sheaves) (Fig. 61).

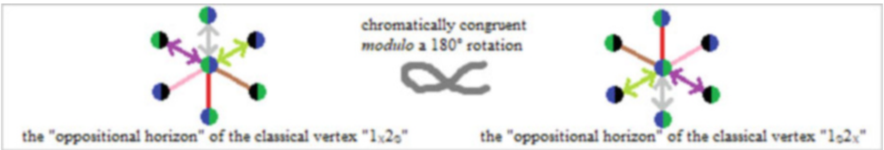


Fig. 61 These two different hexagonal patterns of “oppositional horizon” are in fact equivalent *modulo* a 180° rotation

The second possible hexagonal pattern of oppositional horizon concerns a second pair of vertices, the one expressing the Angot-Pellissierian nonclassical (and centrally symmetrical) numerical sheaves 1_U2_X and 1_X2_U. Again, their two oppositional horizons (i.e., the sets of oppositional colors they have to cross, by an oppositional segment, in order to access to any of the six vertices – i.e., including the possibility of accessing to themselves!) are in fact the same hexagonal pattern of horizon, *modulo*, again, a rotation of 180°: the two horizons are chromatically congruent (Fig. 62).

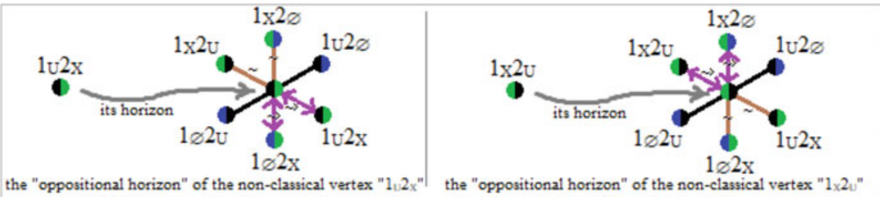


Fig. 62 The “oppositional horizon” of the vertices 1_U2_X and 1_X2_U has the same pattern, *modulo* a rotation of 180°

Finally, the third possible kind of oppositional horizon concerns the pair of the last two vertices (over the six) of the tri-segment: those expressed by the nonclassical (and centrally symmetrical) numerical sheaves $1_\varnothing 2_U$ and $1_U2_\varnothing$. Again, here also the two horizons are in fact the same, *modulo* a rotation of 180° (Fig. 63).

Remark, again, that, in the three cases (i.e., the three pairs of centrally symmetrical vertices), the 180° rotation of the pattern of the hexagonal oppositional horizon comprises also a rotation of the two-color structure of the six vertices (the only thing out of rotation are, again, the algebraic expressions of the numerical sub-sheaves of these vertices, like “1_X2_U”, etc., which do not move: their bicolored points do

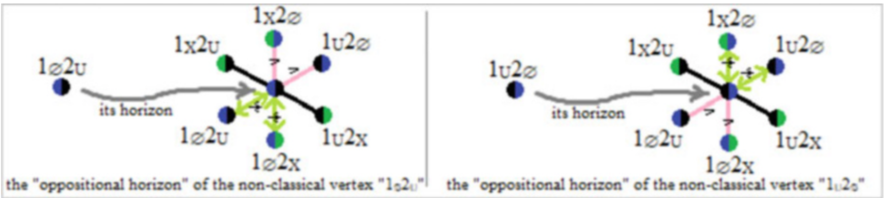


Fig. 63 The "oppositional horizon" of the vertices $1_{\emptyset}2_{U}$ and $1_{U}2_{\emptyset}$ has the same pattern, *modulo* a rotation of 180°

move). This invariance through rotation is related to symmetries in the structure (anchored, among others, in the central symmetry of the sheaves with permuted indexes, which are exactly those of the pairs, like with 1_X2_U and 1_U2_X). These isometries by 180° rotation will reveal themselves quite important later (Sect. 3.6).

Having seen here what appears when a vertex is limited to "what comes next" (and that is by definition its horizon), another internal viewpoint on the complexity of this structure is, conversely, that of the possible "flows" or "inner circuits", made of arbitrarily long concatenations of oppositional segments of similar (if not identical) nature.

3.2 Three Possible Inner Circuits of the Oppositional Tri-segment

The principal possible "flows" inside the tri-segment, meaning by that the concatenations (also by reverse iteration) of oppositional relations of same or similar "quality", happen to be of at least three kinds: the negations, the simplexes (which, in the tri-segment, are reduced to non-simplex avatars of the black simplex), and the implications (arrows). This is of course related to the Aristotelian 3^2-lattice of the tri-simplexes (generated equivalently by the Aristotelian p^q-semantics of Sect. 1.3, Fig. 11, and by Angot-Pellissier's sheaf-theoretical method for the oppositional poly-simplexes of Sect. 1.4, Fig. 18). In fact, it respects the idea that this Aristotelian lattice is made, qualitatively speaking, of three parts: the upper triangular half (kinds of contradictions, i.e., kinds of negations), the horizontal diagonal (kinds of oppositional simplexes, taken in between classical contrariety and subcontrariety), and the lower triangular half (kinds of Smessaertian noncontradictions, classically read as kinds of subalternations, i.e., kinds of implication arrows) (Fig. 64).

So, first of all, if we now focus on "contradictions" (the classical red one as the new pink and brown ones), i.e., the three oppositional relations (among the nine) on the upper part of the Aristotelian 3^2-lattice (Sect. 1.3, Fig. 11), we find that their possible concatenations in the tri-segment constitute together a "subgraph" of the tri-segment (considered not as a solid, but as a graph, [125]), which we propose to call the "contradictions circuit" of the oppositional tri-segment (in graph theory a

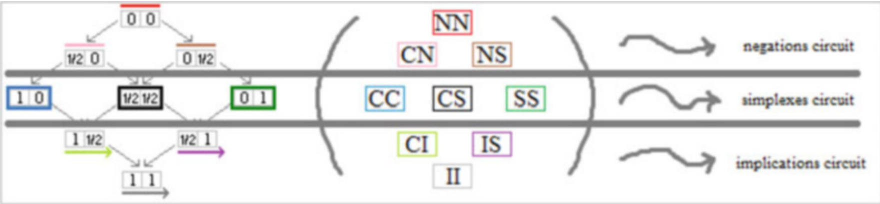


Fig. 64 The structure of the Aristotelian 3^2-lattice generates three “inner circuits” in the oppositional tri-segment

circuit is a closed line). It can be visualized equivalently in at least three different ways, if geometry is “reinjected” in this graph-theoretical structure (Fig. 65).

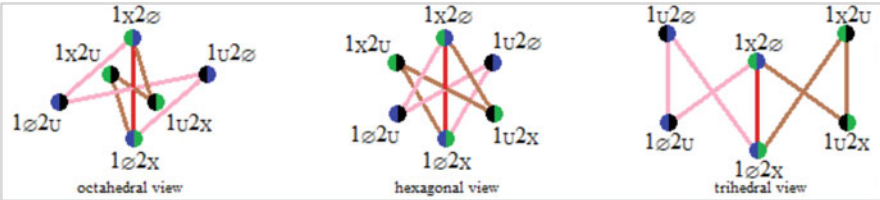


Fig. 65 Three equivalent views of the tri-chromatic “contradictions circuit” of the oppositional tri-segment

What is the use of this? Intuitively it can show, so to say, the *circulation* (hence the dynamic name “circuit” here) that “negation” (in its three different forms) can have between the six vertices of the oppositional tri-segment. As it happens, this circulation can reach all the six vertices, but not through all “passages” (i.e., segments) and not everywhere, in the circuit, by any of the three possible varieties of contradiction (classical, paracomplete, paraconsistent). The importance (if any) of this still *experimental* characterization, that will already play some role in Sect. 3.5, might (and should) become clearer in future studies of higher oppositional poly-*segments* but also, *a fortiori*, from that of any other higher *poly*-simplex. The main idea is that poly-simplexes (poly-triangles, poly-tetrahedra, etc.) grow very fast in mathematical complexity, so this kind of “index” or parameter (inner circuits) can help “navigating” conceptually (without drowning) in this otherwise *discouraging new ocean* of still mysterious shapes.

The second kind of circuit is given by *the global structure of the simplicial relations* in the oppositional tri-segment. As we saw, two of these three simplicial oppositional colors of the *general* tri-simplex (i.e., the blue and the green one) do not emerge at all in the tri-*segment* (Sect. 2.2, Fig. 39). So this “simplicial circuit” here will be not tri- but monochromatic (and more precisely black – to give an idea, in the quadri-segment it becomes bichromatic). It might therefore be called, rather, the “pivotal-simplex circuit” of the oppositional tri-segment. As previously, it can be visualized equivalently in at least three different ways (Fig. 66).

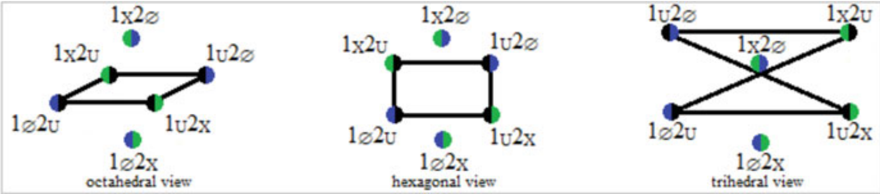


Fig. 66 Three equivalent views of the monochromatic “neo-simplicial circuit” of the oppositional tri-segment

From a graph-theoretical point of view, this subgraph, as the previous one, is a real *circuit*, for it is *closed* and not constituted of *disconnected parts*. Notice however that this time it does not reach all the six vertices of the tri-segment (it misses its two classical vertices). Its intuitive “meaning” (not yet fully clear) should become clearer in the future (in comparison with what happens in higher poly-simplexes, where its shape complexifies growingly fractalwise).

The third and last kind of inner tri-simplicial “circuit” is that of the “subalternations circuit” of the oppositional tri-segment. Remark (Sect. 2.2, Fig. 40) that Smessaert [135] has clarified that in place of subalternation (i.e., implication) one should in fact read “noncontradiction” (the latter being so to say the true “top-down *symmetrical*” *of* “contradiction” in the Aristotelian 2^2-lattice); but, as recalled, subalternation (implication) emerges on that basis as a useful and legitimate restriction of noncontradiction (by it the nondirectional “noncontradiction” relation becomes either directional or bidirectional, Sect. 2.2) and carries as such more interesting mathematical properties (the conditional or the biconditional, classical or nonclassical). So we will continue, with respect to the lower half of the 3^2-lattice, to speak of *implications* (of three different kinds) not forgetting however that these can (and must) also be seen, in some more abstract and general contexts, as three varieties of underlying Smessaertian nondirectional “noncontradiction”. As previously, this third “circuit” can be visualized equivalently in at least three different ways (Fig. 67).

What emerges here is that this three-colored graph (gray, light green, violet), as such, differently from the two previous ones, is a *disconnected* one (i.e., one made of two *separated* parts, an upper and a lower one). Moreover, each of the two disconnected parts fails to be *stricto sensu* (i.e., graph-theoretically) a “circuit”: each is a string (with three loops) with a noncoincident “head” and “tail”. So, *stricto sensu*, it is not a *circuit* (i.e., a graph-theoretical *closed* line), unless we adopt Smessaert’s discovery (related to the just aforementioned one) that the subalternations (i.e., implications) of bi-simplicial oppositional geometry (i.e., of the Aristotelian 2^2-lattice) can systematically also be read as having so to say incorporated in (or associated with) them an invisible but present “reverse

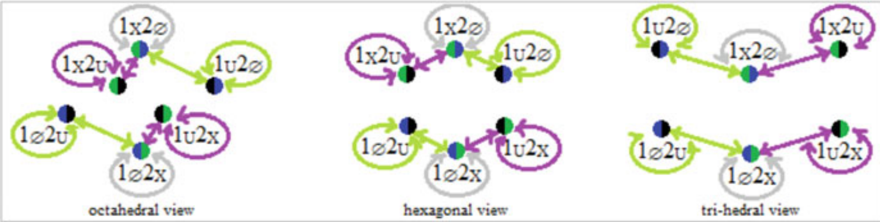


Fig. 67 Three equivalent views of the tri-chromatic “subalternations circuit” of the oppositional tri-segment

implication” (i.e., B “being implied” by A, as a reverse relation of A “implying” B, cf. Sect. 2.2, Fig. 40): for in this case (i.e., if “being implied” is also expressed graphically, near to “implying”), we can see this global graph as *two true circuits* mutually disconnected, which thing justifies (although rather trivially) keeping the term of “inner *circuit*” (Fig. 68).

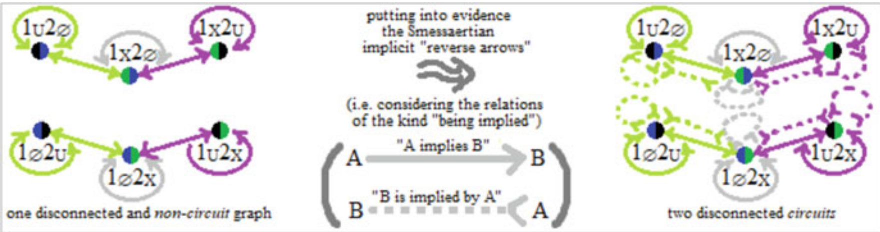


Fig. 68 The *subgraph* of subalternation (of the tri-segment) also viewed as consisting of two disconnected *circuits*

(Given that here the light green and the violet infra-implications are in fact bi-implications, i.e., they are bidirectional, the Smessaertian correlated relation will be it as well.) It will be interesting, in future studies, to examine the evolutions in pattern of this kind of circuit in other poly-simplexes (and first of all in the quadri-segment and in the tri-triangle).

Let us stress once more that these circuits, proposed by us as a new kind of hopefully meaningful oppositional-geometrical parameters of the “inner jungle” (Sect. 3.4) of the tri-segment, should become better understandable (and more clearly useful, if they will) if studied “in the long run”, i.e., considering in a row this kind of features not only in the tri-segment but also in the quadri-segment, in the quinque-segment, etc. (as it happens, some of our ongoing still unpublished draft investigations on higher poly-simplexes already seem to confirm and to establish clearly the robustness of this here only conjectured point).

Having explored some inner patterns, let us now turn back to a comparison of the main different *global* representations of the tri-segment.

3.3 Which Is "the Best" Global Representation of the Tri-segment?

Given that several representations of the tri-segment are possible (Sect. 2.6, Fig. 58), the question arises relatively to knowing whether some of them are better than others or whether they (probably) simply bear different but equivalent qualities and interests, to be alternatively privileged in different contexts of study. Roughly speaking we know (at the moment) at least three ways of expressing geometrically as a whole this structure (the tri-segment) made of six vertices. Let us summarize them.

First of all, there is a 3D representation, as we saw, by means of an octahedron (Sect. 2.6, Fig. 58). The latter can be decomposed in three, two by two 3D orthogonal, inner 2D squares (more precisions on this will be given in Sect. 3.4) (Fig. 69).

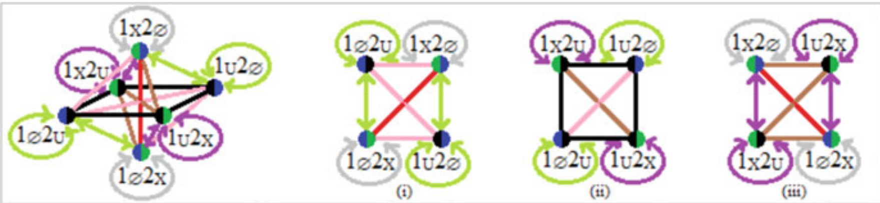


Fig. 69 The 3D expression of the oppositional tri-segment by means of an octahedron (with its three inner squares)

This octahedron-like representation helps highlighting the "oppositional diffraction" of the (here vertical) red bi-simplicial contradiction segment. This expression of the diffraction of contradiction is very symmetric: (1) with the three "negations" (red, pink, brown) in the three mutually orthogonal diagonals (which are the three 1D intersections of the three 3D orthogonal aforementioned 2D squares) (2) and with the horizontal circular "black belt" surrounding the red contradiction as a wheel with its axle. Moreover, the three 2D squares in which the octahedron can be decomposed express also by themselves some nice features, for in some sense they resemble up to a certain extent to the classical bi-simplicial logical square (Sect. 1.1, Fig. 1): in each of them their two diagonals, although different in color, are both negations, and at least four over the six vertical edges (the two light green and the two violet) are kinds of implications, but here the implications (arrows) become biconditionals (double-sided arrows, Sect. 2.2). Notice that in this octahedral representation, there is a central symmetry of the pairs of numeric subsheaves which have permutated indexes (like 1_X2_U with respect to 1_U2_X), which is a generalization that implies as a particular case the classical interpretation (as classical red contradiction) of central symmetry proper of the bi-simplexes (Sect. 2.5, Fig. 51).

Secondly, there is a 2D representation of the oppositional tri-segment, by means of a hexagon, which can also be obtained as a 2D projection of the previous (the 3D octahedron, Sect. 2.6, Fig. 58). Conversely, this hexagon can be pictured in a way (as we do here, playing knot theoretically with 3D chromatic priorities in the crossing of segments) such that it almost expresses (at least to an experimented oppositional geometer's eye) simultaneously the previous 3D octahedron. Among four possible projections of the 3D octahedron into a 2D hexagon, we choose the one which is such that the thus obtained tri-segment hexagon keeps strong enough analogies with the classical (bi-simplicial) logical hexagon (Sect. 1.1, Fig. 2) and, at the same time, it highlights the red segment of classical 2-opposition (i.e., classical contradiction) by putting it as the vertical diagonal of this hexagon. As a consequence, this "tri-segmental hexagon" can also be meaningfully decomposed into its three constitutive rectangles (as can the logical hexagon, Sect. 1.1, Fig. 2), which, as previously (with the octahedron's three squares) are partly analogous to logical squares and in fact are *fully* equivalent (i.e., have each the same four vertices and six segments) to the previous three octahedral squares (i.e., the (i), (ii), and (iii), Fig. 69) (Fig. 70).

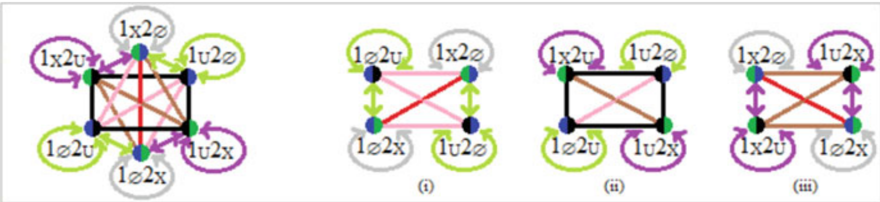


Fig. 70 The 2D expression of the oppositional tri-segment by means of a hexagon (with its three inner rectangles)

This hexagon-like representation helps (better than the 3D octahedron) in making *quick drawings* (which is very helpful to the working oppositional geometer) and as said keeps, *cum grano salis*, the good properties of the previous 3D octahedron-like representation, mainly due to the fact that it keeps the interpretation of central symmetry in terms of permutation of the indexes of the numerical sub-sheaves (e.g., in the central symmetry of $1_X 2_U$ and $1_U 2_X$). In particular, we will meet gratefully enough this helpfulness when considering (in a bunch of coming other studies) higher-order poly-*segments* (and first of all with the quadri-segment and quinque-segment): when unfolding the quadri-segment, it will be very helpful to consider the tri-segments it contains as 2D hexagons.

Thirdly, there is however still another possible 3D representation of the tri-segment, by means this time of a 3D trihedron. It can be remarked that this is the same *geometrical* shape (but not the same *oppositional*-geometrical shape!) as that of our (mistaken) 2009 tentative representation of the tri-segment (Sect. 1.6, Fig. 30) (Fig. 71).

This trihedron-like representation also provides some kind of help at a fundamental level. It loses the nice property of the central symmetry of the permutations

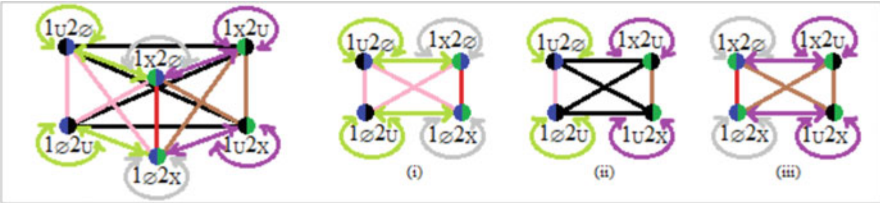


Fig. 71 A 3D expression of the oppositional tri-segment by means of a trihedron (with its three lateral rectangles)

of indexes in the numerical sub-sheaves, but, if one adopts a left-right reversal of each of the three top vertices (i.e., "1_J" and "2_K" switching in "1_J2_K" so to give the equivalent "2_K1_J" – and reversing accordingly the corresponding chromatic representation), it gains the advantage of expressing visually the deep relation existing between the tri-segment as a whole and its originating abstract horizontal 2D section of the Pascalian 3D simplex for tri-simplexes (Sect. 2.5, Fig. 53), by being now *visibly* isomorphic to it and thus giving a deeper visual intuition of the three underlying bi-simplicial "clouds" (for the notion of oppositional cloud, cf. Sects. 1.2, 2.5 and 5.1) (Fig. 72).

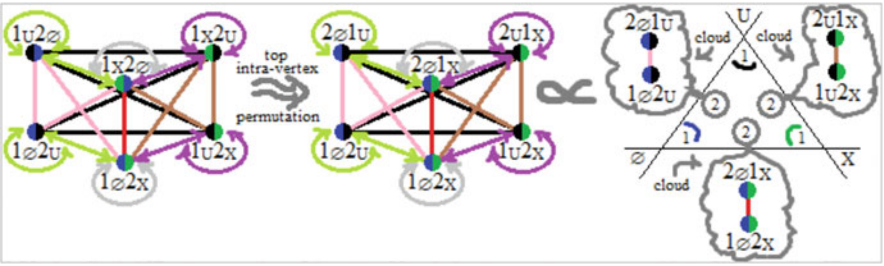


Fig. 72 The trihedral 3D representation of the oppositional tri-segment expresses something of its Pascalian 2D map

Summing up, these three possible global representations of the tri-segment seem to be fundamentally equivalent but bear different geometrical "flavors" and therefore favor different *visual intuitions*, all potentially useful. Having discussed the geometry of the possible shapes of the global structure of the tri-segment, let us now return to a deeper (but still elementary!) look on some of its possible geometrical *sub*structures.

3.4 *Inner Jungle: Possible Hybrid Substructures of the Tri-segment*

We claimed in [101] (and still claim here) that oppositional-geometrical substructures, even “hybrid” (i.e., even infra-simplicial, chromatically irregular), are interesting in the bi-simplicial oppositional geometry: we studied it first of all relatively to the complexity of what we proposed to call the “arrow-hexagons” of B4 (a new kind of generalization of the concept of logical hexagon). For instance, the useful notion of “oppositional shadow” (which appears in B4, e.g., with the counterpart, hybrid, of the non-hybrid B3 “hexagon of linear order” in the non-hybrid B4 “tetrahexahedron of partial order”, Sect. 1.2, Fig. 7) can be conceptualized and studied only once one has a rich, methodical, and exhaustive typology of such hybrid substructures (the goal being of having, as tools, chromatic “markers” for describing oppositional *transformations*). The same seemingly goes for studying “oppositional-geometrical *operations*” on the bi-simplicial space: the result of several such operations (i.e., combinations of previously unrelated oppositional structures, leading then to new ones) is characterized, as by oppositional markers, by very regular hybrid structures, which are like “chromatic signatures” of various kinds of oppositional dynamic phenomena. All these things should help in the future, in building the new study of a whole “oppositional *dynamics*” (Fig. 73).

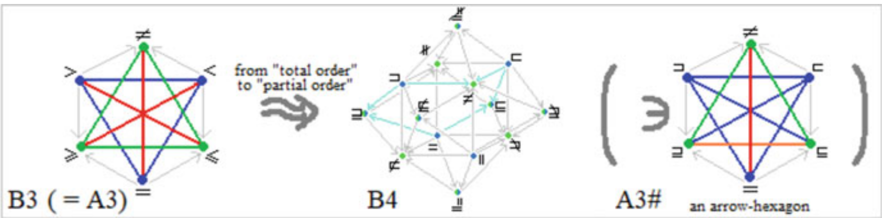


Fig. 73 Oppositional dynamics of the order relations: a B3 becomes B4, leaving a “shadow” of itself inside the B4

Now, there is no reason (other than fear of complexity) not to study this also in the oppositional-geometrical space of the poly-simplexes. And oppositional-geometrically speaking, there seem to be at least two main substructures in the “inner jungle” of the tri-segment (be it expressed as a 2D hexagon, a 3D octahedron, or a 3D trihedron, Sect. 3.3): polychrome (i.e., hybrid) *squares* and *triangles* (the study of its *segments*, even concatenated, has already been quickly evoked, Sect. 3.1 and 3.2, and will be left aside here).

Starting with the squares, we have already evoked three “regular” ones (the (i), (ii), and (iii), Sect. 2.6, Fig. 59, and Sect. 3.3, Figs. 69, 70, and 71) under two different but equivalent presentations (i.e., as contained, respectively, in the hexagon/octahedron or in the trihedron). But in the tri-segment, there are several other oppositional-geometrical squares, which are even more irregular (i.e., hybrid):

in fact, mathematically speaking any possible 4-tuple of vertices, among the six vertices of the oppositional tri-segment, can (and must) be considered an instance of such concept of hybrid square: so, combinatorially speaking, there are exactly $C^4{}_6 = 6!/(6 - 4)!4! = 15$ of them. So, where are (and how do they look like) the remaining 12 squares? We could view them on the trihedron: three are extensions of its “roof triangle” (adding a basement vertex to it), three are extensions of its “basement triangle” (adding a roof vertex to it), and the last six are obtained by combining two vertices of the roof with two vertices of the basement (avoiding the three cases where this gives back the three squares (i)–(iii)). But deriving our missing 12 hybrid squares from the trihedron would break any central symmetry of the numerical sub-sheaves with permuted indices in the so obtained expression of the squares. So it is preferable to derive them from the octahedron, which expresses central symmetries: this property is partly inherited by its parts (among which the squares). In it, a first group of four squares is visible on the upper half of its surface (Fig. 74).

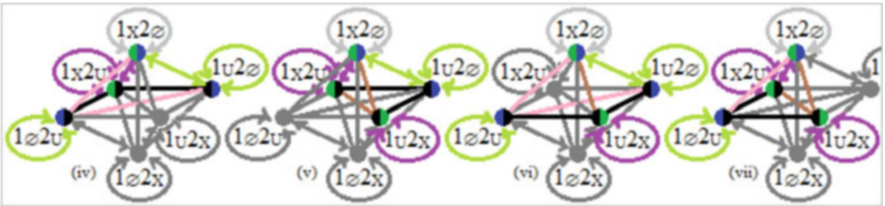


Fig. 74 Viewing 4 of the missing 12 hybrid squares on the top of the surface of the octahedral tri-segment

A second group of four squares can be seen as made of pairs of contiguous triangles such that one is on the upper half and one on the lower half of the octahedron’s surface (Fig. 75).

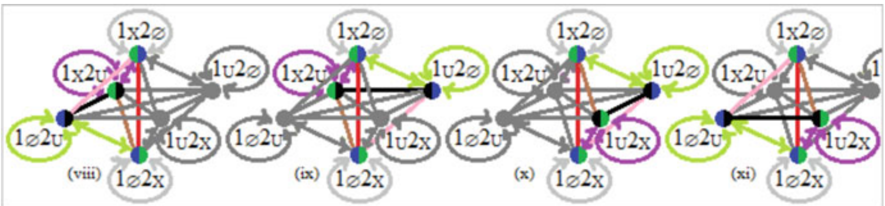


Fig. 75 Viewing 4 of the missing 12 hybrid squares on the vertical quarters of the octahedral tri-segment

And a last group of four squares can be seen on the lower half of the octahedron’s surface (Fig. 76).

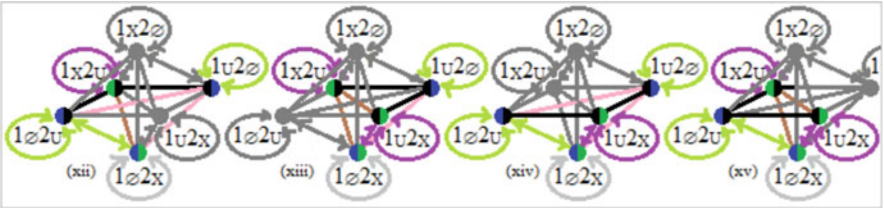


Fig. 76 Viewing 4 of the missing 12 hybrid squares on the bottom of the surface of the octahedral tri-segment

Typologically speaking, the three first squares ((i), (ii), and (iii), Sects. 2.6 and 3.3) are such that all their four vertices are two by two centrally symmetric (this regularity makes them so to say not hybrid). Differently, the 12 more hybrid squares deserve their epithet because they have only two among their four vertices which are centrally symmetric. Accordingly, these 12 can be viewed as forming 3 groups: a first group of four ((viii), (ix), (x), and (xi)) where the unique central symmetry is expressed by one red diagonal, a second group of four ((iv), (vi), (xii), and (xiv)) where the unique central symmetry is expressed by one pink diagonal, and a third group of four ((v), (vii), (xiii), and (xv)) where the unique central symmetry is expressed by one brown diagonal (it seems to be better to represent them rather as lozenges) (Fig. 77).

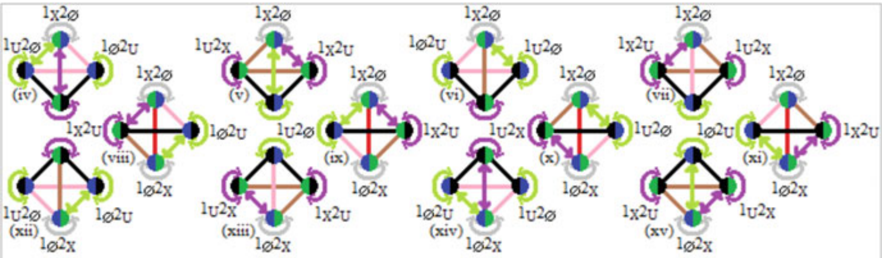


Fig. 77 The 12 broken squares (seen as lozenges) of the octahedral tri-segment

Moreover, these 12 hybrid squares are such that they divide into 6 pairs of chromatically isomorphic squares, and all such pairs of identical squares are centrally symmetric inside the octahedron (beware: their vertices undergo an *indicial* permutation). So, globally there are three different “normal” squares. And then there are six pairs of centrally symmetric hybrid squares: all in all nine different chromatic kinds of squares. These elements might be studied in the future (as “jungle”) relatively to effects of oppositional shadow of the tri-segment with respect to higher tri-simplexes containing tri-segments.

If we now go to the triangles, a similar combinatorial abstract calculation as the previous tells us that there are $C^3{}_6 = 6!/(6 - 3)!3! = 20$ of them. Again, the octahedron helps seeing straightforwardly 8 of the 20: they simply are its eight 2D faces (four on the top half and four on the bottom half of the octahedron's surface) (Fig. 78).

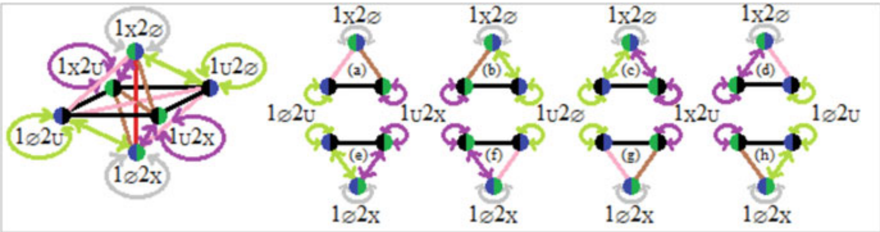


Fig. 78 Eight of the twenty chromatically hybrid (i.e., infra-simplicial) triangles of the oppositional tri-segment

Remark that these eight "surface triangles" are two by two chromatically isomorphic, and this concerns the triangles which are centrally symmetric (the central symmetry of the whole entails the mutual central symmetry of some of its parts – here as well, remember however that their respective vertices undergo an *indicial* permutation). The other 12 triangles over the 20 can be seen easily enough in the octahedron's 3 inner squares ((i), (ii), and (iii)): for there are four triangles in each of these three squares. If these three squares are seen as tetrahedra, the four triangles in each square are the four triangular faces of the equivalent tetrahedron (Fig. 79).

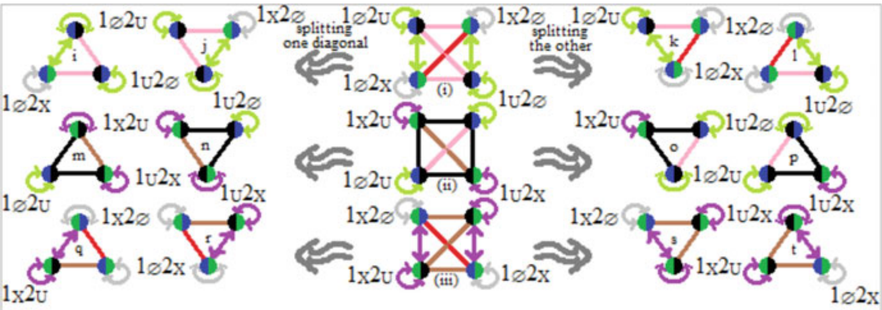


Fig. 79 Twelve chromatically hybrid (i.e., less than regular) triangles of the oppositional tri-segment

Remark that because of the full diagonal central symmetry of the global structure of each of the 3 squares (i)–(iii), these 12 triangles obtained by splitting in 2 ways each of the 3 squares (i)–(iii) let emerge only 6 different chromatic kinds of triangle (the pairs of centrally symmetric such triangles are, here as well, those chromatically isomorphic, *modulo* the reversed indices of their vertices).

As for the final typology of the 20 triangles, their chromaticity can be stressed either relatively to the edges or relatively to the curls. Eight of the 12 triangles inscribed in the 3 central squares (i)–(iii) of the octahedron (i.e., the i, j, m, n, o, p, s, t) are bichromatic with respect to edges, and 4 (i.e., the k, l, q, r) are tri-chromatic. The eight triangles of the octahedron’s surface let emerge four different chromatic kinds of triangle (a and g, b and h, c and e, and d and f; Fig. 78). So, all in all in the tri-segment there are ten different pairs of isomorphic triangles such that four kinds are bichromatic and six are trichromatic with respect to edges. As for the curls, the 8 triangles (a–h) are tri-chromatic (due to the lack of central symmetric vertices), while the 12 triangles (i–t) are bichromatic (due to the presence, in each of these triangles, of two vertices centrally symmetric) (Fig. 80).

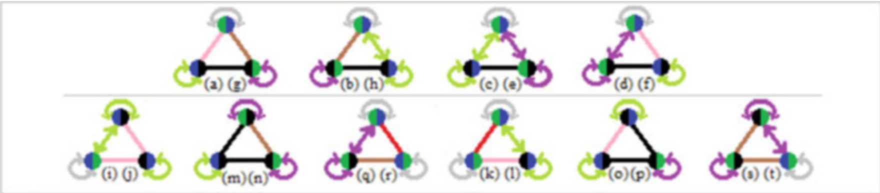


Fig. 80 General typology of the 20 inner triangles of the oppositional tri-segment: there are 10 kinds of them

We cannot say more here. But we will find some use of this already in the next Sect. 3.5: as we are going to see, at least 2 of these 30 triangles (centrally symmetric) seem to be particularly meaningful for the tri-segment taken as a whole.

3.5 *How to Put “Semantic Values” on the Vertices of the Tri-segment?*

We can face now a very important part of our global study of the concept of tri-segment: the question of *meaning*. Clarifying this is absolutely necessary in order to be able to apply this structure to something concrete whatsoever. Remark that the structural or *differential* game (in the sense launched by the structuralist linguist Saussure, cf. [112]) of the oppositional structures is in general so made that it helps *by itself* in building the meaning, by means of the very *system of the oppositions*. But here, with the tri-segment, we are handling very minimal conditions of “opposition” (we have no contrarieties! Cf. Sect. 1.6, Fig. 33), so that the meaning involved seems to have to be quite subtle (related to *varieties of contradiction*) and in that sense harder than usual to figure out.

More precisely, we are looking for something like the “semantic values” of the decorations of the six vertices of the tri-segment. Let us give a concrete example of the problem: given, as our starting point, the semantic value (or meaning) “white” (or, if you prefer, "giraffe", or "running" or anything else), what can (and in fact must) be the semantic values of the other five vertices, so that *together* they build up (without incoherence) an oppositional tri-segment? Remark that the answer seems to be composed of two qualitative halves: three over the six values of the tri-segment (i.e., three over its six vertices) are expected to be more or less "assertoric" (they "affirm"), while the other three are more or less "negative" (they "deny"). This seems clear with respect to the starting point, the classical bi-simplicial (red) segment N of 2-oppositional contradiction (of which the tri-segment is supposed to be a ternary "oppositional diffraction"): it relates two polarities, such that each is the negation of the other, but in a way such that in general (the starting) one is *concrete*, while the other (i.e., the negation of the starting one) is *vague*. So "white" or "giraffe" (or "running" or even "$2 + 2 = 5$" or whatever other possible starting meaning) will be a concrete, non-vague semantic starting point, while the other polarity of the starting (red) segment of contradiction will be the (classical) negation of the first and therefore a vague term (i.e., "all that is not the starting term"!). So far, so good.

Taking off from the 2-oppositional contradiction segment, as we saw (Sects. 1.3 and 2.2), in the oppositional-geometrical space of the tri-simplexes, there are five new colors (in addition to the classical Aristotelian 4 of the bi-simplexes). Putting aside the black one (the new simplex, which is pivotal), the four other new colors, according to the Aristotelian 3^2-lattice (generated equivalently by the Aristotelian 3^2-semantics or by Angot-Pellissier's sheaf-theoretical method), are expected to represent (1) two new forms of contradiction and (2) two new forms of (Smessaertian) noncontradiction (in fact interpretable as implications, Sect. 2.2, Fig. 40 and Sect. 3.2).

Now, as demonstrated by Angot-Pellissier [3], one of these new negations (the pink CN one) is "paracomplete" (it drops "completeness", i.e., it is "intuitionist", and it defies the principle of the excluded middle, by producing situations where you have truth-value "gaps", i.e. holes): it therefore represents a form of negation in some sense *stronger* than the classical one (it goes so far that it so to say "tears" the truth-theoretical space, cf. left side of Fig. 81), such that it is not "involutive" (i.e., with an intuitionist "NoT" negation operator, the formula or meaning "NoT NoT A" is not equivalent, in general, to "A"). The other of these two new negations (the brown NS one), as demonstrated, again by Angot-Pellissier [3], is "paraconsistent" (it drops "consistency", i.e. it is "co-intuitionist", and it defies the principle of noncontradiction, by producing situations where you have truth-value "gluts", i.e., truth-value superpositions of "1" and "0", cf. right side of Fig. 81): it represents a "negation" weaker than the classical one (and *a fortiori* weaker than the paracomplete one), so weak in fact that it might *seem* not to be a negation (cf. Slater, Sect. 1.1) (Fig. 81).

So, in the tri-segment, starting from one of the two |blue-green| vertices (Sect. 2.5, Fig. 51) of its red segment NN of classical 2-oppositional contradiction (say: the green-blue $1_X 2_\varnothing$), taken to be meaning "white" (or "giraffe", "running", etc.), one pink CN segment (of paracomplete negation) leads us therefore to a blue-black vertex ($1_\varnothing 2_U$) meaning "NoT-white" (or "NoT-giraffe", "NoT-running", etc.) and

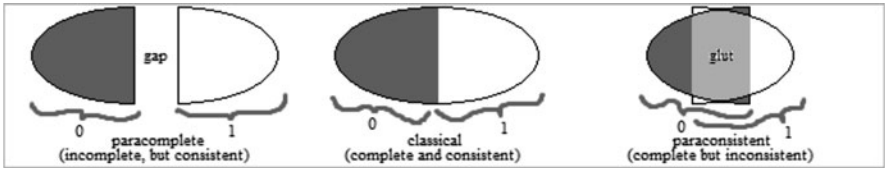


Fig. 81 Negations and truth-values: paracomplete (gaps), classical paraconsistent (gluts)

generating the possibility of a gap (at Angot-Pellissierian level X-U). And starting from the same green-blue classical vertex (i.e., $1_X2_\varnothing$), one brown NS segment (of paraconsistent negation) leads us this time to a black-green vertex (1_U2_X) meaning "nOt-white" (or "nOt-giraffe", "nOt-running", etc.), generating the possibility of a glut (at Angot-Pellissierian level U). Summing up, we thus have so far one starting vertex which expresses a positive meaning and three other vertices, related to it by a red, a pink, and a brown segments, respectively, that express, each, one of three kinds of negation (a classical, a paracomplete, or a paraconsistent one) of the starting concept (or meaning). Remark that in the octahedral tri-segment, together these three vertices, expressing negations (of our starting vertex $1_X2_\varnothing$), form a surface triangle (the "e" in the sense of Sect. 3.4, Fig. 78). What are the mutual relations of these three "negation vertices" in this triangle "e"? The two double arrows CI and IS express a weak form of equivalence between, on one side, the paracomplete and the classical negations of the starting term and, on the other side, the classical and the paraconsistent negations of the starting term. The mutual relation of the two vertices expressing nonclassical negations of the starting meaning is less evident: all we know so far is that the black CS segment means, as its name "CS" says, contrariety at level X-U and subcontrariety at level U.

What said so far covers, relatively to the expression of meaning, four over the six vertices of the tri-segment and the six edges between them: a tetrahedron, or square, the "xi" (Sect. 3.4, Fig. 75). So, what about the remaining two vertices, the numerical sub-sheaves 1_X2_U and $1_U2_\varnothing$? This can be approached in at least two ways: (i) starting from $1_X2_\varnothing$ (as previously) (ii) or starting, this time, from $1_\varnothing2_X$. If we continue starting from $1_X2_\varnothing$, the two remaining vertices are directly accessible from it by two nonclassical forms of biconditional, IS and CI, respectively. This suggests that *the two remaining vertices can be seen as partly equivalent to the starting one*. More precisely, $1_U2_\varnothing$ is "CI-equivalent" (i.e., "paracompletely equivalent") to $1_X2_\varnothing$ at level U, whereas 1_X2_U is "IS-equivalent" (i.e., "paraconsistently equivalent") to $1_X2_\varnothing$ at level X-U. As for the mutual relation of these two assertions (each represented by one of these two vertices), it is expressed by the black segment CS: again, between the two there is contrariety at level X-U and subcontrariety at level U.

Three things at least can be seen at this stage: (1) the Aristotelian 3^2-lattice suggests that they are so to speak "the (vertically) symmetrical" of negations (i.e., they are – as II, CI, and IS – in the lower triangular half of this 3^2-lattice), and, as said, under some "Aristotelian" circumstances (Sect. 2.2, Fig. 40), they even can be read as forms of implications (so is it, for instance, in the logical square and hexagon and in fact in all oppositional geometry). (2) With respect to our starting vertex ($1_X2_\varnothing$), these two remaining vertices can (and must) be read (also, if not only) as "negations of negations (of $1_X2_\varnothing$)". In the classical case (i.e., "¬") this is the classical negation of classical negation and therefore (classical) *affirmation* (or "assertion"). In the other two cases of negation (paracomplete and paraconsistent), this seems to lead, again, to *other varieties of "affirmation"* (an affirmation with a gap and one with a glut, Fig. 81). We thus have three kinds of classical and nonclassical "affirmations (of the starting $1_X2_\varnothing$)". (3) If one reads this centrally symmetric octahedral (or hexagonal) tri-segment "the other way round", i.e., starting from the vertex "$1_\varnothing2_X$", taken (classically) as meaning the (classical) "affirmation of 'the (classical) *negation* of white'" (or of "giraffe", of "running", etc.), the two vertices up to now somehow mysterious, 1_X2_U and $1_U2_\varnothing$, must be read (because of the reasoning above and because of the central symmetry of the structure) as two nonclassical negations of it. So, they are two nonclassical negations of the classical (starting) negation ($1_\varnothing2_X$), therefore they are two nonclassical affirmations of the starting vertex $1_X2_\varnothing$, and together with $1_X2_\varnothing$ itself they therefore seem to be *three forms of "assertion" (or noncontradiction)* of the starting meaning "white" (or of "giraffe", or of "running", etc.). Remark that together the three vertices form a "triangle of assertions" (of the starting meaning): more precisely, the triangle "c" in our notation of Sect. 3.4, Fig. 78, which is centrally symmetric to the previous "triangle of negations", i.e. "e". The 3-oppositional relations of this triangle "c" (of assertions) are, again, CI, IS, and CS. So, in some sense we have reached here two triangles, a "triangle of negations" and a correlated (and centrally symmetric to it) "triangle of assertions" (this is a first example of the apparent usefulness of hybrid substructures, Sect. 3.4). Similarly, the tetrahedron (or "square") "ix" (Fig. 75) is the structure dealing specifically with the vertex $1_\varnothing2_X$ and its three possible negations (Fig. 82).

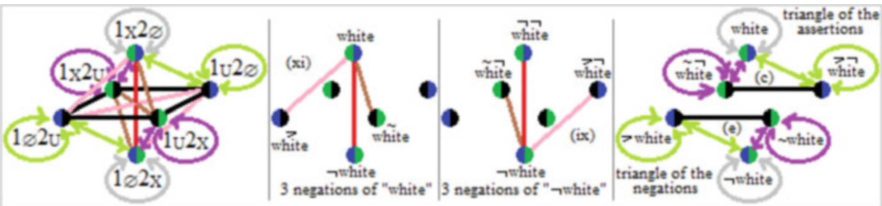


Fig. 82 The basic intuition over the fundamental "semantic values" of the oppositional tri-segment: two triangles

So, remark also that the tri-segment's inner squares reveal potentially meaningful as well in an additional way: this can be seen if one concentrates on the meaning of the three non-hybrid squares (i, ii, iii, Sect. 3.3, Fig. 69). The two vertical squares (of the octahedral tri-segment), i.e., the "i" and the "iii", seem to embody, respectively, the relations between (1) classical negation and paracomplete negation (i) and (2) classical negation and paraconsistent negation (iii). As for the horizontal square (ii), it seems to embody the relations between the four nonclassical meanings: (1) between paracomplete either affirmation or negation *and* paraconsistent affirmation or negation (the four black CS edges), (2) between paracomplete affirmation and paracomplete negation (pink diagonal CN), (3) and between paraconsistent affirmation and paraconsistent negation (brown diagonal NS).

This distribution of the six vertices into two groups of 3 (i.e., the triangles "c" and "e") seems to be *semantically stable*. The "stability" of this distribution may become clearer when one concentrates on the possible (virtual) iterations of negations in the tri-segment (and here we can see somehow at action, in its conceptual potential usefulness, the "circuit of contradictions", Sect. 3.2) (Fig. 83).

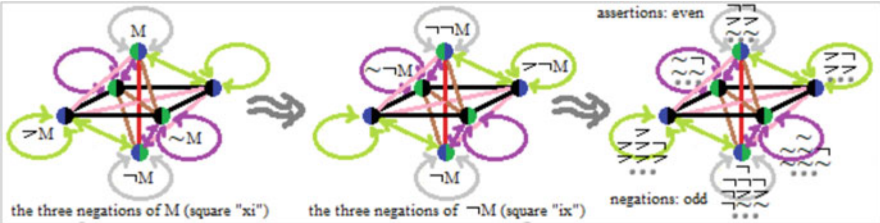


Fig. 83 The possible concatenations of iterations of the three kinds of negations reach a stable "semantic distribution"

The "triangle of assertions" (Fig. 82), with respect to any of its two possible semantic starting points (i.e., $1_X 2_\varnothing$ or $1_\varnothing 2_X$), contains three numerical sub-sheaves which have, prefixed to the semantic starting point, always an *even* number of negation signs (Fig. 83); conversely, the "triangle of denials" (Fig. 82) contains three sub-sheaves such that in each there is, prefixed to the semantic starting point, always an *odd* number of negation signs (Fig. 83). This seemingly distributes meaning over the tri-segment, in a way such that each of classical, paracomplete, and paraconsistent logic/mathematics has two *loca*, a positive (assertion) and a negative (denial) one.

Having already dealt with quite many aspects of the concept of oppositional tri-segment, one last point at least remains nevertheless to be treated before being reasonably able to start using tri-segments "for real" (in applications): understanding how they can be decorated with truth-values, that is, "valuated".

3.6 *Which Possible "Truth-Valuations" of the Global Tri-segment?*

One last crucial problem, seemingly, is that of determining how do function "valuations" (i.e., the attribution of truth-values) for the oppositional tri-segment.

Recall that for a structure of opposition, its strength comes also from coherence in that respect. And that is what works for the logical square, as well as for its many avatars or even its few components: they are useful since they admit coherent patterns of global assignments of truth-values. In order to find a solution for the problem of valuating the tri-segment, which seems to require to reduce combinatorial complexity, as we are going to recall, the inspiring image seems to be the familiar situation with the *bi*-simplexes. For here, *empirically* (cf. Mandelbrot [88]!), there seems to be something like a "law of the two hemispheres" of valuation: one-half of the valuations will be "0"; the other will be "1". This can be seen in the 1D space with the oppositional segment (B2) and in the 2D space with the oppositional hexagon (B3) (Fig. 84).

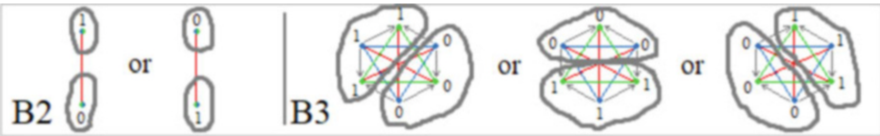


Fig. 84 "Hemisphere theory": the bi-simplexes' (and their closures') *valuation* always cuts their "surface" into two

This hypothetical "*hemi*spheral" behavior of the valuation of the *bi*-simplicial structures can also be seen in the 3D space with the oppositional tetrahexahedron (B4) (Fig. 85).

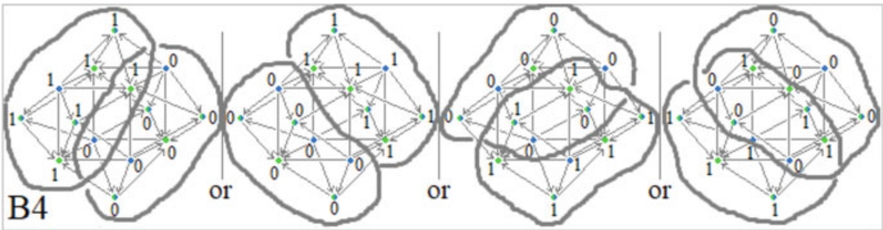


Fig. 85 The four possible valuations of the oppositional tetrahexahedron (the B4, the "closure" of the bi-tetrahedron)

Intuitively, this behavior *seems* to be general for the bi-simplexes and their closures: all the vertices of any blue simplex will have to be valued "0", except one (let us call it the "oppositional hostage"), which will have to be valued "1"; conversely, all the vertices of the correlated, centrally symmetric green simplex will have to be valued "1", except one (also an "oppositional hostage"), which will have to be valued "0" (by construction, these two hostages, the blue and the green one, are mutually centrally symmetric). Now, the structure of the bi-simplex, by construction, is such that it lets emerge (*seemingly* independently from the dimension of the starting simplex) *a partition into two of the resulting bi-simplex's n-dimensional "surface"* (the half around the green hostage, valued "0", will have to be valued "0"; the other half, around the blue hostage, valued "1", will have to be valued "1"), and this *seems* to have to be respected also by the bi-simplex's cloud (this last conjecture is not yet proven for higher B*n*, but this provisory uncertainty seemingly brings not much harm here – but this point will have, of course, to be fully clarified as soon as possible). The number of vertices of the starting simplex (i.e., its dimensionality) determines how many *rotations* of this *valuation pattern* (for that precise bi-simplex and its closure) there can be (this is due to the fact that simplexes, by construction, are symmetric with respect to *all* their vertices and therefore *any* vertex can and must play the oppositional hostage). So, the valuation of bi-simplicial oppositional-geometrical figures (including their closures) admits (i) an abstract *n*-dimensional "hemispheral pattern" and (ii) *n* concrete possibilities of having this hemispheral pattern embodied on the global B*n*: two for the B2 (two vertices), three for the B3 (three vertices), four for the B4 (four vertices), etc.

But our oppositional *tri*-segment is not bi-simplicial, but it is an instance of *tri*-simplex. And, by construction, dealing with tri-simplexes means dealing with *three*-valued logic. How to conceive valuation in this case, then?

Now that we seemingly know with sufficient precision the real structure of the tri-segment (but some surprises wait for us in Sect. 4), the question is: how to handle, with it, this question of its possible global valuations? Remark that from a purely combinatorial viewpoint (six vertices admitting each in abstracto three possible truth-values), we seem to have $3^6 = 729$ different possible valuations (!). But then we must remember that the B2 has *abstractly* $2^2 = 4$ valuations, but *really* it only has 2; similarly, the B3 in abstracto has $2^6 = 64$ possible valuations, but *really* it only has 3; and similarly the B4 has abstractly $2^{14} = 16.384$ valuations (!), whereas *really* it only has 4. So, *it seems natural that oppositional structures reduce drastically, by the constraints they impose by construction, the combinatorial explosiveness*. How to reduce then, comparably (reasoning the safest we can by analogy), the combinatorial complexity of the valuations of the tri-segment? Remark that the structure "tri-segment" has a strong central symmetry, meaning by that that its centrally symmetric parts are oppositionally identical, so the possible cases seem already to be reduced at least by 2. This is not much, but it suggests that symmetries play here as well as in the bi-simplexes: so there is a robust hope of finding here as well drastic reductions of combinatorial complexity. On the other hand, from the viewpoint of *analogy* (with what is known from the bi-simplexes), we have, by construction, three possible truth-values. So, a first temptation, betting on the real existence of a strong analogy with the bi-simplicial case (to be checked now), might be to have us landing here to something like a "surface *tri*partition" (instead of the surface *bi*-partition, or "*hemi*sphere theory", we seemingly have, as we have seen, with the bi-simplexes) (Fig. 86).

This solution, in fact, seems to work. It leads to see three independent "worlds": three bi-simplexes inside the tri-segment, each *bi*-valued (with three possible truth-values *only* at the global scale, *never* in the bi-simplicial substructures of the tri-simplex). And this happens to be made very clear by the "oppositional colors" of

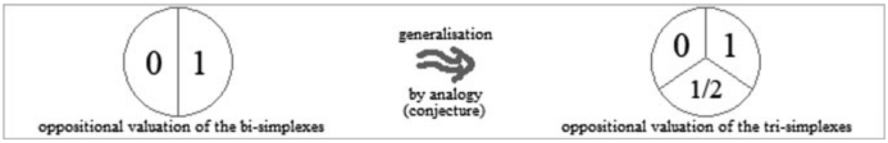


Fig. 86 An intuition on "valuation" in the tri-simplexes: from "hemisphere theory", to "sphere *n*-partition theory"

the vertices (Sect. 2.5)! The rule seems to be that each vertex, in a bi-simplicial pair (i.e., the one comprising the *cloud* where this vertex lives), can receive only either its "natural value" or its "hostage value" (i.e., the "natural value" of its co-simplex in that bi-simplex), which are precisely the two colors of the indexes of this vertex and therefore the two colors of this "point" itself! The same reasoning (with the same two colors) runs for its centrally symmetric mate, and *this gives as a major final result of our reasoning only two possible valuations of the oppositional tri-segment* (Fig. 87).

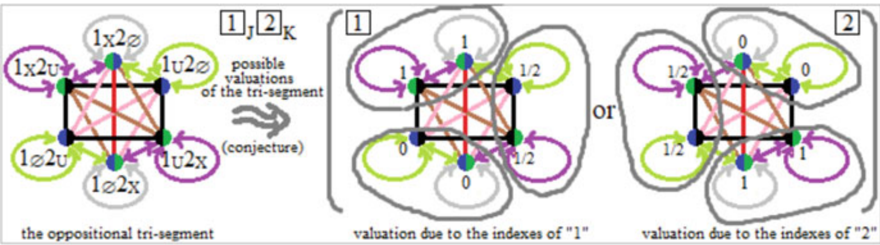


Fig. 87 Sphere *n*-partition: what seem to be the only two possible global valuations of the oppositional tri-segment

More concretely, the combinatorially natural solution to the problem of valuating the tri-segment *seems* to be that (1) |blue-green| vertices (i.e., vertices blue-green or green-blue) can only be valuated "0" or "1"; (2) |blue-black| vertices can only be valuated "1/2" or "0"; (3) |green-black| vertices can only be valuated "1" or "1/2"; (4) there is only one valuation, i.e., *only one kind of valuated pattern* (of the global tri-segment); (5) given the existence of a 2-symmetry by a 180° rotation of the 2D hexagonal representation of the tri-segment, *this valuation (*i.e.*, the unique pattern) can only be "upside" or "down"*; (6) *a posteriori* this matter of fact is pretty analogous to what happens already with the opposition segment (the bi-simplicial counterpart of the tri-segment), so the solution proposed here seems to be a conservative extension; and (7) this behavior is kept in higher poly-*segments* (and in fact, *mutatis mutandis*, in higher poly-*simplexes*), so the solution proposed here seems to be a *generalizable* conservative extension. Remark that the reason why the "hostage rotation" (of the valuations) does not play here is because in the tri-segment there are not yet *simplexes* (an oppositional hostage appears as soon as, but not before, at least one simplicial *triangle* appears). Remark, then, that this seems to be a quite important new element of knowledge (if our starting conjecture holds): it seemingly rules the valuation of *any* poly-simplex! (We will handle this question as soon as we will expound, in another study, the case of the oppositional tri-triangle, of which Angot-Pellissier has opened the sheaf-theoretical exploration in [3], but without reaching its oppositional closure and without affording the question of its global valuation.)

Notice however, as well, that this valuation (Fig. 87) might – and in fact must – perplex us, in the sense that it strongly suggests to see as implication arrows some edges of the tri-segment that we have had no reason so far for seeing as implication arrows. This concerns (1) the four black CS segments, (2) the two light green CI double arrows, and (3) the two violet IS double arrows: each of these three might function – according to the two valuations – as a unidirectional implication arrow (Fig. 88).

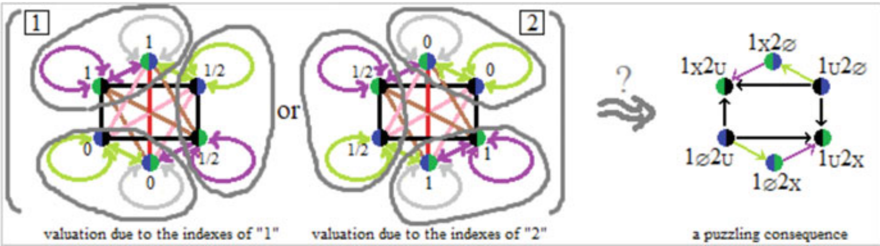


Fig. 88 Some perplexities arising from the valuation (otherwise convincing): it seems to let emerge arrows

So in some sense we have now, unexpectedly, a non-negligible new problem with “arrows”. This seems to be due to the fact that our “Aristotelian strategy” (put into play in Sect. 2.2), aimed at avoiding (lazily) the “heaviness” of a potential Smessaert-like *complete analysis of implication behaviors* (at the non-Smessaertian, tri-simplicial level), possibly did not suffice: our conjecture (and our correlated lazy bet), that just looking for the implication-behavior of the edges admitting some “I” in their code might suffice for dealing, overall, with implication-like relations, was seemingly wrong (and our lazy bet is seemingly lost). So we will now face this very instructive serious problem (remember that we already faced voluntarily an instructive “laziness problem” in Sect. 2.1 and 2.4) by changing now quite radically our strategy on this point of arrows (as we changed it successfully in Sect. 2.4 with respect to trivial extrema), by having, from now on, a direct and systematic “tri-simplicial look” (not yet existing . . .) at the possible Smessaertian-Demeyan “*implication* geometry” of the tri-segment (Sect. 4).

4 There Is Logical Geometry *Inside* the Oppositional Tri-segment!

The result we arrived to at the end of the previous chapter is interesting, but still too puzzling with respect to *implication relations*. And it appears that the instruments used so far do not suffice yet, as we would have liked, to deal in full clarity with this massive "arrows problem". So we turn to the tools offered at the *bi*-simplicial (rather than *tri*-simplicial) level by "logical geometry" (Sect. 2.2, Fig. 40), in so far it contemplates, specifically, the existence of a whole (although small) "*implication* geometry" parallel (in Smessaert and Demey's terms) to "opposition geometry" ("opposition*al* geometry" being seen by them as a bricolage, a non-systematic and unconscious mixture of these two *scientific* characterizations, as we will reexamine in Sects. 4.5 and 4.6). So far, as long as I know, logical geometers have not yet inquired the concept of poly-simplex. Therefore we will have the honor, brave reader, you and me, to open *now* the way of this attempted "junction", right here. And this should close, at a reasonable level of understanding, our complex investigation over the concept of tri-segment, not without some interesting backfire (Sects. 4.6, 5.1 and 5.2).

4.1 The Implication Geometry's 3^2-Semantics/Lattice of the Tri-simplex

We want to explore the "implication geometry" (if, as we think, it exists) correlative of the opposition*al* tri-segment we investigated so far. Our "doubt" is just methodical, since this has never been done before (logical geometers have not yet taken seriously the idea of *poly*-simplex, and for a start they do not seem to favor much – euphemism – the simpler idea of *bi*-simplex). As for us, what we need for that is, first of all, generating a new kind of lattice, comparable to our game-theoretical Aristotelian 3^2-lattice (Sect. 1.3, Fig. 11), for obtaining the "*implication* kinds" of the 21 possible binary relations (edges and curls) holding between the 6 vertices of the tri-segment (Sect. 2.4, Fig. 46). Recall that at the level of the "*bi*-simplicial space" (i.e., two-valued oppositional/logical geometry), such semantics and lattice, devised by Smessaert around 2011 (Sect. 2.2), resulted of two "meta-questions", complementary of the "Aristotelian" ones (proposed by me in 2009) seen so far (Sect. 1.3, Fig. 10), but aimed at generating not "*opposition* relations" but "*implication*-like relations". Truly speaking, Smessaert investigated the strangeness of "subalternation" (i.e., the strangeness of its being, in the classical "Aristotelian quartet" (Sect. 1.1, Fig. 1), the only *asymmetric* relation), and because of his rigorous framing of this question, he discovered, probably unexpected, a whole (although small) "implication geometry" (this is a typical *structuralist* good move). Now, these two Smessaertian meta-questions are (for any pair of things A and B) (Q'1) "Is it possible to have, at the same time, A false and B true?"; (Q'2) "Is it possible to have, at the same time, A true and B false?". As one sees, they inquire not the simultaneous "truth-value *similarity*" (both false, both true) but the simultaneous "truth-value *dissimilarity*" (false the first while true the second, true the first while false the second). For short, Smessaert had the *structuralist* brilliant idea to add the study of dissimilarity to that of similarity (to a continental philosopher's eye that reminds the main methodological lesson of Plato's *Parmenides*, [113]). Now, allowing two kinds of answers, i.e., "0" or "1" (this binarity being the "*bi*-simplicial touch"!), generates four possible pairs of answers ([0|0], [0|1], [1|0], [1|1]) at these two questions (Q'1 and Q'2). These four possible "double answers" ([x|y]) distribute, similarly to what we saw with the Aristotelian 2^2-lattice (Sect. 1.3, Fig. 10), in a "Smessaertian lattice", which gives precisely the four kinds of possible "*implicative* relations" (for the bi-simplicial space) (Fig. 89).

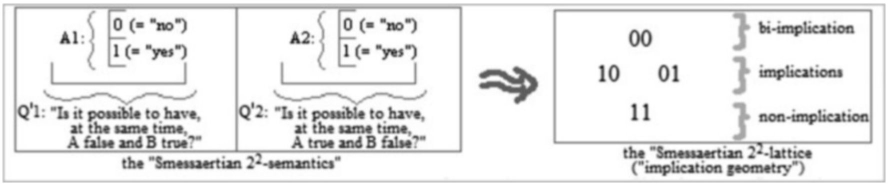


Fig. 89 The Smessaertian 2^2-semantics for the "bi-simplicial" (in our terminology) *implicative geometry*

As we recalled above (Sect. 2.2, Fig. 40), these four kinds of Smessaertian "implication relations" are (i) double-implication ([0|0]), (ii) right-implication ([1|0]), (iii) left-implication ([0|1]), and (iv) no-implication ([1|1]) (Fig. 90).

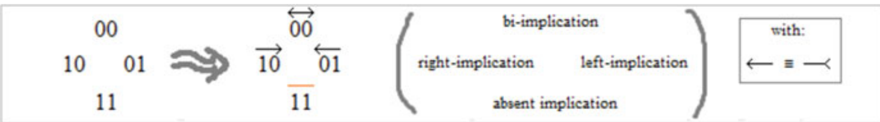


Fig. 90 The meaning of the four Smessaertian kinds of implication relations for bi-simplicial geometry

Let us introduce here, comparably with what I did with respect to Angot-Pellissier (Sect. 2.2, Fig. 37), a small terminological change (aimed at making easier the combination of logical geometry's strategy and conceptuality with our own approach based on simplexes): a useful convention (for what follows) consists in naming with a single letter each of the possible four Smessaertian kinds of implication (so to generate, once one levels up as we will be when we will move from bi-simplicial to tri-simplicial, a "code" made of *two* such literals concatenated), a swift symbolism whose utility should appear soon (Fig. 91).

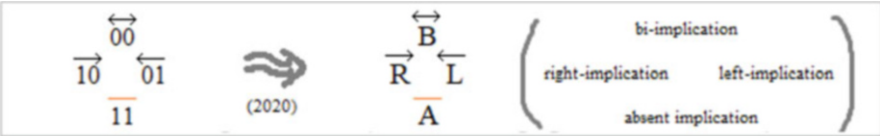


Fig. 91 A useful terminological convention (literals for building higher-level concatenations)

Starting from that, what we need now with respect to our goal (i.e., clarifying the strange "black CS emerging implications" carried by the tri-segment, notably in relation to its two valuations obtained in Sect. 3.6, Fig. 88) is something comparable to Smessaert's 2^2-semantics and its 2^2-lattice, but admitting now not two, but *three* kinds of answers, because of the adoption by us of a third truth-value, "1/2", alongside with the classical "0" and "1" (i.e., so to make this implicative-geometrical meta-lattice match, by the "implication kinds" it generates, the *tri*-simplicial and *three*-valued structure of the oppositive *tri*-segment) (Fig. 92).

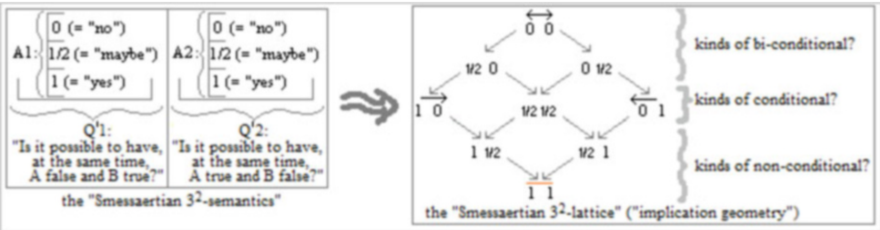


Fig. 92 The "Smessaertian" 3^2-semantics for the "tri-simplicial" implicative geometry

But here we must mention an important problem that we will have to leave open in the rest of this study (but not in a near future), relative to what we called "the *q* parameter" (Sect. 1.3): *some structure will be clearly missing in our approach now, for we are not (yet) able to try to ask all the remaining comparable meta-questions*, like the "Aristotelian" Q3 ("Can two things A and B be ½ together?") or like the "Smessaertian" Q'3 ("Is it possible to have, at the same time, A '0' and B '½'?"). On this we will come back later (Sects. 5.1 and 5.2).

So, back to our introduction, if not of all the possible meta-questions (*q* parameter), at least of a third possible kind of answer (*p* parameter) to them: we see that, as in the case of the Aristotelian 3^2-semantics and 3^2-lattice (Sect. 1.3, Fig. 11), there are, emerging here, five new kinds of possible answers ([1/2|0], [0|1/2], [1/2|1/2], [1|1/2], [1/2|1]), whence, by (inspiring but slippery) analogy, we would expect something like $4 + 5 = 9$ *kinds* of "implicative relations". This can be viewed, from our viewpoint at least, as a form of *tri-simplicial diffraction* of the bi-simplicial Smessaertian "*implication* geometry" (Fig. 93).

The naming, in our convention, of the five new kinds of implicative relations cannot yet be fully clear at this stage. But it should become clearer as soon as we

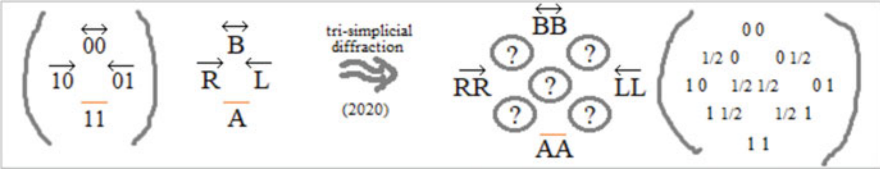


Fig. 93 A first intuition on the “tri-simplicial diffraction” of the (bi-simplicial) Smessaertian “implication geometry”

will start the calculations of the edges carrying these nine (or more?) qualities of implication and when, thus, we will see how these “implications work” (Sect. 4.2). But by analogy with the poly-simplexes, we would expect something comparable with what we have already seen (Sect. 2.2). A still more urgent problem is that of interpreting the *formal meaning* of these five new kinds of implication relations: but, again, a reasoning by analogy, always to be taken carefully (as potentially slippery, of course), can provide already now a *provisory* starting intuition on that (to be further checked by other means) (Fig. 94).

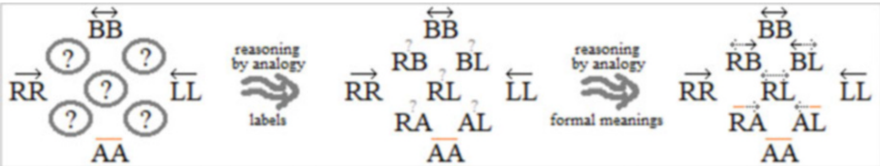


Fig. 94 Guessing, by analogy, the emerging codified labels and the formal meaning of the (at least) five new kinds

Another question arising before we get started is the following: what about the *colors* of “points” (i.e., the vertices), if any in this “implication geometry”? Here, as a provisory methodology, we will keep the “point coloring technique” of the tri-segments (Sect. 2.5). This seems not necessarily too arbitrary since the geometrical shape under examination (the tri-segment, abstraction made of its colours) has been generated by the Pascalian method of Sect. 1.5 (and the Angot-Pellissierian one, isomorphic to it), which seems to have a strong isomorphism with the “coloring of points” we adopted (Sect. 2.5, Figs. 52 and 53). But we will remain ready to revise this provisory methodology (for the coloring of points) as soon as it would appear reasonable to do that. So, given that the vertices of the tri-segment, seen as Angot-Pellissierian numerical sub-sheaves, remain the same as those seen so far (Sects. 2 and 3), we can now go to the question of the *edges*, between any pair of them, of the “implication-geometrical” version of the tri-segment we are trying to study in this Sect. 4.

4.2 *Calculating the 21 Implicative Relations (Edges) of the Tri-segment*

On the basis of the “Smessaertian” 3^2-semantics and 3^2-lattice seen in the previous section (which asks only 2 of the 6 meta-questions it could/should ask, as pointed out in Sect. 4.1), we must now calculate the 21 binary relations holding between the 6 nontrivial vertices of the tri-segment (Sect. 2.4, Fig. 46). This seemingly can be done rather easily relying anew on the Angot-Pellissierian numerical sub-sheaves introduced in the previous Sects. 1, 2, and 3, taken by pairs (hence the number 21: there are 21 possible unordered such pairs). However, an important change here is that since we are mainly dealing with *varieties of implication*, we must now also pay attention to the *order* of each such *unordered* pair of numerical sub-sheaves, i.e., distinguishing AB from BA, so to say (implication is an *order* relation). Combinatorially, taken apart the 6 reflexive pairs (as “1_X2_U and 1_X2_U”), the remaining 15 unordered pairs (as “$|1_X2_U$ and $1_X2_\varnothing|$”) must be examined “both ways”: so all in all $6 + (2 \times 15) = 36$ pairs must be analyzed. So, in principle to each (nonreflexive) *ordered* pair, it should be asked now the four following pairs of Smessaertian questions (Q’1 and Q’2, in both Angot-Pellissierian sheaf-levels U and X): (Q’1/U) “Is it possible to have, at the same time, A *false* and B *true* at level U?”; (Q’2/U) “Is it possible to have, at the same time, A *true* and B *false* at level U?”; (Q’1/X) “Is it possible to have, at the same time, A *false* and B *true* at level X?”; and (Q’2/X) “Is it possible to have, at the same time, A *true* and B *false* at level X?”. In each case, the quartet of answers (A’1/U, A’2/U, A’1/X, and A’2/X) to this general quartet of questions, for each of the 36 ordered pairs of vertices (i.e., sub-sheaves) of the tri-segment, will give us the “implication quality” of this precise ordered pair of vertices (one among the 36), that is the “color” of the corresponding edge of the tri-segment. So let us stress again that this implies that, for each *non-ordered* pair of vertices, the quartet of questions should be applied *two* times (one for each of the *two* possible orders, A-B or B-A, of the unordered pair). But as it happens, the mutually reversed ordered pairs are in fact strictly correlated (as for the quality of implication relation generated by their answers) in a way such that the correlated reversed pairs will turn up to give *globally* (i.e., *modulo* the direction) the same answer, and this will allow in the end a very useful simplification (Sect. 4.3, Fig. 102). For short, reverse pairs just will exchange R with L and vice versa (Fig. 95).

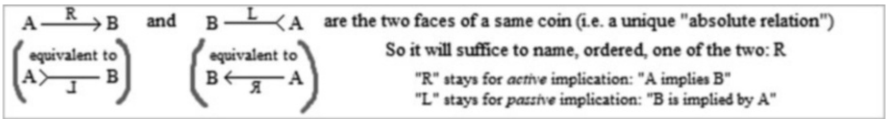


Fig. 95 Possible left-right simplifications of the “implicative relations” (eliminating hidden redundancies)

So it will suffice to calculate one of any pair of correlated pairs (of sub-sheaves), and the other will be obtained by substituting, in the result of the first, any R with L and any L with R. Therefore, we only need to test 21 pairs (instead of 36).

Let us now give just a few examples of calculations, so to nourish the intuition of the reader. Let us first consider a pair which depends on its order, the ordered pair of sub-sheaves “1_X2_U and $1_X2_\varnothing$” (taken as “A and B”). The calculation goes as follows, asking about this ordered pair the four Smessaertian questions: (Q’1/U) “Is it possible to have, at the same time, A *false* and B *true* at level U?”; the answer to this (i.e., A’1/U) is 0. (Q’2/U) “Is it possible to have, at the same time, A *true* and B *false* at level U?”; the answer to this (i.e., A’2/U) is 1. (Q’1/X) “Is it possible to have, at the same time, A *false* and B *true* at level X?”; the answer to this (i.e., A’1/X) is 0. (Q’2/X) “Is it possible to have, at the same time, A *true* and B *false* at level X?”; the answer to this (i.e., A’2/X) is 0. As a result, the string variable *x* (for concatenating orderly the two answers relative to level X, cf. Sect. 2.2), receiving here the value 00, is B (for B is defined, in the Smessaertian 2^2-semantics, as [0|0], Sect. 4.1, Fig. 91), and the variable *u* (for concatenating orderly the two answers relative to the level U, cf. Sect. 2.2), receiving here the value 01, is L (for L, in the Smessaertian 2^2-semantics, is defined as [0|1]), so here the string variable *xu* becomes BL and this is the tri-simplicial “implication kind” holding between the two ordered vertices of the tri-segment (Fig. 96).

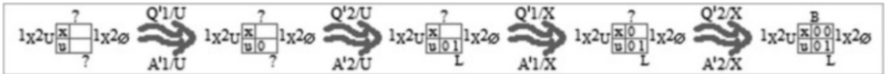


Fig. 96 How to calculate the quality of the “implication relation” of an ordered pair of numerical sub-sheaves

Let us now consider the reverse ordered pair, “$1_X2_\varnothing$ and 1_X2_U” (taken as “A and B”). The calculation for it goes as follows, asking about this ordered pair the four Smessaertian questions: (Q’1/U) “Is it possible to have, at the same time, A *false* and B *true* at level U?”; the answer to this (i.e., A’1/U) is 1. (Q’2/U) “Is it possible to have, at the same time, A *true* and B *false* at level U?”; the answer to this (i.e., A’2/U) is 0. (Q’1/X) “Is it possible to have, at the same time, A *false* and B *true* at level X?”; the answer to this (i.e., A’1/X) is 0. (Q’2/X) “Is it possible to have, at the same time, A *true* and B *false* at level X?”; the answer to this (i.e., A’2/X) is 0. As a result, *x*, being here 00, is B, and *u*, being here 10, is R, so here *xu* is BR and this is the tri-simplicial “implication kind” holding between these two ordered vertices of the tri-segment.

Remark, then, as we already said, that inverting the two vertices here just transforms “BL” in “BR”: which means that the same edge read in one direction gives BR, while read in the other it gives BL. This is because at level X-U (i.e., X minus U) the implicative relation happens here to be B (which is bidirectional), which is therefore not affected by switching the vertices, while at level U the relation happens to be L for the first two ordered vertices and R for their reversal: this means that R or L represent in fact the same "absolute relation", but it specifies (by being R or L) in which of the two possible directions it is *read*; let us call it "|R/L|", which means "L or R, just according the *directional viewpoint*" (Fig. 97).

Fig. 97 The relations "R" and "L" are two directionally opposed *viewpoints* on the same unchanging reality

Let us now take a different example, that of a pair of sub-sheaves of the tri-segment which is invariant with respect to the order of its elements. Let us consider the ordered pair "1_X2_U and 1_U2_X" (taken as "A and B"). Its calculation goes like this, asking about this ordered pair the four Smessaertian questions: (Q'1/U) "Is it possible to have, at the same time, A *false* and B *true* at level U?"; the answer to this (i.e., A'1/U) is 0. (Q'2/U) "Is it possible to have, at the same time, A *true* and B *false* at level U?"; the answer to this (i.e., A'2/U) is 0. (Q'1/X) "Is it possible to have, at the same time, A *false* and B *true* at level X?"; the answer to this (i.e., A'1/X) is 1. (Q'2/X) "Is it possible to have, at the same time, A *true* and B *false* at level X?"; the answer to this (i.e., A'2/X) is 1. As a result, *x*, being here the string 11, is A, and *u*, being here the string 00, is B, so here *xu* is the string AB and this is the "implication kind" holding between the two ordered vertices of the tri-segment.

Let us now consider the same ordered pair, but reversed: "1_U2_X and 1_X2_U" (taken as "A and B"). Its calculation goes like this, asking about this ordered pair the four Smessaertian questions: (Q'1/U) "Is it possible to have, at the same time, A *false* and B *true* at level U?"; the answer to this (i.e., A'1/U) is 0. (Q'2/U) "Is it possible to have, at the same time, A *true* and B *false* at level U?"; the answer to this (i.e., A'2/U) is 0. (Q'1/X) "Is it possible to have, at the same time, A *false* and B *true* at level X?"; the answer to this (i.e., A'1/X) is 1. (Q'2/X) "Is it possible to have, at the same time, A *true* and B *false* at level X?"; the answer to this (i.e., A'2/X) is 1. As a result, *x*, being here the string 11, is A, and *u*, being here the string 00, is B, so *xu* is here the string AB. Remark that in this case the fact of reading this edge in one direction or the other makes no difference: both ways it has to be read AB.

We will not make – we are merciful – all calculations explicitly here. Instead we will give the general resulting calculation of the 21 edges (in only one direction, the result of the reverse calculation can be obtained by substituting the "R" with "L" and vice versa), in a synoptic compact format (Fig. 98).

These 21 calculations give us 11 kinds of implication relations (the BB, BL, AB, LL, AL, AR, LB, AA, LA, RA, BA). But once we exchange L with R (and not the other way round) two become redundant and we obtain thus nine kinds: BB, BR, AB, RR, AR, RB, AA, RA, and BA. If we compare these nine kinds of implication relations obtained here with the nine kinds conjectured by analogy (Sect.

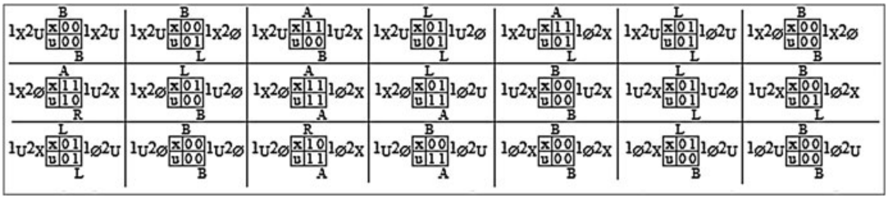


Fig. 98 Synoptic view of the calculations over the 21 ordered pairs of sub-sheaves (i.e., vertices) of the tri-segment

4.1, Fig. 94), we see that the 3^2-lattice for the tri-*segment* must be modified (we will also adopt by the way new formal symbols more fit, notably in terms of color conventions, with those of the opposition tri-segment, as we will see in Sect. 4.5). The result is well-balanced (i.e., fully symmetrical) (Fig. 99).

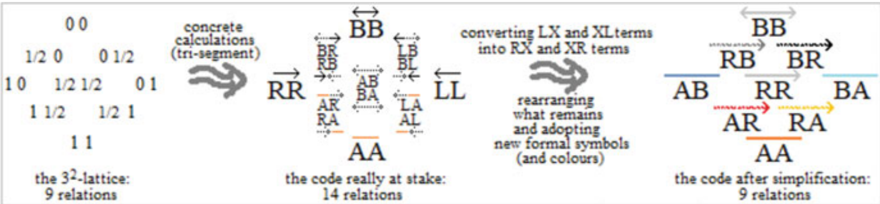


Fig. 99 A simplified version of the 3^2-lattice after the calculation of the 21 edges of the implication tri-*segment*

This change is important and will have to be understood (Sects. 5.1 and 5.2). Relying on this knowledge of the 21 edges (Fig. 98), and based on our new symbolism (cf. Figure 99), we can give now a synoptic view of the “horizons” (Sect. 3.1) of each of the six vertices of the implicative tri-segment (Fig. 100).

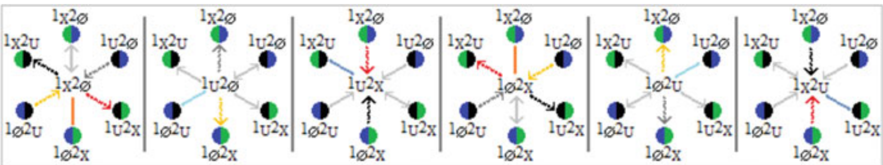


Fig. 100 Synoptical view of the kinds of tri-simplicial implicative relation each vertex has with any possible vertex

Remark that, as in Sect. 3.1, there are here three kinds of horizon, such that the vertices centrally symmetric share the same horizon *modulo* a rotation of 180° of it.

A natural question now, in comparison with what happens with the opposition qualities of the tri-*segment* (less, in number, than those of the general tri-*simplex*:

seven instead of nine, Sect. 2.2, Fig. 39), is that of knowing whether some implicative qualities of the tri-simplex are missing in the tri-segment. We are not yet able to give a systematic *a priori* answer. But this answer could seem to be that no implicative quality among those of the tri-simplex (Sect. 4.1) is missing in the tri-segment. However, comparable calculations done by us on the tri-*triangle* let emerge already two more relations there, absent here: the "LR" and the "RL" (and RL is one which appeared in our previous, conjectural lattice, Fig. 94). This point, which bears no prejudice to the rest of this study, will have to be understood and explained in the future.

Having calculated the edges of the implicative version of the tri-segment (without forgetting however that our current reasoning relies on only two over the six possible "Smessaertian" meta-questions, Sects. 4.1, 5.1, and 5.2), we can now try, in the next Sect. 4.3, to construct a global view of it.

4.3 The Global "Implication Geometry" of the Tri-segment

Having determined the "implicative quality" of each of the 21 possible (so to say absolute, in the sense of "direction independent") binary relations of the tri-segment, let us now try to have a more synoptic view on this. We keep the hexagonal representation of the tri-segment we arrived to in the last Sects. 2 and 3 (Sect. 2.6, Fig. 57). A first step consists in putting, on each of the 21 still colorless edges (6 of which are reflexive curls), its reading (and its color), and this can be of two kinds: (1) either bidirectional and therefore unique (when it is either BB or AA but also AB or BA); or (2) it is unidirectional (i.e., asymmetric) and then the direction in which the edge is read determines two different labels (like in RR and LL, in AR and AL, or in RB and LB, Sect. 4.2, Figs. 95 and 97). So, to begin with, we put on each edge its *two* readings (when there are 2: one with R and one with L), and we try to highlight the direction of the intended segment decoration by the direction (at times a little bit directionally strange) of the letters (Fig. 101).

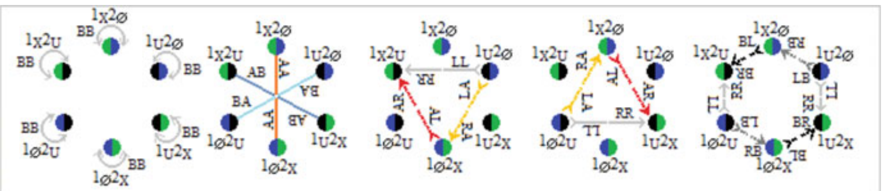


Fig. 101 A global view of the "implicative side" of the tri-segment (with *all* its kinds of "implication relations")

But then it is useful to resort to what we have understood in the previous Sect. 4.2 over the substantial (i.e., regular) coincidence "in the absolute" of all pairs of reversed labels (Figs. 95 and 97). And this will mean that we will now be able to produce another, swifter implicative (but still decomposed) decoration of the tri-segment, with only one-half of its two opposed labels on each of its 21 edges. As a rule, as said, we will privilege the so to say “direct” (or active, Fig. 95) implication (i.e., “A *implies* B”) to its “indirect” (or passive), equivalent counterpart (i.e., “B *is implied* by A”), and this means that we will always choose R instead of L, but we could have done, equivalently, the other way round. So, RR will be preferred to the correlative LL; BR and RB will be preferred to, respectively, their correlative BL and LB; RA and AR will be preferred, respectively, to their correlative LA and AL (Fig. 102).

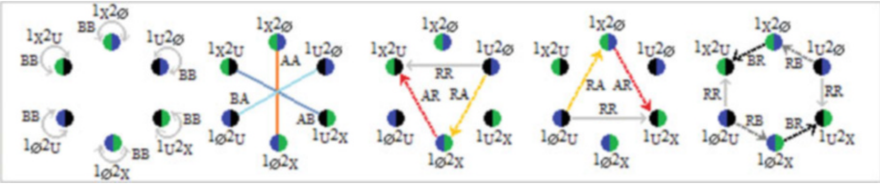


Fig. 102 Global decomposed view of the “implicative side” of the tri-segment, simplified (R and not L)

If we now gather this decomposed view into a whole, this gives the following first global representation(s) of the “implication geometry” version of the tri-segment (Fig. 103).

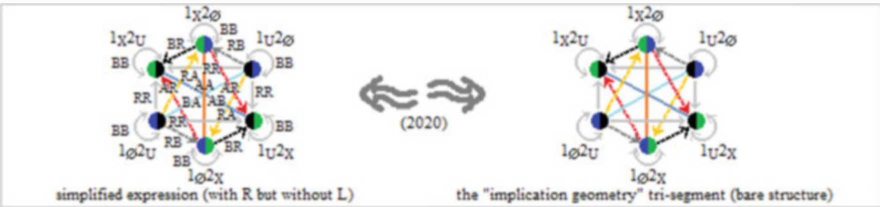


Fig. 103 First two *whole* representations of the “implication geometry” version or the tri-segment

Remark that this seems very interesting: the tri-simplicial diffraction of the contradiction segment gives us this, in some sense unexpected “arrow *complexity*”. Recall: the red segment of 2-opposition already could be seen as “implication geometry” (à la Smessaert, Fig. 104). So, at this so to say “logical-geometry stage”, we have two versions of the tri-segment, one in terms of its “opposition geometry” (on the middle), the other in terms of its “implication geometry” (on the right) (Fig. 104).

This is, so far, the – *cum grano salis* (Sect. 4.1) – complete “logical geometry” view on the tri-segment. We will however afford the question of *unifying* (if it is

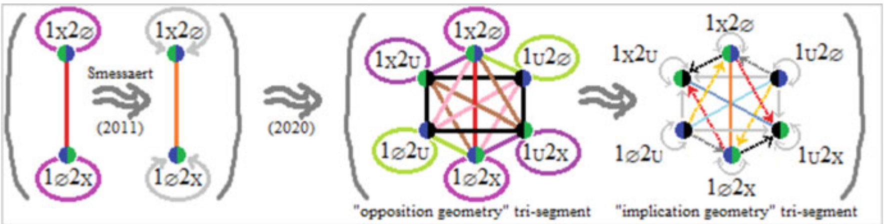


Fig. 104 Tri-simplicial diffraction of the "logical geometry" of the 2-opposition contradiction segment

possible) by a *methodical and reasoned combination* these two representations into just one (Aristotelian) later (Sect. 4.5).

But before that, by analogy with what we did in Sect. 3 (with respect to Sect. 2), we will have now to study at least the basics of the structure of this "implication geometry" tri-segment in some more depth (Sect. 4.4). And, before that, let us make here some more preliminary remarks on this implicative structure taken as a whole. Importantly (for our study), remark that this is more or less precisely what we have been looking for since the end of Sect. 3 (i.e., since the end of Sect. 3.6): finding some kind of *reasoned roadmap of the "implications" of the tri-segment* for helping us in understanding how to judge the otherwise puzzling apparent emergence (from the valuation of Sect. 3.6, Fig. 87) of unexpected *implication relations* (arrows), notably in the four black segments of (non-arrow) simplicial CS relations (Fig. 88). And in fact such a roadmap, that we now have successfully in our hands, provides us immediately at least two very important things: (1) an *exhaustive* list of the kinds of implications at stake in the tri-segment (which appears to be – although not yet the closure: we are not yet able to consider all the 3 + 6 possible meta-questions, Sects. 4.1, 5.1 and 5.2 – quite larger than what we knew and quite larger than what we expected) and (2) a rather univocal and unambiguous indication as to the way to interpret the *valuation* we arrived to (on Sect. 3.6, Figs. 87 and 88). Let us see more in detail these two points.

As for the first point, in our "implication geometry" tri-segment (Fig. 104), there are, at work, three main families – the kinds of nonimplications, the kinds of implications and the kinds of bi-implication – more precisely: (1) five kinds of "nonimplication" (i.e., all the tri-simplicial "implication geometry" relation kinds containing *at least* an occurrence of the bi-simplicial relation "A", as in AR), (2) five kinds of implication (i.e., those containing *at least* an occurrence of R, as in RA), and (3) five kinds of bi-implication (i.e., those containing *at least* one occurrence of B, as in BA). Because of the tri-valuedness of the tri-simplicial space, which implies the existence of two Angot-Pellissierian sheaf-levels U and X (Sects. 1.4 and 2.1), these three kinds are not mutually exclusive any more: their *composition* is now the rule (six over the nine cases); the "pure cases" are just particular cases of composition (three over the nine cases), namely, trivial compositions (or self-compositions), like "RR", "AA", or "BB" (Fig. 105).

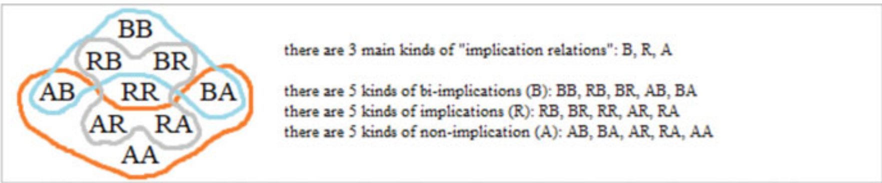


Fig. 105 There are three main "implication relations" kinds, characterized by the fact of containing B, R, and A

Remark *en passant* that the result as for implications seems to confirm that the hexagonal representation that we choose for the tri-segment is *optimal* and this can be seen here in at least three respects: (1) its three diagonals are indeed contradiction diagonals (because they are the "implication geometry" counterpart of this: the nonimplication AA and its two weakenings AB and BA); (2) as we will see in the next Sect. 4.4, Fig. 109, the three main rectangles ("squares") of this implicative hexagonal tri-segment are *very* regular with respect to its "implication geometry" relations; and (3) the balanced character of the hexagonal representation of the tri-segment, still thanks to the unveiling of its arrows, is also confirmed by the "differential topology" viewpoint (Sect. 1.2, Fig. 8, based on [62], p. 52), i.e., by the distribution of the three differential-topological kinds of vertices: two (centrally symmetric) vertices (i.e., the two |black-blue| ones) shoot each four arrows (exhibiting thus a "source" behavior), two (centrally symmetric) vertices (the two |blue-green| ones) shoot each two arrows and receive each two arrows (exhibiting thus a "saddle" behavior), two (centrally symmetric) vertices (the two |black-green| ones) receive each four arrows (exhibiting thus a "sink" behavior) (Fig. 106).

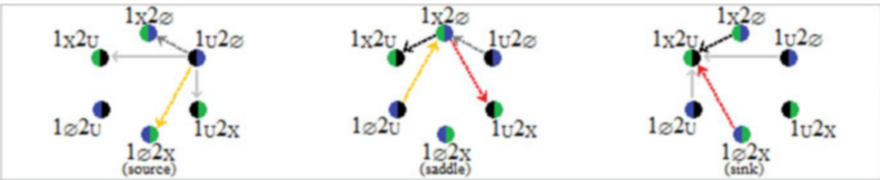


Fig. 106 Reading the "implication geometry" of the tri-segment with "differential-topology"'s eyes

Remark here that the classical position (i.e., the two |blue-green| vertices of the classical bi-simplicial 2-oppositional red segment of contradiction) seems to play a *pivotal role* in between its two symmetric "diffractions" (paracomplete and paraconsistent). In other words, the "tri-simplicial diffraction" opens so to say some kind of left-right "symmetry", orthogonal to the top-bottom affirmative-negative starting (oppositional) "symmetry" of the starting red segment of 2-oppositional contradiction. At the level of the *expression* of the tri-segment by a visual structure

(Sect. 3.3), all its good properties (i.e. the symmetries – among which the three kinds (1)–(3) we just mentioned) seem to be grounded in (and granted by) our starting choice of interpreting central symmetry as a reversal of the indices (Sects. 2.6 and 3.3).

Let us now turn to the second point (valuation and its implications). What we now have clearly confirms some of the arrows we suspected (due to the two valuations we were able to establish, for the tri-segment, at the end of Sect. 3.6, Fig. 88). So let us consider now these two valuations of the tri-segment, but this time under its implicative reading (Fig. 107).

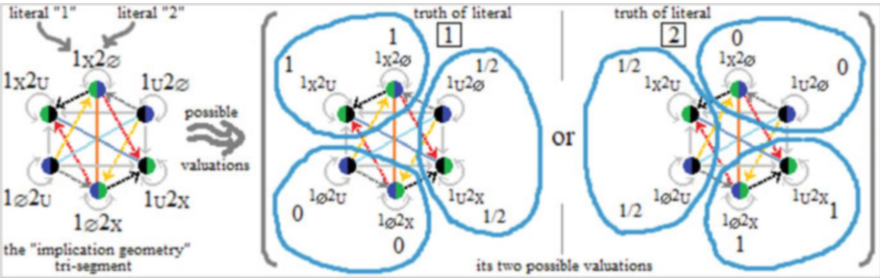


Fig. 107 The two possible valuations of the tri-segment reconsidered from the viewpoint of "implicative relations"

Here there are three remarks. The first thing to be remarked is that the naturalness of these two valuations seems to be *confirmed* here. If the "opposition(al) geometry" tri-segment so to say made clear the meaningfulness of these two valuations with respect to kinds of *negations* (particularly the diagonals), the present "implication geometry" tri-segment so to say makes clear the meaningfulness of these two same valuations with respect to the *kinds of implications*. A second important remark, then, more particularly, is that the "implication geometrical" approach confirms that the black "simplicial relation" CS can (must?) in some sense be read as an implication: no doubt about this seems to remain now. But this Smessaertian (complementary) approach tells even more: the black simplicial relation can be read, seemingly, as a *classical* implication (we will come back to this in Sect. 4.5). Additionally, the oppositional kinds of noncontradiction CI and IS (that we interpreted, à la Aristotle, as biconditionals, given the relevant Angot-Pellissierian sheaf sections, cf. Sect. 2.2, but that the two valuations pushed forward as being, rather, as the CS, possible full-fledged kinds of implications, Sect. 3.6, Fig. 88) can also be read, thanks to this Smessaertian roadmap, as *uni*directional arrows (at the implication-geometrical level): but this time not as classical RR implications (and this is coherent with their oppositional reading: the CI and the IS are, respectively, paracomplete and paraconsistent "implications", Sect. 1.4, Figs. 19 and 20). The "*opposition*-geometrical" light green CI becomes in fact, in the "implication geometry", the "*implication* geometrical" porous (in our representation here) gray implication RB, whereas the "*opposition* geometrical" violet IS becomes the "*implication* geometrical" porous black implication BR. A third and last remark with respect to our rereading of the two valuations of the tri-segment is that two more arrows, in fact, seem to emerge here: the "porous" red arrow AR and the "porous" yellow arrow RA. And this is more surprising: (1) they are parallel to (i.e., they share edges with), respectively, the *non*-arrow *negations* CN (paracomplete) and NS (paraconsistent). (2) Moreover, of all the thus possible five different kinds of implication arrows of the "implication geometry" tri-segment, these two last kinds (embodied each two times) seem rather strange, because sometimes they seem to go beyond the limits imposed by valuation; the AR, taken at face value (i.e., as an arrow), seems to lead to the strange implication (in terms of truth-values) "$1 \rightarrow 1/2$", and the RA seems to lead to the tantamount strange implication "$1/2 \rightarrow 0$". The simple, first explanation is that they are "restricted implications" (cf. [3]): moreover, they are like "water and fire", they join "R" with "A" (but in separated sheaf-levels U and X-U. (3) But for this reason, we could not clearly see them before (*which shows, again, the power and the usefulness of the Smessaertian "implication geometry"*). Remark that, as we mentioned (Sect. 4.2), in the tri-*triangle* we find relations RL and LR generating a similar "strange" issue. We will come back on this important point on Sect. 4.5.

A general final remark here, before going to the next Sect. 4.4, is that the tri-simplicial space confirms here, but also *radicalizes*, what has been seen by Smessaert in the bi-simplicial space, namely, that the relations of the "opposition geometry" and those of the "implication geometry" have partial overlaps (Sect. 4.5). This fact, as we are going to see, is – without exaggeration – hugely important (Sects. 4.5 and 4.6).

4.4 Overview of Some Inner Structures of the Implicative Tri-segment

Before summing up (with rather important consequences at stake, cf. Sects. 4.5 and 4.6), it will be useful to acquire some more understanding of the structure of the "implication geometry" version of the tri-segment (once more: we are working, however, by necessity with a restricted version of it, Sects. 4.1 and 5.1, and 5.2). In some sense, we will just try to repeat for it (quickly!) the kind of tentative categorization we proposed for the "opposition(al) geometry" version of the tri-segment in Sect. 3. We leave aside the kind of characterization in terms of "horizon" made in Sect. 3.1 (for many elements of it are already in Fig. 100 of Sect. 4.2). The question about the "inner circuits" (Sect. 3.2) of the "implication geometry" version of the tri-segment seems potentially interesting (here we will only mention it). The main idea seems to be that there are three main kinds of circuits, reflecting the three main kinds of "implication relations": bi-implication (containing *at least* a B), implication (containing *at least* an R), and nonimplication (containing *at least* an A) (Fig. 108).

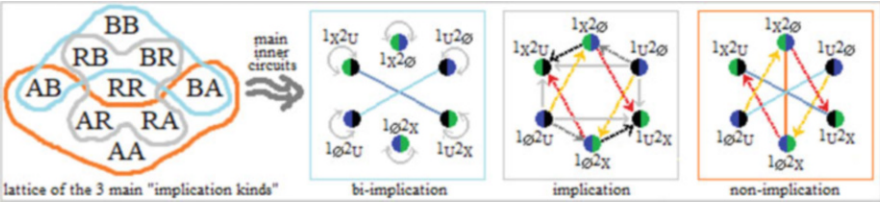


Fig. 108 The possible "inner circuits" of the "implication geometry" version of the tri-segment

A notable difference with the case of the opposition(al) inner circuits (to be reconsidered, then?) is that here, differently from there, the three inner circuits have systematic overlaps.

Another meaningful and potentially useful (although tentative) structural investigation is that consisting in looking for the hybrid "inner squares" (here: rectangles) and triangles of the implicative tri-segment. Again, our methodology here will be just to rely on what seen for the "opposition(al) geometry" of the tri-segment (Sect. 3.4). As for squares, for the same combinatorial reasons put forward in Sect. 3.4, there are here, qualitatively, 3 + 12 = 15 of them. The three main squares (in fact: rectangles!), the (i)–(iii), are very regular and even more mutually similar than what were the three *opposition(al)* rectangles (Fig. 70) (Fig. 109).

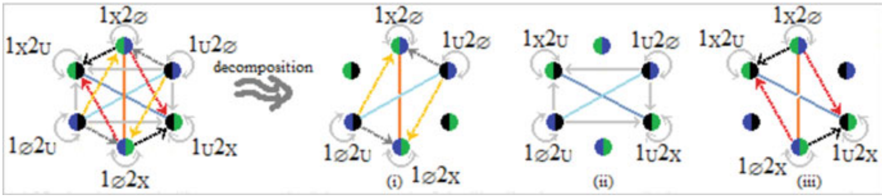


Fig. 109 The three main "inner rectangles" (or squares) of the "implication geometry" tri-segment are very regular

As for the other 12 squares, they seem to be less regular and more "hybrid" (cf. Sect. 3.4, Fig. 77), although their global combinatorial system is very regular (Fig. 110).

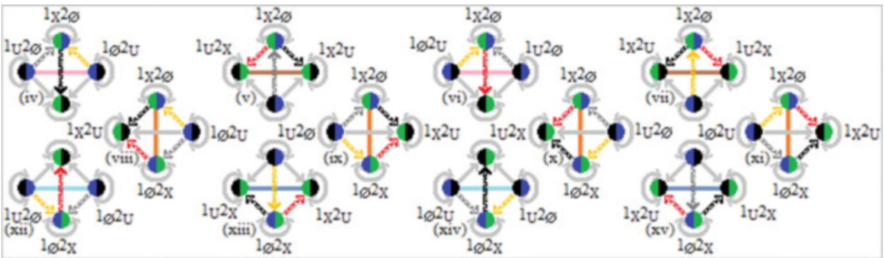


Fig. 110 The "implication geometry" version of the 12 "hybrid squares" (or tetrahedra) of the tri-segment

As for the “inner triangles”, for the same combinatorial reasons seen in the “opposition(al) geometry” tri-segment, there seem to be here 8 + 12 = 20 of them (cf. Sect. 3.4, Fig. 80). Here we only give their qualitative kinds, if needed, and the reader can easily reconstruct the rest, by referring to Sect. 3.4 (in fact the pairs of isomorphic triangles, like (a) and (g), being centrally symmetric, have inverted colors in the *readable* vertices) (Fig. 111).

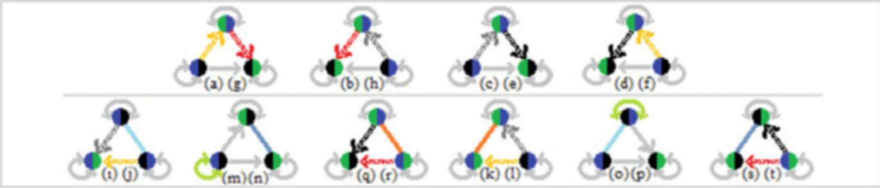


Fig. 111 The qualitative kinds of inner triangles of the “implication geometry” version of the tri-segment

En passant, a possible question is to verify how do look here the two “meaningful triangles” of Sect. 3.5, Fig. 82, recalling that these two triangles seemed meaningful to us since, as we argued, they play some kind of “semantic role” (on the “oppositional roles”, cf. Sect. 3.4): one triangle, namely, the (c), contains the three “affirmative meanings” (of the tri-segment), while the other, namely, the (e), contains the tri-segment’s three “negative meanings”. One sees that each of these two mutually isomorphic (and centrally symmetric) triangles is isomorphic to a “commutative (i.e., transitive) triangle” (*modulo*, however, the fact of having three *different* kinds of implication arrows, instead of a same kind). A general interesting feature expressed by them (and important for the tri-segment, Sect. 3.5, where in some sense we spoke *mistakenly* of “equivalences” or bidirectional light green CI and violet IS arrows) is the “strength order”, expressed by the three implication arrows, “paracomplete > classical > paraconsistent” (Fig. 112).

Fig. 112 “Qui peut le plus, peut le moins”: paracomplete → classical → paraconsistent in the two “semantic triangles”

So, we gained, as expected, at least the first elements of a basic understanding of this new structure (the “implicative tri-segment”). But then, if I am not mistaken (logical geometer friends will tell), we are faced now with a new main problem: some sort of “methodological schizophrenia” (or *“methodological dualism” of* *"logical geometry"*). In other terms, we seemingly paid quite much for obtaining our (very valuable and needed) "implication roadmap": the problem now is that of knowing what we want and/or can do with these *two* logical-geometrical "*twin* tri-segments". This means that there are at least two main possible choices (or issues) in front of us: either (i) accepting, as seem to be strongly suggesting the "logical geometers", as durable and methodologically justified the *separated, parallel twin existence* of the "opposition geometry" version and of the "implication geometry" version of the tri-segment (and this could be seen – at the level of powerful, conscious or unconscious, analogies and/or *fantasies* – as, in quantum physics, with Heisenberg's famous "uncertainty principle": a situation of *structural dualism* forever impossible to get rid of) or (ii) trying to resolutely *systematically combine* the two Smessaertian twin geometries, so to have a swifter, articulated but unique structure for the tri-segment, but also – more importantly – for any future poly-segment and poly-simplex (Sect. 4.6). But then how? Let us now try to see this point.

4.5 *Is an Aristotelian Tri-segment Possible? Yes! Meaningful? Very!*

What said at the end of the previous Sect. 4.4 is equivalent, as Smessaert and Demey in some sense have taught us these last 10 years (in their bi-simplicial-restricted logical-geometrical space), to asking the rhetorical question: "is an "*Aristotelian*" tri-segment possible?" (and their implied answer seems to me to be: "yes, but honestly ... "). Again, the question here means: is it possible to *combine usefully* these two Smessaertian sides of the tri-segment *without losing "logical-geometrical" properties*? (i) if yes: then it would be easier, but maybe even instructive, to use this combination, instead of the two "forever parallel and substantially disjoint sides" of logical geometry; (ii) if not, then we will have to use both, in parallel, without any hope of *finding again the pre-Smessaertian "lost paradise" of a unity of the geometry of oppositions*. The position of Smessaert and Demey, if I am not mistaken, seems to be the second: they consider, *in nuce*, "Aristotelian geometry" a little bit as "logical geometry for dummies". Being a notorious dummy, my position is – alas! – the first.

So, if we now nevertheless afford (as dummies – sorry dear hostage reader) the question of understanding more radically *the* tri-segment, in some sense by looking for its possible (still hypothetical) "Aristotelian" (or maybe Pascalian) version, what we need is a methodical comparison of its two twin *sub*-geometries. In fact, this means that *all* the possible kinds of *edge overlap*, with respect to the two sub-geometries, must be studied and known: and *named* .... The emergence of "Aristotelian simplifications", at this level of complexity, can only be discussed after that (and it is not yet granted *a priori*). So let us now develop some tentative comparative remarks on the two Smessaertian *sub*-geometries of the (*non*-Smessaertian) supposedly unique *Aristotelian* tri-segment.

For that, let us come back, first of all, to what happens, in the tri-segment, to the pink CN and the brown NS relations. And first of all remember that they are supposed to be *oppositionally* very meaningful: they are the tri-simplicial diffractions of the red segment of 2-oppositional (i.e., bi-simplicial) contradiction; they are, so to say, the *ratio essendi* of the *tri*-segment! We just created/discovered it, in thought, with the aim of exploring "contradiction's tri-simplicial diffractions" (Sect. 1.6). So, let us then concentrate on what really happened with them: for, retrospectively, without thematizing it, we were *de facto* surprised to see these two new contradiction kinds, the paracomplete CN and the paraconsistent NS, appear: (1) in the two nonclassical diagonals (this was very satisfactory!) (2) *but also* elsewhere, namely, in the "$1_{\varnothing}2_{U}$—$1_{X}2_{\varnothing}$" and "$1_{U}2_{X}$—$1_{X}2_{\varnothing}$" edges, etc. (this was more disturbing). But then, "implication geometry", under its never seen before *tri-simplicial version*, somehow rescues us, as a roadmap, even in this respect (Sects. 4.1, 4.2, 4.3, and 4.4), by teaching us, *implicitly*, if we now just *explicitly* think of it, that there are in fact, in the tri-*simplexes* in general and in the tri-*segment* in particular, (at least) two different kinds of CN relations and similarly two different kinds of NS relations! They are (i) the CN (resp. the NS) segment *overlapping* with the BA (resp. the AB) diagonal segment (ii) *and* the two CN (resp. the two NS) segments *overlapping*, each one, with one RA (resp. one AR) segment. If one thinks of it, this just means that *"CN" (resp. "NS") means in fact not just one but two different things inside the Aristotelian tri-segment* taken not schizophrenically (Fig. 113).

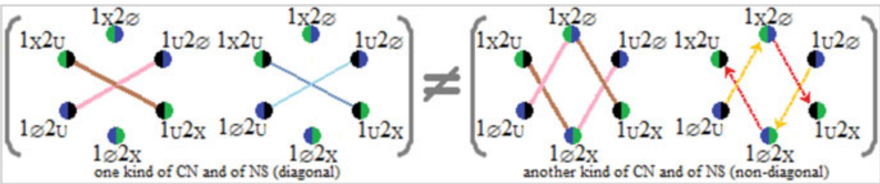


Fig. 113 In *the tri-segment globally taken* (i.e., Aristotelian!), there are, in fact, *two different kinds of CN (and NS)!*

So one can, and in fact must, (1) distinguish, *in the global (Aristotelian) tri-segment*, between (i) the "CN/BA" relation (one diagonal pink-light blue edge of the tri-segment, Fig. 113) and (ii) the "CN/RA" relation (two pink-porous yellow non-diagonal edges of the tri-segment, Fig. 113) (2) distinguish between (i') the "NS/AB" relation (one diagonal brown-ultramarine edge of the tri-segment, Fig. 113) and (ii') the "NS/AR" relation (two non-diagonal brown-porous red edges of the tri-segment, Fig. 113). Consequently, we propose to introduce here *the concept of "Aristotelian combination"* and to apply it in order to "produce" (or rather unfold) two different formal symbols for the two different kinds of CN relations, namely, the "CN/BA" and "CN/RA". For reasons to appear soon, we keep for

CN/BA just the starting representation, unchanged, of CN (i.e., a pink segment) but adopt for CN/RA a new porous *pink* arrow (porous will mean here "beware, this arrow here is *sui generis*") (Fig. 114).

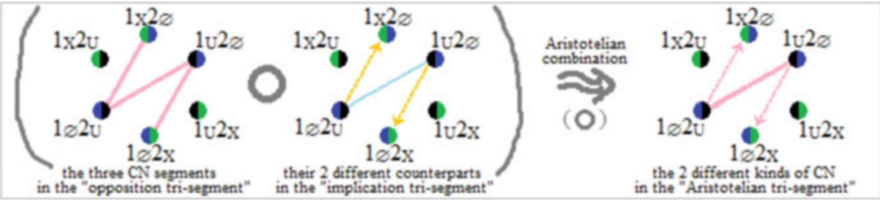


Fig. 114 Toward *the* "Aristotelian tri-segment": the *Aristotelian* composed relations "CN/BA" and "CN/RA"

Similarly, by another "Aristotelian combination", we introduce now two different forms of "NS" relations, namely, the "NS/AB" and the "NS/AR" relations. For reasons to appear soon, we just keep for NS/AB the starting representation of NS (a brown segment) but adopt for NS/AR a new porous *brown* arrow (porous will mean here as well "beware, this arrow here is *sui generis*") (Fig. 115).

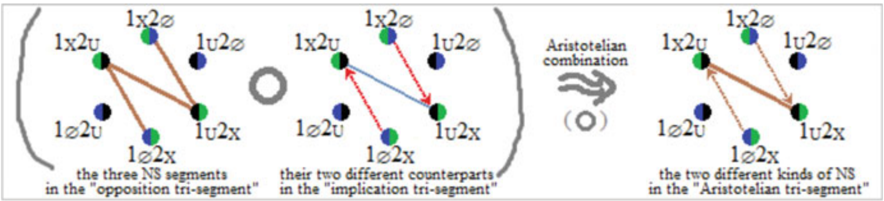


Fig. 115 Toward *the* "Aristotelian tri-segment": the *Aristotelian* composed relations "NS/AB" and "NS/AR"

Second, let us now turn, in our refreshing Aristotelian poly-simplicial dummies-journey, to the CS relation which was our main source of puzzlement about implications (Sect. 3.6, Fig. 88). If one considers the black "opposition geometry" relation CS, which is a *non-arrow* relation, as being directly "challenged" by the "implication geometry" *arrow* relation RR (which in fact seems to be none other than classical implication itself), we see that the two occupy exactly the same four edges of the tri-segment, and this is, again, the first reason of the "buzz" we did with "valuation", starting from Sect. 3.6, and which motivated the δεύτερος πλοῦς (second navigation) of this Sect. 4. In the tri-segment, things are effectively so, but *not so* in the tri-*triangle* (or higher tri-simplicial space)! There, there are two different kinds of CS and at least two different kinds of RR. Therefore, in the Aristotelian combination we operate now, we are better inspired in using, for expressing the relation CS/RR, *not* the classical gray RR arrow (which in the bi-simplicial space of Smessaert is a I/R arrow and in the tri-simplicial space of the tri-*triangle* is a II/RR arrow) but rather a suited new *black* arrow, keeping for it a combination of the arrow *shape* of the RR and of the black *color* of the CS (Fig. 116).

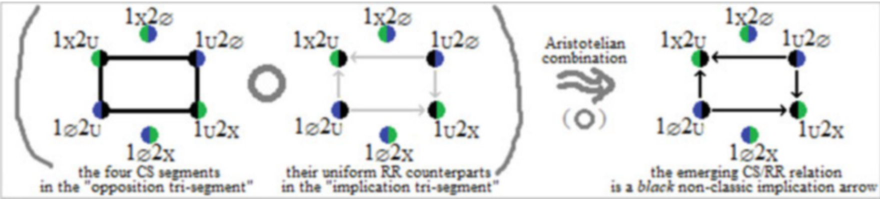


Fig. 116 Toward *the* “Aristotelian tri-segment”: the *Aristotelian* composed relation “CS/RR” and its *black* arrow

Third, still dealing with implication arrows, let us now turn to the “implication geometry” BR and RB, porous black and porous gray arrows (here porous means nothing special). They seem quite interesting: they seem to be compatible with the properties stressed by the “opposition geometry” segments CI and IS (paracompleteness and paraconsistency, one sees this in their two valuations) but express, as for them (i.e., in their “implication geometry”), full-fledged implications. So we operate here a further Aristotelian combination, generating, respectively, the CI/RB and the IS/BR new arrows: each will take the *color* of its “opposition geometry” component and the *arrow* shape (and direction!) of its “implication geometry” component (Fig. 117).

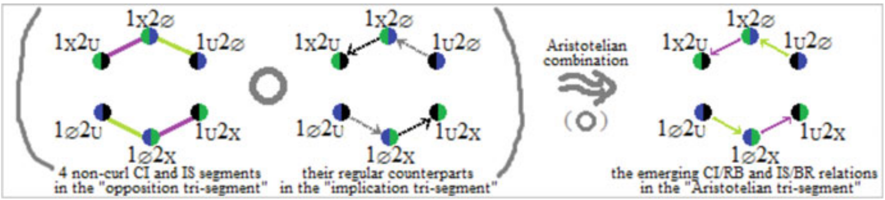


Fig. 117 Toward *the* “Aristotelian tri-segment”: the *Aristotelian* composed relations CI/RB and IS/BR

But here one must remark that if any RB (respectively, any BR) arrow of the tri-*segment* is strictly coupled, as for the edge of the tri-*segment* where it takes place, with a CI (respectively, an IS) relation, *this is not true the other way round*: the CI (respectively, the IS) can also happen to overlap, in form of curls, with a light gray BB curl. This leads us to the next point.

Fourth, in fact, one must remark here that the classical biconditional BB of tri-simplicial “implication geometry” takes place in all the six curls of the tri-segment. But then this seems in some sense quite under-informative with respect to the corresponding colored curls in the Smessaertian twin “opposition geometry”.

The gray classical self-implication, for sure a strong *logical* property, without any other indication, seems to be pretty tautological ("A ↔ A"), and, worse, it erases the oppositive colors, not distinguishing (and not letting distinguish) between the II/BB (the tri-simplicial counterpart of the bi-simplicial case), CI/BB, and IS/BB: it transforms these three into the same (tauto-)*logical* relation. This is no good from a *structuralist* point of view (think of Saussure, but also of Blanché). What must be unfolded and studied systematically is the *differential* (i.e., the structuralist, Saussurian "système des différences"), a.k.a. *oppositional* (. . .), *system* (Blanché's main work [33], where he presented *philosophically* in 1966 the logical hexagon – with its 1967 sequel [34] explicitly directed against the *logicists* and the illogicists – was titled *Structures intellectuelles. Essai sur l'organisation systématique des concepts*). So, being (*oppositionally*, if not more globally *philosophically*) "Aristotelian", we want to show all *different* relations *as different* (Aristotle: "Saying the truth consists in presenting as united what is united, and as separated what is separated", i.e., "truth as adequacy", cf. [5, 144]). So we operate here one more "Aristotelian combination", to the effect of which each curl of the Aristotelian tri-segment will take from its "implication geometry" side, BB, the *shape* (double-sided arrow), but keeping also something of its "opposition geometry" side, namely, the *color* of the nonclassical CI or IS counterparts – remark that we will keep for "II/BB" the classical light gray color of the bi-simplicial I/B (Fig. 118).

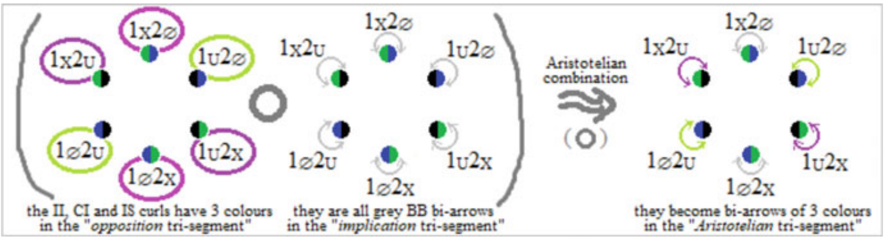


Fig. 118 Toward *the* "Aristotelian tri-segment": the *Aristotelian* composed curls II/BB, CI/BB, IS/BB

The last element to be dealt with, so to end our little journey, is the red classical diagonal of 2-oppositional contradiction (we saw already the two other diagonals, Figs. 114 and 115) which remains here, as NN/AA, exactly as it is in the bi-simplicial *Aristotelian* space, namely, N/A.

So, finally we arrive to something like a systematic "combination table", offering a global view of the 21 combined qualities of the 21 edges of the tri-segment. And this is, *pace Smessaert Demeyque*, the entry gates, if not to paradise (oy!), at least to what seems to be a full-fledged "*Aristotelian* tri-segment" (Fig. 119).

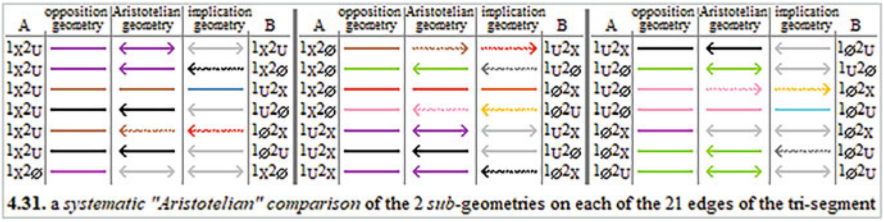


Fig. 119 A *systematic “Aristotelian” comparison* of the 2 *sub*-geometries on each of the 21 edges of the tri-segment

It seems we can therefore arrive, in full *conceptual* rigor (which is something orthogonal to the neo-Scholastic *furor axiomaticus* of the logicists), to an interesting, meaningful, and – most importantly – mathematically quite “natural” (instead of arbitrary, *ad hoc*, suboptimal, *bricolé*, etc.) “*Aristotelian* tri-segment” (Fig. 120).

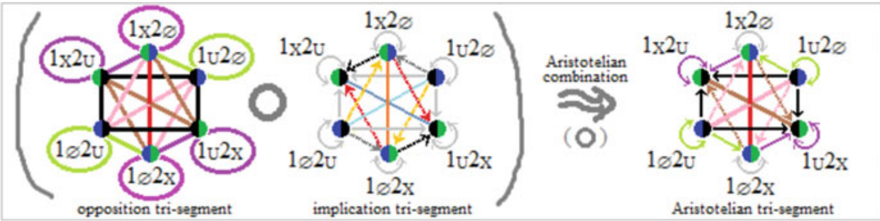


Fig. 120 Out of the two tri-segments there is, emerging, *a mathematically very natural “Aristotelian tri-segment”*!

In order to sum up (for we will spare the reader giving her/it/him a third instance (!) of inner analysis of this structure, we pretty successfully got to), we can have at least a look at the two valuations of this newborn, alive and kicking “Aristotelian tri-segment”, verifying visually that its truth-value relations seem fully reasonable (i.e., conform to the different constraints laid by its two Smessaertian *sub*-geometries and the *non*logical-geometrical composition they induce by it) (Fig. 121).

The final result of this, at this level of our inquiry, is that for *any* of the 21 edges (curls included) of the tri-triangle, it seems we could find a fully reasonable and meaningful combination of its two Smessaertian *sub*-geometries (i.e., the one which was first expounded in Sect. 2.6 and the one which was first expounded in Sect. 4.3), not forgetting that we are working in a fragment: we are considering $2 + 2 = 4$, instead of the total $3 + 6 = 9$, “Aristotelian” (on truth-value identity) and “Smessaertian” (on truth-value difference) meta-questions (Sects. 4.1, 5.1, and 5.2).

If we look now for some provisory, general condensed expression of the “Aristotelian combining methodology” we are *tentatively* proposing, we can consider something like the (informal!) following: (1) one has to play with (i) shapes

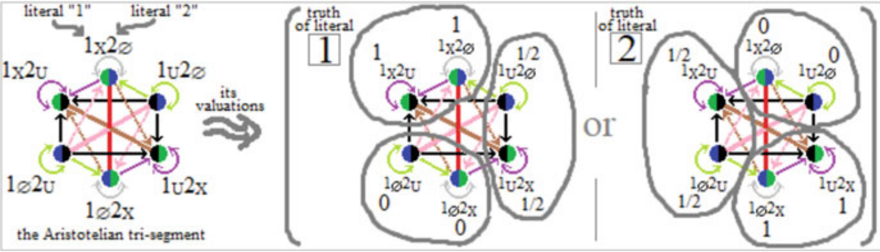


Fig. 121 The two valuations of the "Aristotelian geometry" tri-segment, articulating its two *sub*-geometries

and (ii) colors of any edge (under its expression in both Smessaertian twin and parallel *sub*-geometries); (2) one has to *combine* (rather than *substitute*!) the two versions of each edge, *creating*, for that, new formal arrangements respectful of the starting two not yet combined components; (3) the latter means, rather simply, that *one must tend to distinguish (or separate) what must be distinguished and to combine (or join) what must be combined* (again: the good old wisdom of the *Aristotelian* definition of "truth as adequacy"); (4) also, very importantly, one must prudently (and open to rearrangements) always keep in mind that the exploration of higher levels of the infinite, mathematically complex space of the poly-simplexes can induce unexpected, but meaningful, feedback effects (asking for wise and patient theoretical *re*arrangements), due to unexpected, but *mathematically natural*, theoretical *emergence phenomena* ([145], p. 19, point "d"); (5) the latter is *in line with Gödel's famous anti-formalist and anti-logicist discovery (of 1931,* [104]*) on the impossibility to rule once and forever mathematical serious things (*i.e.*, numbers and higher) by a fixed Russell-Whitehead-style axiomatics* (i.e., *the impossibility of the logicist dream* – for us scary – of a world reducible *de jure* and *de facto* to compositions of "0" and "1", cf. [68]), which is *compatible with a structuralist common sense (*i.e.*, remaining always open to the emergence of unexpected new forms of structure).*

But what we just saw in this Sect. 4.5 seems to be, if one now thinks of it, a small but nice enough *coup de théâtre*: Aristotelian geometry, as we just discovered and (*cum grano salis*) "proved", is not quite much a primitive (and "dummy"), imperfect version of "logical geometry", destined since 2011 to remain forever in the morn and dusty prescientific *past* (and shadows) of the now eternal light of the latter. Rather, "Aristotelian geometry" is, provided it is worked out *properly* (i.e., methodically, i.e., *in primis* with a mathematical – Platonic! – open-minded eye on the poly-simplexes!) and bottom-up (with a stress on "up"!), potentially (if *all* possible "Smessaertian" implication-questions are integrated! Cf. Sect. 4.1) the appealing *mathematical closure* of what Smessaert and Demey call "logical geometry"®. Let us now try to focus, before closing this Sect. 4, on this important question.

4.6 "Logical Geometry" or "Poly-Simplicial Oppositional Geometry"?

Smessaert's discovery, in 2011, of the parallel existence of two geometries, is a major discovery in our common field (be it called "oppositional geometry", "logical geometry", or something else). Our present study (Sect. 4) confirms it, if needed, and at a level of analysis that Smessaert and Demey themselves so far seemingly did not consider: that of the "poly-simplexes". Our present study, notably, seems to confirm Smessaert's and Demey's reasoning to the effect of which the classical "geometry of oppositions" (which *in some sense* contains among others Angot-Pellissier's and myself's "oppositional geometry"), which they call "Aristotelian geometry", is indeed a hybrid mixture of what they call the "opposition geometry" and the "implication geometry" (ch.4). But the *sense* in which such a "hybrid" must be understood is now to be discussed, for it appears to be possibly rather different from what Smessaert and Demey think and teach.

As I understand it, Smessaert and Demey claim that the "Aristotelian" approach to oppositions has mixed (without knowing it) two fundamental geometries (discovered 2.400 years later by Smessaert) and that *any* current researchers developing "avatars" of the traditional Aristotelian structures (such as our own "opposition*al* geometry", with its theory of the *bi*- and of the *poly*-simplexes) keep making the same old "Aristotelian mistake": they do not realize that, truly speaking, there are two rather different geometries at work and that the one relative to opposition (generally the one mainly investigated by naive researchers) is only one of the two. Now, "since 'logic' deals both with *opposition* (negation) and *implication*", Smessaert and Demey propose to baptize the global geometry emerging from their "opposition geometry" and their "implication geometry" with the name "*logical* geometry" (Fig. 122).

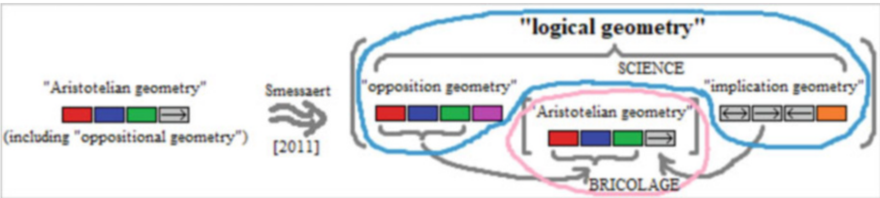


Fig. 122 "Logical geometry" explaining the *bricolage* of "Aristotelian geometry", in all its forms, past *and future*

Thus doing, however, Smessaert and Demey commit in my opinion four non-negligible mistakes: (1) they keep refusing (drastically) to take into account the importance of the presence of "bi-simplexes" (and the *geometrical* consequences this entails) in the "geometry of oppositions" (the refinement goes so far that they honor me writing papers, like [50], partly on my 2004 "*bi-tetrahedron*", "logical cube", i.e., the A4, but always calling it "Moretti's *octagon*"!); (2) they underestimate the idea that from the concept of (oppositional) *bi*-simplex emerges very naturally that of (oppositional) *poly*-simplex (and that therefore its investigations should be worth some attention, some support, or at very least some *mention*); (3) they misunderstand, in some (important!) sense, the nature of the fundamental relation between their own two geometries; (4) they try to impose *urbi et orbi – ESSLLI and JoLLI!* – as common name for the discipline of *anyone* dealing geometrically with oppositions and, as *the* real "scientific standard", the – alas – problematic label "logical geometry". These four non-negligible "mistakes", if I am not mistaken myself, are quite related. Let us try to see why.

A first important starting point is *the non-negligible "extra structure" imposed to oppositiona*l *geometry and/or logical geometry by the fact of going, as we went here, from the bi-simplicial space to the poly-simplicial space*. For, it reveals things (i.e., formal behaviors, mathematical regularities, structures) that seemingly were not so easy to perceive (and in fact seemingly were not perceived!) in the *bi*-simplicial space. But, the latter is – this point is capital and worth repetitions – *the* space where Smessaert and Demey (and therefore "logical geometry") so far remain, without however recognizing overtly that this space where "logical geometry" voluntarily remains is, in some important sense (a Pascalian sense!), a "bi-simplicial space". Why speaking of extra structure? Because our present study – although only the fragment of a future, more complete one (Sects. 4.1, 5.1, and 5.2) – dramatically unveiled that what we proposed to call the "Smessaertian" 3^2-lattice (a *structure*!) is, in fact, way more complex than its "logical-geometrical" ancestor, the official (and unique) Smessaertian 2^2-lattice. If our own simplification (with respect to R and L) is correct, the *simplified* Smessaertian 3^2-lattice at play for the tri-*segment*, as said (Sect. 4.1), is then the following not so simple extra structure (Fig. 123).

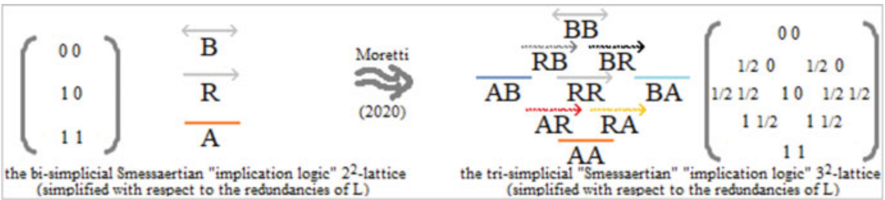


Fig. 123 The unexpected, and very meaningful, exponential growth visible in the "Smessaertian" 3^2-lattice

What one must notice here is that there is a quite big hiatus between the degree of combinatorial complexity of the two "implication geometry" structures (i.e., the 2^2-lattice and the 3^2-lattice): not only we step from three to nine qualitative kinds (once the redundant L simplified), but we pass from a situation where the three kinds (B, R, A) were strictly distinct (2^2-lattice), to a situation where six among the nine kinds are *mixed* (3^2-lattice); in the two constitutive Angot-Pellissierian sheaf-levels (U and X) of the nine kinds of the 3^2-lattice can happen to be put, side by side, very heterogeneous relations (such as "A" and "B" in AB, etc.). This extra structure, which is not even yet the whole story (it does not yet encompass all the possible "Aristotelian" and "Smessaertian" questions, Sects. 4.1, 5.1, and 5.2), introduces already a big *qualitative* jump.

Now, the bi-simplicial 2^2-case (left side of Fig. 123) seemingly gave to Smessaert and Demey the misleading idea of a stability (and simplicity) of the "strange parallelism" of the two Smessaertian *twin* geometries, a strange parallelism which in some sense, at least currently, seems to be *the essence of the methodology of "logical geometry"* (Fig. 122) (to slightly nuance this: maybe Smessaert conceives somewhere the $2 + 2 = 4$ twin questions as a *whole*, dictated by a *global* "logical" combinatorics over the possible truth-values of A and B?). Stability means the same two geometries, more or less, show up always parallel, *always in the same qualitative proportions*, for any (bi-simplicial!) B*n*-structure: red segment B2, logical hexagon B3, logical tetrahexahedron (a.k.a. rhombic dodecahedron) B4, etc. Such an apparent stability (which is a stability of the two twin 2^2-lattices, the Aristotelian and the Smessaertian, both simple and unique!) seemingly suggested them, for short, that the two geometries have no deep relations (other than *mysterious* coexistence) and, most importantly, that it is potentially misleading to "unite" them, given that this happens by "surgery" (i.e., *by mutilation*, as, in fact, in the historically attested bi-simplicial "Aristotelian geometry", Sect. 2.2, Fig. 40 and, here, Fig. 122) into a unique one: for short, *bricolage* is of course useful and up to a certain extent tolerated, but suboptimal with respect to methodical (axiomatic!) science. Remark that this belief, I am ascribing them, in a (deceitful) stability is *apparently* not too harmful to them in so much their very rigorous and valuable study of many other phenomena gives them work enough (and, again, very valuable work). Still, it seems that something precious (Pascalian?) here thus dropped, at least momentarily, out of view (but not only for them: for anyone *following their logicist advice*), with potential harm for our entire discipline.

As we saw, the tri-simplicial case (of which the study is still at the very primordial beginning) reveals however (Sect. 4.5) (1) that *the "strange parallelism" of the twin geometries is not stable at all* (provided one does not stick, somewhat *geometry-blind*, to the *bi*-simplicial space) (2) and that, therefore, it is not the theoretical *terminus ad quem*, of "our common discipline", but rather the *terminus ab quo*: it is not its *all*-encompassing horizon, but just an exciting starting point!

But this, then, means quite much, speaking less abstractly: it means that the higher you go in the poly-simplexes, the more complex are, *there*, the (meaningful!) overlaps of the two Smessaertian twin *sub*-geometries, themselves more and more *geometrically* complex the way up in the higher poly-simplexes. One must stress this point: *both* geometries have, each, an exponentially growing complexity (unknown "by construction" to logical geometry), so that their *combination* (Sect. 4.5) has even more complexity (not only is it a growing series: it is a "*geometrically* growing" series!). This is quite important; it means that (1) the *combinations* of the two "parallel" Smessaertian twin *sub*-geometries seemingly are by no means "forever frozen" (as is the combinatorially morn "Aristotelian combination" proper to the bi-simplicial space, limited to putting right-implication, by "surgery", in the place of noncontradiction, Figs. 40 and 122); (2) *these combinations are, far from it, the general rule of the poly-simplicial space* (for understanding its combined *qualities*); and (3) more precisely, these combinations are the key for understanding the very nature of *each* possible edge (or curl) of *any* general opposition/implication *n*-dimensional polytope! For this reason, it seems to me that you cannot quite *understand* the (very important!) interplay of the "opposition geometry" with the "implication geometry" (preciously offered to us by logical geometry) by remaining in the bi-simplicial space (as until now seem to be doing voluntarily Smessaert and Demey). Because there, in the bi-simplicial space, this interplay not only does not change enough: it simply does not change at all! It remains "forever" at the level of the 2011 "Smessaertian 2^2-semantics" and of the "Smessaertian 2^2-lattice" (echoing the twin "Aristotelian" 2^2-semantics and 2^2-lattice). For this reason, very paradoxically (with respect to their otherwise quite impressive and valuable work), Smessaert and Demey's thinking about "combinations" seemingly has not changed much since its beginning (although they explore 1.000 and 1 varieties of *ad hoc*, suboptimal existing combinations and fragments, from many other – sometimes major, sometimes not – authors past or present, to which they give systematic conceptual and terminological order: but top-down!). For short, you do understand *what is really at stake with Smessaert's groundbreaking discovery* (of the twin geometries), seemingly, only in the poly-simplicial space (or in a structurally similar playground), where this interplay changes, and changes with an exponentially growing complexity!

But this, in turn, means in some sense that what Smessaert and Demey call "Aristotelian geometry" is in fact not, as they think, a *suboptimal ancestor* (because fruit of *unconscious bricolage*) of "logical geometry" (the latter being supposed to be the firm scientific standard of our general discipline) but – rather – the real thing to be studied from a mathematically serious (i.e., nonlogicist) viewpoint! Something like "Aristotelian geometry" (or any comparable equivalent name) appears, paradoxically, to be (Sect. 4.5) not the *limitation*, but the mathematical *limit* (in the powerful, positive meaning of this word), or the "mathematical closure", of what Smessaert and Demey call a little bit recklessly "*logical* geometry" and by no means the other way round! In other terms, there seems to be, here, yet another non-negligible mistake (the third), in my opinion, in Smessaert and Demey's very admirable, but also terminologically dangerously normative program, namely, a confusion between (1) the idea of "*choosing without creating*" (as it seems to be, between the two geometries, in the *bi*-simplicial space) and (2) the idea (not yet clearly assessed by Smessaert and Demey, so far they do not dare enter *poly*-simpliciality) of "*combining methodically bottom-up*", at each poly-simplicial stage (this idea which they seem to miss so far is, on the contrary, in line with Béziau's – structuralist – fertile and open idea of "universal logic", as an intended parallel with the *structuralist* idea, preoccupied of thinking about mathematical "mother structures", of "universal algebra" – "universal" being in the programs of both, universal algebra and universal logic, the fruit of *systematic infinite variations and combinations* – a powerful idea of Bernard Bolzano, cf. [36, 37, 84, 129], and Sect. 5.5). Remark, again, that Smessaert and Demey produce an impressive number of valuable studies over combinations, but so to say always top-down (i.e., *“logical geometry” clarifies with benevolence “suboptimal” materials*, mostly of the past) and not bottom-up, i.e., *not yet investigating new mathematical (oppositional!) spaces* (Fig. 124).

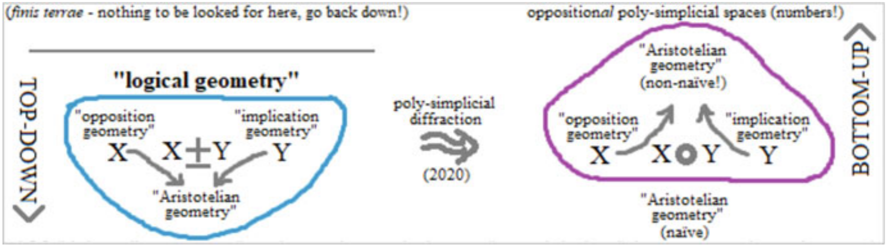


Fig. 124 “Aristotelian geometry”, understood correctly, is not a “merry bricolage”, but a systematic *combination!*

The mistake (the third) – if I am not mistaken myself – on the nature of “combining” the two geometries (grounded, I believe, on mistakes – the first and the second – on the bi- and poly-simplicial nature of the core of the theory of *opposition*, Sect. 1.5) introduces the fourth and last of the non-negligible mistakes we suggested to consider at the beginning of this section: that on the name to be given to our common theory. Naming it “*logical* geometry” is, for the experimented (if not employed) *general* philosopher I am, at least “surprising”, if not clearly shocking. Remember that it has taken a *very* long time before something like the idea of a decent *geometry* of oppositions could be born (around 2004, [93]): and this is, without exaggeration, a major conceptual revolution, since it means that *“opposition”, a very intuitive concept, common to almost all known human cultures, enters thus, unexpectedly, a mathematical legality: opposition is becoming, under our eyes, a mathematical object of its own (and a rich one)*. But one must be aware that historically at least two important thinking schools notoriously refuse(d) the simple possibility of the emergence of such a kind of revolution (for both already fought, mercilessly, against structuralism, which was very close – notably with Blanché [33, 34] – to unfolding such “elementary *structures* of opposition”): these are “dialectics” (Sect. 5.5) and the ambassador of logicism, “analytical philosophy”. The latter did and does it on ground of its founding (and *never* dying) constitutional *logicism*: “things must be reduced to ‘logic,’ and logic is *the* tool, *the* structure, and *the* key (even of mathematics)”. The sterility and harmfulness, for mathematical research, of logicist extremism (a pleonasm) should, in principle, not have to be proven, again and again, in 2020 (Zalamea’s *Synthetic Philosophy of Contemporary Mathematics* of 2009 [146] is, if one looks for one, a masterpiece of intelligence in explaining, painstakingly, the seriousness of this point – but cf. also Mandelbrot [88]). But so it is not: as is known, “analytical philosophy”, which (in good company with the comparable deliria of Hegelianism, positivism, Marxism, phenomenology, and many other past and future) has pretended (and still pretends!) being able to "make of philosophy a science" (!), despite being as such – beyond face lifts – a programmatic "dead horse" (we recall why in [97, 99], relying among others on [67] and [104]), still holds, by inertia and worldwide, an academic strong position of power (currently symbiotic with computer science and the growing *market* of the "smart technologies"), and as such it continues to push forward its "little gray soldiers" (a direct, nondiplomatic, but also non-flat, description of this can be found, among others, in J.-Y. Girard: [68–72]). But then is it also needed here to recall that analytical philosophy and logicism, far from developing themselves (when? how?) the *geometrical* study of "oppositions", hindered it times and times again, and by all means? Analytical philosophy produced rivers (no: oceans!) of ink about "logic", "contradiction", "negation", "implication", "tautology", "truth-values", "possible worlds", and the like. But, unless I am mistaken, one finds hardly *a single word* of Wittgenstein (& Co) about "contrariety" (which is the most characteristic concept of opposition – Sect. 1.6, Fig. 33 – and precisely *the* one requiring, for expressing *n*-contrariety and *n*-opposition, cf. Fig. 9, the concept of simplex!): and this is, paradigmatically, still the position of a Parsons, in the prestigious and "standard" *Stanford Encyclopedia of Philosophy* (Sect. 1.2); in his top-reference paper for the analytical philosophy world on the "logical square" [107], *the existence of the "logical hexagon" (1950) is not even mentioned . . .* (!). So, the problem with logicism is not only "ideological" in a general sense (it hinders "clumsily" the unfolding of fruitful new mathematical – and philosophical! – ideas, cf. Girard [68–72], Mandelbrot [88], and Zalamea [146]) but also in a very concrete sense: in the precise case of oppositional matters, it has proven, very specifically, times and times again, that different forms of logicism have voluntarily "killed in the egg", in a reflex of ("institutional") self-defense, several promising, embryonal developments of the *geometry* of oppositions (cf. [97, 99]). Is it necessary to recall that analytical philosophers have been dismissing (and urging others to dismiss) for more than a century, relentlessly, the square of opposition (Sect. 1.1) – and in more recent times the logical hexagon – notably because of the alleged "paradoxes of existential import" (realizing only in 2013, cf. [43], that this paradox is in fact a pseudo-paradox), which is a mathematical bad joke: *judging normatively mathematical creativity (for, here we are) from the viewpoint of logic, and not the other way round (*i.e.*, judging mathematical logic for its mathematical creativity), is the world upside-down, a very bad joke, historically attested (and persisting), but still devastating*.

For these reasons, and because of its very dubious name and, if it does not change, because of what seems to be its main methodology of "schizophrenic" frozen parallelism of the two twin (micro!) geometries with respect to bottom-up pure geometrical exploration, logical geometry, *nolens volens*, and despite the impressive, increasing crop of its valuable scriptural and conceptual productions in the best journals, runs very seriously the risk of becoming one more logicist machine for killing the radicalness of the emergence of a full-fledged "*oppositional* geometry" (or, if one prefers, of a "geometry of oppositions"). Again, the paradox to be understood, and defeated, is that what Smessaert and Demey call, seemingly with soft irony, "*Aristotelian* geometry" (as meaning "inferior to *logical* geometry") is in fact not a stupid ancestor (or a bizarre fossil of the past), but it is rather the higher-order mathematical methodology (but which possibly should not be mislabeled "Aristotelian" . . .) to be unfolded (through an exploration of the poly-simplicial space) and followed in the future! The future of our common discipline (the geometry of oppositions), at the level of the exploration of *deep mathematical still unknown ideas* if not at that of the "academic *Zeitgeist*", will very seemingly consist, despite all logicist efforts to refrain it, in exploring systematically the overlaps of, among others, the twin *sub*-geometries of the *geometry of oppositions* and in finding techniques for expressing bottom-up (stressing the "up"!) the *autonomous* reality of the *mathematical* (and not "logical") thing. So, in my opinion the question of the name of our general and/or global common theory is very important, and it remains problematically open, "poly-simplicial opposition*al* geometry" seeming so far a much better name.

Back to the successful (although still fragmentary, Sect. 4.1) tri-simplicial diffraction of the 2-opposition*al* red segment of contradiction, if one considers (as we take now the risk of doing here – readers will have to judge) that we seem to have solved satisfactorily the last general important technical question (the tri-segment's two possible global valuations and what these imply, Sects. 3.6, 4.3 and 4.5), it seems that we are now in a position for seriously considering (only sketchily, alas) the question of the possible concrete *applications* of the oppositional tri-segment and that of the consequences that this new possible mathematical structure (to be refined and brought to its closure in the future, Sect. 4.1) has on some well-known other issues related to "contradictions".

5 *Consequences/Applications* of the Tri-segment: Some Remarks

We will, at last, be in a position of making in this concluding Sect. 5 some quick remarks on future possible applications of the tri-segment (and similar poly-segments and tri-simplexes). The concept of contradiction/negation, as distinguished from that of contrariety, has the particularity of being very important in the "exact sciences", but it is also important, in some cases, in the humanities (where contrariety seems, however, much more important). The *oppositional-geometrical* diffraction of the concept of "contradiction", a concept which can happen to generate several misleading *fantasies*, formally speaking may concern, *in primis*, three particular disciplines – many-valued logics, paraconsistent logics, and quantum logics – in so far each of these three pretends to have a very special relation to contradiction/negation. We will try to recall the issues at stake here in Sects. 5.2, 5.3, and 5.4. As for the humanities, contradiction is more or less the focus (or high spot) of at least two among the few major thinking traditions of the last one or two centuries: dialectics and psychoanalysis. We will try to recall this as well, in Sects. 5.5 and 5.6, where we will end by a surprise in the very last pages (the cherry on our oppositional "partitioning cake"). But first we will propose in Sect. 5.1 some preliminary (and necessary) remarks on the general meaning and *limitations* of our present inquiry and on its results and perspectives.

5.1 *Some General Remarks on What Has Been Seen so Far in This Study*

The "Renaissance" of the geometry of oppositions is taking place, mainly thanks to the intellectual and institutional efforts of Jean-Yves Béziau, since nearly 20 years (Sects. 1.1 and 1.2, [28–30]). Since more than 10 years, it has been signaled (by us, [94]) that one of the main issues at stake with the geometry of oppositions (however you prefer to name it) seems to involve oppositional "poly-simplexes" (Sect. 1.3). But for several reasons (among which – but alas not only – a natural, if not glorious, "conceptual inertia"), this message has not been received so far. The present study should have, at least, succeeded, in principle, in making loud and clear that the complexity involved in such poly-simplexes, *a posteriori*, not only exists for real (mathematically speaking) but is in fact much higher than what we perceived and therefore believed in 2009. Far from being a confused fantasy of mine, the *oppositional* poly-simplexes do exist mathematically speaking and are very promising and exciting: and we have, at last, powerful and reliable tools for dealing systematically with them, *in primis* Angot-Pellissier's sheafing technique (Sect. 1.4, [3]). But the game is much more complex, technical and rich than it was perceivable at the beginning (2007–2009). This is first of all true of the global "Pascalian structure" of the general space of the poly-simplexes (Sect. 1.5): this means that, unexpectedly, the poly-simplexes are in fact, so to say, poly-*bi*-simplexes, and this involves that there is much more "structure" than what was thought; for instance, in a tri-simplex, the three main composing bi-simplexes (whatever the *dimension* of the simplex under examination) are such that each involves its own particular instance of simplex for any of the two bi-simplicial *colors* it has, e.g., the "blue simplex" of the "blue and green" bi-simplex is not the same as the "blue simplex" of the "blue and black" bi-simplex, etc. – this is directly readable in the Pascalian roadmap, Sect. 1.5 (provided one learned how to read it), but becomes more concretely clear when one analyses, by humble and down-to-earth calculations, the tri-triangle (or higher). Remark that the natural way for coping with this (i.e., the unexpected "diffraction" of the simplexes) consists, first of all, in introducing a new element in our convention for coloring the vertices (Sect. 2.5): namely, one for coloring, with "oppositional hostages" (Sect. 3.6) inside, the vertices of the *simplexes* (the same technique can represent either the real vertices of a precise simplex or, as in Fig. 125, whole generic simplexes taken as a dimensionally undetermined whole) (Fig. 125).

But this increase in structure is also true relatively to our discovery, made in Sect. 4, that Smessaert and Demey's "logical geometry" intervenes *in* the poly-simplexes and in a way much more rich and complex than what we believed (Sect. 2.2): a

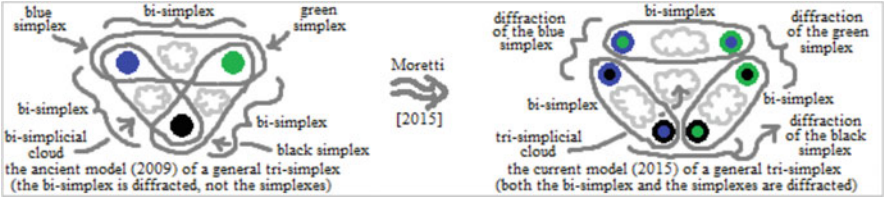


Fig. 125 The poly-simplexes behave as poly-*bi*-simplexes: each simplex becomes "diffracted" by "its" bi-simplexes

way that forced us (Sect. 4.1), from now on, to adopt logical geometry as a very important *part* of oppositional poly-simplicial geometry, namely, as generating one of its two systematic *sub*-geometries. In that respect, our choice in this paper of limiting ourselves to the most "primitive" case of poly-simplex ("poly" ≥3), that of the tri-segment (Sect. 1.5, Fig. 29), *a posteriori* revealed to be a rather wise but also fruitful move. For, the structure we investigated here, the tri-segment, is "simple" (it is a "three-cloud" deprived of simplexes, cf. Fig. 126), but far from trivial, and at least we obtained a rather clear, global understanding of it and, through it, a *starting* global understanding of the more general concept of *tri*-simplex and of poly-*segment* (Fig. 126).

Fig. 126 The "tri-simplicial diffraction" of the red contradiction segment is a "tri-cloud" (having no simplexes)

So this choice paid in at least two respects: (1) we proved the existence (Sect. 2.6) of this until now unknown structure (conjectured by me in 2009, Sect. 1.6), the tri-segment, which is very important (for, it is *the very first full-fledged poly-simplex* and it is *the first mathematical diffraction of contradiction*); and (2) through it we made at least four rather important discoveries about: (i) what must be considered as "oppositionally extremum" (i.e., each n-simplex has not two, but n extrema!) (Sect. 2.4), (ii) how must be treated mathematically, through colors, the *vertices* of general oppositional-geometrical solids (and this is nothing less, if you think of it, than the embryo of a new chapter of graph theory!) (Sect. 2.5), (iii) how to deal successfully, for any poly-simplex, with its "valuations" (which is possibly the embrio of a new chapter of many-valued logics) (Sect. 3.6), and (iv) how must be developed systematically, from its two Smessaertian *sub*-geometries, with the concept of "Aristotelian combination" and against the trap which is laid by the misleading name (and program) of "*logical* geometry" (Sect. 4.6), something like a *non-naïve* "Aristotelian geometry" or more precisely a "poly-simplicial opposition*al* geometry" (Sect. 4.5, and this is possibly a new chapter of ... logical geometry!).

The oppositional poly-simplexes are a successful generalization of the oppositional bi-simplexes (including, this is very important, their closures, the B*n*-structures), which, as we recalled (having learned it first from Angot-Pellissier [111]), are quite important *new* mathematical tools (Sect. 1.2): the oppositional B*n*-structures, generating new kinds – *oppositional* kinds! – of "equivalence classes", allow us naming, measuring, and thus thinking about "oppositional complexity", and they make of "opposition" a new mathematical *object*. The *poly*-simplexes, therefore, should enable us, from now on, to extend this conceptual and formal mathematical new power, relatively to new situations where "oppositional valuations", finer-grained than two-valued, will be needed (and this is what we will try to overview in the next Sects. 5.2, 5.3, 5.4, 5.5 and 5.6).

However, at least three further considerations must now be added to this. First, one important point to be remembered is that, so far, we nevertheless remained strictly *inside* the Aristotelian (and Smessaertian) p^2-lattices (the stress here is on the exponent "2"): this means that the "q" parameter (i.e., the number of meta-*questions*) of the general p^q-lattices (Sect. 1.3) is not yet being explored. The reason is that we still seem to lack, for this, something like an analog of the Angot-Pellissierian successful formal techniques of 2008 and 2013 [3, 111]. It might be the case that $q \geq 3$ shows up impossible. But if it will turn out that the "q parameter" (i.e., $q \geq 3$) corresponds to something mathematically real (as I still believe at the moment, given the promising results found in several draft preliminary investigations), than it seems that, at least in principle, it will be necessary to go patiently through its *a priori* non-easy, full-fledged exploration, (re-)reading *all* the p^2- poly-simplexes one by one (exposed to the risk of a combinatorial explosion ...), in order to get important, still missing insights about the profound meaning of the complex concept of oppositional poly-simplicial space (Fig. 127).

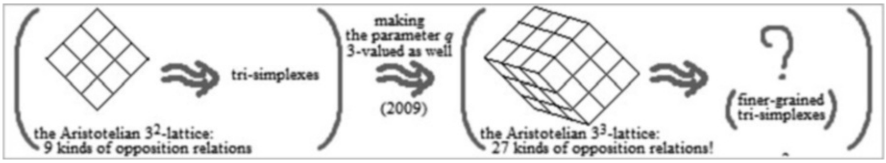


Fig. 127 From the Aristotelian (and Smessaertian) 3^2-lattice to the Aristotelian (and Smessaertian?) 3^3-lattice?

A second important point is that oppositional geometry, as we said (and as we gave further evidence for, in Sect. 1.5, but also in Sects. 2.5 and 4.3, Fig. 106), is at the crossroad of different mathematical distant "spaces". As such, its progressive, deep understanding is conditioned by future works in mathematical directions potentially quite heterogeneous and, generally, rather unpredictable. Remark, in that respect, that the theory of the poly-simplexes, although it belongs to "general mathematics" rather than to "mathematical logic" alone (Sects. 1.2 and 4.6), seems to offer interesting elements of natural conceptual radicalization to Béziau's theoretical framework (which in fact is *structuralist*, in the best sense of the word, since it looks for mathematical "mother structures", suggesting that "logic" might in fact be just *a* new one) of "universal logic" (this appears particularly in Sect. 4.5, with the notion of "Aristotelian *combination*" if it were, for instance, to be seen as a new case of "fusion").

Finally, a third important point, as we said (Sect. 1.6, Fig. 33), is that it must always be remembered here that the higher poly-simplexes will now *really* have to be explored methodically, starting from the tri-triangle (which opens the big jump into "real" poly-*simplexes*) and aiming at, as soon as possible, at least the tri-tetrahedron (so to obtain a first "poly-simplicial diffraction" of B4 (B4 being by now probably the most applied – and in that respect, *cum grano salis*, the most important – of the bi-simplicial oppositional-geometrical structures, Sect. 1.2, Fig. 6) in a way similar to the one successfully followed here for the tri-segment. Remark that we already have important and very encouraging unpublished results: (1) on higher poly-*segments* (quadri-segment and quinque-segment), the series of the poly-segments seems to offer a stable behavior and quite nice formal properties (and is the first "poly-simplicial *series*" which will be reachable soon enough), and (2) and on poly-triangles (we reached the closure of the tri-triangle and of the quadri-triangle, which are both *much* more complex than what appears in Angot-Pellissier's two precious, pioneer draft studies of 2013 [3] and 2014 [4] on the subject).

As said, in the following sections of this last Sect. 5 we are now going to try to give some quick hints and remarks on some possible applications of the tri-segment.

5.2 *The Tri-segment and Many-Valued Logics: Some Remarks*

As we have recalled (Sects. 1.3 and 1.4), the very idea of oppositional *poly*-simplex (of which the "geometry of oppositions" is a classical, *bi*-simplicial case) is strongly linked with the idea of *having more than two truth-values*, so the relations between the *general* theory of opposition and many-valued logics seem to be in some sense quite strong. And with respect to many-valued logics, the main results of the present study (as, for instance, the mathematical birth of the *Aristotelian* tri-segment (Sect. 4.5) but also the fundamental relevance of the Pascalian ND simplex, Sect. 1.5) seem indeed potentially important. The poly-simplexes, as we have seen, seem to open the direct and systematic study, up to now absent, of "many-valued *opposition*". However, as we have stressed several times, oppositions have shown up to be much more generally "mathematical" than specifically "logical" (Sect. 4.6). In this respect, our present inquiry seems to show that, given the now undeniable "Pascalian" side of opposition (Sect. 1.5), which puts forward not only *simplicial geometry* (the Pascalian ND *simplexes*) but also *arithmetic* (Pascal's *triangle*), well-known many-valued issues (e.g., the "MV-algebras", [20]) might have, because of the until now rather hidden or unnoticed presence in their heart of "opposition" (seemingly in terms of the, up to now, invisible *geometry* of the mutual relations of truth-values), to change something relatively to their "mathematical barycenter". For short, many-valued logic might be more "Pascalian" and/or "simplicial" than it was thought/known. In any case, it seems to me by now undisputable that the theory of the oppositional poly-simplexes offers *structure* to many-valued logics.

Currently the relations between many-valued logics and the poly-simplicial space seem, of course, to be at the very beginning of their possible exploration. On one side, precise oppositional-geometrical (poly-simplicial) studies on different known paradigmatic many-valued systems (like those of Łukasiewicz, Bochvar, Kleene ..., cf. [106, 117]) should be carried out step by step. But on the other, for that, more poly-simplexes should also have been studied in *abstracto* extensively, and in particular at least some poly-*triangles*: as we recalled (Sect. 1.6, Fig. 33), the expressive (and conceptual) power of *segments* is nontrivial, but nevertheless comparatively low (with respect to triangles and higher), and higher simplexes are seemingly absolutely needed for that, starting from triangles (which open to "contrariety", a very important feature, absent in the poly-*segments*). Possibly related to these considerations, it seems to be still a little bit too early for studying easily the presence (and the action) of *many-valued connectives* inside poly-simplicial oppositional geometry (by analogy with the important presence and action of two-valued connectives in and for *bi*-simplicial oppositional geometry, cf. Sect. 1.1, Fig. 1). This important basic work still has to be done. Notice however that some non-negligible elements of knowledge in that respect seem, nevertheless, to be already emerging at the basic level of the tri-segment.

One can hope or predict that similar studies will, from now on, be carried also in the direction of what seems to be the mathematical (infinite) horizon of many-valued logics, namely, fuzzy logic (i.e., infinite-valued logic, cf. [20, 38, 75, 106, 117]). Here, several researchers (like, for instance, F. Cavaliere [41], P. Murinová [103], or D. Dubois, H. Prade and A. Rico [59], to name some recent researchers) looking for bridges between fuzziness and oppositions have already proposed many different interesting strategies: but focusing on drastic "shortcuts" (for avoiding a lethal complexity explosion), they do not seem yet to have taken in due consideration the idea (of 2009) that "many-valued oppositions" are (seemingly) to be seen, *as a systematic whole*, as poly-simplexes (Sect. 1.3). Of course, oppositional geometry, in all its variants, seems (so far) committed to *finite* numbers, whereas fuzzy logic, as remembered, is essentially an *infinite*-valued logic. But this openness of opposition theory to finite many-valuedness, through the poly-simplexes, allows, at least in principle, studying their *numerical progressions* and therefore opens, through the concept of *potential* (if not yet actual) *infinite*, the discussion about the geometrical patterns of possible infinite limits of these progressions.

Let us stress, here as well, that an important, and maybe even crucial point on that respect (fuzziness), not to be forgotten, is the potential reference of the poly-simplexes to the parameter "q" (Sects. 1.3, 4.1, and 5.1) of the Aristotelian and "Smessaertian" p^q-semantics. It seems reasonable to think, for instance, that a *real* three-valuedness (i.e., a *radically* three-valued one) would be closer to an Aristotelian (and a Smessaertian?) 3^3-lattice than to 3^2- ones. This parameter “q” ($q \geq 3$) might open to a much finer-grained approach: in a 3^3-lattice (i.e., with $q = 3$), there seem to be 27 instead of only 9 kinds of “opposition (and – *cum grano salis* – implication) qualities” (cf. Fig. 127). So, even with respect to the “spirit of fuzzy logic” (i.e., the idea of getting the more fine-grained you can and finer-grained than “false/true”), an approach to oppositional geometry based on the 3^3-lattice, rather than the 3^2-lattice, would seem, intuitively, more natural and complete. But currently the 3^3-lattice is still being investigated as a *hypothesis*, with no robust founding results already at hand on that so far. As said in Sect. 5.1, what seems to be still lacking us – although several pieces of the “q puzzle” are already there and promising – is *something like an adequate new Angot-Pellissierian mathematical tool able to cope, at the meta-level (the level of the meta-questions, precisely), with truth-values other than 0 and 1*. In fact, a further problem is that the Aristotelian and the Smessaertian possible meta-questions happen to have *different progression rates*: the former deal with “truth-value similarity” (“A and B true together”, etc.), while the latter with “truth-value dissimilarity” (“A false while B true”, etc.). So, to give an example, in a three-valued context (tri-simplexes), there could/should be three Aristotelian meta-questions (our Q1 and Q2, plus the new Q3: “Can A and B be ½ together?”), but six Smessaertian questions (Smessaert’s Q’1 and Q’2, plus the following new four: Q’3 “Is it possible to have A 0 and B ½?”; Q’4 “Is it possible to have A ½ and B 0?”; Q’5 “Is it possible to have A ½ and B 1?”; Q’6 “Is it possible to have A 1 and B ½?”). So, the “Aristotelian” meta-questions (generating the “opposition geometries”) can really be modeled, as we proposed in 2009, by the hypercubic, or measure-polytopic, p^q-lattices (Sect. 1.3, Fig. 12). And, as said, this currently seems to lack a suited Angot-Pellissierian mathematical tool. But, independently from that, the *Smessaertian* meta-questions (generating the “implication geometries”) cannot be modeled as a whole (but maybe as parts?) by a p^q-lattice really parallel (i.e., with the same numerical values of p and q) to the Aristotelian one. The idea, put more clearly, is that intuitively the Aristotelian lattices should be p^p-lattices (same number of qualities of questions and of qualities of answers, given that both depend directly on the truth-values: the Aristotelian meta-questions are reflexive), whereas the Smessaertian lattices should be p^q-lattices where $q = p^2 - p$ (they embody the total number of possible nonredundant binary relations between p truth-values, i.e., p^2, minus the number of the reflexive ones, i.e., p, which are exactly the Aristotelian ones). The overall situation just described can be visualized (and in principle explained, by a simple combinatorial reasoning), again, with still one more instance of the series of the *simplexes* (here in their graph-theoretical suit of “complete graphs”, or “cliques”, cf. [125], p. 7), now interpreted in terms of possible binary relations between pairs of truth-values (Fig. 128).

The result is that if one aims at being “many-valuedly complete” (in the sense of being n-valued also in the meta-level – echoing, maybe, Suszko’s concern about the danger of a “fake many-valuedness”, [138]), this seems to cause a combinatorial explosiveness that might hang upon the researcher’s head like a sword of Damocles.

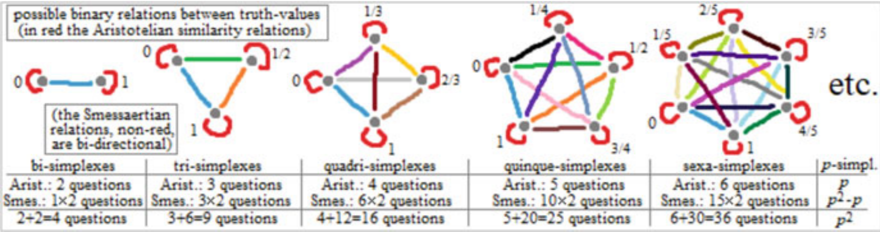


Fig. 128 The possible "meta-questions" conform to the graph-theoretical counterpart of the simplexes, the "cliques"

Again, a viable solution could maybe consist in decomposing the Smessaertian lattice into a multitude of more viable sub-lattices, supposing this is possible and meaningful. In that respect, what we have done in this study resembles a first step in between the current total absence of theory and a theory which could be complete (but at which combinatorial expensiveness' cost is still unclear).

Let us finally recall that, as we saw, tri-simplicial oppositional geometry (starting with the tri-segment) seems to show that there is an *intrinsic* deep link between three-valuedness (and higher!) and the metalogical triad "paracomplete, classical, paraconsistent" (Sects. 1.1 and 3.5, Fig. 81). And in fact, although we must stop here speaking directly about many-valued logics, in the following two sections, we are nevertheless going to have to look more specifically to particular cases of three-valuedness: in paraconsistent logics (Sect. 5.3) and in quantum logics (Sect. 5.4).

5.3 *The Tri-segment and Paraconsistent Logics: Some Remarks*

A second domain of the formal sciences where contradiction/negation is explicitly meant to be of the highest importance is, we have recalled, paraconsistent logics (programmatically "the mathematics of nontrivial self-*contradiction*", i.e., of nontrivial "A∧¬A", for some, but not all A, cf. [22]). And as for the latter, the relevance for it of the present study should already appear clearly in relation to our opening Sect. 1.1 (on the "Slater dispute"), as well as in relation to other comparable considerations we made all over the rest of our study (Sect. 3.5, Fig. 81). Remark that poly-simplicial oppositional geometry, notably its Angot-Pellissierian sheaf-theoretical version (Sect. 1.4, Figs. 19 and 20), seems to *deeply* confirm the rightness of Béziau's fundamental line of defense (2003, [24]) of paraconsistency against the rude charge of Slater (Sect. 1.1, and [132]). Oppositional geometry does it by rediscovering, over and over, the deep mutual relations of paracompleteness and paraconsistency (a.k.a. the relations between intuitionism and co-intuitionism, put into evidence, among others, by Béziau), seen as oppositional diffractions of "classicality": intuitionism being considered as mathematically fully natural (although not as mainstream as classicality), co-intuitionism (i.e., paraconsistency) should be as well (at the Slaterian price, however, of cleaning itself of fantasy elements). Inside (and outside) oppositional geometry, this is done by works like Angot-Pellissier's [1, 2], consisting in putting into precise link "topology" (a fundamental approach to "space" in which "distances" and "shapes rigidity" do not intervene) and (bi-simplicial as poly-simplicial) oppositional geometry, notably with his "topos construction" of tri-simplicial tri-valuedness (Sect. 1.4, Fig. 17 and Sect. 2.1, Fig. 34) and by recalling an old but profound idea (found, e.g., in V.A. Smirnov's [136] commenting N.A. Vasil'ev's [143] – Angot-Pellissier read a draft translation I made of it from Russian) according to which, fundamentally, paracompleteness is proper to any "*open* topology", while paraconsistency, its "dual", is proper to any "*closed* topology" ("open" means "*not possessing* its own frontier", while "closed" means "*comprising in itself* its own frontier"). The tri-segment we arrived at in this study, interestingly, seems to confirm and to summarize these very important ideas by means of the well-displayed interplay of its three diagonals (and of each of the three pairs of numerical sub-sheaves which are these diagonals' vertices): for short, each diagonal embodies one of the three kinds of this fundamental logical-mathematical "trio" (Fig. 129).

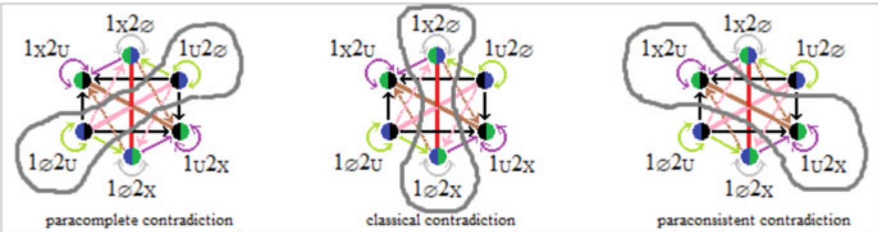


Fig. 129 The Aristotelian tri-segment as a visualization of paracompleteness, classicality and paraconsistency

This deep relation of the tri-segment to this important "metalogical" triad can be suggested more intuitively by combining graphically the two valuations of the Aristotelian tri-segment (Sect. 4.5, Fig. 121) with the symbolic (intuitive) expression we proposed of the concepts of gap and glut (Sect. 3.5, Fig. 81). One sees, then, vertex by vertex, what happens when the two valuations of the tri-segment, i.e., respectively, the supposed truth of the literal "1" and the supposed truth of the literal "2", switch the one into the other. By switching valuations, (i) classicality oscillates between true and false, (ii) whereas paracompleteness oscillates between gap and false, (iii) and paraconsistency oscillates between true and glut. *This seems to be, in some sense, one of the fundamental meanings of the tri-segment: by the tri-simplicial diffraction of the red segment of 2-oppositional contradiction it "opens" the classical concept of contradiction, seen as oscillation between true and false, adding to it two new different ways of oscillating, one paracomplete and the other paraconsistent* (Fig. 130).

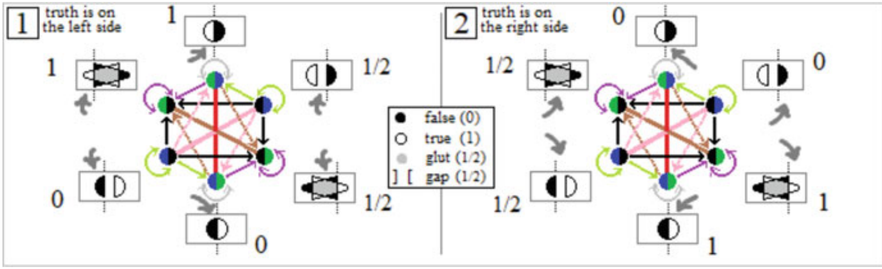


Fig. 130 The two possible valuations of the tri-segment (1 and 2) express the possible plays with "gaps" and "gluts"

This deep, natural link between intuitionism and paraconsistency, as said, had been studied among others by Béziau since years, but principally in modal logic (paradigmatically in [23–25]). He claimed in [23] (seemingly with reason!) that S5 (the "universal system", i.e., the most classical and standard system of modal logic), *very* paradoxically, can be seen as being already a full-fledged "paraconsistent logic" (!), since its apparently innocuous modal operator "◇¬" (traditionally read as "possibly not", equivalent to the negation of necessity, "¬□") in fact also expresses, unseen but real, the gist of "paraconsistent *negation*" (Sect. 1.1, Fig. 3). The emergence of other "demonstrations" of the same fundamental idea seems to appear transversally (throughout logics *and* mathematics) in at least five different domains (and we are probably missing, by ignorance, important others): (1) mathematical logic, (2) modal logic, (3) topology, (4) many-valued logic, (5) and, last but not least (given its high relevance for discussing "contradiction" as such), oppositional geometry. Remark, however, that in some sense *poly-simplicial* oppositional geometry seems even to add to these five domains (which comprise it as their fifth) a sixth domain: (6) for *one, and perhaps even more fundamental, new kind of line of defense of the idea of a mathematical naturalness of paraconsistency, is, I believe, the oppositional "Pascalian ND simplex" itself* (and therefore arithmetic?); inside the "Pascalian roadmap" for the oppositional poly-simplexes (Sect. 1.5), paraconsistency's naturalness becomes even *visible*, in its being, so to say, one of the "fractal branches" – in the sense of Sierpiński's gasket, which is correlated (in several ways) with Pascal's triangle (Sect. 1.5, Fig. 23, cf. also [109, 110]) – relative to the *possible contradiction kinds*, of the global fractal structure. It must be remarked that the fractality lies not only on the very numerical structure of Pascal's 2D triangle but also on the fact that this tri-simplicial behavior (clearly readable, for instance, in the 2D section of the Pascalian 3D simplex, Sect. 2.5, Fig. 53) can be complexified, *n*-simplicially, into infinite by the very simplicial constitutive structure of the Pascalian *ND* simplex (Fig. 131).

Thus doing, poly-simplicial oppositional geometry seems to confirm, over and over, (1) that paraconsistent logics in some sense *must not be overestimated* (by effect of the *dangerous power of the fantasy relative to having "nontrivial classical contradiction",* leading to unjustified fantasies of formal almightiness)

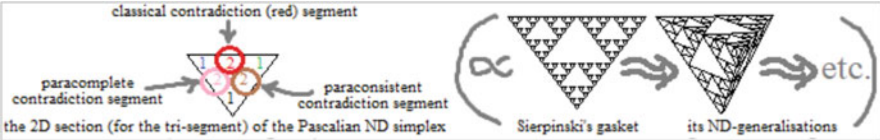


Fig. 131 The 2D section (for the tri-segment) of the Pascalian 3D simplex *shows* the "meta-logical trio" of negations

(this belief or ideal of nontrivial *classical* contradiction, Slater is right at least on this important point, would be a pure, misleading *fantasy*!) and (2) that paraconsistency is, however, indeed an important, natural, and by now even visible (!) feature of the rigorous mathematical approach, not only to negation but more generally (and deeply) to "opposition" (taken as a mathematical object, Sects. 1.2 and 4.6). In this respect, one should never forget, as (logicist) logicians tend to forget it almost "by construction", that in some important mathematical (Pascalian!) sense "contradiction" (i.e., negation) is a very meaningful but nevertheless *particular* case of "opposition" (i.e., not able to erase, or even simply dominate, the *numerical* and therefore Pascalian element of the global theoretical structure of "opposition", Sect. 1.5).

As said for many-valued logics (Sect. 5.2), it must be repeated here that future studies should also try to put at work, rereading humbly and patiently, step by step, "classical" systems and concepts of paraconsistent logic (such as those of Vasil'ev, Jaśkowski, da Costa, Belnap, Routley, Scotch, Batens, Priest, etc.), but this is not yet easy to realize, given also the already mentioned current inexistence of *full* studies of poly-triangles (i.e., published studies determining their oppositional *closure*). And this should also become partly easier than it currently can be, when something more will be understood and known about the seemingly fundamental, but still rather obscure and opaque, relations of poly-simplicial oppositional geometry and many-valued logics (Sect. 5.2).

Let us now turn to the last of the three *formal* approaches to contradiction we consider, one which seems to bear itself many-valued and paraconsistent aspects: quantum logics.

5.4 The Tri-segment and Quantum Logics: Some Remarks

A third domain of the formal sciences strongly interested by contradiction – we enter here, let it be clear, as an amateur – is "quantum logics" (QL). Since it is deeply rooted in "quantum mechanics" (QM), this also touches the question (otherwise left untouched by us in this study) of the importance of contradiction/negation (as different from contrariety) for the *experimental* natural sciences (like, for instance, interestingly, in biology, cf. Figs. 137 and 138 *infra*). As for quantum mechanics, that is microphysics, the problem is the following: this theory is said to be strange, and indeed it is, for it *seems* – as in another way, psychoanalysis (Sect. 5.6) – to challenge fundamentally, and not by choice, but by necessity, the *intuitive* very laws of logic. This is first of all, and notably, the case with "quantum leaps", that is the most elementary and small-scale known causal sequences, which appear to contain in them a strict and intractable *indeterminism* (although these quantic leaps take place in the formal framework – "Schrödinger's equation" – of the strictest statistical *determinism*). Many serious *theoretical*-physics proposals (i.e., nonexperimental, hard to test) have been done, inside physics, for coping with this rationally embarrassing mystery (i.e., the sudden irreversible loss of classical strict causality). This is the case with Everett's famous (but rigorous!) theory of the "parallel universes" (1957), also known as "many worlds (and/or many-minds) interpretation of QM" (on this, cf. [17, 18, 54, 141]), which saves strict causality, abolishes indeterminism, and makes mathematically more symmetrical and less *ad hoc* the von Neumann quantum axiomatics, but at the astonishing price of admitting that each micro-causal sequence (each quantic leap!) makes "split" the universe into two (or more) parallel universes (and this exponentially into infinite): the fractal bushy whole of these almost infinite fractal splits is called the "multiverse" (Fig. 132).

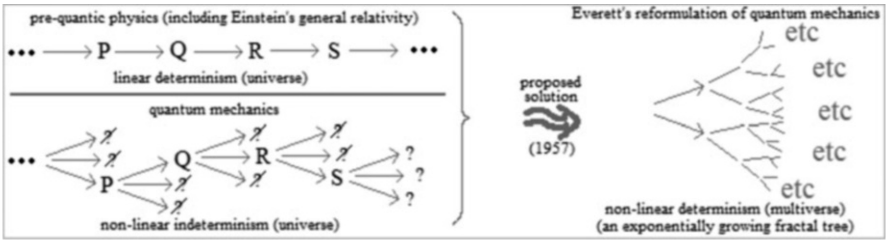


Fig. 132 Linear determinism, branching quantic indeterminism, fractal multi-linear (multi-versum) determinism

Quantic intuitive strangeness – mainly due to the scientific naturalization of some strange "intuitive *contradictions*" – remains in some sense not understood. But it is proven to be "real" (its predictions so far remain unchallenged) and useful in practice (it impacts reality technologically). But as such, by "weird" features like, for instance, "local reversals of time", or the spatial "non-locality", and the like (again, most of which, so to say, justify *intuitive* contradictions), QM becomes the support of many, less serious *fantasies*, generally aiming at justifying "magics" and paranormality (new age, esotericism, religion etc.), like in Jung [78] or Lupasco [86]. In some sense, these fantasies (justified or not) are related to the idea of a physical possibility of having "true *contradictions*" (the dangerous *fantasy* of some paraconsistent logicians: having "nontrivial *classical* contradictions", cf. Sects. 1.1 and 5.3).

With respect to the general theory of *oppositions* (i.e., considering not only contradiction but also contrariety), it must be remarked that microphysics in general is full of entities related with *contrarieties*: *anti*-particles, *anti*-matter, *anti*-energy, etc. One of the fathers of QM, Niels Bohr (1885–1962), is known for having, in that respect, made explicit *philosophical* reference to the Dao's "Yin-Yang" (notably relatively to his formal concept of "quantic *complementarity*"). Consequently, it seems that it would be interesting to look at QM with the new mathematical lens now offered, on "oppositions", by oppositional geometry.

In fact, long before the emergence of oppositional geometry, one way to cope with this lasting and resisting "illogical" strangeness of QM has been to develop, mathematically, something like "quantum logics", a.k.a. "QL" ([60]). Systems of QL have been proposed, at the beginning, by people like Birkhoff and von Neumann in 1936 [32], Destouches-Février in 1937 and 1951 [53], and Reichenbach in 1944 [120], and they are mostly three-valued logical systems (Łukasiewicz, one of the creators of many-valued logics, was among others motivated, by inventing three-valued logic, in modeling physical indeterminism, cf. [75, 106]). This invention/discovery of QL involved, notably, the theorization of new "truth-tables", suited for three-valued propositional connectives, among which are three-valued negations (Fig. 133).

negations			
p	V	F	A
N*p*	F	V	A
~*p*	F	V	V

$\&_c$	V	F	A
V	V	F	A
F	F	F	A
A	A	A	A

$\&_i$	V	F	A
V	A	A	A
F	A	A	A
A	A	A	A

◸	V	F	A
V	A	V	V
F	V	A	F
A	V	F	A

+	V	F	A
V	f	V	V
F	V	A	F
A	V	F	A

v_c	V	F	A
V	V	V	V
F	V	F	F
A	V	F	A

v_i	V	F	A
V	A	V	V
F	V	A	F
A	V	F	A

≡	V	F	A
V	V	F	F
F	F	V	F
A	F	F	V

Fig. 133 Some of the truth-tables of Destouches-Février's three-valued logic for quantum mechanics (1937, 1951)

As we saw, being *three-valued* logical systems, these early formal systems of QL are somehow related to tri-simplexes (Sect. 1.3). Our fresh knowledge of the basic features of the Aristotelian tri-segment (Sects. 2, 3, and 4), even without (the much needed and not yet available) knowledge of tri-triangles and tri-tetrahedra (Sect. 1.6, Fig. 33, Sect. 5.1), allows us, in principle, to try to analyze some features of quantum logic, at least those related to (three-valued) negation.

But QL, together with these rather simple, early three-valued propositional systems, has also resorted (notably with Birkhoff), more abstractly and powerfully, to the then new mathematics of "order" and "lattice theory" (Sect. 1.2, Fig. 7 – cf. [48, 147]): there have been investigations on *nonclassical order-theory* (with structures like "complemented ortholattices") aiming at coping with mathematically strange behaviors, as "non-distributivity" and the like (but there also are rivals to this, namely, mathematically more radical and powerful things, like "noncommutative geometries", cf. Girard [70] and Zalamea [146]). Remark that the lattices of QL, like "orthomodular lattices" and similar, are essentially nonstandard with respect to classical logic (it is precisely by this that they aim relentlessly – but difficultly, as heavily criticized in [70] – at capturing the strangeness of quantic "contradictions") (Fig. 134).

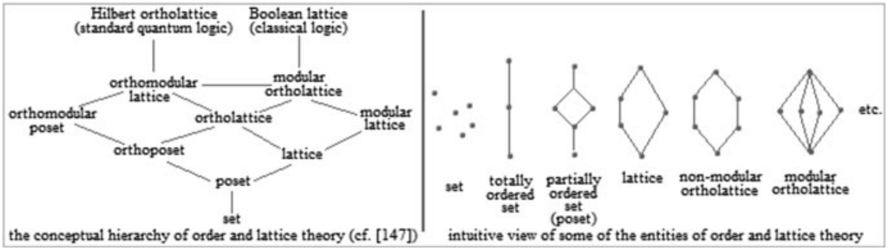


Fig. 134 The "order theory" and "lattice theory" turn of quantum logic

Currently, QL is also the basis of the hope to reach soon "quantic computers" and, through them, "quantic computations" (and, through the latter, "deep AI", cf. [76]), which also opens to some strong and astonishing *fantasies* (like the dream/nightmare of the allegedly imminent "singularity": the *almightiness* of "emergent" artificial ultra-intelligent *agents* . . .).

As said, a further idea is that of studying QM not only through three-valued logic and nonstandard lattices (QL) but also, maybe, through "oppositions". And this is also not entirely new. In the last years, some logicians and epistemologists (like Freytes, de Ronde, Bueno, and others: [19, 52, 66]) have been trying to use the "square of oppositions" (Sect. 1.1, Fig. 1) for inquiring the foundations of QM and of QL. But, *strangely enough, this has been done by them, again and again, without any reference to what is now really known about the structure "logical square"* (Sect. 1.2) – the (seemingly) logicist (Sect. 4.6) still have not understood that something mathematically serious is going on, *outside logic*, with "oppositions" (Sect. 1.5). So this line of researches seems at least suboptimal with respect (1), on one hand, to concepts like the oppositional closures of bi-simplicial *n*-opposition (and for a minimum, nonnegotiable start: the logical hexagon!) (2) and, on the other hand, if reference is done (as we just saw) to the use, by QL, of logical many-valuedness, to the (non) use (with respect to "*microphysical* oppositions") of *poly*-simplicial oppositional geometry (Sect. 1.3)! And, as it happens – as an intriguing example of this strangely underestimated line of thought we are arguing for here since some years – there is already notice (although still "unheard" until now) of at least one striking, possibly interesting similarity, still to be investigated and checked, between existing canonical formulations of QL (viz., those of Pavičić and Megill, cf. [91, 108]) and poly-simplicial oppositional geometry (Fig. 135).

classical implication	$a \to_0 b = a' \cup b$	(classical)
quantum implications	$a \to_1 b = a' \cup (a \cap b)$	(Sasaki)
	$a \to_2 b = b' \to_1 a'$	(Dishkant)
	$a \to_3 b = (a' \cap b) \cup (a' \cap b') \cup (a \to_1 b)$	(Kalmbach)
	$a \to_4 b = b' \to_3 a'$	(non-tollens)
	$a \to_5 b = (a \cap b) \cup (a' \cap b) \cup (a' \cap b')$	(relevance)

Fig. 135 The $1 + 5 = 6$ "implications kinds" mysteriously present in any orthomodular lattice (Pavičić and Megill)

In fact, the 1 + 5 kinds of "quantum implications", put forward by Pavičić and Megill, strongly remind us those, emerging as weakenings of the "II" relation of "opposition geometry", in the Aristotelian 4^2-lattice (i.e., in the quadri-simplexes, Sects. 1.3 and 1.4, Fig. 18) (Fig. 136).

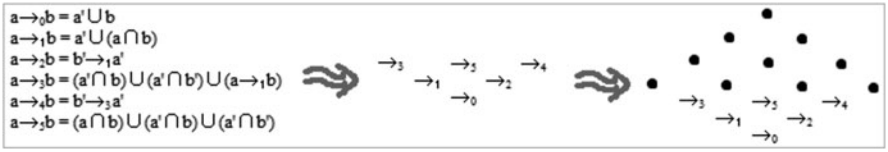


Fig. 136 The six mysterious implications of any OML seem to match quite well those of any oppositional *quadri*-simplex

Our present study, however, has clearly shown (ch. 4) that the complexity of the arrow system of an oppositio*nal*-geometrical poly-simplicial universe is in fact even higher than what shown by Aristotelian p^2-lattices *alone*: the twin Smessaertian p^2-lattices (for "implication geometry"), and maybe even more complex p^{p2-p} ones (Sect. 5.2, Fig. 128), are also needed to have a clear view. So it will be possible to seriously try to study this apparent correspondence between QL and poly-simplicial oppositional geometry (as we hope to do, or to see done, in future researches) only when will be inquired for themselves the *quadri*-simplexes (starting, soon enough, from the quadri-*segment*, which bottom-up unfolds the tri-segment fractally in a very elegant 3D polyhedron containing several interlaced tri-segments).

In order to try to have, nevertheless, at least a sketch of direct application of the tri-segment (and also a first application of it to biology), we can now try to rethink something of the famous thought-experiment of QM known as "Schrödinger's cat". This is a poor furry non-dog quadruped in a dangerous Austrian "black box", who is, paradoxically, provisory "dead AND alive" – because of quantic "superposition" – so long the strange quantic superposition inside the black box is not brutally abolished (quantum leap) by the intervention of an external *observer* of the black box, making a "measure" of what is inside it: a cat that, consequently, suddenly *becomes* either dead or alive, *only* immediately after this "quantic measure" has occurred – the quantic measure triggers (or not: mysteries of the quantum leaps) the opening, inside the box, of a cyanide flask. Now, in some sense a robust reflection on this seemingly requires, as a simplified standard oppositional-geometrical *starting model*: (1) either (standardly) a bi-simplicial *logical triangle* (B3) for opposing as *contraries* (and not as *contradictories*!) "dead", "alive", and "neither alive nor dead", then its tri-simplicial diffraction (the *tri*-triangle) could offer – maybe – some *starting* element of oppositional-geometrical further clarification of the quantic strangeness of the thought-experiment; (2) or (less standardly) a bi-simplicial *oppositional tetrahexahedron* (B4, the closure of the bi-tetrahedron A4) for combining, as orthogonal (and therefore freely combinable), "dead", "alive", and their respective negations, then its tri-simplicial diffraction (the tri-*tetrahedron*) could maybe offer some other *starting* element of clarification (Fig. 137).

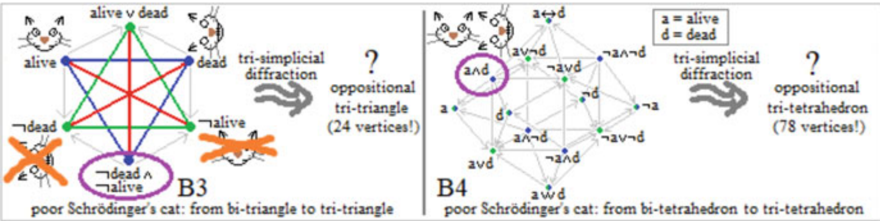


Fig. 137 A B3-structure and a B4-structure for "Schrödinger's (poor) cat"'s three or four possible existential states

Waiting for the tri-triangle (and, still later, for the tri-tetrahedron), the tri-segment can help us having at least some sort of (small and partial) "preview" of the tri-simplicial diffraction, thus investing the starting bi-simplicial oppositional description of Schrödinger's cat's conceptual experiment (but beware: the "tri-triangle", as we will show in another study, has not 6 but 24 nontrivial vertices! – cf. Sect. 1.5, Fig. 29) (Fig. 138).

Fig. 138 "Tri-simplicial diffraction", through the tri-segment, of some "contradictory halves" (!) of Schrödinger's cat

More generally, as already said (Sects. 5.1, 5.2, and 5.3), remaining at the level of three-valued logic, it seems likely that alternative, finer-grained oppositional configurations might appear by resorting not to the Aristotelian *and* the Smessaertian twin 3^2-lattices (as here) but to the Aristotelian *and* the Smessaertian "twin" 3^3-lattices (and, in fact, heavier, Sect. 5.2, Fig. 128).

Having tried to make some remarks on possible (future) applications of the tri-simplicial diffraction of contradiction (i.e., the Aristotelian tri-segment) to some of the formal sciences, les us try to have now a quick comparable look at a possible incidence on some of the humanities: dialectics and psychoanalysis.

5.5 *The Tri-segment and Hegelian-Marxian Dialectics: Some Remarks*

The famous concept of "dialectics" also bears strong "*fantasy* elements" with respect to "contradiction", which is the starting object of this study. Dialectics is in fact supposed (by its partisans) to be "the *science* of contradiction". With respect to all which should be said about "dialectics" – its theory, its history, its issues, etc. (for two good overviews cf. [74, 122]) – we will try to limit ourselves to some of the main points.

First of all, there *is* some interest in considering it, for dialectics (as a doctrine, as well as a symbol) still has some strong impact on reality (notably in politics, but more generally in contemporary philosophy). However, it is notoriously difficult to define properly dialectics: dialecticians themselves (starting from Hegel) justify this as being related to the very subject matter: (1) dialectics is the conceptual and ontological "engine" of everything (according to dialecticians), and (2) it has to do (allegedly) with *the most inner structure of "being" and of "becoming"* (of anything! be it concrete or abstract), and (3) its own structure, it is said, consists mainly in defeating dynamically any *concrete* "structure". Historically, there are essentially two such dialectics (leaving aside other important theories of dialectics, generally of a very different and less "dynamic" nature, e.g., Plato's dialectics [121] or Lautman's dialectics, [16, 80, 145]. First, there is a Hegelian (1770–1831) dialectics (appeared around 1804). Second, derived from it, there is a Marxian (1818–1883) dialectics (appeared around 1841). *The important point for us is that both claim to deal, in their "kernel", with oppositions and contradictions. And both claim to be superior to (in the sense of methodologically and ontologically "more fundamental than", and "irreducible to") mathematics.* This point is crucial: among current philosophers (and activists, etc.), dialectics is still a competitor to mathematics and nourishes, in its partisans, a deep disbelief for mathematics as a reliable source of inspiration for philosophy or action in general. Later (and still nowadays) dialectics has also been put into rivalry with mathematical *logic* – which was born after it, with Boole (1815–1864), around 1847 – but the result of this second confrontation remains rather unclear ([44, 58, 89]). Logicians and analytical philosophers, like Popper (1902–1994), claim to have "demonstrated" that dialectics is unsound. But dialecticians (the remaining few) claim, not without some rigor, that *logic cannot defeat dialectics (it cannot reach it, as a target)* and that this is because logic is a very primitive, too simple thing, in which dialectics just becomes unduly frozen (dialectics is supposed to be more intrinsically lively and fundamental).

In fact, logic cannot "hit" dialectics because, truly speaking, *dialectics (i.e. Hegelian and/or Marxian) should be put into critical comparison not much with logic but with oppositional geometry, for the latter, and not the former, is indeed the "science of opposition", if any* (Sects. 1.2, 4.6). Once this point understood and adopted, the main ulterior point to be seized is that, retrospectively, *dialectics is built on some clear inaugural, deep conceptual mistakes* (inside *philosophy*) about mathematics. This was substantially proven by the great mathematician (also a philosopher, [37]) Bernard Bolzano (1781–1848) – the real discoverer, before Cantor (1845–1918), of the mathematical thinkability of the "actual infinite" [36] and one of the founders, with Cauchy (1789–1857) and Weierstraß (1815–1897), of modern mathematical analysis [129]. But it was not perceived by many, and, very dramatically, not by Marx (most of Bolzano's writings appeared posthumous around 1929, long after his death in 1848). Bolzano demonstrated, by a crystal-clear reasoning, that Kant (1724–1804) made severe mistakes in his theory of "how mathematics can/must function", a crucial theory for his project of a "criticist philosophy", which led him to difficult points ("antinomies") in his "critique of pure reason" [84]. Hegel assumed uncritically the fundamental structure of Kant's philosophy of mathematics (assuming, by that, Kant's mistakes!) but claimed that the difficulties thematized by Kant as "antinomies of pure reason" had to be understood "the other way round": here Hegel used a *binary opposition*, just reversing it (this point is crucial and paradigmatic, cf. Fig. 139), so to suggest that contradictions were not the *bad end* of thought (Kant's antinomies), but its *good beginning*! (i.e. Hegel's dialectical philosophy). On top of this, and using Fichte's (1762–1814) astonishing philosophy of (i) opposition/contradiction and (ii) of the *creation* of the object by the (absolute) subject (!), Hegel elaborated his "logic", of which "dialectics" is the second of three moments (this ternary scheme is supposed to repeat into infinite, thus unfolding *any* reality, from nothingness to "absolute Spirit").

Fig. 139 "Diachronic" and "synchronic" dimensions in Hegel's "logic": the opposition in it is synchronically *binary*

Third, then, it must be understood that one of the fundamental problems with dialectics is that *current dialecticians (and already Hegel and Marx!) seem to have lost a precise intuition of the difference between "contradiction" and "contrariety"* [114]. This oblivion, quite common at the time of the birth of dialectics (and shared by schools of thought quite distant from dialectics, like analytical philosophy, phenomenology, and psychoanalysis – on the last cf. next Sect. 5.6), is conceptually deadly. Dialecticians – as, paradigmatically, Roy Bhaskar (1944–2014), cf. [31] – people obsessed by the concepts of "opposition" and "contradiction", make almost no mention of the "square of opposition", left for dead on the floor! (they discuss Priest's "dialetheism" [115, 116], but not the logical hexagon). But this confusion about contradiction and contrariety has a tremendous impact, lethal for the credibility of dialectics. On the one hand, it shows that *the dialecticians'* *"fundamental bet" (lost!) has been that opposition, at its root, had been theorized once and for all by Hegel (in turn based on the "firm rock" of Kant's theory of mathematics) and that no mathematical new insights on opposition might appear in the future* (Kant's similar tragicomic assertion, about logic admitting no major changes in the future, is famous). *Oppositional geometry, by its simple existence (Sect. 1.2) – and right now by the emergence of the tri-segment! – ruins this insane Hegelian view!* On the other hand, by teaching, since the discovery of the logical hexagon in 1950, that "binary *oppositions*" (i.e., *binary contrarieties*) do not exist (Sect. 1.1, Fig.2), oppositional geometry ruins *de facto* most dialectical reasonings of the past (and of the future!) which can be shown to be crucially grounded on such (alleged) *binary oppositions*: the problem being that *from a binary opposition you can deduce (as the dialecticians), by "reversal", things that you cannot from a ternary or higher contrariety*.

This becomes clearer if one realizes that, in fact, *nolens volens*, Hegelian logic bears, from a structuralist mathematically legitimate viewpoint (cf. [7–9, 112]), a "diachronic" and a "synchronic" dimension: there is, clearly, a diachronic succession of three moments, the second of which is "dialectics" properly said ([130]). But this second moment has also a fundamental "synchronic" dimension: it consists of an *opposition*, and this (Hegelian) opposition, "at some time" (think of it in terms of a mathematical "fixed point"), "exists". But in this necessarily "synchronic" dimension, dialectics commits a mistake with respect to oppositional geometry: it believes in *binary* oppositions, which it uses (by "reversing" them), whereas, as we recalled, *they do not exist* (and thus cannot be *deductively* "binarily reversed")! (Fig. 139).

This confusion appears, even nowadays, in the fact that "true dialecticians" and their admirers (Bhaskar, Ollman, etc.) speak commonly about "two things being in contradiction", whereas what happens is that their *contrariety* (a *fragment* of a larger, at least ternary one, Sect. 1.1, Fig. 2) *implies* two contradictions: each of the two contrary terms also *implies* the contradiction, i.e., the (*vague*!) negation, of the other.

What about the tri-segment, then? One of the most reputed attempts at formalizing Hegelian logic/dialectics, due to Rogowski in 1964 (cf. [124]), uses four-valued logics (the two extra truth-values stay, respectively, for "beginning to be" and "ending to be"): so, in order to see how the "oppositions" truly work in it, we would need *quadri*-simplexes (Sects. 1.3, 1.4, and 1.5). This is out of reach here (but not in the near future). What the oppositional tri-segment already shows, however, and decisively (Sects. 4.5 and 4.6), is, again, that the holy "pure contradiction" *segment* (so, even without speaking of the oppositional-geometrical bi-simplicial – and *a fortiori* poly-simplicial – "fireworks" of *n*-contrariety, starting with *triangles*) is already *mathematizable* and already leads (as draft studies on the quadri- and quinque-segments already establish) to unmistakable *mathematical, infinitely growing, highly structured complexity*, which, again, ruins the "*transcendental* flavor" of Hegel's alleged "*constant ternary* flowing of dialectics" (Fig. 140).

Similar remarks should be done, point by point, for Marx: in his own changes to dialectics (cf. [74, 105, 122]), which are real proposals, he offers new "patches"

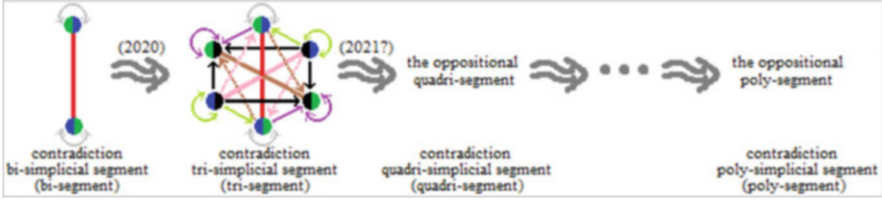


Fig. 140 "Contradiction" is a *mathematical* object: it ruins the Hegelian *anti-mathematical* fantasy on "dialectics"

which – at times – even contain bits of interesting mathematical premonitions (e.g., of catastrophe-theory, system-theory, category-theory, etc. [82, 123]). As such, they are valuable achievements of creative thought. But, they are nothing more than that, and by embedding them into *the deadly fantasy of a unique, almighty scheme, transcendental "dialectics", superior to mathematics*, Marx as well, as already Hegel (*pace* Lawvere), leaves, up to now, dialecticians and "Marxian orphans" blind and hostile to mathematical *non-unifiable* complexity (as intelligently explained, among others, by Badiou [11, 13] and by Mélès [92]).

Having made some minimal remarks on the possible impact of our successful tri-simplicial diffraction of contradiction (the tri-segment) over dialectics, let us now turn, in the last Sect. 5.6, to psychoanalysis.

5.6 *The Tri-segment and Psychoanalysis: Some Remarks*

Last element of our inquiry, the relation to "opposition" of "psychoanalysis" is strong, but complex. Here as well, we can only point to a few general remarks, which we will center on three main theoreticians of psychoanalysis: Freud, Lacan, and Matte Blanco.

Psychoanalysis is a study of the human mind based on the assumption of the existence of "unconscious" mental *dynamics* and "unconscious" mental *structures*, playing a *structuring* role at the level of "meanings" but also at the level of personal identity. From the viewpoint interesting us, the main contribution to this reflection made by Sigmund Freud (1856–1939), the first theoretician of psychoanalysis, can be decomposed critically in three main points. First, he discovers, and synthetically expounds in 1915 [64], that *considerable "negation/opposition problems" appear in the mysterious but real foundations of what he theorizes as the human "metapsychology",* generated by the existence of a quite strange (but observable) "unconscious" mind, of which the metapsychology aims at giving some kind of *conceptual* axiomatics. The existence of the unconscious is shown, convincingly, to be very important, both for explaining "normal mental life" (dreams, parapraxis, acting out, ordinary psychopathology, etc.) and for "mental pathology" properly said (psychosis, severe forms of neurosis, perversion, etc.). But the unconscious is shocking, and one of the five irreducible axioms of the metapsychology says: "In the unconscious, 'negation' seems to be not working". For Freud this is shocking, but at the same time real, and therefore desperately asking for an explanation (Freud himself will never succeed in finding a satisfactory one, as he will admit in his last, posthumous, and unfinished study [65]). Secondly, in order to explore the unconscious, Freud develops a powerful theory of the mental processes and notably, in some sense, a theory of the "mental oppositions" (i.e., "complexes"), both at the individual and at the collective (and historical) level (this results, globally, in a theory of individual and collective "psychogenesis"). This is based on the idea that "mental unity" is the emerging (fragile) property of a constant *process*, rather than a "transcendental", firm starting point (like in the philosophical theories of Kant, Hegel, or Husserl). As such, Freud's theory seems to be "realist" and rather powerful, for it seems to match clinical evidence (in fact Freud's starting point) and everyday life's experience, but at the price of introducing morally shocking elements (like infantile sexuality, constitutive bisexuality, "naturalness" of murder and rape *instincts* or *fantasies*, etc.). Third, however, Freud seemingly confounds at least partly (like most people in his time and most people even now!) "contradiction" (negation) and "contrariety" (i.e., opposition properly said): what Freud talks about when he speaks about "negation not working" (in the "unconscious") seems rather to be, in fact, opposition (which also implies negation) and more precisely "contrariety" (this seems quite clear in his 1910 "remarks on the *oppositions* in primitive languages", [63]). The two (contradiction and contrariety) are of course deeply related (and showing this is one of the main tasks of the square of opposition, Sect. 1.1, Fig. 1, and, thence, of all oppositional geometry, Sect. 1.2) but must not be confused. *As it seems, contemporary psychoanalytical theory, despite its interest for mathematical developments, still has not clarified this important inaugural non-negligible confusion of Freud*. Oppositional geometry in general brings some precious light precisely on this: the articulation and the possible confusion of contrariety and contradiction, at least in the sense that if, as I think after a careful examination of it, what Freud is speaking about – when he speaks, not lightheartedly, of "negation" – is in fact (also) "contrariety", than some important obscurities and difficulties of the theorization of metapsychology by him and by his school seem to disappear, and some new, robust research lines seem to emerge, promisingly enough.

This is more or less, implicitly, the research line of Ignacio Matte Blanco (1908–1995), the author of the part of contemporary psychoanalytical theory that seems to be most deeply (and most promisingly) related to the particular mathematical discoveries of *general* poly-simplicial oppositional geometry. Since 1975 (year of the publication of his *The Unconscious as Infinite Sets. An Essay in Bi-logic*, [90, 119]), the strictly remarkable (and largely underestimated) theory of Matte Blanco proposes concrete theoretical elements for trying to go methodically in this direction. Apparently more modest than the more famous and flamboyant Lacan (cf. *infra*, [55-57, 61]), Matte Blanco has in fact made some very important theoretical proposals, where he substantially claims (convincingly!) that he has unexpectedly (i.e., not too young: aged of 67 years!) solved, through a new sort of mathematical reasoning, the main five metapsychological problems left dramatically unresolved by Freud at his death and until then reputed unsolvable by Matte Blanco himself. For short, since 1975 Matte Blanco tries to draw our attention on the fact that there seem to be important structural links, of an unexpected abstract mathematical kind, between growingly strange but meaningful psychoanalytical oppositions and "symmetrizations" (as in dreams, psychopathology, or psychosis), and the unfolding of a mental kind of hyper-geometry. I must recall that the very invention/discovery of the concept of "oppositional bi-*simplex*" was much helped, in me, by some acquaintance I had with Matte Blanco's psychoanalytical theory of the mind, formulated in terms of geometrical *n*-dimensional *simplexes*. Matte Blanco pedagogically obliges his readers to get acquainted with simplexes of different dimensions and with strange phenomena possibly resulting from the mutual projections of (mental) geometrical spaces and objects of different (perceptual) dimensionality (Fig. 141).

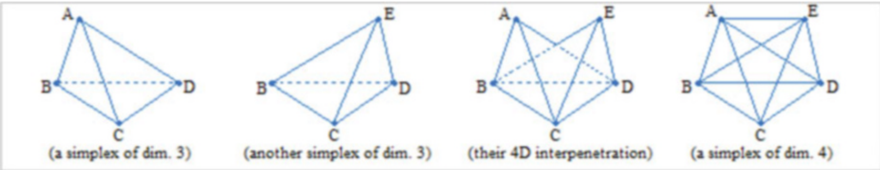


Fig. 141 Matte Blanco's study (*The Unconscious as Infinite Sets*, 1975) of the interplays of some nested simplexes

But since, as Freud, Matte Blanco seems to be much more committed with *contrariety* than with *contradiction*, we must leave an oppositional-geometrical poly-simplicial discussion of his theory (a theory of "bi-logic" and of the "bi-logical *structures*") for a context in which we will have more knowledge about poly-simplexes higher than poly-*segments* (this will start, again, with the tri-*triangle*). Since the tri-segment only deals with "contradiction", we cannot say much more here.

Fundamental metapsychological questions, similar to those explicitly left unresolved by Freud at his death in 1939, are faced with similar radicality by a third major theoretician of psychoanalysis, Jacques Lacan (1901–1981), who innovates mainly by constructing an operative link (through Saussurian differential linguistics) between Freudian investigations of the unconscious and the powerful structuralist interdisciplinary (and among others mathematical) paradigm and methodology [112]. Lacan's theoretical strength (beyond his stylistic "Gongorism", his constant provocations, and his still deranging histrionic outings), among others, seems to have been the deep understanding he progressively gained and defended (against reigning Hegelianism, for instance, but also against Marxism, Heideggerianism, and logicism – not to speak about "psychologisant" psychoanalysis that he, as well as Matte Blanco, fought relentlessly) that mathematics are so to say the most powerful, radical, and relevant key for investigating profound issues about the human mind (Lacan's concept for this is, *in fine*, the "Real", in his fundamental conceptual triad “R.S.I.”, articulating Real, Symbolic, and Imaginary, which *among others* is a powerful tool for *modeling* the complex morphogenesis of *fantasies*). But the mathematics put into play by Lacan – as “mathemes”: inspiring formal *images* – are such that they strongly diverge from the logicist program: they are *structure*-based (instead of *logic*-based or *deduction*-based), highly creative, and in some sense “nonstandard”. In order to model meaningful (mental) “reversals” (contradictions and contrarieties) of all kinds (including meaningful *self*-contradictions), Lacan thus resorts (also) to topological structures typically “strange”, like “Möbius’s stripe” (an open surface, in the 3D space, with only one side!), “Klein’s bottle” (a closed surface, in the 4D space, with only one side!), etc., that is mathematical *structures* such that they can help expressing the most strange and “illogical” *but natural* features of the human mind (and notably those related to the unconscious, “normal”, or pathologic) (Fig. 142).

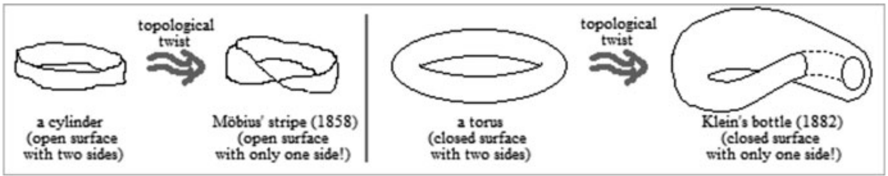


Fig. 142 “Möbius’s stripe” and “Klein’s bottle” provide *topological* intuitions on “opposition subversion”

In particular, one of the most famous (but also one of the most difficult to work out formally) of Lacan’s innovating concepts is that of “pas-tout.e” (in French: “not-all”). Lacan arrives to it by a psychoanalytical complex (and deep) reasoning, about “sexuation”, i.e., “mental gender”, considered as independent from “anatomical destiny”: you can have a penis but “fundamentally be” a woman, etc. (remark that nowadays Lacan’s theory is, despite polemics, very appreciated, debated [14], and used, notably in gender and transgender issues). For modeling this concept of sexuation, in a nutshell, aiming at studying the unconscious relation of “man” and “woman” (as related to concepts as “universality”, “exception”, “enjoyment”, etc.), he deforms the logical square (decorated à la Frege with quantifiers and quantified functional assertions), cutting and extracting from it one of its two red diagonals of contradiction and leaving aside what remains. More precisely, around 1972 [79] he “opens” the so obtained contradiction segment (extracted, as said, from a quantified version of the canonical logical square), by redoubling and renaming each of its two vertices, transforming it into a nonstandard new kind of formal square, his “square of sexuation” (in my opinion of oppositional geometer, one can/must think of it by analogy with the “cut and paste” construction of Möbius’ stripe and Klein’s bottle, Fig. 142, *supra*) (Fig. 143).

As it happens [85], this shape was inspired to Lacan by a reasoning of J. Brunschwig in 1969 [39] on the supposed psychogenetic “difficult origin” of the logical square in Aristotle, where the former argued that, historically, Aristotle

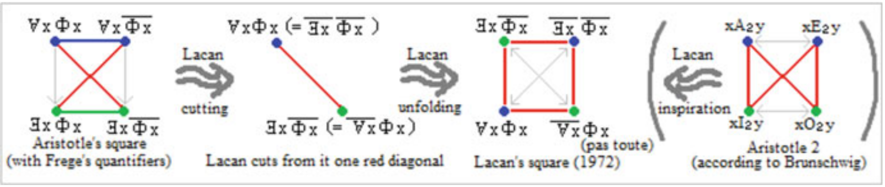


Fig. 143 From Aristotle's to Lacan's square (i.e., the "square of sexuation") through Brunschwig's square (1969)

found difficulties, allegedly "in the very heart of logic" (this is something Lacan liked!), during the invention of the square, and therefore had to hesitate between three successive kinds of (implicit!) formal squares (historically, we still do not know whether Aristotle himself developed *graphically* the full-fledged square (on Aristotle and mathematics, cf. [97], based on [77, 139]; cf. also [142]): he developed at least a preliminary version of it, the ὑπογραφή, in *Peri Hermeneias*, [6], Sect. 13). Brunschwig's square, rigorous *per se*, is nevertheless indigent from the viewpoint of oppositional geometry in so far it is a very *suboptimal* expression of formal properties *optimally* expressed, 20 years before (1950), by the logical *hexagon* (Sect. 1.1, Fig. 2). So, the fact that Lacanians, still now, keep "shielding" their Master, and themselves, with it – as did, for instance, in 2005 Grigg [73] against a benevolent but rigorous Badiou (1992) in his mathematically critical remarks on Lacan [12] – in order to claim (Grigg) that "Lacan's reasoning is *neither* a sophism *nor* a formal mistake, *because* it relies on Brunschwig's square!", is not (yet) a sign of conceptual strength (. . .). More interestingly, one should remark here two things: (1) Lacan aims at proving that "sexual difference" is conceptually deeper than "logical difference" (i.e., contradiction); and (2) by his square, Lacan defines sexuation through (nonstandard) contradiction. The first point is well-known, while the second seems problematic from the viewpoint of oppositional geometry: "female vs. male" is not a *contradiction*, but a *contrariety* (hence the necessity of having a logical *hexagon*, etc.).

But, here – coup de théâtre! – *our tri-segment (at least under the provisory naïve form it took in Sects. 2 and 3) seems, unexpectedly, to be quite interesting and fit, if one thinks of it, for trying to reformulate oppositional-geometrically this otherwise formally strange and perplexing current "square" expression of the Lacanian theory of sexuation* (Fig. 144).

For, not only it bears striking formal similarities with Lacan's square: more deeply, it rescues it from the aforementioned aporia of reducing dangerously "female vs. male" to a *contradiction* (instead of a *contrariety*), by its own structural richness (the tri-segment is a . . . hexagon!). In fact, the Aristotelian tri-segment captures really quite much, but with mathematical rigor, of what Lacan seems "illogically" willing to capture. The tri-segment seems to have at least a quintuple advantage over the previous implicit model of this (i.e., Brunschwig-Lacan's square): (1) it is not arbitrary (while Brunschwig-Lacan's square clearly is); (2)

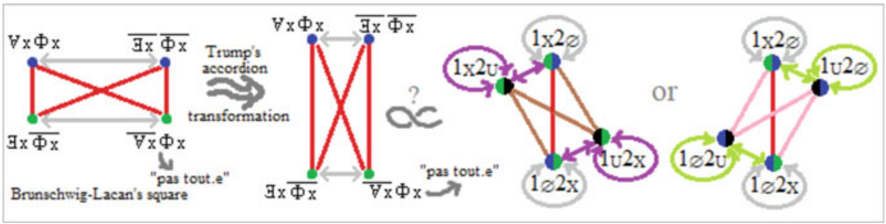


Fig. 144 The “square of sexuation”, or “Lacan’s square”, resembles quite much a tri-segment’s “inner rectangle”!

it has a large and deep *mathematical* theory behind (poly-simplicial oppositional geometry), whereas Lacan’s square has not; (3) it has more structure – the (paracomplete) square, or better rectangle, is only one of its components, but even taken alone as a paracomplete rectangle, it says more (for instance, about the inner structure of the two different “contradiction diagonals” composing it) – (4) it bears *explicit* and *reasoned* reference to intuitionism (i.e., paracompleteness and its gaps), which is a major point advocated (but so far with *problems*, as pointed by Badiou in 1992 [12]) by Lacan (cf. Darmon [47]); and (5) it adds to the reference to paracompleteness a tantamount *explicit* (and intuitively important) reference to its mathematical dual (absent in Lacan!), paraconsistency (i.e., co-intuitionism).

One must remark that this last proposal of application of the tri-segment, although surely strange for some, seems interesting (and paradigmatic): quite many people looked for “Lacan’s square” (including, in some sense, Lacan himself!) and still look for it (as recalled in [128]). So, finding a mathematical solution to the “riddle of the *pas tout*” would be a *fait d’armes*. If seemingly nobody found it so far, despite looking *eagerly* for it, and since years, and with all sort of formal means, this is because nobody had the idea (or, in fact, the means) of *simply looking for a mathematically rigorous, proper diffraction of the oppositional-geometrical concept of contradiction (a diffraction of the red segment)*: in terms of “poly-simplicial diffraction”! This is what we achieved in this study and in some sense seems to be quite close to what Lacan was trying, as he could, to “speak” about (Fig. 145).

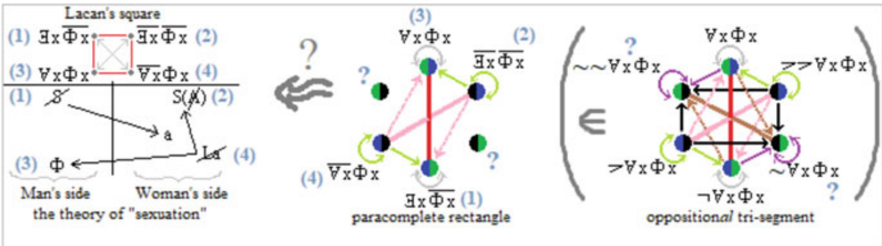


Fig. 145 The Aristotelian tri-segment as mathematical basis of Lacan’s attempted “sexuation square”?

Again, the importance of the "pas-tout.e", in a considerable part of contemporary thought, is such that if it were confirmed (time will tell) that the Aristotelian tri-segment, as in fact I tend to think, is indeed a (serious) model of it, this could be a memorable rather inspiring 2020 result. And a splendid *entrée sur scène* of our newborn, small "formal artist": the first full-fledged oppositional poly-simplex (poly≥3) .... But here we cannot say more (that will be another story) and so leave the consideration of this last suggestion to the future curiosity of our reader.

Here we can close, at last, the long journey, brave reader, which has been our present common inquiry on the poly-simplexes in general and on the tri-segment in particular.

6 Conclusion

Our study was about the mathematical concept of "contradiction" (i.e., negation). In a nutshell, it consisted in showing that the "mathematical nature" of the concept of contradiction/negation is more geometrical (i.e., "simplicial") than "logical". Of this we gave a particularly strong new proof: (1) by establishing an astonishing (and powerful) technical reference to Pascal's triangle (which we generalized to the very useful notion of "Pascalian ND simplex") and (2) by developing a concept of "Aristotelian combination" which proves suboptimal, as for the exploration of "opposition", the logicist program and methodology of Smessaert and Demey's "logical geometry". More specifically, in our study we developed a "tri-simplicial diffraction" of contradiction (seen as, classically, bi-simplicial). This engaged us in recalling first the concept of poly-simplicial space.

We recalled the context of the emergence of the idea of poly-simplicial space that we proposed in our PhD in 2009 and that has not been much developed since. This engaged us in recalling, previously, the concept of "oppositional geometry", which is a powerful framework for explaining the structure of the "logical hexagon" (which explains, since 1950, the otherwise mysterious "logical square"), seen as a "bi-simplex" (viz., a bi-triangle). Importantly, we showed that by now the poly-simplicial space has become explorable, notably (i) thanks to Angot-Pellissier's sheaf-theoretical method for generating vertices and examining the edges between any pair of them (ii) and thanks to our concept, proposed here, of "Pascalian ND simplex", which generalizes Pascal's triangle and provides a quite useful "roadmap", complementary to the sheaf-theoretic method (of which it solves some crucial problems). Through this we explored successfully the simplest of all the poly-simplexes (poly≥3), the "tri-segment": the tri-simplicial diffraction of the classical red "contradiction segment". This turned up to be rich enough as for its structure, which consists of a 2D hexagon (or, equivalently, a 3D octahedron). In order to do that, we had, previously, to solve successfully some intermediary rather difficult steps. We thus discovered (in Sects. 2 and 3) (i) proper treatments for poly-simplicial "extrema", (ii) a general technique for coloring any poly-simplicial *vertex* (and not only the *edges* between them), (iii) and the general combinatorial laws ruling the patterns of the poly-simplicial "valuations" (i.e., the attribution of truth-values to the vertices).

In order to deal with unexpected arrows emerging in the tri-segment from its two valuations, in Sect. 4 we established an important further point: that it is possible and in fact necessary to use, additionally, Smessaert and Demey's concept of "implication geometry" (on top of which they posited the idea of a "logical geometry", encompassing the two twin geometries which are the "opposition geometry" and the "implication geometry", and therefore supposed to be the most general and best theoretical framework for dealing, *among others*, with oppositions). To do that, we developed new, suited versions of Smessaert's starting idea, thus bringing Smessaert and Demey's "logical geometry" to a level of complexity where it had never been before (the level of the poly-simplicial spaces – Smessaert and Demey's logical geometry remains, by construction, bi-simplicial, and in fact the very concept of "simplex", i.e., geometrical number, is banned from the *logicist* vocabulary of logical geometry). And this allowed us making a very interesting discovery: what these two authors take for suboptimal, namely, what they call "Aristotelian geometry" (and which is supposed to contain also our and Angot-Pellissier's "oppositional geometry"), seen as a bricolage, made unconsciously, by conceptual surgery, over the two twin halves of logical geometry (i.e., "opposition geometry" and "implication geometry"), is in fact, in the light of the emerging complexity of the poly-simplicial space, already at the "ground-zero" level of the tri-segment, a necessary and *optimal* transformation (which we baptized "Aristotelian *combination*") – not of *choice*, as they believe, but of *fusion* – allowing methodical complexity reduction. This fact allows understanding that Smessaert and Demey's logical geometry, in its programmatic negation of a mathematical autonomy (with respect to logic) of the "oppositional", is in fact suboptimal. On that respect, we also recalled the deep philosophical reasons why what we therefore take as being their *logicist* posture (inscribed in the very reckless name of their approach) is rather dangerous (and, again, suboptimal as for the exploration bottom-up of the "poly-simplicial space", which is constitutive of the key concept of "*n*-contrariety"). It is notably suboptimal with respect to a *structuralist* approach, very natural for exploring (as Blanché in [33]) the "elementary structures of opposition", but more generally "logical geometry" is suboptimal with respect to any free mathematical approach not submitted to the logicist, mathematically counterproductive agenda.

Finally, in the last part (Sect. 5), we tried to have a prospective first look at possible applications of the successful tri-simplicial diffraction of contradiction, which is the "Aristotelian" tri-segment. Before that, we discussed some current limitations of our approach, which should be, but is not yet, many-valued *also* at the meta-level (technically speaking: at the level of the *number* of the possible "meta-questions") and how it should be tried to overcome these current limitations in a near future. An important point is that, from now on, the higher poly-simplexes (and, for a start, as soon as possible, the "tri-triangle", i.e., the "tri-simplicial diffraction" of the logical hexagon) should be studied in a way comparable to the one we successfully developed in this study for the tri-segment. As for applications, we concentrated on five domains where contradiction/negation seems to play a particularly important role, both positively as a *concept* and also negatively as a *fantasy*: in the exact sciences we proposed to see this in many-valued logics, paraconsistent logics, and quantum logics; in the humanities we proposed to see this in dialectics and psychoanalysis. Among the applications we proposed, the most spectacular one seems to be the (in our view) very convincing formalization of what is traditionally known as "Lacan's square" (1972), or "square of sexuation", a structure and theory much debated and used, notably, in gender and transgender studies and which so far remained a very difficult open problem (despite the many attempts to solve it, notably by some mathematicians). We claim that this famous and strange square, generally reduced – as to its *formal* standard – to "Brunschwig's square" (1969), is in fact, when duly formulated (i.e., better than Lacan and his followers did and still do), a precise fragment (viz., its "*paracomplete* inner rectangle") of the tri-segment, and retrospectively this seems "logical": the tri-segment "diffracts" the contradiction segment, which is precisely what Lacan tried to do, but without an adequate tool. This suggests to use, in the future, the *paraconsistent* "extra structure" of the tri-segment, with respect to the *a posteriori* suboptimal Lacan-Brunschwig's square, for exploring Lacanian (and more generally: gender-theoretical) "sexuation issues" possibly not yet explored by Lacan, the Lacanians, or any of the many working on it.

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