Anthony Bedford

Hamilton's Principle in Continuum Mechanics

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ISBN 978-3-030-90305-3 ISBN 978-3-030-90306-0 (eBook)
https://doi.org/10.1007/978-3-030-90306-0

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

> The good of Hamilton is not in what he has done but in the work (not nearly half done) which he makes other people do. But to understand him you should look him up, and go through all kinds of sciences, then you go back to him, and he tells you a wrinkle.
>
> James Clerk Maxwell

In 1808, when he was 2 years old, William Rowan Hamilton was sent to live with an aunt and uncle, Elizabeth and James Hamilton, in Trim, County Meath. James Hamilton was a classics scholar and graduate of Trinity College Dublin, and was headmaster of a diocesan school for boys. He soon recognized that his nephew showed extraordinary promise, and gave him intensive training in languages and the classics.

While he prepared for entrance to Trinity College, Hamilton became interested in mathematics, particularly analytic geometry. At the age of 17, he was reading *Théorie des Fonctions Analytiques* and *Mecanique Analytique* by Lagrange in addition to the books prescribed for the undergraduate science course at Trinity.

At Trinity College, Hamilton pursued a dual course in science and the classics—although he found it increasingly difficult to maintain his interest in the latter—and also began independent research on geometric optics as a natural extension of his interest in analytic geometry. His work led to a paper, "Theory of Systems of Rays" [37], which he presented to the Royal Irish Academy in April 1827. Primarily on the basis of his original research in optics, he was elected to the position of Andrews Professor of Astronomy at Trinity College in June 1827.

Hamilton's theory of ray optics was a variational theory. It was based on the principle, due to Fermat, that a light ray traveling between two points will follow the path that requires the least time. In the course of his work on optics, he also began to consider the possibility of developing an analogous theory for the dynamics of systems of particles. This resulted, in 1834–1835, in two papers, "On a General Method in Dynamics" [38] and "Second Essay on a General Method in Dynamics" [39]. In the second paper, he presented the result that is known today as Hamilton's principle.

Hamilton's general and elegant work on dynamics was widely quoted but not extensively applied during the remainder of the nineteenth century. However, when quantum mechanics was developed, it was realized that Hamilton's work was the most natural setting for its formulation. In fact, in retrospect, Hamilton's formal analogy between optics and classical mechanics was seen as a precursor of wave mechanics.

A similar historical development has occurred in the field of continuum mechanics. Although formulations of Hamilton's principle for continua began to appear as early as 1839, with the exception of applications to structural analysis, variational methods in continuum mechanics were regarded as academic, because the same results could be obtained using more direct methods. Some modern treatises on continuum mechanics do not mention variational methods. In recent years, however, interest in variational methods has increased markedly. They have been used to obtain approximate solutions, as in the finite element method, and to study the stability of solutions to problems in fluid and solid mechanics. Variational formulations have also been used to develop generalizations of the classical theories of fluid and solid mechanics.

The objective of this monograph is to give a comprehensive account of the use of Hamilton's principle to derive the equations that govern the mechanical behavior of continuous media. The classical theories of fluid and solid mechanics are discussed as well as two generalizations of those theories for which Hamilton's principle is particularly suited—materials with microstructure and mixtures.

These topics are brought together for the first time to acquaint readers who are new to this subject with an interesting and powerful alternative approach to the formulation of continuum theories. Persons interested in fluid and solid mechanics will gain a broadened perspective on those subjects as well as learn the fundamental background required to read the large literature on variational methods in continuum mechanics. For readers who are familiar with these methods, a number of recent results are presented on applications of Hamilton's principle to generalized continua and materials containing singular surfaces. These results are presented in a setting that could encourage generalizations and extensions.

Hamilton's principle was originally expressed in terms of the classical mechanics of systems of particles. The concepts and the terminology involved in applying Hamilton's principle to continuum mechanics are quite similar, and some familiarity with the applications to systems of particles is very helpful in understanding the extension to the case of a continuum. The application of Hamilton's principle to systems of particles is therefore briefly discussed in Chap. 1. This subject provides a simple context in which to introduce the variational ideas underlying Hamilton's principle as well as the method of Lagrange multipliers and the concept of virtual work.

Chapter 2 provides a brief survey of the mathematics and elements of continuum mechanics that are required in the subsequent chapters. Most of this chapter can be skipped by persons familiar with modern continuum mechanics; however, even those who are acquainted with variational methods in continuum mechanics should briefly examine Sect. 2.3 before proceeding to the following chapters.

Applications of Hamilton's principle to a continuous medium are described in Chap. 3. Ideal fluids and elastic solids are treated in Sects. 3.1.1 and 3.1.2. The general case of a continuum that does not exhibit microstructural effects is presented in Sect. 3.1.3. Section 3.2 presents applications of Hamilton's principle to two particular theories of materials with microstructure. These applications illustrate the use of Hamilton's principle to generalize the ordinary theories of fluid and solid mechanics. Persons who are new to this subject may choose to omit this section and the following chapter in a first reading.

As another example of the use of Hamilton's principle to develop generalized continuum theories, applications to mixtures are described in Chap. 4. The fact that the sum of the volume fractions of the constituents of a mixture must equal one at each point can be introduced into Hamilton's principle using the method of Lagrange multipliers. As a result of "wrinkles" such as this, Hamilton's principle provides a simple and elegant way to derive continuum theories of mixtures. A mixture of ideal fluids is discussed in Sect. 4.2. The case of a liquid containing a distribution of gas bubbles is treated as an example, including the microkinetic energy associated with bubble oscillations. In Sect. 4.3, a mixture of an ideal fluid and an elastic material is considered, and it is shown that the equations obtained through Hamilton's principle are equivalent to the Biot equations. A theory of mixtures of materials with microstructure in which the constituents need not be ideal or elastic is presented in Sect. 4.4.

In Chap. 5 a discussion is given of the application of Hamilton's principle to a continuous medium containing a surface across which the fields that characterize the medium, or their derivatives, suffer jump discontinuities. The fundamental results required to include a singular surface in a statement of Hamilton's principle are presented in Sect. 5.1. An elastic fluid is treated as an example in Sect. 5.2, and it is shown that Hamilton's principle yields the jump conditions of momentum and energy across the surface.

The results presented in this monograph are expressed in a modern framework. Persons wishing to gain an impression of Hamilton's research in its original form should consult his collected works [40,41]. The definitive references on Hamilton's life are Graves [32] and Hankins [42]. In Chaps. 1–3, the sources that have been used are cited, but no attempt is made to give complete or original references except for results that are relatively recent. In Chaps. 2 and 3, particular reference is made to works by M. E. Gurtin. The responsibility for errors or misinterpretations of course rests with the author. Chapters 4 and 5 are based in large part on work done by the author in collaboration with D. S. Drumheller and G. Batra. One motivation for writing this monograph was to present these results in their classical context, together with a complete discussion of the foundations.

Hamilton's research anticipated modern trends in mechanics in two respects. He approached problems primarily from the perspective of a mathematician, and he consistently sought the greatest possible generality in his results. It is a measure of his success that, 150 years after the publication of his two great works on mechanics, his results continue to find new and fruitful applications.

Acknowledgments

This monograph is a result of my 15-year professional collaboration with Douglas Drumheller of Sandia National Laboratories. Our work on mixtures led us to become interested in Hamilton's principle, and in addition to his original contributions to the research, Doug helped me understand many of the subtleties of variational methods in continuum mechanics. Chapter 4 is based in large part on our results. Our collaboration took place during periods when I was a temporary staff member and consultant at Sandia. I am grateful for the courteous and generous treatment I received from that organization, and particularly thank Jim Asay, Darrel Munson, and Walt Hermann. The United States Department of Energy supported my work there. Much of the work on which Chap. 5 is based was done in collaboration with Gautam Batra, and the monograph also benefited from his ideas on notation and the organization of the subject. Part of the manuscript was written while I was a summer staff member at Applied Research Laboratories, University of Texas at Austin. Clark Penrod kindly arranged my appointment, and the Office of Naval Research supported my work. Gautam Batra, Ray Bowen, Doug Drumheller, Mort Gurtin, Steve Passman, Morris Stern, and Tim Trucano read an early draft and gave me many helpful suggestions.

Contents

Nomenclature

Definitions of frequently used symbols. The pages on which they first appear are shown in parentheses.

$\mathbf{a}$	Acceleration of a material point (22)
C^N	Indicates a function has N continuous derivatives (1)
$\mathbf{C}$	Left Cauchy–Green strain tensor (25)
cof	Cofactor (17)
det	Determinant (16)
div	Divergence (24)
DIV	Divergence (19)
e	Internal energy (35)
$\mathbf{E}$	Linear strain tensor (25)
$\mathbf{F}$	Deformation gradient (22)
grad	Gradient (23)
GRAD	Gradient (19)
J	Jacobian (22)
L	Lagrangian (7)
$\mathbf{L}$	Velocity gradient (23)
q_k	Generalized coordinate (5)
$\mathbf{S}$	First Piola–Kirchoff stress tensor (42)
T	Kinetic energy (5)
$\mathbf{T}$	Cauchy stress tensor (43)
U	Potential energy (5)
$\mathbf{v}$	Velocity of a material point (22)
x	Real variable (1)
$\mathbf{x}$	Current position of a material point (21)
$\mathbf{X}$	Reference position of a material point (21)
$\boldsymbol{\chi}$	Motion of a continuous medium (21)
$\delta(\cdot)$	Variation of $(\cdot)$ (8)
δ_{km}	Kronecker delta (16)
δW	Virtual work (11)

ε	Scalar parameter (2)
$\eta(x)$	Arbitrary scalar function of x (2)
ϕ	Volume fraction (49)
ρ	Density (23)
$\mathbf{0}$	Zero tensor (15)
$\mathbf{1}$	Identity tensor (15)
$(\cdot)^*$	A comparison function (2)
$[\![\cdot]\!]$	Jump of a quantity (89)

Chapter 1
Mechanics of Systems of Particles

1.1 The First Problem of the Calculus of Variations

Before Hamilton's principle is introduced, some preliminary comments on the calculus of variations are necessary. Hamilton's principle is closely related to what is called the *first problem of the calculus of variations*, which can be introduced by a simple example.

Let x be a real variable, and let the closed interval $x_1 \leq x \leq x_2$ be denoted by $[x_1, x_2]$. A function $y(x)$ is said to be C^N on $[x_1, x_2]$ if the Nth derivative of $y(x)$ exists and is continuous on $[x_1, x_2]$. The value of a derivative at an endpoint is defined to be the limit of the derivative as the endpoint is approached from within the interval.

Let x_1, y_1 and x_2, y_2 be two fixed points in the x-y plane, with $x_1 < x_2$, and let $y(x)$ be a C^1 function on $[x_1, x_2]$ such that $y(x_1) = y_1$ and $y(x_2) = y_2$. Thus $y(x)$ describes a smooth curve that joins the two points, as shown in Fig. 1.1.

The length of the curve joining the two points is

$$L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx, \tag{1.1}$$

where $y' = dy/dx$. Consider the following question: Can a smooth curve joining the two points be found such that its length is a minimum in comparison with other such curves? That is, among functions $y(x)$ that are C^1 on $[x_1, x_2]$ and satisfy the conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, can one be found for which the value of the integral (1.1) is a minimum?

The first problem of the calculus of variations is a generalization of this simple problem. Consider the integral

$$I = \int_{x_1}^{x_2} f\left(x, y, y'\right) dx, \tag{1.2}$$

A. Bedford, *Hamilton's Principle in Continuum Mechanics*,
https://doi.org/10.1007/978-3-030-90306-0_1

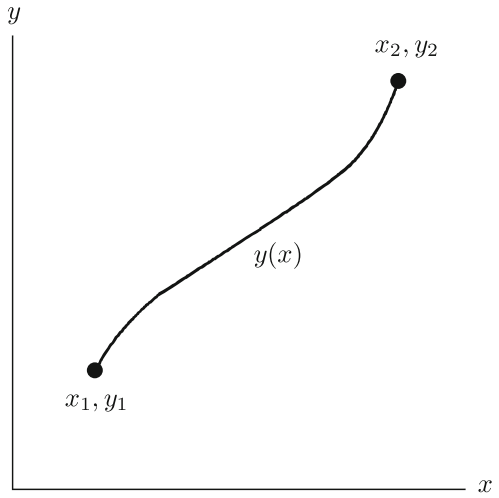

Fig. 1.1 A smooth curve joining two points in the x-y plane

where f is a given function of the arguments x, y and y', and the values $y(x_1) = y_1$ and $y(x_2) = y_2$ are prescribed. The value of the integral (1.2) depends on the function $y(x)$. A scalar-valued function such as this whose argument is itself a function is called a *functional*. As in the previous example, the question is whether a function $y(x)$ can be found such that the value of the integral is a minimum.

Certain restrictions are imposed on the functions $y(x)$ and f by the statement of the problem, the procedures that will be used in seeking its solution, and often by the physical nature of a specific application. Here consideration will be limited to functions $y(x)$ that satisfy the prescribed values at x_1 and x_2 and are C^2 on $[x_1, x_2]$. Functions $y(x)$ having these properties will be called *admissible*. It will also be assumed that the second partial derivatives of the function f exist are continuous on a suitable open domain of the arguments of f. The reasons for these smoothness assumptions will become apparent.

In order to seek an admissible function $y(x)$ for which the value of the integral (1.2) is a minimum, let an admissible *comparison function* be defined by

$$y^*(x, \varepsilon) = y(x) + \varepsilon\eta(x), \tag{1.3}$$

where ε is a parameter and $\eta(x)$ is an arbitrary C^2 function on $[x_1, x_2]$ subject to the requirements that $\eta(x_1) = 0$ and $\eta(x_2) = 0$ (see Fig. 1.2). If the comparison function (1.3) is substituted into the integral (1.2) in place of the function $y(x)$, the integral becomes

$$I^*(\varepsilon) = \int_{x_1}^{x_2} f\left(x, y^*, y^{*\prime}\right) dx, \tag{1.4}$$

where it is indicated that the value of the integral is a function of the parameter ε.

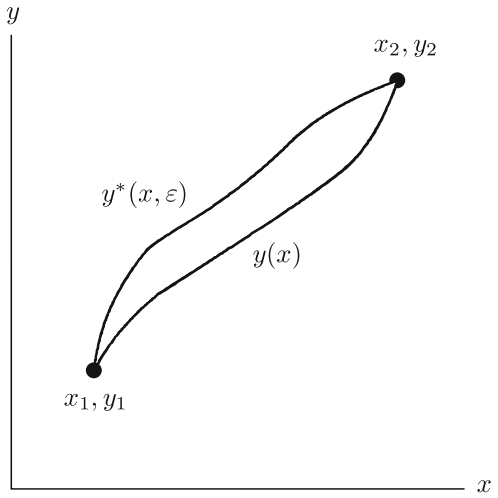

Fig. 1.2 The function $y(x)$ and the comparison function $y^*(x, \varepsilon)$

Now let it be assumed that the value of the integral (1.4) is a minimum when the comparison function $y^*(x, \varepsilon) = y(x)$. That is, the function $I^*(\varepsilon)$ is a minimum when the parameter $\varepsilon = 0$, which implies the necessary condition

$$\left[\frac{dI^*(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = 0. \tag{1.5}$$

The derivative of (1.4) with respect to ε is

$$\begin{aligned}\frac{dI^*(\varepsilon)}{d\varepsilon} &= \int_{x_1}^{x_2}\left(\frac{\partial f^*}{\partial y^*}\frac{\partial y^*}{\partial \varepsilon} + \frac{\partial f^*}{\partial y^{*\prime}}\frac{\partial y^{*\prime}}{\partial \varepsilon}\right)dx \\ &= \int_{x_1}^{x_2}\left(\frac{\partial f^*}{\partial y^*}\eta + \frac{\partial f^*}{\partial y^{*\prime}}\eta'\right)dx \end{aligned} \tag{1.6}$$

where $f^* = f\left(x, y^*, y^{*\prime}\right)$ and $\eta' = d\eta/dx$. Therefore the condition (1.5) states that

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)dx = 0. \tag{1.7}$$

The second term in this expression can be integrated by parts to obtain

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'\,dx = \left[\frac{\partial f}{\partial y'}\eta\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\,dx. \tag{1.8}$$

Using this result and recalling that $\eta(x)$ vanishes at x_1 and x_2, (1.7) can be written

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \eta \, dx = 0. \tag{1.9}$$

Because the function $\eta(x)$ is arbitrary subject to the conditions that it be C^2 on $[x_1, x_2]$ and that it vanish at x_1 and x_2, the expression that multiplies $\eta(x)$ in the integrand of (1.9) must vanish on $[x_1, x_2]$. If this were not the case, a function $\eta(x)$ could be chosen so that (1.9) would be violated.

The formal statement of this result is called the *fundamental lemma of the calculus of variations* (see e.g. Bolza [12], p. 20):

> Suppose that a function $\psi(x)$ is C^0 on $[x_1, x_2]$. If the equation
>
> $$\int_{x_1}^{x_2} \psi(x)\eta(x) \, dx = 0 \tag{1.10}$$
>
> holds for every C^∞ function $\eta(x)$ on $[x_1, x_2]$ that satisfies the conditions $\eta(x_1) = 0$ and $\eta(x_2) = 0$, then $\psi(x)$ must vanish on $[x_1, x_2]$.[1]

Observe that in order to apply this lemma to (1.9), the functions $y(x)$ and f must be smooth enough so that the expression multiplying $\eta(x)$ is continuous on $[x_1, x_2]$. This is the reason for the differentiability requirements that were imposed on these functions. Note that

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = \frac{\partial^2 f}{\partial x \partial y'} + \frac{\partial^2 f}{\partial y \partial y'} y' + \frac{\partial^2 f}{\partial y' \partial y'} y'', \tag{1.11}$$

where $y'' = d^2y/dx^2$. Therefore the second derivative of $y(x)$ and the second partial derivatives of f must exist and be continuous on $[x_1, x_2]$.

On the basis of the fundamental lemma, (1.9) implies that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0 \quad \text{on } [x_1, x_2]. \tag{1.12}$$

This is called the *Euler-Lagrange equation*. It provides a differential equation with which to determine the function $y(x)$. In the case of the simple example (1.1), (1.12) yields the equation

$$y' = \text{constant}, \tag{1.13}$$

which does describe the curve joining the two points that is of minimum length.

[1] A proof of a more general form of this lemma is presented in Sect. 2.4.

The condition (1.5) is obviously only a necessary condition, not a sufficient condition, for the value of the integral (1.4) to be a minimum when $\varepsilon = 0$. This condition is also satisfied if the value of the integral is a maximum or has an inflection point of zero slope at $\varepsilon = 0$. Thus the condition (1.5) and the determined solution $y(x)$ are *necessary conditions given that the value of the integral is stationary in comparison with neighboring admissible functions*.

The open domain on which the second partial derivatives of the function f must be assumed to exist and be continuous can be defined in retrospect. It must encompass the values of the arguments of f associated with the solution $y(x)$ and with comparison functions (1.3) in a neighborhood of the solution.[2]

Recommended references on the calculus of variations include Akhiezer [1], Bliss [11], Bolza [12], Courant and Hilbert [15], Finlayson [28], Gelfand and Fomin [29], Pars [61], Washizu [73], and Weinstock [74].

1.2 Conservative Systems

1.2.1 Hamilton's Principle

Consider a system of particles whose position, or *configuration*, can be described by a set of independent generalized coordinates q_k, $k = 1, 2, \ldots, K$. Let t_1 and t_2 be fixed times, with $t_1 < t_2$, and suppose that the configurations of the system at times t_1 and t_2 are prescribed. An *admissible motion* of the system will be defined to be a set of functions $q_k(t)$, $k = 1, 2, \ldots, K$, which satisfy the prescribed values at t_1 and t_2 and are C^2 on $[t_1, t_2]$.

Let it be assumed that the kinetic energy of the system, T, can be expressed as a function of the generalized coordinates and their time derivatives, $T = T(q_k, \dot{q}_k)$. This expression indicates that T may be a function of q_k and $\dot{q}_k$ for each value of k from 1 to K. It will also be assumed that the system is subject only to conservative forces and that the potential energy of the system, U, can be expressed as a function of the generalized coordinates, $U = U(q_k)$. Each of the second partial derivatives of T and each of the first partial derivatives of U will be assumed to exist and to be continuous.[3]

[2] Henceforth, when a function is said to be continuous with no additional provisos, it will be understood to be continuous on a suitable open domain of its arguments.

[3] In the simplest example, the "system" is a single particle. If there are no geometric constraints on its motion, the generalized coordinates are the three position coordinates of the particle relative to a suitable reference frame. The kinetic energy is $T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}$, where m is the mass of the particle and $\mathbf{v}$ is its velocity vector. The potential energy U is defined such that $dU = -\mathbf{F} \cdot \mathbf{v}\, dt$, where $\mathbf{F}$ is the force vector acting on the particle. If such a function U exists, $\mathbf{F}$ is said to be *conservative*.

What will be called the *first form* of Hamilton's principle for a conservative system of particles states:

> Among admissible motions, the actual motion of a conservative system is such that the value of the integral
>
> $$I = \int_{t_1}^{t_2} (T - U)\, dt \tag{1.14}$$
>
> is stationary in comparison with neighboring admissible motions.

Suppose that the functions $q_k(t)$ describe the actual motion of the system. In analogy with (1.3), an admissible of the system will be defined by

$$q_k{}^*(t, \varepsilon) = q_k(t) + \varepsilon \eta_k(t), \tag{1.15}$$

$k = 1, 2, \ldots, K$, where the $\eta_k(t)$ are arbitrary C^2 functions on $[t_1, t_2]$ subject to the requirements that $\eta_k(t_1) = 0$ and $\eta_k(t_2) = 0$. Upon substituting (1.15) into (1.14) in place of the functions $q_k(t)$, one obtains the integral

$$I^*(\varepsilon) = \int_{t_1}^{t_2} (T^* - U^*)\, dt, \tag{1.16}$$

where $T^* = T\left(q_k^*, \dot{q}_k^*\right)$ and $U^* = U\left(q_k^*\right)$. Hamilton's principle states that the value of this integral is stationary when $q_k^*(t, \varepsilon) = q_k(t)$, which implies that

$$\left[\frac{dI^*(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = 0. \tag{1.17}$$

The derivative of (1.16) with respect to ε is

$$\begin{aligned}\frac{dI^*(\varepsilon)}{d\varepsilon} &= \int_{t_1}^{t_2} \left(\frac{\partial T^*}{\partial q_k^*}\frac{\partial q_k^*}{\partial \varepsilon} + \frac{\partial T^*}{\partial \dot{q}_k^*}\frac{\partial \dot{q}_k^*}{\partial \varepsilon} - \frac{\partial U^*}{\partial q_k^*}\frac{\partial q_k^*}{\partial \varepsilon}\right) dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial T^*}{\partial q_k^*}\eta_k + \frac{\partial T^*}{\partial \dot{q}_k^*}\dot{\eta}_k - \frac{\partial U^*}{\partial q_k^*}\eta_k\right) dt.\end{aligned} \tag{1.18}$$

In this equation, use is made of the *summation convention: Whenever an index appears twice in a single expression, the expression is assumed to be summed over the range of the index.* For example,

$$\frac{\partial T^*}{\partial q_k^*}\frac{\partial q_k^*}{\partial \varepsilon} = \frac{\partial T^*}{\partial q_1^*}\frac{\partial q_1^*}{\partial \varepsilon} + \frac{\partial T^*}{\partial q_2^*}\frac{\partial q_2^*}{\partial \varepsilon} + \cdots + \frac{\partial T^*}{\partial q_K^*}\frac{\partial q_K^*}{\partial \varepsilon}. \tag{1.19}$$

This useful convention will be used throughout this work.

From (1.18), the condition (1.17) is

$$\int_{t_1}^{t_2} \left(\frac{\partial T}{\partial q_k} \eta_k + \frac{\partial T}{\partial \dot{q}_k} \dot{\eta}_k - \frac{\partial U}{\partial q_k} \eta_k \right) dt = 0. \tag{1.20}$$

When the second term is integrated by parts, this equation can be written

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \eta_k \, dt = 0, \tag{1.21}$$

where $L = T - U$ is the *Lagrangian* of the system. Because the functions $\eta_k(t)$ are arbitrary subject to the requirements stated above, they can be assumed to be nonzero on $[t_1, t_2]$ for $k = 1$ only. Equation (1.21) is then of the form (1.10), and the fundamental lemma applies. Repeating this process for each value of k results in the differential equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0 \quad \text{on } [t_1, t_2] \tag{1.22}$$

for each value of k from 1 to K. These are *Lagrange's equations of motion* for the system of particles (see e.g. Goldstein, et al. [30], Chapter 2).

Hamilton's principle is a postulate regarding the motion of the system. It embodies the physics of the problem. The mathematical task is to deduce the equations of motion, which are obtained as necessary conditions implied by the postulate. The number of equations of motion is equal to the number of independent generalized coordinates.

As an illustration, consider the motion of a single particle in the x-y plane. Suppose that the particle is subject only to its own weight and let the y axis be directed upward. The kinetic energy is

$$T = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right), \tag{1.23}$$

where m is the mass of the particle, and the potential energy is

$$U = mgy, \tag{1.24}$$

where g is the acceleration due to gravity (assumed constant). Equation (1.22) yields the equations of motion

$$\begin{aligned} \ddot{x} &= \quad 0, \\ \ddot{y} &= -g. \end{aligned} \tag{1.25}$$

Expressions that depend on the parameter ε have been denoted by an asterisk. In applications of variational methods, derivatives of such expressions with respect

to ε, evaluated at $\varepsilon = 0$, appear frequently. This can be seen, for example, in obtaining (1.20) from (1.16) and (1.17). It is therefore convenient to introduce the notation[4]

$$\delta(\cdot) \equiv \left[\frac{\partial}{\partial \varepsilon}(\cdot)^*\right]_{\varepsilon=0}. \tag{1.26}$$

The symbol $\delta(\cdot)$ is called the *variation* of the expression $(\cdot)$. Observe from (1.15) that

$$\delta q_k = \eta_k. \tag{1.27}$$

Also, From (1.16), the necessary condition (1.17) can be written

$$\int_{t_1}^{t_2} \delta(T - U)\, dt = 0. \tag{1.28}$$

Stating that this equation holds for admissible comparison functions (1.15) is clearly equivalent to the first form of Hamilton's principle for a conservative system of particles. Therefore, what will be called the *second form* of Hamilton's principle for such a system states:

> Among admissible motions, the actual motion of a conservative system is such that (1.28) holds.

This is the form in which the principle was stated in Hamilton's original work [39].

1.2.2 Constraints

Thus far it has been assumed that the generalized coordinates q_k are independent. Suppose instead that they are required to satisfy prescribed equations

$$\alpha_p(q_k) = 0, \tag{1.29}$$

where $p = 1, 2, \ldots, P$, $P < K$. The first partial derivatives of the functions α_p with respect to each of the q_k will be assumed to exist and be continuous.

Hamilton's principle can be stated so that it embodies the *constraints* (1.29) by using the method of Lagrange multipliers (see e.g. Pars [61], Chapter VIII). Let

[4] This notation, which is very common in the literature on variational methods, has acquired a bad reputation in some circles due to a history of vague definitions and a tendency to use it in performing complicated operations that are bewildering to the uninitiated. After initial attempts to write this monograph without using it, the author decided that it is too useful to discard. Throughout this work, this notation should be interpreted *only* as a symbol representing the operation (1.26).

$\pi_p(t)$, $p = 1, 2, ..., P$, denote a set of functions of time, the *Lagrange multipliers*, that are assumed to be C^0 on $[t_1, t_2]$, and define

$$C = \pi_p \alpha_p. \tag{1.30}$$

Then the first form of Hamilton's principle states:

> Among admissible motions, the actual motion of a conservative system subject to the constraints (1.29) is such that the value of the integral
>
> $$I = \int_{t_1}^{t_2} (T - U + C)\, dt \tag{1.31}$$
>
> is stationary in comparison with neighboring admissible motions.

In determining the equations of motion, the generalized coordinates q_k can be treated as if they are *independent*; the constraints (1.29) are accounted for by introducing them into (1.31) together with the Lagrange multipliers.

Substituting the comparison motions (1.15) into (1.31) in place of the functions $q_k(t)$ yields the integral

$$I^*(\varepsilon) = \int_{t_1}^{t_2} (T^* - U^* + C^*)\, dt, \tag{1.32}$$

where $C^* = \pi_p(t)\alpha_p(q_k^*)$. In this case the condition (1.17) is

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \pi_p \frac{\partial \alpha_p}{\partial q_k} \right] \eta_k \, dt = 0, \tag{1.33}$$

and the same argument used to obtain (1.22) results in the differential equations of motion

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \pi_p \frac{\partial \alpha_p}{\partial q_k} = 0 \quad \text{on } [t_1, t_2] \tag{1.34}$$

for each value of k from 1 to K. Equations (1.29) and (1.34) provide $K + P$ equations with which to determine the generalized coordinates $q_k(t)$ and the Lagrange multipliers $\pi_p(t)$.

Returning to the example of the motion of a single particle subject to its own weight, suppose that the particle slides without friction along a wire that constrains its motion to the path $y = x^2$. Then there is a single constraint equation

$$\alpha(x, y) = y - x^2 = 0, \tag{1.35}$$

and the equations of motion obtained from (1.34) are

$$\begin{aligned} m\ddot{x} &= -2x\pi, \\ m\ddot{y} &= \pi - mg. \end{aligned} \tag{1.36}$$

Lagrange multipliers introduced into Hamilton's principle can be interpreted as generalized forces that cause the corresponding constraints to be satisfied. In this example, it is easy to see from the second equation of motion that the Lagrange multiplier π is the vertical component of the force exerted on the particle by the wire.

By substituting (1.32) into the condition (1.17), the second form of Hamilton's principle for a conservative system of particles with constraints is obtained:

> Among admissible motions, the actual motion of a conservative system subject to the constraints (1.29) is such that
>
> $$\int_{t_1}^{t_2} [\delta(T - U) + \delta C]\, dt = 0. \tag{1.37}$$

1.3 Nonconservative Systems

It is a common misconception that variational methods such as Hamilton's principle are only applicable to conservative systems. Because so many interesting problems, including many problems involving continuous media, involve nonconservative forces, this would make the range of applications of Hamilton's principle very limited indeed. One objective of this monograph is to help dispel this myth.

Let the *generalized forces* Q_k be defined by

$$Q_k = -\frac{\partial U}{\partial q_k}. \tag{1.38}$$

Noting that

$$\frac{\partial U^*}{\partial \varepsilon} = \frac{\partial U^*}{\partial q_k}\frac{\partial q_k}{\partial \varepsilon} = \frac{\partial U^*}{\partial q_k}\eta_k \tag{1.39}$$

and using (1.26), (1.27), and (1.38), one obtains

$$\delta U = -Q_k \delta q_k. \tag{1.40}$$

Using this expression, (1.28) assumes the form

$$\int_{t_1}^{t_2} (\delta T + Q_k \delta q_k)\, dt = 0. \tag{1.41}$$

Of course, the system being dealt with is still a conservative one. The only thing that has been done is to introduce the notation (1.38). However, *if Hamilton's principle is postulated in terms of (1.41), it is not necessary to assume that the generalized forces Q_k are conservative.* Thus the form of (1.41) is suggested by Hamilton's

principle for a conservative system, but a new postulate is introduced in the case of a nonconservative system. The term

$$\delta W = Q_k \delta q_k \tag{1.42}$$

is called the *virtual work*.[5] Hamilton's principle for a nonconservative, unconstrained system of particles states:

Among admissible motions, the actual motion of a system is such that

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0. \tag{1.43}$$

Clearly, if the system is conservative this postulate is identical to the statement of the second form of Hamilton's principle on page 8. In that case, the generalized forces are derivable from the potential energy through (1.38). If the system is not conservative, the generalized forces must be prescribed. Two cases occur frequently:

1. The generalized forces are prescribed explicitly as functions of time.
2. The generalized forces are prescribed implicitly through *constitutive equations* in terms of the generalized coordinates and their derivatives.
   Both of these cases will arise in applications of Hamilton's principle to continuous media.

A system may be subjected to both conservative and nonconservative forces, and it is often convenient to introduce the potential energy associated with the conservative forces. In that case, (1.43) is written

$$\int_{t_1}^{t_2} [\delta(T - U) + \delta W]\, dt = 0. \tag{1.44}$$

By using the definition (1.26), it is easy to show that

$$\delta T = \frac{\partial T}{\partial q_k}\delta q_k + \frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k, \quad \delta U = \frac{\partial U}{\partial q_k}\delta q_k, \tag{1.45}$$

so that (1.44) can be written

$$\int_{t_1}^{t_2} \left(\frac{\partial T}{\partial q_k}\delta q_k + \frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k - \frac{\partial U}{\partial q_k}\delta q_k + Q_k \delta q_k \right) dt = 0. \tag{1.46}$$

[5] This notation for the virtual work is entrenched in the literature, although it violates our promise that the symbol δ would only denote the operation (1.26). This inconsistency can be avoided by regarding the notation δW as a single symbol denoting the virtual work.

Integrating the second term by parts and using the fundamental lemma yields the differential equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \quad \text{on } [t_1, t_2] \tag{1.47}$$

for each value of k from 1 to K. These are Lagrange's equations of motion for a system that involves both conservative and nonconservative forces (see e.g. Goldstein, et al. [30], Chapter 2).

The problems addressed in this monograph will involve both nonconservative forces and constraints, and some of them will involve conservative forces as well. This chapter will close with a statement of Hamilton's principle for a system of particles that exhibits each of these characteristics:

Among admissible motions, the actual motion of a system is such that

$$\int_{t_1}^{t_2} [\delta(T - U) + \delta W + \delta C]\, dt = 0. \tag{1.48}$$

The application of Hamilton's principle to systems of particles and rigid bodies is discussed by Goldstein, et al. [30], Hamilton [39,41], Lanczos [50], Torby [68], Weinstock [74], and Whittaker [76].

Chapter 2
Foundations of Continuum Mechanics

2.1 Mathematical Preliminaries

2.1.1 Inner Product Spaces

Many of the variables used in continuum mechanics are assumed to obey the axioms of a finite-dimensional linear vector space with an inner product, which is simply called an *inner product space* (IPS). A result that is stated in terms of an arbitrary IPS can be applied in many contexts, achieving both generality and economy of presentation. The axioms are usually familiar to persons with technical backgrounds because they arise in the study of ordinary vector analysis. The following statement of them is paraphrased from Halmos ([36], pp. 3–14, 118–122). For the purposes of this work, scalars can be assumed to be real numbers.

A *linear vector space* $\mathcal{W}$ is a set of elements called *vectors*. An operation called *addition* is defined that associates with each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{W}$ a vector $\mathbf{x}+\mathbf{y}$ in $\mathcal{W}$ such that[1]

$$\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}, \tag{2.1}$$

and for any three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathcal{W}$,

$$\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}. \tag{2.2}$$

There is a unique vector $\mathbf{o}$ in $\mathcal{W}$ such that, for each vector $\mathbf{x}$ in $\mathcal{W}$,

$$\mathbf{x}+\mathbf{o}=\mathbf{x}. \tag{2.3}$$

[1] Linear vector spaces will be denoted by script capital letters. Vectors will be denoted by bold-face letters, usually lower case, although there will be exceptions that will be defined individually.

A. Bedford, *Hamilton's Principle in Continuum Mechanics*,
https://doi.org/10.1007/978-3-030-90306-0_2

For each vector $\mathbf{x}$ in $\mathcal{W}$, there is a unique vector $-\mathbf{x}$ such that

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{o}. \tag{2.4}$$

An operation called *scalar multiplication* is defined which associates with each scalar α and each vector $\mathbf{x}$ in $\mathcal{W}$ a vector $\alpha\mathbf{x}$ in $\mathcal{W}$ such that, for any scalars α, β and vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathcal{W}$,

$$\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}, \tag{2.5}$$

$$1\mathbf{x} = \mathbf{x}, \tag{2.6}$$

$$\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}, \tag{2.7}$$

$$(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}. \tag{2.8}$$

A finite set of vectors $\{\mathbf{x}_k\} = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ in $\mathcal{W}$ is called *linearly independent* if the equation

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_N\mathbf{x}_N = \alpha_k\mathbf{x}_k = \mathbf{o} \tag{2.9}$$

holds only when $\alpha_k = 0$ for each value of k from 1 to N. If such a set of vectors exists for which each vector $\mathbf{x}$ in $\mathcal{W}$ can be written in the form

$$\mathbf{x} = \beta_k\mathbf{x}_k, \tag{2.10}$$

then $\mathcal{W}$ is said to be of *dimension* N, and $\{\mathbf{x}_k\}$ is called a *basis* for $\mathcal{W}$.

The axioms and definitions stated thus far characterize a finite-dimensional linear vector space. An inner product space is obtained by appending an operation called the *inner product* that associates with each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{W}$ a scalar denoted by $\mathbf{x} \cdot \mathbf{y}$ such that, for any scalars α, β and vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathcal{W}$,

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}, \tag{2.11}$$

$$\mathbf{x} \cdot \mathbf{x} \geq 0, \tag{2.12}$$

where $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{o}$, and

$$(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha(\mathbf{x} \cdot \mathbf{z}) + \beta(\mathbf{y} \cdot \mathbf{z}). \tag{2.13}$$

The *magnitude*, or *norm*, of a vector $\mathbf{x}$ in an IPS is defined to be the scalar

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}. \tag{2.14}$$

The real numbers are an IPS if the inner product is defined to be the usual product of two numbers. It is one dimensional, and any number other than zero is a basis. As a second example, the three-dimensional vectors of ordinary vector analysis

constitute an IPS, with the usual definition of the inner (dot) product. *The symbol $\mathcal{V}$ will be reserved for this particular* IPS. A third example of an IPS that is particularly important in continuum mechanics is the set of linear transformations of $\mathcal{V}$ into $\mathcal{V}$, which will be discussed in the next subsection.

2.1.2 Linear Transformations

Let $\mathcal{U}$ and $\mathcal{W}$ be inner product spaces. A *linear transformation*[2] of $\mathcal{U}$ into $\mathcal{W}$, denoted by $\mathbf{L} : \mathcal{U} \rightarrow \mathcal{W}$, associates with each vector $\mathbf{u}$ in $\mathcal{U}$ a vector $\mathbf{Lu}$ in $\mathcal{W}$ such that, for any scalars α, β and vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathcal{U}$,

$$\mathbf{L}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{L}\mathbf{u} + \beta\mathbf{L}\mathbf{v}. \tag{2.15}$$

The sum of two linear transformations and the product of a scalar and a linear transformation are defined such that, for each scalar α and vector $\mathbf{u}$ in $\mathcal{U}$,

$$(\mathbf{L}_1 + \mathbf{L}_2)\,\mathbf{u} = \mathbf{L}_1\mathbf{u} + \mathbf{L}_2\mathbf{u}, \tag{2.16}$$

$$(\alpha\mathbf{L})\mathbf{u} = \mathbf{L}(\alpha\mathbf{u}). \tag{2.17}$$

Recall that $\mathcal{V}$ denotes the IPS of ordinary three-dimensional vector analysis, and consider linear transformations of $\mathcal{V}$ into itself. The rest of this subsection will be concerned with linear transformations of this kind, which are called *second-order tensors*.Three simple examples are the zero tensor $\mathbf{0}$, the identity tensor $\mathbf{1}$, and the tensor product $\mathbf{u} \otimes \mathbf{v}$, which are defined such that, for any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathcal{V}$,

$$\mathbf{0}\mathbf{v} = \mathbf{o}, \tag{2.18}$$

$$\mathbf{1}\mathbf{v} = \mathbf{v}, \tag{2.19}$$

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w}). \tag{2.20}$$

Let $\{\mathbf{e}_k\} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis for $\mathcal{V}$. Then each vector $\mathbf{v}$ in $\mathcal{V}$ can be written as the linear combination

$$\mathbf{v} = v_k\mathbf{e}_k, \tag{2.21}$$

where the coefficients v_k are called the *components* of $\mathbf{v}$ with respect to $\{\mathbf{e}_k\}$. If $\mathbf{T}$ is a linear tranformation, the equation

$$\mathbf{T}\mathbf{u} = \mathbf{v} \tag{2.22}$$

[2] Linear transformations will be denoted by bold capital letters, with exceptions that will be defined individually.

can be written

$$\mathbf{T}u_k\mathbf{e}_k = v_k\mathbf{e}_k. \tag{2.23}$$

Taking the inner product of this equation with $\mathbf{e}_m$ results in the equation

$$T_{mk}u_k = v_m, \tag{2.24}$$

where the scalars

$$T_{mk} = (\mathbf{T})_{mk} = \mathbf{e}_m \cdot \mathbf{T}\mathbf{e}_k \tag{2.25}$$

are called the *components* of $\mathbf{T}$ with respect to $\{\mathbf{e}_k\}$. For example, the components of the linear transformations $\mathbf{u} \otimes \mathbf{v}$ and $\mathbf{1}$ with respect to $\{\mathbf{e}_k\}$ are easily shown to be

$$(\mathbf{u} \otimes \mathbf{v})_{mk} = u_m v_k, \quad (\mathbf{1})_{mk} = \delta_{mk}, \tag{2.26}$$

where the *Kronecker delta* δ_{mk} is defined by

$$\delta_{mk} = \begin{cases} 1 & \text{if } m = k, \\ 0 & \text{if } m \neq k. \end{cases} \tag{2.27}$$

The *transpose* of a linear transformation $\mathbf{T}$ is defined to be the linear transformation $\mathbf{T}^{\mathrm{t}}$ such that, for any vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathcal{V}$,

$$\mathbf{u} \cdot \mathbf{T}\mathbf{v} = \mathbf{T}^{\mathrm{t}}\mathbf{u} \cdot \mathbf{v}. \tag{2.28}$$

The components of $\mathbf{T}^{\mathrm{t}}$ are

$$T^{\mathrm{t}}_{km} = T_{mk}. \tag{2.29}$$

The *product* of two linear transformations $\mathbf{S}$ and $\mathbf{T}$, denoted by $\mathbf{ST}$, is defined to be the linear transformation

$$\mathbf{STv} = \mathbf{S}(\mathbf{Tv}). \tag{2.30}$$

The components of $\mathbf{ST}$ are easily shown to be

$$(\mathbf{ST})_{km} = S_{kj}T_{jm}. \tag{2.31}$$

The *determinant* of a linear transformation $\mathbf{T}$, denoted by $\det \mathbf{T}$, is defined such that

$$\det \mathbf{T} = \det [T_{km}], \tag{2.32}$$

where $[T_{km}]$ denotes the matrix of the components of $\mathbf{T}$. Two results concerning determinants that will be useful are

$$\frac{\partial(\det \mathbf{T})}{\partial T_{km}} = \operatorname{cof} T_{km}, \quad \delta_{km} \det \mathbf{T} = T_{kj} \operatorname{cof} T_{mj}, \tag{2.33}$$

where cof T_{km} is the cofactor of the element T_{km} of $[T_{km}]$.

The *inverse* of a linear transformation $\mathbf{T}$ is the linear transformation $\mathbf{T}^{-1}$ such that

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{1}. \tag{2.34}$$

The components of $\mathbf{T}^{-1}$ are

$$T^{-1}_{km} = \operatorname{cof} T_{mk} / \det \mathbf{T}. \tag{2.35}$$

The *trace* of a linear transformation $\mathbf{T}$ is defined by[3]

$$\operatorname{tr} \mathbf{T} = T_{kk}, \tag{2.36}$$

and the *inner product* of two linear transformations $\mathbf{T}$ and $\mathbf{S}$ is defined by

$$\mathbf{S} \cdot \mathbf{T} = \operatorname{tr} \left(\mathbf{S}^{\mathrm{t}}\mathbf{T}\right) = S_{km} T_{km}. \tag{2.37}$$

It can be shown that *the set of all linear transformations* $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ *with the inner product (2.37) is an inner product space*.

2.1.3 Functions, Continuity, and Differentiability

Let $\mathcal{U}$ and $\mathcal{W}$ be inner product spaces, and let U be a subset of $\mathcal{U}$. A *function* $\mathbf{f} : U \to \mathcal{W}$ associates with each vector $\mathbf{u}$ in U a vector $\mathbf{f}(\mathbf{u})$ in $\mathcal{W}$. The concept of the magnitude of a vector in an IPS, defined by (2.14), makes it possible to define the limit, continuity, and differentiability of the function $\mathbf{f}(\mathbf{u})$ in a manner entirely analogous to ordinary calculus.

A vector $\mathbf{w}$ in $\mathcal{W}$ is said to be the *limit* of $\mathbf{f}(\mathbf{u})$ at a vector $\mathbf{u}_0$ in $\mathcal{U}$ if, for any positive scalar α, there is a positive scalar β such that $|\mathbf{f}(\mathbf{u}) - \mathbf{w}| < \alpha$ for each vector $\mathbf{u}$ in U that satisfies the relation $0 < |\mathbf{u} - \mathbf{u}_0| < \beta$. The function $\mathbf{f}(\mathbf{u})$ is said to be *continuous* at a vector $\mathbf{u}_0$ in U if the limit $\mathbf{w}$ exists and $\mathbf{f}(\mathbf{u}_0) = \mathbf{w}$, and it is said to be *continuous in* U if it is continuous at each vector in U.

[3] The determinant and trace of a linear transformation can be defined in a way that is independent of any basis (see e.g. Bowen and Wang [14], Section 40). The definitions given here are adequate for the purposes of this monograph.

The set U is called an *open* subset of $\mathcal{U}$ if, for each vector $\mathbf{u}_0$ in U, there is a positive scalar α such that the vector $\mathbf{u}_0 + \mathbf{u}$ is in U for each vector $\mathbf{u}$ in U that satisfies the relation $|\mathbf{u}| < \alpha$.

Let U be an open subset of $\mathcal{U}$. A function $\mathbf{f} : U \to \mathcal{W}$ is said to be *differentiable* at a vector $\mathbf{u}_0$ in U if there is a linear transformation, denoted by $d\mathbf{f}/d\mathbf{u} : \mathcal{U} \to \mathcal{W}$ such that

$$\mathbf{f}(\mathbf{u}_0) - \mathbf{f}(\mathbf{u}) = \frac{d\mathbf{f}}{d\mathbf{u}}(\mathbf{u}_0 - \mathbf{u}) + o\left(|\mathbf{u}_0 - \mathbf{u}|\right). \tag{2.38}$$

The notation $o(\alpha)$ means that $|o(\alpha)/\alpha| \to 0$ as $\alpha \to 0$. The linear transformation $d\mathbf{f}/d\mathbf{u}$ is called the *derivative*[4] of $\mathbf{f}(\mathbf{u})$ at $\mathbf{u}_0$. The function $\mathbf{f}(\mathbf{u})$ is said to be *differentiable in* U if it is differentiable at each vector in U and $d\mathbf{f}/d\mathbf{u}$ is continuous in U.

2.1.4 Fields and the Divergence Theorem

In continuum mechanics the properties of materials are described in terms of piecewise continuous functions called *fields*. Hamilton's principle for a continuous medium will be stated in terms of a prescribed volume of material. Definitions and terminology associated with fields and volumes are introduced in this subsection.

Let a reference point O and an orthonormal basis $\{\mathbf{e}_k\}$ define an inertial reference frame in three-dimensional Euclidean space $\mathcal{E}$, and let the vector $\mathbf{X}$ in $\mathcal{V}$ denote the position vector of a point in $\mathcal{E}$ relative to O. Consider a closed surface ∂B in $\mathcal{E}$. Let B be the interior of the surface ∂B, and let the interior together with its surface (called the *closure*) be denoted by $\bar{B}$.

It will be assumed that B is a *bounded regular region*, and that the surface ∂B may consist of complementary *regular subsurfaces* ∂B_1 and ∂B_2. Precise definitions of bounded regular regions and regular subsurfaces (which insure, for example, that the divergence theorem can be applied) are given by Gurtin ([34], pp. 12–14). A volume that is bounded by a single closed surface consisting of a finite number of smooth subsurfaces, each of which is bounded by a piecewise smooth curve, is a bounded regular region. If the surface of such a volume is divided into two parts by a single piecewise smooth closed curve, the resulting complementary subsurfaces are regular subsurfaces.

Let $\mathcal{W}$ be an inner product space. A *field* $\mathbf{f} : B \to \mathcal{W}$ is a function that associates with each point in B (identified with its position vector $\mathbf{X}$) a vector $\mathbf{f}(\mathbf{X})$ in $\mathcal{W}$. In the cases in which the elements of $\mathcal{W}$ are scalars, vectors, or second-order tensors, $\mathbf{f}(\mathbf{X})$ is called a scalar, vector, or tensor field.

[4] A variety of notations are used for this linear transformation, including $\Delta\mathbf{f}$ and $D\mathbf{f}(\mathbf{u})$. The notation used here was chosen so that it would look familiar to persons used to ordinary derivatives, and also because it makes expressions in which the chain rule is used more intelligible.

As an example, consider a scalar field $\phi(\mathbf{X})$, and let $\mathbf{Z}$ be any vector in $\mathcal{V}$. If $\phi(\mathbf{X})$ is differentiable at a point $\mathbf{X}$ in B, then

$$\frac{d\phi}{d\mathbf{X}}\mathbf{Z} = \text{GRAD}\ \phi \cdot \mathbf{Z}, \tag{2.39}$$

where GRAD ϕ is the familiar gradient

$$\text{GRAD}\ \phi = \frac{\partial \phi}{\partial X_k}\mathbf{e}_k. \tag{2.40}$$

In the case of a vector field $\mathbf{v}(\mathbf{X})$ that is differentiable at a point $\mathbf{X}$ in B, the derivative $d\mathbf{v}/d\mathbf{X}$ is called the gradient of the vector field. In terms of components,

$$\left(\frac{d\mathbf{v}}{d\mathbf{X}}\right)_{km} = \frac{\partial v_k}{\partial X_m}. \tag{2.41}$$

Note that the divergence of the vector field $\mathbf{v}(\mathbf{X})$ is

$$\text{DIV}\ \mathbf{v} = \text{tr}\ \frac{d\mathbf{v}}{d\mathbf{X}} = \frac{\partial v_k}{\partial X_k}. \tag{2.42}$$

The divergence of a tensor field $\mathbf{T}(\mathbf{X})$ that is differentiable at a point $\mathbf{X}$ in B is defined to be the vector DIV $\mathbf{T}$ with the property that, for each vector $\mathbf{Z}$ in $\mathcal{V}$,

$$(\text{DIV}\ \mathbf{T}) \cdot \mathbf{Z} = \text{DIV}\ \left(\mathbf{T}^{\text{t}}\mathbf{Z}\right). \tag{2.43}$$

The components of DIV $\mathbf{T}$ are

$$(\text{DIV}\ \mathbf{T})_k = \frac{\partial T_{km}}{\partial X_m}. \tag{2.44}$$

Let $\mathbf{f}(\mathbf{X})$ be a field that is continuous in B, and let $\mathbf{X}_0$ be a point of the surface ∂B. If the limit of $\mathbf{f}(\mathbf{X})$ as $\mathbf{X} \to \mathbf{X}_0$ exists at each point of ∂B and is continuous on ∂B, then the field $\mathbf{f}(\mathbf{X})$ is said to have a *continuous extension* to the closure $\bar{B}$ if its value at each point $\mathbf{X}_0$ of ∂B is defined to be the value of its limit at that point.

The fields considered in this work will usually be functions of both position and time. A *time-dependent field* $\mathbf{f} : B \times (t_1, t_2) \to \mathcal{W}$ is a function that associates with each point in B and each time in the open interval $t_1 < t < t_2$ a vector $\mathbf{f}(\mathbf{X}, t)$ in $\mathcal{W}$.

A vector $\mathbf{w}$ in $\mathcal{W}$ is said to be the *limit* of $\mathbf{f}(\mathbf{X}, t)$ at the position and time $\mathbf{X}_0, t_0$ in $B \times (t_1, t_2)$ if, for any positive scalar α, there is a positive scalar β such that

$$|\mathbf{f}(\mathbf{X}, t) - \mathbf{w}| < \alpha \tag{2.45}$$

for each $\mathbf{X}, t$ in $B \times (t_1, t_2)$ that satisfy the relation

$$0 < \sqrt{|\mathbf{X} - \mathbf{X}_0|^2 + (t - t_0)^2} < \beta. \tag{2.46}$$

The field $\mathbf{f}(\mathbf{X}, t)$ is said to be *continuous* at $\mathbf{X}_0, t_0$ if the limit $\mathbf{w}$ exists and $\mathbf{f}(\mathbf{X}_0, t_0) = \mathbf{w}$, and it is said to be *continuous in* $B \times (t_1, t_2)$ if it is continuous at each $\mathbf{X}_0, t_0$ in $B \times (t_1, t_2)$.

Let $\partial^n \mathbf{f}/\partial \mathbf{X}^n$ denote the nth derivative of $\mathbf{f}(\mathbf{X}, t)$ holding t fixed. Then $\mathbf{f}(\mathbf{X}, t)$ is said to be C^N in $B \times (t_1, t_2)$ if it is continuous in $B \times (t_1, t_2)$ and the derivatives

$$\frac{\partial^m}{\partial t^m}\left(\frac{\partial^n \mathbf{f}}{\partial \mathbf{X}^n}\right), \quad 0 \le m \le N, \quad 0 \le n \le N, \quad m + n \le N \tag{2.47}$$

exist and are continuous in $B \times (t_1, t_2)$. Such a field is then said to be C^N on $\bar{B} \times (t_1, t_2)$ if these derivatives have continuous extensions to $\bar{B} \times (t_1, t_2)$.

Let ∂B_α be a complementary regular subsurface of B, and let the vector function $\mathbf{N}(\mathbf{X})$ defined on ∂B_α be the outward-directed unit vector normal to ∂B_α at each point $\mathbf{X}$ of ∂B_α. A point $\mathbf{X}$ at which $\mathbf{N}(\mathbf{X})$ is continuous is called a *regular point* of ∂B_α.

A function $\mathbf{f}(\mathbf{X})$ defined on ∂B_α is called *piecewise regular* if it is piecewise continuous on ∂B_α and is continuous at each regular point of ∂B_α. A time-dependent function $\mathbf{f}(\mathbf{X}, t)$ defined on $\partial B_\alpha \times (t_1, t_2)$ is called piecewise regular if it is piecewise continuous on $\partial B_\alpha \times (t_1, t_2)$ and $\mathbf{f}(\mathbf{X}, t_0)$ is piecewise regular on ∂B_α for each fixed time t_0 in $[t_1, t_2]$. A function $\mathbf{f}(\mathbf{X}, t)$ defined on $\partial B_\alpha \times (t_1, t_2)$ is said to be *continuous in time* if, for each fixed point $\mathbf{X}$ of ∂B_α, it is a continuous function of time in $[t_1, t_2]$.

Two functions $\mathbf{f}_1(\mathbf{X}, t)$ and $\mathbf{f}_2(\mathbf{X}, t)$ defined on $\partial B_\alpha \times (t_1, t_2)$ are defined to be equal if, for each time t in $[t_1, t_2]$, they are equal at each regular point of ∂B_α.

The divergence theorem will be used frequently in applying Hamilton's principle to continuous media. The following statement is paraphrased from Gurtin ([34], p. 16): Let $\phi(\mathbf{X})$, $\mathbf{v}(\mathbf{X})$, and $\mathbf{T}(\mathbf{X})$ be scalar, vector, and tensor fields that are continuous on $\bar{B}$ and differentiable in B. Then

$$\int_{\partial B} \phi \mathbf{N}\, dS = \int_B \text{GRAD}\, \phi\, dV, \tag{2.48}$$

$$\int_{\partial B} \mathbf{v} \cdot \mathbf{N}\, dS = \int_B \text{DIV}\, \mathbf{v}\, dV, \tag{2.49}$$

$$\int_{\partial B} \mathbf{T}\mathbf{N}\, dS = \int_B \text{DIV}\, \mathbf{T}\, dV \tag{2.50}$$

when the integrands on the right are piecewise continuous on $\bar{B}$. Recall that $\mathbf{N}$ is the outward-directed unit vector that is normal to ∂B.

Suggested references on the mathematical foundations of continuum mechanics include Bowen and Wang [14], Ericksen [25], Gurtin [35], Halmos [36], Leigh [52], Truesdell and Noll [70], and Truesdell and Toupin [71].

2.2 Motion and Deformation

A *motion* of a material that is modeled as a continuous medium is described by a time-dependent vector field

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \tag{2.51}$$

where $\mathbf{x}$ is the position vector at time t of the *material point* identified with its position vector $\mathbf{X}$ in a reference state, or *reference configuration*. As a simple example, consider a quantity of some malleable material, such as dough, which is at rest. This rest state can be used as the reference configuration. Imagine that a point on the surface or within the material is marked with a pen. Let its position vector be $\mathbf{X}_0$. Then if the material is picked up and deformed, and (2.51) describes its motion, the trajectory in space of the marked point is given by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}_0, t). \tag{2.52}$$

Thus (2.51) describes the motion of each point of the material.

In general, it is not necessary that the reference configuration be one which the material has actually assumed at any time. However, this distinction is not needed for any of the applications to be considered in this monograph. The reference configuration will be assumed to be the configuration of the material at time t_1. That is,

$$\mathbf{X} = \boldsymbol{\chi}(\mathbf{X}, t_1). \tag{2.53}$$

Suppose that in its reference configuration, the material occupies a bounded regular region B with surface ∂B. The motion (2.51) maps the material onto a volume B_t with surface ∂B_t at time t (Fig. 2.1). In keeping with the interpretation of (2.51) as the motion of a material, the mapping of the material points from $\bar{B}$ to $\bar{B}_t$ will be assumed to be *one-one*. That is, if $\mathbf{X}_1$ and $\mathbf{X}_2$ are distinct points of $\bar{B}$, then $\mathbf{x}_1 = \boldsymbol{\chi}(\mathbf{X}_1, t)$ and $\mathbf{x}_2 = \boldsymbol{\chi}(\mathbf{X}_2, t)$ are distinct points of $\bar{B}_t$, and for each point $\mathbf{x}$ of $\bar{B}_t$, there is a point $\mathbf{X}$ of $\bar{B}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. This requirement insures that the *inverse motion*

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t), \tag{2.54}$$

which maps the material points from $\bar{B}_t$ onto $\bar{B}$ at time t, exists and is one-one.

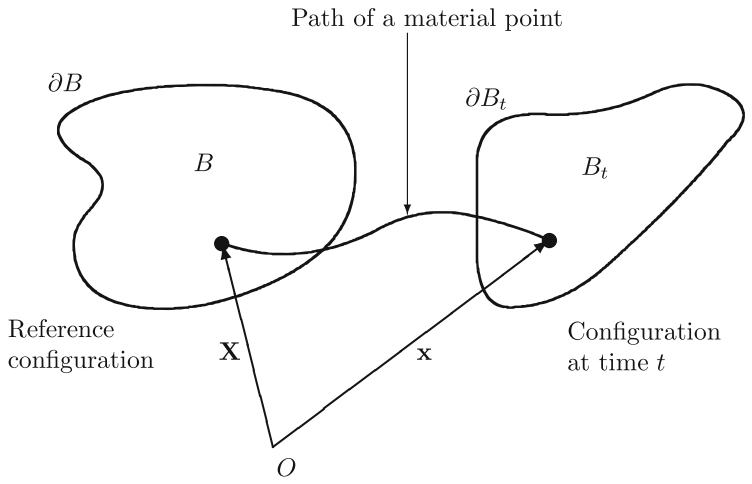

Fig. 2.1 Motion of a material

Suppose that the field (2.51) is C^N on $\bar{B} \times [t_1, t_2]$, $N \geq 1$. The *deformation gradient* $\mathbf{F}$ is the tensor field[5]

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \quad F_{km} = \frac{\partial \chi_k}{\partial X_m}. \tag{2.55}$$

The *Jacobian* of the motion is defined by

$$J = \det \mathbf{F}. \tag{2.56}$$

A necessary condition for (2.51) to describe the motion of a material is that $J \geq 0$ in B. It will be seen that this condition insures that the volume of every element of the material remains positive. Given that it is satisfied, it can be shown (see e.g. Gurtin [35], pp. 60, 65–66) that the inverse motion (2.54) is C^N on $\bar{B}_t \times [t_1, t_2]$.

The interpretation of the motion (2.51) as describing the trajectory of a material point in space motivates the definitions of the *velocity*

$$\mathbf{v} = \frac{\partial}{\partial t} \boldsymbol{\chi}(\mathbf{X}, t) \tag{2.57}$$

and the *acceleration*

$$\mathbf{a} = \frac{\partial^2}{\partial t^2} \boldsymbol{\chi}(\mathbf{X}, t). \tag{2.58}$$

[5] Some expressions will be presented both in direct notation and in terms of components for the sake of clarity.

The inverse motion (2.54) can be used to express the velocity and acceleration as functions of $\mathbf{x}, t$. When the functional dependence of a field is not obvious from the context, a caret $(\hat{\cdot})$ will be used to indicate that it is expressed in terms of $\mathbf{X}, t$. The caret will not be used when the functional dependence is shown explicitly. For example,

$$\mathbf{v}(\mathbf{X}, t) = \hat{\mathbf{v}}(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t) = \mathbf{v}(\mathbf{x}, t), \tag{2.59}$$

$$\mathbf{a} = \frac{\partial}{\partial t}\hat{\mathbf{v}} = \frac{\partial}{\partial t}\mathbf{v} + \mathbf{L}\mathbf{v}, \tag{2.60}$$

where the linear transformation $\mathbf{L}$ is the *velocity gradient*

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \quad L_{km} = \frac{\partial v_k}{\partial x_m}. \tag{2.61}$$

The *material derivative* of a field $\mathbf{f}(\mathbf{X}, t)$ is defined by

$$\dot{\mathbf{f}} = \frac{\partial}{\partial t}\hat{\mathbf{f}}. \tag{2.62}$$

Thus, the material derivative is the time rate of change of a field holding the material point fixed. For example, notice that the acceleration $\mathbf{a} = \dot{\mathbf{v}}$. In the case of a scalar field $\phi(\mathbf{X}, t)$,

$$\dot{\phi} = \frac{\partial}{\partial t}\hat{\phi} = \frac{\partial}{\partial t}\phi + \mathbf{v} \cdot \operatorname{grad} \phi, \tag{2.63}$$

where $\operatorname{grad} \phi = (\partial \phi / \partial x_k)\mathbf{e}_k$.

The motion (2.51) maps a volume element dV of B onto a volume element dV_t of B_t at time t. It can be shown (see e.g. Truesdell and Toupin [71], pp. 247–249) that

$$dV_t = J\, dV. \tag{2.64}$$

The *density* ρ is a scalar field defined such that the mass of each volume element dV_t of B_t is $\rho\, dV_t$. Let the value of ρ at time t_1 be denoted by ρ_R. That is, ρ_R is the density of the reference configuration. Then one form of the equation of *conservation of mass* is

$$\rho\, dV_t = \rho_R\, dV. \tag{2.65}$$

Using (2.64), this equation can be expressed in the form

$$J = \frac{\rho_R}{\rho}. \tag{2.66}$$

The material derivative of the Jacobian is

$$\dot{J} = \frac{\partial(\det \mathbf{F})}{\partial F_{km}} \frac{\partial \hat{F}_{km}}{\partial t}. \tag{2.67}$$

From (2.55),

$$\frac{\partial \hat{F}_{km}}{\partial t} = \frac{\partial^2 \chi_k}{\partial t \partial X_m} = \frac{\partial v_k}{\partial x_p} \frac{\partial x_p}{\partial X_m} = L_{kp} F_{pm}. \tag{2.68}$$

Substituting this result into (2.67) and using (2.33) yields the relation

$$\dot{J} = J \text{ tr } \mathbf{L} = J \text{ div } \mathbf{v}, \tag{2.69}$$

where div $\mathbf{v} = \partial v_k / x_k$. Taking the material derivative of (2.66) and using (2.69) results in the equation of conservation of mass in its more familiar form

$$\dot{\rho} + \rho \text{ div } \mathbf{v} = 0. \tag{2.70}$$

The motion (2.51) maps a surface element dS of ∂B onto a surface element dS_t of ∂B_t at time t. Let the function $\mathbf{n}(\mathbf{x}, t)$ defined on ∂B_t denote the outward-directed unit vector that is normal to ∂B_t, and let $\mathbf{N}(\mathbf{X}) = \hat{\mathbf{n}}(\mathbf{X}, t_1)$. That is, $\mathbf{N}$ is the outward-directed unit vector normal to ∂B. It can be shown (see e.g. Truesdell and Toupin [71], pp. 247–249) that

$$\mathbf{n}\, dS_t = J\, \mathbf{F}^{-\mathrm{t}} \mathbf{N}\, dS, \tag{2.71}$$

where $\mathbf{F}^{-\mathrm{t}} = \left(\mathbf{F}^{-1}\right)^{\mathrm{t}}$.

By means of the relations (2.65) and (2.71), integrals on B and ∂B can be expressed as integrals on B_t and ∂B_t, and *vice versa*. If a field $\mathbf{f}(\mathbf{X}, t)$ is continuous on $\bar{B} \times [t_1, t_2]$, then[6]

$$\int_{B_t} \mathbf{f}\, dV_t = \int_B \mathbf{f} J\, dV. \tag{2.72}$$

Similarly, if a scalar function $\phi(\mathbf{X}, t)$ defined on ∂B is piecewise regular, then

$$\int_{\partial B_t} \phi \mathbf{n}\, dS_t = \int_{\partial B} \phi J\, \mathbf{F}^{-\mathrm{t}} \mathbf{N}\, dS. \tag{2.73}$$

[6] In (2.72) the functional dependence of the field $\mathbf{f}$ is indicated by the context. It must be expressed in terms of $\mathbf{x}, t$ in the left integral and in terms of $\mathbf{X}, t$ in the right integral.

Consider two neighboring material points in B having position vectors $\mathbf{X}$ and $\mathbf{X} + d\mathbf{X}$. The square of the distance separating them is

$$dS^2 = d\mathbf{X} \cdot d\mathbf{X}. \tag{2.74}$$

At time t, the same two material points are separated by the vector

$$\begin{aligned} d\mathbf{x} &= \chi_k \left(X_m + dX_m, t\right) \mathbf{e}_k - \chi_k \left(X_m, t\right) \mathbf{e}_k \\ &= \frac{\partial \chi_k}{\partial X_m} dX_m \mathbf{e}_k \\ &= \mathbf{F}\, d\mathbf{X}, \end{aligned} \tag{2.75}$$

so that the square of the distance separating the points at time t is

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{t}}\mathbf{F}\, d\mathbf{X}. \tag{2.76}$$

Therefore

$$ds^2 - dS^2 = d\mathbf{X} \cdot (\mathbf{C} - \mathbf{1})\, d\mathbf{X}, \tag{2.77}$$

where

$$\mathbf{C} = \mathbf{F}^{\mathrm{t}}\mathbf{F} \tag{2.78}$$

is called the *right Cauchy–Green strain tensor*. Because (2.77) determines the change in the distance between any two neighboring points at time t, the deformation gradient $\mathbf{F}$, or deformation measures that are expressed in terms of $\mathbf{F}$ such as the Cauchy–Green strain tensor, *determines the deformation of the material in the neighborhood of a material point*.

The *displacement* is the vector field

$$\mathbf{u} = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X}. \tag{2.79}$$

It is the displacement vector of a material point relative to its position in the reference configuration. The *displacement gradient* is the tensor field

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{F} - \mathbf{1}, \quad \frac{\partial u_k}{\partial X_m} = F_{km} - \delta_{km}. \tag{2.80}$$

In terms of the displacement gradient, the Cauchy–Green strain tensor is

$$\mathbf{C} = \mathbf{1} + 2\mathbf{E} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^{\mathrm{t}} \frac{\partial \mathbf{u}}{\partial \mathbf{X}}, \tag{2.81}$$

where $\mathbf{E}$ is the *linear strain tensor*

$$\mathbf{E} = \frac{1}{2}\left[\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^{\mathrm{t}}\right], \quad E_{km} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_m} + \frac{\partial u_m}{\partial X_k}\right). \tag{2.82}$$

Recommended references on the motion and deformation of a continuous medium include Eringen [27], Gurtin [35], Leigh [52], Truesdell and Noll [70], and Truesdell and Toupin [71].

2.3 The Comparison Motion

In applying Hamilton's principle to a system of particles, the equations of motion were obtained by introducing a comparison motion (1.15). An identical approach is taken in applying variational methods to a continuous medium.

A motion (2.51) from the reference configuration at time t_1 to a specified configuration at time t_2 will be called *admissible* if it is C^2 on $\bar{B} \times [t_1, t_2]$ and satisfies prescribed boundary conditions on ∂B. An admissible *comparison motion* will be defined by

$$\mathbf{x}^* = \boldsymbol{\chi}(\mathbf{X}, t) + \varepsilon \boldsymbol{\eta}(\mathbf{X}, t). \tag{2.83}$$

Here ε is a parameter and $\boldsymbol{\eta}(\mathbf{X}, t)$ is an arbitrary C^2 vector field on $\bar{B} \times [t_1, t_2]$ subject to the requirements that $\boldsymbol{\eta}(\mathbf{X}, t_1) = \mathbf{o}$ and $\boldsymbol{\eta}(\mathbf{X}, t_2) = \mathbf{o}$. The vector field $\boldsymbol{\eta}$ is also subject to the requirement that the comparison motion must satisfy the prescribed boundary conditions on ∂B.

The comparison motion maps the material from B onto a volume B_t^* with surface ∂B_t^* at time t (see Fig. 2.2). From (2.83), the velocity, gradient, and Jacobian of the comparison motion are

$$\mathbf{v}^* = \mathbf{v} + \varepsilon \dot{\hat{\boldsymbol{\eta}}}, \tag{2.84}$$

$$\mathbf{F}^* = \mathbf{F} + \varepsilon \frac{\partial \hat{\boldsymbol{\eta}}}{\partial \mathbf{X}}, \quad F_{km}^* = F_{km} + \varepsilon \frac{\partial \hat{\eta}_k}{\partial X_m}, \tag{2.85}$$

$$J^* = \det \mathbf{F}^*. \tag{2.86}$$

The derivative of J^* with respect to ε is

$$\frac{\partial J^*}{\partial \varepsilon} = \frac{\partial (\det \mathbf{F}^*)}{\partial F_{km}^*} \frac{\partial F_{km}^*}{\partial \varepsilon} = \frac{\partial (\det \mathbf{F}^*)}{\partial F_{km}^*} \frac{\partial \hat{\eta}_k}{\partial X_m}. \tag{2.87}$$

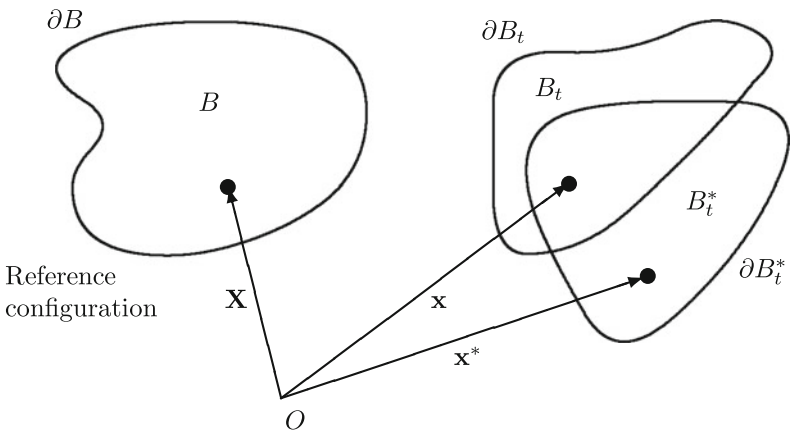

Fig. 2.2 The reference configuration, the configuration at time t, and the configuration resulting from the comparison motion at time t

Recalling the notation

$$\delta(\cdot) \equiv \left[\frac{\partial}{\partial \varepsilon}(\cdot)^*\right]_{\varepsilon=0}, \tag{2.88}$$

(2.33) and (2.87) can be used to obtain the result (see e.g. [28])

$$\delta J = \left(\frac{\partial J^*}{\partial \varepsilon}\right)_{\varepsilon=0} = J \text{ div } \boldsymbol{\eta}. \tag{2.89}$$

Therefore, J^* can be written

$$J^* = J(1 + \varepsilon \text{ div } \boldsymbol{\eta}) + O(\varepsilon^2), \tag{2.90}$$

where the notation $O(\varepsilon^2)$ means that $|O(\varepsilon^2)/\varepsilon| \to 0$ as $\varepsilon \to 0$.

The density of the comparison motion can be determined from the equation of conservation of mass (2.66):

$$\rho^* = \frac{\rho_R}{J^*}. \tag{2.91}$$

Substituting (2.90) into this equation results in the expression

$$\rho^* = \rho(1 - \varepsilon \text{ div } \boldsymbol{\eta}) + O(\varepsilon^2). \tag{2.92}$$

In applications of Hamilton's principle to a continuous medium, it is often convenient to introduce a *comparison field* for the density in the form

$$\rho^* = \rho(\mathbf{X}, t) + \varepsilon r(\mathbf{X}, t). \tag{2.93}$$

The preceding two equations show that, as a consequence of the equation of conservation of mass, the scalar field

$$r = -\rho \operatorname{div} \boldsymbol{\eta} + O(\varepsilon).$$

However, *the fields* $\boldsymbol{\eta}$ *and* r *can be regarded as independent if the equation of conservation of mass is introduced into Hamilton's principle as a constraint* (see the discussion of constraints in Sect. 1.2.2). In such cases, it will be assumed that $r(\mathbf{X}, t)$ is an arbitrary C^1 scalar field on $\bar{B} \times [t_1, t_2]$ such that $r(\mathbf{X}, t_1) = 0$ and $r(\mathbf{X}, t_2) = 0$.

The comparison motion maps a volume element dV of B onto a volume element dV_t^* of B_t^* at time t. Similarly, it maps a surface element dS of ∂B onto a surface element dS_t^* of ∂B_t^* at time t. The relations between these volume and surface elements can be obtained from (2.64) and (2.71):

$$dV_t^* = J^* \, dV, \tag{2.94}$$

$$\mathbf{n}^* dS_t^* = J^* \left(\mathbf{F}^*\right)^{-\mathrm{t}} \mathbf{N} \, dS. \tag{2.95}$$

Here $\mathbf{n}^*$ is the outward directed unit vector that is normal to ∂B_t^*.

As described in Chapter 1, Hamilton's principle is a postulate concerning the mechanical behavior of a system. The equations of motion are derived from the postulate as necessary conditions. Two examples of the types of analysis involved in obtaining equations of motion from statements of Hamilton's principle for a continuous medium will be presented in the remainder of this section. The methods used are quite similar to those that were used in the case of a system of particles.

In analogy with the kinetic energy of a particle, the kinetic energy of the material in an element of volume dV_t of B_t is $\frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} \, dV_t$. Therefore the total kinetic energy of the material occupying the volume B_t is

$$T = \int_{B_t} \tfrac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} \, dV_t = \int_B \tfrac{1}{2}\rho_R \mathbf{v} \cdot \mathbf{v} \, dV. \tag{2.96}$$

Consider the integral of T with respect to time from t_1 to t_2:

$$I = \int_{t_1}^{t_2} T \, dt = \int_{t_1}^{t_2} \int_B \tfrac{1}{2}\rho_R \mathbf{v} \cdot \mathbf{v} \, dV \, dt. \tag{2.97}$$

When it is expressed in terms of the comparison motion (2.83), this integral becomes

$$I^*(\varepsilon) = \int_{t_1}^{t_2} \int_B \tfrac{1}{2}\rho_R \mathbf{v}^* \cdot \mathbf{v}^* \, dV \, dt. \tag{2.98}$$

Taking the derivative of this equation with respect to ε,

$$\frac{dI^*(\varepsilon)}{d\varepsilon} = \int_{t_1}^{t_2} \int_B \rho_R \mathbf{v}^* \cdot \dot{\boldsymbol{\eta}} \, dV \, dt, \tag{2.99}$$

and setting $\varepsilon = 0$ yields

$$\left[\frac{dI^*(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = \int_{t_1}^{t_2}\int_B \rho_R \mathbf{v}\cdot\dot{\boldsymbol{\eta}}\, dV\, dt. \tag{2.100}$$

Integrating the expression on the right by parts with respect to time gives

$$\int_{t_1}^{t_2} \rho_R \mathbf{v}\cdot\dot{\boldsymbol{\eta}}\, dt = \Big[\rho_R \mathbf{v}\cdot\boldsymbol{\eta}\Big]_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho_R \mathbf{a}\cdot\boldsymbol{\eta}\, dt. \tag{2.101}$$

Using this result and recalling that $\boldsymbol{\eta}$ vanishes at t_1 and t_2, (2.100) becomes

$$\left[\frac{dI^*(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = -\int_{t_1}^{t_2}\int_B \rho_R \mathbf{a}\cdot\boldsymbol{\eta}\, dV\, dt = -\int_{t_1}^{t_2}\int_{B_t} \rho \mathbf{a}\cdot\boldsymbol{\eta}\, dV_t\, dt, \tag{2.102}$$

so that

$$\delta\mathbf{T} = -\int_B \rho_R \mathbf{a}\cdot\delta\mathbf{x}\, dV = -\int_{B_t} \rho \mathbf{a}\cdot\delta\mathbf{x}\, dV_t, \tag{2.103}$$

where

$$\delta\mathbf{x} = \boldsymbol{\eta}. \tag{2.104}$$

Note from (2.83) that this definition of $\delta\mathbf{x}$ is consistent with the notation (2.88).

As a second example, consider the integral

$$C = \int_B \pi\left(J - \frac{\rho_R}{\rho}\right) dV = \int_{B_t} \pi\left(1 - \frac{\rho_R}{\rho J}\right) dV_t. \tag{2.105}$$

This expression is the form in which the equation of conservation of mass will be introduced as a constraint in Hamilton's principle for a continuous medium. The scalar field $\pi(\mathbf{X}, t)$, which is assumed to be C^1 on $\bar{B}\times[t_1, t_2]$, is a Lagrange multiplier. Expressed in terms of the comparison motion (2.83) and the comparison field (2.93), this integral becomes

$$C^*(\varepsilon) = \int_B \pi\left(J^* - \frac{\rho_R}{\rho^*}\right) dV. \tag{2.106}$$

Taking the derivative of this equation with respect to ε, setting $\varepsilon = 0$, and using (2.89) to evaluate the derivative of the Jacobian yields

$$\delta C = \int_B \pi J\left(\operatorname{div}\boldsymbol{\eta} + \frac{r}{\rho}\right) dV. \tag{2.107}$$

By using the divergence theorem, this result can be written

$$\delta C = \int_{\partial B_t} \pi \mathbf{n} \cdot \delta \mathbf{x} \, dS_t + \int_{B_t} \left(-\text{grad}\, \pi \cdot \delta \mathbf{x} + \frac{\pi}{\rho} \delta \rho \right) dV_t, \tag{2.108}$$

where $\delta \rho = r$.

2.4 Fundamental Lemmas

The fundamental lemma of the calculus of variations (see Sect. 1.1) is the result required to obtain differential equations that apply *locally* (that is, at a point) from a *global* (that is, expressed in terms of an integral over a volume) variational statement. In this section, extensions of the fundamental lemma are presented that are appropriate for applications of Hamilton's principle to a continuous medium (see Gurtin [34], pp. 20, 244).

Lemma 1 *Let $\mathcal{W}$ be an inner product space, and consider a C^0 field $\mathbf{f} : \bar{B} \times [t_1, t_2] \to \mathcal{W}$. If the equation*

$$\int_{t_1}^{t_2} \int_B \mathbf{f} \cdot \mathbf{w} \, dV \, dt = 0 \tag{2.109}$$

holds for every C^∞ field $\mathbf{w} : \bar{B} \times [t_1, t_2] \to \mathcal{W}$ that vanishes at time t_1, at time t_2, and on ∂B, then $\mathbf{f} = \mathbf{o}$ on $\bar{B} \times [t_1, t_2]$.

Lemma 2 *Suppose that ∂B consists of complementary regular subsurfaces ∂B_1 and ∂B_2. Let $\mathcal{W}$ be an inner product space, and consider a function $\mathbf{f} : \partial B_2 \times [t_1, t_2] \to \mathcal{W}$ that is piecewise regular and continuous in time. If the equation*

$$\int_{t_1}^{t_2} \int_{\partial B_2} \mathbf{f} \cdot \mathbf{w} \, dS \, dt = 0 \tag{2.110}$$

holds for every C^∞ field $\mathbf{w} : \bar{B} \times [t_1, t_2] \to \mathcal{W}$ that vanishes at time t_1, at time t_2, and on ∂B_1, then $\mathbf{f} = \mathbf{o}$ on $\partial B_2 \times [t_1, t_2]$.

These lemmas are important not only because they are used in obtaining the local forms of the equations of motion from Hamilton's principle, but also because they impose smoothness requirements on the fields describing the material. Both of these aspects were illustrated in the case of a system of particles in Chapter 1.

Because these lemmas are so important, a proof of Lemma 1 given by Gurtin ([34], p. 224) will be presented. The proof proceeds by assuming that the field $\mathbf{f}$ does not vanish at some point $\mathbf{X}_0$, t_0 in $B \times (t_1, t_2)$, and then constructing a suitable field $\mathbf{w}$ such that (2.109) is violated.

Let $\{\mathbf{e}_k\}$ be an orthonormal basis for $\mathcal{W}$, so that $\mathbf{f}$ can be written $\mathbf{f} = f_k \mathbf{e}_k$. Assume that $f_k(\mathbf{X}_0, t_0) > 0$ for some value of k and some point $\mathbf{X}_0$, t_0 in $B \times (t_1, t_2)$. Let α be a positive scalar. Denote the open interval of time $(t_0 - \alpha, t_0 + \alpha)$ by T_α, and denote the open region of space $|\mathbf{X}_0 - \mathbf{X}| < \alpha$ by Ω_α. Because of the continuity of

f, there is a value of α such that $f_k(\mathbf{X}, t) > 0$ in $\Omega_\alpha \times T_\alpha$. Define $\beta(t)$ to be a scalar function on $[t_1, t_2]$ that is C^∞ and has the property that $\beta(t) > 0$ if t is in T_α and $\beta(t) = 0$ otherwise. Define $\gamma(\mathbf{X})$ to be a scalar field on $\bar{B}$ that is C^∞ and has the property that $\gamma(\mathbf{X}) > 0$ if $\mathbf{X}$ is in Ω_α and $\gamma(\mathbf{X}) = 0$ otherwise.[7] Then define

$$\mathbf{w} = \beta(t)\gamma(\mathbf{X})\mathbf{e}_k. \tag{2.111}$$

The vector field $\mathbf{w}$ is C^∞ on $\bar{B} \times [t_1, t_2]$ and vanishes at time t_1, at time t_2, and on ∂B. It has been constructed so that

$$\int_{t_1}^{t_2} \int_B \mathbf{f} \cdot \mathbf{w}\, dV\, dt = \int_{T_\alpha} \int_{\Omega_\alpha} f_k \beta(t)\gamma(\mathbf{X})\, dV\, dt > 0, \tag{2.112}$$

which violates (2.109). Therefore **f** must vanish in $B \times (t_1, t_2)$. Because the field **f** is continuous on $\bar{B} \times [t_1, t_2]$, it must vanish on $\bar{B} \times [t_1, t_2]$.

Two slight variations of Lemmas 1 and 2 will also be used:

Lemma 3 *Let the motion (2.51) be C^2 on $\bar{B} \times [t_1, t_2]$. Let $\mathcal{W}$ be an IPS, and consider a C^0 field $\mathbf{f} : \bar{B} \times [t_1, t_2] \to \mathcal{W}$. If the equation*

$$\int_{t_1}^{t_2} \int_{B_t} \mathbf{f} \cdot \mathbf{w}\, dV_t\, dt = 0 \tag{2.113}$$

holds for every C^∞ field $\mathbf{w} : \bar{B} \times [t_1, t_2] \to \mathcal{W}$ that vanishes at time t_1, at time t_2, and on ∂B, then $\mathbf{f}(\mathbf{x}, t) = \mathbf{o}$ on $\bar{B} \times [t_1, t_2]$.

To prove this result, (2.113) can be written

$$\int_{t_1}^{t_2} \int_B J\mathbf{f} \cdot \mathbf{w}\, dV\, dt = 0. \tag{2.114}$$

Because $J\mathbf{f}$ is continuous on $\bar{B} \times [t_1, t_2]$, Lemma 1 requires that $J\mathbf{f} = \mathbf{o}$ on $\bar{B} \times [t_1, t_2]$. Because $J > 0$, the function $\mathbf{f}(\mathbf{X}, t) = \mathbf{o}$ on $\bar{B} \times [t_1, t_2]$, so $\mathbf{f}(\mathbf{x}, t) = \mathbf{o}$ on $\bar{B} \times [t_1, t_2]$.

Lemma 4 *The complementary regular subsurfaces ∂B_1 and ∂B_2 will be mapped onto surfaces ∂B_{t1} and ∂B_{t2} by the motion (2.51). Let the motion (2.51) be C^2 on $\bar{B} \times [t_1, t_2]$. Let $\phi(\mathbf{X}, t)$ be a scalar function defined on $\partial B_2 \times [t_1, t_2]$ that is piecewise regular and continuous in time. If the equation*

$$\int_{t_1}^{t_2} \int_{\partial B_{t2}} \phi \mathbf{n} \cdot \mathbf{w}\, dS_t\, dt = 0 \tag{2.115}$$

*holds for every vector field **w** that is C^∞ on $\bar{B} \times [t_1, t_2]$ and vanishes at time t_1, at time t_2, and on ∂B_1, then $\phi = 0$ on $\partial B_{t2} \times [t_1, t_2]$.*

[7] The existence of functions $\beta(t)$ and $\gamma(\mathbf{X})$ having these properties can be demonstrated (Gurtin [34], p. 19).

The proof is similar to that of Lemma 3. Equation (2.115) can be written

$$\int_{t_1}^{t_2} \int_{\partial B_2} \phi J \mathbf{F}^{-\mathrm{t}} \mathbf{N} \cdot \mathbf{w} \, dS \, dt = 0. \tag{2.116}$$

Because $\phi J \mathbf{F}^{-\mathrm{t}}$ is continuous on $\bar{B} \times [t_1, t_2]$, $\phi J \mathbf{F}^{-\mathrm{t}} \mathbf{N}$ is piecewise regular and continuous in time on $\partial B_2 \times [t_1, t_2]$. Therefore, Lemma 2 requires that $\phi J \mathbf{F}^{-\mathrm{t}} \mathbf{N} = \mathbf{o}$ on $\partial B_2 \times [t_1, t_2]$. From (2.71), this implies that $\phi = 0$ on $\partial B_2 \times [t_1, t_2]$.

Chapter 3
Mechanics of Continuous Media

Applications of Hamilton's principle to deformable continuous media are discussed in this chapter. Statements of the principle for continuous media are formally very similar to those for systems of particles, and certainly were motivated by them. However, it should be emphasized that the statements for continuous media stand as independent postulates; they are not *derived* from Hamilton's principle for a system of particles. It will be shown that the local forms of the equations of motion for continuous media and their associated boundary conditions are obtained as necessary conditions implied by Hamilton's principle. The classical theories of fluid and solid mechanics will be described, and also two recent theories of materials with microstructure. It will be shown that postulates of Hamilton's principle that have been introduced to obtain more general theories are natural and well motivated extensions of the classical theories.

Problems in the mechanics of continuous media usually involve nonconservative forces, and it is frequently convenient to include constraints in statements of Hamilton's principle. Therefore, the postulates that will be introduced in this work will be expressed in the same form as the second form of Hamilton's principle for a system of particles stated on page 12. They will be developed by the heuristic approach of identifying terms associated with the mechanics of continuous media that are analogous to the terms that appear in (1.48). In one example that does not involve nonconservative forces, Hamilton's principle will be expressed in the first form stated on page 6.

A. Bedford, *Hamilton's Principle in Continuum Mechanics*,
https://doi.org/10.1007/978-3-030-90306-0_3

3.1 The Classical Theories

In this section theories are discussed in which the mechanical behavior of a material is completely described by its motion

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t). \tag{3.1}$$

Hamilton's principle will be postulated for a finite amount of material that occupies a bounded regular region B in a prescribed reference configuration at time t_1. As the material undergoes a motion (3.1), it will occupy a volume B_t at each time t. Therefore B_t is called a *material volume*; it contains the same material at each time t.

Throughout this section an *admissible motion* will refer to a motion (3.1) of the material, from the prescribed reference configuration at time t_1 to a prescribed configuration at time t_2, that is C^2 on $\bar{B} \times [t_1, t_2]$ and satisfies prescribed boundary conditions on ∂B. A *comparison motion* will refer to an admissible motion

$$\mathbf{x}^* = \boldsymbol{\chi}(\mathbf{X}, t) + \varepsilon \boldsymbol{\eta}(\mathbf{X}, t), \tag{3.2}$$

where $\boldsymbol{\eta}(\mathbf{X}, t)$ is an arbitrary C^2 vector field on $\bar{B} \times [t_1, t_2]$ subject to the requirements that $\boldsymbol{\eta}(\mathbf{X}, t_1) = \mathbf{o}$ and $\boldsymbol{\eta}(\mathbf{X}, t_2) = \mathbf{o}$.

3.1.1 Ideal Fluids

The terms *ideal* or *inviscid* fluid refer to a model of fluid behavior in which the effects of viscosity are neglected. This is the simplest model for a continuous medium. Two cases, compressible and incompressible fluids, will be treated.

Consider how one might postulate Hamilton's principle for an ideal fluid in a form analogous to the first form for a system of particles stated on page 12. An admissible motion and comparison motion of the fluid are given by (3.1) and (3.2). It will be assumed that there is no geometrical constraint on the motion of the fluid on ∂B.

The density $\rho(\mathbf{X}, t)$ of the fluid will be assumed to be a C^1 scalar field on $\bar{B} \times [t_1, t_2]$. In Sect. 2.3, an admissible *comparison density field* was defined by

$$\rho^* = \rho(\mathbf{X}, t) + \varepsilon r(\mathbf{X}, t), \tag{3.3}$$

where $r(\mathbf{X}, t)$ is an arbitrary C^1 scalar field on $\bar{B} \times [t_1, t_2]$ subject to the requirements that $r(\mathbf{X}, t_1) = 0$ and $r(\mathbf{X}, t_2) = 0$.

Consider the individual terms in (1.48):

- *The potential energy U* If the fluid is compressible, it can store potential energy in the form of energy of deformation (in the same way energy is stored in a deformed spring). The deformation of a fluid is expressed in terms of its change

in density from a reference state. Therefore, let it be assumed that there is a scalar function of the density $e(\rho)$, the *internal energy*, that is defined such that the potential energy of each element dV_t of the fluid contained in B_t is $\rho e(\rho)\, dV_t$. It will be assumed that the second derivative of the function $e(\rho)$ exists and is continuous. In modern terminology, the assumption that the internal energy depends only on the density of the fluid is a *constitutive assumption* that characterizes an *elastic fluid*. The total potential energy of the fluid contained in B_t is

$$U = \int_{B_t} \rho e(\rho)\, dV_t. \tag{3.4}$$

- *The virtual work δW* External forces acting on the fluid will be introduced by means of virtual work terms. Let there be a prescribed vector field $\mathbf{b}(\mathbf{X}, t)$, the *body force*, that is C^0 on $\bar{B} \times [t_1, t_2]$ and defined such that the external force exerted on a volume element dV_t of the fluid contained in B_t is $\rho \mathbf{b}\, dV_t$. This field represents any external forces that are distributed over the volume of the fluid, such as its weight. The virtual work done by this force will be expressed in the form $\rho \mathbf{b}\, dV_t \cdot \delta \mathbf{x}$, which is clearly analogous to (1.42). The virtual work done on the fluid contained in B_t is

$$\int_{B_t} \rho \mathbf{b} \cdot \delta \mathbf{x}\, dV_t. \tag{3.5}$$

It will also be assumed that there is a prescribed scalar field $p_0(\mathbf{X}, t)$, the *external pressure*, that is continuous in time and piecewise regular on $\partial B \times [t_1, t_2]$ and defined such that the external force exerted on an area element dS_t of ∂B_t is $-p_0 \mathbf{n}\, dS_t$. The resulting virtual work will be written $-p_0 \mathbf{n}\, dS_t \cdot \delta \mathbf{x}$, so the virtual work on the fluid contained in B_t is

$$-\int_{\partial B_t} p_0 \mathbf{n} \cdot \delta \mathbf{x}\, dS_t. \tag{3.6}$$

Therefore, the virtual work done on the fluid by external forces is postulated to be of the form

$$\delta W = \int_{B_t} \rho \mathbf{b} \cdot \delta \mathbf{x}\, dV_t - \int_{\partial B_t} p_0 \mathbf{n} \cdot \delta \mathbf{x}\, dS_t. \tag{3.7}$$

- *The constraint C* The motion of the fluid and its density field are related through the equation of conservation of mass. The comparison motion (3.2) and the comparison density field (3.3) can be regarded as independent if the

equation of conservation of mass (2.66) is introduced into Hamilton's principle as a constraint. The constraint will be written in the form

$$C = \int_{B_t} \pi \left(1 - \frac{\rho_R}{\rho J}\right) dV_t, \tag{3.8}$$

where the unknown field $\pi(\mathbf{X}, t)$, which is assumed to be C^1 on $\bar{B} \times [t_1, t_2]$, is a Lagrange multiplier.

- *The kinetic energy T* The kinetic energy of the fluid is

$$T = \int_{B_t} \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV_t. \tag{3.9}$$

Using these definitions, we can state Hamilton's principle for an ideal fluid:

Among comparison motions (3.2) and comparison density fields (3.3), the actual motion and field are such that

$$\int_{t_1}^{t_2} [\delta(T - U) + \delta W + \delta C] \, dt = 0. \tag{3.10}$$

It was shown in Sect. 2.3 [Eqs. (2.103) and (2.108)] that the terms δT and δC can be written

$$\delta T = -\int_{B_t} \rho \mathbf{a} \cdot \delta \mathbf{x} \, dV_t, \tag{3.11}$$

$$\delta C = \int_{\partial B_t} \pi \mathbf{n} \cdot \delta \mathbf{x} \, dS_t + \int_{B_t} \left(-\text{grad}\, \pi \cdot \delta \mathbf{x} + \frac{\pi}{\rho} \delta \rho\right) dV_t. \tag{3.12}$$

The potential energy is

$$U = \int_{B_t} \rho e(\rho) \, dV_t = \int_B \rho_R e(\rho) \, dV. \tag{3.13}$$

In terms of the comparison density field (3.3), this is

$$U^* = \int_B \rho_R e^* \, dV, \tag{3.14}$$

where $e^* = e(\rho^*)$. The derivative of this expression with respect to ε is

$$\frac{dU^*}{d\varepsilon} = \int_B \rho_R \frac{de^*}{d\rho^*} \frac{\partial \rho^*}{\partial \varepsilon} \, dV = \int_B \rho_R \frac{de^*}{d\rho^*} r \, dV, \tag{3.15}$$

so that the variation of the potential energy is

$$\delta U = \int_B \rho_R \frac{de}{d\rho} \delta\rho \, dV = \int_{B_t} \rho \frac{de}{d\rho} \delta\rho \, dV_t. \tag{3.16}$$

Upon substituting (3.7), (3.11), (3.12), and (3.16) into Eq. (3.10), it can be written

$$\begin{aligned} \int_{t_1}^{t_2} \Bigg[\int_{B_t} (-\rho \mathbf{a} - \operatorname{grad} \pi + \rho \mathbf{b}) \cdot \delta\mathbf{x} \, dV_t + \int_{B_t} \left(\frac{\pi}{\rho} - \rho \frac{de}{d\rho} \right) \delta\rho \, dV_t \\ + \int_{\partial B_t} (\pi - p_0) \mathbf{n} \cdot \delta\mathbf{x} \, dS_t \Bigg] dt = 0. \end{aligned} \tag{3.17}$$

The equation of motion and boundary condition for the fluid can be deduced from this equation by applying Lemmas 3 and 4 of Sect. 2.4. Because the fields $\boldsymbol{\eta} = \delta\mathbf{x}$ and $r = \delta\rho$ are arbitrary, it can be assumed that $\delta\rho = 0$ on $\bar{B} \times [t_1, t_2]$ and that $\delta\mathbf{x} = \mathbf{o}$ on $\partial B \times [t_1, t_2]$. As a result, the second and third integrals in (3.17) vanish. Then applying Lemma 3 to the remaining integral yields the equation

$$\rho \mathbf{a} = -\operatorname{grad} \pi + \rho \mathbf{b} \quad \text{on } \bar{B}_t \times [t_1, t_2]. \tag{3.18}$$

This is called the equation of *balance of linear momentum*. Next, assuming that $\delta\mathbf{x} = \mathbf{o}$ on $\bar{B} \times [t_1, t_2]$ and applying Lemma 3 to (3.17) yields the equation

$$\pi = \rho^2 \frac{de}{d\rho} \quad \text{on } \bar{B}_t \times [t_1, t_2]. \tag{3.19}$$

This equation determines the Lagrange multiplier π as a function of the density of the fluid. Finally, applying Lemma 4 to (3.17) provides the boundary condition

$$\pi = p_0 \quad \text{on } \partial B_t \times [t_1, t_2]. \tag{3.20}$$

A physical interpretation of the Lagrange multiplier π can be gained by writing (3.17) in terms of a material volume of fluid B_t' that is contained within B_t (Fig. 3.1):

$$\begin{aligned} \int_{t_1}^{t_2} \Bigg[\int_{B_t'} (-\rho \mathbf{a} - \operatorname{grad} \pi + \rho \mathbf{b}) \cdot \delta\mathbf{x} \, dV_t + \int_{B_t'} \left(\frac{\pi}{\rho} - \rho \frac{de}{d\rho} \right) \delta\rho \, dV_t \\ + \int_{\partial B_t'} (\pi - p) \mathbf{n} \cdot \delta\mathbf{x} \, dS_t \Bigg] dt = 0. \end{aligned} \tag{3.21}$$

Here the term $-p\mathbf{n}$ is the normal traction exerted on the fluid within B_t' by the fluid exterior to B_t'; that is, p is the pressure of the fluid. The function p is not prescribed, but is a *constitutive function* that is assumed to be C^1 on $\bar{B} \times [t_1, t_2]$. By

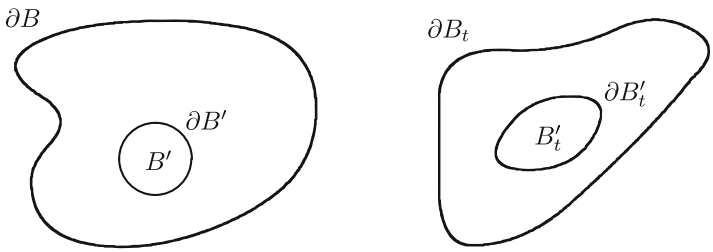

Fig. 3.1 A volume of fluid B' that is contained within B and the corresponding volume B'_t that is contained within B_t at time t

the same procedure that was applied to (3.17), (3.21) implies Eqs. (3.18) and (3.19) on $\bar{B}'_t \times [t_1, t_2]$ and the boundary condition

$$\pi = p \quad \text{on } \partial B'_t \times [t_1, t_2]\,. \tag{3.22}$$

Thus *the Lagrange multiplier π is the pressure of the fluid.* Furthermore, observe from (3.19) that *Hamilton's principle yields the constitutive equation for the pressure of the fluid in terms of the internal energy.*

At this point, the usual method of determining the equation of motion for an ideal fluid will be sketched for the purpose of comparison with Hamilton's principle. The approach used is to write a postulate for an arbitrary material volume of the fluid that is analogous to Newton's second law for a system of particles (see e.g. Gurtin [35], pp. 105–110).

Let B'_t be an arbitrary volume contained within B_t (Fig. 3.1). The *linear momentum* of an element dV_t of B'_t is the product of its mass and velocity, $\rho \mathbf{v}\, dV_t$. It is postulated that, at an arbitrary instant in time, the rate of change of the total linear momentum of the fluid contained within B'_t is equal to the total external force exerted on the fluid:

$$\frac{d}{dt}\int_{B'_t} \rho \mathbf{v}\, dV_t = \int_{B'_t} \rho \mathbf{b}\, dV_t - \int_{\partial B'_t} p\mathbf{n}\, dS_t\,. \tag{3.23}$$

As a consequence of the equation of conservation of mass and Reynolds' transport theorem (see e.g. Gurtin [35], pp. 78–79),

$$\frac{d}{dt}\int_{B'_t} \rho \mathbf{v}\, dV_t = \int_{B'_t} \rho \mathbf{a}\, dV_t\,. \tag{3.24}$$

Using this result and the divergence theorem, (3.23) can be written

$$\int_{B'_t} (\rho \mathbf{a} + \text{grad } p - \rho \mathbf{b}) dV_t = 0. \tag{3.25}$$

Because the volume B_t' is arbitrary, this equation implies that

$$\rho \mathbf{a} = -\text{grad } p + \rho \mathbf{b} \quad \text{on } \bar{B} \times [t_1, t_2]\,. \tag{3.26}$$

It is clear from this derivation why this equation is referred to as the balance of linear momentum.

This *direct* method of obtaining the equation of balance of linear momentum is simpler than Hamilton's principle, although if the derivation of Reynolds' transport theorem is regarded as an integral part of the process, the difference is not so pronounced. Nevertheless, their relative complexity is a criticism that has been made of variational methods in continuum mechanics. Note, however, that the direct method does not yield (3.19). Other advantages of Hamilton's principle, particularly in connection with its ability to incorporate constraints, will be illustrated in subsequent examples. The author regards direct and variational methods as complementary, not competitive. In some cases one method is more advantageous and in some the other, and often both methods lend insight to a given problem.

If the constitutive relation for the internal energy as a function of the density is known, (2.70), (3.18), and (3.19) provide a system of equations with which to determine the density field ρ, velocity field $\mathbf{v}$, and pressure field $\pi = p$. Although this is a simple theory, it is used in the study of aerodynamics to analyze high-speed flows except in regions (such as boundary layers and wakes) where the effects of viscosity cannot be neglected. In this application, (3.19) is usually assumed to be the isentropic relation

$$\frac{p}{\rho^\gamma} = \text{constant}, \tag{3.27}$$

where γ, the *ratio of specific heats*, is assumed to be constant. When it is linearized in terms of small perturbations, this theory is also used in the study of the propagation of acoustic waves. Let

$$\mathbf{v} = \tilde{\mathbf{v}}, \tag{3.28}$$

$$\rho = \rho_R + \tilde{\rho}, \tag{3.29}$$

where $\tilde{\mathbf{v}}$ and $\tilde{\rho}$ are small perturbations and the reference density ρ_R is assumed to be homogeneous. Using (3.27), (2.70) and (3.18) can be written (in the absence of the body force)

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_R \,\text{div}\, \tilde{\mathbf{v}} = 0, \tag{3.30}$$

$$\rho_R \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \frac{\gamma p_R}{\rho_R} \,\text{grad}\, \tilde{\rho} = 0. \tag{3.31}$$

Eliminating $\tilde{\mathbf{v}}$ from these two equations yields the linear wave equation

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = \alpha^2 \nabla^2 \tilde{\rho}, \tag{3.32}$$

where $\alpha = \sqrt{\gamma p_R / \rho_R}$ is the sound speed.

Thus far, the theory of a compressible ideal fluid has been discussed. If the fluid is assumed to be incompressible, $\rho = \rho_R =$ constant. In this case, there is no energy of deformation, so the potential energy $U = 0$. The equation of conservation of mass (2.66) becomes $J = 1$, so the constraint (3.8) reduces to

$$C = \int_{B_t} \pi \left(1 - \frac{1}{J}\right) dV_t. \tag{3.33}$$

The other terms in (3.10) are unchanged. By using the same procedure as in the case of a compressible fluid, (3.10) leads to the equation of balance of linear momentum

$$\rho \mathbf{a} = -\text{grad}\, \pi + \rho \mathbf{b} \quad \text{on } \bar{B}_t \times [t_1, t_2] \tag{3.34}$$

and the boundary condition

$$\pi = p_0 \quad \text{on } \partial B_t \times [t_1, t_2]\,. \tag{3.35}$$

This equation of balance of linear momentum and boundary condition are identical to those obtained in the case of a compressible fluid, but there is no equivalent to (3.19). The pressure π is not a constitutive function of the density. For an incompressible fluid, the equation of conservation of mass (2.70) reduces to

$$\text{div}\, \mathbf{v} = 0. \tag{3.36}$$

Equations (3.34) and (3.36) provide two equations with which to determine the velocity field $\mathbf{v}$ and pressure field π. The theory of incompressible ideal fluids is used in the study of hydraulics and in aerodynamics for the analysis of low-speed flows in regions where viscosity is not important.

Applications of Hamilton's principle to ideal fluids are discussed by Eckart [24], Herivel [43], Lanczos [50], Leech [51], Serrin [65], and Taub [67].

3.1.2 Elastic Solids

An elastic solid can be characterized by the assumption that the internal energy is a function of the deformation gradient $\mathbf{F}$, so that the potential energy of the material contained in B_t is

$$U = \int_{B_t} \rho e(\mathbf{F})\, dV_t = \int_B \rho_R e(\mathbf{F})\, dV. \tag{3.37}$$

It will be assumed that the function $e(\mathbf{F})$ is differentiable on a suitable open domain of its argument and that DIV $(\partial e/\partial \mathbf{F})$ is continuous on $\bar{B} \times [t_1, t_2]$.

Suppose that the surface ∂B consists of complementary regular subsurfaces ∂B_1 and ∂B_2, and that the motion of the material is prescribed on ∂B_1. Let there be no constraint on the motion of the material on ∂B_2.

Let there be a prescribed vector field $\mathbf{t}_0(\mathbf{x}, t)$, the *external traction*, that is defined on $\partial B_{t2} \times [t_1, t_2]$ such that the external force exerted on an element dS_t of ∂B_{t2} is $\mathbf{t}_0\, dS_t$. Then define a vector field $\mathbf{s}_0(\mathbf{X}, t)$ by $\mathbf{s}_0\, dS = \mathbf{t}_0\, dS_t$, and let $\mathbf{s}_0$ be assumed to be continuous in time and piecewise regular on $\partial B_2 \times [t_1, t_2]$. The virtual work done by the external traction is $\mathbf{t}_0\, dS_t \cdot \delta\mathbf{x} = \mathbf{s}_0\, dS \cdot \delta\mathbf{x}$, and the total virtual work done by external forces on the material contained in B_t is

$$\delta W = \int_B \rho_R \mathbf{b} \cdot \delta\mathbf{x}\, dV + \int_{\partial B_2} \mathbf{s}_0 \cdot \delta\mathbf{x}\, dS, \tag{3.38}$$

where the body force $\mathbf{b}$ is defined as in the preceding subsection. Note that $\delta\mathbf{x}$ must vanish on ∂B_1 because the comparison motion (3.2) must satisfy the prescribed boundary conditions on ∂B.

The kinetic energy of the material contained in B_t is

$$T = \int_B \tfrac{1}{2}\rho_R \mathbf{v} \cdot \mathbf{v}\, dV. \tag{3.39}$$

Hamilton's principle for an elastic material states:

Among admissible comparison motions (3.2), the actual motion of the material is such that

$$\int_{t_1}^{t_2} [\delta(T - U) + \delta W]\, dt = 0. \tag{3.40}$$

To determine the variation of the internal energy, it is first expressed in terms of the comparison motion:

$$U^* = \int_B \rho_R e(\mathbf{F}^*)\, dV. \tag{3.41}$$

The derivative of this expression with respect to ε is

$$\frac{\partial U^*}{\partial \varepsilon} = \int_B \rho_R \frac{de^*}{d\mathbf{F}^*} \cdot \frac{\partial \mathbf{F}^*}{\partial \varepsilon}\, dV = \int_B \rho_R \frac{de^*}{d\mathbf{F}^*} \cdot \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{X}}\, dV, \tag{3.42}$$

where $e^* = e(\mathbf{F}^*)$. In terms of components, this equation is

$$\frac{\partial U^*}{\partial \varepsilon} = \int_B \rho_R \frac{\partial e^*}{\partial F^*_{km}} \frac{\partial F^*_{km}}{\partial \varepsilon}\, dV = \int_B \rho_R \frac{\partial e^*}{\partial F^*_{km}} \frac{\partial \eta_k}{\partial X_m}\, dV. \tag{3.43}$$

Therefore, the variation of the potential energy is

$$\delta U = \left[\frac{\partial U^*}{\partial \varepsilon}\right]_{\varepsilon=0} = \int_B \mathbf{S} \cdot \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{X}}\, dV, \tag{3.44}$$

where

$$\mathbf{S} = \rho_R \frac{de}{d\mathbf{F}}, \quad S_{km} = \rho_R \frac{\partial e}{\partial F_{km}}. \tag{3.45}$$

The linear transformation $\mathbf{S}$ is called the *first Piola–Kirchoff stress*. By means of the divergence theorem, (3.44) can be written

$$\delta U = \int_{\partial B_2} \mathbf{SN} \cdot \delta\mathbf{x}\, dS - \int_B \text{DIV}\, \mathbf{S} \cdot \delta\mathbf{x}\, dV. \tag{3.46}$$

Substituting this expression, (2.103), and (3.38) into Eq. (3.40), it assumes the form

$$\begin{aligned} \int_{t_1}^{t_2} \Bigg[\int_B (-\rho_R \mathbf{a} + \text{DIV}\, \mathbf{S} + \rho_R \mathbf{b}) \cdot \delta\mathbf{x}\, dV \\ + \int_{\partial B_2} (\mathbf{s}_0 - \mathbf{SN}) \cdot \delta\mathbf{x}\, dS \Bigg] dt = 0. \end{aligned} \tag{3.47}$$

Invoking Lemmas 1 and 2 of Sect. 2.4, this equation yields the equation of balance of linear momentum

$$\rho_R \mathbf{a} = \text{DIV}\, \mathbf{S} + \rho_R \mathbf{b} \quad \text{on } \bar{B} \times [t_1, t_2] \tag{3.48}$$

and the boundary condition

$$\mathbf{SN} = \mathbf{s}_0 \quad \text{on } \partial B_2 \times [t_1, t_2]\,. \tag{3.49}$$

When the constitutive relation $e(\mathbf{F})$ is specified, (3.45) and (3.48) can be used to determine the displacement field $\mathbf{u}$ and the Piola–Kirchhoff stress $\mathbf{S}$. The constitutive relations for elastic materials are discussed by Gurtin ([34], Chapter C, [35], Chapters IX and X), Truesdell and Noll ([70], Chapters C and D), and Truesdell and Toupin ([71], pp. 723–727). The *linear theory of elasticity* is obtained by assuming that e is a quadratic form in the linear strain tensor $\mathbf{E}$,

$$\rho_R e = \tfrac{1}{2} A_{ijkm} E_{ij} E_{km}, \tag{3.50}$$

where the coefficients A_{ijkm} are constants. If the material is *isotropic*, it can be shown that

$$A_{ijkm} = \lambda \delta_{ij} \delta_{km} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}), \tag{3.51}$$

where λ and μ are the *Lamé constants*. In that case the constitutive equation for the Piola–Kirchhoff stress is

$$\mathbf{S} = \lambda(\text{tr } \mathbf{E})\mathbf{1} + 2\mu\mathbf{E}, \quad S_{km} = \lambda E_{jj}\delta_{km} + 2\mu E_{km}. \tag{3.52}$$

By expressing the variation of the potential energy (3.44) as an integral over B_t, it can be written

$$\delta U = \int_{B_t} \mathbf{T} \cdot \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{x}}\, dV_t, \tag{3.53}$$

where the *Cauchy stress* $\mathbf{T}$ is defined by

$$\mathbf{T} = \frac{1}{J}\mathbf{S}\mathbf{F}^{\mathrm{t}}, \quad T_{km} = \frac{1}{J} S_{kj} \frac{\partial \chi_m}{\partial X_j}. \tag{3.54}$$

By using the expression (3.53) for δU, (3.47) can be written

$$\begin{aligned} \int_{t_1}^{t_2} \Bigg[\int_{B_t} (-\rho\mathbf{a} + \text{div } \mathbf{T} + \rho\mathbf{b}) \cdot \delta\mathbf{x}\, dV_t \\ + \int_{\partial B_{t2}} (\mathbf{t}_0 - \mathbf{T}\mathbf{n}) \cdot \delta\mathbf{x}\, dS_t \Bigg] dt = 0, \end{aligned} \tag{3.55}$$

which yields the equation of balance of linear momentum

$$\rho\mathbf{a} = \text{div } \mathbf{T} + \rho\mathbf{b} \quad \text{on } \bar{B}_t \times [t_1, t_2] \tag{3.56}$$

and the boundary condition

$$\mathbf{T}\mathbf{n} = \mathbf{t}_0 \quad \text{on } \partial B_{t2} \times [t_1, t_2]\,. \tag{3.57}$$

There are some elastic materials, of which rubber is the best known example, for which the assumption that the material is incompressible can be a useful approximation. The equations governing an incompressible elastic material can be obtained by introducing the constraint (3.33) into Hamilton's principle. When the variation of (3.33) is included in (3.55), the resulting equation of balance of linear momentum is

$$\rho\mathbf{a} = \text{div } (-\pi\mathbf{1} + \mathbf{T}) + \rho\mathbf{b} \quad \text{on } \bar{B}_t \times [t_1, t_2] \tag{3.58}$$

and the boundary condition is

$$(-\pi\mathbf{1} + \mathbf{T})\mathbf{n} = \mathbf{t}_0 \quad \text{on } \partial B_{t2} \times [t_1, t_2]\,. \tag{3.59}$$

In this case there is an additional governing equation, the constraint

$$J = 1, \tag{3.60}$$

and an additional unknown field, the *pressure* π.

If the external forces acting on an elastic material are conservative, Hamilton's principle can be stated in a manner analogous to the first form for a system of particles on page 6 (see e.g. Washizu [73]). Suppose that there exist scalar fields $\psi_b(\mathbf{x}, t)$ and $\psi_b(\mathbf{x}, t)$ such that

$$\mathbf{b} = -\text{grad}\ \psi_b, \quad \mathbf{s}_0 = -\text{grad}\ \psi_s. \tag{3.61}$$

Then Hamilton's principle for an elastic material can be stated:

> Among admissible motions, the actual motion of the material is such that the integral
>
> $$I = \int_{t_1}^{t_2} (T - U - U_e)\, dt \tag{3.62}$$
>
> is stationary in comparison with neighboring admissible motions.

The term U_e is defined by

$$U_e = \int_B \rho_R \psi_b\, dV + \int_{\partial B} \psi_s\, dS. \tag{3.63}$$

In terms of the comparison motion, the integral (3.62) is

$$I^*(\varepsilon) = \int_{t_1}^{t_2} (T^* - U^* - U_e^*)\, dt, \tag{3.64}$$

where

$$U_e^* = \int_B \rho_R \psi_b^*\, dV + \int_{\partial B} \psi_s^*\, dS, \tag{3.65}$$

$\psi_b^* = \psi_b(\mathbf{x}^*, t)$, and $\psi_s^* = \psi_s(\mathbf{x}^*, t)$. The derivative of this expression with respect to ε is

$$\begin{aligned} \frac{\partial U_e^*}{\partial \varepsilon} &= \int_B \rho_R \frac{\partial \psi_b^*}{\partial \mathbf{x}^*} \cdot \frac{\partial \mathbf{x}^*}{\partial \varepsilon}\, dV + \int_{\partial B} \frac{\partial \psi_s^*}{\partial \mathbf{x}^*} \cdot \frac{\partial \mathbf{x}^*}{\partial \varepsilon}\, dS \\ &= \int_B \rho_R \frac{\partial \psi_b^*}{\partial \mathbf{x}^*} \cdot \boldsymbol{\eta}\, dV + \int_{\partial B_2} \frac{\partial \psi_s^*}{\partial \mathbf{x}^*} \cdot \boldsymbol{\eta}\, dS, \end{aligned} \tag{3.66}$$

and, using (3.61), the value of this derivative when $\varepsilon = 0$ is

$$\left[\frac{\partial U_e^*}{\partial \varepsilon}\right]_{\varepsilon=0} = -\int_B \rho_R \mathbf{b} \cdot \boldsymbol{\eta}\, dV - \int_{\partial B_2} \mathbf{s}_0 \cdot \boldsymbol{\eta}\, dS. \tag{3.67}$$

Therefore the first form of Hamilton's principle for an elastic material implies that

$$\begin{aligned}\left[\frac{dI^*(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = \int_{t_1}^{t_2}\Big[&-\int_B \rho_R \mathbf{a} \cdot \boldsymbol{\eta}\, dV + \int_B \mathrm{DIV}\, \mathbf{S} \cdot \boldsymbol{\eta}\, dV \\ &-\int_{\partial B_2} \mathbf{SN} \cdot \boldsymbol{\eta}\, dS + \int_B \rho_R \mathbf{b} \cdot \boldsymbol{\eta}\, dV \\ &+\int_{\partial B_2} \mathbf{s}_0 \cdot \boldsymbol{\eta}\, dS\Big]\, dt = 0,\end{aligned} \tag{3.68}$$

which is identical to (3.47).

The application of Hamilton's principle to elastic materials is discussed by Gurtin ([34], pp. 223–226), Love ([53], Chapter VII), Washizu [73], and Weinstock [74].

3.1.3 *Inelastic Materials*

The theories discussed in the preceding two subsections are very special due to the assumptions that were made concerning the functional form of the internal energy. Those assumptions restrict the application of the resulting equations, *a priori*, to elastic materials. For dissipative media, such as viscous fluids, viscoelastic materials, or thermoelastic materials, a more general approach is necessary.

Hamilton's principle can be stated for an *arbitrary* continuous medium, restricted only by the assumption that it does not exhibit microstructural effects. In place of the variation of the internal energy, a virtual work term of the form

$$-\int_B \mathbf{S} \cdot \delta\mathbf{F}\, dV \tag{3.69}$$

is introduced, where the linear tranformation $\mathbf{S}$ is a *constitutive variable* subject only to the requirement that $\mathbf{S}$ and DIV $\mathbf{S}$ be continuous on $\bar{B} \times [t_1, t_2]$. *No assumption is made concerning the dependence of* $\mathbf{S}$ *on the motion or deformation of the material*. It is only assumed that work is done when the deformation gradient of the material changes, and $\mathbf{S}$ is the associated generalized force. An understanding of this point is essential to an appreciation of the applicability of Hamilton's principle to continuum mechanics.

Let the virtual work done on the material contained in B_t be written

$$\delta W = -\int_B \mathbf{S} \cdot \delta \mathbf{F}\, dV + \int_B \rho_R \mathbf{b} \cdot \delta \mathbf{x}\, dV + \int_{\partial B} \mathbf{s}_0 \cdot \delta \mathbf{x}\, dS, \tag{3.70}$$

where the fields $\mathbf{b}$ and $\mathbf{s}_0$ are defined as in the preceding two subsections. It will be assumed that there are no geometric constraints on the motion of the material on ∂B_t.

The kinetic energy of the material contained in B_t is

$$T = \int_B \tfrac{1}{2} \rho_R \mathbf{v} \cdot \mathbf{v}\, dV. \tag{3.71}$$

Hamilton's principle for an arbitrary continuous medium that does not exhibit microstructural effects states:

Among comparison motions (3.2), the actual motion of the material is such that

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0. \tag{3.72}$$

In terms of the comparison motion (3.2), the deformation gradient is

$$\mathbf{F}^* = \mathbf{F} + \varepsilon \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{X}}, \tag{3.73}$$

so the variation of $\mathbf{F}$ is

$$\delta \mathbf{F} = \left[\frac{\partial \mathbf{F}^*}{\partial \varepsilon} \right]_{\varepsilon = 0} = \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{X}}. \tag{3.74}$$

Using this expression and the divergence theorem, the virtual work (3.69) can be written

$$-\int_B \mathbf{S} \cdot \delta \mathbf{F}\, dV = -\int_{\partial B} \mathbf{S}\mathbf{N} \cdot \delta \mathbf{x}\, dS + \int_B \mathrm{DIV}\, \mathbf{S} \cdot \delta \mathbf{x}\, dV. \tag{3.75}$$

Therefore, using the expressions (3.70), (3.71), and (3.75), Eq. (3.72) can be written in the form

$$\begin{aligned} \int_{t_1}^{t_2} \Bigg[\int_B (-\rho_R \mathbf{a} + \mathrm{DIV}\, \mathbf{S} + \rho_R \mathbf{b}) \cdot \delta \mathbf{x}\, dV \\ + \int_{\partial B} (\mathbf{s}_0 - \mathbf{S}\mathbf{N}) \cdot \delta \mathbf{x}\, dS \Bigg] dt = 0. \end{aligned} \tag{3.76}$$

This equation is identical to Eq. (3.47) for an elastic solid. It therefore leads to the same equation of balance of linear momentum

$$\rho_R \mathbf{a} = \text{DIV } \mathbf{S} + \rho_R \mathbf{b} \quad \text{on } \bar{B} \times [t_1, t_2] \tag{3.77}$$

and boundary condition

$$\mathbf{SN} = \mathbf{s}_0 \quad \text{on } \partial B \times [t_1, t_2]\,. \tag{3.78}$$

Alternatively, by using the definition of the Cauchy stress (3.54), (3.72) can be written

$$\int_{t_1}^{t_2} \left[\int_{B_t} (-\rho \mathbf{a} + \text{div } \mathbf{T} + \rho \mathbf{b}) \cdot \delta \mathbf{x}\, dV_t + \int_{\partial B_t} (\mathbf{t}_0 - \mathbf{Tn}) \cdot \delta \mathbf{x}\, dS_t \right] dt = 0, \tag{3.79}$$

which is identical to Eq. (3.55) for an elastic solid and leads to the same equation of balance of linear momentum

$$\rho \mathbf{a} = \text{div } \mathbf{T} + \rho \mathbf{b} \quad \text{on } \bar{B}_t \times [t_1, t_2] \tag{3.80}$$

and boundary condition

$$\mathbf{Tn} = \mathbf{t}_0 \quad \text{on } \partial B_t \times [t_1, t_2]\,. \tag{3.81}$$

Although this statement of Hamilton's principle leads to equations of balance of linear momentum and boundary conditions that are formally identical to those that were obtained in the case of an elastic solid, the crucial difference is that in this case the linear transformations **S** and **T** are constitutive variables. Equations (3.77)–(3.81) apply to an arbitrary continuous medium, subject only to the restriction that the work done by internal forces as the result of a motion of the material is of the form (3.69).[1] However, **S** and **T** are no longer derivable from a potential energy, but must be prescribed through constitutive relations.[2]

Consider an arbitrary volume of material B' contained within B (Fig. 3.1). Let the *heat flux* **q** be a constitutive vector field that is C^1 on $\bar{B} \times [t_1, t_2]$ and defined such that the rate at which heat is lost from the material within B'_t by conduction is

$$\int_{\partial B'_t} \mathbf{q} \cdot \mathbf{n}\, dS_t. \tag{3.82}$$

[1] Theories in which this restriction is relaxed will be described in the next section.

[2] Notice that, because **S** and DIV **S** must be continuous, the form of the constitutive equation for **S** or **T** may impose a more stringent smoothness requirement on the motion of the material.

Let the *external heat supply* s be a prescribed scalar field that is C^0 on $\bar{B} \times [t_1, t_2]$ and defined such that the rate at which heat is added to the material within B'_t by external sources (such as radiation) is

$$\int_{B'_t} \rho s \, dV_t. \tag{3.83}$$

The postulate of *balance of energy* for the material contained within B'_t can be written in the form (see e.g. Leigh [52])

$$\frac{d}{dt} \int_{B'_t} \rho e \, dV_t = \int_{B'_t} \mathbf{T} \cdot \mathbf{L} \, dV_t - \int_{\partial B'_t} \mathbf{q} \cdot \mathbf{n} \, dS_t + \int_{B'_t} \rho s \, dV_t, \tag{3.84}$$

where e is the internal energy and $\mathbf{L} = \partial \mathbf{v} / \partial \mathbf{x}$ is the velocity gradient. Because the volume B'_t is arbitrary, this equation implies the differential equation

$$\rho \dot{e} = \mathbf{T} \cdot \mathbf{L} - \operatorname{div} \mathbf{q} + \rho s \quad \text{on } \partial B \times [t_1, t_2], \tag{3.85}$$

which is called the equation of *balance of energy*.

It is easy to show that

$$\int_{B'_t} \mathbf{T} \cdot \mathbf{L} \, dV_t = \int_{B'} \mathbf{S} \cdot \dot{\mathbf{F}} \, dV. \tag{3.86}$$

This term of the energy balance postulate is called the *mechanical working term*. Observe the correspondence between the form of this term and that of the virtual work (3.69). It will be shown that this correspondence can be used to motivate postulates of balance of energy when Hamilton's principle is used to derive more general theories of continua. Briefly, the mechanical working terms in the balance of energy postulates are deduced from the forms of the virtual work terms in Hamilton's principle.[3] This approach insures that the forms of the equations of balance of linear momentum and balance of energy are mutually consistent.

Thermoelasticity is an example of a theory in which the equation of balance of energy (3.85) is required (see e.g. Nowinski [55]). Let the *absolute temperature* $\theta(\mathbf{X}, t)$ be defined to be a non-negative scalar field that is C^2 on $\bar{B} \times [t_1, t_2]$. A thermoelastic material can be characterized by the constitutive relations ([55], Chapter 4)

$$\begin{aligned} \mathbf{T} &= \mathbf{T}(\mathbf{F}, \theta, \operatorname{grad} \theta), \\ e &= e(\mathbf{F}, \theta, \operatorname{grad} \theta), \\ \mathbf{q} &= \mathbf{q}(\mathbf{F}, \theta, \operatorname{grad} \theta). \end{aligned} \tag{3.87}$$

[3] See Drumheller and Bedford [22]. Similar procedures have been suggested by Ericksen [26] and Serrin ([65], p. 148).

These constitutive relations together with (2.70), (3.80), and (3.85) provide a system of equations with which to determine the density ρ, displacement $\mathbf{u}$, temperature θ, Cauchy stress $\mathbf{T}$, internal energy e, and heat flux $\mathbf{q}$.

3.2 Theories with Microstructure

In the classical theories of fluid and solid mechanics, the mechanical behavior of a material is completely described by its motion (3.1). In a continuum theory with microstructure, new fields are introduced that are independent of the motion and describe mechanical properties of the material that the classical theories are unable to express. Hamilton's principle is a useful technique for obtaining the equations that govern the new fields. In this work, two examples of theories of this type will be described, a theory of granular materials developed by Goodman and Cowin [31] and a general theory of elastic materials with microstructure due to Mindlin [54].

When postulates of Hamilton's principle were first formulated for the classical theories of fluid and solid mechanics, the results being sought were well known. The two examples presented in this section, and the material on mixtures in the following chapter, show how natural extensions of those original postulates can be used to obtain new theories.

3.2.1 Granular Materials

The work of Goodman and Cowin provides an interesting and informative example of the use of Hamilton's principle to derive a relatively simple model of a material with microstructure. Although they did not use Hamilton's principle in developing their theory, it provides a natural and advantageous approach to theories of this type.

They proposed a continuum theory for application to materials consisting of solid grains with interstitial voids. They introduced a field $\phi(\mathbf{X}, t)$, the *volume fraction* of the material, that is a measure of the volume occupied by the grains per unit volume of the material. The volume fraction can vary independently of the motion (3.1) as a result of deformations and reorientations of the grains. Let it be assumed that the motion (3.1) and comparison motion (3.2) are C^3 and that the volume fraction $\phi(\mathbf{X}, t)$ is C^2 on $\bar{B} \times [t_1, t_2]$. A *comparison volume fraction* will be defined by

$$\phi^* = \phi(\mathbf{X}, t) + \varepsilon f(\mathbf{X}, t), \tag{3.88}$$

where $f(\mathbf{X}, t)$ is an arbitrary C^2 scalar field on $\bar{B} \times [t_1, t_2]$ that satisfies the conditions $f(\mathbf{X}, t_1) = 0$ and $f(\mathbf{X}, t_2) = 0$.

The virtual work done on the material by internal forces is postulated to be

$$\int_B [-\mathbf{S} \cdot \delta\mathbf{F} + \rho_R g \, \delta\phi - \mathbf{c} \cdot \delta(\text{GRAD } \phi)] \, dV. \tag{3.89}$$

Comparing this expression to (3.69), the virtual work associated with an ordinary continuous medium is supplemented by two new terms which state that work is done when change occurs in the volume fraction and in the gradient of the volume fraction of the material. It will be assumed that the Piola–Kirchhoff stress $\mathbf{S}$ and DIV $\mathbf{S}$ are continuous on $\bar{B} \times [t_1, t_2]$. The scalar field g is a constitutive function that is assumed to be C^0 on $\bar{B} \times [t_1, t_2]$ and is called the *intrinsic equilibrated body force*. The vector field $\mathbf{c}$ is also a constitutive function and is assumed to be C^1 on $\bar{B} \times [t_1, t_2]$.

The virtual work done by external forces that are distributed over the volume B_t is assumed to be of the form

$$\int_{B_t} (\rho \mathbf{b} \cdot \delta\mathbf{x} + \rho l \ \delta\phi) \, dV_t. \tag{3.90}$$

The prescribed body force $\mathbf{b}$ and the scalar field l are assumed to be continuous on $\bar{B} \times [t_1, t_2]$. The field l is a prescribed function called the *external equilibrated body force*.

Suppose that the surface ∂B consists of complementary regular subsurfaces ∂B_1 and ∂B_2, and that the motion of the material and the volume fraction are prescribed on the surface ∂B_1. (See Sect. 2.4.) The virtual work done by forces distributed on ∂B_2 is postulated in the form

$$\int_{\partial B_{t2}} (\mathbf{t}_0 \cdot \delta\mathbf{x} + H_0 \delta\phi) \, dS. \tag{3.91}$$

The prescribed external traction $\mathbf{t}_0$ and the prescribed scalar function H_0 are assumed to be continuous in time and piecewise regular on $\partial B_{t2} \times [t_1, t_2]$.

The total virtual work done on the material contained in B_t is therefore

$$\begin{aligned} \delta W = &\int_B [-\mathbf{S} \cdot \delta\mathbf{F} + \rho_R g \, \delta\phi - \mathbf{c} \cdot \delta(\text{GRAD } \phi)] \, dV \\ &+ \int_{B_t} (\rho \mathbf{b} \cdot \delta\mathbf{x} + \rho l \ \delta\phi) \, dV_t + \int_{\partial B_{t2}} (\mathbf{t}_0 \cdot \delta\mathbf{x} + H_0 \delta\phi) \, dS. \end{aligned} \tag{3.92}$$

The kinetic energy of the material is written in the form

$$T = \int_{B_t} \left(\tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \tfrac{1}{2} \rho k \dot{\phi}^2 \right) dV_t. \tag{3.93}$$

In comparison to an ordinary continuous medium, an additional kinetic energy expression is introduced which contains the square of the material derivative of

the new independent field, the volume fraction. This term is the kinetic energy associated with the local expansion and contraction of the grains, which can occur independently of the motion (3.1). In general, the coefficient k must be treated as a constitutive function.[4] For simplicity in this discussion, k will be assumed to be a constant.

Observe that the virtual work and kinetic energy expressions that have been defined follow in a natural and systematic way once the new independent field, the volume fraction, was introduced. In addition to the usual kinetic energy due to the translational motion of the material, a new kinetic energy expressed in terms of the rate of change of the volume fraction was included. Similarly, it was assumed that work is done when the volume fraction and its gradient undergo changes.

Hamilton's principle for a Goodman-Cowin material states:

> Among comparison motions (3.1) and comparison volume fraction fields (3.88), the actual motion and volume fraction are such that

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0. \tag{3.94}$$

Notice from (3.88) that

$$\text{GRAD}\, \phi^* = \frac{\partial \phi}{\partial \mathbf{X}} + \varepsilon \frac{\partial f}{\partial \mathbf{X}}, \tag{3.95}$$

so that

$$\delta(\text{GRAD}\, \phi) = \frac{\partial f}{\partial \mathbf{X}}. \tag{3.96}$$

Therefore the third term in the first integral of the virtual work expression (3.92) can be written

$$\int_B \mathbf{c} \cdot \delta(\text{GRAD}\, \phi)\, dV = \int_B \mathbf{c} \cdot \frac{\partial f}{\partial \mathbf{X}}\, dV = \int_{B_t} \mathbf{h} \cdot \frac{\partial F}{\partial \mathbf{x}}\, dV_t, \tag{3.97}$$

where the vector field

$$\mathbf{h} = \frac{1}{J} \mathbf{F} \mathbf{c} \tag{3.98}$$

is called the *equilibrated stress vector*. By applying the divergence theorem, (3.97) can be expressed as

$$\int_B \mathbf{c} \cdot \delta(\text{GRAD}\, \phi)\, dV = \int_{\partial B_2} \mathbf{h} \cdot \mathbf{n}\, \delta\phi\, dS - \int_{B_t} \text{div}\, \mathbf{h}\, \delta\phi\, dV_t, \tag{3.99}$$

where $\delta\phi = f$.

[4] See the related discussion in Sect. 4.4.

The integral with respect to time of the second term in the kinetic energy expression (3.93) is

$$T_2 = \int_{t_1}^{t_2} \int_{B_t} \tfrac{1}{2}\rho k\dot{\phi}^2 \, dV_t \, dt = \int_{t_1}^{t_2} \int_{B} \tfrac{1}{2}\rho_R k\dot{\phi}^2 \, dV \, dt. \tag{3.100}$$

In terms of the comparison volume fraction field (3.88), this is

$$T_2^* = \int_{t_1}^{t_2} \int_{B} \tfrac{1}{2}\rho_R k(\dot{\phi}^*)^2 \, dV \, dt. \tag{3.101}$$

The derivative of this expression with respect to ε is

$$\frac{dT_2^*}{d\varepsilon} = \int_{t_1}^{t_2} \int_{B} \rho_R k\dot{\phi}^* \frac{\partial \dot{\phi}^*}{\partial \varepsilon} \, dV \, dt = \int_{t_1}^{t_2} \int_{B} \rho_R k\dot{\phi}^* \dot{f} \, dV \, dt. \tag{3.102}$$

Integrating this equation by parts with respect to time and evaluating the result when $\varepsilon = 0$ yields

$$\delta T_2 = -\int_{t_1}^{t_2} \int_{B} \rho_R k\ddot{\phi} \, \delta\phi \, dV \, dt = -\int_{t_1}^{t_2} \int_{B_t} \rho k\ddot{\phi} \, \delta\phi \, dV_t \, dt. \tag{3.103}$$

Using (3.99) and (3.103), and expressing the Piola–Kirchhoff stress $\mathbf{S}$ in terms of the Cauchy stress $\mathbf{T}$ through (3.54), (3.94) can be written

$$\begin{aligned} \int_{t_1}^{t_2} \Bigg\{ \int_{B_t} \Big[& (-\rho\mathbf{a} + \operatorname{div}\mathbf{T} + \rho\mathbf{b}) \cdot \delta\mathbf{x} \\ & + (-\rho k\ddot{\phi} + \operatorname{div}\mathbf{h} + \rho l + \rho g)\delta\phi \Big] \, dV_t \\ + \int_{\partial B_{t2}} \Big[& (\mathbf{t}_0 - \mathbf{T}\mathbf{n}) \cdot \delta\mathbf{x} + (H_0 - \mathbf{h}\cdot\mathbf{n})\delta\phi \Big] \, dS_t \Bigg\} \, dt = 0. \end{aligned} \tag{3.104}$$

Due to the independence of the fields $\delta\mathbf{x}$ and $\delta\phi$, Lemmas 3 and 4 of Sect. 2.4 can be applied to (3.104) to obtain the differential equations

$$\left.\begin{aligned} \rho\mathbf{a} &= \operatorname{div}\mathbf{T} + \rho\mathbf{b} \\ \rho k\ddot{\phi} &= \operatorname{div}\mathbf{h} + \rho l + \rho g \end{aligned}\right\} \quad \text{on } \bar{B}_t \times [t_1, t_2] \tag{3.105}$$

and the boundary conditions

$$\left.\begin{aligned} \mathbf{T}\mathbf{n} &= \mathbf{t}_0 \\ \mathbf{h}\cdot\mathbf{n} &= H_0 \end{aligned}\right\} \quad \text{on } \partial B_{t2} \times [t_1, t_2]\,. \tag{3.106}$$

When constitutive relations are specified for the Cauchy stress $\mathbf{T}$, the equilibrated stress vector $\mathbf{h}$, and the intrinsic equilibrated body force g, (2.70) and (3.105) can be used to determine the density ρ, velocity $\mathbf{v}$, and volume fraction ϕ.

Equations (3.105) are identical, with minor changes in notation, to the equations obtained by Goodman and Cowin ([31], Equations (4.7) and (4.10)). Although they did not use Hamilton's principle to obtain these equations, they did use a variational analysis of the static case [17] to motivate them.

This theory has been used to analyze flows of granular materials by Cowin [16], Nunziato et al. [57], and Passman et al. [63], and has been applied to the propagation of waves in granular materials by Cowin and Nunziato [18] and Nunziato and Walsh [58].

Two general observations are illustrated by this example. First, *Hamilton's principle yields an equation for each independent field required to describe the mechanical state of a material.* The independent fields in this example are the motion and the volume fraction. This characteristic of Hamilton's principle makes it particularly advantageous for application to materials with microstructure. Second, *the generalized forces that are introduced into Hamilton's principle as virtual work terms must either be prescribed or must be specified through constitutive relations.* In this example, $\mathbf{b}$, l, $\mathbf{t}$, and H_0 are prescribed, while $\mathbf{T}$, g, and $\mathbf{h}$ are constitutive variables.

The approach described at the end of Sect. 3.1.3 can be used to postulate the equation of balance of energy for a Goodman-Cowin material. Recall the correspondence between the virtual work (3.69) containing the Piola–Kirchhoff stress and the mechanical working term (3.86) that appears in the balance of energy postulate for an ordinary continuous medium. The virtual work done by internal forces in this example is (3.89). The corresponding mechanical working term for an arbitrary volume B' of Goodman-Cowin material is

$$\int_{B'} \left(\mathbf{S} \cdot \dot{\mathbf{F}} - \rho_R g \, \dot{\phi} + \mathbf{c} \cdot \mathrm{GRAD}\, \dot{\phi} \right) dV. \tag{3.107}$$

Equating this expression to the rate of change of the internal energy e of the material within B' and introducing the heat conduction terms (3.82) and (3.83), a postulate of balance of energy for a Goodman-Cowin material is

$$\begin{aligned} \frac{d}{dt} \int_{B'} \rho_R e \, dV = & \int_{B'} \mathbf{S} \cdot \dot{\mathbf{F}} \, dV - \int_{B'_t} \rho g \dot{\phi} \, dV_t \\ & + \int_{B'} \mathbf{c} \cdot \mathrm{GRAD}\, \dot{\phi} \, dV - \int_{\partial B'} \mathbf{q} \cdot \mathbf{n} \, dS_t \\ & + \int_{B'_t} \rho s \, dV_t. \end{aligned} \tag{3.108}$$

The resulting local form of the equation of balance of energy is

$$\rho\dot{e} = \mathbf{T}\cdot\mathbf{L} - \rho g\dot{\phi} + \mathbf{h}\cdot\operatorname{grad}\dot{\phi} - \operatorname{div}\mathbf{q} + \rho s. \tag{3.109}$$

This result is identical to the equation obtained by Goodman and Cowin ([31], Equation (4.11)).

3.2.2 *Elastic Materials with Microstructure*

In the theory described in the previous subsection, a new independent scalar field, the volume fraction, was introduced which provides limited information on the local state of deformation and orientation of the grains in a granular medium. Mindlin [54] used Hamilton's principle to obtain a theory of linear elastic materials with microstructure that contains more extensive information concerning the local state of deformation and orientation of the material. His theory provides a clear illustration of the potential of Hamilton's principle for generating new theories of continuous media.

Mindlin associates with each point of the material a *microelement*. In the case of a granular medium, a microelement could represent a typical grain of the material. The position vector $\mathbf{X}$ of a material point in the reference configuration is assumed to be the position of the center of mass of a microelement in the reference configuration. As the result of a motion (3.1) of the material, the position of the center of mass of the microelement at time t is $\mathbf{x}$. Let the position vector of a material point of the microelement relative to its center of mass in the reference configuration be $\boldsymbol{\Xi}$. The position vector of this material point relative to the center of mass at time t is denoted by $\boldsymbol{\xi}$ (Fig. 3.2). The *microdisplacement* of the material point of the microelement is defined by

$$\bar{u} = \boldsymbol{\xi} - \boldsymbol{\Xi}. \tag{3.110}$$

It is then assumed that for each material point of the microelement,

$$\bar{u} = \boldsymbol{\psi}^t\boldsymbol{\xi}, \tag{3.111}$$

where the linear transformation $\boldsymbol{\psi}(\mathbf{X}, t)$ is called the *microdeformation*. *The microdeformation is the new independent field of the theory*. It describes the state of strain of the microelement associated with each material point. Let $\boldsymbol{\psi}(\mathbf{X}, t)$ be C^2 on $\bar{B}\times[t_1, t_2]$, and define the *microdeformation comparison field* by

$$\boldsymbol{\psi}^* = \boldsymbol{\psi}(\mathbf{X}, t) + \varepsilon\mathbf{R}(\mathbf{X}, t), \tag{3.112}$$

where the linear transformation $\mathbf{R}$ is an arbitrary C^2 field on $\bar{B}\times[t_1, t_2]$ subject to the conditions that $\mathbf{R}(\mathbf{X}, t_1) = \mathbf{0}$ and $\mathbf{R}(\mathbf{X}, t_2) = \mathbf{0}$.

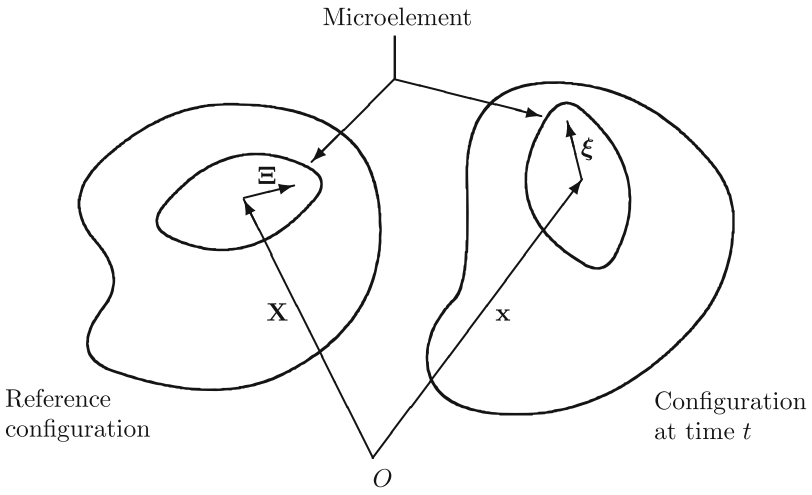

Fig. 3.2 A microelement in the reference configuration and at time t

The strain measures of the theory are the usual linear strain $\mathbf{E}$, the *relative deformation*

$$\boldsymbol{\gamma} = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^t - \boldsymbol{\psi}, \quad \gamma_{km} = \frac{\partial u_m}{\partial X_k} - \psi_{km}, \tag{3.113}$$

where $\mathbf{u}$ is the usual displacement, and the *microdeformation gradient*

$$\boldsymbol{\kappa} = \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{X}}, \quad \kappa_{kmn} = \frac{\partial \psi_{km}}{\partial X_n}. \tag{3.114}$$

Mindlin developed a theory for an elastic material with microstructure by introducing an internal energy e that is a function of the strain measures $\mathbf{E}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\kappa}$. The total potential energy of the material contained in B_t is written

$$U = \int_B \rho_R e(\mathbf{E}, \boldsymbol{\gamma}, \boldsymbol{\kappa})\, dV. \tag{3.115}$$

It will be assumed that the terms

$$\begin{aligned}
\boldsymbol{\tau} &= \rho_R \frac{\partial e}{\partial \mathbf{E}}, \ \tau_{km} = \rho_R \frac{\partial e}{\partial E_{km}}, \\
\boldsymbol{\sigma} &= \rho_R \frac{\partial e}{\partial \boldsymbol{\gamma}}, \ \sigma_{km} = \rho_R \frac{\partial e}{\partial \gamma_{km}}, \\
\boldsymbol{\mu} &= \rho_R \frac{\partial e}{\partial \boldsymbol{\kappa}}, \ \mu_{kmn} = \rho_R \frac{\partial e}{\partial \kappa_{kmn}},
\end{aligned} \tag{3.116}$$

exist and are continuous on suitable open domains of their arguments, and that the fields DIV $\boldsymbol{\tau}$, DIV $\boldsymbol{\sigma}$, and DIV $\boldsymbol{\mu}$, where

$$(\text{DIV } \boldsymbol{\mu})_{km} = \frac{\partial \mu_{kmn}}{\partial X_n}, \tag{3.117}$$

are continuous on $\bar{B} \times [t_1, t_2]$.

It will be assumed that on the surface ∂B there are no geometrical constraints on the motion of the material or on the value of the microdeformation $\boldsymbol{\psi}$. The virtual work done by external forces is postulated in the form

$$\delta W = \int_B \rho_R (\mathbf{b} \cdot \delta \mathbf{x} + \mathbf{D} \cdot \delta \boldsymbol{\psi})\, dV + \int_{\partial B} (\mathbf{s}_0 \cdot \delta \mathbf{x} + \mathbf{M}_0 \cdot \delta \boldsymbol{\psi})\, dS. \tag{3.118}$$

The body force $\mathbf{b}$ and the linear transformation $\mathbf{D}$ are assumed to be prescribed and continuous on $\bar{B} \times [t_1, t_2]$. The field $\mathbf{D}$ is called the double force per unit mass. The surface traction $\mathbf{s}_0$ and the linear transformation $\mathbf{M}_0$ are also prescribed and are assumed to be continuous in time and piecewise regular on $\partial B \times [t_1, t_2]$. The field $\mathbf{M}_0$ is called the double force per unit area. Compare (3.118) to the virtual work expressions (3.89) and (3.91) used in the theory of granular materials due to Goodman and Cowin. In the latter theory, the supplementary independent field was a scalar, the volume fraction. In the present theory, the supplementary independent field is a linear transformation, the microdeformation. Thus the present theory contains much more information about the local motion and deformation of the microelements.

The kinetic energy of the material is postulated in the form

$$T = \int_B (\tfrac{1}{2} \rho_R \mathbf{v} \cdot \mathbf{v} + \tfrac{1}{6} \rho'_R \dot{\boldsymbol{\psi}} \cdot \mathbf{Q} \dot{\boldsymbol{\psi}})\, dV, \tag{3.119}$$

where

$$\dot{\boldsymbol{\psi}} \cdot \mathbf{Q} \dot{\boldsymbol{\psi}} = Q_{ijkm} \dot{\psi}_{ij} \dot{\psi}_{km}. \tag{3.120}$$

The scalar ρ'_R and linear transformation $\mathbf{Q}$ are constants which are determined by the distribution of mass within the microelement in the reference configuration. The second term in (3.119) is the kinetic energy associated with the rotation and rate of deformation of the microelement. Compare (3.119) with the expression (3.93) for a Goodman-Cowin material, in which the additional kinetic energy term was due to the dilatational motion of the microelement.

Hamilton's principle for a Mindlin material states:

> Among comparison motions (3.2) and microdeformation comparison fields (3.112), the actual fields are such that
>
> $$\int_{t_1}^{t_2} [\delta(T - U) + \delta W]\, dt = 0. \tag{3.121}$$

In terms of the comparison motion and the comparison microdeformation field, the potential energy (3.115) is

$$U = \int_B \rho_R e(\mathbf{E}^*, \boldsymbol{\gamma}^*, \boldsymbol{\kappa}^*)\, dV. \tag{3.122}$$

The derivative of this expression with respect to ε is

$$\frac{\partial U^*}{\partial \varepsilon} = \int_B \left(\boldsymbol{\tau}^* \cdot \frac{\partial \mathbf{E}^*}{\partial \varepsilon} + \boldsymbol{\sigma}^* \cdot \frac{\partial \boldsymbol{\gamma}^*}{\partial \varepsilon} + \boldsymbol{\mu}^* \cdot \frac{\partial \boldsymbol{\kappa}^*}{\partial \varepsilon} \right) dV, \tag{3.123}$$

where

$$\boldsymbol{\mu}^* \cdot \frac{\partial \boldsymbol{\kappa}^*}{\partial \varepsilon} = \mu^*_{kmn} \frac{\partial \kappa^*_{kmn}}{\partial \varepsilon}. \tag{3.124}$$

The second term in (3.123) can be written

$$\begin{aligned} \int_B \boldsymbol{\sigma}^* \cdot \frac{\partial \boldsymbol{\gamma}^*}{\partial \varepsilon}\, dV &= \int_B \left[(\boldsymbol{\sigma}^{\mathrm{t}})^* \cdot \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{X}} - \boldsymbol{\sigma}^* \cdot \mathbf{R} \right] dV \\ &= \int_{\partial B} (\boldsymbol{\sigma}^{\mathrm{t}})^* \mathbf{N} \cdot \boldsymbol{\eta}\, dS - \int_B \mathrm{DIV}\, (\boldsymbol{\sigma}^{\mathrm{t}})^* \cdot \boldsymbol{\eta}\, dV \\ &\quad - \int_B \boldsymbol{\sigma}^* \cdot \mathbf{R}\, dV. \end{aligned} \tag{3.125}$$

By performing similar manipulations on the other terms in (3.123), the variation of the potential energy can be expressed as

$$\begin{aligned} \delta U = &\int_{\partial B} \boldsymbol{\tau} \mathbf{n} \cdot \delta \mathbf{x}\, dS - \int_B \mathrm{DIV}\, \boldsymbol{\tau} \cdot \delta \mathbf{x}\, dV \\ &+ \int_{\partial B} \boldsymbol{\sigma}^{\mathrm{t}} \mathbf{N} \cdot \delta \mathbf{x}\, dS - \int_B \mathrm{DIV}\, \boldsymbol{\sigma}^{\mathrm{t}} \cdot \delta \mathbf{x}\, dV \\ &- \int_B \boldsymbol{\sigma} \cdot \delta \boldsymbol{\psi}\, dV + \int_{\partial B} \boldsymbol{\mu} \mathbf{N} \cdot \delta \boldsymbol{\psi}\, dS \\ &- \int_B \mathrm{DIV}\, \boldsymbol{\mu} \cdot \delta \boldsymbol{\psi}\, dV, \end{aligned} \tag{3.126}$$

where $\delta \boldsymbol{\psi} = \mathbf{R}$ and

$$(\boldsymbol{\mu} \mathbf{N})_{km} = \mu_{kmn} N_n. \tag{3.127}$$

The integral from t_1 to t_2 of the second term in the kinetic energy (3.119) is

$$I = \int_{t_1}^{t_2} \int_B \tfrac{1}{6}\rho_R' \dot{\boldsymbol{\psi}} \cdot \mathbf{Q}\dot{\boldsymbol{\psi}}\, dV\, dt. \tag{3.128}$$

Expressing this equation in terms of the comparison microdeformation field and taking the derivative with respect to ε yields

$$\frac{dI^*}{d\varepsilon} = \int_{t_1}^{t_2} \int_B \tfrac{1}{3}\rho_R' \mathbf{Q}\dot{\boldsymbol{\psi}}^* \cdot \dot{\mathbf{R}}\, dV\, dt. \tag{3.129}$$

This equation can be integrated by parts to obtain

$$\delta I = -\int_{t_1}^{t_2} \int_B \tfrac{1}{3}\rho_R' \mathbf{Q}\ddot{\boldsymbol{\psi}} \cdot \mathbf{R}\, dV\, dt. \tag{3.130}$$

Using this equation together with the expressions (3.118) and (3.126), (3.121) can be written

$$\begin{aligned} \int_{t_1}^{t_2} \Big\{ & \int_B [-\rho_R \mathbf{a} + \mathrm{DIV}(\boldsymbol{\tau} + \boldsymbol{\sigma}^{\mathrm{t}}) + \rho_R \mathbf{b}] \cdot \delta\mathbf{x}\, dV \\ & + \int_B (-\tfrac{1}{3}\rho_R' \mathbf{Q}\ddot{\boldsymbol{\psi}} + \boldsymbol{\sigma} + \mathrm{DIV}\, \boldsymbol{\mu} + \rho_R \mathbf{D}) \cdot \delta\boldsymbol{\psi}\, dV \\ & + \int_{\partial B} [\mathbf{s}_0 - (\boldsymbol{\tau} + \boldsymbol{\sigma}^{\mathrm{t}})\mathbf{N}] \cdot \delta\mathbf{x}\, dS \\ & + \int_{\partial B} (\mathbf{M}_0 + \boldsymbol{\mu}\mathbf{N}) \cdot \delta\boldsymbol{\psi}\, dS \Big\}\, dt = 0. \end{aligned} \tag{3.131}$$

Due to the independence of the fields $\delta\mathbf{x}$ and $\delta\boldsymbol{\psi}$, this equation yields the differential equations

$$\left.\begin{aligned} \rho_R \mathbf{a} &= \mathrm{DIV}(\boldsymbol{\tau} + \boldsymbol{\sigma}^{\mathrm{t}}) + \rho_R \mathbf{b} \\ \tfrac{1}{3}\rho_R' \mathbf{Q}\ddot{\boldsymbol{\psi}} &= \boldsymbol{\sigma} + \mathrm{DIV}\, \boldsymbol{\mu} + \rho_R \mathbf{D} \end{aligned}\right\} \quad \text{on } \bar{B} \times [t_1, t_2]\,, \tag{3.132}$$

and the boundary conditions

$$\left.\begin{aligned} (\boldsymbol{\tau} + \boldsymbol{\sigma}^{\mathrm{t}})\mathbf{N} &= \mathbf{s}_0 \\ \boldsymbol{\mu}\mathbf{N} &= \mathbf{M}_0 \end{aligned}\right\} \quad \text{on } \partial B \times [t_1, t_2]\,. \tag{3.133}$$

Thus, in addition to the equation of balance of linear momentum, Hamilton's principle leads to an equation of motion for the microdeformation $\boldsymbol{\psi}$. Notice that the two equations of motion are coupled through the term $\boldsymbol{\sigma}$. The fact that this coupling

is derived explicitly is one of the strengths of Hamilton's principle. It would be difficult to simply postulate Eqs. (3.132).

When the constitutive equation for the internal energy e is specified, (3.116) and (3.132) can be used to determine the fields $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\mu}$, the displacement field $\mathbf{u}$, and the microdeformation field $\boldsymbol{\psi}$. Mindlin [54] obtained a linear theory by expressing the internal energy as a second order expansion in it arguments and used the theory to analyze the propagation of harmonic waves.

Chapter 4
Mechanics of Mixtures

Blood is a mixture of a liquid, called the plasma, and particles, primarily erythrocytes, or red cells. In an *erythrocyte sedimentation test*, a vertical tube of anticoagulated blood is allowed to stand at rest. The cells, being slightly denser than the plasma, settle to the bottom of the tube. The rate at which the upper cell boundary falls is a standard clinical test for disease. When a leak occurs in the cooling system of a nuclear reactor (known as a *loss of coolant accident*), vapor bubbles appear in the suddenly depressurized coolant fluid, and the bubbly liquid flows rapidly toward the leak. It was the study of these two very disparate phenomena that resulted in the applications of Hamilton's principle to the continuum theory of mixtures described in this chapter.

When the volume fraction of one constituent of a binary mixture (i.e. the volume occupied by that constituent per unit volume of the mixture) changes, the volume fraction of the other constituent must adjust accordingly. This *volume fraction constraint* can be introduced into a postulate of Hamilton's principle for the mixture by the method of Lagrange multipliers. When a bubble of gas in a liquid expands or contracts, it induces a radial motion in the surrounding fluid. The inertia associated with this radial motion can be introduced into Hamilton's principle by including, in addition to the kinetic energy of translational motion of the constituents, a kinetic energy expressed in terms of the rate of change of the density of the gas. These ideas suggested that Hamilton's principle could be a useful method for deriving theories of mixtures.

A. Bedford, *Hamilton's Principle in Continuum Mechanics*,
https://doi.org/10.1007/978-3-030-90306-0_4

4.1 Motions and Comparison Motions of a Mixture

4.1.1 Motions

Consider a mixture of two constituents (a binary mixture), such as a fluid containing a distribution of particles or bubbles, or a porous solid saturated by a fluid. In general, the two constituents of the mixture can flow relative to one another. Their individual motions can be described by modeling the constituents as two superimposed continuous media. Let the symbol $\mathcal{C}_\xi$ denote the ξth constituent. A motion of $\mathcal{C}_\xi$ is the vector field

$$\mathbf{x} = \boldsymbol{\chi}_\xi(\mathbf{X}_\xi, t), \tag{4.1}$$

where $\mathbf{x}$ is the position vector at time t of the material point of $\mathcal{C}_\xi$ whose position vector is $\mathbf{X}_\xi$ in a prescribed reference configuration.[1] The *inverse motion* of $\mathcal{C}_\xi$ is

$$\mathbf{X}_\xi = \boldsymbol{\chi}_\xi^{-1}(\mathbf{x}, t). \tag{4.2}$$

Consider a finite amount of the mixture that occupies a bounded regular region B in a prescribed reference configuration at time t_1. In general, the individual motions (4.1) would cause the constituents to occupy different regions at time t. To prevent the constituents from moving apart during the time interval $[t_1, t_2]$, it will be assumed that the displacement of each constituent vanishes on ∂B, or, in the case of an ideal fluid constituent, it will be assumed that the normal component of the velocity vanishes on ∂B. This is equivalent to assuming that the mixture is bounded by a rigid wall.[2] As a result, both constituents occupy a single volume $B_t = B$ with a single surface $\partial B_t = \partial B$ at each time t. This assumption is not merely a theoretical convenience. At a free surface of the mixture, the constituents could actually separate as shown in Fig. 4.1. Then two types of surface result, a free surface of a single constituent, and a surface that is a boundary of one constituent but not of the other. A systematic study of boundary conditions at the latter type of surface would be possible using the methods to be described in Chap. 5.

Throughout this chapter the motions (4.1) will be assumed to be C^3 on $\bar{B} \times [t_1, t_2]$. The description of the kinematics and deformation of $\mathcal{C}_\xi$ in terms of its motion is identical to that for a single continuous medium presented in Sect. 2.2.

[1] See the discussion of the motion of a continuous medium in Sect. 2.2.

[2] An alternative approach would be to express Hamilton's principle in terms of a fixed spatial volume through which the constituents are allowed to diffuse. However, Leech [51] observes that "Many investigators…have tried to derive by application of Hamilton's principle the momentum equation using the so-called Eulerian coordinate system. They have applied the principle using a fixed (control) volume. This is not Hamilton's principle, which for a continuum must be associated with a fixed aggregate or control mass."

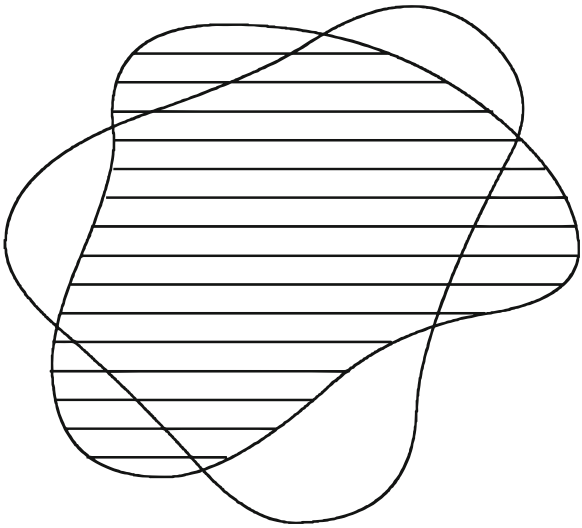

Fig. 4.1 Two constituents diffusing relative to one another at the boundaries

The velocity, acceleration, deformation gradient, Jacobian, displacement, and linear strain of $\mathcal{C}_\xi$ are defined by

$$\begin{aligned}
\mathbf{v}_\xi &= \frac{\partial}{\partial t}\boldsymbol{\chi}_\xi(\mathbf{X}_\xi, t),\\
\mathbf{a}_\xi &= \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}_\xi(\mathbf{X}_\xi, t),\\
\mathbf{F}_\xi &= \frac{\partial}{\partial \mathbf{X}_\xi}\boldsymbol{\chi}_\xi(\mathbf{X}_\xi, t),\\
J_\xi &= \det \mathbf{F}_\xi,\\
\mathbf{u}_\xi &= \boldsymbol{\chi}_\xi(\mathbf{X}_\xi, t) - \mathbf{X}_\xi,\\
\mathbf{E}_\xi &= \frac{1}{2}\left[\frac{\partial \mathbf{u}_\xi}{\partial \mathbf{X}_\xi} + \left(\frac{\partial \mathbf{u}_\xi}{\partial \mathbf{X}_\xi}\right)^{\mathrm{t}}\right].
\end{aligned} \tag{4.3}$$

The material derivative of a field $\mathbf{f}_\xi(\mathbf{X}_\xi, t)$ is defined by

$$\dot{\mathbf{f}}_\xi = \frac{\partial}{\partial t}\mathbf{f}_\xi(\mathbf{X}_\xi, t) = \frac{\partial}{\partial t}\hat{\mathbf{f}}_\xi. \tag{4.4}$$

The inverse motion (4.2) maps an element dV_t of B_t at time t onto an element dV_ξ in the reference configuration. These volume elements are related by (see (2.64))

$$dV_t = J_\xi dV_\xi. \tag{4.5}$$

Let the part of dV_t occupied by the constituent $\mathcal{C}_\xi$ be $dV_{\xi t}$, and let the mass of $\mathcal{C}_\xi$ contained in $dV_{\xi t}$ be dm_ξ. The *partial density* of $\mathcal{C}_\xi$ is defined by $\rho_\xi = dm_\xi / dV_t$. The *material density* of $\mathcal{C}_\xi$ is defined by $\bar{\rho}_\xi = dm_\xi / dV_{\xi t}$, and the *volume fraction* of $\mathcal{C}_\xi$ is defined by $\phi_\xi = dV_{\xi t} / dV_t$. Therefore

$$\rho_\xi = \phi_\xi \bar{\rho}_\xi. \tag{4.6}$$

The partial density ρ_ξ is the mass of $\mathcal{C}_\xi$ per unit volume of the mixture at time t. The material density $\bar{\rho}_\xi$ is the mass of $\mathcal{C}_\xi$ per unit volume of $\mathcal{C}_\xi$ at time t, and the volume fraction ϕ_ξ is the volume of $\mathcal{C}_\xi$ per unit volume of the mixture at time t. The fields $\rho_\xi(\mathbf{X}_\xi, t)$, $\bar{\rho}_\xi(\mathbf{X}_\xi, t)$, and $\phi_\xi(\mathbf{X}_\xi, t)$ will be assumed to be C^2 on $\bar{B} \times [t_1, t_2]$.

One form of the equation of conservation of mass for $\mathcal{C}_\xi$ is

$$J_\xi = \frac{\rho_{\xi R}}{\rho_\xi} = \frac{\phi_{\xi R} \bar{\rho}_{\xi R}}{\phi_\xi \bar{\rho}_\xi}, \tag{4.7}$$

where $\rho_{\xi R} = \rho_\xi(\mathbf{X}_\xi, t_1)$, $\phi_{\xi R} = \phi_\xi(\mathbf{X}_\xi, t_1)$, and $\bar{\rho}_{\xi R} = \bar{\rho}_\xi(\mathbf{X}_\xi, t_1)$ are the values of the partial density, volume fraction, and material density in the reference configuration. The equation of conservation of mass for $\mathcal{C}_\xi$ can also be expressed in the form

$$\dot{\rho}_\xi + \rho_\xi \operatorname{div} \mathbf{v}_\xi = 0. \tag{4.8}$$

In this work, consideration will be limited to mixtures for which

$$\sum_\xi \phi_\xi(\mathbf{x}, t) = 1, \tag{4.9}$$

where the notation $\sum_\xi$ denotes summation over the constituents of the mixture. That is, it will be assumed that the constituents of the mixture occupy all of the volume B_t at each time t; there are no voids. Equation (4.9) is called the *volume fraction constraint*. It plays a central role in the theories to be discussed in this chapter.

4.1.2 Comparison Fields

A motion (4.1) of $\mathcal{C}_\xi$ will be called *admissible* if it is C^3 on $\bar{B} \times [t_1, t_2]$ and it satisfies the prescribed boundary condition on ∂B. An admissible of $\mathcal{C}_\xi$ is defined in analogy with (2.83),

$$\begin{aligned} \mathbf{x}_\xi^* &= \boldsymbol{\chi}_\xi(\mathbf{X}_\xi, t) + \varepsilon \boldsymbol{\eta}_\xi(\mathbf{X}_\xi, t) \\ &= \mathbf{K}_\xi(\mathbf{X}_\xi, t, \varepsilon), \end{aligned} \tag{4.10}$$

where ε is a parameter and $\boldsymbol{\eta}_\xi(\mathbf{X}_\xi, t)$ is an arbitrary C^3 vector field on $\bar{B} \times [t_1, t_2]$ subject to the conditions that $\boldsymbol{\eta}_\xi(\mathbf{X}_\xi, t_1) = \mathbf{o}$, $\boldsymbol{\eta}_\xi(\mathbf{X}_\xi, t_2) = \mathbf{o}$, and (4.10) satisfies the prescribed boundary condition on ∂B. The inverse of the comparison motion is

$$\mathbf{X}_\xi = \mathbf{K}_\xi^{-1}(\mathbf{x}_\xi^*, t, \varepsilon). \tag{4.11}$$

The inverse motion gives the position vector in the reference configuration of the material point of $\mathcal{C}_\xi$ whose position is $\mathbf{x}_\xi^*$ at time t.

In addition to the comparison motion, *comparison material density* and *comparison volume fraction* fields are defined by

$$\begin{aligned} \bar{\rho}_\xi^* &= \bar{\rho}_\xi(\mathbf{X}_\xi, t) + \varepsilon \bar{r}_\xi(\mathbf{X}_\xi, t), \\ \phi_\xi^* &= \phi_\xi(\mathbf{X}_\xi, t) + \varepsilon f_\xi(\mathbf{X}_\xi, t), \end{aligned} \tag{4.12}$$

where $\bar{r}_\xi(\mathbf{X}_\xi, t)$ and $f_\xi(\mathbf{X}_\xi, t)$ are arbitrary fields subject to the conditions that they are C^2 on $\bar{B} \times [t_1, t_2]$ and vanish at times t_1 and t_2.

Let the volume fraction of $\mathcal{C}_\xi$ be expressed as a function of $\mathbf{X}_\xi, t$:

$$\phi_\xi = \hat{\phi}_\xi(\mathbf{X}_\xi, t). \tag{4.13}$$

Using this expression and the inverse motion (4.2), the volume fraction constraint (4.9) can be written

$$\sum_\xi \hat{\phi}_\xi(\boldsymbol{\chi}_\xi^{-1}(\mathbf{x}, t), t) = 1. \tag{4.14}$$

This equation can be written in terms of the inverse of the comparison motion (4.11) and the comparison volume fraction field $(4.12)_1$ to obtain the relation

$$\sum_\xi \hat{\phi}_\xi^*(\mathbf{K}_\xi^{-1}(\mathbf{x}, t, \varepsilon), t, \varepsilon) = 1. \tag{4.15}$$

The derivative of this equation with respect to ε is

$$\sum_\xi \left(\frac{\partial \hat{\phi}_\xi^*}{\partial \mathbf{K}_\xi^{-1}} \cdot \frac{\partial \mathbf{K}_\xi^{-1}}{\partial \varepsilon} + \frac{\partial \hat{\phi}_\xi^*}{\partial \varepsilon} \right) = 0. \tag{4.16}$$

To evaluate the partial derivative of $\mathbf{K}_\xi^{-1}$ with respect to ε that appears in this expression, the differential of (4.11) can be taken while holding $\mathbf{X}_\xi$ and t fixed to obtain

$$\mathbf{o} = \frac{\partial \mathbf{K}_\xi^{-1}}{\partial \mathbf{x}_\xi^*} \, d\mathbf{x}_\xi^* + \frac{\partial \mathbf{K}_\xi^{-1}}{\partial \varepsilon} \, d\varepsilon. \tag{4.17}$$

Therefore

$$\frac{\partial \mathbf{K}_\xi^{-1}}{\partial \varepsilon} = -\frac{\partial \mathbf{K}_\xi^{-1}}{\partial \mathbf{x}_\xi^*}\left[\frac{d\mathbf{x}_\xi^*}{d\varepsilon}\right]_{\mathbf{X}_\xi,t} = -\frac{\partial \mathbf{K}_\xi^{-1}}{\partial \mathbf{x}_\xi^*}\boldsymbol{\eta}_\xi. \tag{4.18}$$

Substituting this result into (4.16) and evaluating the resulting equation when $\varepsilon = 0$, one obtains [5]

$$\sum_\xi (\text{grad}\ \phi_\xi \cdot \delta\mathbf{x}_\xi - \delta\phi_\xi) = 0, \tag{4.19}$$

where $\delta\mathbf{x}_\xi = \boldsymbol{\eta}_\xi$ and $\delta\phi_\xi = f_\xi$. *Eq. (4.19) is a constraint imposed on the variations* $\delta\mathbf{x}_\xi$ *and* $\delta\phi_\xi$ *by the volume fraction constraint*. It will be introduced into statements of Hamilton's principle for mixtures in the form

$$\int_{B_t} \lambda \sum_\xi (\text{grad}\ \phi_\xi \cdot \delta\mathbf{x}_\xi - \delta\phi_\xi)\, dV_t = 0, \tag{4.20}$$

where the scalar field $\lambda(\mathbf{x}, t)$ is a Lagrange multiplier that is assumed to be C^1 on $\bar{B}_t \times [t_1, t_2]$.

The equations of conservation of mass of the constituents (4.7) will also be introduced as constraints into statements of Hamilton's principle for mixtures, in the same form as in the case of a single continuous medium (see (2.105)):

$$\begin{aligned} &\sum_\xi \int_{B_t} \pi_\xi \left(1 - \frac{\phi_{\xi R}\bar{\rho}_{\xi R}}{\phi_\xi \bar{\rho}_\xi J_\xi}\right) dV_t \\ &= \sum_\xi \int_B \pi_\xi \left(J_\xi - \frac{\phi_{\xi R}\bar{\rho}_{\xi R}}{\phi_\xi \bar{\rho}_\xi}\right) dV, \end{aligned} \tag{4.21}$$

where the scalar fields $\pi_\xi(\mathbf{X}_\xi, t)$ are Lagrange multipliers that are assumed to be C^1 on $\bar{B} \times [t_1, t_2]$. To determine the variation of this expression, it can be written in terms of the comparison motion (4.10) and the comparison fields (4.12):

$$\sum_\xi \int_B \pi_\xi \left(J_\xi^* - \frac{\phi_{\xi R}\bar{\rho}_{\xi R}}{\phi_\xi^* \bar{\rho}_\xi^*}\right) dV. \tag{4.22}$$

Taking the derivative with respect to ε and setting $\varepsilon = 0$ yields

$$\begin{aligned} &\sum_\xi \int_B \pi_\xi J_\xi \left(\text{div}\ \boldsymbol{\eta}_\xi + \frac{f_\xi}{\phi_\xi} + \frac{\bar{r}_\xi}{\bar{\rho}_\xi}\right) dV \\ &= \sum_\xi \int_{B_t} \left[-\text{grad}\ \pi_\xi \cdot \delta\mathbf{x}_\xi + \pi_\xi \left(\frac{\delta\phi_\xi}{\phi_\xi} + \frac{\delta\bar{\rho}_\xi}{\bar{\rho}_\xi}\right)\right] dV_t \end{aligned} \tag{4.23}$$

where $\delta\bar{\rho}_\xi = \bar{r}_\xi$.

The postulates of Hamilton's principle for mixtures that will be introduced in the following sections are closely analogous to those for a single material described in Chap. 3. The volume fraction constraint is a new element, and new degrees of freedom will be seen to arise in comparison with the theories of single materials without microstructure. The formulations for mixtures have some elements in common with the theory for granular materials discussed in Sect. 3.2.

4.2 Mixtures of Ideal Fluids

4.2.1 Compressible Fluids

It will be assumed that each constituent $\mathcal{C}_\xi$ has an internal energy per unit mass $e_\xi(\bar{\rho}_\xi)$ that is a function only of the material density of that constituent.[3] The second derivatives of these functions are assumed to be continuous. The total potential energy of the mixture contained in B_t is assumed to be the sum of the potential energies of the constituents:

$$U = \sum_\xi \int_{B_t} \rho_\xi e_\xi(\bar{\rho}_\xi)\, dV_t. \tag{4.24}$$

The virtual work done on the mixture by external forces is postulated in the form

$$\delta W = \sum_\xi \int_{B_t} (\rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi) \cdot \delta\mathbf{x}\, dV_t. \tag{4.25}$$

In this expression the external force on each constituent is decomposed into two parts. The body force $\mathbf{b}_\xi$ is the force per unit mass exerted on $\mathcal{C}_\xi$ by external agencies, such as gravity. It is assumed to be prescribed. The *interaction force*, or *drag* $\mathbf{d}_\xi$ is the force per unit volume exerted on $\mathcal{C}_\xi$ by the other constituent of the mixture.[4] The vector fields $\mathbf{b}_\xi(\mathbf{X}_\xi, t)$ and $\mathbf{d}_\xi(\mathbf{X}_\xi, t)$ will be assumed to be C^0 on $\bar{B} \times [t_1, t_2]$. Recall that the mixture is assumed to be bounded by a rigid wall. As a result, no virtual work is done by external forces at the surface ∂B_t.

[3] This assumption, like the assumptions made in Sect. 3.1 that led to theories of elastic fluids and elastic solids, will obviously result in a very special theory.

[4] Although the constituents are here being treated as inviscid with regard to their macroscopic behavior, it is nevertheless assumed that they may exert drag forces on one another. This is a common assumption in mixture theories for processes in which macroscopic viscous effects may be neglected (see e.g. Bowen [13]).

The comparison fields (4.10) and (4.12) are subject to the constraints (4.20) and (4.23) arising from the volume fraction constraint and the equations of conservation of mass. Therefore, the constraint term

$$\begin{aligned}\delta C = &\sum_{\xi} \int_{B_t} \lambda(\operatorname{grad} \phi_\xi \cdot \delta \mathbf{x}_\xi - \delta\phi_\xi)\, dV_t \\ &+ \sum_{\xi} \int_{B_t} \left[-\operatorname{grad} \pi_\xi \cdot \delta \mathbf{x}_\xi + \pi_\xi \left(\frac{\delta\phi_\xi}{\phi_\xi} + \frac{\delta\bar{\rho}_\xi}{\bar{\rho}_\xi} \right) \right] dV_t\end{aligned} \tag{4.26}$$

will be included in Hamilton's principle.

The total kinetic energy of the mixture in B_t will be assumed to be the sum of the kinetic energies due to the translational motions of the constituents:

$$T = \sum_{\xi} \int_{B_t} \tfrac{1}{2} \rho_\xi \mathbf{v}_\xi \cdot \mathbf{v}_\xi \, dV_t. \tag{4.27}$$

Based on the expressions (4.24)–(4.27), a postulate of Hamilton's principle for a mixture of elastic ideal fluids states [5]:

> Among comparison motions (4.10) and comparison fields (4.12), the actual fields are such that
>
> $$\int_{t_1}^{t_2} [\delta(T - U) + \delta C + \delta W]\, dt = 0. \tag{4.28}$$

Substituting (4.24)–(4.27), this equation can be written

$$\begin{aligned}\sum_{\xi} \int_{t_1}^{t_2} \int_{B_t} \Big[&(-\rho_\xi \mathbf{a}_\xi + \rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi - \operatorname{grad} \pi_\xi + \lambda \operatorname{grad} \phi_\xi) \cdot \delta \mathbf{x} \\ &+ \left(-\rho_\xi \frac{de_\xi}{d\bar{\rho}_\xi} + \frac{\pi_\xi}{\bar{\rho}_\xi} \right) \delta\bar{\rho}_\xi + \left(\frac{\pi_\xi}{\phi_\xi} - \lambda \right) \delta\phi_\xi \Big] dV_t\, dt = 0.\end{aligned} \tag{4.29}$$

Because of the independence of the fields $\delta\mathbf{x}_\xi$, $\delta\bar{\rho}_\xi$, and $\delta\phi_\xi$ for each constituent, Lemma 3 of Sect. 2.4 can be applied to (4.29) to obtain the equations

$$\left.\begin{aligned}\rho_\xi \mathbf{a}_\xi &= \rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi - \operatorname{grad} \pi_\xi + \lambda \operatorname{grad} \phi_\xi, \\ \pi_\xi &= \phi_\xi \bar{\rho}_\xi^2 \frac{de_\xi}{d\bar{\rho}_\xi}, \\ \pi_\xi &= \phi_\xi \lambda\end{aligned}\right\} \quad \text{on } \bar{B}_t \times [t_1, t_2]\,. \tag{4.30}$$

When constitutive relations are specified for the internal energies $e_\xi(\bar{\rho}_\xi)$ and the drag terms $\mathbf{d}_\xi$, the three Eqs. (4.30) together with (4.6), (4.8), and (4.9) can be used to determine the fields ρ_ξ, $\bar{\rho}_\xi$, ϕ_ξ, λ, $\mathbf{v}_\xi$, and π_ξ.

Notice from (3.19) and the discussion following it that the term $\bar{\rho}_\xi^2 de_\xi/d\bar{\rho}_\xi$ is the pressure of the constituent $\mathcal{C}_\xi$. From (4.30),

$$\lambda = \bar{\rho}_\xi^2 \frac{de_\xi}{d\bar{\rho}_\xi}. \tag{4.31}$$

Thus a consequence of this postulate of Hamilton's principle for a mixture is that *the pressures of the constituents are equal*. This condition is often introduced as an assumption in theoretical studies of multiphase flow. Models for mixtures in which this condition does not hold will be discussed later in this section. Also, observe that $(4.30)_1$ can be written

$$\rho_\xi \mathbf{a}_\xi = \rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi - \phi_\xi \text{ grad } \lambda. \tag{4.32}$$

The form of the last term in this equation has been a subject of some controversy among those interested in theoretical models for mixtures. The form that appears here is a consequence of including the volume fraction constraint in Hamilton's principle.[5]

4.2.2 Incompressible Fluids

If both constituents of a binary mixture are incompressible ($\bar{\rho}_\xi = \bar{\rho}_{\xi R} =$ constant), (4.28) assumes the form

$$\int_{t_1}^{t_2} (\delta T + \delta C + \delta W)\, dt = 0. \tag{4.33}$$

Equations (4.25) and (4.27) for δW and T are unchanged. The only change in Eq. (4.26) for δC is that $\delta\bar{\rho}_\xi = 0$. The equations resulting from Hamilton's principle are

$$\left.\begin{aligned} \rho_\xi \mathbf{a}_\xi = \rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi - \text{grad } \pi_\xi + \lambda \text{ grad } \phi_\xi, \\ \pi_\xi = \phi_\xi \lambda \end{aligned}\right\} \text{ on } \bar{B}_t \times [t_1, t_2]. \tag{4.34}$$

Eliminating the Lagrange multipliers π_ξ yields the equations of balance of linear momentum

$$\rho_\xi \mathbf{a}_\xi = \rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi - \phi_\xi \text{ grad } \lambda. \tag{4.35}$$

[5] The comments in this paragraph have been discussed at length by Bedford and Drumheller [7].

For incompressible constituents, the equations of conservation of mass (4.8) can be written

$$\dot{\phi}_\xi + \phi_\xi \operatorname{div} \mathbf{v}_\xi = 0. \tag{4.36}$$

If constitutive relations are specified for the drag terms $\mathbf{d}_\xi$, Eqs. (4.9), (4.35), and (4.36) can be used to determine the fields λ, $\mathbf{v}_\xi$, and ϕ_ξ. The Lagrange multiplier λ is the pressure of the constituents.

The statement of Hamilton's principle for a granular solid described in Sect. 3.2.1 contained a virtual work term expressed in terms of the variation of the volume fraction of the material. Would the physics of the problem justify the inclusion of such terms in a theory for a mixture of ideal fluids? Suppose that the theory is used to model a fluid containing a distribution of particles. If the particles are sufficiently small, they will undergo mutual impacts as a result of their Brownian motions. This *diffusive effect* of particle impacts is analogous to the ordinary pressure which arises in a fluid due to impacts on the molecular scale. The particles can also exert forces on one another through hydrodynamic interactions when the mixture is in motion. Either of these phenomena will result in work being done when the volume fraction of the particles changes.

The postulates of Hamilton's principle that have been stated for a mixture of ideal fluids can be extended to the case of a mixture of ideal fluids with diffusivity by adding to (4.25) a virtual work term of the form [47][6]

$$-\sum_\xi \int_{B_t} P_\xi \frac{\delta \phi_\xi}{\phi_\xi}\, dV_t. \tag{4.37}$$

In the case of incompressible constituents, the equations resulting from Hamilton's principle are

$$\left.\begin{aligned} \rho_\xi \mathbf{a}_\xi = \rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi - \operatorname{grad} \pi_\xi + \lambda \operatorname{grad} \phi_\xi, \\ \pi_\xi = \phi_\xi \lambda + P_\xi \end{aligned}\right\} \quad \text{on } \bar{B}_t \times [t_1, t_2]\,. \tag{4.38}$$

Eliminating the Lagrange multipliers λ results in the equations of balance of linear momentum

$$\rho_\xi \mathbf{a}_\xi = \rho_\xi \mathbf{b}_\xi + \mathbf{d}_\xi - \phi_\xi \operatorname{grad} \lambda - \operatorname{grad} P_\xi. \tag{4.39}$$

In addition to the gradient of the pressure λ appearing in the equations of motion, the gradients of the *diffusive pressures* P_ξ also appear.

[6] The generalized force in this expression is written in the form P_ξ/ϕ_ξ because it results in simpler equations. This does not imply an *a priori* assumption of the functional form of the diffusive force, because the terms P_ξ are assumed to be constitutive functions of the volume fractions.

To apply this theory to the sedimentation of a distribution of rigid particles in an incompressible fluid, Hill et al. [47] postulated the constitutive relations

$$\begin{aligned} \mathbf{d}_\xi &= \mathbf{d}_\xi(\phi_\xi, \mathbf{v}_\xi - \mathbf{v}_\gamma), \\ P_\xi &= P_\xi(\phi_\xi, \mathbf{v}_\xi - \mathbf{v}_\gamma), \end{aligned} \tag{4.40}$$

where $\xi \neq \gamma$. Notice that because the volume fractions are related through the volume fraction constraint, it is not necessary to assume that the constitutive relations are functions of both volume fractions. Hill et al. further assumed that these constitutive relations are isotropic and linear in the relative velocity, which implies that they must be of the forms

$$\begin{aligned} \mathbf{d}_\xi &= \alpha_\xi(\mathbf{v}_\xi - \mathbf{v}_\gamma), \\ P_\xi &= \beta_\xi, \end{aligned} \tag{4.41}$$

where α_ξ and β_ξ are scalar functions of ϕ_ξ.

For purposes of comparison with this derivation, equivalent theories have been derived by two other approaches. Craine [19] used postulated equations of motion and introduced the volume fraction constraint into the second law of thermodynamics (the Clausius–Duhem inequality) for the mixture. Drew [20] used an averaging approach.

Equations (4.9), (4.36), (4.39), and (4.41) have been applied to the erythrocyte sedimentation test described in the introduction to this chapter by Hill and Bedford [46] and Hill et al. [47]. Figures 4.2 and 4.3 compare their numerical solutions to experimental measurements made using anticoagulated human whole blood by Whelan et al. [75]. In Fig. 4.2, the predicted distribution of the cell volume fraction as a function of height in the vertical tube is compared to measurements made at several times. An empirical expression for the "drag coefficient" α_ξ was used, and the constitutive coefficients were chosen to obtain the best agreement with the data at 2 h. The coefficients were then held fixed while the computations were extended to 4 h and 8.5 h. In Fig. 4.3, the predicted position of the upper cell boundary as a function of time, which is used as a clinical indicator of disease, is compared to the observed position.

This theory has also been used to study the stability of steady sedimentation of a uniform distribution of particles in a fluid by Hill [44] and Hill and Bedford [45].

4.2.3 Fluids with Microinertia

Suppose that there is a spherical bubble of gas in an unbounded incompressible liquid. Let the radius of the bubble be R, and let the densities of the gas and the liquid

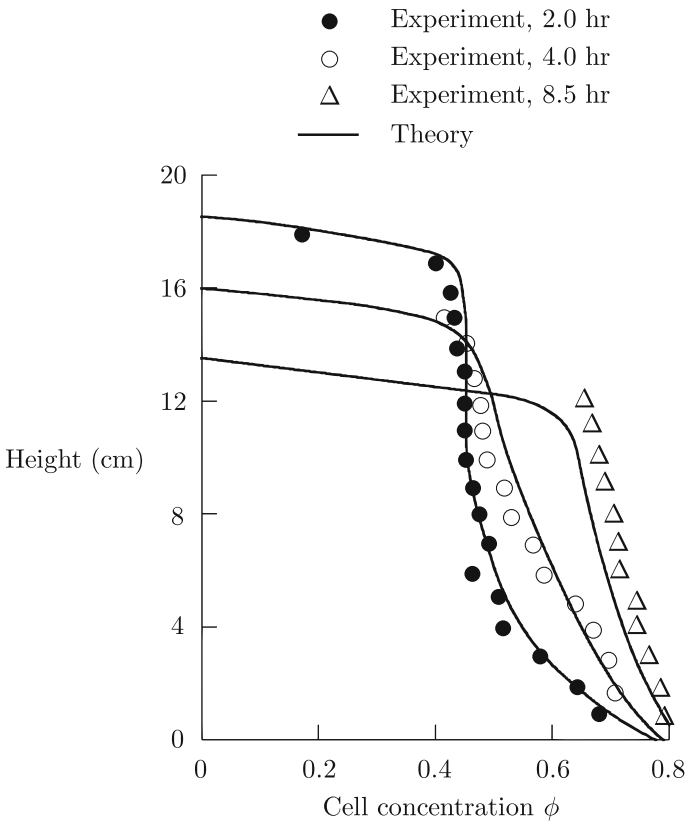

Fig. 4.2 Comparison of the mixture theory with cell concentration profiles measured in blood sedimentation

be $\bar{\rho}_g$ and $\bar{\rho}_f$. If the bubble expands or contracts, it will induce a radial velocity distribution in the liquid. The velocity of the liquid at a distance r from the center of the bubble is

$$v_f = \frac{R^2}{r^2}\dot{R}, \tag{4.42}$$

where the dot denotes the derivative with respect to time. The resulting kinetic energy of the liquid surrounding the bubble is

$$\int_R^\infty \tfrac{1}{2}\bar{\rho}_f 4\pi r^2 v_f^2 \, dr. \tag{4.43}$$

Substituting the velocity distribution (4.42), this integral can be evaluated to obtain

$$2\pi\bar{\rho}_f R^3 \dot{R}^2. \tag{4.44}$$

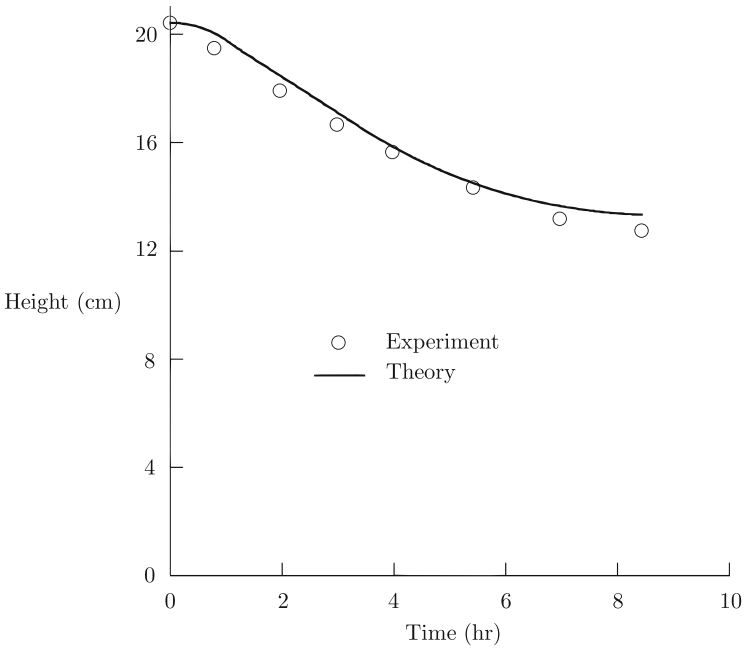

Fig. 4.3 Comparison of the mixture theory with the upper cell interface measured in blood sedimentation

Because the mass of the gas within the bubble, $(4/3)\pi R^3 \bar{\rho}_g$, is constant, the kinetic energy (4.44) can be expressed in terms of the rate of change of the gas density:

$$\left(\frac{2\pi \bar{\rho}_f \bar{\rho}_{gR}^{5/3} R_R^5}{9\bar{\rho}_g^{11/3}} \right) \dot{\bar{\rho}}_g^2, \tag{4.45}$$

where $\bar{\rho}_{gR}$ and R_R are reference values.

Now consider a liquid containing a dilute distribution of bubbles, and suppose that in a prescribed reference configuration the bubbles are uniformly distributed and each has radius R_R and density $\bar{\rho}_{gR}$. In a motion of this bubble liquid, the bubbles will undergo volumetric oscillations and induce local radial motions of the liquid. If it is assumed that the kinetic energy of the liquid surrounding each bubble can be approximated by the expression (4.45), the kinetic energy per unit volume of the mixture due to radial motions of the bubbles can be obtained by multiplying (4.45) by the number of bubbles per unit volume. The number of bubbles per unit volume is $\phi_g/(4/3)\pi R^3$, where ϕ_g is the volume fraction of the gas. The product of this expression with (4.45) can be written

$$\rho_g \left(\frac{\bar{\rho}_f \bar{\rho}_{gR}^{2/3} R_R^2}{6\bar{\rho}_g^{11/3}} \right) \dot{\bar{\rho}}_g^2. \tag{4.46}$$

This *microkinetic energy* due to bubble oscillations can have a dominant effect on the dynamic behavior of bubbly liquids (see van Wijngaarden [72]). The equations governing the motion of a bubbly liquid can be obtained by introducing the microkinetic energy into Hamilton's principle [21].

For simplicity in this presentation, the relative motion between the bubbles and the liquid will be neglected. This is an acceptable assumption in many applications because of the relatively small mass of the bubbles. Thus the mixture will be assumed to have a single motion (3.1). The microkinetic energy of the mixture contained in B_t will be expressed in the form

$$T_m = \int_{B_t} \tfrac{1}{2}\rho_g I_g(\bar{\rho}_\gamma)\dot{\bar{\rho}}_g^2 \, dV_t = \int_B \tfrac{1}{2}\rho_{gR} I_g(\bar{\rho}_\gamma)\dot{\bar{\rho}}_g^2 \, dV \tag{4.47}$$

The term $I_g(\bar{\rho}_\gamma)$ is a constitutive function that depends on each of the constituent densities. This expression for the microkinetic energy is motivated by (4.46) and includes it as a special case. The integral of (4.47) with respect to time from t_1 to t_2 is

$$I = \int_{t_1}^{t_2} T_m \, dt = \int_{t_1}^{t_2} \int_B \tfrac{1}{2}\rho_{gR} I_g(\bar{\rho}_\gamma)\dot{\bar{\rho}}_g^2 \, dV \, dt. \tag{4.48}$$

Proceeding in the now familiar way to determine the variation, this equation is written in terms of the comparison material density field $(4.12)_1$ to obtain

$$I^*(\varepsilon) = \int_{t_1}^{t_2} \int_B \tfrac{1}{2}\rho_{gR} I_g(\bar{\rho}_\gamma^*)(\dot{\bar{\rho}}_g^*)^2 \, dV \, dt. \tag{4.49}$$

The derivative of this equation with respect to ε is

$$\frac{dI^*(\varepsilon)}{d\varepsilon} = \int_{t_1}^{t_2} \int_B \left[\rho_{gR} I_g^* \dot{\bar{\rho}}_g^* \dot{\bar{r}}_g + \tfrac{1}{2}\rho_{gR} \left(\sum_\gamma \frac{\partial I_g^*}{\partial \bar{\rho}_\gamma^*} \bar{r}_\gamma \right) (\dot{\bar{\rho}}_g^*)^2 \right] dV \, dt, \tag{4.50}$$

where $I_g^* = I_g(\bar{\rho}_\gamma^*)$. Integrating the first term by parts with respect to time and setting $\varepsilon = 0$ yields the variation

$$\delta T_m = \int_{B_t} \left[-\rho_g \overline{I_g \dot{\bar{\rho}}_g} \delta\bar{\rho}_g + \tfrac{1}{2}\rho_g \left(\sum_\gamma \frac{\partial I_g}{\partial \bar{\rho}_\gamma} \delta\bar{\rho}_\gamma \right) \dot{\bar{\rho}}_g^2 \right] dV_t. \tag{4.51}$$

The total kinetic energy of the mixture contained in B_t is

$$T = \int_{B_t} \left[\tfrac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} + \tfrac{1}{2}\rho_g I_g(\bar{\rho}_\gamma)\dot{\bar{\rho}}_g^2 \right] dV_t, \tag{4.52}$$

where $\rho = \rho_f + \rho_g$ is the density of the mixture.

The potential energy of the mixture is postulated to be the total internal energy for a mixture of two compressible ideal fluids:

$$U = \sum_{\xi} \int_{B_t} \rho_\xi e_\xi(\bar{\rho}_\xi)\, dV_t. \tag{4.53}$$

Because there is not relative motion between the constituents, no work is done by drag forces and the only virtual word done is that due to the external body force:

$$\delta W = \int_{B_t} \rho \mathbf{b} \cdot \delta \mathbf{x}\, dV_t. \tag{4.54}$$

The constraint arising from the equations of balance of mass and the volume fraction constraint, (4.26), is altered only by the fact that there is a single motion:

$$\begin{aligned} \delta C = &\sum_{\xi} \int_{B_t} \lambda(\operatorname{grad} \phi_\xi \cdot \delta \mathbf{x} - \delta\phi_\xi)\, dV_t \\ &+ \sum_{\xi} \int_{B_t} \left[-\operatorname{grad} \pi_\xi \cdot \delta \mathbf{x} + \pi_\xi \left(\frac{\delta\phi_\xi}{\phi_\xi} + \frac{\delta\bar{\rho}_\xi}{\bar{\rho}_\xi} \right) \right] dV_t. \end{aligned} \tag{4.55}$$

Based on (4.52)–(4.55), a statement of Hamilton's principle for an ideal compressible liquid containing a distribution of bubbles of an ideal gas is [23]:

Among comparison motions (3.2) and comparison fields (4.12), the actual fields are such that

$$\int_{t_1}^{t_2} [\delta(T - U) + \delta C + \delta W]\, dt = 0. \tag{4.56}$$

Substituting (4.52)–(4.55) and using the result (4.51), (4.56) can be written

$$\begin{aligned} \int_{t_1}^{t_2} \int_{B_t} \Big\{ &[-\rho \mathbf{a} + \rho \mathbf{b} - \operatorname{grad}(\pi_f + \pi_g)] \cdot \delta \mathbf{x} \\ &+ \left(-\rho_g \overline{I_g \dot{\bar{\rho}}_g} + \tfrac{1}{2} \rho_g \frac{\partial I_g}{\partial \bar{\rho}_g} \dot{\bar{\rho}}_g^2 - \rho_g \frac{de_g}{d\bar{\rho}_g} + \frac{\pi_g}{\bar{\rho}_g} \right) \delta \bar{\rho}_g \\ &+ \left(\tfrac{1}{2} \rho_g \frac{\partial I_g}{\partial \bar{\rho}_f} \dot{\bar{\rho}}_g^2 - \rho_f \frac{de_f}{d\bar{\rho}_f} + \frac{\pi_f}{\bar{\rho}_f} \right) \delta \bar{\rho}_f \\ &+ \left(\frac{\pi_g}{\phi_g} - \lambda \right) \delta \phi_g + \left(\frac{\pi_f}{\phi_f} - \lambda \right) \delta \phi_f \Big\} dV_t\, dt = 0, \end{aligned} \tag{4.57}$$

which yields the equations

$$\left.\begin{aligned} \rho\mathbf{a} &= \rho\mathbf{b} - \operatorname{grad}(\pi_f + \pi_g), \\ \rho_g \overset{\cdot}{\overline{I_g \dot{\bar{\rho}}_g}} - \tfrac{1}{2}\rho_g \frac{\partial I_g}{\partial \bar{\rho}_g}\dot{\bar{\rho}}_g^2 &= -\rho_g \frac{de_g}{d\bar{\rho}_g} + \frac{\pi_g}{\bar{\rho}_g}, \\ -\tfrac{1}{2}\rho_g \frac{\partial I_g}{\partial \bar{\rho}_f}\dot{\bar{\rho}}_g^2 &= -\rho_f \frac{de_f}{d\bar{\rho}_f} + \frac{\pi_f}{\bar{\rho}_f}, \\ \pi_g &= \phi_g \lambda, \\ \pi_f &= \phi_f \lambda \end{aligned}\right\} \quad \text{on } \bar{B} \times [t_1, t_2]\,. \tag{4.58}$$

The last two equations can be used to eliminate the Lagrange multipliers π_f and π_g, resulting in the three equations

$$\begin{aligned} \rho\mathbf{a} &= \rho\mathbf{b} - \operatorname{grad}\lambda, \\ \bar{\rho}_g^2 \overset{\cdot}{\overline{I_g \dot{\bar{\rho}}_g}} - \tfrac{1}{2}\bar{\rho}_g \left(\bar{\rho}_g \frac{\partial I_g}{\partial \bar{\rho}_g} - \frac{\phi_g}{\phi_f}\bar{\rho}_f \frac{\partial I_g}{\partial \bar{\rho}_f}\right)\dot{\bar{\rho}}_g^2 &= p_f - p_g, \\ \lambda &= p_f - \tfrac{1}{2}\bar{\rho}_g \frac{\phi_f}{\phi_f}\bar{\rho}_f \frac{\partial I_g}{\partial \bar{\rho}_f}\dot{\bar{\rho}}_g^2, \end{aligned} \tag{4.59}$$

where the constituent pressures p_ξ are

$$p_\xi = \bar{\rho}_\xi^2 \frac{de_\xi}{d\bar{\rho}_\xi}. \tag{4.60}$$

When constitutive relations are specified for the internal energies e_ξ and the coefficient I_g, the equations of conservation of mass

$$\dot{\rho}_\xi + \rho_\xi \operatorname{div}\mathbf{v} = 0 \tag{4.61}$$

together with (4.6), (4.9), (4.59), and (4.60) provide a system of equations with which to determine the fields ϕ_ξ, λ, $\mathbf{v}$, $\bar{\rho}_\xi$, ρ_ξ, and p_ξ.

Observe from $(4.59)_2$ that the pressures of the liquid and gas are not generally equal, which is a consequence of introducing the microkinetic energy. It is the difference in the liquid and gas pressures that drives the bubble oscillations. The pressures are equal when the mixture is in a state of equilibrium. Also, notice from $(4.59)_3$ that the term λ, whose gradient appears in the equation of balance of linear momentum $(4.59)_1$, is *not* in general equal to the pressure of the liquid.

These equations have been compared to experimental data on wave propagation in bubbly liquids by Bedford and Stern [8] and Drumheller et al. [23]. To do so, it was necessary to account for the effects of heat transfer between the gas and liquid in determining the constitutive equation for the pressure of the gas (see e.g. Drumheller and Bedford [21]). The coefficient I_g was evaluated using the expression (4.46).

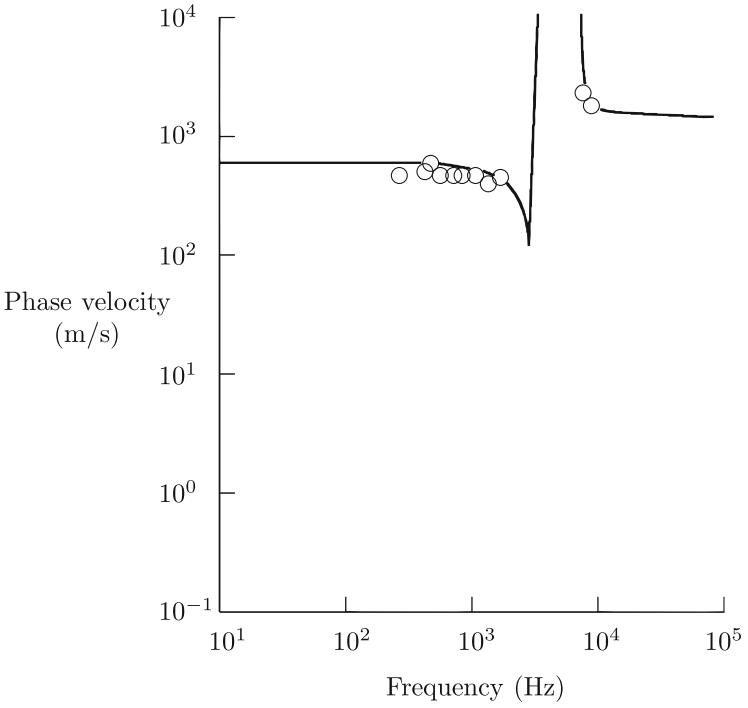

Fig. 4.4 Phase velocity of acoustic waves in water containing air bubbles

This seems contradictory since the liquid is here being assumed to be compressible whereas (4.46) was derived under the assumption that it is incompressible. In using this procedure, it is being assumed that the spatial variation of the density of the liquid is small in the neighborhood of a bubble. That is, wavelengths must be large in comparison to the bubble diameter.

In Figs. 4.4 and 4.5, the predicted phase velocity and attenuation of plane acoustic waves are compared to measurements made by Silberman [66] for air bubbles in water. The gas volume fraction was $\phi_{gR} = 3.77(10^{-4})$ and the bubble radius was 1.01 mm. The peak in the attenuation occured at the resonance frequency for bubble oscillations.

In Fig. 4.6, a numerical solution of the gas pressure p_g resulting from an impulsively applied pressure is compared to data obtained using a shock tube by Kuznetsov et al. [49]. The mixture consisted of carbon dioxide bubbles in a water-glycerine solution. The gas volume fraction was $\phi_{gR} = 0.01$ and the bubble radius was 0.5 mm. The "ringing" observed in the pressure history results from bubble oscillations.

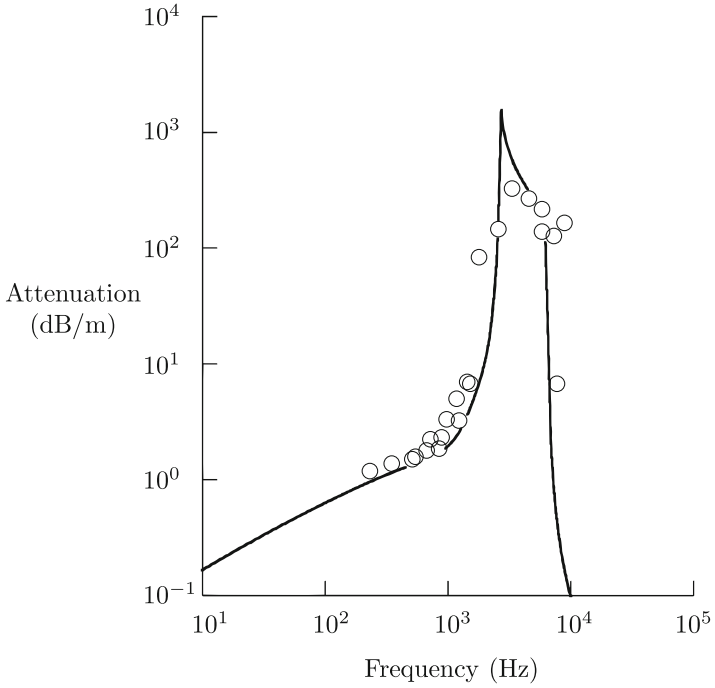

Fig. 4.5 Attenuation of acoustic waves in water containing air bubbles

If the liquid is assumed to be incompressible, the coefficient I_g is evaluated using (4.46), and $(4.59)_2$ is expressed in terms of the bubble radius R instead of $\bar{\rho}_g$, Eqs. (4.59) become

$$\begin{aligned} \rho\mathbf{a} &= \rho\mathbf{b} - \operatorname{grad}\lambda, \\ R\ddot{R} + \tfrac{3}{2}\left(1 - \frac{\phi_g}{\phi_f}\right)\dot{R}^2 &= \frac{p_g - p_f}{\bar{\rho}_f}, \\ \lambda &= p_f - \tfrac{3}{2}\frac{\phi_g}{\phi_f}\bar{\rho}_f\dot{R}^2. \end{aligned} \tag{4.62}$$

In the limit $\phi_g \to 0$, these equations reduce to

$$\begin{aligned} \rho\mathbf{a} &= \rho\mathbf{b} - \operatorname{grad} p_f, \\ R\ddot{R} + \tfrac{3}{2}\dot{R}^2 &= \frac{p_g - p_f}{\bar{\rho}_f}. \end{aligned} \tag{4.63}$$

Equation $(4.63)_2$ is the *Rayleigh-Plesset equation* for the dilatational motion of a single bubble in an unbounded incompressible liquid. It was pointed out by Drumheller et al. [23] and independently by Passman et al. [64] that this equation

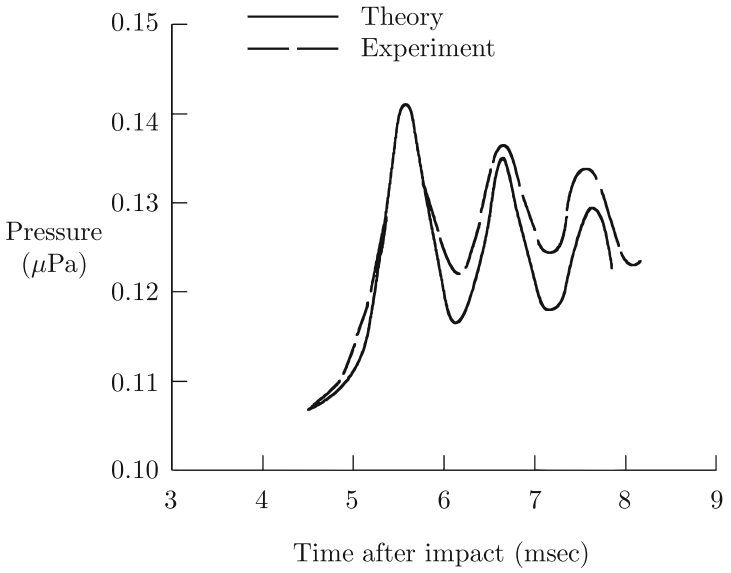

Fig. 4.6 Transient pressure history in a bubbly liquid compared with shock tube data

arises from a mixture theory of this type. The approach that has traditionally been used to model the dynamics of bubbly liquids is to adopt $(4.63)_1$, which is simply the equation of balance of linear momentum for an ideal fluid, and to assume that $(4.63)_2$ applies (see e.g. van Wijngaarden [72]). Thus this model is recovered from the equations obtained from Hamilton's principle in the limit as the bubble volume fraction approaches zero.

Hamilton's principle has been used to obtain the governing equations for a bubbly liquid in which there is relative motion between the liquid and bubbles by Drumheller and Bedford [21].

4.3 Mixture of an Ideal Fluid and an Elastic Solid

The application of Hamilton's principle to a binary mixture of an elastic ideal fluid and an elastic solid is discussed in this section. Although this case requires only a minor extension of the formulation for mixtures of ideal fluids, it provides an introduction to the more general solution that is discussed in the next section. Another reason this case deserves attention is that it leads to Biot's theory for a fluid saturated porous elastic material, which is one of the most widely accepted and applied theories of mixtures.

Let the fluid and solid constituents be denoted by subscripts f and s respectively. The potential energy of the mixture contained in B_t will be expressed in the form

$$U = \int_{B_t} [\rho_f e_f(\bar{\rho}_f) + \rho_s e_s(\bar{\rho}_s, \mathbf{E}_s)]\, dV_t, \tag{4.64}$$

where $\mathbf{E}_s$ is the linear strain of the elastic material. The material density $\bar{\rho}_s$ of a porous material can vary independently of $\mathbf{E}_s$, and is therefore included as an argument in the internal energy of the material. The derivatives $de_f/d\bar{\rho}_f$, $de_s/d\bar{\rho}_s$, and $\partial e_s/\partial \mathbf{E}_s$ will be assumed to exist and be continuous, and the fields grad $(de_f/d\bar{\rho}_f)$, grad $(de_s/d\bar{\rho}_s)$, and grad $(\partial e_s/\partial \mathbf{E}_s)$ will be assumed to be continuous on $\bar{B} \times [t_1, t_2]$.

To determine the variation of the potential energy, (4.64) must be expressed in terms of the comparison motion (4.10) and the comparison material density $(4.12)_1$. Upon taking the derivative of the result with respect to ε and setting $\varepsilon = 0$, the variation is

$$\delta U = \int_{B_t} \left[\rho_f \frac{de_f}{d\bar{\rho}_f} \delta\bar{\rho}_f + \rho_s \frac{\partial e_s}{\partial \bar{\rho}_s} \delta\bar{\rho}_s - \text{div} \left(\rho_s \frac{\partial e_s}{\partial \mathbf{E}_s} \mathbf{F}^{\text{t}} \right) \cdot \delta\mathbf{x}_s \right] dV_t. \tag{4.65}$$

Using this expression together with the same expressions used in the case of a mixture of compressible fluids for the virtual work δW (4.25), the constraint term δC (4.26), and the kinetic energy T (4.27), Hamilton's principle for a mixture of an elastic ideal fluid and an elastic solid states [6]:

> Among comparison motions (4.10) and comparison fields (4.12), the actual fields are such that
>
> $$\int_{t_1}^{t_2} [\delta(T - U) + \delta W + \delta C]\, dt = 0. \tag{4.66}$$

In the present case this equation results in an expression that is identical to (4.29) except for the replacement of the expression for the variation of the potential energy by (4.65). It is therefore easy to show that the equations resulting from Hamilton's principle in this case are

$$\begin{aligned}
\rho_s \mathbf{a}_s &= \rho_s \mathbf{b}_s + \mathbf{d}_s - \text{grad}\, \pi_s + \lambda\, \text{grad}\, \phi_s + \text{div} \left(\rho_s \frac{\partial e_s}{\partial \mathbf{E}_s} \mathbf{F}^{\text{t}} \right), \\
\rho_f \mathbf{a}_f &= \rho_f \mathbf{b}_f + \mathbf{d}_f - \text{grad}\, \pi_f + \lambda\, \text{grad}\, \phi_f, \\
\pi_s &= \phi_s \bar{\rho}_s^2 \frac{\partial e_s}{\partial \bar{\rho}_s}, \\
\pi_f &= \phi_f \bar{\rho}_f^2 \frac{\partial e_f}{\partial \bar{\rho}_f}, \\
\pi_s &= \phi_s \lambda, \\
\pi_f &= \phi_f \lambda
\end{aligned} \tag{4.67}$$

on $\bar{B} \times [t_1, t_2]$. Upon using the last two equations to eliminate π_s and π_f, the remaining equations can be written

$$\begin{aligned} \rho_s \mathbf{a}_s &= \rho_s \mathbf{b}_s + \mathbf{d}_s - \phi_s \operatorname{grad} \lambda + \operatorname{div}\left(\rho_s \frac{\partial e_s}{\partial \mathbf{E}_s} \mathbf{F}^{\mathrm{t}}\right), \\ \rho_f \mathbf{a}_f &= \rho_f \mathbf{b}_f + \mathbf{d}_f - \phi_f \operatorname{grad} \lambda, \\ \lambda &= \bar{\rho}_s^2 \frac{\partial e_s}{\partial \bar{\rho}_s} = \bar{\rho}_f^2 \frac{\partial e_f}{\partial \bar{\rho}_f}. \end{aligned} \tag{4.68}$$

Let the material densities and volume fractions be expressed as sums of their reference values and small perturbations:

$$\begin{aligned} \bar{\rho}_\xi &= \bar{\rho}_{\xi R} + \tilde{\bar{\rho}}_\xi, \\ \phi_\xi &= \phi_{\xi R} + \tilde{\phi}_\xi. \end{aligned} \tag{4.69}$$

The resulting linearized forms of the equations of conservation of mass (4.7) are

$$\frac{\tilde{\phi}_\xi}{\phi_{\xi R}} + \frac{\tilde{\bar{\rho}}_\xi}{\bar{\rho}_{\xi R}} + \operatorname{tr} \mathbf{E}_\xi = 0, \tag{4.70}$$

and the linearized form of the volume fraction constraint (4.9) is

$$\sum_\xi \tilde{\phi}_\xi = 0. \tag{4.71}$$

The two volume fractions can be eliminated from the three Eqs. (4.70) and (4.71) to obtain the single equation

$$\phi_{sR}\left(\frac{\tilde{\bar{\rho}}_s}{\bar{\rho}_{sR}} + \operatorname{tr} \mathbf{E}_s\right) + \phi_{fR}\left(\frac{\tilde{\bar{\rho}}_f}{\bar{\rho}_{fR}} + \operatorname{tr} \mathbf{E}_f\right) = 0. \tag{4.72}$$

Now let the internal energies of the constituents be expressed as isotropic second-order expansions in their arguments:

$$\begin{aligned} \rho_s e_s &= \tfrac{1}{2}\bar{c}\,(\operatorname{tr} \mathbf{E}_s)^2 + \bar{d}\,\mathbf{E}_s \cdot \mathbf{E}_s + \bar{f}\,\tilde{\bar{\rho}}_s \operatorname{tr} \mathbf{E}_s + \tfrac{1}{2}\bar{g}\,\tilde{\bar{\rho}}_s^2, \\ \rho_f e_f &= \tfrac{1}{2}\bar{h}\,\tilde{\bar{\rho}}_f^2, \end{aligned} \tag{4.73}$$

where $\bar{c}$, $\bar{d}$, $\bar{f}$, $\bar{g}$, and $\bar{h}$ are constitutive constants. Using these expressions, the linearized forms of (4.68) can be written (with external body forces neglected)

$$\begin{aligned}
\rho_{sR}\ddot{\mathbf{u}}_s &= \mathbf{d}_s - \phi_{sR}\ \text{grad}\ \lambda + (\bar{c} + 2\bar{d})\ \text{grad div}\ \mathbf{u}_s \\
&\quad -\bar{d}\ \text{curl curl}\ \mathbf{u}_s + \bar{f}\ \text{grad}\ \tilde{\tilde{\rho}}_s, \\
\rho_{fsR}\ddot{\mathbf{u}}_f &= \mathbf{d}_f - \phi_{fR}\ \text{grad}\ \lambda, \\
\lambda &= \frac{\bar{\rho}_{sR}}{\phi_{sR}}(\bar{f}\ \text{tr}\ \mathbf{E}_s + \bar{g}\tilde{\tilde{\rho}}_s), \\
\lambda &= \frac{\bar{\rho}_{fR}}{\phi_{fR}}\bar{h}\tilde{\tilde{\rho}}_f.
\end{aligned} \tag{4.74}$$

If linear constitutive relations are specified for the drag terms $\mathbf{d}_\xi$, (4.72) and (4.74) provide a system of linear equations with which to determine the fields $\mathbf{u}_s$, $\mathbf{u}_f$, $\tilde{\tilde{\rho}}_s$, $\tilde{\tilde{\rho}}_f$, and λ.

Equation (4.72) and the last two of Eqs. (4.74) can be solved for the variables $\tilde{\tilde{\rho}}_s$, $\tilde{\tilde{\rho}}_f$, and λ in terms of tr $\mathbf{E}_s$ and tr $\mathbf{E}_f$. When the resulting expressions are substituted into the first two of Eqs. (4.74), they can be written

$$\begin{aligned}
\rho_{sR}\ddot{\mathbf{u}}_s &= \mathbf{d}_s + (\bar{P} + 2\bar{N})\ \text{grad div}\ \mathbf{u}_s \\
&\quad -\bar{N}\ \text{curl curl}\ \mathbf{u}_s + \bar{Q}\ \text{grad div}\ \mathbf{u}_f, \\
\rho_{fR}\ddot{\mathbf{u}}_f &= \mathbf{d}_f + \bar{Q}\ \text{grad div}\ \mathbf{u}_s + \bar{R}\ \text{grad div}\ \mathbf{u}_f,
\end{aligned} \tag{4.75}$$

where $\bar{P}$, $\bar{Q}$, $\bar{R}$, and $\bar{N}$ are constants. These two linear equations for the displacement fields $\mathbf{u}_s$ and $\mathbf{u}_f$ are the *Biot equations* [10].

The application of Hamilton's principle to a mixture of an elastic ideal fluid and an elastic solid described in this section has been extended to include microkinetic energy of the constituents by Bedford and Drumheller [6]. This approach has been used to develop a theory of a porous elastic material containing a bubbly liquid by Bedford and Stern [8], and it has been used to obtain a theory of a saturated porous medium with microstructure and nonlinear material behavior by Berryman and Thigpen [9].

4.4 A Theory of Mixtures with Microstructure

The theories discussed in Sects. 4.2 and 4.3 are very special for the same reason that the theories of elastic ideal fluids and elastic solids described in Sects. 3.1.1 and 3.1.2 were special: Internal energies were introduced which were assumed to depend only on the states of deformation of the constituents. This restriction can be removed in the case of a mixture by using the same approach that was used in

Sect. 3.1.3 for a single material. That is, internal forces can be expressed through virtual work terms rather than by internal energies. In this section, an illustration will be given of the use of Hamilton's principle to derive a quite general theory of mixtures with microkinetic energy in which the constituents are not constrained to be ideal or elastic [22]. The theory will include the results of the preceding two sections as special cases.

Microkinetic Energy Consider a homogeneous sphere of radius R and density $\bar{\rho}$. If the sphere expands homogeneously, its kinetic energy relative to the center of the sphere is

$$\tfrac{2}{5}\pi\bar{\rho}R^3\dot{R}^2, \tag{4.76}$$

where the dot denotes the time derivative. Because the radius of the sphere and the density of the homogeneous material are related by $\bar{\rho}R^3 = \text{ constant}$, the kinetic energy can be expressed in terms of the density as

$$\frac{2\pi\bar{\rho}_R^{5/3}R_R^5}{45\bar{\rho}^{8/3}}\dot{\bar{\rho}}^2, \tag{4.77}$$

where $\bar{\rho}_R$ and R_R are reference values.

Suppose that a constituent of a mixture consists of a distribution of such spheres, and let ϕ be the volume fraction of the constituent. Multiplying (4.77) by the number of spheres per unit volume $\phi/(4/3)\pi R^3$, the kinetic energy per unit volume due to expansion or contraction of the spheres is

$$\tfrac{1}{2}\rho\left(\frac{\bar{\rho}_R^{2/3}R_R^2}{15\bar{\rho}^{8/3}}\right)\dot{\bar{\rho}}^2, \tag{4.78}$$

which is of the same functional form as (4.46). Therefore, two examples of microkinetic energy, the energy of the liquid surrounding a distribution of oscillating bubbles, discussed in Sect. 4.2.3, and the energy due to the homogeneous expansion and contraction of a distribution of particles, can be included in the present model if it is assumed that each constituent $\mathcal{C}_\xi$ has a microkinetic energy per unit volume of the form

$$\tfrac{1}{2}\rho_\xi I_\xi(\bar{\rho}_\gamma)\dot{\bar{\rho}}_\xi^2. \tag{4.79}$$

The terms $I_\xi(\bar{\rho}_\gamma)$ are constitutive functions that are assumed to depend on the material density of each constituent. Their second partial derivatives will be assumed to exist and be continuous. Therefore, the total kinetic energy of the mixture contained in B_t will be expressed in the form

$$T = \sum_\xi \int_{B_t} \tfrac{1}{2}\rho_\xi(\mathbf{v}_\xi\cdot\mathbf{v}_\xi + I_\xi\dot{\bar{\rho}}_\xi^2)\,dV_t. \tag{4.80}$$

Virtual Work The virtual work done on the mixture contained in B_t by internal forces will be assumed to have the form

$$-\sum_{\xi}\int_{B}\mathbf{S}_{\xi}\cdot\delta\mathbf{F}_{\xi}\,dV-\sum_{\xi}\int_{B_t}\frac{\phi_{\xi}}{\bar{\rho}_{\xi}}\,p_{\xi}\,\delta\bar{\rho}_{\xi}\,dV_t. \tag{4.81}$$

That is, it is assumed that work is done on $\mathcal{C}_{\xi}$ when its deformation gradient changes and when its material density changes. These two variables can change independently of one another if a constituent consists of, for example, a porous medium or a distribution of particles. The form of (4.81) is motivated by the form of the virtual work done by internal forces in a single elastic material (3.69) and by the independent variables which appear in the internal energy for a mixture of an ideal fluid and an elastic solid (4.64). The generalized forces, which are the tensor fields $\mathbf{S}_{\xi}$ and the scalar fields p_{ξ}, are constitutive variables. It will be assumed that $\mathbf{S}\xi$, DIV $\mathbf{S}_{\xi}$, p_{ξ}, and GRAD p_{ξ} are continuous on $\bar{B}\times[t_1,t_2]$. Adding to (4.81) the virtual work done by external forces in the form (4.25), the total virtual work on the mixture contained in B_t is

$$\begin{aligned}\delta W=&-\sum_{\xi}\int_{B}\mathbf{S}_{\xi}\cdot\delta\mathbf{F}_{\xi}\,dV-\sum_{\xi}\int_{B_t}\frac{\phi_{\xi}}{\bar{\rho}_{\xi}}\,p_{\xi}\,\delta\bar{\rho}_{\xi}\,dV_t\\&+\sum_{\xi}\int_{B_t}(\rho_{\xi}\mathbf{b}_{\xi}+\mathbf{d}_{\xi})\cdot\delta\mathbf{x}_{\xi}\,dV_t.\end{aligned} \tag{4.82}$$

Constraints The motions, volume fractions, and material densities of the constituents are subject to the constraints (4.20) and (4.23) arising from the volume fraction constraint and the equations of conservation of mass respectively. Therefore, the constraint term (4.26),

$$\begin{aligned}\delta C=&\sum_{\xi}\int_{B_t}\lambda(\operatorname{grad}\phi_{\xi}\cdot\delta\mathbf{x}_{\xi}-\delta\phi_{\xi})\,dV_t\\&+\sum_{\xi}\int_{B_t}\left[-\operatorname{grad}\pi_{\xi}\cdot\delta\mathbf{x}_{\xi}+\pi_{\xi}\left(\frac{\delta\phi_{\xi}}{\phi_{\xi}}+\frac{\delta\bar{\rho}_{\xi}}{\bar{\rho}_{\xi}}\right)\right]dV_t,\end{aligned} \tag{4.83}$$

will be included in Hamilton's principle.

Hamilton's Principle Based on the expressions (4.80), (4.82), and (4.83), Hamilton's principle for a mixture of materials with microkinetic energy states:

> Among comparison motions (4.10) and comparison fields (4.12), the actual fields are such that
>
> $$\int_{t_1}^{t_2}(\delta T+\delta W+\delta C)\,dt=0. \tag{4.84}$$

Substituting (4.80), (4.82), and (4.83), and through the use of steps that are familiar from previous sections, (4.84) can be written

$$\begin{aligned}\sum_{\xi}\int_{t_1}^{t_2}\int_{B_t}\Big[&(-\rho_\xi \mathbf{a}_\xi + \operatorname{div}\mathbf{T}_\xi - \operatorname{grad}\pi_\xi\\ &+\lambda\operatorname{grad}\phi_\xi + \rho_\xi\mathbf{b}_\xi + \mathbf{d}_\xi)\cdot\delta\mathbf{x}_\xi\\ &+\Big(-\rho_\xi\overset{\cdot}{\overline{I_\xi\dot{\bar\rho}_\xi}} + \sum_\gamma \tfrac{1}{2}\rho_\gamma\frac{\partial I_\gamma}{\partial\bar\rho_\xi}\dot{\bar\rho}_\gamma^2 + \frac{\pi_\xi}{\bar\rho_\xi} - \phi_\xi\frac{p_\xi}{\bar\rho_\xi}\Big)\delta\bar\rho_\xi\\ &+\Big(\frac{\pi_\xi}{\phi_\xi} - \lambda\Big)\delta\phi_\xi\Big]\,dV_t\,dt = 0,\end{aligned} \tag{4.85}$$

where

$$\mathbf{T}_\xi = \frac{1}{J_\xi}\mathbf{S}_\xi\mathbf{F}_\xi^{\mathrm{t}} \tag{4.86}$$

is the Cauchy stress of $\mathcal{C}_\xi$. Applying the fundamental lemmas to (4.85), the resulting equations are

$$\left.\begin{aligned}\rho_\xi\mathbf{a}_\xi &= \operatorname{div}\mathbf{T}_\xi - \operatorname{grad}\pi_\xi + \lambda\operatorname{grad}\phi_\xi\\ &\quad +\rho_\xi\mathbf{b}_\xi + \mathbf{d}_\xi,\\ \rho_\xi\overset{\cdot}{\overline{I_\xi\dot{\bar\rho}_\xi}} - \sum_\gamma \tfrac{1}{2}\rho_\gamma\frac{\partial I_\gamma}{\partial\bar\rho_\xi}\dot{\bar\rho}_\gamma^2 &= \frac{\pi_\xi}{\bar\rho_\xi} - \phi_\xi\frac{p_\xi}{\bar\rho_\xi},\\ \pi_\xi &= \phi_\xi\lambda\end{aligned}\right\}\quad \text{on}\quad \bar B\times[t_1,t_2]\,. \tag{4.87}$$

Using the last equation to eliminate π_ξ, these equations reduce to

$$\begin{aligned}\rho_\xi\mathbf{a}_\xi &= \operatorname{div}\mathbf{T}_\xi - \phi_\xi\operatorname{grad}\lambda + \rho_\xi\mathbf{b}_\xi + \mathbf{d}_\xi,\\ \bar\rho_\xi^2\overset{\cdot}{\overline{I_\xi\dot{\bar\rho}_\xi}} - \sum_\gamma \tfrac{1}{2}\rho_\gamma\frac{\bar\rho_\xi}{\phi_\xi}\frac{\partial I_\gamma}{\partial\bar\rho_\xi}\dot{\bar\rho}_\gamma^2 &= \lambda - p_\xi.\end{aligned} \tag{4.88}$$

To obtain a complete mechanical theory, constitutive relations must be postulated for the generalized forces $\mathbf{T}_\xi$, $\mathbf{d}_\xi$, and p_ξ and for the microkinetic energy coefficients I_ξ. Then (4.6), (4.8), (4.9), and (4.88) provide a system of equations with which to determine the fields ρ_ξ, $\bar\rho_\xi$, ϕ_ξ, λ, and $\mathbf{v}_\xi$.

Balance of Energy A postulate of the equations of balance of energy for the mixture can be motivated using the method described in Sect. 3.1.3. Consider an arbitrary volume B_t' contained within B_t (Fig. 3.1). The part of $\mathcal{C}_\xi$ that is contained

in B'_t at time t occupies a volume B'_ξ in the reference configuration. Recall the correspondence between the form of the virtual work term (3.69) and that of the mechanical working term (3.86) that appears in the global form of the equation of balance of energy for an ordinary continuous medium. In the case of the mixture under consideration, the corresponding virtual work term is (4.81). From the form of this term, it can be deduced that the mechanical working term for the part of $\mathcal{C}_\xi$ contained in B'_t is

$$\int_{B'_t} \mathbf{S}_\xi \cdot \dot{\mathbf{F}}_\xi \, dV_\xi + \int_{B'_t} \frac{\phi_\xi}{\bar{\rho}_\xi} p_\xi \dot{\bar{\rho}}_\xi \, dV_t. \tag{4.89}$$

Equating this expression to the rate of change of the internal energy of $\mathcal{C}_\xi$ within B'_t and introducing heat conduction terms analogous to (3.82) and (3.83), the balance of energy postulate for $\mathcal{C}_\xi$ is

$$\begin{aligned} \frac{d}{dt} \int_{B'_t} \rho_\xi e_\xi \, dV_t = &\int_{B'_t} \mathbf{T}_\xi \cdot \mathbf{L}_\xi \, dV_t + \int_{B'_t} \frac{\phi_\xi}{\bar{\rho}_\xi} p_\xi \dot{\bar{\rho}}_\xi \, dV_t \\ &- \int_{\partial B'_t} \mathbf{q}_\xi \cdot \mathbf{n} \, dS_t + \int_{B'_t} \rho_\xi s_\xi \, dV_t, \end{aligned} \tag{4.90}$$

where (3.86) has been used. The heat flux $\mathbf{q}_\xi$ is assumed to be C^1 and the heat supply s_ξ is assumed to be C^0 on $\bar{B} \times [t_1, t_2]$. Here the heat supply s_ξ is defined to be the rate at which heat is added to $\mathcal{C}_\xi$ both by external sources and by the other constituent of the mixture. The local form of the equation of balance of energy for $\mathcal{C}_\xi$ obtained from the postulate (4.90) is

$$\rho_\xi \dot{e}_\xi = \mathbf{T}_\xi \cdot \mathbf{L}_\xi + \frac{\phi_\xi}{\bar{\rho}_\xi} p_\xi \dot{\bar{\rho}}_\xi - \operatorname{div} \mathbf{q}_\xi + \rho_\xi s_\xi. \tag{4.91}$$

Let the field $\theta_\xi(\mathbf{X}_\xi, t)$ denote the absolute temperature of $\mathcal{C}_\xi$. Then if constitutive relations are postulated for $\mathbf{T}_\xi$, $\mathbf{d}_\xi$, p_ξ, I_ξ, e_ξ, $\mathbf{q}_\xi$, and s_ξ, Eqs. (4.6), (4.8), (4.9), (4.88), and (4.91) can be used to determine the fields ρ_ξ, $\bar{\rho}_\xi$, ϕ_ξ, λ, $\mathbf{v}_\xi$, and θ_ξ, yielding a *thermomechanical theory of mixtures with microkinetic energy*.

Nunziato et al. [59,62,64] have developed a theory of mixtures with microstructure that shares many elements with this one. Their theory was motivated by the theory of granular solids due to Goodman and Cowin that is described in Sect. 3.2.1. They adopted (3.105) and (3.109) for each constituent of the mixture, then introduced appropriate terms to account for the interchanges of momentum and energy between constituents.

Chapter 5
Discontinuous Fields

5.1 Singular Surfaces

The objective is to apply Hamilton's principle to a continuous medium containing a surface of discontinuity, such as a boundary or wave front. How can such a surface be described? Assume that the motion of a continuous medium

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \tag{5.1}$$

is one-one and C^0 on $\bar{B} \times [t_1, t_2]$.[1] Let Σ denote a fixed, plane, open surface in $\mathcal{E}$, and define a function

$$\mathbf{z} = \boldsymbol{\zeta}(\mathbf{W}, t) \tag{5.2}$$

that maps Σ onto a surface Σ_t that intersects $\bar{B}_t$ at time t (Fig. 5.1). The vector $\mathbf{W}$ denotes the position vector of a point of Σ (a *surface point*). The vector $\mathbf{z}$ is the position vector of the surface point $\mathbf{W}$ at time t. The mapping (5.2) will be assumed to be C^2 on $\Sigma \times [t_1, t_2]$.

Let the intersection of Σ_t with $\bar{B}_t$ be denoted by $\mathcal{S}_t$. The surface Σ does not necessarily represent a physical surface, but simply provides a means to describe the motion of the surface $\mathcal{S}_t$. The surface $\mathcal{S}_t$ may represent a wave front or other surface of interest in the material at time t. The surface $\mathcal{S}_t$ divides B_t into two parts that will be called B_t^+ and B_t^-. Let the field $\mathbf{n}(\mathbf{x}, t)$ defined on $\mathcal{S}_t$ be the unit vector normal to $\mathcal{S}_t$ that points into B_t^+ (Fig. 5.2).

Because the motion of the material is assumed to be one-one and continuous on $\bar{B} \times [t_1, t_2]$, a unique material point is located at a given point $\mathbf{z}$ of $\mathcal{S}_t$ at time t. The

[1] See the discussion of the motion in Sect. 2.2.

A. Bedford, *Hamilton's Principle in Continuum Mechanics*,
https://doi.org/10.1007/978-3-030-90306-0_5

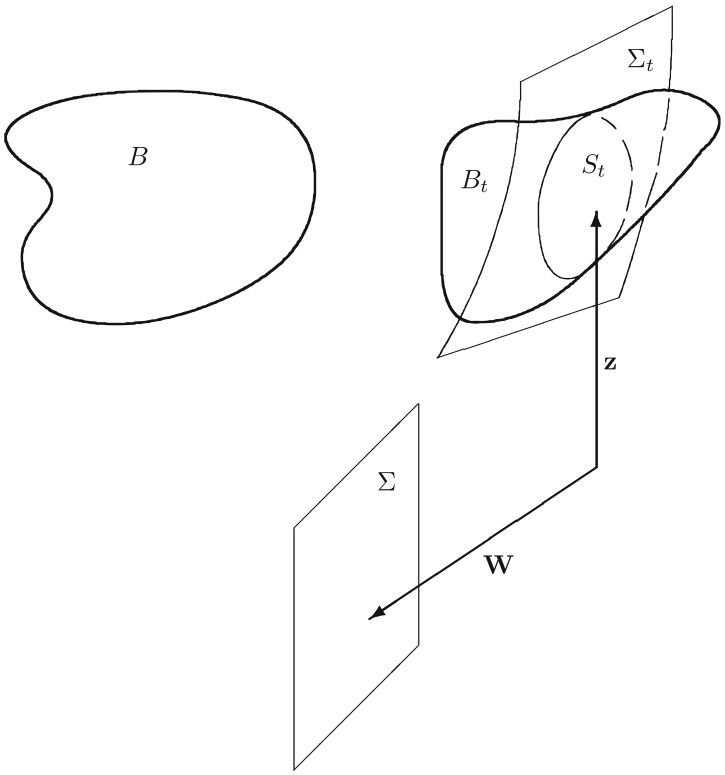

Fig. 5.1 The surfaces Σ, Σ_t, and $\mathcal{S}_t$

position of this material point in the reference configuration, denoted by **Z**, is given by the inverse motion:

$$\mathbf{Z} = \boldsymbol{\chi}^{-1}(\mathbf{z}, t). \tag{5.3}$$

This function maps the surface $\mathcal{S}_t$ onto a surface $\mathcal{S}$ in the reference configuration (Fig. 5.2). The surface $\mathcal{S}$ is called the *image surface*; it is the locus in the reference configuration of the material points that coincide with the surface $\mathcal{S}_t$ at time t. The image surface divides the reference configuration into two parts B^+ and B^-. The function $\mathbf{N}(\mathbf{X}, t)$ defined on $\mathcal{S}$ will denote the unit vector normal to $\mathcal{S}$ that points into B^+.

Let ∂B^+ be the outer surface of B^+, and let $\bar{B}^+$ denote the closure of B^+; that is, B^+ together with its surface $\partial B^+ + \mathcal{S}$. The notations ∂B^- and $\bar{B}^-$ are defined correspondingly. The motion (5.1) will be assumed to be C^2 on $\bar{B}^+ \times [t_1, t_2]$ and on $\bar{B}^- \times [t_1, t_2]$. Thus *the motion of the material is assumed to be C^2 on each part of B, but is merely assumed to be continuous across the surface $\mathcal{S}$.*

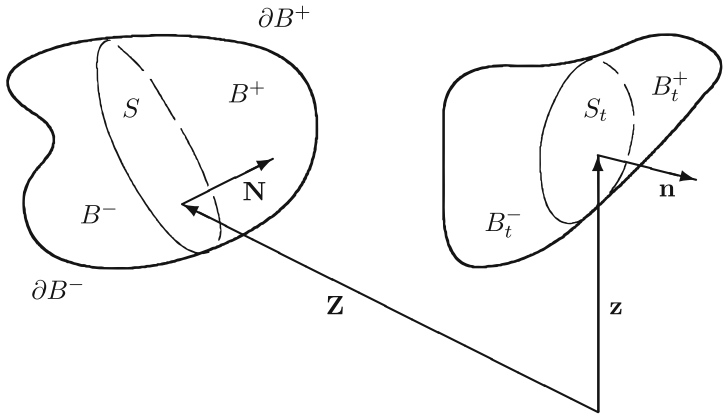

Fig. 5.2 The surface $\mathcal{S}_t$ and the image surface $\mathcal{S}$

Consider a field $\mathbf{f}(\mathbf{X}, t)$, and define $\mathbf{f}^+$ by

$$\mathbf{f}^+ = \lim_{\mathbf{X}\to\mathbf{Z}} \mathbf{f}(\mathbf{X}, t), \tag{5.4}$$

where the limit is taken as $\mathbf{X}$ approaches $\mathbf{Z}$ along a smooth path within B^+. The notation $\mathbf{f}^-$ is defined correspondingly. With some exceptions that will be obvious from their contexts, the superscripts $^+$ and $^-$ will refer to these limits. The *jump* of $\mathbf{f}(\mathbf{X}, t)$ across $\mathcal{S}$ is defined by

$$[\![\mathbf{f}]\!] = \mathbf{f}^+ - \mathbf{f}^-. \tag{5.5}$$

In terms of the mapping (5.2), the velocity of the surface point $\mathbf{W}$ is

$$\dot{\mathbf{z}} = \frac{\partial}{\partial t}\boldsymbol{\zeta}(\mathbf{W}, t). \tag{5.6}$$

The normal component $\dot{\mathbf{z}} \cdot \mathbf{n}$ is the speed of the surface $\mathcal{S}_t$. It is called the *speed of displacement* ([71], p. 499). From (5.3), the velocity of the surface point $\mathbf{W}$ relative to the reference configuration is

$$\begin{aligned}\dot{\mathbf{Z}} &= \frac{\partial}{\partial t}\boldsymbol{\chi}^{-1}(\boldsymbol{\zeta}(\mathbf{W}, t), t) \\ &= \left(\frac{\partial \boldsymbol{\chi}^{-1}}{\partial \mathbf{x}}\right)^+ \frac{\partial \boldsymbol{\zeta}}{\partial t} + \left(\frac{\partial \boldsymbol{\chi}^{-1}}{\partial t}\right)^+ \\ &= \left(\frac{\partial \boldsymbol{\chi}^{-1}}{\partial \mathbf{x}}\right)^- \frac{\partial \boldsymbol{\zeta}}{\partial t} + \left(\frac{\partial \boldsymbol{\chi}^{-1}}{\partial t}\right)^-.\end{aligned} \tag{5.7}$$

Notice that

$$d\mathbf{X} = \frac{\partial \boldsymbol{\chi}^{-1}}{\partial \mathbf{x}}\, d\mathbf{x} + \frac{\partial \boldsymbol{\chi}^{-1}}{\partial t}\, dt, \tag{5.8}$$

so the partial derivative of the inverse motion with respect to time holding $\mathbf{X}$ fixed can be written in terms of the inverse of the deformation gradient and the velocity of the material:

$$\frac{\partial \boldsymbol{\chi}^{-1}}{\partial t} = -\frac{\partial \boldsymbol{\chi}^{-1}}{\partial \mathbf{x}}\left[\frac{d\mathbf{x}}{dt}\right]_{\mathbf{X}} = -\mathbf{F}^{-1}\mathbf{v}. \tag{5.9}$$

Substituting this result into (5.7) yields a relation between the velocity of the material point $\mathbf{W}$ and the velocity of its image in the reference configuration:

$$\begin{aligned} \dot{\mathbf{Z}} &= (\mathbf{F}^{-1})^{+}(\dot{\mathbf{z}} - \mathbf{v}^{+}) \\ &= (\mathbf{F}^{-1})^{-}(\dot{\mathbf{z}} - \mathbf{v}^{-}). \end{aligned} \tag{5.10}$$

The normal component $\dot{\mathbf{Z}} \cdot \mathbf{N}$ is the speed of the image surface $\mathcal{S}$ relative to the reference configuration. It is called the *speed of propagation* ([71], p. 508). Equation (5.10) yields the result

$$[\![\mathbf{F}^{-1}(\dot{\mathbf{z}} - \mathbf{v})]\!] = \mathbf{o}. \tag{5.11}$$

This equation results from the assumed continuity of the motion (5.1) across $\mathcal{S}$. Taking the inner product of (5.11) with $\mathbf{N}$ and using (2.66) and (2.71) yields the *Stokes-Christoffel condition* ([71], p. 522)

$$[\![\rho(\dot{\mathbf{z}} - \mathbf{v}) \cdot \mathbf{n}]\!] = 0. \tag{5.12}$$

This jump condition insures conservation of mass of the material across $\mathcal{S}_t$.

To apply Hamilton's principle, a comparison motion of the material is defined by[2]

$$\begin{aligned} \mathbf{x}^{*} &= \boldsymbol{\chi}(\mathbf{X}, t) + \varepsilon \boldsymbol{\eta}(\mathbf{X}, t) \\ &= \mathbf{K}(\mathbf{X}, t, \varepsilon). \end{aligned} \tag{5.13}$$

The vector field $\boldsymbol{\eta}(\mathbf{X}, t)$ is arbitrary subject to the conditions that it be C^2 on $\bar{B}^{+} \times [t_1, t_2]$ and on $\bar{B}^{-} \times [t_1, t_2]$, that $\boldsymbol{\eta}(\mathbf{X}, t_1) = \mathbf{o}$ and $\boldsymbol{\eta}(\mathbf{X}, t_2) = \mathbf{o}$, and that (5.13) satisfy prescribed boundary conditions on ∂B.[3] The field $\boldsymbol{\eta}(\mathbf{X}, t)$ is *not* assumed to be continuous across $\mathcal{S}$.

[2] See the discussion of the comparison motion in Sect. 2.3.

[3] Because the example to be presented is an ideal fluid, and there will be no concern with boundary conditions, it will be assumed henceforth that $\boldsymbol{\eta} \cdot \mathbf{n} = 0$ on ∂B_t.

A comparison density field is defined by

$$\rho^* = \rho(\mathbf{X}, t) + \varepsilon r(\mathbf{X}, t), \tag{5.14}$$

where $r(\mathbf{X}, t)$ is an arbitrary scalar field subject to the conditions that it be C^1 on $\bar{B}^+ \times [t_1, t_2]$ and on $\bar{B}^- \times [t_1, t_2]$ and that $r(\mathbf{X}, t_1) = 0$ and $r(\mathbf{X}, t_2) = 0$. The field $r(\mathbf{X}, t)$ is not assumed to be continuous across $\mathcal{S}$.

In analogy with (5.13), a comparison motion of the surface $\mathcal{S}_t$ is defined by [67]

$$\mathbf{z}^* = \boldsymbol{\zeta}(\mathbf{W}, t) + \varepsilon \boldsymbol{\mu}(\mathbf{W}, t), \tag{5.15}$$

where $\boldsymbol{\mu}(\mathbf{W}, t)$ is an arbitrary C^2 function on $\Sigma \times [t_1, t_2]$ such that $\boldsymbol{\mu}(\mathbf{W}, t_1) = \mathbf{o}$ and $\boldsymbol{\mu}(\mathbf{W}, t_2) = \mathbf{o}$.

As a result of the comparison motions (5.13) and (5.15), the position in the reference configuration of the material point that is located at $\mathbf{z}^*$ at time t is (Fig. 5.3)

$$\mathbf{Z}^* = \mathbf{K}^{-1}(\mathbf{z}^*, t, \varepsilon). \tag{5.16}$$

Expanding this expression with respect to ε yields

$$\begin{aligned} \mathbf{Z}^* &= \mathbf{Z} + \left[\left(\frac{\partial \mathbf{K}^{-1}}{\partial \mathbf{z}^*} \right)^+ \frac{\partial \mathbf{z}^*}{\partial \varepsilon} + \left(\frac{\partial \mathbf{K}^{-1}}{\partial \varepsilon} \right)^+ \right]_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \\ &= \mathbf{Z} + (\mathbf{F}^{-1})^+ (\boldsymbol{\mu} - \boldsymbol{\eta}^+)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{5.17}$$

where the result (4.18) has been used. This equation also holds when the $+$ superscripts are replaced by $-$. Introducing the notation (2.88), (5.17) yields the

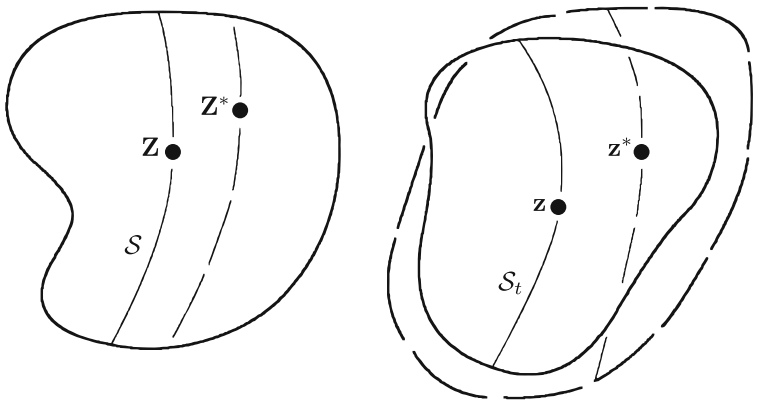

Fig. 5.3 The points $\mathbf{Z}^*$ and $\mathbf{z}^*$

result

$$\begin{aligned}\delta\mathbf{Z} &= (\mathbf{F}^{-1})^{+}(\delta\mathbf{z} - \delta\mathbf{x}^{+}) \\ &= (\mathbf{F}^{-1})^{-}(\delta\mathbf{z} - \delta\mathbf{x}^{-}),\end{aligned} \tag{5.18}$$

where $\delta\mathbf{z} = \boldsymbol{\mu}$. Therefore,

$$[\![\mathbf{F}^{-1}(\delta\mathbf{z} - \delta\mathbf{x})]\!] = \mathbf{o}. \tag{5.19}$$

This equation is a constraint imposed on the comparison motions (5.13) and (5.15) by the continuity of the motion of the material at the surface $\mathcal{S}$. Compare Eqs. (5.18) and (5.19) to Eqs. (5.10) and (5.11).

Let $\mathbf{f}(\mathbf{X}, t)$ be a field that is continuous on $\bar{B}^{+} \times [t_1, t_2]$ and on $\bar{B}^{-} \times [t_1, t_2]$, and let $\mathbf{f}^{*}(\mathbf{X}, t, \varepsilon)$ be its associated comparison field. Consider the integral

$$I = \int_{B_t^{\pm}} \rho\mathbf{f}\, dV_t = \int_{B^{\pm}} \rho_R\mathbf{f}\, dV, \tag{5.20}$$

where the notation $B_t^{\pm}$ means the sum of the integrals over B_t^{+} and B_t^{-}. The value of this integral when it is expressed in terms of the comparison field $\mathbf{f}^{*}(\mathbf{X}, t, \varepsilon)$ and the comparison motions (5.13) and (5.15) is (see Fig. 5.3)

$$I^{*}(\varepsilon) = \int_{B^{\pm}} \rho_R\mathbf{f}^{*}\, dV - \int_{\mathcal{S}} [\![\rho_R\mathbf{f}[(\mathbf{Z}^{*} - \mathbf{Z}) \cdot \mathbf{N}]\,]\!]\, dS + O(\varepsilon^2). \tag{5.21}$$

The second integral in this expression is due to the displacement of the surface $\mathcal{S}$. Taking the derivative of (5.21) with respect to ε and equating ε to zero, the variation of the integral is

$$\delta I = \int_{B^{\pm}} \rho_R\, \delta\mathbf{f}\, dV - \int_{\mathcal{S}} [\![\rho_R\mathbf{f}(\delta\mathbf{Z} \cdot \mathbf{N})]\!]\, dS. \tag{5.22}$$

By using the results (2.66), (2.71), and (5.18), this variation can be expressed in terms of integrals over B_t and $\mathcal{S}_t$:

$$\delta I = \int_{B_t^{\pm}} \rho\, \delta\mathbf{f}\, dV_t - \int_{\mathcal{S}_t} [\![\rho\mathbf{f}(\delta\mathbf{z} - \delta\mathbf{x}) \cdot \mathbf{n}]\!]\, dS_t. \tag{5.23}$$

As an example of the application of these results, consider the kinetic energy of the material contained in B_t[4]

$$T = \int_{B_t^{\pm}} \tfrac{1}{2}\rho\mathbf{v} \cdot \mathbf{v}\, dV_t = \int_{B^{\pm}} \tfrac{1}{2}\rho_R\mathbf{v} \cdot \mathbf{v}\, dV. \tag{5.24}$$

[4] See the treatment of this example in Sect. 2.3.

The integral of the kinetic energy with respect to time from t_1 to t_2 is

$$I = \int_{t_1}^{t_2} T\, dt = \int_{t_1}^{t_2} \int_{B^\pm} \tfrac{1}{2}\rho_R \mathbf{v}\cdot\mathbf{v}\, dV\, dt. \tag{5.25}$$

From (5.22), the variation of this integral is

$$\delta I = \int_{t_1}^{t_2} \left[\int_{B^\pm} \rho_R \mathbf{v}\cdot\dot{\boldsymbol{\eta}}\, dV - \int_{\mathcal{S}} [\![\tfrac{1}{2}\rho_R (\mathbf{v}\cdot\mathbf{v})(\delta\mathbf{Z}\cdot\mathbf{N})]\!]\, dS \right] dt, \tag{5.26}$$

where $\boldsymbol{\eta} = \delta\mathbf{x}$. To integrate the first term in this expression by parts, the motion of the image surface $\mathcal{S}$ must be accounted for. This can be done by first evaluating the derivative

$$\begin{aligned} \frac{d}{dt}\int_{B^\pm} \rho_R \mathbf{v}\cdot\boldsymbol{\eta}\, dV = &\int_{B^\pm} \rho_R \mathbf{a}\cdot\boldsymbol{\eta}\, dV + \int_{B^\pm} \rho_R \mathbf{v}\cdot\dot{\boldsymbol{\eta}}\, dV \\ &- \int_{\mathcal{S}} [\![\rho_R (\mathbf{v}\cdot\boldsymbol{\eta})(\dot{\mathbf{Z}}\cdot\mathbf{N})]\!]\, dS. \end{aligned} \tag{5.27}$$

Integrating this equation with respect to time from t_1 to t_2 and noting that $\boldsymbol{\eta}$ vanishes at t_1 and t_2 yields the desired integration by parts:

$$\begin{aligned} \int_{t_1}^{t_2} \int_{B^\pm} \rho_R \mathbf{v}\cdot\dot{\boldsymbol{\eta}}\, dV\, dt = \int_{t_1}^{t_2} \Bigg[&- \int_{B^\pm} \rho_R \mathbf{a}\cdot\boldsymbol{\eta}\, dV \\ &+ \int_{\mathcal{S}} [\![\rho_R (\mathbf{v}\cdot\boldsymbol{\eta})(\dot{\mathbf{Z}}\cdot\mathbf{N})]\!]\, dS \Bigg] dt. \end{aligned} \tag{5.28}$$

Using this result, the variation of the kinetic energy is

$$\begin{aligned} \delta T = &- \int_{B^\pm} \rho_R \mathbf{a}\cdot\delta\mathbf{x}\, dV + \int_{\mathcal{S}} [\![(\rho_R \mathbf{v}\otimes\dot{\mathbf{Z}})\mathbf{N}\cdot\delta\mathbf{x}]\!]\, dS \\ &- \int_{\mathcal{S}} [\![\tfrac{1}{2}\rho_R (\mathbf{v}\cdot\mathbf{v})\mathbf{N}\cdot\delta\mathbf{Z}]\!]\, dS \\ = &- \int_{B_t^\pm} \rho \mathbf{a}\cdot\delta\mathbf{x}\, dV_t + \int_{\mathcal{S}_t} [\![[\rho\mathbf{v}\otimes(\dot{\mathbf{z}} - \mathbf{v})]\mathbf{n}\cdot\delta\mathbf{x}]\!]\, dS_t \\ &- \int_{\mathcal{S}_t} [\![\tfrac{1}{2}\rho (\mathbf{v}\cdot\mathbf{v})\mathbf{n}\cdot(\delta\mathbf{z} - \delta\mathbf{x})]\!]\, dS_t, \end{aligned} \tag{5.29}$$

where the relations (2.71), (5.10), and (5.18) have been used.

As a second example, consider the constraint term associated with the equation of conservation of mass[5]

$$C = \int_{B^{\pm}} \pi \left(J - \frac{\rho_R}{\rho} \right) dV = \int_{B_t^{\pm}} \pi \left(1 - \frac{\rho_R}{\rho J} \right) dV_t. \tag{5.30}$$

From (5.22), the variation is

$$\delta C = \int_{B^{\pm}} \pi J \left(\operatorname{div} \boldsymbol{\eta} + \frac{r}{\rho} \right) dV = \int_{B_t^{\pm}} \pi \left(\operatorname{div} \boldsymbol{\eta} + \frac{r}{\rho} \right) dV_t, \tag{5.31}$$

where $r = \delta\rho$. To apply the divergence theorem to the first terms in the integrands, the presence of the singular surface must be taken into account. When this is done, the variation can be written

$$\delta C = \int_{B_t^{\pm}} \pi \left(-\operatorname{grad} \pi \cdot \delta\mathbf{x} + \frac{\pi}{\rho} \delta\rho \right) dV_t - \int_{\mathcal{S}_t} [\![\pi \mathbf{n} \cdot \delta\mathbf{x}]\!] \, dS_t. \tag{5.32}$$

The results discussed in this section are quite general and could be applied to any of the examples in Chap. 3. In the next section their use will be illustrated using the specific case of an elastic ideal fluid.

5.2 An Ideal Fluid Containing a Singular Surface

Consider an elastic fluid that occupies a bounded regular region B at time t_1.[6] Let it be assumed that during the time interval $[t_1, t_2]$ the volume B_t is divided into two parts B_t^+ and B_t^- by a singular surface $\mathcal{S}_t$. Hamilton's principle states:

> Among comparison motions (5.13), comparison density fields (5.14), and comparison motions (5.15) of the singular surface, the actual fields are such that
>
> $$\int_{t_1}^{t_2} [\delta(T - U) + \delta W + \delta C] \, dt = 0, \tag{5.33}$$

where

$$\begin{aligned} T &= \int_{B_t^{\pm}} \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV_t, \\ U &= \int_{B_t^{\pm}} \rho e(\rho) \, dV_t, \\ \delta W &= \int_{B_t^{\pm}} \rho \mathbf{b} \cdot \delta\mathbf{x} \, dV_t, \end{aligned} \tag{5.34}$$

[5] See the treatment of this example in Sect. 2.3.

[6] See the discussion of ideal fluids in Sect. 3.1.1.

$$\delta C = \int_{B_t^{\pm}} \left(-\text{grad}\, \pi \cdot \delta \mathbf{x} + \frac{\pi}{\rho} \delta \rho \right) dV_t - \int_{\mathcal{S}_t} [\![\pi \mathbf{n} \cdot \delta \mathbf{x}]\!]\, dS_t$$
$$+ \int_{\mathcal{S}_t} \nu [\![\rho (\delta \mathbf{z} - \delta \mathbf{x}) \cdot \mathbf{n}]\!]\, dS_t + \int_{\mathcal{S}_t} \kappa [\![\delta \rho (\dot{\mathbf{z}} - \mathbf{v}) \cdot \mathbf{n}]\!]\, dS_t .$$

Here the constraint term δC contains both the constraint (5.32) arising from the conservation of mass and the constraint (5.19) imposed by the continuity of the motion at $\mathcal{S}_t$. The scalar fields $\nu(\mathbf{z}, t)$ and $\kappa(\mathbf{z}, t)$ are Lagrange multipliers that are assumed to be continuous on $\mathcal{S}_t \times [t_1, t_2]$.

By using the result (5.22) and the expression (5.29), (5.33) can be written

$$\begin{aligned}
\int_{t_1}^{t_2} \Big\{ & - \int_{B_t^{\pm}} \rho \mathbf{a} \cdot \delta \mathbf{x}\, dV_t + \int_{\mathcal{S}_t} [\![[\rho \mathbf{v} \otimes (\dot{\mathbf{z}} - \mathbf{v})] \mathbf{n} \cdot \delta \mathbf{x}]\!]\, dS_t \\
& - \int_{\mathcal{S}_t} [\![\tfrac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{n} \cdot (\delta \mathbf{z} - \delta \mathbf{x})]\!]\, dS_t \\
& - \int_{B_t^{\pm}} \rho \frac{de}{d\rho} \delta \rho\, dV_t + \int_{\mathcal{S}_t} [\![\rho e (\delta \mathbf{z} - \delta \mathbf{x}) \cdot \mathbf{n}]\!]\, dS_t \\
& + \int_{B_t^{\pm}} \rho \mathbf{b} \cdot \delta \mathbf{x}\, dV_t + \int_{B_t^{\pm}} \left(-\text{grad}\, \pi \cdot \delta \mathbf{x} + \frac{\pi}{\rho} \delta \rho \right) dV_t \\
& - \int_{\mathcal{S}_t} [\![\pi \mathbf{n} \cdot \delta \mathbf{x}]\!]\, dS_t + \int_{\mathcal{S}_t} \nu [\![\rho (\delta \mathbf{z} - \delta \mathbf{x}) \cdot \mathbf{n}]\!]\, dS_t \\
& + \int_{\mathcal{S}_t} \kappa [\![\delta \rho (\dot{\mathbf{z}} - \mathbf{v}) \cdot \mathbf{n}]\!]\, dS_t \Big\} dt = 0.
\end{aligned} \tag{5.35}$$

If it is assumed that $\delta \mathbf{z} = \mathbf{o}$ and that the variations $\delta \mathbf{x}^{\pm}$ and $\delta \rho$ vanish on $\mathcal{S}_t$, (5.35) reduces to the case considered in Sect. 3.1.1 and yields the equation of balance of linear momentum (3.18) on $\bar{B}^+ \times [t_1, t_2]$ and on $\bar{B}^- \times [t_1, t_2]$. As a consequence, only the terms involving integrals over $\mathcal{S}_t$ remain in (5.35). Assuming that the other variations vanish and that $\delta \rho$ is arbitrary on $\mathcal{S}_t$ merely leads to the conclusion that the Lagrange multiplier $\kappa = 0$. Next, let $\delta \mathbf{x}^{\pm} = \mathbf{o}$ in (5.35) while $\delta \mathbf{z}$ is permitted to be arbitrary on $\mathcal{S}_t$. This results in the jump condition

$$[\![-\tfrac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{n} + \rho e \mathbf{n} + \nu \rho \mathbf{n}]\!] = \mathbf{o} \quad \text{on } \mathcal{S}_t \times [t_1, t_2] . \tag{5.36}$$

Finally, permitting the variations $\delta \mathbf{x}^+$ and $\delta \mathbf{x}^-$ to be arbitrary in (5.35) yields the two equations

$$\{ [\rho \mathbf{v} \otimes (\dot{\mathbf{z}} - \mathbf{v})] \mathbf{n} + \tfrac{1}{2} \rho (\mathbf{v} \cdot \mathbf{v}) \mathbf{n} - \rho e \mathbf{n} - \pi \mathbf{n} - \nu \rho \mathbf{n} \}^{\pm} = \mathbf{o}. \tag{5.37}$$

Subtracting the $-$ equation from the $+$ equation and adding the result to (5.36) results in the jump condition

$$[\![[\rho \mathbf{v} \otimes (\dot{\mathbf{z}} - \mathbf{v})]\mathbf{n} - \pi \mathbf{n}]\!] = \mathbf{o} \quad \text{on } \mathcal{S}_t \times [t_1, t_2]\,. \tag{5.38}$$

This is the *momentum jump condition*. It insures conservation of linear momentum of the material across $\mathcal{S}_t$ (see e.g. [27], pp. 104–106).

Taking the dot product of the + Eq. (5.37) with $\mathbf{v}^+$, the dot product of the − equation with $\mathbf{v}^-$, subtracting the − equation from the + equation, and using the definition of the tensor product results in the jump condition

$$[\![\rho(\mathbf{v} \cdot \mathbf{v})(\dot{\mathbf{z}} - \mathbf{v}) \cdot \mathbf{n} + [\tfrac{1}{2}\rho(\mathbf{v} \cdot \mathbf{v}) - \rho e - \pi - \rho\nu](\mathbf{v} \cdot \mathbf{n})]\!] = \mathbf{o}. \tag{5.39}$$

Taking the dot product of (5.36) with $\dot{\mathbf{z}}$ gives the jump condition

$$[\![[-\tfrac{1}{2}\rho(\mathbf{v} \cdot \mathbf{v}) + \rho e + \nu\rho](\dot{\mathbf{z}} \cdot \mathbf{n})]\!] = \mathbf{o}. \tag{5.40}$$

Summing (5.39) and (5.40) and using (5.12) results in the usual form of the *energy jump condition*

$$[\![\rho(e + \tfrac{1}{2}\mathbf{v} \cdot \mathbf{v})[(\dot{\mathbf{z}} - \mathbf{v}) \cdot \mathbf{n}] - \pi(\mathbf{v} \cdot \mathbf{n})]\!] = 0 \quad \text{on } \mathcal{S}_t \times [t_1, t_2]\,. \tag{5.41}$$

This equation insures conservation of energy of the material across $\mathcal{S}_t$ (see e.g. [27], pp. 121–123). This derivation of the energy jump condition did not include terms associated with heat conduction.

Thus Hamilton's principle yields both the linear momentum and energy jump conditions for the fluid. This procedure has been extended to mixtures of fluids and elastic materials by Batra [3] and Batra, et al. [4]. It could potentially be extended to other generalized theories of continuous media.

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